

Demand Functions for Reverse Logistic Model

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Abstract

The optimal control theory is applied to get the optimal paths of manufacturing, remanufacturing and disposal rates for reverse logistic model using Pontryagin's principle. The total cost of the system consists of the sums of the holding cost of inventory levels in two different stores, the manufacturing cost, the remanufacturing cost and the disposal cost. The general solution of the controlled system is obtained analytically. Finally, the numerical examples will be presented to demonstrate the effectiveness of the suggested control with different types of demand and return rates.

Keywords: Pontryagin's principle; Optimal control problem; Controlled system; Deterioration; Sensitivity analysis; Demand rates; Return rates.

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1 Introduction

This research is a realistic example for using an applied mathematics that treats with some associated processes of storage and manufacturing. As well as the use of numerical programming methods of solving nonlinear equations and solving differential equations, enabling us to achieve the desired goal of the decision-maker optimally. We have focused this application in the field of problems inventory systems that contain multiple goods may be manufactured for use if exposed or damaged through time has recycled later.

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Briefly, in the reverse logistic systems containing goods damage through the time or the friction. The goal of optimal control in this model is to reduce the cost to as little as possible, as well as to obtain the optimum rates to levels of storage, manufacturing and remanufacturing rate and the disposal rate, which ensures optimum cost in the lowest level. Reverse logistic is a term for logistic environments with reuse of products and materials. In these systems products can be manufactured and the returned units from the market are remanufactured and then can be considered as new items.

The demand is to be satisfied with new manufactured and the remanufactured products. Then, there is no difference between manufactured and remanufactured items. The operation of such a system is described by the return process of the used products and the disposal activity. It is a good opportunity to reuse the used items, if it is economical. Reuse products and materials is not a new phenomenon. Metal scrap brokers, waste paper recycling and deposit systems for soft drink bottles are all examples that have been around for a long time. In these cases, recovery of the used products is economically more attractive than disposal. Early in 1960, Pontryagin and his Russian colleagues published a general principle (referred to as either a "maximum" or "minimum" principle) which deals not only with continuous controls, but also with bounded and possibly discontinuous controls, Sethi and Thompson [9]. Many authors have discussed the reverse logistic models such Dobos [2,3] who has investigated the optimal inventory policies in a reverse logistic system with special structure. He assumed that the demand rate is known as a continuous function in a given planning horizon and return rate of used items is a given function. This model is considered as a generalization of the model of Holt et al. [6]. Dobos and Kistner [4] have presented the optimal production-inventory strategies for a reverse logistics system where the costs of this system consist of the linear holding costs for two stores and the convex non-decreasing manufacturing and remanufacturing costs, and there is no delay between the using and return process. The results of these examinations are that the optimal manufacturing-remanufacturing strategy is extreme. It means that the optimal trajectory contains always constraints either on the inventory levels or on the control variables. After characterizing the optimal path, a forward algorithm was determined to construct the optimal manufacturing-inventory control system. A similar result was shown by Kleber et al. [8] for a multiple reuse model with linear cost structure. The first contribution examines a reverse logistic model without disposal and have generalizes the convex model with disposal activity. This model can be interpreted as a multi-product generalization of the Arrow-Karlin model [1]. Fleischmann [5] has presented the recently emerged field of reverse logistics. Foul et

al. [7] presented the optimal control of HMMS reverse logistic model with deteriorating items. This paper developed the last one, Foul et al. [7], and presented a new formulations for the demand and return rates and got some conclusion.

The present paper is organized as follows: Section (2) is devoted to obtain the mathematical formulation of model. Also, the optimal manufacturing, remanufacturing and disposal rates and optimal inventory levels in two stores are derived in this section. The condition of non-negativity of optimal manufacturing, optimal remanufacturing and optimal disposal rates will be investigated. The solution of the controlled system will be presented analytically. In section (3) some numerical examples will be offered using different types of the demand and return rates. Finally, in section (4) conclusions of the results will be presented.

2 Reverse Logistic Model

Many books devoted with the optimal control problems, such Sethi and Thompson [9] In this section, we will be concerned with mathematical formulation and the optimal control of reverse logistic model. There is a constant delay between the using and return process. In the inventory management, the decay of the items plays an important role. In reality, some of the items are either damaged, decayed, vaporized or affected by some other factors, these are not in a perfect condition to satisfy the demand. The rate of the deterioration of an item is either constant, time dependent or stock dependent. Some items which are made of glass, China clay or ceramic break during their storage period for which the deterioration rate depends upon the size of the total inventory.

The decaying item such as photographic film, electronic goods, fruits and vegetables gradually loose their utility with time. In this paper, we investigate two stores inventory model. The demand is satisfied from the first store, where the manufactured items are stored. The returned products are collected in the second store and then remanufactured or disposed off. The costs of this system consist of the holding costs for these two stores, the manufacturing cost, the remanufacturing cost and the disposal cost. The model is represented as an optimal control problem with two state variables: the inventory status in the first store and the second store and with three control variables: the manufacturing, remanufacturing and disposal rates. The objective is to minimize the sum of the quadratic deviation from described inventory levels in two stores and from described manufacturing, remanufacturing and disposal rates.

The numerical examples include three different cases of the demand and return rates

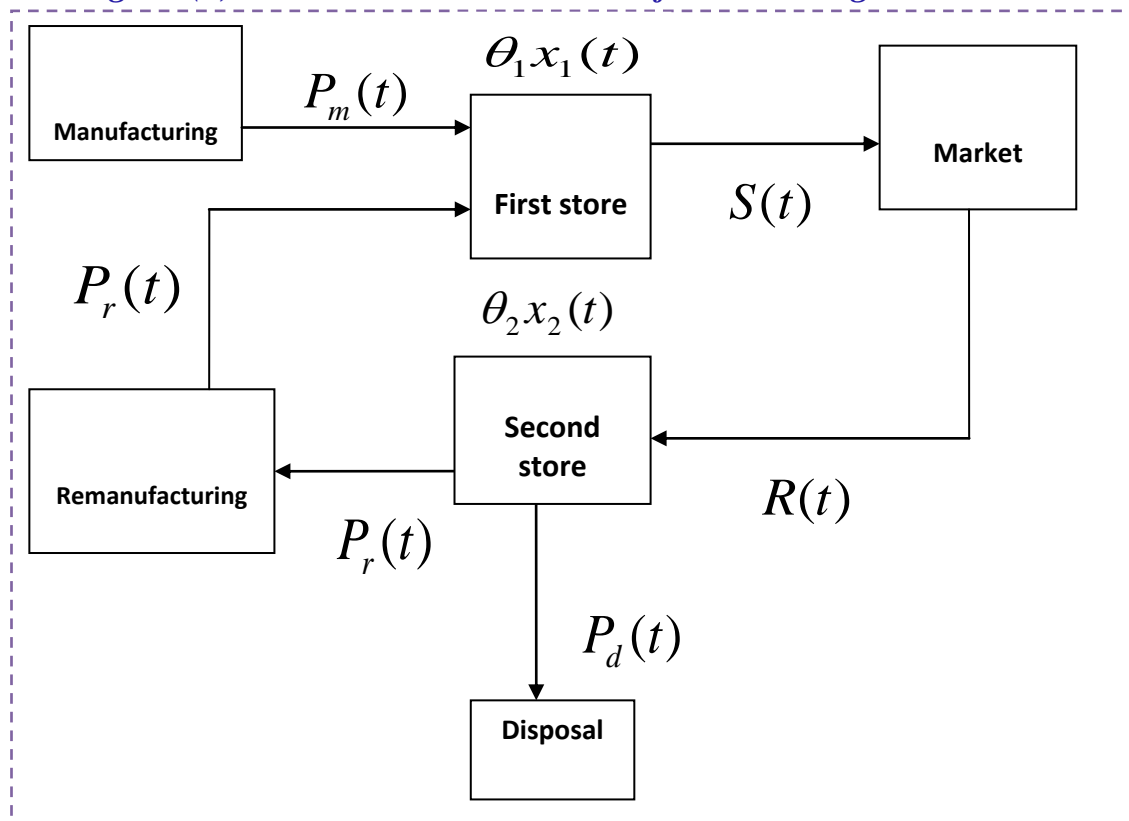
which are:

- 1) Linear functions of time.
- 2) Quadratic functions of time.
- 3) Periodic functions of time.

This subsection is devoted to discuss the model assumptions. We assumed that the deterioration rates in two stores are constant ratios of the on-hand inventory levels. Also, the inventory goal levels in two stores, the goal manufacturing rate, the goal remanufacturing rate and the goal disposal rate, are constants. There is a constant delay between using and return process. The solution of this system will be used for three different types of the demand and the return rates which will be indicated in the numerical examples.

We can explain material and cost flow of the reverse logistics model in figure (1) as follows:

Figure (1) : Material and Cost Flow of Reverse Logistic Model



The following parameters are used in the mathematical formulation of the model:

T	: The length of the planning horizon.
$S(t)$: The rate of demand at time t .
τ	: The delay of the return, it is non-negative.
r	: The proportion of return in the second store , $0 \leq r \leq 1$.
$R(t)$: The rate of return at time t .
\bar{x}_1	: The inventory goal level in the first store.
\bar{x}_2	: The inventory goal level in the second store.
\bar{P}_m	: The manufacturing goal rate.
\bar{P}_r	: The remanufacturing goal rate.
\bar{P}_d	: The disposal goal rate.
h_1	: The inventory holding cost coefficient in the first store.
h_2	: The inventory holding cost coefficient in the second store.
c_m	: The production cost coefficient for the manufacturing.
c_r	: The production cost coefficient for the remanufacturing.
c_d	: The production cost coefficient for the disposal.
θ_1	: The deterioration rate of $x_1(t)$.
θ_2	: The deterioration rate of $x_2(t)$.
$x_1(t)$: The inventory level in the first store at time t .
$x_2(t)$: The inventory level in the second store at time t .
$P_m(t)$: The rate of manufacturing at time t .
$P_r(t)$: The rate of remanufacturing at time t .
$P_d(t)$: The rate of disposal at time t .

The inventory goal levels are safety stock levels that the company wants to keep on hand. Also, the manufacturing, remanufacturing and disposal goal rates are the most efficient rates at which they are described to run the factory. Next we will use the

above assumptions and Pontryagin minimum principle to describe the problem of optimal control of a HMMS-reverse logistic model.

2.1 Optimal Control Problem

The main objective of this subsection is to derive the optimal inventory level, the optimal manufacturing rate, the optimal remanufacturing rate and the optimal disposal rate with finite horizon of time. Now, the optimal control problem is defined to determine the manufacturing, remanufacturing and disposal rates which minimize the total cost given by:

$$2J = \underset{P_m, P_r, P_d \geq 0}{\text{Min}} \int_0^T \left\{ h_1(x_1 - \bar{x}_1)^2 + h_2(x_2 - \bar{x}_2)^2 + c_m(P_m - \bar{P}_m)^2 + c_r(P_r - \bar{P}_r)^2 + c_d(P_d - \bar{P}_d)^2 \right\} dt, \quad (1)$$

Subject to:

$$\dot{x}_1 = P_m(t) + P_r(t) - S(t) - \theta_1 x_1(t), \quad (2)$$

$$\dot{x}_2 = -P_r(t) - P_d(t) + R(t) - \theta_2 x_2(t),$$

and

$$x_1(t) \geq 0, \quad x_2(t) \geq 0, \quad P_m(t) \geq 0, \quad P_r(t) \geq 0, \quad P_d(t) \geq 0, \quad (3)$$

where, $h_1 > 0, h_2 > 0, c_m > 0, c_r > 0$ and $c_d > 0$

The economic interpretation of the objective function (1) is that we want to keep the inventory levels of two stores $[x_1(t), x_2(t)]$ as close as possible to their goal levels (\bar{x}_1, \bar{x}_2) and also keep the manufacturing, the remanufacturing and the disposal rates $[P_m(t), P_r(t), P_d(t)]$ as close as possible to their goal rates $(\bar{P}_m, \bar{P}_r, \bar{P}_d)$ respectively.

The integral in (1) represents the total cost of the model, since the quadratic terms inside this integral represent the imposed penalties for having either $[x_1(t), x_2(t)]$ or $[P_m(t), P_r(t), P_d(t)]$ not being close to their corresponding goal levels (\bar{x}_1, \bar{x}_2) and goal rates $(\bar{P}_m, \bar{P}_r, \bar{P}_d)$ respectively.

Now we replace the cost integral by introducing an additional state variable which satisfies the state equation

$$2\dot{x}_0(t) = h_1(x_1 - \bar{x}_1)^2 + h_2(x_2 - \bar{x}_2)^2 + c_m(P_m - \bar{P}_m)^2 + c_r(P_r - \bar{P}_r)^2 + c_d(P_d - \bar{P}_d)^2, \quad (4)$$

with initial value and final condition $x_0(0) = 0, x_0(T) = J$, respectively.

Now, we introduce the co-state variables λ_0, λ_1 and λ_2 corresponding to the state variables x_0, x_1 and x_2 respectively. From (3) and (4), we can write the Hamiltonian function as follows:

$$H = \lambda_0 \dot{x}_0 + \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2, \quad (5)$$

In addition, to obtain the co-state equations and the Lagrange multipliers associated with the constraints (3), we form the Lagrange function as follows:

$$L = H + \mu_1(t)x_1 + \mu_2(t)x_2 + \mu_3(t)P_m + \mu_4(t)P_r + \mu_5(t)P_d, \quad (6)$$

where $\mu_1(t), \mu_2(t), \mu_3(t), \mu_4(t)$ and $\mu_5(t)$ are known as Lagrange multipliers.

These Lagrange multipliers satisfy the complementary conditions.

$$\begin{aligned} \mu_1(t) \geq 0, \quad \mu_2(t) \geq 0, \quad \mu_3(t) \geq 0, \quad \mu_4(t) \geq 0, \quad \mu_5(t) \geq 0, \\ \mu_1 x_1(t) = 0, \quad \mu_2 x_2(t) = 0, \quad \mu_3 P_m(t) = 0, \quad \mu_4 P_r(t) = 0, \quad \mu_5 P_d(t) = 0, \end{aligned} \quad (7)$$

From (6) we can easily obtain the co-state equations

$$\dot{\lambda}_i(t) = -\frac{\partial L}{\partial x_i}, \quad i = 0, 1, 2, \quad (8)$$

then,

$$\dot{\lambda}_0(t) = -\frac{\partial L}{\partial x_0} = 0, \quad \dot{\lambda}_1(t) = -\frac{\partial L}{\partial x_1}, \quad \dot{\lambda}_2(t) = -\frac{\partial L}{\partial x_2}, \quad (9)$$

The first equation of the system (9) shows that the co-state variable $\lambda_0(t)$ is constant along the optimal trajectory and the Pontryagin maximum principle requires that this constant should be negative, without loss of generality, we can choose

$$\lambda_0(t) = -1, \quad (10)$$

Substituting from (2), (5), (9) and (10) in (6) we can write L in the form

$$\begin{aligned} L = & -\frac{1}{2}[h_1(x_1 - \bar{x}_1)^2 + h_2(x_2 - \bar{x}_2)^2 + c_m(P_m - \bar{P}_m)^2 + c_r(P_r - \bar{P}_r)^2 + c_d(P_d - \bar{P}_d)^2] \\ & + \lambda_1(P_m + P_r - S - \theta_1 x_1) + \lambda_2(-P_r - P_d + R - \theta_2 x_2) \\ & + \mu_1 x_1 + \mu_2 x_2 + \mu_3 P_m + \mu_4 P_r + \mu_5 P_d, \end{aligned} \quad (11)$$

From conditions (3) and (7), we get

$$\mu_1(t) = \mu_2(t) = \mu_3(t) = \mu_4(t) = \mu_5(t) = 0 \quad (12)$$

Substituting from (11) and (12) into (9) we get

$$\dot{\lambda}_1 = \lambda_1 \theta_1 + h_1(x_1 - \bar{x}_1), \quad (13)$$

$$\dot{\lambda}_2 = \lambda_2 \theta_2 + h_2(x_2 - \bar{x}_2), \quad (14)$$

with boundary conditions

$$\lambda_i(T) = 0, \quad i = 1, 2 \quad (15)$$

To obtain the optimal manufacturing rate, the optimal remanufacturing rate and the optimal disposal rate, we differentiate the Lagrange function (11) with respect to P_m, P_r, P_d respectively and putting

$$\frac{\partial L}{\partial P_m} = 0, \quad \frac{\partial L}{\partial P_r} = 0, \quad \frac{\partial L}{\partial P_d} = 0, \quad (16)$$

From (16), we can write the co-state variables $\lambda_1(t)$ and $\lambda_2(t)$ as follows:

$$\left. \begin{aligned} \lambda_1(t) &= c_m(P_m(t) - \bar{P}_m) \\ \lambda_2(t) &= -c_d(P_d(t) - \bar{P}_d) \\ \lambda_1(t) - \lambda_2(t) &= c_r(P_r(t) - \bar{P}_r) \end{aligned} \right\} \quad (17)$$

Using the system (17) with respect to the manufacturing, remanufacturing and disposal rates, we get

$$P_m(t) = \frac{\lambda_1(t)}{c_m} + \bar{P}_m, \quad P_m(t) \geq 0 \quad (18)$$

$$P_d(t) = -\frac{\lambda_2(t)}{c_d} + \bar{P}_d, \quad P_d(t) \geq 0 \quad (19)$$

$$P_r(t) = \frac{\lambda_1(t) - \lambda_2(t)}{c_r} + \bar{P}_r, \quad P_r(t) \geq 0 \quad (20)$$

The condition of non-negativity of $P_m(t), P_r(t)$ and $P_d(t)$ can be obtained in the form:

$$c_d \bar{P}_d + c_m \bar{P}_m \geq c_r \bar{P}_r$$

This is realistic, where the economical theory says: it is not economical that the remanufacturing cost be more than the sum of manufacturing and disposal costs.

From (15), (18), (19) and (20) we get

$$P_m(T) = \bar{P}_m, \quad P_r(T) = \bar{P}_r, \quad P_d = \bar{P}_d, \quad (21)$$

From (2), (13), (14), (18), (19) and (20), we can get the following system of linear differential equations.

$$\left. \begin{aligned} \dot{x}_1 &= -\theta_1 x_1(t) + \frac{c_m + c_r}{c_m c_r} \lambda_1(t) - \frac{\lambda_2(t)}{c_r} + \bar{P}_m + \bar{P}_r - S(t) \\ \dot{x}_2 &= -\theta_2 x_2(t) + \frac{c_r + c_d}{c_r c_d} \lambda_2(t) - \frac{\lambda_1(t)}{c_r} - \bar{P}_d - \bar{P}_r + R(t) \\ \dot{\lambda}_1 &= \theta_1 \lambda_1(t) + h_1(x_1(t) - \bar{x}_1) \\ \dot{\lambda}_2 &= \theta_2 \lambda_2(t) + h_2(x_2(t) - \bar{x}_2) \end{aligned} \right\}, \quad (22)$$

We can put the system (22) in the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} -\theta_1 & 0 & \frac{c_m + c_r}{c_m c_r} & -\frac{1}{c_r} \\ 0 & -\theta_2 & -\frac{1}{c_r} & \frac{c_r + c_d}{c_r c_d} \\ h_1 & 0 & \theta_1 & 0 \\ 0 & h_2 & 0 & \theta_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} \bar{P}_m + \bar{P}_r - S(t) \\ -\bar{P}_d - \bar{P}_r + R(t) \\ -h_1 \bar{x}_1 \\ -h_2 \bar{x}_2 \end{pmatrix}, \quad (23)$$

with initial and terminal conditions

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ \lambda_1(T) \\ \lambda_2(T) \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \\ 0 \\ 0 \end{pmatrix}.$$

Then, the equations (23) can be presented in the matrix form as

$$\dot{X} = AX + F, \quad (24)$$

where,

$$A = \begin{pmatrix} -\theta_1 & 0 & \frac{c_m + c_r}{c_m c_r} & -\frac{1}{c_r} \\ 0 & -\theta_2 & -\frac{1}{c_r} & \frac{c_r + c_d}{c_r c_d} \\ h_1 & 0 & \theta_1 & 0 \\ 0 & h_2 & 0 & \theta_2 \end{pmatrix},$$

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} \bar{P}_m + \bar{P}_r - S(t) \\ -\bar{P}_d - \bar{P}_r + R(t) \\ -h_1 \bar{x}_1 \\ -h_2 \bar{x}_2 \end{pmatrix}.$$

The matrix A has four eigenvalues $\gamma_i (i=1,2,3,4)$ that can be obtained from the solution of the matrix equation $|A - \gamma_i I| = 0$, which are given by

$$\gamma_i = \pm \sqrt{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}, \quad (25)$$

where,

$$a = 1,$$

$$b = -[\theta_1^2 + \theta_2^2 + h_1 \left(\frac{c_m + c_r}{c_m c_r}\right) + h_2 \left(\frac{c_r + c_d}{c_r c_d}\right)],$$

$$c = \theta_1 \theta_2 + \theta_2^2 h_1 \left(\frac{c_m + c_r}{c_m c_r}\right) + \theta_1^2 h_2 \left(\frac{c_r + c_d}{c_r c_d}\right) + h_1 h_2 \left(\frac{c_m^2 c_r c_d + c_m c_r^2 c_d + c_m c_r c_d^2}{c_m^2 c_r^2 c_d^2}\right).$$

The corresponding eigenvectors $V_i (i=1,2,3,4)$ are given by the equation

$$[A - \gamma_i I]V_i = 0,$$

that is

$$\begin{pmatrix} -\theta_1 - \gamma_i & 0 & \frac{c_m + c_r}{c_m c_r} & -\frac{1}{c_r} \\ 0 & -\theta_2 - \gamma_i & -\frac{1}{c_r} & \frac{c_r + c_d}{c_r c_d} \\ h_1 & 0 & \theta_1 - \gamma_i & 0 \\ 0 & h_2 & 0 & \theta_2 - \gamma_i \end{pmatrix} \begin{pmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \\ v_{i4} \end{pmatrix} = 0.$$

We can easily get the following linear system of algebraic equations for the eigenvalues $\gamma_i (i=1,2,3,4)$.

$$\left. \begin{aligned} & -(\gamma_i + \theta_1)v_{i1} + \frac{c_m + c_r}{c_m c_r}v_{i3} - \frac{v_{i4}}{c_r} = 0 \\ & -(\gamma_i + \theta_2)v_{i2} + \frac{c_r + c_d}{c_r c_d}v_{i4} - \frac{v_{i3}}{c_r} = 0 \\ & h_1 v_{i1} + (\theta_1 - \gamma_i)v_{i3} = 0 \\ & h_2 v_{i2} + (\theta_2 - \gamma_i)v_{i4} = 0 \end{aligned} \right\} \quad (26)$$

By solving (26) algebraically, we get

$$\begin{aligned}
v_{i1} &= 1 \\
v_{i2} &= -\frac{-h_1(c_r + c_d + c_m) + c_m c_r \gamma_i^2 + c_m c_d \gamma_i^2 - c_m c_r \theta_1^2 - c_m c_d \theta_1^2}{c_m c_d (-\gamma_i \theta_1 - \theta_1 \theta_2 + \gamma_i^2 + \gamma_i \theta_2)}, \\
v_{i3} &= \frac{h_1}{\gamma_i - \theta_1}, \\
v_{i4} &= -\frac{-h_1(c_r + c_m) + c_m c_r \gamma_i^2 - c_m c_r \theta_1^2}{c_m (\gamma_i - \theta_1)}.
\end{aligned}$$

We need to find a particular solution X_p of system (26) and add it to the general solution X_c of the associated homogeneous system. Since the general solution consists of the sum of the general solution of the associated homogeneous and a particular solution of the system, we will use the variation of parameters technique to find the particular solution X_p .

Let us define

$$X_c = \begin{pmatrix} v_{11} & v_{21} & v_{31} & v_{41} \\ v_{12} & v_{22} & v_{32} & v_{42} \\ v_{13} & v_{23} & v_{33} & v_{43} \\ v_{14} & v_{24} & v_{34} & v_{44} \end{pmatrix} \begin{pmatrix} c_1 e^{\gamma_1 t} \\ c_2 e^{\gamma_2 t} \\ c_3 e^{\gamma_3 t} \\ c_4 e^{\gamma_4 t} \end{pmatrix}, \quad (27)$$

where $c_i (i=1,2,3,4)$ are constants, and

$$X_p = \begin{pmatrix} v_{11} & v_{21} & v_{31} & v_{41} \\ v_{12} & v_{22} & v_{32} & v_{42} \\ v_{13} & v_{23} & v_{33} & v_{43} \\ v_{14} & v_{24} & v_{34} & v_{44} \end{pmatrix} \begin{pmatrix} c_1(t) e^{\gamma_1 t} \\ c_2(t) e^{\gamma_2 t} \\ c_3(t) e^{\gamma_3 t} \\ c_4(t) e^{\gamma_4 t} \end{pmatrix}, \quad (28)$$

From (27), (28) and substitution into (24), we get

$$\begin{pmatrix} v_{11} & v_{21} & v_{31} & v_{41} \\ v_{12} & v_{22} & v_{32} & v_{42} \\ v_{13} & v_{23} & v_{33} & v_{43} \\ v_{14} & v_{24} & v_{34} & v_{44} \end{pmatrix} \begin{pmatrix} \dot{c}_1(t) e^{\gamma_1 t} \\ \dot{c}_2(t) e^{\gamma_2 t} \\ \dot{c}_3(t) e^{\gamma_3 t} \\ \dot{c}_4(t) e^{\gamma_4 t} \end{pmatrix} = \begin{pmatrix} \bar{P}_m + \bar{P}_r - S(t) \\ -\bar{P}_d - \bar{P}_r + R(t) \\ -h_1 \bar{x}_1 \\ -h_2 \bar{x}_2 \end{pmatrix}, \quad (29)$$

$$\begin{pmatrix} \zeta_1(t)e^{\gamma_1 t} \\ \zeta_2(t)e^{\gamma_2 t} \\ \zeta_3(t)e^{\gamma_3 t} \\ \zeta_4(t)e^{\gamma_4 t} \end{pmatrix} = \begin{pmatrix} v_{11} & v_{21} & v_{31} & v_{41} \\ v_{12} & v_{22} & v_{32} & v_{42} \\ v_{13} & v_{23} & v_{33} & v_{43} \\ v_{14} & v_{24} & v_{34} & v_{44} \end{pmatrix}^{-1} \begin{pmatrix} \bar{P}_m + \bar{P}_r - S(t) \\ -\bar{P}_d - \bar{P}_r + R(t) \\ -h_1 \bar{x}_1 \\ -h_2 \bar{x}_2 \end{pmatrix}, \quad (30)$$

For $(i, j = 1, 2, 3, 4)$ denote by b_{ij} the elements of the matrix V^{-1} , then

$$\begin{pmatrix} \zeta_1(t)e^{\gamma_1 t} \\ \zeta_2(t)e^{\gamma_2 t} \\ \zeta_3(t)e^{\gamma_3 t} \\ \zeta_4(t)e^{\gamma_4 t} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} \begin{pmatrix} \bar{P}_m + \bar{P}_r - S(t) \\ -\bar{P}_d - \bar{P}_r + R(t) \\ -h_1 \bar{x}_1 \\ -h_2 \bar{x}_2 \end{pmatrix}, \quad (31)$$

From (31) we get $\zeta_i(t)$ ($i = 1, 2, 3, 4$)

$$\zeta_i(t) = e^{-\gamma_i t} (b_{i1}[\bar{P}_m + \bar{P}_r - S(t)] + b_{i2}[-\bar{P}_d - \bar{P}_r + R(t)] - b_{i3}h_1\bar{x}_1 - b_{i4}h_2\bar{x}_2), \quad (32)$$

By integration (32) we get $c_i(t)$ ($i = 1, 2, 3, 4$)

$$c_i(t) = \int_0^t e^{-\gamma_i s} (b_{i1}[\bar{P}_m + \bar{P}_r - S(s)] + b_{i2}[-\bar{P}_d - \bar{P}_r + R(s)] - b_{i3}h_1\bar{x}_1 - b_{i4}h_2\bar{x}_2) ds. \quad (33)$$

From (24), (27), (28) and (33), we can get the optimal solution $X^* = X_c + X_p$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} = \begin{pmatrix} v_{11} & v_{21} & v_{31} & v_{41} \\ v_{12} & v_{22} & v_{32} & v_{42} \\ v_{13} & v_{23} & v_{33} & v_{43} \\ v_{14} & v_{24} & v_{34} & v_{44} \end{pmatrix} \left[\begin{pmatrix} c_1 e^{\gamma_1 t} \\ c_2 e^{\gamma_2 t} \\ c_3 e^{\gamma_3 t} \\ c_4 e^{\gamma_4 t} \end{pmatrix} + \begin{pmatrix} c_1(t) e^{\gamma_1 t} \\ c_2(t) e^{\gamma_2 t} \\ c_3(t) e^{\gamma_3 t} \\ c_4(t) e^{\gamma_4 t} \end{pmatrix} \right], \quad (34)$$

therefore,

$$x_1^*(t) = \sum_{i=1}^4 e^{\gamma_i t} \left(\int_0^t e^{-\gamma_i s} (b_{i1}[\bar{P}_m + \bar{P}_r - S(s)] + b_{i2}[-\bar{P}_d - \bar{P}_r + R(s)] - b_{i3}h_1\bar{x}_1 - b_{i4}h_2\bar{x}_2) ds + c_i \right), \quad (35)$$

$$\begin{aligned} x_2^*(t) &= \sum_{i=1}^4 -\frac{-h_1(c_r + c_d + c_m) + c_m c_r \gamma_i^2 + c_m c_d \gamma_i^2 - c_m c_r \theta_1^2 - c_m c_d \theta_1^2}{c_m c_d (-\gamma_i \theta_1 - \theta_1 \theta_2 + \gamma_i^2 + \gamma_i \theta_2)} \\ &\times e^{\gamma_i t} \left(\int_0^t e^{-\gamma_i s} (b_{i1}[\bar{P}_m + \bar{P}_r - S(s)] + b_{i2}[-\bar{P}_d - \bar{P}_r + R(s)] - b_{i3}h_1\bar{x}_1 - b_{i4}h_2\bar{x}_2) ds + c_i \right), \end{aligned} \quad (36)$$

$$\lambda_1(t) = \sum_{i=1}^4 \frac{h_1}{\gamma_i - \theta_1} e^{\gamma_i t} \left(\int_0^t e^{-\gamma_i s} (b_{i1}[\bar{P}_m + \bar{P}_r - S(s)] + b_{i2}[-\bar{P}_d - \bar{P}_r + R(s)] - b_{i3}h_1\bar{x}_1 - b_{i4}h_2\bar{x}_2) ds + c_i \right)$$

(37)

$$\lambda_2(t) = \sum_{i=1}^4 -\frac{-h_1(c_r + c_m) + c_m c_r \gamma_i^2 - c_m c_r \theta_1^2}{c_m(\gamma_i - \theta_1)} e^{\gamma_i t} \left(\int_0^t e^{-\gamma_i s} (b_{i1}[\bar{P}_m + \bar{P}_r - S(s)] \right. \\ \left. + b_{i2}[-\bar{P}_d - \bar{P}_r + R(s)] - b_{i3}h_1\bar{x}_1 - b_{i4}h_2\bar{x}_2) ds + c_i \right) \quad (38)$$

where the arbitrary constants $c_i (i=1,2,3,4)$ are determined using initial conditions $x_1(0) = x_{10}$, $x_2(0) = x_{20}$ and terminal conditions $\lambda_1(T) = 0$, $\lambda_2(T) = 0$.

These constraints should enable us to find the optimal solution to our problem. The optimal manufacturing, the optimal remanufacturing and the optimal disposal rates can be written as

$$P_m^*(t) = \max\left[0, \frac{\lambda_1(t)}{c_m} + \bar{P}_m\right], \quad (39)$$

$$P_r^*(t) = \max\left[0, \frac{\lambda_1(t) - \lambda_2(t)}{c_r} + \bar{P}_r\right], \quad (40)$$

and

$$P_d^*(t) = \max\left[0, -\frac{\lambda_2(t)}{c_d} + \bar{P}_d\right], \quad (41)$$

Therefore, the system of equations from (39), (40) and (41) gives the optimal solution of the controlled system (22).

3 Numerical Examples

In this section, we will present some numerical examples for the previous optimal solution. The numerical examples of the system will be discussed for three different types of the demand and return rates:

1. Both of the demand and return rates are linear functions of time:

$$S(t) = (w_1 + w_2)t, \quad R(t) = rw_2(t - \tau) + \alpha.$$

2. Both of the demand and return rates are quadratic functions of time:

$$S(t) = w_1 + w_2 t^2, \quad R(t) = rw_2(t^2 - \tau) + \alpha.$$

3. Both of the demand and return rates are periodic functions of time:

$$S(t) = 1 + \cos(t), \quad R(t) = r[1 + \cos(t - \tau)] + \alpha.$$

where w_1, w_2 and α are positive constants.

Table (1) presents the values of system parameters and the initial states which are used in the numerical examples as follow:

Table (1): Initial states of system parameters

c_m	c_r	c_d	\bar{P}_m	\bar{P}_r	\bar{P}_d	h_1	h_1	α	r
9	10	15	15	13	11	10	12	15	0.8
T	\bar{x}_1	\bar{x}_2	x_{10}	x_{20}	θ_1	θ_2	w_1	w_2	τ
5	40	25	10	15	0.02	0.03	5	1	1

We can summarize the values of the optimal inventory levels, the optimal manufacturing rate, the optimal remanufacturing rate and the optimal disposal rate at end of the planning horizon period, and the objective function in Table 2 as follows:

Table (2) : Optimal solution

Demand and Return Rates	$x_1^*(T)$	$x_2^*(T)$	$P_m^*(T)$	$P_r^*(T)$	$P_d^*(T)$	J^*
Linear	39	20	15	13	11	4156
Quadratic	43	30	15	13	11	4540
Periodic	58	23	15	13	11	6731

From previous examples and from the results of Table 2, we have found that:

- The optimal inventory levels are different in three types of the demand and return rates. But the maximum value of these levels are achieved in the periodic case in the first store. The minimum value of these levels are achieved in the linear case in the second store.
- The optimal manufacturing, the optimal remanufacturing and the optimal disposal rates are the same values in three types for the demand and return rates.
- The aim is to minimize the total cost and by comparing the total cost in all three types of demand and return rates, we have found that the linear rates achieved the minimum value of the total cost at end of the planning horizon period , $T=5$, and the

maximum value of the total cost is achieved in the periodic rates.

4 Conclusions

The problem of the optimal control of reverse logistic model is studied. The condition of non-negativity of the optimal rates and how to agree with the economic theory is discussed. The behavior of the optimal inventory rates and the total cost at the end of planning period are investigated with different types of the demand and return rates. Note that: The inventory levels in two cases reflect the reason of decreasing or increasing the total cost.

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