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**Global existence and exponential stability for some
dynamic contact problems**

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Dedication

I dedicate this work to ...

my great parents

my sister and brothers

all my family

my friends

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LIST OF SYMBOLS

$\mathbb{R}$	: the set of all real numbers.
$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$	: the extended real line.
$\mathbb{R}^d$	: the Euclidean space of dimension d .
$\mathbb{S}^d$	: the space of symmetric tensors of second-order on $\mathbb{R}^d$.
$\mathbb{R}_+$	: the set of all nonnegative real numbers.
Ω	: an open bounded and connected set in the space $\mathbb{R}^d$ with Lipschitz boundary.
$\Gamma = \partial\Omega$	: the boundary of Ω such that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where Γ_1, Γ_2 and Γ_3 are three disjoint measurable parts with $\text{meas } \Gamma_1 > 0$.
Γ_i	: a part of Γ which is the boundary of Ω .
$f', f'', f^{(i)}$	: the derivatives of the function denoted by f with respect to the time .
f_t, f_{tt}	: are partial derivative with respect to the time .
$u \cdot v$	: the canonical inner product of the vectors $u, v \in \mathbb{R}^d$.
$\sigma \cdot \tau$	: the canonical inner product of the tensors $\sigma, \tau \in \mathbb{S}^d$.
ν	: the unit outward normal to the boundary Γ .
u_ν	: the normal component of u and we have $u_\nu = u \cdot \nu$.
u_τ	: the tangential component of u and we have $u_\tau = u - u_\nu \nu$.
σ_ν	: the normal component of σ and we have $\sigma_\nu = \sigma \nu \cdot \nu$.
σ_τ	: the tangential component of σ and we have $\sigma_\tau = \sigma \nu - \sigma_\nu \nu$.
Div	: divergence operator for tensors.
∇	: gradient operator.
I	: identity operator.
$L^p(\Omega)$	: the space of Lebesgue measurable and p -integrable functions for $1 \leq p < \infty$.
$L^\infty(\Omega)$	: the space of Lebesgue measurable functions u in Ω such that $\exists c > 0 : u < c, a.e. \text{ in } \Omega$.
$C^m(\overline{\Omega})$	: the space of continuous functions whose derivatives of order from 1 to m are continuous in $\overline{\Omega}$.
$C_0^\infty(\Omega)$	: the space of all infinitely differentiable functions whose support in Ω is compact.
$W^{k,p}(\Omega)$	: the Sobolev space of all functions which are p -integrable on Ω and their weak derivatives of order less than k are also p -integrable on Ω .

$H^k(\Omega)$	$\equiv W^{k,2}(\Omega).$
$W_0^{k,p}(\Omega)$	: the closure of the space $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega).$
$H_0^k(\Omega)$	$\equiv W_0^{k,2}(\Omega).$
H	$= \left\{ v = (v_1, \dots, v_d)^T : v_i \in L^2(\Omega), 1 \leq i \leq d \right\} = L^2(\Omega)^d.$
$\mathcal{H}$	$= \left\{ \tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega), 1 \leq i, j \leq d \right\}.$
H_1	$= \left\{ v = (v_1, \dots, v_d)^T : v_i \in H^1(\Omega), 1 \leq i \leq d \right\}.$
$\mathcal{H}_1$	$= \left\{ \tau \in \mathcal{H} : \text{Div } \tau \in H \right\}.$
V	$= \left\{ v \in H^1(\Omega)^d : v = 0 \text{ a.e. on } \Gamma_1 \right\}.$
X	: denotes a Hilbert space with the inner product $(\cdot, \cdot)_X$ or a Banach space with norm $\ \cdot\ _X.$
X'	: denotes the dual of $X.$
$\langle u, v \rangle_{X \times X'}$	: the duality paring between the spaces X and $X'.$
V'	: the dual of $V.$
$\langle u, v \rangle_{V \times V'}$	: the duality paring between the spaces V and $V'.$
$\langle u, v \rangle_H$	: inner product of $H.$
$C^m([0, T]; X)$	$= \left\{ v \in C([0, T]; X) : v^{(j)} \in C([0, T]; X), j = 1, \dots, m \right\}.$
$L^p(0, T; X)$	$= \left\{ v : [0, T] \rightarrow X \text{ measurable} : \ v\ _{L^p(0, T; X)} < \infty \right\}.$
$W^{k,p}(0, T; X)$	$= \left\{ v \in L^p(0, T; X) : \ v^{(j)}\ _{L^p(0, T; X)} < \infty, \forall j \leq k \right\}.$
$H^k(0, T; X)$	$\equiv W^{k,2}(0, T; X) .$
r_+	$= \max\{r, 0\} : \text{positive part of } r.$
a.e.	: almost everywhere.
$\forall$	: for all.
$\mathcal{L} \sim E$	: $\mathcal{L}$ and E are equivalent.

LIST OF PAPERS

Chapter 1

- I. Ouakil, B. Benabderrahmane and Y. Boukhatem. Well-posedness and exponential stability of a dynamic frictionless contact problem with normal compliance and infinite memory. *Applied Mathematics-A Journal of Chinese Universities*. Accepted, 2023.

Chapter 2

- I. Ouakil, B. Benabderrahmane, Y. Boukhatem and B. Feng. On a dynamic frictional contact problem with normal damped response and long-term memory. *Mathematics and Mechanics of Solids*, 29(5):924-943, 2024. Doi: 10.1177/10812865231218458.
- I. Ouakil, B. Benabderrahmane, Y. Boukhatem and B. Feng. Exponential stabilization of a dynamic frictional contact problem for viscoelastic materials with normal damped response and infinite memory. *Mathematical Control and Related Fields*, 2024. Doi: 10.3934/mcrf.2024011.

Chapter 3

- I. Ouakil, B. Benabderrahmane and Y. Boukhatem. Global existence and exponential stability of a dynamic frictionless contact problem with infinite memory and adhesion. *International Journal of Applied and Computational Mathematics*, In Production, 2023. Doi: 10.21203/rs.3.rs-2664363/v1.

Chapter 4

- I. Ouakil, B. Benabderrahmane and Y. Boukhatem. Global existence and exponential stability for a type III thermo-viscoelastic frictional contact problem with infinite memory and normal damped response. *Mathematics in Engineering, Science and Aerospace (MESA)*. In production, 2023.

GENERAL INTRODUCTION

The phenomena of contact involving deformable bodies is of particular interest in a variety of industries and everyday life. It is considered as the study of deformation of bodies that touch each other at one or more points. Typical examples of contact problems are the contact wheel rail, the automotive industry more precisely the adhesion between tires and curbs, crash tests. Civil engineering is another field of application of contact problems in the case where we estimate the forces exerted by the materials of bridges and skyscrapers. In other side, contact phenomena are included in metal working where we shape and reshape metals to make useful objects, parts and assemblies.

The studies in the area of contact mechanics started in 1882 with Hertz's publication [37]. Signorini was the first to give a formulation of the problem of unilateral contact with friction in 1933 [71]. This was followed by Fichera who solved the Signorini problem using techniques of variational elliptic inequalities in [27]. The works of Duvaut and Lions [48] represent the first treatment of contact problems using variational formulations. They also proved some existence and uniqueness results. Among the early relevant contributions to contact problems, the following can be enumerated Duvaut [20], Necas et al. [54], Jarusek [39], Cocu [14] and Kato [41].

There are three kinds of processes: dynamic process modeled by the equation of motion see [15, 23]. Quasistatic processes modeled by an equilibrium equation and at least one of the derivatives of the unknowns displacement and stress field is found in the rest of equations or in the boundary conditions see [2, 50] and static in the opposite case see [21, 40]. Diverse contact conditions have been taken into account to model contact problems, such as the normal damped response condition. It represents the contact with a foundation which is lubricated. Further, a reactive normal pressure depending on normal velocity is assigned by normal damped response on contact surface see for instance [5, 47, 51, 67] and the references therein. Another condition is also employed the so-called normal compliance condition. The term "normal compliance" was first considered in [42, 43]. In [42], Klarbring et al. showed the result of existence of a weak solution to a frictional contact with normal compliance. This condition was first introduced by Martins and Oden in [55] in the study of dynamic problems. It characterizes a deformable foundation. Further, a reactive normal pressure depending on the interpenetration of both the asperities on the body surface and those on the foundation is assigned by normal compliance condition. We refer the reader to [34, 40, 74, 76, 77].

In the presence of frictional effects, contact problem between deformable bodies will be frictional. Classical results for a frictional contact problem are found in the papers written by F.W. Carter

and H. Fromm. They independently used Coulomb's dry friction law to present creep versus creep force relation for not only a cylinder on a plane but also for two cylinders in steady rolling contact see [44]. Frictional contact problems are concerned with a large range of applications such as the bouncing on a surface of a rubber ball, skydiver, making fire, climber plants etc. Because of the importance of contact problems with friction, a significant effort has been made in their modeling, their numerical simulations and mathematical analysis [4,5,68].

Processes of adhesion are of major importance in many industrial settings which have non-metallic parts glued together. For the fundamental modeling, we refer the reader to [25,26]. Analysis for adhesive contact problems can be found in [13,46,69] and in the monographs [70,72]. The surface internal variable or the bonding field is denoted by β . It was first introduced by M. Fremond (see [25], [24]). The bonding field describes the fractional density of active bonds on the contact surface. The values of the surface internal variable β are between the two constants one and zero. According to the different values taken by the variable β , we distinguish three cases: in the case where β equals to 1, then all bonds are active and adhesion is complete. If β equals to zero, all bonds are inactive, severed and no adhesion exists. In the case where $0 < \beta < 1$, then only a part of the bond is active and so adhesion is partial see for instance [12,17,33,46,61,79].

The basis of the thermoelasticity model is the Fourier's law given by

$$q = -\kappa \nabla \theta, \quad (1)$$

where q is the heat flux vector, $\nabla \theta$ is the temperature gradient and κ denotes the heat conductivity.

The un-physical property presented by (1) shows that if a surprising temperature changes at some point. This will be felt everywhere instantly. To finish this paradox, the two researchers Green and Naghdi introduced models of thermoelasticity of type II and III ([30,31]). The nature of systems of thermoelasticity of type III is dissipative and this type of thermoelasticity has been studied intensively ([45,49,52]).

The goal of this thesis is to present some dynamic viscoelastic contact problems between a deformable body and foundation. We deal with the global existence of solutions and their asymptotic behavior. In this way, we investigate several nonlinear dynamic contact problems. The contact is described with a normal compliance, normal damped response and a version of a normal compliance with adhesion. This thesis is based mainly on results of nonlinear semigroup, operator theory, fixed point argument and Lyapunov stability. A modeling of process of contact is also presented more precisely we provide a description of the physical setting, the material's behavior and the contact conditions. Our purpose is to establish the global existence and exponential decay results.

This thesis is divided into four chapters. A brief description follows.

- ✎ The first chapter deals with a contact problem in viscoelasticity with a normal compliance. The constitutive law is assumed to be with infinite long memory term. We prove the global existence of solution using nonlinear semigroup theory then we establish the exponential stability by applying energy method. The global existence and stability result for this kind of problems are first studied in this chapter. The obtained results are also the subject of an article accepted for publication in [58].
- ✎ Chapter 2 is devoted to studying the global existence of a unique solution to a contact problem with normal damped response and its exponential stability. In the case of finite memory, we apply a theory of nonlinear operator, a fixed point argument and an extension procedure to deal with the issue of global existence. Then, we show that the obtained solution decays exponentially using Lyapunov functional with some modifications related to the nature of the studied contact problem. In the case of infinite memory, we prove the global existence of the solution by utilizing nonlinear semigroup theory and by constructing a Lyapunov functional, we establish the exponential decay of solution. From what we know, there are no results for the considered systems. The issues included in this chapter have been published in [59,60].
- ✎ In Chapter 3, a viscoelastic contact problem with adhesion is considered. We introduce the surface internal variable β . We show that the solution exists for all time using nonlinear semigroup method and we derive the exponential decay rate. In the presence of adhesion in the current chapter, we can say that our results extend the various well-posedness and stability results obtained in Chapter 1. The outcomes of the current chapter have been submitted for publication in [57].
- ✎ In Chapter 4, we study a thermoviscoelastic contact problem with friction. A thermoelasticity of type III is considered. A thermal effect is involved in the constitutive law with long-term memory. The global existence of solution is established by exploiting nonlinear semigroup theory combined with arguments of nonlinear operators. By applying energy method, we obtain the exponential decay of energy solution. This work generalizes the results obtained in Chapter 2 in the case of the presence of thermal effect. The obtained results are also a subject of an article submitted for publication in [56].

We end our thesis with a general conclusion in which we discuss the achieved results and give perspectives on various future works, followed by an appendix where we present basic notation and necessary results of contact mechanics, some results of nonlinear analysis, some fundamental inequalities and mathematical modeling of the process of contact.

A DYNAMIC CONTACT PROBLEM WITH INFINITE MEMORY AND NORMAL COMPLIANCE

In this chapter, we show the existence and uniqueness of a global solution for a contact problem with normal compliance. The proof of global existence is based on nonlinear semigroup theory. Next, we investigate the exponential stability of the system by using energy method and under suitable assumptions on problem data.

Contents in Brief

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1.1 Problem statement

In this work, we look into a solution for the following nonlinear frictionless contact problem

$$\sigma = A\varepsilon(u') + G\varepsilon(u) - \int_0^{+\infty} g(s)B\varepsilon(u(t-s))ds \quad \text{in } \Omega \times (0, \infty), \quad (1.1)$$

$$\rho u'' = \text{Div} \sigma + f_0 \quad \text{in } \Omega \times (0, \infty), \quad (1.2)$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, \infty) \quad (1.3)$$

$$\sigma \nu = f_2 \quad \text{on } \Gamma_2 \times (0, \infty), \quad (1.4)$$

$$-\sigma_\nu = p_\nu(u_\nu) \quad \text{on } \Gamma_3 \times (0, \infty) \quad (1.5)$$

$$\sigma_\tau = 0 \quad \text{on } \Gamma_3 \times (0, \infty), \quad (1.6)$$

$$u(-t) = u_0(t) \quad \text{in } \Omega \times [0, \infty), \quad (1.7)$$

$$u'(0) = u_1 \quad \text{in } \Omega. \quad (1.8)$$

1.2 Assumptions

To study contact problems and to formulate global existence and stability results, we shall need some assumptions in all parts of our thesis.

$$\left\{ \begin{array}{l} \text{(a) } A : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d \text{ is a nonlinear viscosity operator;} \\ \text{(b) There exists } L_A > 0 \text{ such that} \\ \|A(x, \varepsilon_1) - A(x, \varepsilon_2)\| \leq L_A \|\varepsilon_1 - \varepsilon_2\| \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(c) There exists } m_A > 0 \text{ such that} \\ (A(x, \varepsilon_1) - A(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_A \|\varepsilon_1 - \varepsilon_2\|^2 \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(d) } x \mapsto A(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega; \\ \text{(e) } A(x, 0) = 0 \text{ a.e. } x \in \Omega. \end{array} \right. \quad (1.9)$$

$$\left\{ \begin{array}{l} \text{(a) } G : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d \text{ is a nonlinear elasticity operator;} \\ \text{(b) There exists } L_G > 0 \text{ such that} \\ \|G(x, \varepsilon_1) - G(x, \varepsilon_2)\| \leq L_G \|\varepsilon_1 - \varepsilon_2\| \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(c) There exists } m_G > 0 \text{ such that} \\ (G(x, \varepsilon_1) - G(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_G \|\varepsilon_1 - \varepsilon_2\|^2 \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(d) } x \mapsto G(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega; \\ \text{(e) } G(x, 0) = 0 \text{ a.e. } x \in \Omega. \end{array} \right. \quad (1.10)$$

$$\left\{ \begin{array}{l} \text{(a) } B : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d \text{ is a nonlinear operator;} \\ \text{(b) There exists } L_B > 0 \text{ such that} \\ \|B(x, \varepsilon_1) - B(x, \varepsilon_2)\| \leq L_B \|\varepsilon_1 - \varepsilon_2\| \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d \text{ a.e. } x \in \Omega; \\ \text{(c) There exists } m_B > 0 \text{ such that} \\ (B(x, \varepsilon_1) - B(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_B \|\varepsilon_1 - \varepsilon_2\|^2 \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d \text{ a.e. } x \in \Omega; \\ \text{(d) } x \mapsto B(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega; \\ \text{(e) } B(x, 0) = 0 \text{ a.e. } x \in \Omega. \end{array} \right. \quad (1.11)$$

$$\left\{ \begin{array}{l} \text{(a) } p_\nu : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R} \text{ is a nonlinear contact function;} \\ \text{b) There exists a constant } L_\nu > 0 \text{ such that} \\ |p_\nu(x, r_1) - p_\nu(x, r_2)| \leq L_\nu |r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\ \text{c) } (p_\nu(x, r_1) - p_\nu(x, r_2))(r_1 - r_2) \geq 0 \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\ \text{d) The mapping } x \mapsto p_\nu(x, r) \text{ is Lebesgue measurable on } \Gamma_3 \text{ for any } r \in \mathbb{R}; \\ \text{e) } p_\nu(x, r) = 0 \quad \forall r \leq 0, \text{ a.e. } x \in \Gamma_3. \end{array} \right. \quad (1.12)$$

We suppose that the body forces and the tractions satisfy the following regularity assumption

$$f_0 \in W^{1,1}(\mathbb{R}_+; H), \quad f_2 \in W^{1,1}(\mathbb{R}_+; L^2(\Gamma_2)^d). \quad (1.13)$$

The mapping $f : \mathbb{R}_+ \rightarrow V'$ is given by

$$(f(t), v)_{V' \times V} = \int_{\Omega} f_0(t) \cdot v dx + \int_{\Gamma_2} f_2(t) \cdot v da, \quad (1.14)$$

for all $v \in V, t \text{ a.e. } \in \mathbb{R}_+$.

From assumption (1.13), we have

$$f \in W^{1,1}(\mathbb{R}_+; V'). \quad (1.15)$$

(G1) The function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is nonincreasing, class C^1 function satisfying

$$g(0) > 0, \quad g_0 = \int_0^\infty g(s) ds < \frac{m_G}{L_B}. \quad (1.16)$$

(G2) There exists a constant $\alpha > 0$ such that

$$g'(t) \leq -\alpha g(t), \quad \forall t \geq 0. \quad (1.17)$$

(G3) There exists a constant $\zeta_1 > 0$ such that

$$\|f'(t)\|_{V'} \leq \zeta_1 \|f(t)\|_{V'}, \quad \forall t \geq 0, \quad \forall x \in \Omega. \quad (1.18)$$

Remark 1.1. For functions $f \in W^{1,1}(\mathbb{R}_+; V')$ satisfying (G3), we can take

$$f(t) = c_1 e^{-\zeta_1 t} h,$$

where c_1 is a constant and the function h belongs to V' .

In what follows, $(\text{number of system})_i, i = 1, 2, 3, 4, 5, 6$ refers to the i th equation in the considered system.

1.3 Global existence

Our main result, in this section, is dedicated to studying the global existence of a unique solution to the preceding system.

Let define the auxiliary variable denoted by η as in [16]

$$\eta(x, s, t) = u(x, t) - u(x, t - s), \quad x \in \Omega, \quad t, s \geq 0. \quad (1.19)$$

From (1.19), we obtain

$$\int_0^\infty g(s) B\varepsilon(u(t - s)) ds = \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s)) ds.$$

Then, system (1.1)-(1.8) becomes

$$\begin{aligned} \rho u'' - \text{Div} \sigma &= f_0 && \text{in } \Omega \times (0, \infty), \\ \sigma &= A\varepsilon(u') + G\varepsilon(u) - \int_0^\infty g(s) B\varepsilon(u - \eta(s)) ds && \text{in } \Omega \times (0, \infty), \\ \eta_t &= u' - \eta_s && \text{in } \Omega \times (0, \infty) \times (0, \infty), \\ u(x, t) &= 0 && \text{on } \Gamma_1 \times (0, \infty), \\ \eta(x, s, t) &= 0 && \text{on } \Gamma_1 \times (0, \infty) \times (0, \infty), \\ \sigma \nu &= f_2 && \text{on } \Gamma_2 \times (0, \infty), \\ -\sigma_\nu &= p(u_\nu) && \text{on } \Gamma_3 \times (0, \infty), \\ \sigma_\tau &= 0 && \text{on } \Gamma_3 \times (0, \infty), \\ u(-t) &= u_0(t) && \text{in } \Omega \times [0, \infty), \\ u'(0) &= u_1 && \text{in } \Omega, \\ \eta(s, 0) &= u_0(0) - u_0(s) = \eta_0(s) && \text{for } s \in (0, \infty). \end{aligned} \quad (1.20)$$

First, we introduce the state vector $U = (u, v, \eta)^T$ with $v = u'$. Then, we consider the following Hilbert space $\mathcal{H}$

$$\mathcal{H} = V \times H \times \mathcal{M}. \quad (1.21)$$

Here, the set $\mathcal{M}$ is given by

$$\mathcal{M} = \left\{ \chi : \mathbb{R}_+ \rightarrow V, \int_0^{+\infty} g(s) \|\chi(s)\|_V^2 ds < +\infty \right\}, \quad (1.22)$$

with the inner product

$$\langle \chi, \phi \rangle_{\mathcal{M}} = \int_0^{+\infty} g(s) \langle \varepsilon(\chi(s)), \varepsilon(\mathcal{W}(s)) \rangle_{\mathcal{H}} ds. \quad (1.23)$$

The inner product associated to $\mathcal{H}$ is

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} = \langle u, \tilde{u} \rangle_V + \langle v, \tilde{v} \rangle_H + \langle \eta, \tilde{\eta} \rangle_{\mathcal{M}}, \quad (1.24)$$

for every $U = (u, v, \eta)^T$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\eta})^T$ in $\mathcal{H}$.

Assuming that σ is a smooth function (say $\sigma \in C^1(\bar{\Omega}; \mathbb{S}^d)$), for $U = (u, v, \eta)^T$ with $v = u'$, we can rewrite the problem (1.20) as

$$\begin{cases} U_t(t) + \mathcal{A}U(t) = F(t), & \forall t > 0, \\ U(0) = U_0 = (u_0, u_1, \eta_0)^T, \end{cases} \quad (1.25)$$

here the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$, its domain $\mathcal{D}(\mathcal{A})$ and the mapping $F(t)$ are respectively given as follows

$$\mathcal{A} \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} -v \\ -\frac{1}{\rho} \text{Div}(A\varepsilon(v) + G\varepsilon(u)) + \frac{1}{\rho} \text{Div}(\int_0^\infty g(s) B\varepsilon(u - \eta(s)) ds) \\ \eta_s - v \end{pmatrix}, \quad (1.26)$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} (u, v, \eta)^T \in \mathcal{H}; \ v \in V; \ \eta_s \in \mathcal{M}; \\ \sigma\nu = f_2 \text{ on } \Gamma_2; \\ -\sigma_\nu = p_\nu(u_\nu) \text{ on } \Gamma_3; \\ \sigma_\tau = 0 \text{ on } \Gamma_3 \end{array} \right\}, \quad (1.27)$$

and

$$F(t) = \begin{pmatrix} 0 \\ \frac{f_0(t)}{\rho} \\ 0 \end{pmatrix}. \quad (1.28)$$

The result of the global existence is guaranteed by the following

Theorem 1.1. *Under the assumptions (G1) and (1.9)-(1.12), for any initial datum $U_0 \in \overline{\mathcal{D}(\mathcal{A})}$ and $F \in L^1(\mathbb{R}_+; \mathcal{H})$, there exists a unique mild solution of the system (1.25) satisfying*

$$U \in C(\mathbb{R}_+; \mathcal{H}). \quad (1.29)$$

Proof. To prove Theorem 1.1, we verify the m-accretivity of the operator $\mathcal{A} + \omega_{\mathcal{A}}I$ for some $\omega_{\mathcal{A}} > 0$, see Definition A.3. It follows from Theorem A.3, $U_0 \in \mathcal{D}(\mathcal{A}) \subset \overline{\mathcal{D}(\mathcal{A})}$ and the regularity $F \in L^1(\mathbb{R}_+; \mathcal{H})$ that there exists a unique mild solution U of the system (1.20) satisfying the regularity (1.29).

Let $U_i = (u_i, v_i, \eta_i) \in \mathcal{D}(\mathcal{A})$, $i = 1, 2$. So, we have

$$\begin{aligned}
 \langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} &= -\langle v_1 - v_2, u_1 - u_2 \rangle_V \\
 &\quad - \left\langle \frac{1}{\rho} \operatorname{Div} (A\varepsilon(v_1)) - A\varepsilon(v_2), v_1 - v_2 \right\rangle_H \\
 &\quad - \left\langle \frac{1}{\rho} \operatorname{Div} (G\varepsilon(u_1) - G\varepsilon(u_2)), v_1 - v_2 \right\rangle_H \\
 &\quad + \left\langle \frac{1}{\rho} \operatorname{Div} \int_0^\infty g(s) B\varepsilon(u_1 - \eta_1(s)) ds, v_1 - v_2 \right\rangle_H \\
 &\quad - \left\langle \frac{1}{\rho} \operatorname{Div} \int_0^\infty g(s) B\varepsilon(u_2 - \eta_2(s)) ds, v_1 - v_2 \right\rangle_H \\
 &\quad - \langle -\eta_{1s} + \eta_{2s} + v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} \\
 &\quad + \omega_{\mathcal{A}} \langle U_1 - U_2, U_1 - U_2 \rangle_{\mathcal{H}}.
 \end{aligned}$$

Using formula of Green and boundary conditions to get

$$\begin{aligned}
 \langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} &= -\langle v_1 - v_2, u_1 - u_2 \rangle_V \\
 &\quad + \langle A\varepsilon(v_1) - A\varepsilon(v_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\
 &\quad + \langle G\varepsilon(u_1) - G\varepsilon(u_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\
 &\quad - \left\langle \int_0^\infty g(s) B\varepsilon(u_1 - \eta_1(s)) ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}} \\
 &\quad + \left\langle \int_0^\infty g(s) B\varepsilon(u_2 - \eta_2(s)) ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}} \\
 &\quad + \int_{\Gamma_3} (p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})) (v_{1\nu} - v_{2\nu}) da \\
 &\quad - \frac{1}{2} \int_0^\infty g'(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds \\
 &\quad - \langle v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} + \omega_{\mathcal{A}} \|U_1 - U_2\|_{\mathcal{H}}^2.
 \end{aligned}$$

The use of inequalities of Cauchy- Schwarz and Young, (G1), (b) of (1.10), (1.12) and (1.11), (c) of (1.9) and (A.6) leads to

$$\begin{aligned}
 \langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} &\geq - \left(\frac{3}{2\delta_*} - m_A \right) \|v_1 - v_2\|_V^2 \\
 &\quad - \delta_* \left(1 + L_G^2 + g_0^2 L_B^2 + L_\nu^2 C_0^4 \right) \|u_1 - u_2\|_V^2 \\
 &\quad - \delta_* \left(L_B^2 g_0 + g_0 \right) \int_0^\infty g(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds \\
 &\quad + \omega_{\mathcal{A}} \|U_1 - U_2\|_{\mathcal{H}}^2.
 \end{aligned}$$

Thus, by choosing $\delta_* > \frac{3}{2m_A}$, we get

$$\langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \geq (\omega_{\mathcal{A}} - L) \|U_1 - U_2\|_{\mathcal{H}}^2,$$

where $L = \max(\delta_* (1 + L_G^2 + g_0^2 L_B^2 + L_\nu^2 C_0^4), \delta_* (L_B^2 g_0 + g_0))$. Thus, $\mathcal{A} + \omega_{\mathcal{A}} I$ is accretive when

$\omega_{\mathcal{A}} > L$.

Next, we exhibit that $\mathcal{A} + \lambda I$ is surjective for some constant $\lambda > 0$. In other words, for a given $(h_1, h_2, h_3)^T \in \mathcal{H}$, we should obtain $U = (u, v, \eta)^T \in \mathcal{D}(\mathcal{A})$ such that

$$(\lambda I + \mathcal{A})(u, v, \eta)^T = (h_1, h_2, h_3)^T,$$

thus, the previous equation reads

$$\begin{aligned} \lambda u - v &= h_1 \\ \lambda v - \frac{1}{\rho} \operatorname{Div} (G\varepsilon(u) + A\varepsilon(v)) + \frac{1}{\rho} \operatorname{Div} \left(\int_0^\infty g(s) B\varepsilon(u - \eta(s)) ds \right) &= h_2 \\ \lambda \eta + \eta_s - v &= h_3. \end{aligned} \quad (1.30)$$

We Suppose that v is found in V . Thus, using (1.30)₁ gives

$$u = \frac{h_1 + v}{\lambda}. \quad (1.31)$$

Then,

$$u \in V$$

Note that

$$\eta(s) = e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau, \quad s \in \mathbb{R}_+. \quad (1.32)$$

is the unique solution of (1.30)₃ with $\eta(0) = 0$.

The system (1.30) is equivalent to

$$\begin{aligned} \lambda v &- \frac{1}{\rho} \operatorname{Div} \left(G\varepsilon \left(\frac{h_1 + v}{\lambda} \right) + A\varepsilon(v) \right) \\ &+ \frac{1}{\rho} \operatorname{Div} \left(\int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds \right) \\ &= h_2. \end{aligned} \quad (1.33)$$

We define the operator $\mathcal{T} : V \rightarrow V'$ by

$$\begin{aligned} \langle \mathcal{T}(v), v \rangle_{V' \times V} &= \langle \lambda v, v \rangle_H + \left\langle G\varepsilon \left(\frac{h_1 + v}{\lambda} \right) + A\varepsilon(v), \varepsilon(v) \right\rangle_{\mathcal{H}} \\ &- \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) \right\rangle_{\mathcal{H}} \\ &+ \int_{\Gamma_3} p_\nu \left(\frac{h_1 + v_\nu}{\lambda} \right) v_\nu da - \int_{\Gamma_2} f_2 v da. \end{aligned} \quad (1.34)$$

Let prove that $\mathcal{T} : V \rightarrow V'$ is a surjective mapping. From Corollary A.1, we must prove that the mapping $\mathcal{T}$ is coercive and maximal monotone.

Let show that $\mathcal{T} : V \rightarrow V'$ is maximal monotone. Applying Theorem A.1, it's enough to prove the monotonicity and hemicontinuity of $\mathcal{T}$.

To justify the monotonicity of $\mathcal{T}$. Let $v, \tilde{v} \in V$. Thus, we obtain

$$\begin{aligned}
& \langle \mathcal{T}v - \mathcal{T}\tilde{v}, v - \tilde{v} \rangle_{V' \times V} \\
&= \langle \lambda v - \lambda \tilde{v}, v - \tilde{v} \rangle_H + \langle G\varepsilon \left(\frac{h_1 + v}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} \right), \varepsilon(v) - \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \\
&+ \langle A\varepsilon(v) - A\varepsilon(\tilde{v}), \varepsilon(v) - \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \\
&- \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&+ \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + \tilde{v}) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&+ \int_{\Gamma_3} \left(p_\nu \left(\frac{h_1 + v_\nu}{\lambda} \right) - p_\nu \left(\frac{h_1 + \tilde{v}_\nu}{\lambda} \right) \right) (v_\nu - \tilde{v}_\nu) da.
\end{aligned}$$

Using (c) of (1.10) yields

$$\left\langle G\varepsilon \left(\frac{h_1 + v}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} \right), \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \geq \frac{1}{\lambda} m_G \|v - \tilde{v}\|_V^2. \quad (1.35)$$

From (c) of (1.9), we have

$$\langle A\varepsilon(v) - A\varepsilon(\tilde{v}), \varepsilon(v) - \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \geq m_A \|v - \tilde{v}\|_V^2. \quad (1.36)$$

Using inequality of Cauchy-Schwarz, (b) of (1.11) and (G1), we have

$$\begin{aligned}
& - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
& + \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + \tilde{v}) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
& \geq - \frac{g_0 L_B}{\lambda} \|v - \tilde{v}\|_V^2.
\end{aligned} \quad (1.37)$$

From (c) of (1.12), we obtain

$$\int_{\Gamma_3} \left(p_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) - p_\nu \left(\frac{h_{1\nu} + \tilde{v}_\nu}{\lambda} \right) \right) (v_\nu - \tilde{v}_\nu) da \geq 0. \quad (1.38)$$

Combining (1.35)-(1.38), we obtain

$$\langle \mathcal{T}v - \mathcal{T}\tilde{v}, v - \tilde{v} \rangle_{V' \times V} \geq \left(\frac{m_G - g_0 L_B}{\lambda} \right) \|v - \tilde{v}\|_V^2.$$

It follows from assumption (1.16) that the operator $\mathcal{T}$ is strongly monotone. The strong mono-

tonicity of $\mathcal{T}$ implies also the coercivity of $\mathcal{T}$.

Next, let prove the hemicontinuity of $\mathcal{T}$. Using Definition A.2, it suffices to prove that

$$\mathcal{T}(v + \zeta v_1) \xrightarrow{w} \mathcal{T}(v), \quad \forall v, v_1 \in V.$$

when $\zeta \rightarrow 0$.

Let $\tilde{v} \in V$ and $\lambda > 0$, then

$$\begin{aligned} & \langle \mathcal{T}(v + \zeta v_1), \tilde{v} \rangle_{V' \times V} - \langle \mathcal{T}(v), \tilde{v} \rangle_{V' \times V} \\ = & \zeta \langle \lambda v_1, \tilde{v} \rangle_H + \left\langle G\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + v}{\lambda} \right), \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \langle A\varepsilon(v + \zeta v_1) - A\varepsilon(v), \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \\ & - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v + \zeta v_1) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \int_{\Gamma_3} \left(p_\nu \left(\frac{h_{1\nu} + v_\nu + \zeta v_{1\nu}}{\lambda} \right) - p_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right) \tilde{v}_\nu da \end{aligned} \quad (1.39)$$

It follows from inequality of Cauchy-Schwarz and (A.9) that

$$\begin{aligned} |\zeta \langle \lambda v_1, \tilde{v} \rangle_H| & \leq |\zeta \lambda| \|v_1\|_H \|\tilde{v}\|_H \\ & \leq |\zeta \lambda| d^2 \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \end{aligned} \quad (1.40)$$

as $\zeta \rightarrow 0$. Exploiting inequality of Cauchy-Schwarz and (b) of (1.10), we get

$$\begin{aligned} \left| \left\langle G\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + v}{\lambda} \right), \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \right| & \leq \frac{|\zeta|}{\lambda} L_G \|v_1\|_V \|\tilde{v}\|_V, \\ & \rightarrow 0, \end{aligned} \quad (1.41)$$

as $\zeta \rightarrow 0$. Similar to (1.41) and by using assumption (b) of (1.9) instead of (1.10), we obtain

$$|\langle A\varepsilon(v + \zeta v_1) - A\varepsilon(v), \varepsilon(\tilde{v}) \rangle_{\mathcal{H}}| \leq |\zeta| L_A \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \quad (1.42)$$

as $\zeta \rightarrow 0$. The use of inequality of Cauchy-Schwarz, (G1), (b) of (1.11) give

$$\begin{aligned} & \left| - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v + \zeta v_1) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \right. \\ & \quad \left. + \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \right| \\ & \leq \frac{|\zeta| L_B g_0}{\lambda} \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \end{aligned} \quad (1.43)$$

Applying the inequality of Cauchy-Schwarz and using (b) of (1.12) and (A.6), we have

$$\begin{aligned}
& \left| \int_{\Gamma_3} \left(p_\nu \left(\frac{h_{1\nu} + v_\nu + \zeta v_{1\nu}}{\lambda} \right) - p_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right) \widetilde{v}_\nu da \right| \\
& \leq \left\| p_\nu \left(\frac{h_{1\nu} + v_\nu + \zeta v_{1\nu}}{\lambda} \right) - p_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right\|_{L^2(\Gamma_3)} \|\widetilde{v}_\nu\|_{L^2(\Gamma_3)} \\
& \leq \frac{L_\nu |\zeta|}{\lambda} \|v_1\|_{L^2(\Gamma_3)^d} \|\widetilde{v}\|_{L^2(\Gamma_3)^d} \\
& \leq \frac{L_\nu C_0^2 |\zeta|}{\lambda} \|v_1\|_V \|\widetilde{v}\|_V \rightarrow 0,
\end{aligned} \tag{1.44}$$

as $\zeta \rightarrow 0$. Therefore, combining (1.40)-(1.44) yields

$$|\langle \mathcal{T}(v + \zeta v_1), \widetilde{v} \rangle_{V' \times V} - \langle \mathcal{T}(v), \widetilde{v} \rangle_{V' \times V}| \rightarrow 0, \tag{1.45}$$

as $\zeta \rightarrow 0$. Thus, the operator $\mathcal{T} : V \rightarrow V'$ is hemicontinuous.

We have shown that the operator $\mathcal{T}$ is coercive, monotone and hemicontinuous which implies that $\mathcal{T}$ is both coercive and maximal monotone. Therefore, we deduce that $\mathcal{T}$ is surjective. Thus, for every $(h_1, h_2, h_3)^T \in \mathcal{H}$, we can find $v \in V$ satisfying the equation (1.33).

It follows From (1.32) that there exists $C_1 > 0$ such that

$$\|\eta\|_{\mathcal{M}}^2 \leq C_1 (\|h_3\|_{\mathcal{M}}^2 + \|v\|_V^2). \tag{1.46}$$

Then, we obtain that $\eta \in \mathcal{M}$.

As a consequence of the use of (1.30)₃ there exists $C_2 > 0$ such that

$$\|\eta_s\|_{\mathcal{M}}^2 \leq C_2 (\|h_3\|_{\mathcal{M}}^2 + \|v\|_V^2 + \|\eta\|_{\mathcal{M}}^2). \tag{1.47}$$

Hence, we obtain $\eta_s \in \mathcal{M}$.

At last, we have found $U \in \mathcal{D}(\mathcal{A})$ which verifies (1.30) and thus $\mathcal{A} + \lambda I$ is surjective for some constant $\lambda > 0$. This achieves the proof of Theorem 1.1. $\square$

1.4 Exponential stability

In the current section, we are concerned with the exponential stability result associated to the energy solution of contact problem (1.25). Assuming that σ is smooth, we are going to establish technical lemmas to prove our result after defining the energy functional E .

Now, we introduce the following energy functional E

$$E(t) = \frac{1}{2} \|U(t)\|_{\mathcal{H}}^2 = \frac{1}{2} \left(\|u'(t)\|_H^2 + \|u(t)\|_V^2 + \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \right), \quad \forall t \in \mathbb{R}_+. \tag{1.48}$$

Lemma 1.1. Assume that (G1) and (1.9)-(1.13) hold and $U_0 \in \mathcal{D}(\mathcal{A})$. Then, the energy functional defined by (1.48) satisfies for $\lambda_1 > 0$, for all $t \geq 0$

$$\begin{aligned} E'(t) \leq & -(m_A - K_1) \|u'(t)\|_V^2 + \lambda_1 K_2 \|u(t)\|_V^2 + \lambda_1 K_3 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ & + \frac{1}{2} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds + \lambda_1 \|f(t)\|_{V'}^2, \end{aligned} \quad (1.49)$$

where

$$K_1 = \frac{7}{4\lambda_1}, K_2 = L_G^2 + g_0^2 L_B^2 + 1 + L_V^2 C_0^4, K_3 = g_0 + g_0 L_B^2.$$

Proof. we differentiate (1.48) with respect to the variable t . Then, by using (1.20)₁ and (1.20)₃ and integrating over Ω using formula of Green and boundary conditions, one has

$$\begin{aligned} E'(t) &= \langle u''(t), u'(t) \rangle_H + \langle u'(t), u(t) \rangle_V + \int_0^{+\infty} g(s) \langle \eta_t(s, t), \eta(s, t) \rangle_V ds \\ &= -\langle A\varepsilon(u'(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} - \langle G\varepsilon(u(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} \\ &\quad + \left\langle \int_0^{+\infty} g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u'(t)) \right\rangle_{\mathcal{H}} - \int_{\Gamma_3} p_\nu(u_\nu(t)) u'_\nu(t) da \\ &\quad + \langle f(t), u'(t) \rangle_{V' \times V} + \langle u'(t), u(t) \rangle_V + \int_0^{+\infty} g(s) \langle u'(t), \eta(s, t) \rangle_V ds \\ &\quad + \frac{1}{2} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds. \end{aligned}$$

The use of (G1), (1.9) – (1.12), inequalities of Cauchy-Schwarz and Young leads to the estimate (1.49). □

Remark 1.2. From (1.49), it is clear that the functional E is not decreasing in general.

Lemma 1.2. Assume that (G1), (G3) and (1.9)-(1.13) hold and $U_0 \in \mathcal{D}(\mathcal{A})$. Then the functional

$$\mathcal{F}_1(t) = \langle u'(t), u(t) \rangle_H - e^{3\zeta_1 t} \|f(t)\|_{V'}^2, \quad (1.50)$$

satisfies for $\lambda_2 > 0$, for all $t \geq 0$

$$\begin{aligned} \mathcal{F}'_1(t) \leq & (-l + 3\lambda_2) \|u(t)\|_V^2 + K_4 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds + K_5 \|u'(t)\|_V^2 \\ & + \left(\frac{1}{4\lambda_2} - \lambda_0 \right) \|f(t)\|_{V'}^2, \end{aligned} \quad (1.51)$$

where

$$K_4 = \frac{g_0 L_B^2}{4\lambda_2}, K_5 = \frac{L_A^2}{4\lambda_2} + d^2.$$

Proof. We differentiate (1.50) with respect to t to get

$$\mathcal{F}'_1(t) = \|u'(t)\|_H^2 + \langle u''(t), u(t) \rangle_H - 3\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2 - 2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'}.$$

Next, we multiply (1.20)₁ by $u(t)$ and also integrate over the domain Ω using formula of Green, boundary conditions and (1.10), we arrive at

$$\begin{aligned} \mathcal{F}'_1(t) \leq & \|u'(t)\|_H^2 - \langle A\varepsilon(u'(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} - m_G \|u(t)\|_V^2 \\ & + \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} - \int_{\Gamma_3} p_\nu(u_\nu(t)) u_\nu(t) da \\ & + \langle f(t), u(t) \rangle_{V' \times V} - 3\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2 - 2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'}. \end{aligned} \quad (1.52)$$

By exploiting inequalities of Cauchy-Schwarz and Young and (1.9), we get

$$\begin{aligned} \langle A\varepsilon(u'(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} & \leq \|A\varepsilon(u'(t))\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\ & \leq \frac{L_A^2}{4\lambda_2} \|u'(t)\|_V^2 + \lambda_2 \|u(t)\|_V^2. \end{aligned} \quad (1.53)$$

Using inequalities of Cauchy-Schwarz and Young, (1.11) and (G1), we obtain

$$\begin{aligned} \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} & \leq \left\| \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds \right\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\ & \leq L_B \int_0^\infty g(s) (\|u(t)\|_V + \|\eta(s, t)\|_V) ds \|u(t)\|_V \\ & \leq g_0 L_B \|u(t)\|_V^2 + \frac{L_B^2 g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\ & \quad + \lambda_2 \|u(t)\|_V^2. \end{aligned} \quad (1.54)$$

Exploiting inequalities of Cauchy-Schwarz and Young, we get

$$\langle f(t), u(t) \rangle_{V' \times V} \leq \frac{1}{4\lambda_2} \|f(t)\|_{V'}^2 + \lambda_2 \|u(t)\|_V^2. \quad (1.55)$$

Using inequality of Cauchy-Schwarz and (G3), we have

$$\begin{aligned} -2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'} & \leq 2e^{3\zeta_1 t} \|f'(t)\|_{V'} \|f(t)\|_{V'} \\ & \leq 2\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2. \end{aligned} \quad (1.56)$$

Inserting (1.53)-(1.56) in (1.52) and using (A.9) and (c) of (1.12), we get

$$\begin{aligned} \mathcal{F}'_1(t) \leq & (-l + 3\lambda_2) \|u(t)\|_V^2 + \frac{g_0 L_B^2}{4\lambda_2} \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ & + \left(\frac{L_A^2}{4\lambda_2} + d^2 \right) \|u'(t)\|_V^2 + \left(\frac{1}{4\lambda_2} - \lambda_0 \right) \|f(t)\|_{V'}^2, \end{aligned}$$

where $l = m_G - g_0 L_B$ and $\lambda_0 > 0$ satisfying $-\zeta_1 e^{3\zeta_1 t} \leq -\lambda_0$. $\square$

Lemma 1.3. Assume that (G1) and (1.9)-(1.13) hold and let $U_0 \in \mathcal{D}(\mathcal{A})$. Then the functional

$$\mathcal{F}_2(t) = - \left\langle u'(t), \int_0^{+\infty} g(s) \eta(s, t) ds \right\rangle_H, \quad (1.57)$$

satisfies for $\lambda_2 > 0$, for all $t \geq 0$

$$\begin{aligned} \mathcal{F}_2'(t) \leq & -g_0 \|u'(t)\|_H^2 + K_6 \lambda_2 \|u'(t)\|_V^2 + K_7 \lambda_2 \|u\|_V^2 + K_8 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ & - K_9 \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds + \lambda_2 \|f(t)\|_{V'}^2, \end{aligned} \quad (1.58)$$

where

$$K_6 = L_A^2 + d^2, K_7 = L_G^2 + L_B^2 + L_\nu^2 C_0^4, K_8 = \frac{4g_0}{4\lambda_2} + \frac{g_0^3}{4\lambda_2} + L_B g_0, K_9 = \frac{g(0)d^2}{4\lambda_2}.$$

Proof. We differentiate (1.57) with respect to variable t . Then, we exploit (1.20)₃ to obtain

$$\mathcal{F}_2'(t) = -\langle u''(t), \int_0^\infty g(s) \eta(s, t) ds \rangle_H - g_0 \|u'(t)\|_H^2 + \langle u'(t), \int_0^\infty g(s) \eta_s(s, t) ds \rangle_H.$$

We integrate the term $\langle u'(t), \int_0^\infty g(s) \eta_s(s, t) ds \rangle_H$ with respect to the variable s and thanks to the fact that $\eta(0) = 0$, $\lim_{s \rightarrow +\infty} g(s) = 0$, we find

$$\mathcal{F}_2'(t) = -\langle u''(t), \int_0^\infty g(s) \eta(s, t) ds \rangle_H - g_0 \|u'(t)\|_H^2 - \langle u'(t), \int_0^\infty g'(s) \eta(s, t) ds \rangle_H.$$

Replace (1.20)₁ in the above equality, integrating over domain Ω and using formula of Green, boundary conditions, one has

$$\begin{aligned} \mathcal{F}_2'(t) = & \langle A\varepsilon(u'(t)), \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \rangle_{\mathcal{H}} + \langle G\varepsilon(u(t)), \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \rangle_{\mathcal{H}} \\ & - \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\rangle_{\mathcal{H}} \\ & + \int_{\Gamma_3} p_\nu(u_\nu(t)) \int_0^\infty g(s) \eta_\nu(s, t) ds da - \langle f(t), \int_0^\infty g(s) \eta(s, t) ds \rangle_{V' \times V} \\ & - g_0 \|u'(t)\|_H^2 - \langle u'(t), \int_0^\infty g'(s) \eta(s, t) ds \rangle_H. \end{aligned} \quad (1.59)$$

By using inequalities of Cauchy-Schwarz and Young, (G1) and (1.9), we see that

$$\begin{aligned} & \left\langle A\varepsilon(u'(t)), \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\rangle_{\mathcal{H}} \\ & \leq \|A\varepsilon(u'(t))\|_{\mathcal{H}} \left\| \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\|_{\mathcal{H}} \\ & \leq \lambda_2 L_A^2 \|u'(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds. \end{aligned} \quad (1.60)$$

Similar as (1.60) by using (1.10) instead of (1.9), we get

$$\begin{aligned}
& \left\langle G\varepsilon(u(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
& \leq \|G\varepsilon(u(t))\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\
& \leq \lambda_2 L_G^2 \|u(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{1.61}$$

Thanks to inequalities of Cauchy-Schwarz and Young, (G1) and (1.11), it follows that

$$\begin{aligned}
& - \left\langle \int_0^\infty g(s)B\varepsilon(u(t) - \eta(s,t))ds, \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
& \leq \left\| \int_0^\infty g(s)B\varepsilon(u(t) - \eta(s,t))ds \right\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\
& \leq L_B \int_0^\infty g(s) \|u(t) - \eta(s,t)\|_V ds \int_0^\infty g(s) \|\eta(s,t)\|_V ds \\
& \leq \lambda_2 L_B^2 \|u(t)\|_V^2 + \left(\frac{g_0^3}{4\lambda_2} + L_B g_0 \right) \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{1.62}$$

Applying inequalities of Cauchy-Schwarz and Young and using (1.12), (G1), (A.6), we obtain

$$\begin{aligned}
& \int_{\Gamma_3} p_\nu(u_\nu(t)) \left(\int_0^\infty g(s)\eta_\nu(s,t)ds \right) da \\
& \leq \left(\int_{\Gamma_3} p_\nu^2(u_\nu(t))da \right)^{\frac{1}{2}} \left(\int_{\Gamma_3} \left(\int_0^\infty g(s)\eta_\nu(s,t)ds \right)^2 da \right)^{\frac{1}{2}} \\
& \leq \lambda_2 L_\nu^2 C_0^4 \|u(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{1.63}$$

Using inequalities of Cauchy-Schwarz and Young, we get

$$- \left\langle f(t), \int_0^\infty g(s)\eta(s,t)ds \right\rangle_{V' \times V} \leq \lambda_2 \|f(t)\|_{V'}^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds. \tag{1.64}$$

The use of inequalities of Cauchy-Schwarz and Young, (G1) and (A.9) leads to

$$\begin{aligned}
& - \left\langle u'(t), \int_0^\infty g'(s)\eta(s,t)ds \right\rangle_H \\
& \leq \|u'(t)\|_H \left\| \int_0^\infty g'(s)\eta(s,t)ds \right\|_H \\
& \leq \lambda_2 \|u'(t)\|_H^2 - \frac{g(0)}{4\lambda_2} \left(\int_0^\infty g'(s) \|\eta(s,t)\|_H^2 ds \right) \\
& \leq d^2 \lambda_2 \|u'(t)\|_V^2 - \frac{g(0)d^2}{4\lambda_2} \left(\int_0^\infty g'(s) \|\eta(s,t)\|_V^2 ds \right).
\end{aligned} \tag{1.65}$$

Combining the estimates (1.60)-(1.65), we obtain (1.58). $\square$

Now, we consider the following Lyapunov functional $\mathcal{L}$

$$\mathcal{L}(t) = E(t) + \epsilon(N\mathcal{F}_1(t) + \mathcal{F}_2(t)), \quad (1.66)$$

where N, ϵ are positive constants that will be chosen later.

Theorem 1.2. Assume that (G1)-(G3) and (1.9)-(1.13) hold. Suppose that m_G is such that

$$m_G > \frac{3}{4\lambda_0} + g_0 L_B, \quad (1.67)$$

and there exists δ_0 a positive constant independent of m_A such that

$$m_A > \delta_0, \quad (1.68)$$

thus, for every $U_0 \in \mathcal{D}(\mathcal{A})$, there exist positive constants δ_1 and δ_2 such that

$$\|U(t)\|_{\mathcal{H}}^2 \leq \delta_2 e^{-\delta_1 t}. \quad (1.69)$$

Proof. We differentiate (1.66) with respect to the variable t . Then, we use (1.49), (1.51), (1.58) to obtain for all $t \geq 0$

$$\begin{aligned} \mathcal{L}'(t) &= E'(t) + \epsilon(N\mathcal{F}'_1(t) + \mathcal{F}'_2(t)) \\ &\leq -\epsilon \left(g_0 \|u'(t)\|_H^2 + \left(N(l - 3\lambda_2) - \lambda_2 K_7 - \frac{K_2 \lambda_1}{\epsilon} \right) \|u(t)\|_V^2 \right) \\ &\quad -\epsilon \left(K_3 \left(N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3 \lambda_1}{\epsilon} \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\ &\quad + (K_1 - m_A + \epsilon(K_5 N + \lambda_2 K_6)) \|u'(t)\|_V^2 + \left(\frac{1}{2} - \epsilon K_9 \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds \\ &\quad + \epsilon K_{10} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds + \left(\lambda_1 + \epsilon \left(N \left(\frac{1}{4\lambda_2} - \lambda_0 \right) + \lambda_2 \right) \right) \|f(t)\|_{V'}^2, \end{aligned} \quad (1.70)$$

where $K_{10} = NK_4 + K_8 + K_3 \left(N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right)$.

Now, using (1.67), we can take λ_2 such that

$$\frac{1}{4\lambda_0} < \lambda_2 < \frac{l}{3}, \quad (1.71)$$

and therefore N large enough such that

$$\begin{aligned} N(l - 3\lambda_2) - \lambda_2 K_7 &> 0, \\ N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 &> 0. \end{aligned} \quad (1.72)$$

From (G2), we get

$$\left(\frac{1}{2} - \epsilon K_9 \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds + \epsilon K_{10} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \leq \left(\frac{1}{2} - \epsilon K_{11} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds,$$

where $K_{11} = K_9 + \frac{K_{10}}{\alpha}$.

Inserting the preceding inequality in (1.70), we get

$$\begin{aligned}
\mathcal{L}'(t) &= E'(t) + \epsilon(N\mathcal{F}'_1(t) + \mathcal{F}'_2(t)) \\
&\leq -\epsilon \left(g_0 \|u'(t)\|_H^2 + \left(N(l - 3\lambda_2) - \lambda_2 K_7 - \frac{K_2 \lambda_1}{\epsilon} \right) \|u(t)\|_V^2 \right) \\
&\quad -\epsilon \left(K_3 \left(N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3 \lambda_1}{\epsilon} \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\
&\quad + (K_1 - m_A + \epsilon(K_5 N + \lambda_2 K_6)) \|u'(t)\|_V^2 + \left(\frac{1}{2} - \epsilon K_{11} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds \\
&\quad + \left(\lambda_1 + \epsilon \left(N \left(\frac{1}{4\lambda_2} - \lambda_0 \right) + \lambda_2 \right) \right) \|f(t)\|_{V'}^2.
\end{aligned} \tag{1.73}$$

Next, we assume that the constant m_A satisfies (1.68) if we choose δ_0 such that

$$\delta_0 = \max \left\{ \frac{\lambda_1 (K_5 N + \lambda_2 K_6)}{N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2}, \frac{K_2 \lambda_1 (K_5 N + \lambda_2 K_6)}{N(l - 3\lambda_2) - \lambda_2 K_7} \right\} + K_1, \tag{1.74}$$

By definition of E , $\mathcal{F}_1$ and $\mathcal{F}_2$ and applying inequalities of Cauchy-Schwarz and Young and using (G1) and (A.9), we have

$$\begin{aligned}
|\mathcal{F}_1(t)| &= |\langle u'(t), u(t) \rangle_H - e^{3\varsigma_1 t} \|f\|_{V'}^2| \\
&\leq \frac{1}{2} \|u'(t)\|_H^2 + \frac{1}{2} \|u(t)\|_H^2 \\
&\leq \max(1, d^2) E(t),
\end{aligned} \tag{1.75}$$

and

$$\begin{aligned}
|\mathcal{F}_2(t)| &= \left| \left\langle u'(t), \int_0^{+\infty} g(s) \eta(s, t) ds \right\rangle_H \right| \\
&\leq \frac{1}{2} \left(\|u'(t)\|_H^2 + g_0 d^2 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \right) \\
&\leq \max(1, g_0 d^2) E(t).
\end{aligned} \tag{1.76}$$

Combining (1.66), (1.48), (1.75), (1.76), we get

$$|\mathcal{L}(t) - E(t)| \leq \epsilon \varsigma_3 E(t), \tag{1.77}$$

where ς_3 is a positive constant given by

$$\varsigma_3 = N \max(1, d^2) + \max(1, g_0 d^2),$$

Using (1.77), we get

$$(1 - \epsilon \varsigma_3) E(t) \leq \mathcal{L}(t) \leq (1 + \epsilon \varsigma_3) E(t), \tag{1.78}$$

then by choosing $\epsilon < \frac{1}{\varsigma_3}$, we have $\mathcal{L} \sim E$.

Besides, we suppose that λ_1 satisfies

$$\lambda_1 < \min \left\{ \frac{\left(\lambda_0 - \frac{1}{4\lambda_2}\right) N - \lambda_2}{\varsigma_3}, \frac{\left(\lambda_0 - \frac{1}{4\lambda_2}\right) N - \lambda_2}{2K_{11}}, \frac{N(l - 3\lambda_2) - \lambda_2 K_7}{K_2 \varsigma_3}, \frac{N(l - 3\lambda_2) - \lambda_2 K_7}{2K_2 K_{11}} \right\}, \quad (1.79)$$

From (1.79), it follows that

$$\frac{\lambda_1}{\left(\lambda_0 - \frac{1}{4\lambda_2}\right) N - \lambda_2} < \min \left\{ \frac{1}{\varsigma_3}, \frac{1}{2K_{11}} \right\}, \quad (1.80)$$

and

$$\frac{K_2 \lambda_1}{N(l - 3\lambda_2) - \lambda_2 K_7} < \min \left\{ \frac{1}{\varsigma_3}, \frac{1}{2K_{11}} \right\}. \quad (1.81)$$

From (1.74), we have

$$\frac{\lambda_1}{\left(\lambda_0 - \frac{1}{4\lambda_2}\right) N - \lambda_2} < \frac{m_A - K_1}{K_5 N + \lambda_2 K_6}, \quad (1.82)$$

and

$$\frac{K_2 \lambda_1}{N(l - 3\lambda_2) - \lambda_2 K_7} < \frac{m_A - K_1}{K_5 N + \lambda_2 K_6}. \quad (1.83)$$

Therefore, combining (1.80)-(1.83), we can fix ϵ like this

$$\max \left\{ \frac{\lambda_1}{\left(\lambda_0 - \frac{1}{4\lambda_2}\right) N - \lambda_2}, \frac{K_2 \lambda_1}{N(l - 3\lambda_2) - \lambda_2 K_7} \right\} < \epsilon < \min \left\{ \frac{m_A - K_1}{K_5 N + \lambda_2 K_6}, \frac{1}{2K_{11}}, \frac{1}{\varsigma_3} \right\}. \quad (1.84)$$

Consequently, we obtain

$$\mathcal{L}'(t) \leq -\epsilon K_{12} E(t) + \left(\frac{1}{2} - \epsilon K_{11}\right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds,$$

where

$$K_{12} = 2 \min \left(g_0, N(l - 3\lambda_2) - \lambda_2 K_7 - \frac{K_2 \lambda_1}{\epsilon}, K_3 \left(N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3 \lambda_1}{\epsilon} \right).$$

Using (G1), (1.84) and (1.78), we arrive at

$$\mathcal{L}'(t) \leq \frac{-\epsilon K_{12}}{1 + \epsilon \varsigma_3} \mathcal{L}(t). \quad (1.85)$$

We integrate (1.85) from 0 to t , we get

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\delta_1 t}, \quad \forall t \in \mathbb{R}_+, \quad (1.86)$$

where $\delta_1 = \frac{\epsilon K_{12}}{1 + \epsilon \varsigma_3}$

From (1.78) and (1.86), we have

$$\|U(t)\|_{\mathcal{H}}^2 = 2E(t) \leq \frac{2}{1 - \epsilon\zeta_3} \mathcal{L}(t) \leq \frac{2}{1 - \epsilon\zeta_3} \mathcal{L}(0)e^{-\delta_1 t},$$

which yields (1.69) with $\delta_2 = \frac{2}{1 - \epsilon\zeta_3} \mathcal{L}(0)$. Therefore, the proof of Theorem 1.2 is completed. $\square$

A DYNAMIC FRICTIONAL CONTACT PROBLEM WITH NORMAL DAMPED RESPONSE

The results of this chapter are dedicated to studying global existence and stability of a unique solution for contact problem with normal damped response and friction. Arguments of nonlinear evolution equations, operator theory, fixed point and extension procedure are used to show the existence of solution of the problem with finite memory. Nonlinear semigroup arguments are used to prove the global existence of mild solution of the problem with infinite memory. The construction of a suitable Lyapunov functional allows to establish the stability result in the two cases.

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2.1 Frictional contact problem with finite memory

2.1.1 Problem statement

Consider the following frictional contact problem with normal damped response and finite memory

$$\sigma = A\varepsilon(u') + G\varepsilon(u) - \int_0^t g(t-s)B\varepsilon(u(s)) ds \quad \text{in } \Omega \times (0, \infty), \quad (2.1)$$

$$\rho u'' = \text{Div} \sigma + f_0 \quad \text{in } \Omega \times (0, \infty), \quad (2.2)$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \quad (2.3)$$

$$\sigma \nu = f_2 \quad \text{on } \Gamma_2 \times (0, \infty), \quad (2.4)$$

$$-\sigma_\nu = p_\nu(u'_\nu) \quad \text{on } \Gamma_3 \times (0, \infty), \quad (2.5)$$

$$-\sigma_\tau = p_\tau(u'_\tau) \quad \text{on } \Gamma_3 \times (0, \infty), \quad (2.6)$$

$$u(0) = u_0, u'(0) = u_1 \quad \text{in } \Omega. \quad (2.7)$$

2.1.2 Additional assumption

In addition to assumptions (1.9)-(1.15) and (G2)-(G3) defined on Sect. 1.2, we present the following additional assumptions

$$\left\{ \begin{array}{l} \text{(a) } p_\tau : \Gamma_3 \times \mathbb{R}^d \rightarrow \mathbb{R}^d \text{ is a nonlinear contact function;} \\ \text{(b) There exists a constant } L_\tau \text{ such that} \\ \quad \|p_\tau(x, r_1) - p_\tau(x, r_2)\| \leq L_\tau \|r_1 - r_2\|, \forall r_1, r_2 \in \mathbb{R}^d, \text{ a.e. } x \in \Gamma_3; \\ \text{(c) } (p_\tau(x, r_1) - p_\tau(x, r_2)) \cdot (r_1 - r_2) \geq 0, \forall r_1, r_2 \in \mathbb{R}^d, \text{ a.e. } x \in \Gamma_3; \\ \text{(d) The mapping } x \mapsto p_\tau(x, r) \text{ is Lebesgue measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}^d; \\ \text{(e) } p_\tau(x, r) = 0 \text{ if } r = 0_{\mathbb{R}^d}, \text{ a.e. } x \in \Gamma_3. \end{array} \right. \quad (2.8)$$

An example of p_τ is

$$p_\tau = \mu r$$

where $\mu > 0$ denotes the friction's coefficient.

(G4) The kernel function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is class C^1 , nonincreasing function satisfying

$$g(0) > 0, \quad l := 1 - \int_0^\infty g(s)ds > 0. \quad (2.9)$$

The initial data satisfy

$$u_0 \in V, \quad u_1 \in H. \quad (2.10)$$

The functional $j : V \times V \rightarrow \mathbb{R}$ is defined by

$$j(v, w) = \int_{\Gamma_3} p_\nu(v_\nu)w_\nu da + \int_{\Gamma_3} p_\tau(v_\tau) \cdot w_\tau da, \quad \forall v, w \in V. \quad (2.11)$$

2.1.3 Global existence

This subsection is dedicated to the global existence result for the problem (2.1)-(2.7).

Supposing that (u, σ) are sufficiently smooth, then, by using formula of Green, boundary conditions and also the given constitutive law (2.1), we get the coming variational formulation of the system (2.1)-(2.7), in terms of displacement field.

Problem $\mathcal{P}_V$ Find a displacement $u : \mathbb{R}_+ \rightarrow V$ such that

$$\begin{aligned} & \langle u''(t), w \rangle_{V' \times V} + \langle A\varepsilon(u'(t)), \varepsilon(w) \rangle_{\mathcal{H}} + \langle G\varepsilon(u(t)), \varepsilon(w) \rangle_{\mathcal{H}} \\ & - \langle \int_0^t g(t-s)B\varepsilon(u(s))ds, \varepsilon(w) \rangle_{\mathcal{H}} + j(u'(t), w) \\ & = \langle f(t), w \rangle_{V' \times V} \quad \forall w \in V, \quad \forall t > 0, \end{aligned} \quad (2.12)$$

$$u(0) = u_0, \quad u'(0) = u_1. \quad (2.13)$$

The existence of a local solution to the variational problem (2.12)-(2.13) is established by the following theorem

Theorem 2.1. *Suppose that the assumptions (G4), (1.9)-(1.13), (2.8) and (2.10) hold. Then, the problem (2.12)-(2.13) has a unique solution u defined on $[0, T]$ satisfying*

$$u \in W^{1,2}(0, T; V) \cap C^1(0, T, H), \quad u'' \in L^2(0, T; V'). \quad (2.14)$$

Several lemmas are used to prove Theorem 2.1.

Let $\eta \in L^2(0, T; V')$ be given. Our starting point is the introduction of the following variational problem

Problem $\mathcal{PV}_\eta$ Find a displacement $u_\eta : [0, T] \rightarrow V$ such that

$$\begin{aligned} & \langle u_\eta''(t), w \rangle_{V' \times V} + \langle A\varepsilon(u_\eta'(t)), \varepsilon(w) \rangle_{\mathcal{H}} + \langle \eta(t), w \rangle_{V' \times V} + j(u_\eta'(t), w) \\ & = \langle f(t), w \rangle_{V' \times V} \quad \forall w \in V, \text{ a.e. } t \in (0, T), \end{aligned} \quad (2.15)$$

$$u_\eta(0) = u_0, u_\eta'(0) = u_1. \quad (2.16)$$

Lemma 2.1. *Assume that assumptions of Theorem 2.1 hold. Then, the problem $\mathcal{PV}_\eta$ admits a unique solution satisfying the regularity (2.14). Moreover, if we denote by u_i the solution of problem PV_η for $\eta = \eta_i \in L^2(0, T; V')$, $i = 1, 2$, then, there exists a constant $c > 0$ such that*

$$\int_0^t \|u_1'(s) - u_2'(s)\|_V^2 ds \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}^2 ds, \quad \forall t \in [0, T]. \quad (2.17)$$

Proof. We define the operator $\mathcal{T} : V \rightarrow V'$ by

$$\langle \mathcal{T}v, w \rangle_{V' \times V} = \langle A\varepsilon(v), \varepsilon(w) \rangle_{\mathcal{H}} + j(v, w) \quad \forall v, w \in V. \quad (2.18)$$

Recalling inequality of Cauchy-Schwarz and exploiting (A.6), (b) of (1.9), (1.12), (2.8), we get

$$\|\mathcal{T}v - \mathcal{T}w\|_{V'} \leq (L_A + L_\nu C_0^2 + L_\tau C_0^2) \|v - w\|_V \quad \forall v, w \in V, \quad (2.19)$$

consequently, the hemicontinuity of $\mathcal{T} : V \rightarrow V'$ follows.

Using (c) of (1.9), (1.12) and (2.8), we obtain

$$\langle \mathcal{T}v - \mathcal{T}w, v - w \rangle_{V' \times V} \geq m_A \|v - w\|_V^2 \quad \forall v, w \in V, \quad (2.20)$$

then, we deduce that $\mathcal{T}$ is monotone. Thus, by taking $w = 0_V$ in (2.20) and using (e) of (1.9), (1.12) and (2.8), we arrive at

$$\langle \mathcal{T}v, v \rangle_{V' \times V} \geq m_A \|v\|_V^2 \quad (2.21)$$

From (2.19), we deduce that

$$\|\mathcal{T}v\|_{V'} \leq (L_A + L_\nu C_0^2 + L_\tau C_0^2) \|v\|_V, \quad \forall v \in V. \quad (2.22)$$

Thanks to (1.15) and (2.10), we have

$$f - \eta \in L^2(0, T; V') \text{ and } u_1 \in H. \quad (2.23)$$

Having in mind that $\mathcal{T}$ is monotone, hemicontinuous and employing (2.21)-(2.23), we conclude from Theorem A.2 the existence of a function v_η that it is unique and satisfies

$$v_\eta \in L^2(0, T; V) \cap C(0, T; H), \quad v_\eta' \in L^2(0, T; V'), \quad (2.24)$$

$$v'_\eta + \mathcal{T}v_\eta + \eta(t) = f(t) \quad \text{a.e. } t \in (0, T), \quad (2.25)$$

$$v_\eta(0) = u_1. \quad (2.26)$$

Let $u_\eta : [0, T] \rightarrow V$ be the function given by

$$u_\eta = \int_0^t v_\eta(s) ds + u_0, \quad t \in [0, T]. \quad (2.27)$$

Exploiting (2.18) and (2.24)-(2.27), we conclude that the function u_η is a solution of the variational problem PV_η satisfying the regularity (2.14). Because of the uniqueness of v_η , the uniqueness of u_η follows.

Let $\eta_1, \eta_2 \in L^2(0, T; V')$ and denote $u_i = u_{\eta_i}$, $v_i = v_{\eta_i} = u'_{\eta_i}$, $i = 1, 2$.

From (2.15), we obtain

$$\begin{aligned} & \langle v'_1 - v'_2, v_1 - v_2 \rangle_{V' \times V} + \langle A\varepsilon(v_1) - A\varepsilon(v_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\ & + j(v_1, v_1 - v_2) - j(v_2, v_1 - v_2) + \langle \eta_1 - \eta_2, v_1 - v_2 \rangle_{V' \times V} \end{aligned} \quad (2.28)$$

$$= 0, \quad (2.29)$$

a.e. on $(0, T)$. Let $t \in [0, T]$. Integrating (2.28) with respect to time variable t , using initial conditions $v_1(0) = v_2(0) = u_1$ and using (c) of (1.9), (1.12) and (2.8), one has

$$m_A \int_0^t \|v_1(s) - v_2(s)\|_V^2 ds \leq - \int_0^t \langle \eta_1(s) - \eta_2(s), v_1(s) - v_2(s) \rangle_{V' \times V} ds. \quad (2.30)$$

Making use of Young inequality, we obtain

$$\begin{aligned} & - \int_0^t \langle \eta_1(s) - \eta_2(s), v_1(s) - v_2(s) \rangle_{V' \times V} ds \\ & \leq \frac{1}{m_A} \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}^2 ds + \frac{m_A}{4} \int_0^t \|v_1(s) - v_2(s)\|_V^2 ds. \end{aligned} \quad (2.31)$$

Combining (2.30)-(2.31), we get

$$\int_0^t \|v_1(s) - v_2(s)\|_V^2 ds \leq \frac{4}{3m_A^2} \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}^2 ds,$$

which implies (2.17) with $c = \frac{4}{3m_A^2}$. □

We consider the operator $\mathcal{J} : L^2(0, T; V') \rightarrow L^2(0, T; V')$ defined by

$$\langle \mathcal{J}\eta(t), w \rangle_{V' \times V} = \langle G\varepsilon(u_\eta(t)), \varepsilon(w) \rangle_{\mathcal{H}} - \left\langle \int_0^t g(t-s)B\varepsilon(u_\eta(s))ds, \varepsilon(w) \right\rangle_{\mathcal{H}}, \quad (2.32)$$

for all $w \in V$, for all $t \in [0, T]$.

Lemma 2.2. *The operator $\mathcal{J}$ has a unique fixed point $\eta^* \in L^2(0, T; V')$.*

Proof. Assume that $\eta_1, \eta_2 \in L^2(0, T; V')$ and $t \in [0, T]$. We denote $u_i = u_{\eta_i}$ for $i = 1, 2$. It follows from (2.32), (1.10), (1.11) and elementary manipulations that

$$\begin{aligned} & | \langle \mathcal{J}\eta_1(t) - \mathcal{J}\eta_2(t), w \rangle_{V' \times V} | \\ & \leq \left(L_G \|u_1(t) - u_2(t)\|_V + L_B \int_0^t g(t-s) \|u_1(s) - u_2(s)\|_V ds \right) \|w\|_V. \end{aligned} \quad (2.33)$$

Having in mind $u_1(0) = u_2(0) = u_0$, one has

$$\|u_1(t) - u_2(t)\|_V \leq \int_0^t \|u'_1(s) - u'_2(s)\|_V ds. \quad (2.34)$$

Exploiting (2.33), (2.34), (G4) and (2.17) we get

$$\|\mathcal{J}\eta_1(t) - \mathcal{J}\eta_2(t)\|_{V'}^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}^2 ds \quad \forall t \in [0, T]. \quad (2.35)$$

Reiterating the inequality (2.35) m times, we obtain

$$\|\mathcal{J}^m \eta_1 - \mathcal{J}^m \eta_2\|_{L^2(0, T; V')}^2 \leq \frac{c^m T^m}{m!} \|\eta_1 - \eta_2\|_{L^2(0, T; V')}^2. \quad (2.36)$$

By taking m sufficiently large, $\mathcal{J}^m$ is a contraction on $L^2(0, T; V')$. Thus, there exists a unique $\eta^* \in L^2(0, T; V')$ satisfying $\mathcal{J}^m \eta^* = \eta^*$. Besides, the operator $\mathcal{J}$ defined in (2.32) has a unique fixed point η^* . $\square$

Now, we turn to prove our Theorem 2.1

Proof. Assume that η^* is the fixed point of $\mathcal{J}$ defined in (2.32). Let denote

$$u^* = u_{\eta^*}. \quad (2.37)$$

And we have

$$\langle \eta^*(t), w \rangle_{V' \times V} = \langle G\varepsilon(u^*(t)), \varepsilon(w) \rangle_{\mathcal{H}} - \left\langle \int_0^t g(t-s) B\varepsilon(u^*(s)) ds, \varepsilon(w) \right\rangle_{\mathcal{H}}. \quad (2.38)$$

We show that u^* satisfies (2.12)-(2.13) for $t \in [0, T]$.

We write (2.15) for $\eta = \eta^*$ and using (2.37), (2.38) to get that (2.12) is satisfied for $t \in (0, T)$.

From (2.16), it follows that (2.13) holds. The regularity (2.14) results from Lemma 2.1.

The uniqueness result is proved by employing the fact that the operator $\mathcal{J}$ has a unique fixed point. $\square$

Now, we consider the following energy functional

$$E(t) = \frac{1}{2} \|u'(t)\|_H^2 + \frac{1}{2} \left(1 - \int_0^t g(s) ds \right) \|u(t)\|_V^2 + \frac{1}{2} (g \circ u)(t), \quad \forall t \geq 0, \quad (2.39)$$

where

$$(g \circ u)(t) = \int_0^t g(t-s) \|u(t) - u(s)\|_V^2 ds.$$

Lemma 2.3. *Assume that (G4), (1.9)-(1.13), (2.8) and (2.10) are satisfied. Then the energy functional (2.39) satisfies for $\lambda_1 > 0$, for all $t \geq 0$*

$$\begin{aligned} E'(t) \leq & - (m_A - K_1) \|u'(t)\|_V^2 + \lambda_1 K_2 \|u(t)\|_V^2 + \lambda_1 K_3 (g \circ u)(t) \\ & + \frac{1}{2} (g' \circ u)(t) + \lambda_1 \|f(t)\|_{V'}^2, \end{aligned} \quad (2.40)$$

where

$$K_1 = \frac{7}{4\lambda_1}, K_2 = L_G^2 + 1 + (1-l)^2(1 + L_B^2), K_3 = (1-l)(L_B^2 + 1).$$

Proof. Differentiating the equation (2.39) with respect to time variable t , exploiting (2.1)-(2.2), and we integrate over domain Ω , using formula of Green and boundary conditions, it results that

$$\begin{aligned} E'(t) &= \langle u''(t), u'(t) \rangle_H - \frac{1}{2} g(t) \|u(t)\|_V^2 + \left(1 - \int_0^t g(s) ds\right) \langle u'(t), u(t) \rangle_V \\ &\quad + \frac{1}{2} (g' \circ u)(t) + \int_0^t g(t-s) \langle u'(t), u(t) - u(s) \rangle_V ds \\ &= -\langle A\varepsilon(u'(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} - \langle G\varepsilon(u(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} \\ &\quad + \left\langle \int_0^t g(t-s) (B\varepsilon(u(s)) - B\varepsilon(u(t))) ds, \varepsilon(u'(t)) \right\rangle_{\mathcal{H}} \\ &\quad + \left\langle \int_0^t g(t-s) B\varepsilon(u(t)) ds, \varepsilon(u'(t)) \right\rangle_{\mathcal{H}} \\ &\quad - \int_{\Gamma_3} p_\nu(u'_\nu(t)) u'_\nu(t) da - \int_{\Gamma_3} p_\tau(u'_\tau(t)) \cdot u'_\tau(t) da \\ &\quad + \langle f(t), u'(t) \rangle_{V' \times V} - \frac{1}{2} g(t) \|u(t)\|_V^2 + \left(1 - \int_0^t g(s) ds\right) \langle u'(t), u(t) \rangle_V \\ &\quad + \int_0^t g(t-s) \langle u'(t), u(t) - u(s) \rangle_V ds + \frac{1}{2} (g' \circ u)(t). \end{aligned}$$

From (c) of (1.9), we get

$$-\langle A\varepsilon(u'(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} \leq -m_A \|u'(t)\|_V^2. \quad (2.41)$$

Using inequalities of Cauchy-Schwarz, Young and (b) of (1.10), we obtain

$$\langle G\varepsilon(u(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} \leq \lambda_1 L_G^2 \|u(t)\|_V^2 + \frac{1}{4\lambda_1} \|u'(t)\|_V^2. \quad (2.42)$$

Exploiting inequalities of Cauchy-Schwarz and Young, (b) of (1.11) and (G4), we obtain

$$\begin{aligned} & \left\langle \int_0^t g(t-s) (B\varepsilon(u(s)) - B\varepsilon(u(t))) ds, \varepsilon(u'(t)) \right\rangle_{\mathcal{H}} \\ & \leq \lambda_1 L_B^2 \left(\int_0^t \sqrt{g(t-s)} \sqrt{g(t-s)} \|u(t) - u(s)\|_V ds \right)^2 + \frac{1}{4\lambda_1} \|u'(t)\|_V^2 \\ & \leq \lambda_1 L_B^2 (1-l)(g \circ u)(t) + \frac{1}{4\lambda_1} \|u'(t)\|_V^2. \end{aligned} \quad (2.43)$$

Recalling inequalities of Cauchy-Schwarz and Young, it results that

$$\langle f(t), u'(t) \rangle_{V' \times V} \leq \lambda_1 \|f(t)\|_{V'}^2 + \frac{1}{4\lambda_1} \|u'(t)\|_V^2. \quad (2.44)$$

From inequalities of Cauchy-Schwarz and Young and assumption (G4), we get

$$\left(1 - \int_0^t g(s) ds\right) \langle u'(t), u(t) \rangle_V \leq \lambda_1 \left(1 + (1-l)^2\right) \|u(t)\|_V^2 + \frac{1}{2\lambda_1} \|u'(t)\|_V^2, \quad (2.45)$$

and

$$\int_0^t g(t-s) \langle u'(t), u(t) - u(s) \rangle_V ds \leq (1-l)\lambda_1 (g \circ u)(t) + \frac{1}{4\lambda_1} \|u'(t)\|_V^2. \quad (2.46)$$

The use of inequalities of Cauchy-Schwarz and Young, (b) of (1.11) and (G4) leads to

$$\left\langle \int_0^t g(t-s) B\varepsilon(u(t)) ds, \varepsilon(u'(t)) \right\rangle_{\mathcal{H}} \leq L_B^2 (1-l)^2 \lambda_1 \|u(t)\|_V^2 + \frac{1}{4\lambda_1} \|u'(t)\|_V^2. \quad (2.47)$$

Combining the estimates (2.41)-(2.47), using (c) of (1.12) and (2.8) and (G4), we get (2.40). $\square$

Using continuation principle [66] and techniques found in [32, 78], we will show the global existence under the assumption (G3).

We consider the following modified energy functional

$$F(t) = \frac{1}{2} \left(\|u'(t)\|_H^2 + \left(1 - \int_0^t g(s) ds\right) \|u(t)\|_V^2 + (g \circ u)(t) + \|f(t)\|_{V'}^2 \right). \quad (2.48)$$

Theorem 2.2. *Let $u(x, t)$ the weak solution of system (2.1)-(2.7) such that $u \in W^{1,2}(0, T; V) \cap C^1(0, T, H)$, assumptions (1.9)-(1.13), (2.8), (2.10), (G4) and (G3) hold. Then, $u(x, t)$ exists globally in time.*

Proof. From (2.39) and (2.48), we get

$$F(t) = E(t) + \frac{1}{2} \|f(t)\|_{V'}^2. \quad (2.49)$$

Differentiating (2.49) with respect to time variable t and exploiting (2.40) and (G4), we arrive at

$$\begin{aligned} F'(t) \leq & -(m_A - K_1) \|u'(t)\|_V^2 + \lambda_1 K_2 \|u(t)\|_V^2 + \lambda_1 K_3 (g \circ u)(t) \\ & + \lambda_1 \|f(t)\|_{V'}^2 + \langle f'(t), f(t) \rangle_{V' \times V}. \end{aligned} \quad (2.50)$$

Using the inequality of Cauchy-Schwarz and (G3), it results that

$$\langle f'(t), f(t) \rangle_{V'} \leq \zeta_1 \|f(t)\|_{V'}^2. \quad (2.51)$$

Combining (2.50)-(2.51) and picking λ_1 big enough such that

$$m_A - \frac{7}{4\lambda_1} > 0,$$

we arrive at

$$F'(t) \leq K_4 \left(\|u(t)\|_V^2 + (g \circ u)(t) + \|f(t)\|_{V'}^2 \right), \quad (2.52)$$

where $K_4 = \max \{ \lambda_1 K_2, \lambda_1 K_3, \zeta_1 + \lambda_1 \}$.

Moreover, from (2.48) and (G4), we get

$$F(t) \geq K_5 \left(\|u(t)\|_V^2 + (g \circ u)(t) + \|f(t)\|_{V'}^2 \right), \quad (2.53)$$

where $K_5 = \min \left\{ \frac{1}{2}, \frac{l}{2} \right\}$. Therefore, using (2.52), (2.53), we get

$$F'(t) \leq K_6 F(t), \quad (2.54)$$

where $K_6 = \frac{K_4}{K_5}$. A simple integration of (2.54) yields

$$F(t) \leq F(0) e^{K_6 t}. \quad (2.55)$$

Making use of estimate (2.55) with continuation principle ends the proof. $\square$

2.1.4 Exponential stability

To exhibit the exponential stability of solution, we introduce some technical lemmas needed for the proof of our main result.

We consider the following Lyapunov functional $\mathcal{L}$

$$\mathcal{L}(t) = E(t) + \epsilon(N\mathcal{F}_1(t) + \mathcal{F}_2(t)), \quad (2.56)$$

where N, ϵ are positive constants needed to be chosen later and the functionals $\mathcal{F}_1$ and $\mathcal{F}_2$ are respectively given by

$$\mathcal{F}_1(t) = \langle u'(t), u(t) \rangle_H - e^{3\zeta_1 t} \|f(t)\|_{V'}^2, \quad \forall t \geq 0, \quad (2.57)$$

$$\mathcal{F}_2(t) = - \left\langle u'(t), \int_0^t g(t-s) (u(t) - u(s)) ds \right\rangle_H, \quad \forall t \geq 0. \quad (2.58)$$

Lemma 2.4. *Assume that assumptions (G4), (G3), (1.9)-(1.13), (2.8) and (2.10) are satisfied. Thus, the functional $\mathcal{F}_1$ given by (2.57) satisfies for $\lambda_2 > 0$, for all $t \geq 0$*

$$\begin{aligned} \mathcal{F}'_1(t) \leq & (-l_0 + 5\lambda_2) \|u(t)\|_V^2 + K_7(g \circ u)(t) + K_8 \|u'(t)\|_V^2 \\ & + \left(\frac{1}{4\lambda_2} - \gamma_0 \right) \|f(t)\|_{V'}^2, \end{aligned} \quad (2.59)$$

where

$$K_7 = \frac{(1-l)L_B^2}{4\lambda_2}, K_8 = \frac{L_A^2}{4\lambda_2} + \frac{L_\tau^2 C_0^4}{4\lambda_2} + \frac{L_\tau^2 C_0^4}{4\lambda_2} + d^2.$$

Proof. We differentiate (2.57) with respect to time variable t , one has

$$I'_1(t) = \|u'(t)\|_H^2 + \langle u''(t), u(t) \rangle_H - 3\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2 - 2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'}.$$

Multiplying (2.2) by $u(t)$, using (2.1), integrating by parts over domain Ω , recalling formula of Green, boundary conditions and assumption (1.10), the following is immediate

$$\begin{aligned} \mathcal{F}'_1(t) \leq & \|u'(t)\|_H^2 - \langle A\varepsilon(u'(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} - m_G \|u(t)\|_V^2 \\ & + \left\langle \int_0^t g(t-s) (B\varepsilon(u(s)) - B\varepsilon(u(t))) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \\ & + \left\langle \int_0^t g(t-s) B\varepsilon(u(t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} - \int_{\Gamma_3} p_\nu(u'_\nu(t)) u_\nu(t) da \\ & - \int_{\Gamma_3} p_\tau(u'_\tau(t)) u_\tau(t) da + \langle f(t), u(t) \rangle_{V' \times V} - 3\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2 \\ & - 2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'}. \end{aligned} \quad (2.60)$$

Applying inequalities of Cauchy-Schwarz and Young and using (1.9), it holds that

$$\begin{aligned} \langle A\varepsilon(u'(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} & \leq \|A\varepsilon(u'(t))\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\ & \leq \frac{L_A^2}{4\lambda_2} \|u'(t)\|_V^2 + \lambda_2 \|u(t)\|_V^2. \end{aligned}$$

By exploiting again inequalities of Cauchy-Schwarz and Young, taking into account (1.11) and

(G4), it results that

$$\begin{aligned}
 & \left\langle \int_0^t g(t-s) (B\varepsilon(u(s)) - B\varepsilon(u(t))) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \\
 & \leq \frac{L_B^2}{4\lambda_2} \left(\int_0^t g(t-s) \|u(s) - u(t)\|_V ds \right)^2 + \lambda_2 \|u(t)\|_V^2 \\
 & \leq \frac{(1-l)L_B^2}{4\lambda_2} (g \circ u)(t) + \lambda_2 \|u(t)\|_V^2.
 \end{aligned}$$

It follows from assumption (b) of (1.11) and (G4) that

$$\left\langle \int_0^t g(t-s) B\varepsilon(u(t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \leq L_B(1-l) \|u(t)\|_V^2$$

Making use of inequalities of Cauchy-Schwarz and Young, (1.12) and (A.6), we obtain

$$\begin{aligned}
 \int_{\Gamma_3} p_\nu(u'_\nu(t)) u_\nu(t) da & \leq \left(\int_{\Gamma_3} |p_\nu(u'_\nu(t))|^2 da \right)^{\frac{1}{2}} \left(\int_{\Gamma_3} |u_\nu(t)|^2 da \right)^{\frac{1}{2}} \\
 & \leq L_\nu \|u'(t)\|_{(L^2(\Gamma_3))^d} \|u(t)\|_{(L^2(\Gamma_3))^d} \\
 & \leq \frac{L_\nu^2 C_0^4}{4\lambda_2} \|u'(t)\|_V^2 + \lambda_2 \|u(t)\|_V^2.
 \end{aligned}$$

It follows from inequalities of Cauchy-Schwarz and Young, (2.8) and (A.6) that

$$\begin{aligned}
 \int_{\Gamma_3} p_\tau(u'_\tau(t)) u_\tau(t) da & \leq \left(\int_{\Gamma_3} \|p_\tau(u'_\tau(t))\|^2 da \right)^{\frac{1}{2}} \left(\int_{\Gamma_3} \|u_\tau(t)\|^2 da \right)^{\frac{1}{2}} \\
 & \leq L_\tau \|u'(t)\|_{(L^2(\Gamma_3))^d} \|u(t)\|_{(L^2(\Gamma_3))^d} \\
 & \leq \frac{L_\tau^2 C_0^4}{4\lambda_2} \|u'(t)\|_V^2 + \lambda_2 \|u(t)\|_V^2.
 \end{aligned}$$

Applying inequalities of Cauchy-Schwarz and Young, it holds that

$$\langle f(t), u(t) \rangle_{V' \times V} \leq \frac{1}{4\lambda_2} \|f(t)\|_{V'}^2 + \lambda_2 \|u(t)\|_V^2.$$

Using the inequality of Cauchy-Schwarz and taking into account (G3), we obtain

$$\begin{aligned}
 -2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'} & \leq 2e^{3\zeta_1 t} \|f'(t)\|_{V'} \|f(t)\|_{V'} \\
 & \leq 2\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2.
 \end{aligned}$$

Substituting the previous seven inequalities into (2.60) and using (A.9), we get

$$\begin{aligned} \mathcal{F}'_1(t) \leq & (-l_0 + 5\lambda_2) \|u(t)\|_V^2 + \frac{(1-l)L_B^2}{4\lambda_2} (g \circ u)(t) \\ & + \left(\frac{L_A^2}{4\lambda_2} + \frac{L_\nu^2 C_0^4}{4\lambda_2} + \frac{L_\tau^2 C_0^4}{4\lambda_2} + d^2 \right) \|u'(t)\|_V^2 + \left(\frac{1}{4\lambda_2} - \gamma_0 \right) \|f(t)\|_{V'}^2, \end{aligned}$$

where $l_0 = m_G - L_B(1-l)$ and the constant $\gamma_0 > 0$ such that $-\zeta_1 e^{3\zeta_1 t} \leq -\gamma_0$. $\square$

Lemma 2.5. Assume that (G4), and (1.9)-(1.13), (2.8) and (2.10) are satisfied. Then the functional $\mathcal{F}_2$ given by (2.58) satisfies for $\lambda_2 > 0$, for all $t \geq 0$

$$\begin{aligned} \mathcal{F}'_2(t) \leq & - \int_0^t g(s) ds \|u'(t)\|_H^2 + K_9 \lambda_2 \|u'(t)\|_V^2 + K_{10} \lambda_2 \|u(t)\|_V^2 \\ & + K_{11} (g \circ u)(t) - K_{12} (g' \circ u)(t) + \lambda_2 \|f(t)\|_{V'}^2, \end{aligned} \quad (2.61)$$

where

$$K_9 = L_A^2 + L_\nu^2 C_0^4 + L_\tau^2 C_0^4 + d^2, \quad K_{10} = L_G^2 + L_B^2,$$

and

$$K_{11} = \frac{5(1-l)}{4\lambda_2} + \frac{(1-l)^3}{4\lambda_2} + L_B(1-l), \quad K_{12} = \frac{g(0)d^2}{4\lambda_2}.$$

Proof. By taking the derivative of (2.58) with respect to time variable t , we get

$$\begin{aligned} \mathcal{F}'_2(t) = & - \left\langle u''(t), \int_0^t g(t-s)(u(t) - u(s)) ds \right\rangle_H \\ & - \left\langle u'(t), \int_0^t g'(t-s)(u(t) - u(s)) ds \right\rangle_H \\ & - \langle u'(t), \int_0^t g(t-s)u'(t) ds \rangle_H. \end{aligned}$$

Using (2.1)-(2.2), integrating over domain Ω and recalling the formula of Green, boundary

conditions, one has

$$\begin{aligned}
\mathcal{F}'_2(t) = & \left\langle A\varepsilon(u'(t)), \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\
& + \left\langle G\varepsilon(u(t)), \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\
& + \left\langle \int_0^t g(t-s)(B\varepsilon(u(t)) - B\varepsilon(u(s)))ds, \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\
& - \left\langle \int_0^t g(t-s)B\varepsilon(u(t))ds, \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\
& + \int_{\Gamma_3} p_\nu(u'_\nu(t)) \int_0^t g(t-s)(u_\nu(t) - u_\nu(s))dsda \\
& + \int_{\Gamma_3} p_\tau(u'_\tau(t)) \cdot \int_0^t g(t-s)(u_\tau(t) - u_\tau(s))dsda \\
& - \left\langle f(t), \int_0^t g(t-s)(u(t) - u(s))ds \right\rangle_{V' \times V} \\
& - \left\langle u'(t), \int_0^t g'(t-s)(u(t) - u(s))ds \right\rangle_H - \int_0^t g(s)ds \|u'(t)\|_H^2.
\end{aligned} \tag{2.62}$$

Exploiting inequalities of Cauchy-Schwarz and Young, (G4) and (1.9), we see that

$$\begin{aligned}
& \left\langle A\varepsilon(u'(t)), \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\
& \leq \lambda_2 L_A^2 \|u'(t)\|_V^2 + \frac{(1-l)}{4\lambda_2} (g \circ u)(t).
\end{aligned} \tag{2.63}$$

Using again inequalities of Cauchy-Schwarz and Young and taking into account (G4) and (1.10), we get

$$\begin{aligned}
& \left\langle G\varepsilon(u(t)), \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\
& \leq \lambda_2 L_G^2 \|u(t)\|_V^2 + \frac{(1-l)}{4\lambda_2} (g \circ u)(t).
\end{aligned} \tag{2.64}$$

Applying inequality of Cauchy-Schwarz, (G4) and (1.11), it results that

$$\begin{aligned}
& \left\langle \int_0^t g(t-s)(B\varepsilon(u(s)) - B\varepsilon(u(t)))ds, \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\
& \leq L_B(1-l)(g \circ u)(t).
\end{aligned} \tag{2.65}$$

Recalling inequalities of Cauchy-Schwarz and Young, (G4) and (1.11), we have

$$\begin{aligned}
& - \left\langle \int_0^t g(t-s)B\varepsilon(u(t))ds, \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\
& \leq \lambda_2 L_B^2 \|u(t)\|_V^2 + \frac{(1-l)^3}{4\lambda_2} (g \circ u)(t).
\end{aligned} \tag{2.66}$$

Thanks to inequalities of Cauchy-Schwarz and Young, (1.12), (G4) and (A.6), it follows that

$$\begin{aligned} & \int_{\Gamma_3} p_\nu(u'_\nu(t)) \left(\int_0^t g(t-s) (u_\nu(t) - u_\nu(s)) ds \right) da \\ & \leq \lambda_2 L_\nu^2 C_0^4 \|u'(t)\|_V^2 + \frac{(1-l)}{4\lambda_2} (g \circ u)(t). \end{aligned} \quad (2.67)$$

The use of inequalities of Cauchy-Schwarz and Young, (2.8), (G4) and (A.6) leads to the following estimate

$$\begin{aligned} & \int_{\Gamma_3} p_\tau(u'_\tau(t)) \cdot \left(\int_0^t g(t-s) (u_\tau(t) - u_\tau(s)) ds \right) da \\ & \leq \lambda_2 L_\tau^2 C_0^4 \|u'(t)\|_V^2 + \frac{(1-l)}{4\lambda_2} (g \circ u)(t). \end{aligned} \quad (2.68)$$

The following term is estimated as follows

$$-\left\langle f(t), \int_0^t g(t-s) (u(t) - u(s)) ds \right\rangle_{V' \times V} \leq \lambda_2 \|f(t)\|_{V'}^2 + \frac{(1-l)}{4\lambda_2} (g \circ u)(t). \quad (2.69)$$

It follows from inequalities of Cauchy-Schwarz and Young, (G4) and (A.9) that

$$\begin{aligned} & -\left\langle u'(t), \int_0^t g'(t-s) (u(t) - u(s)) ds \right\rangle_H \\ & \leq \|u'(t)\|_H \left\| \int_0^t g'(t-s) (u(t) - u(s)) ds \right\|_H \\ & \leq d^2 \lambda_2 \|u'(t)\|_V^2 - \frac{g(0)d^2}{4\lambda_2} (g' \circ u)(t). \end{aligned} \quad (2.70)$$

Substituting the estimates (2.63)-(2.70) in (2.62), we obtain (2.61). $\square$

Lemma 2.6. *Assume that assumption (G4) is satisfied. Then, there exist positive constants ς_1 and ς_2 such that*

$$\varsigma_1 E(t) \leq \mathcal{L}(t) \leq \varsigma_2 E(t). \quad (2.71)$$

Proof. Applying inequalities of Cauchy-Schwarz and Young and (A.9), it holds that

$$\begin{aligned} |\mathcal{F}_1(t)| &= |\langle u'(t), u(t) \rangle_H - e^{3\varsigma_1 t} \|f(t)\|_{V'}^2| \leq |\langle u'(t), u(t) \rangle_H| \\ &\leq \max\left(1, \frac{d^2}{l}\right) E(t), \end{aligned} \quad (2.72)$$

and

$$\begin{aligned} |\mathcal{F}_2(t)| &= \left| \left\langle u'(t), \int_0^t g(t-s) (u(t) - u(s)) ds \right\rangle_H \right| \\ &\leq \max(1, (1-l)d^2) E(t). \end{aligned} \quad (2.73)$$

Combining (2.39), (2.56), (2.72) and (2.73), we obtain

$$|\mathcal{L}(t) - E(t)| \leq \epsilon \varsigma_3 E(t),$$

where

$$\varsigma_3 = N \max \left(1, \frac{d^2}{l} \right) + \max \left(1, (1-l)d^2 \right) > 0.$$

We notice that if

$$\epsilon < \frac{1}{\varsigma_3}, \quad (2.74)$$

it holds that $\mathcal{L} \sim E$ where $\varsigma_1 = 1 - \epsilon \varsigma_3$ and $\varsigma_2 = 1 + \epsilon \varsigma_3$. $\square$

The following theorem states the exponential stability result.

Theorem 2.3. *Assume that (G2)-(G3), (G4) and (1.9)-(1.13), (2.8) and (2.10) hold. Suppose that m_G is such that*

$$m_G > \frac{5}{4\gamma_0} + L_B(1-l), \quad (2.75)$$

and there exists δ_0 a positive constant independent of m_A satisfying

$$m_A > \delta_0, \quad (2.76)$$

thus, there exist positive constants δ_1 and δ_2 such that

$$E(t) \leq \delta_2 e^{-\delta_1 t}, \quad \forall t \geq t_0. \quad (2.77)$$

Proof. It follows from the positivity of g that for every $t_0 > 0$

$$\int_0^t g(s) ds \geq \int_0^{t_0} g(s) ds := g_0 > 0, \quad t \geq t_0.$$

A differentiation of (2.56) and use of (2.40), (2.59), (2.61) leads for all $t \geq t_0$ to

$$\begin{aligned} \mathcal{L}'(t) &= E'(t) + \epsilon(N\mathcal{F}'_1(t) + \mathcal{F}'_2(t)) \\ &\leq -\epsilon \left(g_0 \|u'(t)\|_H^2 + \left(N(l_0 - 5\lambda_2) - \lambda_2 K_{10} - \frac{K_2 \lambda_1}{\epsilon} \right) \|u(t)\|_V^2 \right) \\ &\quad -\epsilon \left(K_3 \left(N \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3 \lambda_1}{\epsilon} \right) (g \circ u)(t) \\ &\quad + (K_1 - m_A + \epsilon(K_8 N + \lambda_2 K_9)) \|u'(t)\|_V^2 \\ &\quad + \left(\frac{1}{2} - \epsilon K_{12} \right) (g' \circ u)(t) + \epsilon K_{13} (g \circ u)(t) \\ &\quad + \left(\lambda_1 + \epsilon \left(N \left(\frac{1}{4\lambda_2} - \gamma_0 \right) + \lambda_2 \right) \right) \|f(t)\|_{V'}^2, \end{aligned} \quad (2.78)$$

where $K_{13} = NK_7 + K_{11} + K_3 \left(N \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right)$.

Using (2.75), we can take λ_2 such that

$$\frac{1}{4\gamma_0} < \lambda_2 < \frac{l_0}{5}, \quad (2.79)$$

and thus N large enough such that

$$\begin{aligned} N(l_0 - 5\lambda_2) - \lambda_2 K_{10} &> 0, \\ N \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 &> 0. \end{aligned} \quad (2.80)$$

The use of (G2) leads to

$$\begin{aligned} &\left(\frac{1}{2} - \epsilon K_{12} \right) (g' \circ u)(t) + \epsilon K_{13} (g \circ u)(t) \\ &\leq \left(\frac{1}{2} - \epsilon K_{14} \right) (g' \circ u)(t), \end{aligned}$$

where $K_{14} = K_{12} + \frac{K_{13}}{\alpha}$.

Using the above inequality in (2.78), we obtain

$$\begin{aligned} \mathcal{L}'(t) \leq & -\epsilon \left(g_0 \|u'(t)\|_H^2 + \left(N(l_0 - 5\lambda_2) - \lambda_2 K_{10} - \frac{K_2 \lambda_1}{\epsilon} \right) \|u(t)\|_V^2 \right) \\ & -\epsilon \left(K_3 \left(N \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3 \lambda_1}{\epsilon} \right) (g \circ u)(t) \\ & + (K_1 - m_A + \epsilon(K_8 N + \lambda_2 K_9)) \|u'(t)\|_V^2 \\ & + \left(\frac{1}{2} - \epsilon K_{14} \right) (g' \circ u)(t) \\ & + \left(\lambda_1 + \epsilon \left(N \left(\frac{1}{4\lambda_2} - \gamma_0 \right) + \lambda_2 \right) \right) \|f(t)\|_{V'}^2. \end{aligned} \quad (2.81)$$

Additionally, we assume that the constant m_A satisfies (2.76) if we choose δ_0 in the following form

$$\delta_0 = \max \left\{ \frac{\lambda_1 (K_8 N + \lambda_2 K_9)}{N \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2}, \frac{K_2 \lambda_1 (K_8 N + \lambda_2 K_9)}{N(l_0 - 5\lambda_2) - \lambda_2 K_{10}} \right\} + K_1, \quad (2.82)$$

and λ_1 satisfies

$$\begin{aligned} \lambda_1 < \min & \left\{ \frac{\left(\gamma_0 - \frac{1}{4\lambda_2} \right) N - \lambda_2}{\varsigma_3}, \frac{\left(\gamma_0 - \frac{1}{4\lambda_2} \right) N - \lambda_2}{2K_{14}}, \right. \\ & \left. \frac{N(l_0 - 5\lambda_2) - \lambda_2 K_{10}}{K_2 \varsigma_3}, \frac{N(l_0 - 5\lambda_2) - \lambda_2 K_{10}}{2K_2 K_{14}} \right\}. \end{aligned} \quad (2.83)$$

From (2.83), we have

$$\frac{\lambda_1}{\left(\gamma_0 - \frac{1}{4\lambda_2}\right)N - \lambda_2} < \min \left\{ \frac{1}{\varsigma_3}, \frac{1}{2K_{14}} \right\}, \quad (2.84)$$

and

$$\frac{K_2\lambda_1}{N(l_0 - 5\lambda_2) - \lambda_2K_{10}} < \min \left\{ \frac{1}{\varsigma_3}, \frac{1}{2K_{14}} \right\}. \quad (2.85)$$

It follows from (2.82) that

$$\frac{\lambda_1}{\left(\gamma_0 - \frac{1}{4\lambda_2}\right)N - \lambda_2} < \frac{m_A - K_1}{K_8N + \lambda_2K_9}, \quad (2.86)$$

and

$$\frac{K_2\lambda_1}{N(l_0 - 5\lambda_2) - \lambda_2K_{10}} < \frac{m_A - K_1}{K_8N + \lambda_2K_9}. \quad (2.87)$$

Therefore, combining (2.74) and (2.84)-(2.87), ϵ can be fixed as

$$\begin{aligned} & \max \left\{ \frac{\lambda_1}{\left(\gamma_0 - \frac{1}{4\lambda_2}\right)N - \lambda_2}, \frac{K_2\lambda_1}{N(l_0 - 5\lambda_2) - \lambda_2K_{10}} \right\} \\ & < \epsilon < \min \left\{ \frac{m_A - K_1}{K_8N + \lambda_2K_9}, \frac{1}{2K_{14}}, \frac{1}{\varsigma_3} \right\}. \end{aligned} \quad (2.88)$$

Thus, we get

$$\mathcal{L}'(t) \leq -\epsilon K_{15}E(t) + \left(\frac{1}{2} - \epsilon K_{14}\right)(g' \circ u)(t), \quad \forall t \geq t_0,$$

where

$$K_{15} = 2\min \left(g_0, \frac{N(l_0 - 5\lambda_2) - \lambda_2K_{10} - \frac{K_2\lambda_1}{\epsilon}}{1 - g_0}, K_3 \left(N \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3\lambda_1}{\epsilon} \right).$$

Using (G4), (2.88) and (2.71), we get

$$\mathcal{L}'(t) \leq \frac{-\epsilon K_{15}}{1 + \epsilon \varsigma_3} \mathcal{L}(t). \quad (2.89)$$

Integrating (2.89) on $[t_0, t]$, we arrive at

$$\mathcal{L}(t) \leq \mathcal{L}(t_0)e^{-\delta_1(t-t_0)}, \quad \forall t \geq t_0, \quad (2.90)$$

where $\delta_1 = \frac{\epsilon K_{15}}{1 + \epsilon \varsigma_3}$

Using (2.71) and (2.90), we get

$$\begin{aligned} E(t) & \leq \frac{1}{1 - \epsilon \varsigma_3} \mathcal{L}(t) \\ & \leq \frac{1}{1 - \epsilon \varsigma_3} \mathcal{L}(t_0)e^{\delta_1 t_0} e^{-\delta_1 t}, \end{aligned}$$

which yields (2.77) with $\delta_2 = \frac{1}{1-\epsilon\zeta_3}\mathcal{L}(t_0)e^{\delta_1 t_0}$. □

2.2 Frictional contact problem with infinite memory

2.2.1 Problem statement

In this section, we consider the following frictional contact problem with infinite memory and normal damped response

$$\sigma = A\varepsilon(u') + G\varepsilon(u) - \int_0^\infty g(s)B\varepsilon(u(t-s))ds \quad \text{in } \Omega \times (0, \infty), \quad (2.91)$$

$$\rho u'' = \text{Div} \sigma + f_0 \quad \text{in } \Omega \times (0, \infty), \quad (2.92)$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \quad (2.93)$$

$$\sigma \nu = f_2 \quad \text{on } \Gamma_2 \times (0, \infty), \quad (2.94)$$

$$-\sigma_\nu = p_\nu(u'_\nu) \quad \text{on } \Gamma_3 \times (0, \infty), \quad (2.95)$$

$$-\sigma_\tau = p_\tau(u'_\tau) \quad \text{on } \Gamma_3 \times (0, \infty), \quad (2.96)$$

$$u(-t) = u_0(t) \quad \text{in } \Omega \times [0, \infty), \quad (2.97)$$

$$u'(0) = u_1 \quad \text{in } \Omega. \quad (2.98)$$

2.2.2 Global existence

Introducing the variable η as in ([16])

$$\eta(x, s, t) = u(x, t) - u(x, t - s), \quad x \in \Omega, \quad t, s \in \mathbb{R}_+,$$

the dynamic contact problem (2.91)-(2.98) takes the following form

$$\begin{aligned}
\rho u''(t) - \operatorname{Div} \sigma &= f_0 && \text{in } \Omega \times (0, \infty), \\
\sigma &= A\varepsilon(u') + G\varepsilon(u) - \int_0^\infty g(s)B\varepsilon(u - \eta(s))ds && \text{in } \Omega \times (0, \infty), \\
\eta_t &= u' - \eta_s && \text{in } \Omega \times (0, \infty) \times (0, \infty), \\
u(x, t) &= 0 && \text{on } \Gamma_1 \times (0, \infty), \\
\eta(x, s, t) &= 0 && \text{on } \Gamma_1 \times (0, \infty) \times (0, \infty), \\
\sigma \nu &= f_2 && \text{on } \Gamma_2 \times (0, \infty), \\
-\sigma_\nu &= p_\nu(u'_\nu) && \text{on } \Gamma_3 \times (0, \infty), \\
-\sigma_\tau &= p_\tau(u'_\tau) && \text{on } \Gamma_3 \times (0, \infty), \\
u(-t) &= u_0(t) && \text{in } \Omega \times [0, \infty), \\
u'(0) &= u_1 && \text{in } \Omega, \\
\eta(s, 0) &= u_0(0) - u_0(s) = \eta_0(s) && \text{for } s \in (0, \infty).
\end{aligned} \tag{2.99}$$

Assuming that the function σ is smooth (say $\sigma \in C^1(\overline{\Omega}; \mathbb{S}^d)$), thus, the system (2.99) is viewed as

$$\begin{cases} U_t(t) + \mathcal{A}U(t) = F(t), & \forall t > 0, \\ U(0) = U_0 = (u_0, u_1, \eta_0)^T, \end{cases} \tag{2.100}$$

where the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$, the domain $\mathcal{D}(\mathcal{A})$ and the mapping $F(t)$ are correspondingly given by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} -v \\ -\frac{1}{\rho} \operatorname{Div}(A\varepsilon(v) + G\varepsilon(u)) + \frac{1}{\rho} \operatorname{Div}(\int_0^\infty g(s)B\varepsilon(u - \eta(s))ds) \\ \eta_s - v \end{pmatrix}, \tag{2.101}$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} (u, v, \eta)^T \in \mathcal{H}; \ v \in V; \ \eta_s \in \mathcal{M}; \\ \sigma \nu = f_2 \text{ on } \Gamma_2; \\ -\sigma_\nu = p_\nu(v_\nu) \text{ on } \Gamma_3; \\ -\sigma_\tau = p_\tau(v_\tau) \text{ on } \Gamma_3 \end{array} \right\}, \tag{2.102}$$

where

$$\mathcal{H} = V \times H \times \mathcal{M}, \tag{2.103}$$

and

$$F(t) = \begin{pmatrix} 0 \\ \frac{f_0(t)}{\rho} \\ 0 \end{pmatrix}. \tag{2.104}$$

The following theorem ensures the main of the current section

Theorem 2.4. *Under the assumptions (G1), (1.9)-(1.12) and (2.8), for any initial datum*

$U_0 \in \overline{\mathcal{D}(\mathcal{A})}$ and $F \in L^1(\mathbb{R}_+; \mathcal{H})$, there exists a unique mild solution U of the system (2.100) satisfying

$$U \in C(\mathbb{R}_+; \mathcal{H}), \quad (2.105)$$

Proof. To prove Theorem 2.4, we need to show the m-accretivity of the operator $\mathcal{A} + \omega_{\mathcal{A}}I$ for some $\omega_{\mathcal{A}} > 0$.

First, we prove the accretivity of the operator $\mathcal{A} + \omega_{\mathcal{A}}I$ for some $\omega_{\mathcal{A}} > 0$.

Let $U_i = (u_i, v_i, \eta_i) \in \mathcal{D}(\mathcal{A})$, $i = 1, 2$. Then, we obtain

$$\begin{aligned} \langle (\mathcal{A} + \omega_{\mathcal{A}}I)U_1 - (\mathcal{A} + \omega_{\mathcal{A}}I)U_2, U_1 - U_2 \rangle_{\mathcal{H}} &= -\langle v_1 - v_2, u_1 - u_2 \rangle_V \\ &\quad - \left\langle \frac{1}{\rho} \operatorname{Div} (A\varepsilon(v_1) - A\varepsilon(v_2)), v_1 - v_2 \right\rangle_H \\ &\quad - \left\langle \frac{1}{\rho} \operatorname{Div} (G\varepsilon(u_1) - G\varepsilon(u_2)), v_1 - v_2 \right\rangle_H \\ &\quad + \left\langle \frac{1}{\rho} \operatorname{Div} \int_0^\infty g(s)B\varepsilon(u_1 - \eta_1(s))ds, v_1 - v_2 \right\rangle_H \\ &\quad - \left\langle \frac{1}{\rho} \operatorname{Div} \int_0^\infty g(s)B\varepsilon(u_2 - \eta_2(s))ds, v_1 - v_2 \right\rangle_H \\ &\quad - \langle -\eta_{1s} + \eta_{2s} + v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} \\ &\quad + \omega_{\mathcal{A}} \langle U_1 - U_2, U_1 - U_2 \rangle_{\mathcal{H}}. \end{aligned}$$

Using the formula of Green and boundary conditions, one has

$$\begin{aligned} \langle (\mathcal{A} + \omega_{\mathcal{A}}I)U_1 - (\mathcal{A} + \omega_{\mathcal{A}}I)U_2, U_1 - U_2 \rangle_{\mathcal{H}} &= -\langle v_1 - v_2, u_1 - u_2 \rangle_V \\ &\quad + \langle A\varepsilon(v_1) - A\varepsilon(v_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\ &\quad + \langle G\varepsilon(u_1) - G\varepsilon(u_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\ &\quad - \left\langle \int_0^\infty g(s)B\varepsilon(u_1 - \eta_1(s))ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}} \\ &\quad + \left\langle \int_0^\infty g(s)B\varepsilon(u_2 - \eta_2(s))ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} (p_\nu(v_{1\nu}) - p_\nu(v_{2\nu})) (v_{1\nu} - v_{2\nu}) da \\ &\quad + \int_{\Gamma_3} (p_\tau(v_{1\tau}) - p_\tau(v_{2\tau})) \cdot (v_{1\tau} - v_{2\tau}) da \\ &\quad - \frac{1}{2} \int_0^\infty g'(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds \\ &\quad - \langle v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} \\ &\quad + \omega_{\mathcal{A}} \langle U_1 - U_2, U_1 - U_2 \rangle_{\mathcal{H}}. \end{aligned}$$

Also using inequalities of Cauchy- Schwarz and Young, (G1), (b) of (1.10) and (1.11), (c) of

1.9), (1.12) and (2.8), we have

$$\begin{aligned} \langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} &\geq - \left(\frac{5}{4\delta_*} - m_A \right) \|v_1 - v_2\|_V^2 \\ &\quad - \delta_* \left(1 + L_G^2 + g_0^2 L_B^2 \right) \|u_1 - u_2\|_V^2 \\ &\quad - \delta_* \left(L_B^2 g_0 + g_0 \right) \int_0^\infty g(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds \\ &\quad + \omega_{\mathcal{A}} \|U_1 - U_2\|_{\mathcal{H}}^2. \end{aligned}$$

Then, by picking $\delta_* > \frac{5}{4m_A}$, we get

$$\langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \geq (\omega_{\mathcal{A}} - L) \|U_1 - U_2\|_{\mathcal{H}}^2,$$

where $L = \max(\delta_*(1 + L_G^2 + g_0^2 L_B^2), \delta_*(L_B^2 g_0 + g_0))$. So, the operator $\mathcal{A} + \omega_{\mathcal{A}} I$ is accretive if $\omega_{\mathcal{A}} > L$.

Second, let establish the surjectivity of the operator $\mathcal{A} + \lambda I$ for some constant $\lambda > 0$.

Let $(h_1, h_2, h_3)^T \in \mathcal{H}$, we need to show the existence of $U = (u, v, \eta)^T \in \mathcal{D}(\mathcal{A})$ such that

$$(\lambda I + \mathcal{A})(u, v, \eta)^T = (h_1, h_2, h_3)^T,$$

this last can be read as

$$\begin{aligned} \lambda u - v &= h_1 \\ \lambda v - \frac{1}{\rho} \text{Div}(G\varepsilon(u) + A\varepsilon(v)) + \frac{1}{\rho} \text{Div} \left(\int_0^\infty g(s) B\varepsilon(u - \eta(s)) ds \right) &= h_2 \\ \lambda \eta + \eta_s - v &= h_3. \end{aligned} \tag{2.106}$$

Suppose that v is found in V . Then, using (2.106)₁, we get

$$u = \frac{h_1 + v}{\lambda}, \tag{2.107}$$

hence,

$$u \in V.$$

Next, (2.106)₃ with $\eta(0) = 0$ admits the following unique solution

$$\eta(s) = e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau, \quad s \in \mathbb{R}_+. \tag{2.108}$$

Note that (2.106) is equivalent to

$$\begin{aligned} & \lambda v - \frac{1}{\rho} \operatorname{Div} \left(G \varepsilon \left(\frac{h_1 + v}{\lambda} \right) + A \varepsilon(v) \right) \\ & + \frac{1}{\rho} \operatorname{Div} \left(\int_0^\infty g(s) B \varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds \right) \\ & = h_2. \end{aligned} \quad (2.109)$$

We define the operator $\mathcal{T} : V \rightarrow V'$ by

$$\begin{aligned} \langle \mathcal{T}(v), v \rangle_{V' \times V} &= \langle \lambda v, v \rangle_H + \left\langle G \varepsilon \left(\frac{h_1 + v}{\lambda} \right) + A \varepsilon(v), \varepsilon(v) \right\rangle_{\mathcal{H}} \\ &- \left\langle \int_0^\infty g(s) B \varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) \right\rangle_{\mathcal{H}} \\ &+ \int_{\Gamma_3} p_\nu(v_\nu) v_\nu da + \int_{\Gamma_3} p_\tau(v_\tau) \cdot v_\tau da - \int_{\Gamma_2} f_2 v da. \end{aligned} \quad (2.110)$$

We exhibit that the mapping $\mathcal{T} : V \rightarrow V'$ is surjective. It follows from Corollary A.1 that we must prove that the operator $\mathcal{T}$ is coercive and maximal monotone.

Using Theorem A.1, we prove that $\mathcal{T} : V \rightarrow V'$ is maximal monotone by verifying that $\mathcal{T}$ is both monotone and hemicontinuous.

In order to demonstrate the monotonicity of the operator $\mathcal{T}$, let $v, \tilde{v} \in V$. Then, we have

$$\begin{aligned} \langle \mathcal{T}v - \mathcal{T}\tilde{v}, v - \tilde{v} \rangle_{V' \times V} &= \langle \lambda v - \lambda \tilde{v}, v - \tilde{v} \rangle_H + \left\langle G \varepsilon \left(\frac{h_1 + v}{\lambda} \right) - G \varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} \right), \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ &+ \langle A \varepsilon(v) - A \varepsilon(\tilde{v}), \varepsilon(v) - \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \\ &- \left\langle \int_0^\infty g(s) B \varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ &+ \left\langle \int_0^\infty g(s) B \varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + \tilde{v}) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ &+ \int_{\Gamma_3} (p_\nu(v_\nu) - p_\nu(\tilde{v}_\nu)) (v_\nu - \tilde{v}_\nu) da \\ &+ \int_{\Gamma_3} (p_\tau(v_\tau) - p_\tau(\tilde{v}_\tau)) \cdot (v_\tau - \tilde{v}_\tau) da. \end{aligned}$$

From (c) of (1.12), we have

$$\int_{\Gamma_3} (p_\nu(v_\nu) - p_\nu(\tilde{v}_\nu)) (v_\nu - \tilde{v}_\nu) da \geq 0, \quad (2.111)$$

and from (c) of (2.8), we get

$$\int_{\Gamma_3} (p_\tau(v_\tau) - p_\tau(\tilde{v}_\tau)) \cdot (v_\tau - \tilde{v}_\tau) da \geq 0. \quad (2.112)$$

Applying the same arguments as in (1.35)-(1.37) and using (2.111)-(2.112), we get

$$\langle \mathcal{T}v - \mathcal{T}\tilde{v}, v - \tilde{v} \rangle_{V \times V'} \geq \left(\frac{m_G - g_0 L_B}{\lambda} \right) \|v - \tilde{v}\|_V^2.$$

It follows from assumption (1.16) that the operator $\mathcal{T}$ is strongly monotone. The strong monotonicity of $\mathcal{T}$ implies also the coercivity of $\mathcal{T}$.

Next, let prove the hemicontinuity of $\mathcal{T}$. Using Definition A.2, it suffices to prove that

$$\mathcal{T}(v + \zeta v_1) \xrightarrow{w} \mathcal{T}(v), \quad \forall v, v_1 \in V.$$

when $\zeta \rightarrow 0$.

Let $\tilde{v} \in V$ and $\lambda > 0$, then

$$\begin{aligned} & \langle \mathcal{T}(v + \zeta v_1), \tilde{v} \rangle_{V' \times V} - \langle \mathcal{T}(v), \tilde{v} \rangle_{V' \times V} \\ = & \zeta \langle \lambda v_1, \tilde{v} \rangle_H + \left\langle G\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + v}{\lambda} \right), \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \langle A\varepsilon(v + \zeta v_1) - A\varepsilon(v), \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \\ & - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v + \zeta v_1) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \int_{\Gamma_3} (p_\nu(v_\nu + \zeta v_{1\nu}) - p_\nu(v_\nu)) \tilde{v}_\nu da + \int_{\Gamma_3} (p_\tau(v_\tau + \zeta v_{1\tau}) - p_\tau(v_\tau)) \cdot \tilde{v}_\tau da. \end{aligned} \quad (2.113)$$

Applying again inequality of Cauchy-Schwarz and using (b) of (1.12) and (A.6), one has

$$\begin{aligned} \left| \int_{\Gamma_3} (p_\nu(v_\nu + \zeta v_{1\nu}) - p_\nu(v_\nu)) \tilde{v}_\nu da \right| & \leq \|p_\nu(v_\nu + \zeta v_{1\nu}) - p_\nu(v_\nu)\|_{L^2(\Gamma_3)} \|\tilde{v}_\nu\|_{L^2(\Gamma_3)} \\ & \leq L_\nu |\zeta| \|v_1\|_{L^2(\Gamma_3)^d} \|\tilde{v}\|_{L^2(\Gamma_3)^d} \\ & \leq L_\nu C_0^2 |\zeta| \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0. \end{aligned} \quad (2.114)$$

as $\zeta \rightarrow 0$. From Cauchy-Schwarz's inequality, (b) of (2.8) and (A.6), we have

$$\begin{aligned} & \left| \int_{\Gamma_3} (p_\tau(v_\tau + \zeta v_{1\tau}) - p_\tau(v_\tau)) \cdot \tilde{v}_\tau da \right| \\ & \leq \|p_\tau(v_\tau + \zeta v_{1\tau}) - p_\tau(v_\tau)\|_{L^2(\Gamma_3)^d} \|\tilde{v}_\tau\|_{L^2(\Gamma_3)^d} \\ & \leq L_\tau C_0^2 |\zeta| \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \end{aligned} \quad (2.115)$$

as $\zeta \rightarrow 0$. Therefore, by applying similar procedure of (1.40)-(1.43) and exploiting (2.114)-(2.115), it holds that

$$|\langle \mathcal{T}(v + \zeta v_1), \tilde{v} \rangle_{V' \times V} - \langle \mathcal{T}(v), \tilde{v} \rangle_{V' \times V}| \rightarrow 0, \quad (2.116)$$

as $\zeta \rightarrow 0$. As a deduction, $\mathcal{T} : V \rightarrow V'$ is hemicontinuous.

We have proved that $\mathcal{T}$ is coercive, monotone and hemicontinuous and so $\mathcal{T}$ is coercive and maximal monotone. Thus, the surjectivity of the operator $\mathcal{T}$ follows. Therefore, for every $(h_1, h_2, h_3)^T \in \mathcal{H}$, we can find $v \in V$ satisfying the equation (2.109).

Similar procedure as in (1.46) and (1.47), we get

$$(\eta, \eta_s) \in \mathcal{M} \times \mathcal{M}$$

Therefore $U = (u, v, \eta)^T \in \mathcal{D}(\mathcal{A})$ and thus $\mathcal{A} + \lambda I$ is surjective operator for some positive λ . . It results from Theorem A.3 and $F \in L^1(\mathbb{R}_+; \mathcal{H})$ that there exists a unique mild solution U with regularity (2.105) satisfying the system (2.100) which completes the proof of Theorem 2.4. $\square$

2.2.3 Exponential stability

In the current subsection, assuming that the function σ is smooth, we expose some technical lemmas to exhibit our main results. Then, we establish our main exponential stability result.

Now, we set the energy functional E

$$E(t) = \frac{1}{2} \|U(t)\|_{\mathcal{H}}^2 = \frac{1}{2} \left(\|u'(t)\|_H^2 + \|u(t)\|_V^2 + \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \right), \quad \forall t \in \mathbb{R}_+. \quad (2.117)$$

Lemma 2.7. *Assume that (G1), (1.9)-(1.13) and (2.8) hold and $U_0 \in \mathcal{D}(\mathcal{A})$. Then, the energy functional defined by (2.117) satisfies for $\lambda_1 > 0$, for all $t \geq 0$*

$$\begin{aligned} E'(t) \leq & -(m_A - K_1) \|u'(t)\|_V^2 + \lambda_1 K_2 \|u(t)\|_V^2 + \lambda_1 K_3 \int_0^{+\infty} g(s) \|\eta(s)\|_V^2 ds \\ & + \frac{1}{2} \int_0^{+\infty} g'(s) \|\eta(s)\|_V^2 ds + \lambda_1 \|f(t)\|_{V'}^2, \end{aligned} \quad (2.118)$$

where

$$K_1 = \frac{3}{2\lambda_1}, K_2 = L_G^2 + g_0^2 L_B^2 + 1, K_3 = g_0 + g_0 L_B^2.$$

Proof. Differentiating the energy functional (2.117) with respect to time variable t . Then, using (2.99)₁ and (2.99)₃, integrating over domain Ω , using formula of Green and boundary conditions,

it results

$$\begin{aligned}
E'(t) &= \langle u''(t), u'(t) \rangle_H + \langle u'(t), u(t) \rangle_V + \int_0^{+\infty} g(s) \langle \eta_t(s, t), \eta(s, t) \rangle_V ds \\
&= -\langle A\varepsilon(u'(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} - \langle G\varepsilon(u(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} + \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u'(t)) \right\rangle_{\mathcal{H}} \\
&\quad - \int_{\Gamma_3} p_\nu(u'_\nu(t)) u'_\nu(t) da - \int_{\Gamma_3} p_\tau(u'_\tau(t)) u'_\tau(t) da + \langle f(t), u'(t) \rangle_{V' \times V} + \langle u'(t), u(t) \rangle_V \\
&\quad + \int_0^\infty g(s) \langle u'(t), \eta(s, t) \rangle_V ds + \frac{1}{2} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds.
\end{aligned}$$

Using (G1), (1.9) – (1.12), (2.8), inequalities of Cauchy-Schwarz and Young, we get (2.118). $\square$

Remark 2.1. From (2.118), we note that the functional E is not decreasing in general.

We start by establishing several lemmas to prove our primary result.

Lemma 2.8. Assume that (G1), (G3), (1.9)-(1.13) and (2.8) hold and $U_0 \in \mathcal{D}(\mathcal{A})$. Then the functional

$$\mathcal{F}_1(t) = \langle u'(t), u(t) \rangle_H - e^{3\zeta_1 t} \|f(t)\|_{V'}^2, \quad (2.119)$$

satisfies for $\lambda_2 > 0$, for all $t \geq 0$

$$\mathcal{F}'_1(t) \leq (-l + 5\lambda_2) \|u(t)\|_V^2 + K_4 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds + K_5 \|u'(t)\|_V^2 + \left(\frac{1}{4\lambda_2} - \lambda_0 \right) \|f(t)\|_{V'}^2, \quad (2.120)$$

where

$$K_4 = \frac{g_0 L_B^2}{4\lambda_2}, K_5 = \frac{L_A^2}{4\lambda_2} + \frac{L_B^2 C_0^4}{4\lambda_2} + \frac{L_\tau^2 C_0^4}{4\lambda_2} + d^2.$$

Proof. Differentiating (2.119) with respect to t , one has

$$\mathcal{F}'_1(t) = \|u'(t)\|_H^2 + \langle u''(t), u(t) \rangle_H - 3\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2 - 2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'}.$$

After that, we multiply (2.99)₁ by $u(t)$, integrating over domain Ω , using formula of Green, boundary conditions and (1.10), we arrive at

$$\begin{aligned}
\mathcal{F}'_1(t) &\leq \|u'(t)\|_H^2 - \langle A\varepsilon(u'(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} - m_G \|u(t)\|_V^2 + \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \\
&\quad - \int_{\Gamma_3} p_\nu(u'_\nu(t)) u_\nu(t) da - \int_{\Gamma_3} p_\tau(u'_\tau(t)) u_\tau(t) da + \langle f(t), u(t) \rangle_{V' \times V} - 3\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2 \\
&\quad - 2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'}.
\end{aligned} \quad (2.121)$$

Thanks to inequalities of Cauchy-Schwarz and Young, (1.12) and (A.6), it follows that

$$\begin{aligned} \int_{\Gamma_3} p_\nu(u'_\nu(t))u_\nu(t)da &\leq \left(\int_{\Gamma_3} |p_\nu(u'_\nu(t))|^2 da \right)^{\frac{1}{2}} \left(\int_{\Gamma_3} |u_\nu(t)|^2 da \right)^{\frac{1}{2}} \\ &\leq L_\nu \|u'(t)\|_{(L^2(\Gamma_3))^d} \|u(t)\|_{(L^2(\Gamma_3))^d} \\ &\leq \frac{L_\nu^2 C_0^4}{4\lambda_2} \|u'(t)\|_V^2 + \lambda_2 \|u(t)\|_V^2. \end{aligned} \quad (2.122)$$

Recalling inequalities of Cauchy-Schwarz and Young, (2.8) and (A.6), one has

$$\begin{aligned} \int_{\Gamma_3} p_\tau(u'_\tau(t))u_\tau(t)da &\leq \left(\int_{\Gamma_3} |p_\tau(u'_\tau(t))|^2 da \right)^{\frac{1}{2}} \left(\int_{\Gamma_3} |u_\tau(t)|^2 da \right)^{\frac{1}{2}} \\ &\leq L_\tau \|u'(t)\|_{(L^2(\Gamma_3))^d} \|u(t)\|_{(L^2(\Gamma_3))^d} \\ &\leq \frac{L_\tau^2 C_0^4}{4\lambda_2} \|u'(t)\|_V^2 + \lambda_2 \|u(t)\|_V^2. \end{aligned} \quad (2.123)$$

Applying the same arguments of (1.53)–(1.56), using (A.9), (2.122) and (2.123), we get

$$\begin{aligned} \mathcal{F}'_1(t) &\leq (-l + 5\lambda_2) \|u(t)\|_V^2 + \frac{g_0 L_B^2}{4\lambda_2} \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ &\quad + \left(\frac{L_A^2}{4\lambda_2} + \frac{L_\nu^2 C_0^4}{4\lambda_2} + \frac{L_\tau^2 C_0^4}{4\lambda_2} + d^2 \right) \|u'(t)\|_V^2 + \left(\frac{1}{4\lambda_2} - \lambda_0 \right) \|f(t)\|_{V'}^2, \end{aligned}$$

where $l = m_G - g_0 L_B$ and the constant $\lambda_0 > 0$ is such that $-\zeta_1 e^{3\zeta_1 t} \leq -\lambda_0$. $\square$

Lemma 2.9. Assume that (G1), (1.9)–(1.13) and (2.8) hold and $U_0 \in \mathcal{D}(\mathcal{A})$. Then, the functional

$$\mathcal{F}_2(t) = - \left\langle u'(t), \int_0^{+\infty} g(s) \eta(s, t) ds \right\rangle_H, \quad (2.124)$$

satisfies for $\lambda_2 > 0$, for all $t \geq 0$

$$\begin{aligned} \mathcal{F}'_2(t) &\leq -g_0 \|u'(t)\|_H^2 + K_6 \lambda_2 \|u'(t)\|_V^2 + K_7 \lambda_2 \|u(t)\|_V^2 + K_8 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ &\quad - K_9 \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds + \lambda_2 \|f(t)\|_{V'}^2, \end{aligned} \quad (2.125)$$

where

$$K_6 = L_A^2 + L_\nu^2 C_0^4 + L_\tau^2 C_0^4 + d^2, K_7 = L_G^2 + L_B^2, K_8 = \frac{5g_0}{4\lambda_2} + \frac{g_0^3}{4\lambda_2} + L_B g_0, K_9 = \frac{g(0)d^2}{4\lambda_2}.$$

Proof. The Differentiation of (2.124) with respect to the variable t and the use of (2.99)₃ lead to

$$\mathcal{F}'_2(t) = - \left\langle u''(t), \int_0^\infty g(s) \eta(s, t) ds \right\rangle_H - g_0 \|u'(t)\|_H^2 + \left\langle u'(t), \int_0^\infty g(s) \eta_s(s, t) ds \right\rangle_H.$$

Integrating with respect to the variable s the term $\langle u'(t), \int_0^\infty g(s) \eta_s(s, t) ds \rangle_H$ and using $\eta(0) =$

0, $\lim_{s \rightarrow +\infty} g(s) = 0$, we find

$$\mathcal{F}'_2(t) = - \left\langle u''(t), \int_0^\infty g(s) \eta(s, t) ds \right\rangle_H - g_0 \|u'(t)\|_H^2 - \left\langle u'(t), \int_0^\infty g'(s) \eta(s, t) ds \right\rangle_H.$$

Exploiting (2.99)₁, integrating over domain Ω and using formula of Green, boundary conditions, one obtains

$$\begin{aligned} \mathcal{F}'_2(t) &= \left\langle A\varepsilon(u'(t)), \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\rangle_{\mathcal{H}} + \left\langle G\varepsilon(u(t)), \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\rangle_{\mathcal{H}} \\ &\quad - \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\rangle_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} p_\nu(u'_\nu(t)) \int_0^\infty g(s) \eta_\nu(s, t) ds da + \int_{\Gamma_3} p_\tau(u'_\tau(t)) \int_0^\infty g(s) \eta_\tau(s, t) ds da \\ &\quad - \left\langle f(t), \int_0^\infty g(s) \eta(s, t) ds \right\rangle_{V' \times V} - g_0 \|u'(t)\|_H^2 \\ &\quad - \left\langle u'(t), \int_0^\infty g'(s) \eta(s, t) ds \right\rangle_H. \end{aligned} \quad (2.126)$$

Applying inequalities of Cauchy-Schwarz and Young and using (1.12), (G1), (A.6), we obtain

$$\begin{aligned} \int_{\Gamma_3} p_\nu(u'_\nu(t)) \left(\int_0^\infty g(s) \eta_\nu(s, t) ds \right) da &\leq \left(\int_{\Gamma_3} p_\nu^2(u'_\nu(t)) da \right)^{\frac{1}{2}} \left(\int_{\Gamma_3} \left(\int_0^\infty g(s) \eta_\nu(s, t) ds \right)^2 da \right)^{\frac{1}{2}} \\ &\leq \lambda_2 L_\nu^2 C_0^4 \|u'(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds. \end{aligned} \quad (2.127)$$

Recalling inequalities of Cauchy-Schwarz and Young and using (1.12), (G1), (A.6), we obtain

$$\begin{aligned} \int_{\Gamma_3} p_\tau(u'_\tau(t)) \left(\int_0^\infty g(s) \eta_\tau(s, t) ds \right) da &\leq \left(\int_{\Gamma_3} p_\tau^2(u'_\tau(t)) da \right)^{\frac{1}{2}} \left(\int_{\Gamma_3} \left(\int_0^\infty g(s) \eta_\tau(s, t) ds \right)^2 da \right)^{\frac{1}{2}} \\ &\leq \lambda_2 L_\tau^2 C_0^4 \|u'(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds. \end{aligned} \quad (2.128)$$

Combining the estimates (1.60)-(1.62), (1.64)-(1.65) and (2.127)-(2.128), we obtain (2.125). $\square$

We consider a Lyapunov functional $\mathcal{L}$

$$\mathcal{L}(t) = E(t) + \epsilon(N\mathcal{F}_1(t) + \mathcal{F}_2(t)), \quad (2.129)$$

where N, ϵ are positive constants that will be chosen later.

Lemma 2.10. *Assume that (G1) hold, there exist two positive constants ς_1 and ς_2 such that*

$$\varsigma_1 E(t) \leq \mathcal{L}(t) \leq \varsigma_2 E(t). \quad (2.130)$$

Proof. Using inequalities of Cauchy-Schwarz and Young and (A.9), we have

$$\begin{aligned} |\mathcal{F}_1(t)| &= |\langle u'(t), u(t) \rangle_H - e^{3\varsigma_1 t} \|f(t)\|_{V'}^2| \\ &\leq \frac{1}{2} \|u'(t)\|_H^2 + \frac{1}{2} \|u(t)\|_H^2 \\ &\leq \max(1, d^2) E(t), \end{aligned} \quad (2.131)$$

and

$$\begin{aligned} |\mathcal{F}_2(t)| &= \left| \left\langle u'(t), \int_0^{+\infty} g(s) \eta(s, t) ds \right\rangle_H \right| \\ &\leq \frac{1}{2} \left(\|u'(t)\|_H^2 + g_0 d^2 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \right) \\ &\leq \max(1, g_0 d^2) E(t). \end{aligned} \quad (2.132)$$

Combining (2.129), (2.117), (2.131) and (2.132), we get

$$|\mathcal{L}(t) - E(t)| \leq \epsilon \varsigma_3 E(t),$$

where the constant $\varsigma_3 > 0$ is such that

$$\varsigma_3 = N \max(1, d^2) + \max(1, g_0 d^2),$$

then by choosing

$$\epsilon < \frac{1}{\varsigma_3}, \quad (2.133)$$

we have $\mathcal{L} \sim E$. □

Theorem 2.5. Assume that (G1)-(G3), (1.9)-(1.13) and (2.8) hold. Suppose that m_G is such that

$$m_G > \frac{5}{4\lambda_0} + g_0 L_B, \quad (2.134)$$

and there exists δ_0 a positive constant independant of m_A satisfying

$$m_A > \delta_0, \quad (2.135)$$

thus, for every $U_0 \in \mathcal{D}(\mathcal{A})$, there exist positive constants δ_1 and δ_2 such that

$$\|U(t)\|_{\mathcal{H}}^2 \leq \delta_2 e^{-\delta_1 t}. \quad (2.136)$$

Proof. Combining (2.118), (2.120) and (2.125), we have for all $t \geq 0$

$$\begin{aligned}
\mathcal{L}'(t) &= E'(t) + \epsilon(N\mathcal{F}'_1(t) + \mathcal{F}'_2(t)) \\
&\leq -\epsilon \left(g_0 \|u'(t)\|_H^2 + \left(N(l - 5\lambda_2) - \lambda_2 K_7 - \frac{K_2 \lambda_1}{\epsilon} \right) \|u(t)\|_V^2 \right) \\
&\quad -\epsilon \left(\left(K_3 \left(N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3 \lambda_1}{\epsilon} \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \right) \\
&\quad + (K_1 - m_A + \epsilon(K_5 N + \lambda_2 K_6)) \|u'(t)\|_V^2 + \left(\frac{1}{2} - \epsilon K_9 \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds \\
&\quad + \epsilon K_{10} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds + \left(\lambda_1 + \epsilon \left(N \left(\frac{1}{4\lambda_2} - \lambda_0 \right) + \lambda_2 \right) \right) \|f(t)\|_{V'}^2, \quad (2.137)
\end{aligned}$$

where $K_{10} = NK_4 + K_8 + K_3 \left(N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right)$.

Using (2.134), we can pick the constant λ_2 such that

$$\frac{1}{4\lambda_0} < \lambda_2 < \frac{l}{5}, \quad (2.138)$$

and therefore N large enough such that

$$\begin{aligned}
N(l - 5\lambda_2) - \lambda_2 K_7 &> 0, \\
N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 &> 0. \quad (2.139)
\end{aligned}$$

Using (G2), we obtain

$$\left(\frac{1}{2} - \epsilon K_9 \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds + \epsilon K_{10} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \leq \left(\frac{1}{2} - \epsilon K_{11} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds, \quad (2.140)$$

where $K_{11} = K_9 + \frac{K_{10}}{\alpha}$.

Inserting (2.140) in (2.137), we get

$$\begin{aligned}
\mathcal{L}'(t) &\leq -\epsilon \left(g_0 \|u'(t)\|_H^2 + \left(N(l - 5\lambda_2) - \lambda_2 K_7 - \frac{K_2 \lambda_1}{\epsilon} \right) \|u(t)\|_V^2 \right) \\
&\quad -\epsilon \left(\left(K_3 \left(N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3 \lambda_1}{\epsilon} \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \right) \\
&\quad + (K_1 - m_A + \epsilon(K_5 N + \lambda_2 K_6)) \|u'(t)\|_V^2 + \left(\frac{1}{2} - \epsilon K_{11} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds \\
&\quad + \left(\lambda_1 + \epsilon \left(N \left(\frac{1}{4\lambda_2} - \lambda_0 \right) + \lambda_2 \right) \right) \|f(t)\|_{V'}^2. \quad (2.141)
\end{aligned}$$

Afterward, we assume that the constant m_A satisfies (2.135) if we choose δ_0 such that

$$\delta_0 = \max \left\{ \frac{\lambda_1 (K_5 N + \lambda_2 K_6)}{N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2}, \frac{K_2 \lambda_1 (K_5 N + \lambda_2 K_6)}{N(l - 4\lambda_2) - \lambda_2 K_7} \right\} + K_1, \quad (2.142)$$

and λ_1 satisfies

$$\lambda_1 < \min \left\{ \frac{\left(\lambda_0 - \frac{1}{4\lambda_2}\right) N - \lambda_2}{\varsigma_3}, \frac{\left(\lambda_0 - \frac{1}{4\lambda_2}\right) N - \lambda_2}{2K_{11}}, \frac{N(l - 5\lambda_2) - \lambda_2 K_7}{K_2 \varsigma_3}, \frac{N(l - 5\lambda_2) - \lambda_2 K_7}{2K_2 K_{11}} \right\}. \quad (2.143)$$

From (2.143), we have

$$\frac{\lambda_1}{\left(\lambda_0 - \frac{1}{4\lambda_2}\right) N - \lambda_2} < \min \left\{ \frac{1}{2K_{11}}, \frac{1}{\varsigma_3} \right\}, \quad (2.144)$$

and

$$\frac{K_2 \lambda_1}{N(l - 5\lambda_2) - \lambda_2 K_7} < \min \left\{ \frac{1}{2K_{11}}, \frac{1}{\varsigma_3} \right\}. \quad (2.145)$$

Using (2.142), we get

$$\frac{\lambda_1}{\left(\lambda_0 - \frac{1}{4\lambda_2}\right) N - \lambda_2} < \frac{m_A - K_1}{K_5 N + \lambda_2 K_6}, \quad (2.146)$$

and

$$\frac{K_2 \lambda_1}{N(l - 5\lambda_2) - \lambda_2 K_7} < \frac{m_A - K_1}{K_5 N + \lambda_2 K_6}. \quad (2.147)$$

Combining (2.144)-(2.147) and (2.133), we can fix ϵ in this way

$$\max \left\{ \frac{\lambda_1}{\left(\lambda_0 - \frac{1}{4\lambda_2}\right) N - \lambda_2}, \frac{K_2 \lambda_1}{N(l - 5\lambda_2) - \lambda_2 K_7} \right\} < \epsilon < \min \left\{ \frac{m_A - K_1}{K_5 N + \lambda_2 K_6}, \frac{1}{2K_{11}}, \frac{1}{\varsigma_3} \right\}. \quad (2.148)$$

Therefore, we arrive at

$$\mathcal{L}'(t) \leq -\epsilon K_{12} E(t) + \left(\frac{1}{2} - \epsilon K_{11} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds,$$

where

$$K_{12} = 2 \min \left(g_0, N(l - 5\lambda_2) - \lambda_2 K_7 - \frac{K_2 \lambda_1}{\epsilon}, K_3 \left(N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3 \lambda_1}{\epsilon} \right).$$

From (G1), (2.148) and (2.130), we get

$$\mathcal{L}'(t) \leq \frac{-\epsilon K_{12}}{1 + \epsilon \varsigma_3} \mathcal{L}(t). \quad (2.149)$$

Integrating (2.149) over $(0, t)$, we obtain

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\delta_1 t}, \quad \forall t \in \mathbb{R}_+, \quad (2.150)$$

where $\delta_1 = \frac{\epsilon K_{12}}{1 + \epsilon \varsigma_3}$

From (2.130) and (2.150), we have

$$\begin{aligned}
\|U(t)\|_{\mathcal{H}}^2 &= 2E(t) \\
&\leq \frac{2}{1 - \epsilon_{\zeta_3}} \mathcal{L}(t) \\
&\leq \frac{2}{1 - \epsilon_{\zeta_3}} \mathcal{L}(0) e^{-\delta_1 t},
\end{aligned}$$

which yields (2.136) with $\delta_2 = \frac{2}{1 - \epsilon_{\zeta_3}} \mathcal{L}(0)$. □

A DYNAMIC CONTACT PROBLEM WITH ADHESION

The fundamental objective of this chapter is to study the global existence of a solution for a dynamic contact problem with normal compliance and adhesion and establish its exponential decay. The nonlinear semigroup theory is used to show the existence of solution and the exponential decay is investigated by setting a suitable Lyapunov functional.

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3.1 Problem statement

First, we state a model for nonlinear dynamic contact problem where the contact is assumed to be adhesive and modeled with normal compliance. Then, assumptions on problem data are listed. Consider the following nonlinear adhesive contact problem with normal compliance

$$\sigma = A\varepsilon(u') + G\varepsilon(u) - \int_0^\infty g(s)B\varepsilon(u(t-s))ds \quad \text{in } \Omega \times (0, \infty), \quad (3.1)$$

$$\rho u'' = \text{Div} \sigma + f_0 \quad \text{in } \Omega \times (0, \infty), \quad (3.2)$$

$$u = 0 \quad \text{in } \Gamma_1 \times (0, \infty), \quad (3.3)$$

$$\sigma \nu = f_2 \quad \text{in } \Gamma_2 \times (0, \infty), \quad (3.4)$$

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu \beta^2 R_\nu(u_\nu) \quad \text{in } \Gamma_3 \times (0, \infty), \quad (3.5)$$

$$-\sigma_\tau = p_\tau(\beta) R_\tau(u_\tau) \quad \text{in } \Gamma_3 \times (0, \infty), \quad (3.6)$$

$$\beta' = - \left(\beta \left(\gamma_\nu (R_\nu(u_\nu))^2 + \gamma_\tau \|R_\tau(u_\tau)\|^2 \right) - \varepsilon_a \right)_+ \quad \text{in } \Gamma_3 \times (0, \infty), \quad (3.7)$$

$$\beta(0) = \beta_0 \quad \text{in } \Gamma_3, \quad (3.8)$$

$$u(-t) = u_0(t) \quad \text{in } \Omega \times [0, \infty), \quad (3.9)$$

$$u'(0) = u_1 \quad \text{in } \Omega. \quad (3.10)$$

Here, the internal surface β is governed by the differential equation (3.7) combined with initial condition (3.8) where $R_\nu : \mathbb{R} \rightarrow \mathbb{R}_+$ is the following truncation operator

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0, \end{cases} \quad (3.11)$$

and $R_\tau : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the truncation operator that follows

$$R_\tau(v) = \begin{cases} v & \text{if } \|v\| \leq L, \\ L \frac{v}{\|v\|} & \text{if } \|v\| > L. \end{cases} \quad (3.12)$$

$L > 0$ denotes the specific length of the bond.

Remark 3.1. The two operators R_ν and R_τ are Lipschitz continuous where the Lipschitz constants are denoted by L_{R_ν} and L_{R_τ} respectively.

Using the variable η defined by Dafermos in [16], we get

$$\eta(x, s, t) = u(x, t) - u(x, t - s), \quad x \in \Omega, \quad t, s \in \mathbb{R}_+.$$

Note that

$$\eta_t(x, s, t) = u'(x, t) - u'(x, t - s) \text{ and } \eta_s(x, s, t) = u'(x, t - s),$$

thus

$$\eta_t(x, s, t) + \eta_s(x, s, t) = u'(x, t).$$

Then, the system (3.1)-(3.10) becomes

$$\begin{aligned} \rho u'' - \operatorname{Div} \sigma &= f_0 && \text{in } \Omega \times (0, \infty), \\ \sigma &= A\varepsilon(u') + G\varepsilon(u) - \int_0^\infty g(s)B\varepsilon(u - \eta(s))ds && \text{in } \Omega \times (0, \infty), \\ \eta_t &= u' - \eta_s && \text{in } \Omega \times (0, \infty) \times (0, \infty), \\ \beta' &= -(\beta(\gamma_\nu(R_\nu(u_\nu))^2 + \gamma_\tau\|R_\tau(u_\tau)\|^2) - \varepsilon_a)_+ && \text{on } \Gamma_3 \times (0, \infty), \\ u &= \eta(s) = 0 && \text{on } \Gamma_1 \times (0, \infty) \times (0, \infty), \\ \sigma\nu &= f_2 && \text{on } \Gamma_2 \times (0, \infty), \\ -\sigma_\nu &= p_\nu(u_\nu) - \gamma_\nu\beta^2 R_\nu(u_\nu) && \text{on } \Gamma_3 \times (0, \infty), \\ -\sigma_\tau &= p_\tau(\beta)R_\tau(u_\tau) && \text{in } \Gamma_3 \times (0, \infty), \\ u(-t) &= u_0(t) && \text{in } \Omega \times [0, \infty), \\ u'(0) &= u_1 && \text{in } \Omega, \\ \beta(0) &= \beta_0 && \text{on } \Gamma_3, \\ \eta(s, 0) &= u_0(0) - u_0(s) = \eta_0(s) && \text{for } s \in (0, \infty). \end{aligned} \tag{3.13}$$

3.2 Additional assumptions

In addition to assumptions (1.9)-(1.15), (G1)-(G3) defined on Sect. 1.2, we present the following additional assumptions

$$\left\{ \begin{array}{l} \text{(a) } p_\tau : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+; \\ \text{(b) There exists a constant } L_\tau > 0 \text{ such that} \\ \quad |p_\tau(x, \beta_1) - p_\tau(x, \beta_2)| \leq L_\tau |\beta_1 - \beta_2|, \forall \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\ \text{(c) } (p_\tau(x, \beta_1) - p_\tau(x, \beta_2))(\beta_1 - \beta_2) \geq 0, \forall \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\ \text{(d) The mapping } x \mapsto p_\tau(x, \beta) \text{ is Lebesgue measurable on } \Gamma_3, \text{ for any } \beta \in \mathbb{R}; \\ \text{(e) } p_\tau(x, \beta) = 0 \text{ if } \beta = 0, \text{ a.e. } x \in \Gamma_3. \end{array} \right. \tag{3.14}$$

We assume that the adhesion's coefficients γ_ν, γ_τ and the bound limit ε_a satisfy

$$\gamma_\nu, \gamma_\tau, \varepsilon_a \in L^\infty(\Gamma_3), \gamma_\nu, \gamma_\tau, \varepsilon_a \geq 0 \text{ in } \Gamma_3. \tag{3.15}$$

3.3 Global existence

This section is devoted to the existence and uniqueness of a global solution of system (3.1)-(3.10). We denote by $U = (u, v, \eta, \beta)^T$ with $v = u'$ and we assume that σ is a smooth function (say $\sigma \in C^1(\bar{\Omega}; \mathbb{S}^d)$), let consider the following Hilbert space

$$\mathcal{H} = V \times H \times \mathcal{M} \times L^2(\Gamma_3). \quad (3.16)$$

The Hilbert space $\mathcal{H}$, defined before, is equipped with an inner product as follows

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} = \langle u, \tilde{u} \rangle_V + \langle v, \tilde{v} \rangle_H + \langle \eta, \tilde{\eta} \rangle_{\mathcal{M}} + \langle \beta, \tilde{\beta} \rangle_{L^2(\Gamma_3)}, \quad (3.17)$$

for every $U = (u, v, \eta, \beta)^T$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\eta}, \tilde{\beta})^T$ in $\mathcal{H}$.

The system (3.13) can be rewritten as

$$\begin{cases} U_t(t) + \mathcal{A}U(t) = F(t), & \forall t > 0, \\ U(0) = U_0 = (u_0, u_1, \eta_0, \beta_0)^T, \end{cases} \quad (3.18)$$

where the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ and the domain $\mathcal{D}(\mathcal{A})$ are given by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ \eta \\ \beta \end{pmatrix} = \begin{pmatrix} -v \\ -\frac{1}{\rho} \text{Div}(A\varepsilon(v) + G\varepsilon(u)) + \frac{1}{\rho} \text{Div}(\int_0^\infty g(s)B\varepsilon(u - \eta(s))ds) \\ \eta_s - v \\ (\beta(\gamma_\nu(R_\nu(u_\nu))^2 + \gamma_\tau\|R_\tau(u_\tau)\|^2) - \varepsilon_a)_+ \end{pmatrix}, \quad (3.19)$$

and

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} (u, v, \eta, \beta)^T \in \mathcal{H}; \ v \in V; \ \eta_s \in \mathcal{M}; \\ \sigma_\nu = f_2 \text{ on } \Gamma_2; \\ -\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu \beta^2 R_\nu(u_\nu) \text{ on } \Gamma_3; \\ -\sigma_\tau = p_\tau(\beta) R_\tau(u_\tau) \text{ on } \Gamma_3; \\ 0 \leq \beta \leq 1 \end{array} \right\}, \quad (3.20)$$

the function $F(t)$ is given by

$$F(t) = \begin{pmatrix} 0 \\ \frac{f_0(t)}{\rho} \\ 0 \\ 0 \end{pmatrix}.$$

Our main existence result is the following

Theorem 3.1. *Under the assumptions (G1), (1.9)-(1.12), (3.14) (3.15) and for any initial datum $U_0 \in \mathcal{D}(\mathcal{A})$ and $F \in L^1(\mathbb{R}_+; \mathcal{H})$, there exists a unique mild solution U of the system (3.18) satisfying*

$$U \in C(\mathbb{R}_+; \mathcal{H}). \quad (3.21)$$

Proof. To prove Theorem 3.1, we need to exhibit that the operator $\mathcal{A} + \omega_{\mathcal{A}} I$ is m-accretive for some $\omega_{\mathcal{A}} > 0$. It follows from Theorem A.3, $U_0 \in \mathcal{D}(\mathcal{A}) \subset \overline{\mathcal{D}(\mathcal{A})}$ and the regularity $F \in L^1(\mathbb{R}_+; \mathcal{H})$ that there exists a unique mild solution U of system (3.18) satisfying the regularity (3.21).

The operator $\mathcal{A} + \omega_{\mathcal{A}} I$ is accretive for some $\omega_{\mathcal{A}} > 0$, that is,

$$\langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \geq 0, \text{ for all } U_1, U_2 \in \mathcal{D}(\mathcal{A}).$$

Let $U_i = (u_i, v_i, \eta_i, \beta_i)^T \in \mathcal{D}(\mathcal{A})$, $i = 1, 2$. By using the inner product (3.17), we get

$$\begin{aligned} & \langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \\ &= -\langle v_1 - v_2, u_1 - u_2 \rangle_V \\ & \quad - \left\langle \frac{1}{\rho} \operatorname{Div} (A\varepsilon(v_1) - A\varepsilon(v_2)), v_1 - v_2 \right\rangle_H \\ & \quad - \left\langle \frac{1}{\rho} \operatorname{Div} (G\varepsilon(u_1) - G\varepsilon(u_2)), v_1 - v_2 \right\rangle_H \\ & \quad + \left\langle \frac{1}{\rho} \operatorname{Div} \int_0^\infty g(s) B\varepsilon(u_1 - \eta_1(s)) ds, v_1 - v_2 \right\rangle_H \\ & \quad - \left\langle \frac{1}{\rho} \operatorname{Div} \int_0^\infty g(s) B\varepsilon(u_2 - \eta_2(s)) ds, v_1 - v_2 \right\rangle_H \\ & \quad + \langle \eta_{1s} - \eta_{2s}, \eta_1 - \eta_2 \rangle_{\mathcal{M}} \\ & \quad - \langle v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} \\ & \quad + \left\langle \left(\beta_1 \left(\gamma_\nu (R_\nu(u_{1\nu}))^2 + \gamma_\tau \|R_\tau(u_{1\tau})\|^2 \right) - \varepsilon_a \right)_+, \beta_1 - \beta_2 \right\rangle_{L^2(\Gamma_3)} \\ & \quad - \left\langle \left(\beta_2 \left(\gamma_\nu (R_\nu(u_{2\nu}))^2 + \gamma_\tau \|R_\tau(u_{2\tau})\|^2 \right) - \varepsilon_a \right)_+, \beta_1 - \beta_2 \right\rangle_{L^2(\Gamma_3)} \\ & \quad + \omega_{\mathcal{A}} \langle U_1 - U_2, U_1 - U_2 \rangle_{\mathcal{H}}. \end{aligned}$$

We apply Green's formula and use boundary conditions. Then, we integrate the term $\langle \eta_{1s} - \eta_{2s}, \eta_1 - \eta_2 \rangle_{\mathcal{M}}$ with respect to s and use $\eta(0) = 0$, $\lim_{s \rightarrow +\infty} g(s) = 0$ to get

$$\begin{aligned}
& \langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \\
= & -\langle v_1 - v_2, u_1 - u_2 \rangle_V \\
& + \langle A\varepsilon(v_1) - A\varepsilon(v_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\
& + \langle G\varepsilon(u_1) - G\varepsilon(u_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\
& - \left\langle \int_0^\infty g(s) B\varepsilon(u_1 - \eta_1(s)) ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}} \\
& + \left\langle \int_0^\infty g(s) B\varepsilon(u_2 - \eta_2(s)) ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}} \\
& + \int_{\Gamma_3} \left(p_\nu(u_{1\nu}) - \gamma_\nu \beta_1^2 R_\nu(u_{1\nu}) \right) (v_{1\nu} - v_{2\nu}) da \\
& - \int_{\Gamma_3} \left(p_\nu(u_{2\nu}) - \gamma_\nu \beta_2^2 R_\nu(u_{2\nu}) \right) (v_{1\nu} - v_{2\nu}) da \\
& + \int_{\Gamma_3} \left(p_\tau(\beta_1) R_\tau(u_{1\tau}) - p_\tau(\beta_2) R_\tau(u_{2\tau}) \right) \cdot (v_{1\tau} - v_{2\tau}) da \\
& - \frac{1}{2} \int_0^\infty g'(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds - \langle v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} \\
& + \left\langle \left(\beta_1 \left(\gamma_\nu (R_\nu(u_{1\nu}))^2 + \gamma_\tau \|R_\tau(u_{1\tau})\|^2 \right) - \varepsilon_a \right)_+, \beta_1 - \beta_2 \right\rangle_{L^2(\Gamma_3)} \\
& - \left\langle \left(\beta_2 \left(\gamma_\nu (R_\nu(u_{2\nu}))^2 + \gamma_\tau \|R_\tau(u_{2\tau})\|^2 \right) - \varepsilon_a \right)_+, \beta_1 - \beta_2 \right\rangle_{L^2(\Gamma_3)} \\
& + \omega_{\mathcal{A}} \langle U_1 - U_2, U_1 - U_2 \rangle_{\mathcal{H}}.
\end{aligned}$$

Thanks to the inequalities of Cauchy-Schwarz and Young, we obtain

$$-\langle v_1 - v_2, u_1 - u_2 \rangle_V \geq -\frac{1}{4\delta_*} \|v_1 - v_2\|_V^2 - \delta_* \|u_1 - u_2\|_V^2. \quad (3.22)$$

The use of (1.9) leads to

$$\langle A\varepsilon(v_1) - A\varepsilon(v_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \geq m_A \|v_1 - v_2\|_V^2. \quad (3.23)$$

It follows from (1.10), inequalities of Cauchy-Schwarz and Young that

$$\langle G\varepsilon(u_1) - G\varepsilon(u_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \geq -\frac{1}{4\delta_*} \|v_1 - v_2\|_V^2 - \delta_* L_G^2 \|u_1 - u_2\|_V^2. \quad (3.24)$$

Applying the inequalities of Cauchy-Schwarz and Young, using (1.11) and (G1), we obtain

$$\begin{aligned}
& - \left\langle \int_0^\infty g(s) B \varepsilon(u_1 - \eta_1(s)) ds - \int_0^\infty g(s) B \varepsilon(u_2 - \eta_2(s)) ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}} \\
& \geq -L_B \int_0^{+\infty} g(s) \|u_1 - u_2\|_V ds \|v_1 - v_2\|_V - L_B \int_0^{+\infty} g(s) \|\eta_1(s) - \eta_2(s)\|_V ds \|v_1 - v_2\|_V \\
& \geq -\frac{1}{2\delta_*} \|v_1 - v_2\|_V^2 - \delta_* L_B^2 g_0^2 \|u_1 - u_2\|_V^2 - \delta_* L_B^2 g_0 \int_0^{+\infty} g(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds. \quad (3.25)
\end{aligned}$$

Using inequalities of Cauchy-Schwarz and Young, (1.12) and (A.6), we obtain

$$\begin{aligned}
\int_{\Gamma_3} (p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})) (v_{1\nu} - v_{2\nu}) da & \geq -L_\nu \|u_{1\nu} - u_{2\nu}\|_{L^2(\Gamma_3)} \|v_{1\nu} - v_{2\nu}\|_{L^2(\Gamma_3)} \\
& \geq -\delta_* L_\nu^2 C_0^4 \|u_1 - u_2\|_V^2 - \frac{1}{4\delta_*} \|v_1 - v_2\|_V^2. \quad (3.26)
\end{aligned}$$

Exploiting the inequalities of Cauchy-Schwarz and Young and using (3.15), $\beta_i \leq 1, i = 1, 2$, the Lipschitz continuity of R_ν and (A.6), we get

$$\begin{aligned}
& - \int_{\Gamma_3} \gamma_\nu (\beta_1^2 R_\nu(u_{1\nu}) - \beta_2^2 R_\nu(u_{2\nu})) (v_{1\nu} - v_{2\nu}) da \\
& \geq -\|\gamma_\nu\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} ((\beta_1^2 - \beta_2^2) R_\nu(u_{1\nu}) + \beta_2^2 (R_\nu(u_{1\nu}) - R_\nu(u_{2\nu}))) (v_{1\nu} - v_{2\nu}) da \\
& \geq -2L \|\gamma_\nu\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} (\beta_1 - \beta_2) (v_{1\nu} - v_{2\nu}) da - L_{R_\nu} \|\gamma_\nu\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} (u_{1\nu} - u_{2\nu}) (v_{1\nu} - v_{2\nu}) da \\
& \geq -\frac{1}{2\delta_*} \|v_1 - v_2\|_V^2 - 4\delta_* \|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 C_0^2 L^2 \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)}^2 \\
& \quad - \delta_* \|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 L_{R_\nu}^2 C_0^4 \|u_1 - u_2\|_V^2. \quad (3.27)
\end{aligned}$$

It follows from the inequalities of Cauchy-Schwarz and Young, (3.14), (A.6), the Lipschitz continuity of the operators R_τ and $\|R_\tau\| \leq L$ that

$$\begin{aligned}
& \int_{\Gamma_3} (p_\tau(\beta_1) R_\tau(u_{1\tau}) - p_\tau(\beta_2) R_\tau(u_{2\tau})) \cdot (v_{1\tau} - v_{2\tau}) da \\
& \geq -\|(p_\tau(\beta_1) - p_\tau(\beta_2)) R_\tau(u_{1\tau}) - p_\tau(\beta_2) (R_\tau(u_{2\tau}) - R_\tau(u_{1\tau}))\|_{L^2(\Gamma_3)^d} \|v_{1\tau} - v_{2\tau}\|_{L^2(\Gamma_3)^d} \\
& \geq -\delta_* C_0^2 L^2 L_\tau^2 \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)}^2 - \frac{1}{2\delta_*} \|v_1 - v_2\|_V^2 - L_\tau^2 L_{R_\tau}^2 C_0^4 \|u_1 - u_2\|_V^2. \quad (3.28)
\end{aligned}$$

Applying the inequalities of Cauchy-Schwarz and Young and (G1) leads to

$$\begin{aligned}
-\langle v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} & \geq -\|v_1 - v_2\|_V \int_0^{+\infty} g(s) \|\eta_1(s) - \eta_2(s)\|_V ds \\
& \geq -\frac{1}{4\delta_*} \|v_1 - v_2\|_V^2 - \delta_* g_0 \int_0^{+\infty} g(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds. \quad (3.29)
\end{aligned}$$

The use of (A.12), $\beta_1 = \beta_1 - \beta_2 + \beta_2$, inequalities of Cauchy-Schwarz and Young, the Lipschitz

continuity of the operators R_ν and R_τ , (3.15), (A.6), $\|R_\nu\| \leq L$, $\|R_\tau\| \leq L$ and $\beta_i \leq 1$, we get

$$\begin{aligned}
& \left\langle \left(\beta_1 \left(\gamma_\nu (R_\nu(u_{1\nu}))^2 + \gamma_\tau \|R_\tau(u_{1\tau})\|^2 \right) - \varepsilon_a \right)_+, \beta_1 - \beta_2 \right\rangle_{L^2(\Gamma_3)} \\
& - \left\langle \left(\beta_2 \left(\gamma_\nu (R_\nu(u_{2\nu}))^2 + \gamma_\tau \|R_\tau(u_{2\tau})\|^2 \right) - \varepsilon_a \right)_+, \beta_1 - \beta_2 \right\rangle_{L^2(\Gamma_3)} \\
& \geq -L^2 \left(\|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)} \right) \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)}^2 \\
& - \|\gamma_\nu\|_{L^\infty(\Gamma_3)} \left(\|R_\nu(u_{1\nu})\|^2 - \|R_\nu(u_{2\nu})\|^2 \right) + \gamma_\tau \left(\|R_\tau(u_{1\tau})\|^2 - \|R_\tau(u_{2\tau})\|^2 \right) \|_{L^2(\Gamma_3)} \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)} \\
& \geq - \left(L^2 \left(\|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)} \right) + \frac{1}{2\delta_*} \right) \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)}^2 \\
& - 4\delta_* L^2 C_0^2 \left(\|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 L_{R_\nu}^2 + \|\gamma_\tau\|_{L^\infty(\Gamma_3)}^2 L_{R_\tau}^2 \right) \|u_1 - u_2\|_V^2.
\end{aligned} \tag{3.30}$$

Combining the estimations (3.22)-(3.30), we obtain

$$\begin{aligned}
\langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} & \geq \left(m_A - \frac{5}{2\delta_*} \right) \|v_1 - v_2\|_V^2 - C_1 \|u_1 - u_2\|_V^2 \\
& - C_2 \int_0^{+\infty} g(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds \\
& - \frac{1}{2} \int_0^{+\infty} g'(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds \\
& - C_3 \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)} + \omega_{\mathcal{A}} \|U_1 - U_2\|_{\mathcal{H}}^2,
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= \delta_* \left(1 + L_G^2 + L_B^2 g_0^2 + L_\nu C_0^4 + L_\tau^2 L_{R_\tau}^2 C_0^4 + \|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 L_{R_\nu}^2 C_0^4 \right) \\
&+ 4\delta_* L^2 C_0^2 \left(\|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 L_{R_\nu}^2 + \|\gamma_\tau\|_{L^\infty(\Gamma_3)}^2 L_{R_\tau}^2 \right), \\
C_2 &= \delta_* \left(L_B^2 g_0 + g_0 \right), \\
C_3 &= \delta_* \left(4\|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 L^2 C_0^2 + L^2 L_\tau^2 C_0^2 \right) + L^2 \left(\|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)} \right) + \frac{1}{2\delta_*}.
\end{aligned}$$

Then, taking $\delta_* > \frac{5}{2m_A}$ and using (G1) lead to

$$\langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \geq (\omega_{\mathcal{A}} - C) \|U_1 - U_2\|_{\mathcal{H}}^2,$$

where $C = \max(C_1, C_2, C_3)$. Therefore, the operator $\mathcal{A} + \omega_{\mathcal{A}} I$ is accretive if $\omega_{\mathcal{A}} > C$.

The operator $\mathcal{A} + \lambda I$ is surjective. Indeed, for given $(h_1, h_2, h_3, h_4)^T \in \mathcal{H}$, we need to find

$U = (u, v, \eta, \beta)^T \in \mathcal{D}(\mathcal{A})$ such that

$$\lambda u - v = h_1 \quad (3.31)$$

$$\lambda v - \frac{1}{\rho} \text{Div}(G\varepsilon(u) + A\varepsilon(v)) + \frac{1}{\rho} \text{Div}\left(\int_0^\infty g(s)B\varepsilon(u - \eta(s))ds\right) = h_2 \quad (3.32)$$

$$\lambda\eta + \eta_s - v = h_3 \quad (3.33)$$

$$\lambda\beta + \left(\beta\left(\gamma_\nu(R_\nu(u_\nu))^2 + \gamma_\tau\|R_\tau(u_\tau)\|^2\right) - \varepsilon_a\right)_+ = h_4. \quad (3.34)$$

Suppose that v is found in V . So, (3.31) yields to

$$u = \frac{h_1 + v}{\lambda}. \quad (3.35)$$

Thus,

$$u \in V.$$

From (3.33) and $\eta(0) = 0$, we can determine

$$\eta(s) = e^{-\lambda s} \int_0^s e^{\lambda\tau} (h_3(\tau) + v) d\tau, \quad s \in \mathbb{R}_+. \quad (3.36)$$

We define the operator $\mathcal{S} : L^2(\Gamma_3) \rightarrow L^2(\Gamma_3)$ by

$$\langle \mathcal{S}(\beta), \beta \rangle_{L^2(\Gamma_3)} = \langle \lambda\beta, \beta \rangle_{L^2(\Gamma_3)} + \left\langle \left(\beta \left(\gamma_\nu \left(R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right)^2 + \gamma_\tau \left\| R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) \right\|^2 \right) - \varepsilon_a \right)_+, \beta \right\rangle_{L^2(\Gamma_3)}.$$

To prove the surjectivity of the mapping $\mathcal{S} : L^2(\Gamma_3) \rightarrow L^2(\Gamma_3)$. we need to adduce that $\mathcal{S}$ is coercive and maximal monotone (see Corollary A.1).

Let prove the maximal monotonicity of $\mathcal{S} : L^2(\Gamma_3) \rightarrow L^2(\Gamma_3)$. For this objective, we use Theorem A.1, it follows that is enough to find that $\mathcal{S}$ is monotone and hemicontinuous.

First, $\mathcal{S}$ is monotone. Indeed, Let $\beta, \tilde{\beta} \in L^2(\Gamma_3)$

$$\begin{aligned} & \langle \mathcal{S}(\beta) - \mathcal{S}(\tilde{\beta}), \beta - \tilde{\beta} \rangle_{L^2(\Gamma_3)} \\ &= \langle \lambda\beta - \lambda\tilde{\beta}, \beta - \tilde{\beta} \rangle_{L^2(\Gamma_3)} \\ &+ \left\langle \left(\beta \left(\gamma_\nu \left(R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right)^2 + \gamma_\tau \left\| R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) \right\|^2 \right) - \varepsilon_a \right)_+, \beta - \tilde{\beta} \right\rangle_{L^2(\Gamma_3)} \\ &- \left\langle \left(\tilde{\beta} \left(\gamma_\nu \left(R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right)^2 + \gamma_\tau \left\| R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) \right\|^2 \right) - \varepsilon_a \right)_+, \beta - \tilde{\beta} \right\rangle_{L^2(\Gamma_3)}. \end{aligned}$$

The use of Cauchy- Schwarz's inequality, (A.12), (3.15), $|R_\nu| \leq L$ and $\|R_\tau\| \leq L$ leads to

$$\begin{aligned} & \langle \mathcal{S}(\beta) - \mathcal{S}(\tilde{\beta}), \beta - \tilde{\beta} \rangle_{L^2(\Gamma_3)} \\ & \geq \lambda \|\beta - \tilde{\beta}\|_{L^2(\Gamma_3)}^2 - \left\| \left(\beta - \tilde{\beta} \right) \left(\gamma_\nu \left| R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right|^2 + \gamma_\tau \left\| R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) \right\|^2 \right) \right\|_{L^2(\Gamma_3)} \|\beta - \tilde{\beta}\|_{L^2(\Gamma_3)} \\ & \geq \left(\lambda - L^2 \left(\|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)} \right) \right) \|\beta - \tilde{\beta}\|_{L^2(\Gamma_3)}^2. \end{aligned}$$

By choosing $\lambda > L^2 \left(\|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)} \right)$, we see that $\mathcal{S}$ is strongly monotone. The strong monotonicity of $\mathcal{S}$ implies the monotonicity and the coercivity of the operator $\mathcal{S}$.

Let prove that $\mathcal{S}$ is hemicontinuous (see Definition A.2) that is

$$\mathcal{S}(\beta + \zeta\beta_1) \xrightarrow{w} \mathcal{S}(\beta), \forall \beta, \beta_1 \in L^2(\Gamma_3),$$

when $\zeta \rightarrow 0$.

Let $\tilde{\beta} \in L^2(\Gamma_3)$ and $\lambda > L^2 \left(\|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)} \right)$, then

$$\begin{aligned} & \langle \mathcal{S}(\beta + \zeta\beta_1), \tilde{\beta} \rangle_{L^2(\Gamma_3)} - \langle \mathcal{S}(\beta), \tilde{\beta} \rangle_{L^2(\Gamma_3)} \\ & = \langle \lambda\zeta\beta_1, \tilde{\beta} \rangle_{L^2(\Gamma_3)} \\ & + \left\langle \left((\beta + \zeta\beta_1) \left(\gamma_\nu \left(R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right)^2 + \gamma_\tau \left\| R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) \right\|^2 \right) - \varepsilon_a \right) \right\rangle_{+} \tilde{\beta} \Bigg\rangle_{L^2(\Gamma_3)} \\ & - \left\langle \left(\beta \left(\gamma_\nu \left(R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right)^2 + \gamma_\tau \left\| R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) \right\|^2 \right) - \varepsilon_a \right) \right\rangle_{+} \tilde{\beta} \Bigg\rangle_{L^2(\Gamma_3)}. \end{aligned}$$

Applying inequality of Cauchy-Schwarz, we get

$$\left| \langle \lambda\zeta\beta_1, \tilde{\beta} \rangle_{L^2(\Gamma_3)} \right| \leq \lambda |\zeta| \|\beta_1\|_{L^2(\Gamma_3)} \|\tilde{\beta}\|_{L^2(\Gamma_3)} \rightarrow 0, \quad (3.37)$$

as $\zeta \rightarrow 0$. Also, exploiting inequality of Cauchy-Schwarz, (A.12), $|R_\nu| \leq L$, $\|R_\tau\| \leq L$ and (3.15), one has

$$\begin{aligned} & \left| \left\langle \left((\beta + \zeta\beta_1) \left(\gamma_\nu \left(R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right)^2 + \gamma_\tau \left\| R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) \right\|^2 \right) - \varepsilon_a \right) \right\rangle_{+} \tilde{\beta} \right\rangle_{L^2(\Gamma_3)} \\ & - \left\langle \left(\beta \left(\gamma_\nu \left(R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right)^2 + \gamma_\tau \left\| R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) \right\|^2 \right) - \varepsilon_a \right) \right\rangle_{+} \tilde{\beta} \right\rangle_{L^2(\Gamma_3)} \right| \\ & \leq \left\| \zeta\beta_1 \left(\gamma_\nu \left(R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right)^2 + \gamma_\tau \left\| R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) \right\|^2 \right) \right\|_{L^2(\Gamma_3)} \|\tilde{\beta}\|_{L^2(\Gamma_3)} \\ & \leq L^2 \left(\|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)} \right) |\zeta| \|\tilde{\beta}\|_{L^2(\Gamma_3)} \|\beta_1\|_{L^2(\Gamma_3)} \rightarrow 0, \quad (3.38) \end{aligned}$$

as $\zeta \rightarrow 0$. Combining (3.37) and (3.38), we get

$$\left| \langle \mathcal{S}(\beta + \zeta \beta_1), \tilde{\beta} \rangle_{L^2(\Gamma_3)} - \langle \mathcal{S}(\beta), \tilde{\beta} \rangle_{L^2(\Gamma_3)} \right| \rightarrow 0, \quad (3.39)$$

as $\zeta \rightarrow 0$. Thus, the operator $\mathcal{S} : L^2(\Gamma_3) \rightarrow L^2(\Gamma_3)$ is hemicontinuous and by employing the fact that $\mathcal{S}$ is monotone and coercive, we deduce that $\mathcal{S}$ is coercive and maximal monotone. Thus, $\mathcal{S}$ is surjective which implies that we can find $\beta \in L^2(\Gamma_3)$ satisfying (3.34) for every $(h_1, h_4) \in V \times L^2(\Gamma_3)$.

It follows from (3.7) and (3.8) that $\beta(t) \leq \beta_0$. Using the fact that $U_0 \in \mathcal{D}(\mathcal{A})$, then $0 \leq \beta_0 \leq 1$. Consequently, $\beta(t) \leq 1$ for $t \geq 0$, a.e. $x \in \Gamma_3$. Moreover, if $\beta(t_*) = 0$ at $t = t_*$, then, from eq. (3.7), it holds that $\beta'(t) = 0$ for all $t \geq t_*$ a.e. on Γ_3 . Thus, $\beta(t) = 0$ for all $t \geq t_*$ a.e. $x \in \Gamma_3$. We deduce that $0 \leq \beta(t) \leq 1, \forall t \geq 0$.

The system (3.31)-(3.34) can be seen as

$$\begin{aligned} & \lambda v - \frac{1}{\rho} \text{Div} \left(G \varepsilon \left(\frac{h_1 + v}{\lambda} \right) + A \varepsilon(v) \right) \\ & + \frac{1}{\rho} \text{Div} \left(\int_0^\infty g(s) B \varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds \right) \\ & = h_2. \end{aligned} \quad (3.40)$$

We define the operator $\mathcal{T} : V \rightarrow V'$ by

$$\begin{aligned} \langle \mathcal{T}(v), v \rangle_{V' \times V} &= \langle \lambda v, v \rangle_H + \left\langle G \varepsilon \left(\frac{h_1 + v}{\lambda} \right) + A \varepsilon(v), \varepsilon(v) \right\rangle_{\mathcal{H}} \\ &- \left\langle \int_0^\infty g(s) B \varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) \right\rangle_{\mathcal{H}} \\ &+ \int_{\Gamma_3} p_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) v_\nu da - \int_{\Gamma_3} \gamma_\nu \beta^2 R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) v_\nu da \\ &+ \int_{\Gamma_3} p_\tau(\beta) R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) \cdot v_\tau da - \int_{\Gamma_2} f_2 v da. \end{aligned} \quad (3.41)$$

Now, We proceed to show the surjectivity of $\mathcal{T} : V \rightarrow V'$. Using Corollary A.1, we prove that the defined operator $\mathcal{T}$ is coercive and maximal monotone.

To check that $\mathcal{T}$ is monotone, let $v, \tilde{v} \in V$

$$\begin{aligned}
\langle \mathcal{T}v - \mathcal{T}\tilde{v}, v - \tilde{v} \rangle_{V' \times V} &= \langle \lambda v - \lambda \tilde{v}, v - \tilde{v} \rangle_H + \left\langle G\varepsilon \left(\frac{h_1 + v}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} \right), \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}}, \\
&+ \langle A\varepsilon(v) - A\varepsilon(\tilde{v}), \varepsilon(v) - \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \\
&- \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&+ \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + \tilde{v}) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&+ \int_{\Gamma_3} \left(p_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) - p_\nu \left(\frac{h_{1\nu} + \tilde{v}_\nu}{\lambda} \right) \right) (v_\nu - \tilde{v}_\nu) da \\
&- \int_{\Gamma_3} \gamma_\nu \beta^2 \left(R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) - R_\nu \left(\frac{h_{1\nu} + \tilde{v}_\nu}{\lambda} \right) \right) (v_\nu - \tilde{v}_\nu) da \\
&+ \int_{\Gamma_3} p_\tau(\beta) \left(R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) - R_\tau \left(\frac{h_{1\tau} + \tilde{v}_\tau}{\lambda} \right) \right) \cdot (v_\tau - \tilde{v}_\tau) da.
\end{aligned}$$

From (c) of (1.10), one has

$$\left\langle G\varepsilon \left(\frac{h_1 + v}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} \right), \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \geq \frac{1}{\lambda} m_G \|v - \tilde{v}\|_V^2, \quad (3.42)$$

and (c) of (1.9) yields

$$\langle A\varepsilon(v) - A\varepsilon(\tilde{v}), \varepsilon(v) - \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \geq m_A \|v - \tilde{v}\|_V^2. \quad (3.43)$$

Exploiting the inequality of Cauchy-Schwarz, (b) of (1.11) and (G1), we obtain

$$\begin{aligned}
&- \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&+ \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + \tilde{v}) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&\geq -\frac{g_0 L_B}{\lambda} \|v - \tilde{v}\|_V^2.
\end{aligned} \quad (3.44)$$

The use of assumption (c) of (1.12) leads to

$$\int_{\Gamma_3} \left(p_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) - p_\nu \left(\frac{h_{1\nu} + \tilde{v}_\nu}{\lambda} \right) \right) (v_\nu - \tilde{v}_\nu) da \geq 0. \quad (3.45)$$

Using inequality of Cauchy-Schwarz, (3.15), (A.6), $\beta \leq 1$ and the Lipschitz continuity of R_ν ,

we get

$$\begin{aligned} & - \int_{\Gamma_3} \gamma_\nu \beta^2 \left(R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) - R_\nu \left(\frac{h_{1\nu} + \tilde{v}_\nu}{\lambda} \right) \right) (v_\nu - \tilde{v}_\nu) da \\ & \geq \frac{-\|\gamma_\nu\|_{L^\infty(\Gamma_3)} L_{R_\nu} C_0^2}{\lambda} \|v - \tilde{v}\|_V^2. \end{aligned} \quad (3.46)$$

Using the monotonicity of R_τ and the fact that $p_\tau(\beta) \geq 0$, one has

$$\int_{\Gamma_3} p_\tau(\beta) \left(R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) - R_\tau \left(\frac{h_{1\tau} + \tilde{v}_\tau}{\lambda} \right) \right) \cdot (v_\tau - \tilde{v}_\tau) da \geq 0. \quad (3.47)$$

Combining the estimations (3.42)-(3.47), we get

$$\langle Tv - T\tilde{v}, v - \tilde{v} \rangle_{V' \times V} \geq \frac{(m_G - g_0 L_B)}{\lambda} \|v - \tilde{v}\|_V^2 + \left(m_A - \frac{\|\gamma_\nu\|_{L^\infty(\Gamma_3)} L_{R_\nu} C_0^2}{\lambda} \right) \|v - \tilde{v}\|_V^2.$$

Using assumption (1.16) and taking $\lambda > \frac{\|\gamma_\nu\|_{L^\infty(\Gamma_3)} L_{R_\nu} C_0^2}{m_A}$, we see that the operator $\mathcal{T}$ is strongly monotone and so the coercivity and the monotonicity of $\mathcal{T}$ follows.

Next, to show the hemicontinuity of $\mathcal{T}$, we should prove the following

$$\mathcal{T}(v + \zeta v_1) \xrightarrow{w} \mathcal{T}(v), \quad \forall v, v_1 \in V,$$

when $\zeta \rightarrow 0$.

Let $\tilde{v} \in V$, and $\lambda > \max \left\{ L^2 \left(\|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)} \right), \frac{\|\gamma_\nu\|_{L^\infty(\Gamma_3)} L_{R_\nu} C_0^2}{m_A} \right\}$, then

$$\begin{aligned} & \langle \mathcal{T}(v + \zeta v_1), \tilde{v} \rangle_{V' \times V} - \langle \mathcal{T}(v), \tilde{v} \rangle_{V' \times V} \\ & = \zeta \langle \lambda v_1, \tilde{v} \rangle_H \\ & + \left\langle G\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + v}{\lambda} \right), \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \langle A\varepsilon(v + \zeta v_1) - A\varepsilon(v), \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \\ & - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v + \zeta v_1) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \int_{\Gamma_3} \left(p_\nu \left(\frac{h_{1\nu} + v_\nu + \zeta v_{1\nu}}{\lambda} \right) - p_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right) \tilde{v}_\nu da \\ & - \int_{\Gamma_3} \gamma_\nu \beta^2 \left(R_\nu \left(\frac{h_{1\nu} + v_\nu + \zeta v_{1\nu}}{\lambda} \right) - R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right) \tilde{v}_\nu da \\ & + \int_{\Gamma_3} p_\tau(\beta) \left(R_\tau \left(\frac{h_{1\tau} + v_\tau + \zeta v_{1\tau}}{\lambda} \right) - R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) \right) \cdot \tilde{v}_\tau da. \end{aligned} \quad (3.48)$$

The use of inequality of Cauchy-Schwarz and (A.9) leads to

$$\begin{aligned} |\zeta \langle \lambda v_1, \tilde{v} \rangle_H| &\leq |\zeta| \lambda \|v_1\|_H \|\tilde{v}\|_H \\ &\leq |\zeta| \lambda d^2 \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \end{aligned} \quad (3.49)$$

as $\zeta \rightarrow 0$. By applying inequality of Cauchy-Schwarz and using (b) of (1.10), we get

$$\left| \left\langle G\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + v}{\lambda} \right), \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \right| \leq \frac{|\zeta|}{\lambda} L_G \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \quad (3.50)$$

as $\zeta \rightarrow 0$. Using again the inequality of Cauchy-Schwarz and (b) of (1.9), one has

$$|\langle A\varepsilon(v + \zeta v_1) - A\varepsilon(v), \varepsilon(\tilde{v}) \rangle_{\mathcal{H}}| \leq |\zeta| L_A \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \quad (3.51)$$

as $\zeta \rightarrow 0$. From inequality of Cauchy-Schwarz, (G1) and (b) of (1.11), we obtain

$$\begin{aligned} &\left| \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v + \zeta v_1) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \right. \\ &\quad \left. - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \right| \\ &\leq \frac{|\zeta| L_B g_0}{\lambda} \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \end{aligned} \quad (3.52)$$

as $\zeta \rightarrow 0$. The use of inequality of Cauchy-Schwarz, (b) of (1.12) and (A.6) leads to

$$\begin{aligned} &\left| \int_{\Gamma_3} \left(p_\nu \left(\frac{h_{1\nu} + v_\nu + \zeta v_{1\nu}}{\lambda} \right) - p_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right) \tilde{v}_\nu da \right| \\ &\leq \left\| p_\nu \left(\frac{h_{1\nu} + v_\nu + \zeta v_{1\nu}}{\lambda} \right) - p_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right\|_{L^2(\Gamma_3)} \|\tilde{v}_\nu\|_{L^2(\Gamma_3)} \\ &\leq \frac{L_\nu |\zeta|}{\lambda} \|v_1\|_{L^2(\Gamma_3)^d} \|\tilde{v}\|_{L^2(\Gamma_3)^d} \\ &\leq \frac{L_\nu C_0^2 |\zeta|}{\lambda} \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \end{aligned} \quad (3.53)$$

as $\zeta \rightarrow 0$. It results from (3.15), (A.6), $\beta \leq 1$, the Lipschitz continuity of R_ν that

$$\begin{aligned} &\left| \int_{\Gamma_3} \gamma_\nu \beta^2 \left(R_\nu \left(\frac{h_{1\nu} + v_\nu + \zeta v_{1\nu}}{\lambda} \right) - R_\nu \left(\frac{h_{1\nu} + v_\nu}{\lambda} \right) \right) \tilde{v}_\nu da \right| \\ &\leq \frac{|\zeta| \|\gamma_\nu\|_{L^\infty(\Gamma_3)} L_{R_\nu} C_0^2}{\lambda} \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \end{aligned} \quad (3.54)$$

as $\zeta \rightarrow 0$. Exploiting inequality of Cauchy-Schwarz, the Lipschitz continuity of R_ν , (3.14), $\beta \leq 1$,

(A.6), we obtain

$$\left| \int_{\Gamma_3} p_\tau(\beta) \left(R_\tau \left(\frac{h_{1\tau} + v_\tau + \zeta v_{1\tau}}{\lambda} \right) - R_\tau \left(\frac{h_{1\tau} + v_\tau}{\lambda} \right) \right) \cdot \tilde{v}_\tau da \right| \leq \frac{L_\tau L_{R_\tau} C_0^2 |\zeta|}{\lambda} \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \quad (3.55)$$

as $\zeta \rightarrow 0$. Combining (3.49)-(3.55), we get

$$|\langle \mathcal{T}(v + \zeta v_1), \tilde{v} \rangle_{V' \times V} - \langle \mathcal{T}(v), \tilde{v} \rangle_{V' \times V}| \rightarrow 0, \quad (3.56)$$

as $\zeta \rightarrow 0$. Therefore, we conclude that $\mathcal{T} : V \rightarrow V'$ is hemicontinuous and by employing the fact that $\mathcal{T}$ is monotone and coercive, one can deduce that $\mathcal{T}$ is coercive and maximal monotone. Then, the surjectivity of $\mathcal{T} : V \rightarrow V'$ follows. Thus, given any $(h_1, h_2, h_3, h_4)^T \in \mathcal{H}$, we can find an element $v \in V$ satisfying equation (3.40). Similar procedure as in (1.46) and (1.47), we get

$$(\eta, \eta_s) \in \mathcal{M} \times \mathcal{M}$$

Therefore, $U = (u, v, \eta, \beta)^T \in \mathcal{D}(\mathcal{A})$ and thus $\mathcal{A} + \lambda I$ is surjective for some constant $\lambda > 0$. This finishes the proof of Theorem 3.1. □

3.4 Exponential stability

In the current section, we are concerned with the exponential decay of solution for a contact problem with normal compliance and adhesion. Assuming that σ is a smooth function, we will construct a Lyapunov functional with which we can establish the exponential decay.

The energy functional E is prescribed by

$$E(t) = \frac{1}{2} \|U(t)\|_{\mathcal{H}}^2 = \frac{1}{2} \left(\|u'(t)\|_H^2 + \|u(t)\|_V^2 + \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds + \|\beta(t)\|_{L^2(\Gamma_3)}^2 \right), \quad \forall t \in \mathbb{R}_+. \quad (3.57)$$

Lemma 3.1. *Assume that (G1), (1.9)-(1.13) and (3.14) hold and let $U_0 \in \mathcal{D}(\mathcal{A})$. Then, the energy functional defined by (3.57) satisfies for $\lambda_1 > 0$, for all $t \geq 0$*

$$\begin{aligned} E'(t) \leq & -(m_A - K_1) \|u'(t)\|_V^2 + \lambda_1 K_2 \|u(t)\|_V^2 + \lambda_1 K_3 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ & + \frac{1}{2} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds - \|\beta(t)\|_{L^2(\Gamma_3)}^2 + K_4 + \lambda_1 \|f(t)\|_{V'}^2, \end{aligned} \quad (3.58)$$

where

$$K_1 = \frac{9}{4\lambda_1}, \quad K_2 = L_G^2 + g_0^2 L_B^2 + 1 + L_\nu^2 C_0^4 + L_{R_\nu}^2 \|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 C_0^4 + L_\tau^2 L_{R_\tau}^2 C_0^4, \quad K_3 = g_0 + g_0 L_B^2, \\ K_4 = \left(1 + \|\varepsilon_a\|_{L^\infty(\Gamma_3)}\right) \text{meas}(\Gamma_3).$$

Proof. Differentiation of eq.(3.57) with respect to the variable t , the use of (3.13)₁, (3.13)₃ and (3.13)₄ integration over Ω and use of formula of Green and boundary conditions lead to

$$\begin{aligned} E'(t) &= \langle u''(t), u'(t) \rangle_H + \langle u'(t), u(t) \rangle_V + \int_0^{+\infty} g(s) \langle \eta_t(s, t), \eta(s, t) \rangle_V ds + \langle \beta'(t), \beta(t) \rangle_{L^2(\Gamma_3)} \\ &= -\langle A\varepsilon(u'(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} - \langle G\varepsilon(u(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} + \left\langle \int_0^{+\infty} g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u'(t)) \right\rangle_{\mathcal{H}} \\ &\quad - \int_{\Gamma_3} p_\nu(u_\nu(t)) u'_\nu(t) da + \int_{\Gamma_3} \gamma_\nu \beta^2(t) R_\nu(u_\nu(t)) u'_\nu(t) da - \int_{\Gamma_3} p_\tau(\beta(t)) R_\tau(u_\tau(t)) \cdot u'_\tau(t) da \\ &\quad + \int_0^{+\infty} g(s) \langle u'(t), \eta(s, t) \rangle_V ds + \frac{1}{2} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds + \langle u'(t), u(t) \rangle_V \\ &\quad - \left\langle \left(\beta(t) \left(\gamma_\nu (R_\nu(u_\nu(t)))^2 + \gamma_\tau \|R_\tau(u_\tau(t))\|^2 \right) - \varepsilon_a \right)_+, \beta(t) \right\rangle_{L^2(\Gamma_3)} + \langle f(t), u'(t) \rangle_{V' \times V}. \quad (3.59) \end{aligned}$$

It follows from assumption (c) of (1.9) that

$$-\langle A\varepsilon(u'(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} \leq -m_A \|u'(t)\|_V^2. \quad (3.60)$$

Thanks to inequalities of Cauchy-Schwarz and Young and (b) of (1.10), we obtain

$$\begin{aligned} -\langle G\varepsilon(u(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} &\leq L_G \|u(t)\|_V \|u'(t)\|_V \\ &\leq \lambda_1 L_G^2 \|u(t)\|_V^2 + \frac{1}{4\lambda_1} \|u'(t)\|_V^2. \quad (3.61) \end{aligned}$$

Applying inequalities of Cauchy-Schwarz and Young and using (1.11) and (G1), one has

$$\begin{aligned} \left\langle \int_0^{+\infty} g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u'(t)) \right\rangle_{\mathcal{H}} &\leq \left\| \int_0^{+\infty} g(s) B\varepsilon(u(t) - \eta(s, t)) ds \right\|_{\mathcal{H}} \|\varepsilon(u'(t))\|_{\mathcal{H}} \\ &\leq L_B \int_0^{+\infty} g(s) (\|u(t)\|_V + \|\eta(s, t)\|_V) ds \|u'(t)\|_V \\ &\leq \lambda_1 g_0^2 L_B^2 \|u(t)\|_V^2 + \frac{1}{2\lambda_1} \|u'(t)\|_V^2 \\ &\quad + \lambda_1 L_B^2 g_0 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds. \quad (3.62) \end{aligned}$$

Using inequalities of Cauchy-Schwarz and Young, (1.12) and (A.6), we get

$$\begin{aligned}
-\int_{\Gamma_3} p_\nu(u_\nu(t))u'_\nu(t)da &\leq \left(\int_{\Gamma_3} |p_\nu(u_\nu(t))|^2 da\right)^{\frac{1}{2}} \left(\int_{\Gamma_3} |u'_\nu(t)|^2 da\right)^{\frac{1}{2}} \\
&\leq L_\nu \|u(t)\|_{(L^2(\Gamma_3))^d} \|u'(t)\|_{(L^2(\Gamma_3))^d} \\
&\leq \lambda_1 L_\nu^2 C_0^4 \|u(t)\|_V^2 + \frac{1}{4\lambda_1} \|u'(t)\|_V^2.
\end{aligned} \tag{3.63}$$

It follows from inequalities of Cauchy-Schwarz and Young, $\beta(t) \leq 1$, the Lipschitz continuity of R_ν , $R_\nu(0) = 0$, (3.15) and (A.6) that

$$\begin{aligned}
\int_{\Gamma_3} \gamma_\nu \beta^2(t) R_\nu(u_\nu(t)) u'_\nu(t) da &\leq \|\gamma_\nu R_\nu(u_\nu(t))\|_{L^2(\Gamma_3)} \|u'_\nu(t)\|_{L^2(\Gamma_3)} \\
&\leq L_{R_\nu} \|\gamma_\nu\|_{L^\infty(\Gamma_3)} \|u_\nu(t)\|_{L^2(\Gamma_3)} \|u'_\nu(t)\|_{L^2(\Gamma_3)} \\
&\leq \lambda_1 L_{R_\nu}^2 \|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 C_0^4 \|u(t)\|_V^2 + \frac{1}{4\lambda_1} \|u'(t)\|_V^2.
\end{aligned} \tag{3.64}$$

The use of inequalities of Cauchy-Schwarz and Young, (3.14), $\beta(t) \leq 1$, the Lipschitz continuity of R_τ , $R_\tau(0) = 0$, (3.15) and (A.6) leads to

$$\begin{aligned}
-\int_{\Gamma_3} p_\tau(\beta(t)) R_\tau(u_\tau) \cdot u'_\tau da &\leq L_\tau \int_{\Gamma_3} |\beta(t)| |R_\tau(u_\tau(t)) \cdot u'_\tau(t)| da \\
&\leq L_\tau L_{R_\tau} \|u_\tau(t)\|_{L^2(\Gamma_3)^d} \|u'_\tau(t)\|_{L^2(\Gamma_3)^d} \\
&\leq \lambda_1 L_\tau^2 L_{R_\tau}^2 C_0^4 \|u(t)\|_V^2 + \frac{1}{4\lambda_1} \|u'(t)\|_V^2.
\end{aligned} \tag{3.65}$$

Exploiting inequalities of Cauchy-Schwarz and Young and (G1), we get

$$\int_0^\infty g(s) \langle u'(t), \eta(s, t) \rangle_V ds \leq \frac{1}{4\lambda_1} \|u'(t)\|_V^2 + \lambda_1 g_0 \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds. \tag{3.66}$$

Applying inequalities of Cauchy-Schwarz and Young, we arrive at

$$\langle u', u(t) \rangle_V \leq \frac{1}{4\lambda_1} \|u'(t)\|_V^2 + \lambda_1 \|u(t)\|_V^2. \tag{3.67}$$

Using the inequality (A.13), $\beta(t) \leq 1$ and (3.15), we obtain

$$\begin{aligned}
&-\left\langle \left(\beta(t) \left(\gamma_\nu (R_\nu(u_\nu(t)))^2 + \gamma_\tau \|R_\tau(u_\tau(t))\|^2 \right) - \varepsilon_a \right)_+, \beta(t) \right\rangle_{L^2(\Gamma_3)} \\
&\leq -\int_{\Gamma_3} \left(\beta(t) \left(\gamma_\nu (R_\nu(u_\nu(t)))^2 + \gamma_\tau \|R_\tau(u_\tau(t))\|^2 \right) - \varepsilon_a \right) \beta(t) da \\
&\leq -\int_{\Gamma_3} |\beta(t)|^2 \left(\gamma_\nu (R_\nu(u_\nu(t)))^2 + \gamma_\tau \|R_\tau(u_\tau(t))\|^2 + 1 \right) da + \int_{\Gamma_3} |\beta(t)|^2 da + \int_{\Gamma_3} \varepsilon_a \beta(t) da \\
&\leq -\|\beta(t)\|_{L^2(\Gamma_3)}^2 + \left(1 + \|\varepsilon_a\|_{L^\infty(\Gamma_3)} \right) meas(\Gamma_3).
\end{aligned} \tag{3.68}$$

From inequalities of Cauchy-Schwarz and Young, one has

$$\langle f(t), u'(t) \rangle_{V' \times V} \leq \lambda_1 \|f(t)\|_{V'}^2 + \frac{1}{4\lambda_1} \|u'(t)\|_V^2. \quad (3.69)$$

Combining (3.60)-(3.69) and using (3.59), we get (3.58). $\square$

Remark 3.2. It follows from the right-hand side of (3.58) that the energy functional (3.57) is not decreasing in general.

Now, we turn to establishing lemmas that we need to exhibit our stability result.

Lemma 3.2. Assume that (G1), (G3), (1.9)-(1.13) and (3.14) hold $U_0 \in \mathcal{D}(\mathcal{A})$. Then the functional

$$\mathcal{F}_1(t) = \langle u'(t), u(t) \rangle_H - \frac{K_4}{N\epsilon} t - e^{3\zeta_1 t} \|f(t)\|_{V'}^2, \quad (3.70)$$

satisfies for $\lambda_2 > 0$, for all $t \geq 0$

$$\begin{aligned} \mathcal{F}'_1(t) \leq & (-l + 4\lambda_2) \|u(t)\|_V^2 + K_5 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds + K_6 \|u'(t)\|_V^2 + K_7 \|\beta(t)\|_{L^2(\Gamma_3)}^2 \\ & + \left(\frac{1}{4\lambda_2} - \lambda_0 \right) \|f(t)\|_{V'}^2 - \frac{K_4}{N\epsilon}, \end{aligned} \quad (3.71)$$

where

$$K_4 = (1 + \|\varepsilon_a\|_{L^\infty(\Gamma_3)}) \text{meas}(\Gamma_3), \quad K_5 = \frac{g_0 L_B^2}{4\lambda_2}, \quad K_6 = \frac{L_A^2}{4\lambda_2} + d^2, \quad K_7 = \frac{L^2 \|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 C_0^2}{4\lambda_2}.$$

N, ϵ are positive constants that will be chosen later.

Proof. Differentiating (3.70) with respect to time variable t , we get

$$\mathcal{F}'_1(t) = \|u'(t)\|_H^2 + \langle u''(t), u(t) \rangle_H - \frac{K_4}{N\epsilon} - 3\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2 - 2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'}.$$

Next, a multiplication of (3.13)₁ by $u(t)$, an integration over the domain Ω , use of formula of Green with boundary conditions and (1.10) lead to

$$\begin{aligned} \mathcal{F}'_1(t) \leq & \|u'(t)\|_H^2 - \langle A\varepsilon(u'(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} - m_G \|u(t)\|_V^2 + \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \\ & - \int_{\Gamma_3} p_\nu(u_\nu(t)) u_\nu(t) da + \int_{\Gamma_3} \gamma_\nu \beta^2(t) R_\nu(u_\nu(t)) u_\nu(t) da - \int_{\Gamma_3} p_\tau(\beta(t)) R_\tau(u_\tau(t)) \cdot u_\tau(t) da \\ & + \langle f(t), u(t) \rangle_{V' \times V} - \frac{K_4}{N\epsilon} - 3\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2 - 2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'}. \end{aligned} \quad (3.72)$$

It follows from inequalities of Cauchy-Schwarz and Young, (1.9) that

$$\begin{aligned} -\langle A\varepsilon(u'(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} & \leq \|A\varepsilon(u'(t))\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\ & \leq \frac{L_A^2}{4\lambda_2} \|u'(t)\|_V^2 + \lambda_2 \|u(t)\|_V^2. \end{aligned} \quad (3.73)$$

Applying inequalities of Cauchy-Schwarz and Young, using (1.11) and (G1), we obtain

$$\begin{aligned}
 \left\langle \int_0^\infty g(s) B \varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} &\leq \left\| \int_0^\infty g(s) B \varepsilon(u(t) - \eta(s, t)) ds \right\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\
 &\leq L_B \int_0^\infty g(s) (\|u(t)\|_V + \|\eta(s, t)\|_V) ds \|u(t)\|_V \\
 &\leq g_0 L_B \|u(t)\|_V^2 + \frac{L_B^2 g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\
 &\quad + \lambda_2 \|u(t)\|_V^2.
 \end{aligned} \tag{3.74}$$

Thanks to (1.12), we have

$$- \int_{\Gamma_3} p_\nu(u_\nu(t)) u_\nu(t) da \leq 0. \tag{3.75}$$

The use of $\beta(t) \leq 1$, (3.15), $|R_\nu| \leq L$, inequalities of Cauchy-Schwarz and Young and (A.6) leads to

$$\begin{aligned}
 \int_{\Gamma_3} \gamma_\nu \beta^2(t) R_\nu(u_\nu(t)) u_\nu(t) da &\leq \int_{\Gamma_3} \beta(t) \gamma_\nu R_\nu(u_\nu(t)) u_\nu(t) da \\
 &\leq \|\gamma_\nu\|_{L^\infty(\Gamma_3)} L C_0 \|\beta(t)\|_{L^2(\Gamma_3)} \|u(t)\|_V \\
 &\leq \lambda_2 \|u(t)\|_V^2 + \frac{L^2 \|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 C_0^2}{4\lambda_2} \|\beta(t)\|_{L^2(\Gamma_3)}^2.
 \end{aligned} \tag{3.76}$$

Using (3.14) and the fact that $R_\tau(u_\tau) \cdot u_\tau \geq 0$, one has

$$- \int_{\Gamma_3} p_\tau(\beta(t)) R_\tau(u_\tau(t)) \cdot u_\tau(t) da \leq 0. \tag{3.77}$$

From inequalities of Cauchy-Schwarz and Young, we have

$$\langle f(t), u(t) \rangle_{V' \times V} \leq \frac{1}{4\lambda_2} \|f(t)\|_{V'}^2 + \lambda_2 \|u(t)\|_V^2. \tag{3.78}$$

From the inequality of Cauchy-Schwarz and (G3), we obtain

$$\begin{aligned}
 -2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'} &\leq 2e^{3\zeta_1 t} \|f'(t)\|_{V'} \|f(t)\|_{V'} \\
 &\leq 2\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2.
 \end{aligned} \tag{3.79}$$

Inserting the inequalities (3.73)-(3.79) in (3.72) and using (A.9), we get

$$\begin{aligned}
 \mathcal{F}'_1(t) &\leq (-l + 4\lambda_2) \|u(t)\|_V^2 + \frac{g_0 L_B^2}{4\lambda_2} \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds + \left(\frac{L_A^2}{4\lambda_2} + d^2 \right) \|u'(t)\|_V^2 \\
 &\quad + \frac{L^2 \|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 C_0^2}{4\lambda_2} \|\beta(t)\|_{L^2(\Gamma_3)}^2 + \left(\frac{1}{4\lambda_2} - \lambda_0 \right) \|f(t)\|_{V'}^2 - \frac{K_4}{N\epsilon},
 \end{aligned}$$

where $l = m_G - g_0 L_B$ and the constant $\lambda_0 > 0$ satisfying $-\zeta_1 e^{3\zeta_1 t} \leq -\lambda_0$. □

Lemma 3.3. Assume that (G1), (1.9)-(1.13) and (3.14) hold and $U_0 \in \mathcal{D}(\mathcal{A})$. Then the functional

$$\mathcal{F}_2(t) = - \left\langle u'(t), \int_0^{+\infty} g(s) \eta(s, t) ds \right\rangle_H, \quad (3.80)$$

satisfies for $\lambda_2 > 0$, for all $t \geq 0$

$$\begin{aligned} \mathcal{F}_2'(t) \leq & -g_0 \|u'(t)\|_H^2 + K_8 \lambda_2 \|u'(t)\|_V^2 + K_9 \lambda_2 \|u(t)\|_V^2 + K_{10} \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ & - K_{11} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds + \lambda_2 \|f(t)\|_{V'}^2, \end{aligned} \quad (3.81)$$

where

$$K_8 = L_A^2 + d^2, \quad K_9 = L_G^2 + L_B^2 + L_\nu^2 C_0^4 + \|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 L_{R_\nu}^2 C_0^4 + L_\tau^2 L_{R_\tau}^2 C_0^4, \quad K_{10} = \frac{6g_0}{4\lambda_2} + \frac{g_0^3}{4\lambda_2} + L_B g_0, \quad K_{11} = \frac{g(0)d^2}{4\lambda_2}.$$

Proof. We differentiate (3.80) with respect to the variable t . Then, employing (3.13)₃, we obtain

$$\mathcal{F}_2'(t) = - \langle u''(t), \int_0^\infty g(s) \eta(s, t) ds \rangle_H - g_0 \|u'(t)\|_H^2 + \left\langle u'(t), \int_0^\infty g(s) \eta_s(s, t) ds \right\rangle_H. \quad (3.82)$$

Integrating with respect to the variable s the term $\langle u'(t), \int_0^\infty g(s) \eta_s(s, t) ds \rangle_H$ and using the fact that $\eta(0) = 0$, $\lim_{s \rightarrow +\infty} g(s) = 0$, we get

$$\mathcal{F}_2'(t) = - \langle u''(t), \int_0^\infty g(s) \eta(s, t) ds \rangle_H - g_0 \|u'(t)\|_H^2 - \left\langle u'(t), \int_0^\infty g'(s) \eta(s, t) ds \right\rangle_H.$$

Replacing the term u'' by (3.13)₁, integrating over the domain Ω and using formula of Green with boundary conditions, it results that

$$\begin{aligned} \mathcal{F}_2'(t) = & \left\langle A\varepsilon(u'(t)), \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\rangle_{\mathcal{H}} + \left\langle G\varepsilon(u(t)), \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\rangle_{\mathcal{H}} \\ & - \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\rangle_{\mathcal{H}} \\ & + \int_{\Gamma_3} p_\nu(u_\nu(t)) \int_0^\infty g(s) \eta_\nu(s, t) ds da \\ & - \int_{\Gamma_3} \gamma_\nu \beta^2(t) R_\nu(u_\nu(t)) \int_0^\infty g(s) \eta_\nu(s, t) ds da \\ & + \int_{\Gamma_3} p_\tau(\beta(t)) R_\tau(u_\tau(t)) \cdot \int_0^\infty g(s) \eta_\tau(s, t) ds da \\ & - \left\langle f(t), \int_0^\infty g(s) \eta(s, t) ds \right\rangle_{V' \times V} - g_0 \|u'(t)\|_H^2 - \left\langle u'(t), \int_0^\infty g'(s) \eta(s, t) ds \right\rangle_H. \end{aligned} \quad (3.83)$$

Using inequalities of Cauchy-Schwarz and Young, (G1) and (1.9), we obtain

$$\begin{aligned} \left\langle A\varepsilon(u'(t)), \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\rangle_{\mathcal{H}} & \leq \|A\varepsilon(u'(t))\|_{\mathcal{H}} \left\| \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\|_{\mathcal{H}} \\ & \leq \lambda_2 L_A^2 \|u'(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds. \end{aligned} \quad (3.84)$$

Applying inequalities of Cauchy-Schwarz and Young, using (G1) and (1.10), we arrive at

$$\begin{aligned} \left\langle G\varepsilon(u(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} &\leq \|G\varepsilon(u(t))\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\ &\leq \lambda_2 L_G^2 \|u(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s)\|\eta(s,t)\|_V^2 ds. \end{aligned} \quad (3.85)$$

Exploiting inequalities of Cauchy-Schwarz and Young, (G1) and (1.11), we get

$$\begin{aligned} & - \left\langle \int_0^\infty g(s)B\varepsilon(u(t) - \eta(s,t))ds, \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\ & \leq \left\| \int_0^\infty g(s)B\varepsilon(u(t) - \eta(s,t))ds \right\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\ & \leq L_B \int_0^\infty g(s)\|u(t) - \eta(s,t)\|_V ds \int_0^\infty g(s)\|\eta(s,t)\|_V ds \\ & \leq \lambda_2 L_B^2 \|u(t)\|_V^2 + \left(\frac{g_0^3}{4\lambda_2} + L_B g_0 \right) \int_0^\infty g(s)\|\eta(s,t)\|_V^2 ds. \end{aligned} \quad (3.86)$$

It follows from inequalities of Cauchy-Schwarz and Young, (G1), (1.12), (A.6) that

$$\begin{aligned} \int_{\Gamma_3} p_\nu(u_\nu(t)) \left(\int_0^\infty g(s)\eta_\nu(s,t)ds \right) da &\leq \left(\int_{\Gamma_3} p_\nu^2(u_\nu(t))da \right)^{\frac{1}{2}} \left(\int_{\Gamma_3} \left(\int_0^\infty g(s)\eta_\nu(s,t)ds \right)^2 da \right)^{\frac{1}{2}} \\ &\leq \lambda_2 L_\nu^2 C_0^4 \|u(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s)\|\eta(s,t)\|_V^2 ds. \end{aligned} \quad (3.87)$$

The use of inequalities of Cauchy-Schwarz and Young, $\beta(t) \leq 1$, Lipschitz continuity of R_ν , $R_\nu(0) = 0$, (3.15), (G1) and (A.6) leads to

$$\begin{aligned} & - \int_{\Gamma_3} \gamma_\nu \beta^2(t) R_\nu(u_\nu(t)) \int_0^\infty g(s)\eta_\nu(s,t)ds da \\ & \leq \int_{\Gamma_3} \gamma_\nu R_\nu(u_\nu(t)) \int_0^\infty g(s)\eta_\nu(s,t)ds da \\ & \leq \|\gamma_\nu\|_{L^\infty(\Gamma_3)} L_{R_\nu} \|u_\nu(t)\|_{L^2(\Gamma_3)} \left\| \int_0^\infty g(s)\eta(s,t)ds \right\|_{L^2(\Gamma_3)} \\ & \leq \lambda_2 \|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 L_{R_\nu}^2 C_0^4 \|u(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s)\|\eta(s,t)\|_V^2 ds. \end{aligned} \quad (3.88)$$

From (3.14), Lipschitz continuity of R_τ , $R_\tau(0) = 0$, $\beta(t) \leq 1$, (A.6), inequalities of Cauchy-Schwarz and Young, we have

$$\begin{aligned} & \int_{\Gamma_3} p_\tau(\beta(t)) R_\tau(u_\tau(t)) \cdot \int_0^\infty g(s)\eta_\tau(s,t)ds da \\ & \leq L_\tau \int_{\Gamma_3} |\beta(t)| |R_\tau(u_\tau(t))| \cdot \int_0^\infty g(s)\eta_\tau(s,t)ds da \\ & \leq L_\tau L_{R_\tau} \|u_\tau(t)\|_{L^2(\Gamma_3)^d} \left\| \int_0^\infty g(s)\eta_\tau(s,t)ds \right\|_{L^2(\Gamma_3)^d} \\ & \leq \lambda_2 L_\tau^2 L_{R_\tau}^2 C_0^4 \|u(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s)\|\eta(s,t)\|_V^2 ds. \end{aligned} \quad (3.89)$$

The following term is estimated by

$$-\left\langle f(t), \int_0^\infty g(s)\eta(s,t)ds \right\rangle_{V' \times V} \leq \lambda_2 \|f(t)\|_{V'}^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds. \quad (3.90)$$

Exploiting inequalities of Cauchy-Schwarz and Young, (G1) and (A.9), we obtain

$$\begin{aligned} -\left\langle u'(t), \int_0^\infty g'(s)\eta(s,t)ds \right\rangle_H &\leq \|u'(t)\|_H \left\| \int_0^\infty g'(s)\eta(s,t)ds \right\|_H \\ &\leq \lambda_2 \|u'(t)\|_H^2 - \frac{g(0)}{4\lambda_2} \left(\int_0^\infty g'(s) \|\eta(s,t)\|_H^2 ds \right) \\ &\leq d^2 \lambda_2 \|u'(t)\|_V^2 - \frac{g(0)d^2}{4\lambda_2} \left(\int_0^\infty g'(s) \|\eta(s,t)\|_V^2 ds \right). \end{aligned} \quad (3.91)$$

Combining the estimates (3.84)-(3.91) and using (3.83), we get

$$\begin{aligned} \mathcal{F}_2'(t) &\leq -g_0 \|u'(t)\|_H^2 + \lambda_2 (L_A^2 + d^2) \|u'(t)\|_V^2 \\ &\quad + \lambda_2 (L_G^2 + L_B^2 + L_\nu^2 C_0^4 + \|\gamma_\nu\|_{L^\infty(\Gamma_3)}^2 L_{R_\nu}^2 C_0^4 + L_\tau^2 L_{R_\tau}^2 C_0^4) \|u(t)\|_V^2 + \lambda_2 \|f(t)\|_{V'}^2 \\ &\quad - \frac{g(0)d^2}{4\lambda_2} \int_0^{+\infty} g'(s) \|\eta(s,t)\|_V^2 ds + \left(\frac{6g_0}{4\lambda_2} + \frac{g_0^3}{4\lambda_2} + L_B g_0 \right) \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds. \end{aligned}$$

□

The Lyapunov functional $\mathcal{L}$ is defined as follows

$$\mathcal{L}(t) = E(t) + \epsilon(N\mathcal{F}_1(t) + \mathcal{F}_2(t)), \quad (3.92)$$

where N, ϵ are positive constants that will be chosen later.

Lemma 3.4. Assume that (G1) hold, then, there exist two positive constants ς_1 and ς_2 such that

$$\varsigma_1 E(t) \leq \mathcal{L}(t) \leq \varsigma_2 E(t). \quad (3.93)$$

Proof. Applying the inequalities of Cauchy-Schwarz and Young and (A.9), we get

$$\begin{aligned} |\mathcal{F}_1(t)| &= \left| \langle u'(t), u(t) \rangle_H - \frac{K_4}{N\epsilon} t - e^{3\varsigma_1 t} \|f\|_{V'}^2 \right| \leq |\langle u'(t), u(t) \rangle_H| \leq \frac{1}{2} \|u'(t)\|_H^2 + \frac{1}{2} \|u(t)\|_H^2 \\ &\leq \max(1, d^2) E(t), \end{aligned} \quad (3.94)$$

and

$$\begin{aligned} |\mathcal{F}_2(t)| &= \left| \left\langle u'(t), \int_0^{+\infty} g(s)\eta(s,t)ds \right\rangle_H \right| \leq \frac{1}{2} \left(\|u'(t)\|_H^2 + g_0 d^2 \int_0^{+\infty} g(s) \|\eta(s,t)\|_V^2 ds \right) \\ &\leq \max(1, g_0 d^2) E(t). \end{aligned} \quad (3.95)$$

Combining (3.92), (3.57), (3.94), (3.95), we get

$$|\mathcal{L}(t) - E(t)| \leq \epsilon \varsigma_3 E(t), \quad (3.96)$$

where the positive constant ς_3 is defined by

$$\varsigma_3 = N \max(1, d^2) + \max(1, g_0 d^2),$$

then by choosing

$$\epsilon < \frac{1}{\varsigma_3}, \quad (3.97)$$

we have $\mathcal{L} \sim E$. □

Theorem 3.2. Assume that (G1)-(G3) and (1.9)-(1.13), (3.14) hold. Suppose that m_G satisfies

$$m_G > \frac{1}{\lambda_0} + g_0 L_B, \quad (3.98)$$

and there exists δ_0 a positive constant independent of m_A satisfying

$$m_A > \delta_0, \quad (3.99)$$

thus, for every $U_0 \in \mathcal{D}(\mathcal{A})$, there exist positive constants δ_1 and δ_2 such that

$$\|U(t)\|_{\mathcal{H}}^2 \leq \delta_2 e^{-\delta_1 t}. \quad (3.100)$$

Proof. Combining (3.58), (3.71), (3.81), we have for all $t \geq 0$

$$\begin{aligned} \mathcal{L}'(t) &= E'(t) + \epsilon(N\mathcal{F}'_1(t) + \mathcal{F}'_2(t)) \\ &\leq -\epsilon \left(g_0 \|u'(t)\|_H^2 + \left(N(l - 4\lambda_2) - K_9\lambda_2 - \frac{K_2\lambda_1}{\epsilon} \right) \|u(t)\|_V^2 \right) \\ &\quad -\epsilon \left(\left(K_3 \left(N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3\lambda_1}{\epsilon} \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds + \left(\frac{1}{\epsilon} - NK_7 \right) \|\beta(t)\|_{L^2(\Gamma_3)} \right) \\ &\quad + (K_1 - m_A + \epsilon(K_6N + K_8\lambda_2)) \|u'(t)\|_V^2 + \left(\frac{1}{2} - \epsilon K_{11} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds \\ &\quad + \epsilon K_{12} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds + \left(\lambda_1 + \epsilon \left(N \left(\frac{1}{4\lambda_2} - \lambda_0 \right) + \lambda_2 \right) \right) \|f(t)\|_{V'}^2, \end{aligned} \quad (3.101)$$

where $K_{12} = NK_5 + K_{10} + K_3 \left(N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right)$.

From assumption (3.98), we can choose λ_2 as follows

$$\frac{1}{4\lambda_0} < \lambda_2 < \frac{l}{4}, \quad (3.102)$$

and thus N large enough such that

$$\begin{aligned} N(l - 4\lambda_2) - K_9\lambda_2 &> 0, \\ N\left(\lambda_0 - \frac{1}{4\lambda_2}\right) - \lambda_2 &> 0. \end{aligned} \quad (3.103)$$

Using (G2), one has

$$\left(\frac{1}{2} - \epsilon K_{11}\right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds + \epsilon K_{12} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \leq \left(\frac{1}{2} - \epsilon K_{13}\right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds,$$

where $K_{13} = K_{11} + \frac{K_{12}}{\alpha}$.

Exploiting the previous inequality, we get

$$\begin{aligned} \mathcal{L}'(t) \leq & -\epsilon \left(g_0 \|u'(t)\|_H^2 + \left(N(l - 4\lambda_2) - K_9\lambda_2 - \frac{K_2\lambda_1}{\epsilon} \right) \|u(t)\|_V^2 \right) \\ & -\epsilon \left(\left(K_3 \left(N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3\lambda_1}{\epsilon} \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds + \left(\frac{1}{\epsilon} - NK_7 \right) \|\beta(t)\|_{L^2(\Gamma_3)} \right) \\ & + (K_1 - m_A + \epsilon(K_6N + K_8\lambda_2)) \|u'(t)\|_V^2 + \left(\frac{1}{2} - \epsilon K_{13} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds \\ & + \left(\lambda_1 + \epsilon \left(N \left(\frac{1}{4\lambda_2} - \lambda_0 \right) + \lambda_2 \right) \right) \|f(t)\|_{V'}^2. \end{aligned} \quad (3.104)$$

Next, we assume that the constant m_A satisfies assumption (3.99) if we choose δ_0 in the following form

$$\delta_0 = \max \left\{ \frac{\lambda_1 (K_6N + K_8\lambda_2)}{N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2}, \frac{K_2\lambda_1 (K_6N + K_8\lambda_2)}{N(l - 4\lambda_2) - K_9\lambda_2} \right\} + K_1, \quad (3.105)$$

and λ_1 satisfies

$$\begin{aligned} \lambda_1 < \min \left\{ \frac{\left(\lambda_0 - \frac{1}{4\lambda_2} \right) N - \lambda_2}{\varsigma_3}, \frac{\left(\lambda_0 - \frac{1}{4\lambda_2} \right) N - \lambda_2}{2K_{13}}, \frac{N(l - 4\lambda_2) - K_9\lambda_2}{K_2\varsigma_3}, \right. \\ \left. \frac{N(l - 4\lambda_2) - K_9\lambda_2}{2K_2K_{13}}, \frac{N(l - 4\lambda_2) - K_9\lambda_2}{NK_7K_2}, \frac{\left(\lambda_0 - \frac{1}{4\lambda_2} \right) N - \lambda_2}{NK_7} \right\}. \end{aligned} \quad (3.106)$$

It follows from (3.106) that

$$\frac{\lambda_1}{\left(\lambda_0 - \frac{1}{4\lambda_2} \right) N - \lambda_2} < \min \left\{ \frac{1}{2K_{13}}, \frac{1}{\varsigma_3}, \frac{1}{NK_7} \right\}, \quad (3.107)$$

$$\frac{K_2\lambda_1}{N(l - 4\lambda_2) - K_9\lambda_2} < \min \left\{ \frac{1}{2K_{13}}, \frac{1}{\varsigma_3}, \frac{1}{NK_7} \right\}, \quad (3.108)$$

and from (3.105), we get

$$\frac{\lambda_1}{\left(\lambda_0 - \frac{1}{4\lambda_2} \right) N - \lambda_2} < \frac{m_A - K_1}{K_6N + K_8\lambda_2}, \quad (3.109)$$

and

$$\frac{K_2\lambda_1}{N(l-4\lambda_2)-K_9\lambda_2} < \frac{m_A-K_1}{K_6N+K_8\lambda_2}. \quad (3.110)$$

Using (3.97) and (3.107)-(3.110), ϵ can be fixed as

$$\max \left\{ \frac{\lambda_1}{\left(\lambda_0 - \frac{1}{4\lambda_2}\right)N - \lambda_2}, \frac{K_2\lambda_1}{N(l-4\lambda_2)-K_9\lambda_2} \right\} < \epsilon < \min \left\{ \frac{m_A-K_1}{K_6N+K_8\lambda_2}, \frac{1}{2K_{13}}, \frac{1}{\varsigma_3}, \frac{1}{NK_7} \right\}. \quad (3.111)$$

Therefore, we arrive at

$$\mathcal{L}'(t) \leq -\epsilon K_{14}E(t) + \left(\frac{1}{2} - \epsilon K_{13}\right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds,$$

where

$$K_{14} = 2\min \left(g_0, N(l-4\lambda_2)-K_9\lambda_2 - \frac{K_2\lambda_1}{\epsilon}, K_3 \left(N \left(\lambda_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3\lambda_1}{\epsilon}, \frac{1}{\epsilon} - NK_7 \right).$$

From (G1), (3.111) and (3.96), we get

$$\mathcal{L}'(t) \leq \frac{-\epsilon K_{14}}{1 + \epsilon \varsigma_3} \mathcal{L}(t). \quad (3.112)$$

Integrating (3.112) from $(0, t)$, we obtain

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\delta_1 t}, \quad \forall t \in \mathbb{R}_+, \quad (3.113)$$

where $\delta_1 = \frac{\epsilon K_{14}}{1 + \epsilon \varsigma_3}$.

From (3.96) and (3.113), we have

$$\begin{aligned} \|U(t)\|_{\mathcal{H}}^2 &= 2E(t) \\ &\leq \frac{2}{1 - \epsilon \varsigma_3} \mathcal{L}(t) \\ &\leq \frac{2}{1 - \epsilon \varsigma_3} \mathcal{L}(0)e^{-\delta_1 t}, \end{aligned}$$

which yields (3.100) with $\delta_2 = \frac{2}{1 - \epsilon \varsigma_3} \mathcal{L}(0)$. □

A DYNAMIC FRICTIONAL CONTACT PROBLEM WITH THERMOELASTICITY OF TYPE III

Our main goal in this chapter is to investigate a thermoviscoelastic contact problem to show the global existence of the solution of the system using nonlinear semigroup theory combined with arguments of operator theory. An exponential decay estimate for the energy is established under some hypotheses on the problem data.

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4.1 Problem statement

In the current chapter, we study the following thermoviscoelastic contact problem

$$\sigma = A\varepsilon(u') + G\varepsilon(u) - \int_0^\infty g(s)B\varepsilon(u(t-s))ds + C\theta \quad \text{in } \Omega \times (0, \infty), \quad (4.1)$$

$$\rho u'' = \text{Div}\sigma + f_0 \quad \text{in } \Omega \times (0, \infty), \quad (4.2)$$

$$\theta'' - \kappa\Delta\theta - \delta\Delta\theta' + \beta \text{div } u'' = 0 \quad \text{in } \Omega \times (0, \infty), \quad (4.3)$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \quad (4.4)$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0, \infty), \quad (4.5)$$

$$\theta = 0 \quad \text{on } \Gamma \times (0, \infty), \quad (4.6)$$

$$-\sigma_\nu = p_\nu(u'_\nu) \quad \text{on } \Gamma_3 \times (0, \infty), \quad (4.7)$$

$$-\sigma_\tau = p_\tau(u'_\tau) \quad \text{on } \Gamma_3 \times (0, \infty), \quad (4.8)$$

$$u(-t) = u_0(t) \quad \text{in } \Omega \times [0, \infty), \quad (4.9)$$

$$u'(0) = u_1, \theta(0) = \theta_0, \theta'(0) = \theta_1 \quad \text{in } \Omega. \quad (4.10)$$

Here, Eq.(4.1) represents the thermo-viscoelastic law with long memory where C is a nonlinear operator, θ is the temperature which is governed by equation (4.3) in which the constants κ, δ and β are positive. θ_0 and θ_1 are initial conditions.

4.2 An additional assumption

In addition to assumptions (1.9)-(1.15), (G1)-(G3) defined on Sect. 1.2 and assumption (2.8) defined on Subsect.2.1.2, the following additional assumption is needed

$$\left\{ \begin{array}{l} \text{(a) } C : \Omega \times \mathbb{R} \rightarrow \mathbb{S}^d \text{ is a nonlinear thermal expansion operator;} \\ \text{(b) There exists } L_C > 0 \text{ such that} \\ \|C(x, \theta_1) - C(x, \theta_2)\| \leq L_C |\theta_1 - \theta_2| \quad , \forall \theta_1, \theta_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega; \\ \text{(c) There exists } m_C > 0 \text{ such that} \\ (C(x, \theta_1) - C(x, \theta_2)) \cdot (\theta_1 - \theta_2) \geq m_C |\theta_1 - \theta_2|^2 \quad , \forall \theta_1, \theta_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega; \\ \text{(d) } x \mapsto C(x, \theta) \text{ is Lebesgue measurable on } \Omega; \\ \text{(e) } C(x, 0) = 0 \text{ a.e. } x \in \Omega. \end{array} \right. \quad (4.11)$$

4.3 Global existence

In the current section, we give the global existence result for the contact problem (4.1)-(4.10). For this purpose, the new variable ψ considered in [80] is introduced

$$\psi(x, t) = \int_0^t \theta(x, s) ds + \Phi(x), \quad \forall x \in \Omega, \quad \forall t \geq 0. \quad (4.12)$$

where $\Phi \in Y$ solves

$$\begin{cases} -\kappa \Delta \Phi = \delta \Delta \theta_0 - \theta_1 - \beta \operatorname{div} u_1 & \text{in } \Omega, \\ \Phi = 0 & \text{on } \Gamma. \end{cases} \quad (4.13)$$

We integrate (4.3) with respect to t . By taking into account (4.12) and (4.13), we get

$$\psi'' - \kappa \Delta \psi - \delta \Delta \psi' + \beta \operatorname{div} u' = 0.$$

Moreover, as in [16], we use the new variable

$$\eta(x, s, t) = u(x, t) - u(x, t - s), \quad x \in \Omega, \quad t, s \in \mathbb{R}_+.$$

Consequently, the system (4.1)-(4.10) takes the form

$$\begin{aligned} \rho u'' - \operatorname{Div} \sigma &= f_0 & \text{in } \Omega \times (0, \infty), \\ \sigma &= A\varepsilon(u') + G\varepsilon(u) - \int_0^\infty g(s) B\varepsilon(u - \eta(s)) ds + C\psi' & \text{in } \Omega \times (0, \infty), \\ \eta_t &= u' - \eta_s & \text{in } \Omega \times (0, \infty) \times (0, \infty), \\ \psi'' - \kappa \Delta \psi - \delta \Delta \psi' + \beta \operatorname{div} u' &= 0 & \text{in } \Omega \times (0, \infty), \\ u &= 0 & \text{on } \Gamma_1 \times (0, \infty), \\ \eta &= 0 & \text{on } \Gamma_1 \times (0, \infty) \times (0, \infty), \\ \psi &= 0 & \text{on } \Gamma \times (0, \infty), \\ \sigma \nu &= f_2 & \text{on } \Gamma_2 \times (0, \infty), \\ -\sigma_\nu &= p_\nu(u'_\nu) & \text{on } \Gamma_3 \times (0, \infty), \\ -\sigma_\tau &= p_\tau(u'_\tau) & \text{on } \Gamma_3 \times (0, \infty), \\ u(-t) &= u_0(t), & \text{in } \Omega \times [0, \infty), \\ u'(0) &= u_1 & \text{in } \Omega, \\ \eta(s, 0) &= u_0(0) - u_0(s) = \eta_0(s) & \text{for } s \in (0, \infty), \\ \psi(0) &= \psi_0, \quad \psi'(0) = \psi_1 & \text{in } \Omega \times [0, \infty). \end{aligned} \quad (4.14)$$

For $U = (u, v, \eta, \psi, \phi)^T$ with $v = u'$ and $\phi = \psi'$. Assuming that σ is smooth function, the system (4.14) can be rewritten

$$\begin{cases} U_t(t) + \mathcal{A}U(t) = F(t), & \forall t > 0, \\ U(0) = U_0 = (u_0, u_1, \eta_0, \psi_0, \psi_1)^T, \end{cases} \quad (4.15)$$

where the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ and the domain $\mathcal{D}(\mathcal{A})$ are given by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ \eta \\ \psi \\ \phi \end{pmatrix} = \begin{pmatrix} -v \\ -\frac{1}{\rho} \text{Div}(A\varepsilon(v) + G\varepsilon(u) + C\phi) + \frac{1}{\rho} \text{Div}(\int_0^\infty g(s)B\varepsilon(u - \eta(s))ds) \\ \eta_s - v \\ -\phi \\ -\kappa\Delta\psi - \delta\Delta\phi + \beta \text{div } v \end{pmatrix}, \quad (4.16)$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} (u, v, \eta, \psi, \phi)^T \in \mathcal{H}; v \in V; \phi \in Y; \eta_s \in \mathcal{M}; \\ \kappa\psi + \delta\phi \in H^2(\Omega); \\ \sigma\nu = f_2 \text{ on } \Gamma_2; \\ -\sigma_\nu = p_\nu(v_\nu) \text{ on } \Gamma_3; \\ -\sigma_\tau = p_\tau(v_\tau) \text{ on } \Gamma_3 \end{array} \right\}, \quad (4.17)$$

where

$$\mathcal{H} = V \times H \times \mathcal{M} \times Y \times L^2(\Omega). \quad (4.18)$$

and the function $F(t)$ is as follows

$$F(t) = \begin{pmatrix} 0 \\ \frac{f_0(t)}{\rho} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.19)$$

The inner product on the Hilbert space $\mathcal{H}$ follows

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} = \langle u, \tilde{u} \rangle_V + \langle v, \tilde{v} \rangle_H + \langle \eta, \tilde{\eta} \rangle_{\mathcal{M}} + \kappa \langle \psi, \tilde{\psi} \rangle_Y + \langle \phi, \tilde{\phi} \rangle_{L^2(\Omega)}, \quad (4.20)$$

for every $U = (u, v, \eta, \psi, \phi)^T$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\eta}, \tilde{\psi}, \tilde{\phi})^T$ in $\mathcal{H}$.

The main result of this section is the following

Theorem 4.1. *Under the assumptions (G1) and (1.9)-(1.12), (2.8), for any initial datum $U_0 \in \overline{\mathcal{D}(\mathcal{A})}$ and $F \in L^1(\mathbb{R}_+; \mathcal{H})$, there exists a unique solution U of the system (4.15) satisfying*

$$U \in C(\mathbb{R}_+; \mathcal{H}). \quad (4.21)$$

Proof. We adduce that the operator $\mathcal{A} + \omega_{\mathcal{A}}I$ is m-accretive for some $\omega_{\mathcal{A}} > 0$ (see Definition

A.3). For any $U_i = (u_i, v_i, \eta_i, \psi_i, \phi_i)^T \in \mathcal{D}(\mathcal{A})$, $i = 1, 2$, we have

$$\begin{aligned}
& \langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \\
= & -\langle v_1 - v_2, u_1 - u_2 \rangle_V \\
& - \left\langle \frac{1}{\rho} \operatorname{Div} (A\varepsilon(v_1) - A\varepsilon(v_2)), v_1 - v_2 \right\rangle_H \\
& - \left\langle \frac{1}{\rho} \operatorname{Div} (G\varepsilon(u_1) - G\varepsilon(u_2)), v_1 - v_2 \right\rangle_H \\
& - \left\langle \frac{1}{\rho} \operatorname{Div} (C\phi_1 - C\phi_2), v_1 - v_2 \right\rangle_H \\
& + \left\langle \frac{1}{\rho} \operatorname{Div} \int_0^\infty g(s) B\varepsilon(u_1 - \eta_1(s)) ds, v_1 - v_2 \right\rangle_H \\
& - \left\langle \frac{1}{\rho} \operatorname{Div} \int_0^\infty g(s) B\varepsilon(u_2 - \eta_2(s)) ds, v_1 - v_2 \right\rangle_H \\
& - \langle -\eta_{1s} + \eta_{2s} + v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} \\
& - \kappa \langle \phi_1 - \phi_2, \psi_1 - \psi_2 \rangle_Y \\
& - \kappa \langle \Delta \psi_1 - \Delta \psi_2, \phi_1 - \phi_2 \rangle_{L^2(\Omega)} \\
& - \delta \langle \Delta \phi_1 - \Delta \phi_2, \phi_1 - \phi_2 \rangle_{L^2(\Omega)} \\
& + \beta \langle \operatorname{div} v_1 - \operatorname{div} v_2, \phi_1 - \phi_2 \rangle_{L^2(\Omega)} \\
& + \omega_{\mathcal{A}} \langle U_1 - U_2, U_1 - U_2 \rangle_{\mathcal{H}}.
\end{aligned}$$

Thanks to Green's formulas (A.4)-(A.5) and boundary conditions, it holds that

$$\begin{aligned}
& \langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \\
= & -\langle v_1 - v_2, u_1 - u_2 \rangle_V \\
& + \langle A\varepsilon(v_1) - A\varepsilon(v_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\
& + \langle G\varepsilon(u_1) - G\varepsilon(u_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\
& + \langle C\phi_1 - C\phi_2, \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\
& - \left\langle \int_0^\infty g(s) B\varepsilon(u_1 - \eta_1(s)) ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}} \\
& + \left\langle \int_0^\infty g(s) B\varepsilon(u_2 - \eta_2(s)) ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}} \\
& + \int_{\Gamma_3} (p_\nu(v_{1\nu}) - p_\nu(v_{2\nu})) (v_{1\nu} - v_{2\nu}) da \\
& + \int_{\Gamma_3} (p_\tau(v_{1\tau}) - p_\tau(v_{2\tau})) \cdot (v_{1\tau} - v_{2\tau}) da \\
& - \frac{1}{2} \int_0^\infty g'(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds \\
& - \langle v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} \\
& - \kappa \langle \phi_1 - \phi_2, \psi_1 - \psi_2 \rangle_Y \\
& + \kappa \langle \nabla \psi_1 - \nabla \psi_2, \nabla \phi_1 - \nabla \phi_2 \rangle_{L^2(\Omega)^d} \\
& + \delta \langle \nabla \phi_1 - \nabla \phi_2, \nabla \phi_1 - \nabla \phi_2 \rangle_{L^2(\Omega)^d} \\
& - \beta \langle v_1 - v_2, \nabla \phi_1 - \nabla \phi_2 \rangle_{L^2(\Omega)^d} \\
& + \omega_{\mathcal{A}} \langle U_1 - U_2, U_1 - U_2 \rangle_{\mathcal{H}}.
\end{aligned}$$

Inequalities of Cauchy- Schwarz and Young, (G1), (b) of (1.10), (1.11) and (4.11), (c) of (1.9), (1.12) and (2.8), (A.7), (A.8) imply that

$$\begin{aligned}
\langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \geq & - \left(\frac{6}{4\delta_*} - m_A \right) \|v_1 - v_2\|_V^2 \\
& - \delta_* \left(1 + L_G^2 + g_0^2 L_B^2 \right) \|u_1 - u_2\|_V^2 \\
& - \delta_* \left(L_B^2 g_0 + g_0 \right) \int_0^\infty g(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds \\
& - \left(\frac{1}{4\delta_*} - \delta \right) \|\phi_1 - \phi_2\|_Y^2 \\
& - \delta_* L_C^2 \|\phi_1 - \phi_2\|_{L^2(\Omega)}^2 \\
& - \frac{\delta_* \beta^2}{\rho_*^2} \|v_1 - v_2\|_H^2 \\
& + \omega_{\mathcal{A}} \|U_1 - U_2\|_{\mathcal{H}}^2.
\end{aligned}$$

Therefore, if we take $\delta_* > \max\left(\frac{3}{2m_A}, \frac{1}{4\delta}\right)$, we get

$$\langle (\mathcal{A} + \omega_{\mathcal{A}} I) U_1 - (\mathcal{A} + \omega_{\mathcal{A}} I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \geq (\omega_{\mathcal{A}} - L) \|U_1 - U_2\|_{\mathcal{H}}^2,$$

where

$$L = \max \left(\delta_* \left(1 + L_G^2 + g_0^2 L_B^2 \right), \delta_* \left(L_B^2 g_0 + g_0 \right), \delta_* L_C^2, \frac{\delta_* \beta^2}{\rho_*^2} \right),$$

which implies that $\mathcal{A} + \omega_{\mathcal{A}} I$ is accretive if we take $\omega_{\mathcal{A}} > L$.

Next, we show that $\mathcal{A} + \lambda I$ is surjective for $\lambda > 0$. So, we assume that $(h_1, h_2, h_3, h_4, h_5)^T \in \mathcal{H}$ and we look for $U = (u, v, \eta, \psi, \phi)^T \in \mathcal{D}(\mathcal{A})$ such that

$$(\lambda I + \mathcal{A})(u, v, \eta, \psi, \phi)^T = (h_1, h_2, h_3, h_4, h_5)^T,$$

this last can be seen as

$$\begin{aligned} \lambda u - v &= h_1 \\ \lambda v - \frac{1}{\rho} \operatorname{Div} (G\varepsilon(u) + A\varepsilon(v) + C\phi) + \frac{1}{\rho} \operatorname{Div} \left(\int_0^\infty g(s) B\varepsilon(u - \eta(s)) ds \right) &= h_2 \\ \lambda \eta + \eta_s - v &= h_3 \\ \lambda \psi - \phi &= h_4 \\ \lambda \phi - \kappa \Delta \psi - \delta \Delta \phi + \beta \operatorname{div} v &= h_5. \end{aligned} \tag{4.22}$$

Suppose that the couple (v, ϕ) is found in $V \times Y$. Then, from (4.22)₁ and (4.22)₄, we have

$$u = \frac{h_1 + v}{\lambda}, \tag{4.23}$$

$$\psi = \frac{h_4 + \phi}{\lambda}. \tag{4.24}$$

Therefore,

$$(u, \psi) \in V \times Y.$$

We integrate (4.22)₃ and employ $\eta(0) = 0$ to obtain

$$\eta(s) = e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau, \quad s \in \mathbb{R}_+. \tag{4.25}$$

(4.22)₅ is equivalent to

$$\lambda \phi - \frac{\kappa \Delta \phi}{\lambda} - \delta \Delta \phi + \beta \operatorname{div} v = h_5 + \frac{\kappa \Delta h_4}{\lambda}.$$

The operator $\mathcal{R} : Y \rightarrow Y'$ is defined by

$$\begin{aligned} \langle \mathcal{R}(\phi), \phi \rangle_{Y' \times Y} &= \langle \lambda \phi, \phi \rangle_{L^2(\Omega)} + \frac{\kappa}{\lambda} \langle \nabla \phi, \nabla \phi \rangle_{L^2(\Omega)} + \delta \langle \nabla \phi, \nabla \phi \rangle_{L^2(\Omega)} \\ &\quad - \beta \langle v, \nabla \phi \rangle_{L^2(\Omega)} \end{aligned}$$

Now, we need to establish the surjectivity of $\mathcal{R} : Y \rightarrow Y'$. The result follows from Corollary A.1 provided that we prove that $\mathcal{R}$ is coercive and maximal monotone. By Theorem A.1, we prove that $\mathcal{R}$ is monotone, coercive and hemicontinuous.

Let $\phi, \tilde{\phi} \in Y$. Then, we obtain

$$\begin{aligned} \langle \mathcal{R}\phi - \mathcal{R}\tilde{\phi}, \phi - \tilde{\phi} \rangle_{Y' \times Y} &= \langle \lambda\phi - \lambda\tilde{\phi}, \phi - \tilde{\phi} \rangle_{L^2(\Omega)} \\ &\quad + \left(\frac{\kappa}{\lambda} + \delta \right) \langle \nabla\phi - \nabla\tilde{\phi}, \nabla\phi - \nabla\tilde{\phi} \rangle_{L^2(\Omega)^d} \\ &\geq \left(\frac{\kappa}{\lambda} + \delta \right) \|\phi - \tilde{\phi}\|_Y^2. \end{aligned}$$

Thus, the operator $\mathcal{R}$ is strongly monotone for $\lambda > 0$ and then, $\mathcal{R}$ is monotone and coercive. The operator $\mathcal{R}$ is hemicontinuous. From Definition A.2, we show the following

$$\mathcal{R}(\phi + \zeta\phi_1) \xrightarrow{w} \mathcal{R}(\phi), \quad \forall \phi, \phi_1 \in Y.$$

when $\zeta \rightarrow 0$.

Let $\tilde{\phi} \in Y$ and $\lambda > 0$, so

$$\begin{aligned} &\langle \mathcal{R}(\phi + \zeta\phi_1), \tilde{\phi} \rangle_{Y' \times Y} - \langle \mathcal{R}(\phi), \tilde{\phi} \rangle_{Y' \times Y} \\ &= \zeta \langle \lambda\phi_1, \tilde{\phi} \rangle_{L^2(\Omega)} \\ &\quad + \zeta \left(\frac{\kappa}{\lambda} + \delta \right) \langle \nabla\phi_1, \nabla\tilde{\phi} \rangle_{L^2(\Omega)^d}. \end{aligned}$$

Considering the inequalities of Cauchy-Schwarz and Poincaré, we obtain

$$\begin{aligned} \left| \zeta \langle \lambda\phi_1, \tilde{\phi} \rangle_{L^2(\Omega)} \right| &\leq |\zeta| \lambda C_*^2 \|\phi_1\|_Y \|\tilde{\phi}\|_Y \\ &\rightarrow 0, \end{aligned} \tag{4.26}$$

as $\zeta \rightarrow 0$. Recalling the inequality of Cauchy-Schwarz to get

$$\begin{aligned} \left| \zeta \left(\frac{\kappa}{\lambda} + \delta \right) \langle \nabla\phi_1, \nabla\tilde{\phi} \rangle_{L^2(\Omega)^d} \right| &\leq |\zeta| \left(\frac{\kappa}{\lambda} + \delta \right) \|\phi_1\|_Y \|\tilde{\phi}\|_Y \\ &\rightarrow 0, \end{aligned} \tag{4.27}$$

as $\zeta \rightarrow 0$. Combining (4.26)-(4.27) yields

$$\left| \langle \mathcal{R}(\phi + \zeta\phi_1), \tilde{\phi} \rangle_{Y' \times Y} - \langle \mathcal{R}(\phi), \tilde{\phi} \rangle_{Y' \times Y} \right| \rightarrow 0,$$

as $\zeta \rightarrow 0$. Then, the hemicontinuity of $\mathcal{R}$ follows.

Consequently, there exists $\phi \in Y$ satisfying (4.22)₅. Also, we have

$$\begin{aligned} &\langle \lambda\phi, \phi \rangle_{L^2(\Omega)} + \frac{\kappa}{\lambda} \langle \nabla\phi, \nabla\phi \rangle_{L^2(\Omega)} + \delta \langle \nabla\phi, \nabla\phi \rangle_{L^2(\Omega)} - \beta \langle v, \nabla\phi \rangle_{L^2(\Omega)} \\ &= \langle h_5 + \frac{\kappa}{\lambda} \Delta h_4, \phi \rangle_{L^2(\Omega)}, \end{aligned}$$

for all $\phi \in Y$. Then,

$$\frac{\kappa}{\lambda} \Delta \phi + \delta \Delta \phi + \frac{\kappa \Delta h_4}{\lambda} = \lambda \phi - h_5 + \beta \operatorname{div} v \in L^2(\Omega).$$

The regularity theory of linear elliptic equation implies that

$$\left(\frac{\kappa}{\lambda} + \delta \right) \phi + \frac{\kappa}{\lambda} h_4 \in H^2(\Omega).$$

Having in mind $h_4 = \lambda \psi - \phi$, then, $\kappa \psi + \delta \phi \in H^2(\Omega)$.

Note that (4.22) is equivalent to

$$\begin{aligned} & \lambda v - \frac{1}{\rho} \operatorname{Div} \left(G \varepsilon \left(\frac{h_1 + v}{\lambda} \right) + A \varepsilon(v) + C \phi \right) \\ & + \frac{1}{\rho} \operatorname{Div} \left(\int_0^\infty g(s) B \varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds \right) \\ & = h_2. \end{aligned} \quad (4.28)$$

We define the operator $\mathcal{J} : V \rightarrow V'$ by

$$\begin{aligned} \langle \mathcal{J}(v), v \rangle_{V' \times V} &= \langle \lambda v, v \rangle_H + \left\langle G \varepsilon \left(\frac{h_1 + v}{\lambda} \right) + A \varepsilon(v) + C \phi, \varepsilon(v) \right\rangle_{\mathcal{H}} \\ & - \left\langle \int_0^\infty g(s) B \varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) \right\rangle_{\mathcal{H}} \\ & + \int_{\Gamma_3} p_\nu(v_\nu) v_\nu da + \int_{\Gamma_3} p_\tau(v_\tau) \cdot v_\tau da - \int_{\Gamma_2} f_2 v da. \end{aligned} \quad (4.29)$$

We prove that $\mathcal{J} : V \rightarrow V'$ is surjective by repeating the same arguments used to prove the surjectivity of the operator $\mathcal{R}$.

$\mathcal{J}$ is monotone. Indeed, let $v, \tilde{v} \in V$. Then, we arrive at

$$\begin{aligned} & \langle \mathcal{J}v - \mathcal{J}\tilde{v}, v - \tilde{v} \rangle_{V' \times V} \\ &= \langle \lambda v - \lambda \tilde{v}, v - \tilde{v} \rangle_H + \left\langle G \varepsilon \left(\frac{h_1 + v}{\lambda} \right) - G \varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} \right), \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \langle A \varepsilon(v) - A \varepsilon(\tilde{v}), \varepsilon(v) - \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \\ & - \left\langle \int_0^\infty g(s) B \varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \left\langle \int_0^\infty g(s) B \varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + \tilde{v}) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \int_{\Gamma_3} (p_\nu(v_\nu) - p_\nu(\tilde{v}_\nu)) (v_\nu - \tilde{v}_\nu) da + \int_{\Gamma_3} (p_\tau(v_\tau) - p_\tau(\tilde{v}_\tau)) \cdot (v_\tau - \tilde{v}_\tau) da. \end{aligned}$$

Assumption (c) of (1.10) implies that

$$\left\langle G\varepsilon\left(\frac{h_1+v}{\lambda}\right) - G\varepsilon\left(\frac{h_1+\tilde{v}}{\lambda}\right), \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \geq \frac{1}{\lambda} m_G \|v - \tilde{v}\|_V^2. \quad (4.30)$$

Assumption (c) of (1.9) yields

$$\langle A\varepsilon(v) - A\varepsilon(\tilde{v}), \varepsilon(v) - \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \geq m_A \|v - \tilde{v}\|_V^2. \quad (4.31)$$

Employing inequality of Cauchy-Schwarz, (b) of (1.11) and (G1), we get

$$\begin{aligned} & - \left\langle \int_0^\infty g(s) B\varepsilon\left(\frac{h_1+v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda\tau} (h_3(\tau) + v) d\tau\right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \left\langle \int_0^\infty g(s) B\varepsilon\left(\frac{h_1+\tilde{v}}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda\tau} (h_3(\tau) + \tilde{v}) d\tau\right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & \geq -\frac{g_0 L_B}{\lambda} \|v - \tilde{v}\|_V^2. \end{aligned} \quad (4.32)$$

From (c) of (1.12), we have

$$\int_{\Gamma_3} (p_\nu(v_\nu) - p_\nu(\tilde{v}_\nu)) (v_\nu - \tilde{v}_\nu) da \geq 0, \quad (4.33)$$

and thanks to (c) of (2.8), we have

$$\int_{\Gamma_3} (p_\tau(v_\tau) - p_\tau(\tilde{v}_\tau)) \cdot (v_\tau - \tilde{v}_\tau) da \geq 0. \quad (4.34)$$

Now, combining (4.30)-(4.34), we obtain

$$\langle \mathcal{J}v - \mathcal{J}\tilde{v}, v - \tilde{v} \rangle_{V' \times V} \geq \left(\frac{m_G - g_0 L_B}{\lambda} \right) \|v - \tilde{v}\|_V^2.$$

In view of (1.16), we deduce that the operator $\mathcal{J}$ is strongly monotone for $\lambda > 0$ and then, $\mathcal{J}$ is monotone and coercive.

Next, $\mathcal{J}$ is hemicontinuous that is

$$\mathcal{J}(v + \zeta v_1) \xrightarrow{w} \mathcal{J}(v), \quad \forall v, v_1 \in V.$$

as $\zeta \rightarrow 0$.

Let $\tilde{v} \in V$ and $\lambda > 0$, so

$$\begin{aligned}
& \langle \mathcal{J}(v + \zeta v_1), \tilde{v} \rangle_{V' \times V} - \langle \mathcal{J}(v), \tilde{v} \rangle_{V' \times V} \\
&= \zeta \langle \lambda v_1, \tilde{v} \rangle_H + \left\langle G\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + v}{\lambda} \right), \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&\quad + \langle A\varepsilon(v + \zeta v_1) - A\varepsilon(v), \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \\
&\quad - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v + \zeta v_1) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&\quad + \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&\quad + \int_{\Gamma_3} (p_\nu(v_\nu + \zeta v_{1\nu}) - p_\nu(v_\nu)) \tilde{v}_\nu da + \int_{\Gamma_3} (p_\tau(v_\tau + \zeta v_{1\tau}) - p_\tau(v_\tau)) \cdot \tilde{v}_\tau da. \tag{4.35}
\end{aligned}$$

Exploiting inequality of Cauchy-Schwarz and (A.9), we get

$$\begin{aligned}
|\zeta \langle \lambda v_1, \tilde{v} \rangle_H| &\leq |\zeta| \lambda \|v_1\|_H \|\tilde{v}\|_H \\
&\leq |\zeta| \lambda d^2 \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \tag{4.36}
\end{aligned}$$

as $\zeta \rightarrow 0$. Considering inequality of Cauchy-Schwarz and using (b) of (1.10), we get

$$\begin{aligned}
\left| \left\langle G\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + v}{\lambda} \right), \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \right| &\leq \frac{|\zeta|}{\lambda} L_G \|v_1\|_V \|\tilde{v}\|_V, \\
&\rightarrow 0, \tag{4.37}
\end{aligned}$$

as $\zeta \rightarrow 0$. Similar to (4.37) and thanks to (b) of (1.9) instead of (1.10), we get

$$|\langle A\varepsilon(v + \zeta v_1) - A\varepsilon(v), \varepsilon(\tilde{v}) \rangle_{\mathcal{H}}| \leq |\zeta| L_A \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \tag{4.38}$$

as $\zeta \rightarrow 0$. Making use of inequality of Cauchy-Schwarz, (G1), (b) of (1.11), we obtain

$$\begin{aligned}
& \left| - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v + \zeta v_1}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v + \zeta v_1) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \right. \\
& \quad \left. + \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \right| \\
&\leq \frac{|\zeta| L_B g_0}{\lambda} \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \tag{4.39}
\end{aligned}$$

as $\zeta \rightarrow 0$. Recalling again inequality of Cauchy-Schwarz, (b) of (1.12) and (A.6), one has

$$\begin{aligned}
\left| \int_{\Gamma_3} (p_\nu(v_\nu + \zeta v_{1\nu}) - p_\nu(v_\nu)) \tilde{v}_\nu da \right| &\leq \|p_\nu(v_\nu + \zeta v_{1\nu}) - p_\nu(v_\nu)\|_{L^2(\Gamma_3)} \|\tilde{v}_\nu\|_{L^2(\Gamma_3)} \\
&\leq L_\nu |\zeta| \|v_1\|_{L^2(\Gamma_3)^d} \|\tilde{v}\|_{L^2(\Gamma_3)^d} \\
&\leq L_\nu C_0^2 |\zeta| \|v_1\|_V \|\tilde{v}\|_V \rightarrow 0, \tag{4.40}
\end{aligned}$$

as $\zeta \rightarrow 0$. Employing inequality of Cauchy-Schwarz, (b) of (2.8) and (A.6), we have

$$\begin{aligned} \left| \int_{\Gamma_3} (p_\tau(v_\tau + \zeta v_{1\tau}) - p_\tau(v_\tau)) \cdot \widetilde{v}_\tau da \right| &\leq \|p_\tau(v_\tau + \zeta v_{1\tau}) - p_\tau(v_\tau)\|_{L^2(\Gamma_3)^d} \|\widetilde{v}_\tau\|_{L^2(\Gamma_3)^d} \\ &\leq L_\tau C_0^2 |\zeta| \|v_1\|_V \|\widetilde{v}\|_V \rightarrow 0, \end{aligned} \quad (4.41)$$

as $\zeta \rightarrow 0$. Combining (4.36)-(4.41) yields

$$|\langle \mathcal{J}(v + \zeta v_1), \widetilde{v} \rangle_{V' \times V} - \langle \mathcal{J}(v), \widetilde{v} \rangle_{V' \times V}| \rightarrow 0, \quad (4.42)$$

as $\zeta \rightarrow 0$. Therefore, $\mathcal{J} : V \rightarrow V'$ is hemicontinuous.

Corollary A.1 guarantees that $\mathcal{J}$ is surjective. Then, we can find $v \in V$ which satisfies eq. (4.28). Similar procedure as in (1.46) and (1.47), we get

$$(\eta, \eta_s) \in \mathcal{M} \times \mathcal{M}$$

Therefore, $U = (u, v, \eta, \psi, \phi)^T \in \mathcal{D}(\mathcal{A})$ which satisfies (4.22). Consequently, the surjectivity of $\mathcal{A} + \lambda I$ follows for some positive constant λ .

Theorem A.3 ensures that there exists a unique solution U of (4.15) satisfying (4.21) if $F \in L^1(\mathbb{R}_+; \mathcal{H})$ and $U_0 \in \overline{\mathcal{D}(\mathcal{A})}$. This ends the proof of Theorem 4.1. $\square$

4.4 Exponential stability

Assuming the smoothness of σ , Let prove the result of the exponential decay of solution. The Lyapunov functional denoted by $\mathcal{L}$ is defined as follows

$$\mathcal{L}(t) = E(t) + \epsilon (N_1 \mathcal{F}_1(t) + \mathcal{F}_2(t) + \mathcal{F}_3(t)), \quad (4.43)$$

where N_1 and ϵ are positive constants which will be fixed properly afterwards and

$$\begin{aligned} E(t) &= \frac{1}{2} \|U(t)\|_{\mathcal{H}}^2, \\ &= \frac{1}{2} \left(\|u'(t)\|_H^2 + \|u(t)\|_V^2 + \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \right. \\ &\quad \left. + \kappa \|\psi(t)\|_Y^2 + \|\psi'(t)\|_{L^2(\Omega)}^2 \right), \forall t \in \mathbb{R}_+, \end{aligned} \quad (4.44)$$

$$\mathcal{F}_1(t) = \langle u'(t), u(t) \rangle_H - e^{3\zeta_1 t} \|f(t)\|_{V'}^2, \forall t \in \mathbb{R}_+, \quad (4.45)$$

$$\mathcal{F}_2(t) = \langle \psi'(t), \psi(t) \rangle_{L^2(\Omega)} + \frac{\delta}{2} \|\nabla \psi(t)\|_{L^2(\Omega)}^2, \forall t \in \mathbb{R}_+, \quad (4.46)$$

$$\mathcal{F}_3(t) = - \left\langle u'(t), \int_0^{+\infty} g(s) \eta(s, t) ds \right\rangle_H, \forall t \in \mathbb{R}_+. \quad (4.47)$$

Lemma 4.1. Assume that (G1), (1.9)-(1.13) and (2.8) hold and $U_0 \in \mathcal{D}(\mathcal{A})$. Then, the energy functional E defined by (4.44) satisfies for $\lambda_1 > 0$, for all $t \geq 0$

$$\begin{aligned} E'(t) \leq & -(m_A - K_1) \|u'(t)\|_V^2 + K_2 \lambda_1 \|u(t)\|_V^2 + K_3 \lambda_1 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ & + \frac{1}{2} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds - (\delta - K_4 \lambda_1) \|\psi'(t)\|_Y^2 \\ & + K_5 \lambda_1 \|\psi(t)\|_Y^2 + \lambda_1 \|f(t)\|_{V'}^2, \end{aligned} \quad (4.48)$$

where

$$K_1 = \frac{8}{4\lambda_1}, K_2 = L_G^2 + g_0^2 L_B^2 + 1, K_3 = g_0 + g_0 L_B^2, K_4 = L_C^2 C_*^2, K_5 = \beta^2.$$

Proof. Differentiating (4.44) with respect to time variable t , employing (4.14)₁, (4.14)₃ and (4.14)₄, integrating over domain Ω and making use of formulas of Green (A.4)-(A.5), boundary conditions and (1.14), it holds that

$$\begin{aligned} E'(t) &= \langle u''(t), u'(t) \rangle_H + \langle u'(t), u(t) \rangle_V + \int_0^{+\infty} g(s) \langle \eta_t(s, t), \eta(s, t) \rangle_V ds \\ &\quad + \langle \psi''(t), \psi'(t) \rangle_{L^2(\Omega)} + \kappa \langle \psi'(t), \psi(t) \rangle_Y \\ &= -\langle A\varepsilon(u'(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} - \langle G\varepsilon(u(t)), \varepsilon(u'(t)) \rangle_{\mathcal{H}} - \langle C\psi'(t), \varepsilon(u'(t)) \rangle_{\mathcal{H}} \\ &\quad + \left\langle \int_0^{+\infty} g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u'(t)) \right\rangle_{\mathcal{H}} - \int_{\Gamma_3} p_\nu(u'_\nu(t)) u'_\nu(t) da \\ &\quad - \int_{\Gamma_3} p_\tau(u'_\tau) \cdot u'_\tau da + \langle f(t), u'(t) \rangle_{V' \times V} + \langle u'(t), u(t) \rangle_V \\ &\quad + \int_0^{+\infty} g(s) \langle u'(t), \eta(s, t) \rangle_V ds + \frac{1}{2} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds \\ &\quad - \kappa \langle \nabla \psi, \nabla \psi' \rangle_{L^2(\Omega)^d} - \delta \langle \nabla \psi', \nabla \psi' \rangle_{L^2(\Omega)^d} + \beta \langle u', \nabla \psi' \rangle_{L^2(\Omega)} \\ &\quad + \kappa \langle \nabla \psi, \nabla \psi' \rangle_{L^2(\Omega)^d}. \end{aligned}$$

Considering (G1), (1.9)–(1.12), (2.8), inequalities of Cauchy-Schwarz, Young and Poincaré, we obtain the estimate (4.48). □

Remark 4.1. From the inequality (4.48), we can infer that the energy functional (4.44) is not decreasing generally.

We recall the following technical lemmas to prove the stability result.

Lemma 4.2. Assume that assumptions (G1), (G3), (1.9)-(1.13) and (2.8) hold and let $U_0 \in \mathcal{D}(\mathcal{A})$. Then, the functional $\mathcal{F}_1$ defined by (4.45) satisfies for $\lambda_2 > 0$, for all $t \geq 0$

$$\begin{aligned} \mathcal{F}'_1(t) \leq & (-l + 6\lambda_2) \|u(t)\|_V^2 + K_6 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds + K_7 \|u'(t)\|_V^2 \\ & + K_8 \|\psi'(t)\|_Y^2 + \left(\frac{1}{4\lambda_2} - \gamma_0 \right) \|f(t)\|_{V'}^2, \end{aligned} \quad (4.49)$$

where

$$K_6 = \frac{g_0 L_B^2}{4\lambda_2}, K_7 = \frac{L_A^2}{4\lambda_2} + \frac{L_\nu^2 C_0^4}{4\lambda_2} + \frac{L_\tau^2 C_0^4}{4\lambda_2} + d^2, K_8 = \frac{L_C^2 C_*^2}{4\lambda_2}.$$

Proof. We differentiate (4.45) with respect to time variable t , thus

$$\mathcal{F}'_1(t) = \|u'(t)\|_H^2 + \langle u''(t), u(t) \rangle_H - 3\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2 - 2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'}.$$

Multiplying (4.14)₁ by $u(t)$. After that, integrating over domain Ω , recalling formula of Green (A.5), boundary conditions and (1.10), we arrive at

$$\begin{aligned} \mathcal{F}'_1(t) \leq & \|u'(t)\|_H^2 - \langle A\varepsilon(u'(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} - m_G \|u(t)\|_V^2 - \langle C\psi'(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} \\ & + \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} - \int_{\Gamma_3} p_\nu(u'_\nu(t)) u_\nu(t) da \\ & - \int_{\Gamma_3} p_\tau(u'_\tau(t)) \cdot u_\tau(t) da + \langle f(t), u(t) \rangle_{V' \times V} - 3\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2 \\ & - 2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'}. \end{aligned} \quad (4.50)$$

Considering inequalities of Cauchy-Schwarz, Young and having in mind (1.9), we get

$$\begin{aligned} \langle A\varepsilon(u'(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} & \leq \|A\varepsilon(u'(t))\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\ & \leq \frac{L_A^2}{4\lambda_2} \|u'(t)\|_V^2 + \lambda_2 \|u(t)\|_V^2. \end{aligned}$$

Thanks to inequalities of Cauchy-Schwarz, Young, Poincaré and in view of (4.11), we obtain

$$\begin{aligned} \langle C\psi'(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} & \leq \|C\psi'(t)\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\ & \leq \frac{L_C^2 C_*^2}{4\lambda_2} \|\psi'(t)\|_Y^2 + \lambda_2 \|u(t)\|_V^2. \end{aligned}$$

Inequalities of Cauchy-Schwarz, Young, (1.11) and (G1) imply that

$$\begin{aligned} & \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \\ & \leq \left\| \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds \right\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\ & \leq L_B \int_0^\infty g(s) (\|u(t)\|_V + \|\eta(s, t)\|_V) ds \|u(t)\|_V \\ & \leq (g_0 L_B + \lambda_2) \|u(t)\|_V^2 + \frac{L_B^2 g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds. \end{aligned}$$

Utilizing inequalities of Cauchy-Schwarz and Young, (1.12) and (A.6), one has

$$\begin{aligned} \int_{\Gamma_3} p_\nu(u'_\nu(t))u_\nu(t)da &\leq \left(\int_{\Gamma_3} |p_\nu(u'_\nu(t))|^2 da \right)^{\frac{1}{2}} \left(\int_{\Gamma_3} |u_\nu(t)|^2 da \right)^{\frac{1}{2}} \\ &\leq L_\nu \|u'(t)\|_{L^2(\Gamma_3)^d} \|u(t)\|_{L^2(\Gamma_3)^d} \\ &\leq \frac{L_\nu^2 C_0^4}{4\lambda_2} \|u'(t)\|_V^2 + \lambda_2 \|u(t)\|_V^2. \end{aligned}$$

From inequalities of Cauchy-Schwarz and Young, (2.8) and (A.6), we arrive at

$$\begin{aligned} \int_{\Gamma_3} p_\tau(u'_\tau) \cdot u_\tau(t)da &\leq \left(\int_{\Gamma_3} \|p_\tau(u'_\tau(t))\|^2 da \right)^{\frac{1}{2}} \left(\int_{\Gamma_3} \|u_\tau(t)\|^2 da \right)^{\frac{1}{2}} \\ &\leq L_\tau \|u'(t)\|_{L^2(\Gamma_3)^d} \|u(t)\|_{L^2(\Gamma_3)^d} \\ &\leq \frac{L_\tau^2 C_0^4}{4\lambda_2} \|u'(t)\|_V^2 + \lambda_2 \|u(t)\|_V^2. \end{aligned}$$

Employing again inequalities of Cauchy-Schwarz and Young to get

$$\langle f(t), u(t) \rangle_{V' \times V} \leq \frac{1}{4\lambda_2} \|f(t)\|_{V'}^2 + \lambda_2 \|u(t)\|_V^2.$$

Using the inequality of Cauchy-Schwarz and considering assumption (G3), we obtain

$$\begin{aligned} -2e^{3\zeta_1 t} \langle f'(t), f(t) \rangle_{V'} &\leq 2e^{3\zeta_1 t} \|f'(t)\|_{V'} \|f(t)\|_{V'} \\ &\leq 2\zeta_1 e^{3\zeta_1 t} \|f(t)\|_{V'}^2. \end{aligned}$$

Substituting the above seven inequalities into (4.50) and using (A.9), we get

$$\begin{aligned} \mathcal{F}'_1(t) &\leq (-l + 6\lambda_2) \|u(t)\|_V^2 + \frac{g_0 L_B^2}{4\lambda_2} \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ &\quad + \left(\frac{L_A^2}{4\lambda_2} + \frac{L_\nu^2 C_0^4}{4\lambda_2} + \frac{L_\tau^2 C_0^4}{4\lambda_2} + d^2 \right) \|u'(t)\|_V^2 + \frac{L_C^2 C_*^2}{4\lambda_2} \|\psi'(t)\|_Y^2 \\ &\quad + \left(\frac{1}{4\lambda_2} - \gamma_0 \right) \|f(t)\|_{V'}^2, \end{aligned}$$

where $l = m_G - g_0 L_B$ and γ_0 is such that $\gamma_0 > 0$ and $-\zeta_1 e^{3\zeta_1 t} \leq -\gamma_0$. □

Lemma 4.3. *Let $U_0 \in \mathcal{D}(\mathcal{A})$. Thus, the functional $\mathcal{F}_2$ defined by (4.46) satisfies for $\lambda_3 > 0$, for all $t \geq 0$*

$$\mathcal{F}'_2(t) \leq K_9 \|u'(t)\|_V^2 - (\kappa - \lambda_3) \|\psi(t)\|_Y^2 + K_{10} \|\psi'(t)\|_Y^2, \quad (4.51)$$

where

$$K_9 = \frac{\beta^2 d^2}{4\lambda_3 \rho_*^2}, K_{10} = C_*^2.$$

Proof. Differentiating (4.46) with respect to the time variable t , one has

$$\mathcal{F}'_2(t) = \|\psi'(t)\|_{L^2(\Omega)}^2 + \langle \psi''(t), \psi(t) \rangle_{L^2(\Omega)} + \delta \langle \nabla \psi'(t), \nabla \psi(t) \rangle_{L^2(\Omega)}.$$

Multiplying (4.14)₄ by $\psi(t)$. Then, integrating over domain Ω , making use of Green's formula (A.4) and boundary conditions, it follows that

$$\mathcal{F}'_2(t) \leq \|\psi'(t)\|_{L^2(\Omega)}^2 - \kappa \|\psi(t)\|_Y^2 + \beta \langle u'(t), \nabla \psi(t) \rangle_{L^2(\Omega)}. \quad (4.52)$$

Employing inequalities of Cauchy-Schwarz, Young and Poincaré, (A.7) and (A.8), we get

$$\beta \langle u'(t), \nabla \psi(t) \rangle_{L^2(\Omega)} \leq \frac{\beta^2 d^2}{4\lambda_3 \rho_*^2} \|u'(t)\|_V^2 + \lambda_3 \|\psi(t)\|_Y^2.$$

Substituting the above inequality in (4.52) and exploiting inequality of Poincaré, we obtain

$$\mathcal{F}'_2(t) \leq \frac{\beta^2 d^2}{4\lambda_3 \rho_*^2} \|u'(t)\|_V^2 - (\kappa - \lambda_3) \|\psi(t)\|_Y^2 + C_*^2 \|\psi'(t)\|_Y^2$$

□

Lemma 4.4. Assume that (G1), (1.9)-(1.13) and (2.8) hold and $U_0 \in \mathcal{D}(\mathcal{A})$. Then the functional $\mathcal{F}_3$ defined by (4.47) satisfies for $\lambda_2 > 0$, for all $t \geq 0$

$$\begin{aligned} \mathcal{F}'_3(t) \leq & -g_0 \|u'(t)\|_H^2 + K_{11} \lambda_2 \|u'(t)\|_V^2 + K_{12} \lambda_2 \|u(t)\|_V^2 + K_{13} \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ & - K_{14} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds + \lambda_2 \|f(t)\|_{V'}^2 + K_{15} \lambda_2 \|\psi'(t)\|_Y^2, \end{aligned} \quad (4.53)$$

where

$$K_{11} = L_A^2 + L_\nu^2 C_0^4 + L_\tau^2 C_0^4 + d^2, K_{12} = L_G^2 + L_B^2, K_{13} = \frac{6g_0}{4\lambda_2} + \frac{g_0^3}{4\lambda_2} + L_B g_0, K_{14} = \frac{g(0)d^2}{4\lambda_2}, K_{15} = L_C^2 C_*^2.$$

Proof. We differentiate (4.47) with respect to time variable t and considering (4.14)₃, we get

$$\mathcal{F}'_3(t) = - \left\langle u''(t), \int_0^\infty g(s) \eta(s, t) ds \right\rangle_H - g_0 \|u'(t)\|_H^2 + \left\langle u'(t), \int_0^\infty g(s) \eta_s(s, t) ds \right\rangle_H.$$

Integrating the term $\langle u'(t), \int_0^\infty g(s) \eta_s(s, t) ds \rangle_H$ with respect to the variable s and exploiting $\eta(0) = 0$, $\lim_{s \rightarrow +\infty} g(s) = 0$, we obtain

$$\mathcal{F}'_3(t) = - \left\langle u''(t), \int_0^\infty g(s) \eta(s, t) ds \right\rangle_H - g_0 \|u'(t)\|_H^2 - \left\langle u'(t), \int_0^\infty g'(s) \eta(s, t) ds \right\rangle_H.$$

Making use of (4.14)₁, integrating over domain Ω and employing formula of Green (A.5),

boundary conditions lead to

$$\begin{aligned}
\mathcal{F}'_3(t) = & \left\langle A\varepsilon(u'(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} + \left\langle G\varepsilon(u(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
& - \left\langle \int_0^\infty g(s)B\varepsilon(u(t) - \eta(s,t))ds, \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
& + \left\langle C\psi'(t), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
& + \int_{\Gamma_3} p_\nu(u'_\nu(t)) \int_0^\infty g(s)\eta_\nu(s,t)dsda + \int_{\Gamma_3} p_\tau(u'_\tau(t)) \cdot \int_0^\infty g(s)\eta_\tau(s,t)dsda \\
& - \left\langle f(t), \int_0^\infty g(s)\eta(s,t)ds \right\rangle_{V' \times V} - \left\langle u'(t), \int_0^\infty g'(s)\eta(s,t)ds \right\rangle_H \\
& - g_0 \|u'(t)\|_H^2.
\end{aligned} \tag{4.54}$$

By inequalities of Cauchy-Schwarz and Young, (G1) and (1.9), we see that

$$\begin{aligned}
& \left\langle A\varepsilon(u'(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
& \leq \|A\varepsilon(u'(t))\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\
& \leq \lambda_2 L_A^2 \|u'(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{4.55}$$

Employing again inequalities of Cauchy-Schwarz and Young, (G1) and (1.10), we get

$$\begin{aligned}
& \left\langle G\varepsilon(u(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
& \leq \|G\varepsilon(u(t))\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\
& \leq \lambda_2 L_G^2 \|u(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{4.56}$$

Making use of inequalities of Cauchy-Schwarz and Young, (G1) and (1.11), it follows that

$$\begin{aligned}
& - \left\langle \int_0^\infty g(s)B\varepsilon(u(t) - \eta(s,t))ds, \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
& \leq \left\| \int_0^\infty g(s)B\varepsilon(u(t) - \eta(s,t))ds \right\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\
& \leq \lambda_2 L_B^2 \|u(t)\|_V^2 + \left(\frac{g_0^3}{4\lambda_2} + L_B g_0 \right) \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{4.57}$$

From inequalities of Cauchy-Schwarz, Young and Poincaré, (4.11), we have

$$\begin{aligned}
\left\langle C\psi'(t), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} & \leq L_C^2 \lambda_2 \|\psi'(t)\|_{L^2(\Omega)}^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds \\
& \leq L_C^2 C_*^2 \lambda_2 \|\psi'(t)\|_Y^2 \\
& \quad + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{4.58}$$

Considering inequalities of Cauchy-Schwarz and Young, (1.12), (G1) and (A.6), we obtain

$$\begin{aligned}
& \int_{\Gamma_3} p_\nu(u'_\nu(t)) \left(\int_0^\infty g(s) \eta_\nu(s, t) ds \right) da \\
& \leq \left(\int_{\Gamma_3} p_\nu^2(u'_\nu(t)) da \right)^{\frac{1}{2}} \left(\int_{\Gamma_3} \left(\int_0^\infty g(s) \eta_\nu(s, t) ds \right)^2 da \right)^{\frac{1}{2}} \\
& \leq \lambda_2 L_\nu^2 C_0^4 \|u'(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds.
\end{aligned} \tag{4.59}$$

Applying inequalities of Cauchy-Schwarz and Young, (2.8), (G1) and (A.6), we have

$$\begin{aligned}
& \int_{\Gamma_3} p_\tau(u'_\tau(t)) \cdot \left(\int_0^\infty g(s) \eta_\tau(s, t) ds \right) da \\
& \leq \left(\int_{\Gamma_3} p_\tau^2(u'_\tau(t)) da \right)^{\frac{1}{2}} \left(\int_{\Gamma_3} \left(\int_0^\infty g(s) \eta_\tau(s, t) ds \right)^2 da \right)^{\frac{1}{2}} \\
& \leq \lambda_2 L_\tau^2 C_0^4 \|u'(t)\|_V^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds.
\end{aligned} \tag{4.60}$$

Utilizing inequalities of Cauchy-Schwarz and Young and assumption (G1), it follows

$$-\left\langle f(t), \int_0^\infty g(s) \eta(s, t) ds \right\rangle_{V' \times V} \leq \lambda_2 \|f(t)\|_{V'}^2 + \frac{g_0}{4\lambda_2} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds. \tag{4.61}$$

Exploiting inequalities of Cauchy-Schwarz and Young, (G1) and (A.9), it holds that

$$\begin{aligned}
& -\left\langle u'(t), \int_0^\infty g'(s) \eta(s, t) ds \right\rangle_H \\
& \leq \|u'(t)\|_H \left\| \int_0^\infty g'(s) \eta(s, t) ds \right\|_H \\
& \leq \lambda_2 \|u'(t)\|_H^2 - \frac{g(0)}{4\lambda_2} \left(\int_0^\infty g'(s) \|\eta(s, t)\|_H^2 ds \right) \\
& \leq d^2 \lambda_2 \|u'(t)\|_V^2 - \frac{g(0)d^2}{4\lambda_2} \left(\int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds \right).
\end{aligned} \tag{4.62}$$

Substituting the estimates (4.55)-(4.62) in (4.54), we get

$$\begin{aligned}
\mathcal{F}'_3(t) & \leq -g_0 \|u'(t)\|_H^2 + \lambda_2 \left(L_A^2 + L_\nu^2 C_0^4 + L_\tau^2 C_0^4 + d^2 \right) \|u'(t)\|_V^2 \\
& + \lambda_2 \left(L_G^2 + L_B^2 \right) \|u(t)\|_V^2 + \lambda_2 \|f(t)\|_{V'}^2 - \frac{g(0)d^2}{4\lambda_2} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds \\
& + \left(\frac{6g_0}{4\lambda_2} + \frac{g_0^3}{4\lambda_2} + L_B g_0 \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds + \lambda_2 L_C^2 C_*^2 \|\psi'(t)\|_Y^2.
\end{aligned}$$

□

Lemma 4.5. Assume that assumption (G1) is satisfied. Then, there exist two positive constants ς_1 and ς_2

such that

$$\varsigma_1 E(t) \leq \mathcal{L}(t) \leq \varsigma_2 E(t). \quad (4.63)$$

Proof. From inequalities of Cauchy-Schwarz, Young and Poincaré and (A.9), we obtain

$$\begin{aligned} |\mathcal{F}_1(t)| &= |\langle u'(t), u(t) \rangle_H - e^{3\varsigma_1 t} \|f(t)\|_{V'}^2| \leq |\langle u'(t), u(t) \rangle_H| \\ &\leq \max(1, d^2) E(t), \end{aligned} \quad (4.64)$$

$$\begin{aligned} |\mathcal{F}_2(t)| &= \left| \langle \psi'(t), \psi(t) \rangle_{L^2(\Omega)} + \frac{\delta}{2} \|\nabla \psi(t)\|_{L^2(\Omega)}^2 \right| \\ &\leq \max(1, C_*^2 + \delta) E(t), \end{aligned} \quad (4.65)$$

$$\begin{aligned} |\mathcal{F}_3(t)| &= \left| \left\langle u'(t), \int_0^{+\infty} g(s) \eta(s, t) ds \right\rangle_H \right| \\ &\leq \max(1, g_0 d^2) E(t). \end{aligned} \quad (4.66)$$

Combining (4.43), (4.44), (4.64)-(4.66), we obtain

$$|\mathcal{L}(t) - E(t)| \leq \epsilon \varsigma_3 E(t),$$

where the positive constant ς_3 is defined by

$$\varsigma_3 = N_1 \max(1, d^2) + \max(1, C_*^2 + \delta) + \max(1, g_0 d^2).$$

Note that, thanks to

$$\epsilon < \frac{1}{\varsigma_3}, \quad (4.67)$$

we get $\mathcal{L} \sim E$ with $\varsigma_1 = 1 - \epsilon \varsigma_3$ and $\varsigma_2 = 1 + \epsilon \varsigma_3$. $\square$

The following theorem provides the exponential stability issue

Theorem 4.2. Assume that (G1)-(G3) and (1.9)-(1.13), (2.8) hold. Suppose that m_G is such that

$$m_G > \frac{6}{4\gamma_0} + g_0 L_B, \quad (4.68)$$

and there exists δ_0 a positive constant independent of m_A such that

$$m_A > \delta_0, \quad (4.69)$$

thus, for every $U_0 \in \mathcal{D}(\mathcal{A})$, there exist positive constants δ_2 and δ_3 such that

$$\|U(t)\|_{\mathcal{H}}^2 \leq \delta_3 e^{-\delta_2 t}. \quad (4.70)$$

Proof. A differentiation of (4.43) and use of (4.48), (4.49), (4.51) and (4.53) give us for all $t \geq 0$

$$\begin{aligned}
\mathcal{L}'(t) &= E'(t) + \epsilon(N_1\mathcal{F}'_1(t) + \mathcal{F}'_2(t) + \mathcal{F}'_3(t)) \\
&\leq -\epsilon \left(g_0 \|u'(t)\|_H^2 + \left(N_1(l - 6\lambda_2) - K_{12}\lambda_2 - \frac{K_2\lambda_1}{\epsilon} \right) \|u(t)\|_V^2 \right) \\
&\quad -\epsilon \left(K_3 \left(N_1 \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3\lambda_1}{\epsilon} \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\
&\quad -\epsilon \left(\frac{\delta - K_4\lambda_1}{\epsilon} - K_8N_1 - K_{15}\lambda_2 - K_{10} \right) \|\psi'(t)\|_Y^2 \\
&\quad -\epsilon \left(\kappa - \lambda_3 - \frac{K_5\lambda_1}{\epsilon} \right) \|\psi(t)\|_Y^2 \\
&\quad + (K_1 - m_A + \epsilon(N_1K_7 + K_9 + K_{11}\lambda_2)) \|u'(t)\|_V^2 \\
&\quad + \left(\frac{1}{2} - \epsilon K_{14} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds + \epsilon K_{16} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\
&\quad + \left(\lambda_1 + \epsilon \left(N_1 \left(\frac{1}{4\lambda_2} - \gamma_0 \right) + \lambda_2 \right) \right) \|f(t)\|_{V'}^2,
\end{aligned} \tag{4.71}$$

where $K_{16} = N_1K_6 + K_{13} + K_3 \left(N_1 \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right)$.

Using (4.68), λ_2 can be chosen such that

$$\frac{1}{4\gamma_0} < \lambda_2 < \frac{l}{6}, \tag{4.72}$$

and therefore N_1 big enough such that

$$\begin{aligned}
N_1(l - 6\lambda_2) - K_{12}\lambda_1 &> 0, \\
N_1 \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 &> 0.
\end{aligned} \tag{4.73}$$

Moreover, we have to choose λ_3 such that

$$\lambda_3 < \kappa. \tag{4.74}$$

From (G2), one has

$$\begin{aligned}
&\left(\frac{1}{2} - \epsilon K_{14} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds + \epsilon K_{16} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\
&\leq \left(\frac{1}{2} - \epsilon K_{17} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds,
\end{aligned}$$

where $K_{17} = K_{14} + \frac{K_{16}}{\alpha}$.

It follows from the previous equality that

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\epsilon \left(g_0 \|u'(t)\|_H^2 + \left(N_1 (l - 6\lambda_2) - K_{12}\lambda_2 - \frac{K_2\lambda_1}{\epsilon} \right) \|u(t)\|_V^2 \right) \\
& -\epsilon \left(K_3 \left(N_1 \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3\lambda_1}{\epsilon} \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\
& -\epsilon \left(\frac{\delta - K_4\lambda_1}{\epsilon} - N_1 K_8 - K_{15}\lambda_2 - K_{10} \right) \|\psi'(t)\|_Y^2 \\
& -\epsilon \left(\kappa - \lambda_3 - \frac{K_5\lambda_1}{\epsilon} \right) \|\psi(t)\|_Y^2 \\
& + (K_1 - m_A + \epsilon (N_1 K_7 + K_9 + K_{11}\lambda_2)) \|u'(t)\|_V^2 \\
& + \left(\frac{1}{2} - \epsilon K_{17} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds \\
& + \left(\lambda_1 + \epsilon \left(N_1 \left(\frac{1}{4\lambda_2} - \gamma_0 \right) + \lambda_2 \right) \right) \|f(t)\|_{V'}^2,
\end{aligned} \tag{4.75}$$

We assume that the constant m_A satisfies (4.69) if we choose δ_0 in the following form

$$\begin{aligned}
\delta_0 = \max \left\{ \frac{\lambda_1 K_5 (K_7 N_1 + K_9 + K_{11}\lambda_2)}{\kappa - \lambda_3}, \frac{\lambda_1 K_2 (K_7 N_1 + K_9 + K_{11}\lambda_2)}{N_1 (l - 6\lambda_2) - K_{12}\lambda_2}, \right. \\
\left. \frac{\lambda_1 (K_7 N_1 + K_9 + K_{11}\lambda_2)}{N_1 \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2} \right\} + K_1,
\end{aligned} \tag{4.76}$$

Moreover, λ_1 is assumed to satisfy

$$\begin{aligned}
\lambda_1 < \min \left\{ \frac{\delta (N_1 (l - 6\lambda_2) - K_{12}\lambda_2)}{K_2 (K_8 N_1 + K_{15}\lambda_2 + K_{10}) + K_4 (N_1 (l - 6\lambda_2) - K_{12}\lambda_2)}, \right. \\
\frac{\delta (\kappa - \lambda_3)}{K_5 (K_8 N_1 + K_{15}\lambda_2 + K_{10}) + K_4 (\kappa - \lambda_3)}, \\
\frac{\delta \left(N_1 \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right)}{(K_8 N_1 + K_{15}\lambda_2 + K_{10}) + K_4 \left(N_1 \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right)}, \\
\frac{N_1 (l - 6\lambda_2) - K_{12}\lambda_2}{2K_2 K_{17}}, \frac{N_1 \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2}{2K_{17}}, \\
\frac{\kappa - \lambda_3}{2K_{17} K_5}, \frac{N_1 (l - 6\lambda_2) - K_{12}\lambda_2}{K_2 \varsigma_3}, \frac{N_1 \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2}{\varsigma_3}, \\
\left. \frac{\kappa - \lambda_3}{\varsigma_3 K_5}, \frac{\delta}{K_4} \right\}.
\end{aligned} \tag{4.77}$$

From (4.77), we have

$$\frac{K_5\lambda_1}{\kappa - \lambda_3} < \min \left\{ \frac{\delta - K_4\lambda_1}{K_8 N_1 + K_{15}\lambda_2 + K_{10}}, \frac{1}{2K_{17}}, \frac{1}{\varsigma_3} \right\}, \tag{4.78}$$

$$\frac{\lambda_1}{N_1 \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2} < \min \left\{ \frac{\delta - K_4\lambda_1}{K_8N_1 + K_{15}\lambda_2 + K_{10}}, \frac{1}{2K_{17}}, \frac{1}{\varsigma_3} \right\}, \quad (4.79)$$

$$\frac{K_2\lambda_1}{N_1(l - 6\lambda_2) - K_{12}\lambda_2} < \min \left\{ \frac{\delta - K_4\lambda_1}{K_8N_1 + K_{15}\lambda_2 + K_{10}}, \frac{1}{2K_{17}}, \frac{1}{\varsigma_3} \right\}, \quad (4.80)$$

and

$$\delta - K_4\lambda_1 > 0. \quad (4.81)$$

It follows from (4.76) that

$$\frac{K_5\lambda_1}{\kappa - \lambda_3} < \frac{m_A - K_1}{K_7N_1 + K_9 + K_{11}\lambda_2}, \quad (4.82)$$

$$\frac{\lambda_1}{N_1 \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2} < \frac{m_A - K_1}{K_7N_1 + K_9 + K_{11}\lambda_2}, \quad (4.83)$$

and

$$\frac{K_2\lambda_1}{N(l - 6\lambda_2) - K_{12}\lambda_2} < \frac{m_A - K_1}{K_7N_1 + K_9 + K_{11}\lambda_2}, \quad (4.84)$$

Therefore, combining (4.67) and (4.78)-(4.84), ϵ is fixed as

$$K_{18} < \epsilon < K_{19}, \quad (4.85)$$

where

$$K_{18} = \max \left\{ \frac{K_5\lambda_1}{\kappa - \lambda_3}, \frac{\lambda_1}{N_1 \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2}, \frac{K_2\lambda_1}{N(l - 6\lambda_2) - K_{12}\lambda_2} \right\},$$

$$K_{19} = \min \left\{ \frac{\delta - K_4\lambda_1}{K_8N_1 + K_{15}\lambda_2 + K_{10}}, \frac{m_A - K_1}{K_7N_1 + K_9 + K_{11}\lambda_2}, \frac{1}{2K_{17}}, \frac{1}{\varsigma_3} \right\}.$$

Using (4.85) and inequality of Poincaré, we get

$$\begin{aligned} & -\epsilon \left(\frac{\delta - K_4\lambda_1}{\epsilon} - N_1K_8 - K_{15}\lambda_2 - K_{10} \right) \|\psi'(t)\|_Y^2 \\ & \leq -\frac{\epsilon}{C_2^*} \left(\frac{\delta - K_4\lambda_1}{\epsilon} - N_1K_8 - K_{15}\lambda_2 - K_{10} \right) \|\psi'(t)\|_{L^2(\Omega)}^2. \end{aligned}$$

Thus, we arrive at

$$\mathcal{L}'(t) \leq -\epsilon K_{20}E(t) + \left(\frac{1}{2} - \epsilon K_{17} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds,$$

where

$$K_{20} = 2 \min \left\{ g_0, N_1(l - 6\lambda_2) - K_{12}\lambda_2 - \frac{K_2\lambda_1}{\epsilon}, \right. \\ \left. K_3 \left(N_1 \left(\gamma_0 - \frac{1}{4\lambda_2} \right) - \lambda_2 \right) - \frac{K_3\lambda_1}{\epsilon}, \right. \\ \left. \frac{1}{\kappa} \left(\kappa - \lambda_3 - \frac{K_5\lambda_1}{\epsilon} \right), \frac{1}{C_2^*} \left(\frac{\delta - K_4\lambda_1}{\epsilon} - N_1K_8 - K_{15}\lambda_2 - K_{10} \right) \right\}.$$

Using (G1), (4.85) and (4.63), we obtain

$$\mathcal{L}'(t) \leq \frac{-\epsilon K_{20}}{1 + \epsilon \varsigma_3} \mathcal{L}(t). \quad (4.86)$$

Integrating (4.86) from 0 to t yields

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\delta_2 t}, \quad \forall t \in \mathbb{R}_+, \quad (4.87)$$

where $\delta_2 = \frac{\epsilon K_{20}}{1 + \epsilon \varsigma_3}$

Using (4.63) and (4.87), we get

$$\begin{aligned} \|U(t)\|_{\mathcal{H}}^2 &= 2E(t) \\ &\leq \frac{2}{1 - \epsilon \varsigma_3} \mathcal{L}(t) \\ &\leq \frac{2}{1 - \epsilon \varsigma_3} \mathcal{L}(0)e^{-\delta_2 t}, \end{aligned}$$

which yields (4.70) with $\delta_3 = \frac{2}{1 - \epsilon \varsigma_3} \mathcal{L}(0)$. □

GENERAL CONCLUSIONS AND FUTURE RESEARCH

In this work, we study the global existence, uniqueness and exponential decay of a solution for nonlinear contact problems. The process of contact is dynamic and the constitutive law is assumed to be viscoelastic or thermo-viscoelastic with term memory. The considered contact conditions are: normal compliance condition, normal damped response and normal compliance with adhesion.

First, we prove the global existence of a unique solution for frictionless viscoelastic dynamic contact problem with normal compliance using nonlinear semigroup theory. The stability result for the model is established using the energy method and by supposing that the relaxation function g decreases exponentially. Second, we show the global existence of a solution for a frictional dynamic contact problem with normal damped response condition and in the case where the memory term in the constitutive law is finite and infinite. In the case of finite memory, the proof of the global existence of a unique solution is based on arguments of nonlinear operator theory, fixed point argument and extension procedure. In the other case, the global existence is proved using nonlinear semigroup theory. In addition, we prove the exponential decay of the solution in the two cases. Next, we introduce the internal variable surface called the bonding field β and we consider a contact problem with normal compliance and adhesion. The evolution of the variable β is described by some nonlinear differential equation. Assuming that the frictional tangential traction is negligible, we prove the global existence of a unique solution using nonlinear semigroup theory and under some assumptions on initial problem data. Using again energy method and by constructing an appropriate Lyapunov functional, we prove the stability result for this adhesive contact problem. Finally, we consider a dynamic thermo-viscoelastic contact problem. The thermal effect is taken into account together with the frictional effect. By exploiting arguments of nonlinear semigroup, we establish the global solvability of the considered problem. The exponential decay of its solution is shown by applying energy method.

Completing this work and as a research perspective. It seems to be interesting to achieve the following objectives

- To extend the results of global existence and stability to study electro-viscoelastic contact problems, contact problems with damage and viscoplastic contact problems.
- To establish the general decay for contact problems considered in this thesis when the relaxation function g satisfying a general assumption.
- To weaken assumptions on problem data more precisely the regularity of f and the assumptions on positive constants m_G and m_A and to prove the global existence and exponential decay of solutions.

Appendices



PRELIMINARY MATERIAL AND MODELING

In this appendix, we introduce some basic preliminaries from functional analysis that we need subsequently and we present the mathematical modeling for the processes of contact. First, we review some Sobolev spaces for contact mechanics. Then, we recall some standard results on nonlinear analysis that will be used repeatedly in proving the existence of a unique solution for contact problems. Next, we list several inequalities that we need in our study. Finally, the second section is devoted to the mathematical modeling of contact problems. For further details about the different concepts found in this appendix, we refer the reader to the references [1, 8, 10, 51].

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A.1 Preliminaries on functional analysis

A.1.1 Sobolev spaces in contact mechanics

Let Ω be a bounded domain of $\mathbb{R}^d$ ($d = 2, 3$) with a Lipschitz boundary denoted by Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where Γ_1, Γ_2 and Γ_3 are disjoint measurable parts with $\text{meas } \Gamma_1 > 0$.

The inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$ are

$$u \cdot v = u_i v_i, \quad \|v\| = (v \cdot v)^{1/2} \quad \text{for all } u = (u_i), v = (v_i) \in \mathbb{R}^d, \quad (\text{A.1})$$

$$\sigma \cdot \tau = \sigma_{ij} \tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{1/2} \quad \text{for all } \sigma = (\sigma_{ij}), \tau = (\tau_{ij}) \in \mathbb{S}^d. \quad (\text{A.2})$$

The two indices i and j range from 1 to d . Moreover, summation over repeated indices is implied. Assuming that X denotes a real Banach space, we shall consider the different spaces that follow

$$\begin{aligned} H &= \{u = (u_i) \mid u_i \in L^2(\Omega)\}, \\ \mathcal{H} &= \{\sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \\ H_1 &= \{u = (u_i) \mid u_i \in H^1(\Omega)\}, \\ \mathcal{H}_1 &= \{\sigma \in \mathcal{H} \mid \text{Div } \sigma \in H\}, \\ V &= \{v \in H_1 \mid v = 0 \text{ on } \Gamma_1\}, \\ Y &= \{y \in H^1(\Omega) \mid y = 0 \text{ on } \Gamma\}, \end{aligned}$$

where $\text{Div} : \mathcal{H}_1 \longrightarrow H$ denotes the divergence operator and we have

$$\text{Div } \sigma = (\sigma_{ij,j}),$$

where $\sigma_{ij,j} = \sum_{j=1}^d \frac{\partial \sigma_{ij}}{\partial x_j}$.

The canonical inner product of the previous real Hilbert spaces $H, \mathcal{H}, H_1, \mathcal{H}_1, V$ and Y are

$$\begin{aligned} \langle u, v \rangle_{L^2(\Omega)^d} &= \int_{\Omega} u_i v_i \, dx, \\ \langle \sigma, \tau \rangle_{\mathcal{H}} &= \int_{\Omega} \sigma_{ij} \tau_{ij} \, dx, \\ \langle u, v \rangle_{H_1} &= \langle u, v \rangle_{L^2(\Omega)^d} + \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}}, \end{aligned}$$

and

$$\begin{aligned}\langle \sigma, \tau \rangle_{\mathcal{H}_1} &= (\sigma, \tau)_{\mathcal{H}} + (\operatorname{Div} \sigma, \operatorname{Div} \tau)_H, \\ \langle u, v \rangle_V &= (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \\ \langle y, z \rangle_Y &= \langle \nabla y, \nabla z \rangle_{L^2(\Omega)^d},\end{aligned}$$

here $\varepsilon : H_1 \longrightarrow \mathcal{H}$ denotes the deformation operator stated by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2} (u_{i,j} + u_{j,i}),$$

where $u_{i,j} = \frac{\partial u_i}{\partial x_j}$.

The norms defined on the spaces $\mathcal{H}$, H_1 , $\mathcal{H}_1$, V and Y are represented respectively by $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}_1}$, $\|\cdot\|_V$ and $\|\cdot\|_Y$.

Using the fact that $\operatorname{meas} \Gamma_1 > 0$, we get the following Korn's inequality

$$\|\varepsilon(v)\|_{\mathcal{H}} \geq c_K \|v\|_{H_1}, \quad \forall v \in V, \quad (\text{A.3})$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1 (see for instance [22]).

The vector ν is the unit outer normal on Γ . We denote by v the trace of vector v on Γ for every element $v \in H_1$. v_ν denotes the normal component of the element v on Γ . Furthermore, v_τ denotes the tangential component of the element v on Γ . The two components are given by

$$v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu.$$

Denote by σ_ν the normal trace of $\sigma \in \mathcal{H}_1$. Otherwise, the notation σ_τ represents the tangential trace of $\sigma \in \mathcal{H}_1$. If σ is a regular tensor field, say $\sigma \in C^1(\overline{\Omega}; \mathbb{S}^d)$, thus we have

$$\sigma_\nu = (\sigma \nu) \cdot \nu, \quad \sigma_\tau = \sigma \nu - \sigma_\nu \nu.$$

To derive the variational formulation for the contact problem, we apply the following Green formulas

$$\int_{\Omega} (u \operatorname{div} v + \nabla u \cdot v) dx = \int_{\Gamma} u (v \cdot \nu) da, \quad \forall u \in H^1(\Omega), v \in H_1 \quad (\text{A.4})$$

and

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (\operatorname{Div} \sigma, v)_H = \int_{\Gamma} \sigma \nu \cdot v da, \quad \forall v \in H_1, \forall \sigma \in C^1(\overline{\Omega}; \mathbb{S}^d). \quad (\text{A.5})$$

There exists a constant $C_0 > 0$ that depends only on Ω , Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq C_0 \|v\|_V, \quad \forall v \in V. \quad (\text{A.6})$$

Next, we assume that the space H is equipped with the modified inner product

$$\langle u, v \rangle_H = \langle \rho u, v \rangle_{L^2(\Omega)^d}, \quad \forall u, v \in H, \quad (\text{A.7})$$

where the density of mass ρ satisfies the following assumption

$$\rho \in L^\infty(\Omega), \text{ there exists } \rho^* > 0 \text{ such that } \rho(x) \geq \rho^* \text{ a.e. } x \in \Omega, \quad (\text{A.8})$$

$\|\cdot\|_H$ represents the associated norm of the space H .

The notation V' represents the dual of space V . $\|\cdot\|_{V'}$ stands for its corresponding norm. Otherwise, the duality pairing the two spaces V' and V is denoted by $\langle \cdot, \cdot \rangle_{V' \times V}$.

The triple of spaces (V, H, V') is called Gelfand triple where

$$V \subset H \subset V'.$$

The embedding of the space V into the space H is continuous and V is dense in H . Then, there exists $d > 0$ such that

$$\|u\|_H \leq d\|u\|_V, \quad \forall u \in V, \quad (\text{A.9})$$

$(Y, L^2(\Omega), Y')$ is another Gelfand triple, where Y' is the dual space of Y .

Through this thesis, we consider

$$\langle u, v \rangle_{V' \times V} = \langle u, v \rangle_H = \langle \rho u, v \rangle_{L^2(\Omega)^d}, \quad \forall u \in H, \forall v \in V, \quad (\text{A.10})$$

$$\langle y, z \rangle_{Y' \times Y} = \langle y, z \rangle_{L^2(\Omega)}, \quad \forall y \in L^2(\Omega), \forall z \in Y. \quad (\text{A.11})$$

We denote by r_+ the positive part of an extended real number r . Then

$$r_+ := \max\{r, 0\}.$$

Next, for every extended real numbers m and n , we have the two following inequalities

$$|m_+ - n_+| \leq |m - n|, \quad (\text{A.12})$$

and

$$(m - n)_+ \geq m - n. \quad (\text{A.13})$$

In what follows, we suppose that X is a real Banach space. Its corresponding norm is $\|\cdot\|_X$. X' denotes the dual of X and $\langle \cdot, \cdot \rangle_{X' \times X}$ represents the duality pairing between the spaces X and X' .

A.1.2 Elements of nonlinear analysis

Definition A.1. A sequence $\{x_n\} \subset X$ converges weakly to $x \in X$, if

$$\langle x_n, l \rangle_{X \times X'} \rightarrow \langle x, l \rangle_{X \times X'}, \quad \forall l \in X'.$$

In this case, x denotes the weak limit of the sequence $\{x_n\}$. The notation $x_n \xrightarrow{w} x$ is used to denote this kind of convergence.

Definition A.2. The single-valued operator $A : D(A) \subset X \rightarrow X'$ is called

(a) monotone, if

$$\langle Au - Av, u - v \rangle_{X' \times X} \geq 0 \quad \forall u, v \in D(A).$$

(b) strongly monotone, if there exists a constant $c > 0$ and $p > 1$ such that

$$\langle Au - Av, u - v \rangle_{X' \times X} \geq c \|u - v\|_X^p \quad \forall u, v \in D(A).$$

(c) Maximal monotone, if A is monotone, $x^* \in X'$ and $x \in X$

$$\langle x^* - Av, u - v \rangle_{X' \times X} \geq 0 \quad \forall u, v \in D(A) \Rightarrow u \in D(A) \text{ and } x^* = Au.$$

(d) hemicontinuous, if $D(A) = X$ and for all $u, v \in X$, we have

$$A(u + \zeta v) \xrightarrow{w} A(u)$$

as $\zeta \rightarrow 0$.

(d) Lipschitz continuous, if there exists $L > 0$ such that

$$\|Au - Av\|_X \leq L \|u - v\|_X \quad \forall u, v \in X.$$

Definition A.3. Let H be a Hilbert space, The single valued operator $A : D(A) \subset H \rightarrow H$ is called

(a) accretive, if

$$\langle Au - Av, u - v \rangle_H \geq 0 \quad \forall u, v \in D(A).$$

(b) m-accretive, if A is an accretive operator and

$$R(I + A) = H,$$

where $R(I + A)$ denotes the range of the operator $I + A$.

(c) ω -m-accretive where $\omega \in \mathbb{R}$, if $A + \omega I$ is m-accretive.

Theorem A.1. Assume that X is a reflexive Banach space and the operator $A : X \rightarrow X'$ is monotone and hemicontinuous. Then, A is maximal monotone .

Corollary A.1. Assume that X is reflexive Banach space and A is coercive and maximal monotone. Then A is surjective.

Theorem A.2. Let $V \subset H \subset V'$ a Gelfand triple and $A : V \rightarrow V'$ a monotone and hemicontinuous operator satisfying

$$\langle Au, u \rangle_{V' \times V} \geq \omega |u|_V^2 + \lambda \quad \forall u \in V,$$

$$\|Au\|_{V'} \leq C(|u|_V + 1) \quad \forall u \in V,$$

where $\omega > 0$, $C > 0$ and $\lambda \in \mathbb{R}$.

Given $u_0 \in H$ et $f \in L^2(0, T; V')$, then, there exists a unique function $u(t)$ satisfying

$$u \in L^2(0, T; V) \cap C(0, T; H), \quad u' \in L^2(0, T; V'),$$

$$\begin{aligned} u' + Au &= f(t) \quad \text{a.e. } t \in (0, T), \\ u(0) &= u_0. \end{aligned}$$

Definition A.4. Let C be a subset of a Banach space X . A semigroup on C is a group $\{S(t) : t \geq 0\}$ of a self-maps defined on the subset C which satisfies the following properties

- $S(0) = I$,
- $S(t+s)x = S(t)S(s)x, \forall t, s \geq 0, x \in C$.

Definition A.5. A semi-group $S(t)$ is said to be continuous if the mapping $g : [0, \infty) \times C \rightarrow C$ defined by $g(t, x) = S(t)x$ is continuous when the subset C carries the norm topology of the element x .

Definition A.6. A continuous semigroup S which satisfies

$$\|S(t)x - S(t)y\|_X \leq \|x - y\|_X, \quad \forall t \geq 0, x, y \in C,$$

is a semigroup of contractions.

Definition A.7. The semigroup S is said to be a quasicontractive on C if there is a constant ω such that

$$\|S(t)x - S(t)y\|_X \leq e^{\omega t} \|x - y\|_X \quad \forall t \geq 0, x, y \in C.$$

The notion of semigroup is considered as an abstraction of "a uniquely solvable initial-value problem" of the following form

$$\begin{cases} \frac{\partial u(t)}{\partial t} + Au(t) = 0, & \forall t > 0, \\ u(0) = x, \end{cases} \quad (\text{A.14})$$

where $A : D(A) \subseteq X \rightarrow X$ is a (nonlinear) operator. If for all $x \in C$, the problem (A.14) has a unique solution $u(t)$ on $[0, \infty)$, then putting $S(t)x = u(t)$ should be a semigroup on X .

Theorem A.3. *Let A be a single valued operator in a reflexive Banach space X , $f \in L^1(\mathbb{R}_+; X)$ and $x_0 \in \overline{D(A)}$. If $A + \omega I$ is m -accretive, then the problem*

$$\begin{cases} u'(t) + Au(t) = f(t), & \forall t > 0, \\ u(0) = x_0. \end{cases} \quad (\text{A.15})$$

has a unique mild solution $u \in C(\mathbb{R}_+; X)$.

Remark A.1. This theorem will be used in the particular case where X is a Hilbert space.

A .1.3 Principal inequalities

Notation

Let $1 \leq p \leq +\infty$ and $1 \leq q \leq +\infty$. Then, p and q are conjugate exponents if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Theorem A.4. (Hölder's inequality) *Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then, fg belongs to $L^1(\Omega)$ and we have*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Corollary A.2. (Cauchy-Schwarz's inequality) *In the case where $p = q = 2$, we get*

$$\|fg\|_1 \leq \|f\|_2 \|g\|_2.$$

Theorem A.5. (Young's inequality) *Let $a, b \geq 0$. Then, for any $\eta > 0$*

$$ab \leq \eta a^2 + \frac{b^2}{4\eta}.$$

Lemma A.1. *For all $a, b \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Theorem A.6. (*Poincaré's inequality*). Let $1 \leq p \leq +\infty$ and Ω be a bounded (at least in one direction) subset. Then there exists $C_* > 0$ that depends on Ω and p satisfying

$$\|\varphi\|_p \leq C_* \|\nabla \varphi\|_p, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

A.2 Modeling

This section is devoted to the mathematical modeling of some contact problems. We provide a presentation of physical setting, the variables of the state of the considered system, the behavior of material and the boundary conditions of the studied system.

A.2.1 Physical setting

A deformable body occupies a domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ that is bounded and with a Lipschitz boundary Γ that is separated into three disjoint and measurable parts Γ_1, Γ_2 and Γ_3 satisfying $\text{meas } \Gamma_1 > 0$.

The deformable body is fixed on Γ_1 and the surface tractions with density f_2 act on the measurable part Γ_2 . The contact between the deformable body and an obstacle named foundation is on Γ_3 .

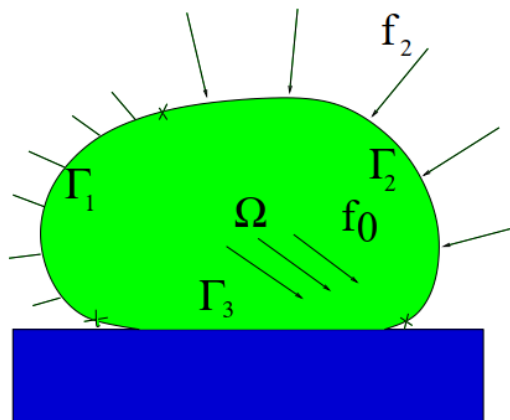


Figure A.1: The physical setting

This thesis is intended to describe mainly the evolution of the mechanical state of the deformable body where the time interval that interests us is $[0, +\infty)$. Let denote by $u = (u_i(x, t)) = u(x, t)$ the

displacement field and by $\sigma = (\sigma_{ij}(x, t)) = \sigma(x, t)$ the stress field where x denotes the position vector and t represents the time variable. The unknowns of the considered contact problems are the two functions $u : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{S}^d$.

In the case of the presence of adhesion or temperature effect, we add to the previous unknowns the function $\beta : \mathbb{R}_+ \times \Omega \rightarrow [0, 1]$ which represents the bonding field or $\theta : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ which stands for the temperature.

The mechanical state of the deformable body is described by the following equation of motion

$$\rho u'' = \text{Div} \sigma + f_0 \quad \text{in } \Omega \times \mathbb{R}_+, \quad (\text{A.16})$$

where ρ denotes the mass density, u'' represents the acceleration field and f_0 denotes the density of applied forces. The notation "Div" represents the divergence operator for tensors.

The processes modeled by Eq. (A.16) are called dynamic processes.

The body is fixed on Γ_1 . Therefore, we get

$$u = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \quad (\text{A.17})$$

where u stands for the displacement field.

The following equation represents the traction boundary condition

$$\sigma \nu = f_2 \quad \text{on } \Gamma_2 \times \mathbb{R}_+. \quad (\text{A.18})$$

Later, we will complete the equations (A.16)-(A.18) with the contact conditions on the boundary Γ_3 .

The previous equations are insufficient to describe the considered problem. we need to take into account the description of the behavior of the material body. For this purpose, we will talk about the constitutive laws in the next subsection.

A.2.2 Constitutive laws

The constitutive laws are the relationships between stresses and strains. They describe the deformation of the body. The researchers made series of tests and experiments to establish these particular laws more precisely they mostly used testing machines and rheological properties of material to do so. We refer the reader to [18, 19, 35, 38, 75].

A .2.2.1 Elastic constitutive laws.

We say that a body has a property of elasticity when it can resist a distorting influence. Moreover, it returns to its original shape and size when the applied force or influence is removed. A general form of elastic constitutive law is

$$\sigma = G\varepsilon(u), \quad (\text{A.19})$$

where G denotes the elasticity operator.

A .2.2.2 Viscoelastic constitutive laws

We say that a body is viscous if it deforms slowly when we apply an external force on it. On another side, a material is said to be viscoelastic if it becomes viscous and elastic when a force is applied to it.

A general viscoelastic constitutive law with short memory is

$$\sigma = A\varepsilon(u') + G\varepsilon(u), \quad (\text{A.20})$$

where A is the viscosity operator, G is the elasticity operator, u' denotes the velocity field. A general viscoelastic constitutive law with a long memory is given by

$$\sigma = A\varepsilon(u') + G\varepsilon(u) + \int_0^t B(t-s, \varepsilon(u(s)))ds, \quad (\text{A.21})$$

where A is the viscosity operator, G is the elasticity operator and B is the relaxation operator. In the case when the viscoelastic properties depend on temperature. Constitutive law is assumed to be thermo-viscoelastic and we have

$$\sigma = A\varepsilon(u') + G\varepsilon(u) + \int_0^t B(t-s, \varepsilon(u(s)))ds + C\theta, \quad (\text{A.22})$$

here θ is the temperature and the nonlinear operator C is the thermal expansion operator. The constitutive laws with long-term memory used in this thesis are given by

$$\sigma = A\varepsilon(u') + G\varepsilon(u) - \int_0^t g(s)B\varepsilon(u(t-s))ds \quad \text{in } \Omega \times (0, \infty), \quad (\text{A.23})$$

$$\sigma = A\varepsilon(u') + G\varepsilon(u) - \int_0^\infty g(s)B\varepsilon(u(t-s))ds \quad \text{in } \Omega \times (0, \infty), \quad (\text{A.24})$$

and

$$\sigma = A\varepsilon(u') + G\varepsilon(u) - \int_0^\infty g(s)B\varepsilon(u(t-s))ds + C\theta \quad \text{in } \Omega \times (0, \infty), \quad (\text{A.25})$$

where $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ stands for the kernel of the memory term.

A.2.3 Contact conditions

On the boundary Γ_3 , we distinguish two kinds of conditions: conditions in the normal direction (contact conditions) and conditions in the tangential direction (friction laws).

Let's describe the conditions on the normal direction used in this thesis. First, the normal damped response condition is given by

$$-\sigma_\nu = p_\nu(u'_\nu) \quad \text{on } \Gamma_3 \times \mathbb{R}_+, \quad (\text{A.26})$$

where σ_ν refers to the normal stress, the normal velocity is expressed by u'_ν and p_ν is the normal damped function. It is a specific function that vanishes in the case of a negative argument.

An example of function p_ν is the following

$$p_\nu(r) = c_\nu r_+,$$

where c_ν is a positive constant which represents the coefficient of damping resistance.

The normal damped response presents the contact with some lubricated foundation. Besides, it assigns a normal pressure that is reactive and depends on normal velocity.

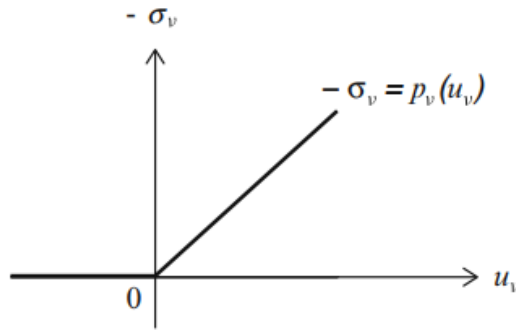
Second, we consider the normal compliance condition. It takes the following form

$$-\sigma_\nu = p_\nu(u_\nu) \quad \text{on } \Gamma_3 \times \mathbb{R}_+, \quad (\text{A.27})$$

where σ_ν refers to the normal stress, the normal displacement is expressed by u_ν and p_ν is the normal compliance function. It is a specific function that vanishes in the case of a negative argument.

An example of the normal compliance function p_ν is

$$p_\nu(r) = r_+,$$

Figure A.2: Graphic representation of p_ν

More generally, we can take

$$p_\nu(r) = (r_+)^n,$$

where $n > 0$ and $r_+ = \max\{0, r\}$.

The normal compliance condition presents a deformable foundation. In addition, it assigns a normal pressure that it is reactive and depends on the asperities's interpenetration on the surface of the body and those on the considered foundation.

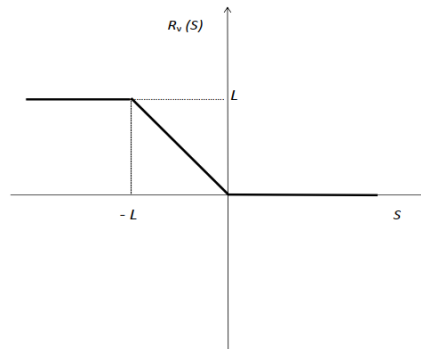
Next, Let's describe the condition of normal compliance with adhesion given by

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu \beta^2 R_\nu(u_\nu) \quad \text{on } \Gamma_3 \times (0, \infty), \quad (\text{A.28})$$

where σ_ν denotes the normal stress, u_ν stands for the normal displacement, p_ν refers to the normal compliance function. It equals to zero when its argument is negative and $\gamma_\nu \geq 0$ is the adhesion coefficient. $R_\nu : \mathbb{R} \rightarrow \mathbb{R}_+$ is the following truncation operator

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0, \end{cases} \quad (\text{A.29})$$

here $L > 0$ denotes the specific length of the bond.

Figure A.3: Graphic representation of the truncation operator R_ν

Now, we present the internal surface variable $0 \leq \beta \leq 1$. If the variable β equals to 1, then all bonds are active and adhesion is complete. If β equals to zero, all bonds are inactive, severed and no adhesion exists. In the case where $0 < \beta < 1$, then only a part of the bond is active and adhesion is partial.

The condition of normal compliance with adhesion (A.28) can be found in [12, 36, 73].

We assume that the variable β is governed by the following differential equation

$$\beta' = - \left(\beta \left(\gamma_\nu (R_\nu(u_\nu))^2 + \gamma_\tau \|R_\tau(u_\tau)\|^2 \right) - \varepsilon_a \right)_+ \quad \text{in } \Gamma_3 \times (0, \infty), \quad (\text{A.30})$$

$$\beta(0) = \beta_0 \quad \text{in } \Gamma_3, \quad (\text{A.31})$$

where γ_ν , γ_τ are the adhesion's coefficients and ε_a represents the limit bound, u_τ refers to the tangential displacement and β_0 is the initial adhesion such that

$$0 \leq \beta_0 \leq 1 \text{ a.e. } x \in \Gamma_3,$$

and $R_\tau : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the truncation operator that follows

$$R_\tau(v) = \begin{cases} v & \text{if } \|v\| \leq L, \\ L \frac{v}{\|v\|} & \text{if } \|v\| > L. \end{cases} \quad (\text{A.32})$$

A.2.4 Friction laws

Now, we turn to the conditions with the tangential directions called also friction laws. We distinguish two cases.

In the case where the friction force vanishes then we obtain

$$\sigma_\tau = 0 \text{ on } \Gamma_3 \times \mathbb{R}_+,$$

where σ_τ denotes the tangential stress. In the study of a contact problem with normal compliance and adhesion, we assume that σ_τ satisfies

$$-\sigma_\tau = p_\tau(\beta) R_\tau(u_\tau) \text{ on } \Gamma_3 \times (0, \infty), \quad (\text{A.33})$$

where $p_\tau : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+$ is a function of tangential contact. $p_\tau(\beta)$ serves like a spring constant. It increases with the bonding field β . The traction and the displacement are opposite in direction. It follows from $\beta \leq 1$ and $\|R_\tau\| \leq L$ that the maximal modulus which can be taken by the tangential traction is $p_\tau(1)L$.

Assuming that we neglect the frictional tangential traction. It follows from (A.33) that the considered contact is frictionless. If we don't neglect the frictional tangential traction as in [64, 65].

The contact will be frictional with adhesion.

In the case where the contact is frictional, we assume for example that σ_τ satisfies

$$-\sigma_\tau = p_\tau(u'_\tau) \text{ on } \Gamma_3 \times (0, \infty), \quad (\text{A.34})$$

where $p_\tau : \Gamma_3 \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the tangential function.

As an example of p_τ , we can take

$$p_\tau = \mu r,$$

where $\mu > 0$ denotes the coefficient of friction.

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الملخص : في هذه الرسالة، ندرس بعض مسائل التماس الديناميكي، مع الاحتكاك أو بدون، بين جسم قابل للتشوه وقاعدة. نعتبر قانونا تأسيسيا غير خطي ذي ذاكرة طويلة للمواد اللزجة المرنة أو اللزجة المرنة الحرارية. لا يتم إهمال تسارع النظام ومن ثم تكون العمليات النمذجة ديناميكية. تم نمذجة التماس بشروط مختلفة وهي التوافق الناظمي، الاستجابة اللحظية الناظرية والتوافق الناظمي مع الالتصاق. علاوة على ذلك، فإن التماس المدروس ثنائي لأنه يكون محفوظا طوال الوقت. النتائج التي حصلنا عليها تخص الوجود الإجمالي ووحداية الحلول المعتدلة. تعتمد التقنيات التي استخدمناها على نظرية المؤثرات، النقطة الثابتة و حجج نصف الزمرة الغير خطية. نهدف أيضا إلى إثبات الاستقرار الأساسي للحلول التي تم الحصول عليها باستخدام طريقة الطاقة مع وضع بعض الفرضيات على معطيات المسألة.

الكلمات المفتاحية: السلوك المقارب، الوجود الإجمالي، مبدد، مسألة التماس، نصف الزمر.

Abstract: In this thesis, we study some dynamic contact problems, with or without friction, between a deformable body and a foundation. We consider a nonlinear constitutive law with long memory term for viscoelastic or thermo-viscoelastic materials. The acceleration of the system is not neglected and then the modeled processes are dynamic. The contact is modeled with different conditions including normal compliance, normal damped response and normal compliance with adhesion. Moreover, the considered contact is bilateral because it is maintained at any time. The results we obtain concern the global existence and uniqueness of mild solutions. The techniques that we used are based on the theory of operators, fixed point and a nonlinear semigroup arguments. We also aim to establish the exponential stability of the obtained solutions using energy method and under some assumptions on problem data.

Keywords : Asymptotic behavior, Contact problem, Dissipative, Global existence, Semigroup.

Résumé: Dans cette thèse, nous étudions quelques problèmes de contact dynamique, avec ou sans frottement, entre un corps déformable et une fondation. On considère une loi de comportement non linéaire avec mémoire longue pour des matériaux viscoélastiques ou thermo-viscoélastiques. L'accélération du système n'est pas négligée et alors les processus modélisés sont dynamiques. Le contact est modélisé avec différentes conditions y compris la compliance normale, la réponse normale instantanée et la compliance normale avec adhérence. De plus, le contact considéré est bilatéral car il est maintenu à tout moment. Les résultats que nous avons obtenu concernent l'existence globale et l'unicité des solutions allégées. Les techniques que nous avons utilisés sont basées sur la théorie des opérateurs, point fixe et des arguments de semi-groupe non linéaire. Nous visons également à établir la stabilité exponentielle des solutions obtenues en utilisant la méthode d'énergie et sous certaines hypothèses sur les données du problème.

Mots clés : Comportement asymptotique, Dissipatif, Existence globale, Problème de Contact, Semi-groupe.