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Neutrosophic sets

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Introduction

Florentin smarandache introdued neutrosophic sets theory in 1995 , it an extension of a fuzzy sets introduced by L.A Zadah in 1965.Neutrosophic set is a part of neutrosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra . Neutrsophic set is a powerful general formal framework that has been recently proposed. In neutrosophic set, indeterminacy by the evident is quantified explicitly and in this concept membership , indeterminacy membership and non-membership functional values are independent. Where membership, indeterminacy membership and non-membership functional values are real standard or non-standard subsets of $[0, 1]$.
This memory is divided into three chapters.
In the first chapter , we will mention the notion of fuzzy sets , the basic operations and characteristic on fuzzy sets , the definition of Cartesian product on fuzzy set , the definition of triangular norm and triangular conorm ,we give the definition of fuzzy relations , also the operations on fuzzy relations and we will add the composition of fuzzy relations , and we give definition of fuzzy order relations and we give the definition fuzzy equivalent relations .
In the second chapter , we give the definition intuitionistic fuzzy sets ,the basic operations and characteristic on intuitionistic fuzzy sets , the definition of Cartesian product on intuitionistic fuzzy set , we give the definition of intuitionistic fuzzy relations , also the operations on intuitionistic fuzzy relations and we will add the composition of intuitionistic fuzzy relations , and we give definition of intuitionistic fuzzy order relations and we give the definition intuitionistic fuzzy equivalent relations .
In the last chapter ,we give the definition of neutrosophic sets , the the basic operations and characteristic on neutrosophic sets, the definition of Cartesian product on neutrosohic set , we give the definition of neutrosophic relations , also the operations on neutrosophic relations and we will add the composition of neutrosophic relations , and we give definition of neutrosophic fu order relations and we give the definition neutrosophic equivalent relations ,with a comparison between fuzzy sets , intuionistic sets and neutrossophic sets.

CHAPTER 1

GENERALITIES ON FUZZY SETS

1.1 Classical Sets

Crisp set is an unodered collection of different elements.

(i) Enumerating its elements $a_1, a_2, ..., a_n$ are the element of set A, it is represented as follows $A = \{a_1, a_2, ..., a_n\}$;

(ii) specifying the conditions of elements i.e., $A = \{x \mid P(x)\}$;

(iii) Membership function of A is a function on X.

$$\chi_A : X \longrightarrow \{0, 1\}$$
$$x \longmapsto \begin{cases} 0 & if \quad x \notin A; \\ 1 & if \quad x \in A. \end{cases}$$

1.2 Fuzzy sets

Definition 1.1 *Let X be a non empty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for each $x \in X$.*

Example 1.1 *(1) Let $X = \{a, b, c\}$ be universal set. $A = \{(a, 0.2), (b, 0.8), (c, 1)\}$ a fuzzy subset in X ;*

(2) Let $X = [0, 10]$, and A fuzzy subset in X, defined by :

$$\mu_A(x) = \frac{1}{1 + x}$$

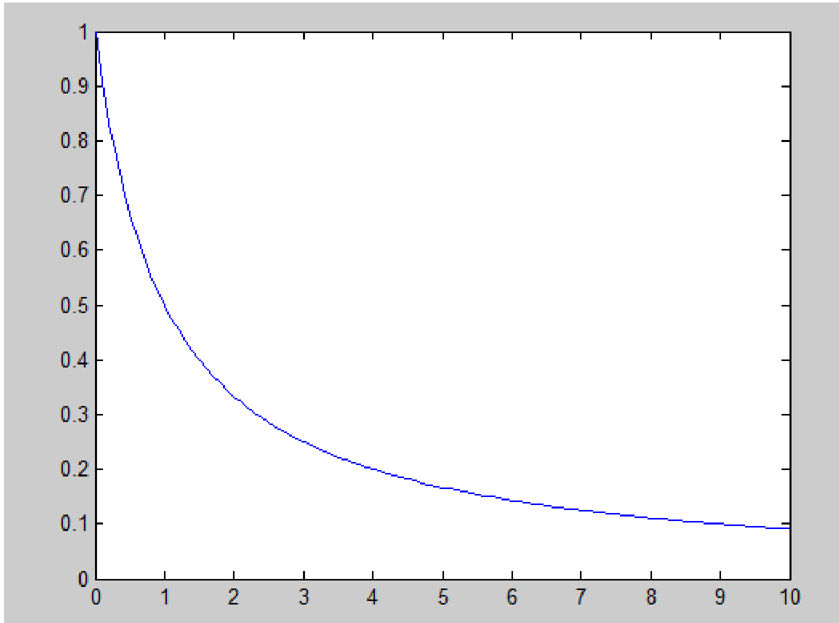


Figure 1.1: *
graph of μ_A

1.2.1 Operations of fuzzy set

In this section we will give definitions for Operations of fuzzy set : equality, inclusion, intersection, union, sum and product of two fuzzy subsets, and complement of a fuzzy set, and we will give an example.

Definition 1.2 ***(Equality)*** *Let X be a non empty set and let A and B two fuzzy subsets, we say that $A = B$, if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$.*

Definition 1.3 ***(Inclusion)*** *Let X be a non empty set and let A and B two fuzzy subsets, we say that $A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$ for all x in X.*

Definition 1.4 ***(Intersection)*** *Let X be a non empty set and let A and B two fuzzy subsets, the intersection defined by for all $x \in X$*

$$\mu_{A\cap B}(x) = min\left\{\mu_A(x), \mu_B(x)\right\} = \mu_A(x) \wedge \mu_B(x)$$

Definition 1.5 ***(Union)*** *Let X be a non empty set and let A and B two fuzzy subsets, the union defined by for all $x \in X$*

$$\mu_{A\cup B}(x) = max\left\{\mu_A(x), \mu_B(x)\right\} = \mu_A(x) \vee \mu_B(x)$$

Definition 1.6 ***(Complement)*** *The complement of a fuzzy set A is de noted by $C(A)$ and is defined by : for all $x \in X$*

$$\mu_{C(A)}(x) = 1 - \mu_A(x)$$

Definition 1.7 ***(Sum)*** *Let X be a non empty set and let A and B two fuzzy subsets, the sum defined by for all $x \in X$*

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$$

Definition 1.8 ***(Product)*** *Let X be a non empty set and let A and B two fuzzy subsets, the product defined by for all $x \in X$*

$$\mu_{A \times B}(x) = \mu_A(x)\mu_B(x)$$

Example 1.2 *Let $X = \{a, b, c\}$,and let $A = \{(a, 0.7), (b, 0.3), (c, 0.9), (d, 0.1)\}$, and $B = \{(a, 0.2), (b, 0.5), (c, 0, 7)(d, 0.4)\}$ we have :*

1. $A \cap B = \{(a, 0.2), (b, 0.3), (c, 0.7)(d, 0, 1)\}$
2. $A \cup B = \{(a, 0.7), (b, 0.5), (c, 0, 9)(d, 0, 4)\}$
3. $A \times B = \{(a, 0.14), (b, 0.15), (c, 0.63)(d, 0.04)\}$
4. $A + B = \{(a, 0.76), (b, 0.65), (c, 0.97)(d, 0.54)\}$
5. $C(A) = \{(a, 0.3), (b, 0.7), (c, 0.1)(d, 0.9)\}$

Property 1.1 *Considering the basic conectives in fuzzy set theory, the following properties hold true:*

1. *Associativity*
$$A \cap (B \cap C) = (A \cap B) \cap C$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$
2. *Commutativity*
$$A \cup B = B \cup A$$
$$A \cup B = B \cup A$$
3. *Identity*
$$A \cap X = x \cap A = A$$
$$A \cup \varnothing = \varnothing \cup A = A$$
$$A \cup X = X \cup A = X$$
$$A \cap \varnothing = \varnothing \cap A = \varnothing$$
4. *Idempotence*
$$A \cap A = A$$
$$A \cup A = A$$
5. *Absorption by $\varnothing$ and X*
$$A \cap \varnothing = \varnothing$$
$$A \cup X = X$$
6. *De Morgan Laws*
$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$
$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

7. *Distributivity*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

8. *Ivolution*

$$\bar{\bar{A}} = A$$

9. *Absorption*

$$A \cap (A \cup B) = A$$
$$A \cup (A \cap B) = A$$

Property 1.2 *if A is a non-classical fuzzy set $A : X \to [0,1]$ (i.e., there exists $x \in X$ with$A(x) \notin 0,1$) then*

$$A \cap \bar{A} \neq \varnothing$$
$$A \cup \bar{A} \neq X$$

1.3 Cartesian product and projection on fuzzy set

1.3.1 Cartesian product on fuzzy set

The cartesian product of the fuzzy subsets is the minimum of these degrees of belonging.

Definition 1.9 *The cartesian product applied to n fuzzy sets can be defined as follows : Let $\mu_{A_1}, \mu_{A_2}, ...\mu_{A_n}$, be membership functions of $A_1, A_2, ..., A_n$. Then, the membership degree of $(x_1, x_2, ...x_n) \in X_1 \times X_2 \times ... \times X_n$ on the fuzzy set $A_1 \times A_2 \times ... \times A_n$ is ,*

$$\mu_{A_1 \times A_{2 \times ... \times A_n}}(x_1, x_2, ..., x_n) = min \{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\}.$$

Example 1.3 *Let $X_1 = \{a, b, c, \}$, $X_2 = \{\alpha, \beta\}$ and let A_1, A_2 two fuzzy subset respectively defined on X_1 and X_2 given by:*
$A_1 = \{(a, 0.1), (b, 0.4), (c, 0.8)\}$.
$A_2 = \{(\alpha, 0.2), (\beta, 0.6)\}$.
So, we get:

$$A_1 \times A_2 = \{((a, \alpha), 0.1), ((a, \beta), 0.1), ((b, \alpha), 0.2), ((b, \beta), 0.4), ((c, \alpha), 0.2), ((c, \beta), 0.6)\}.$$

1.3.2 Projection on fuzzy set

The projections is the maximum of these cartesian products.

Definition 1.10 *The projection on X_1 of the fuzzy set A of $X_1 \times X_2 \times ... \times X_n$ is the fuzzy set $Proj_{X_1}(A)$ of X_1, whose membership function is defined by: for any $x_1 \in X_1$,*

$$\mu_{ProjX_{1(A)}}(x_1) = sup_{x_2 \in X_2, x_3 \in X_3, ..., x_n \in X_n}(\mu_A(x_1, x_2, ..., x_n)).$$

Example 1.4 *Let* $X = X_1 \times X_2$ *the set of reference such that* X_1 *and* X_2 *two sets, we consider* $A_1 \times A_2 = A$ *given by:*
$A = \{((a,\alpha),0.1),((a,\beta),0.1),((b,\alpha),0.2),((b,\beta),0.4),((c,\alpha),0.2),((c,\beta),0.6)\}$.
So, we get:

$$\begin{aligned} Proj_{X_1}(A) &= \{(a, max(0.1,0.1)),(b, max(0.2,0.4)),(c, max(0.2,0.6))\}; \\ &= \{(a,0.1),(b,0.4),(c,0.6)\}. \end{aligned}$$

1.4 Characteristics of fuzzy set

In this section we will give definitions for characteristics of fuzzy set : support, ker, height and cardinality of a fuzzy set, and we will give an example and proposition with proof.

Definition 1.11 ***(α-cuts)*** *Let* A *be a fuzzy set in* X *and let* $\alpha \in]0,1]$*, The* α*-cut of* A*, denoted* A_α*. we mean all elements of* X *that belong to* A *to a degree of at least* α*. That is* A_α *is a classical set defined by*

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

Property 1.3 (Basic properties of α-cuts)*Let* A*,* B *are two fuzzy subsets on a universe* X *and* $\alpha, \beta \in [0,1]$

1. *if* $\alpha \leq \beta$ *then* $A_\beta \subset A_\alpha$.
2. $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.
3. $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.
4. $A_0 = X$.
5. $A_1 = ker(A)$.

Preuve.

1. *Let* $x \in A_\beta$ *i.e* $\mu_A(x) \geq \beta$
   $\mu_A(x) \geq \beta \Rightarrow \mu_A(x) \geq \alpha$ *because*$\alpha \leq \beta$
   Then $x \in A_\alpha$*. Finally* $A_\beta \subseteq A_\alpha$.

2. $(A \cap B)_\alpha = \{x \in X : \mu_{A\cap B}(x) \geq \alpha\}$
   $= \{x \in X : min\{\mu_A(x), \mu_B(x)\} \geq \alpha\}$
   $= \{x \in X : \mu_A(x) \geq \alpha \wedge \mu_B(x) \geq \alpha\}$
   $= \{x \in X : \mu_A(x) \geq \alpha\} \cap \{x \in X : \mu_B(x) \geq \alpha\}$
   $= A_\alpha \cap B_\alpha$.

3. $(A \cup B)_\alpha = \{x \in X : \mu_{A\cup B}(x) \geq \alpha\}$
   $= \{x \in X : max\{\mu_A(x), \mu_B(x)\} \geq \alpha\}$
   $= \{x \in X : \mu_A(x) \geq \alpha \vee \mu_B(x) \geq \alpha\}$
   $= \{x \in X : \mu_A(x) \geq \alpha\} \cup \{x \in X : \mu_B(x) \geq \alpha\}$
   $= A_\alpha \cup B_\alpha$

4. $A_0 = \{x \in X : \mu_A(x) \geq 0\} = X$.

5. $A_1 = \{x \in X : \mu_A(x) \geq 1\} = \{x \in X : \mu_A(x) = 1\} = ker(A)$.

■

Definition 1.12 ***(The strong α-cuts)*** *For any α of $[0,1]$ we defined the strong α-cuts of the fuzzy subset A as the subset*

$$A_\alpha = \{x \in X | \mu_A(x) > \alpha\}.$$

Definition 1.13 ***(Support)*** *The support of a fuzzy set A, denoted by $Supp(A)$, we mean all elements of X that belong to a nonzero degree. That is $S(A)$is a calssical set defined by*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}$$

Definition 1.14 ***(Kernel)*** *The ker of a fuzzy set A, denoted by $ker(A)$, we mean all elements of X that belong to a equal one. That is $ker(A)$is a calssical set defined by*

$$ker(A) = \{x \in X \mid \mu_A(x) = 1\}$$

Definition 1.15 ***(Height)*** *The height of a fuzzy set A is the largest membership grade of any element in A.*

$$H(A) = Max\mu_A(x)$$

Definition 1.16 ***(Cardinality)*** *Cardinality of a finite fuzzy set A, denoted $\mid A \mid$ is defined as*

$$\mid A \mid = \sum_{x \in X} \mu_A(x).$$

Example 1.5 *Let $X = \{x_1, x_2, x_3, x_4\}$, and $A = \{(x_1, 0.6), (x_2, 1), (x_3, 0.3), (x_4, 0)\}$*
$A_{0.5} = \{x_1, x_2\}$*(α-cuts).*
$A_{0.2} = \{x_1, x_2, x_3\}$ *(The strong $\alpha - cuts$).*
$Supp(A) = \{x_1, x_2, x_3\}$.
$ker(A) = \{x_2\}$.
$H(A) = 1$.
$|A| = 1.9$.

1.5 T-norm and T-conorm

In the fuzzy sets theory, there are two types of operators: T-norm and T-conorm, they are often called Triangular norm and Triangular conorm respectively.

Definition 1.17 ***(Triangular norm).*** *Triangular norm is a binary operation T on the unit interval $[0,1]$, i.e., it is a function $T : [0,1]^2 \longrightarrow [0,1]$: the following four axioms are satisfied for all x, y, z and $w \in [0,1]$:*

(T1) Commutativity i.e., $T(x, y) = T(y, x)$.

(T2) Associativity i.e., $T(x, T(y, z)) = T(T(x, y), z)$.

(T3) Monotonicity i.e., $T(x, y) \leq T(z, w)$ *whenever* $x \leq z$ *and* $y \leq w$*.*

(T4) Boundary condition i.e., $T(x, 1) = T(1, x) = x$*.*

Example 1.6 (1) *Minimum t-norm* $T_M = min(x, y)$

(2) *Lukasiewiez t-norm* $T_L = max(x + y - 1, 0)$

(3) *Product t-norm* $T_P = xy$

(4) *Einstein t-norm* $T_E = \dfrac{xy}{2 - x - y + xy)}$

(5) *Drastic product:*

$$T_D(x, y) = \begin{cases} x & if \;\; y = 1 \\ y & if \;\; x = 1 \\ 0 & if \;\; x, y < 1. \end{cases}$$

Definition 1.18 ***(Triangular conorm).*** *A triangular conorm is a binary operation* S *on the unit interval* $[0, 1]$*, i.e., it is a function* $S : [0, 1]^2 \longrightarrow [0, 1]$ *: the following four axioms are satisfied for all* x, y, z *and* $w \in [0, 1]$*:*

(S1) Commutativity : $S(x, y) = S(y, x)$ *.*

(S2) Associativity : $S(x, S(y, z)) = S(S(x, y), z)$ *.*

(S3) Monotonicity : $S(x, y) \leq S(z, w)$ *whenever* $x \leq z$ *and* $y \leq w$ *.*

(S4) Boundary condition : $S(x, 0) = S(0, x) = x$*.*

Example 1.7 .

(1) Maximum t-conorm $S_M = max(x, y)$*.*

(2) Lukasiewiez t-conorm $S_L = min(x + y, 1)$*.*

(3) Probabilistic sum $S_P = x + y - xy$

(4) Einstein t-conorm $S_E = \frac{x+y}{1+xy}$*.*

(T8) *Drastic sum:*

$$S_D(x, y) = \begin{cases} 1, & if \;\; (x, y) \in [0, 1)^2 \\ \max\{x, y\}, & otherwise \end{cases}$$

Property 1.4 *A T-norm and T-conorm are dual if and only if:*

(1) $1 - T(x, y) = S(1 - x, 1 - y)$ *.*

(2) $1 - S(x, y) = T(1 - x, 1 - y)$ *.*

Preuve.

$$
\begin{aligned}
1 - T(x,y) &= 1 - 1 \times \begin{cases} x & if x \leq y \\ y & if x > y \end{cases} \\
&= 1 + \begin{cases} -x & if x \leq y \\ -y & if x > y \end{cases} \\
&= 1 + \begin{cases} -x & if -x \geq -y \\ -y & if -x < -y \end{cases} \\
&= \begin{cases} 1-x & if 1-x \geq 1-y \\ 1-y & if 1-x < 1-y \end{cases}
\end{aligned}
$$

Then $1 - T(x,y) = max(1-x, 1-y)$
Hense $1 - T(x,y) = S(1-x, 1-y)$.

$$
\begin{aligned}
1 - S(x,y) &= 1 - 1 \times \begin{cases} x & if x \geq y \\ y & if x < y \end{cases} \\
&= 1 + \begin{cases} -x & if x \geq y \\ -y & if x < y \end{cases} \\
&= 1 + \begin{cases} -x & if -x \leq -y \\ -y & if -x > -y \end{cases} \\
&= \begin{cases} 1-x & if 1-x \leq 1-y \\ 1-y & if 1-x > 1-y \end{cases}
\end{aligned}
$$

Then $1 - S(x,y) = min(1-x, 1-y)$
Hense $1 - S(x,y) = T(1-x, 1-y)$. ∎

1.6 Fuzzy relations

In the section, we introduce the definition of fuzzy relations, examples and their basic properties.

Definition 1.19 *Fuzzy relation from X to Y is a fuzzy subset of $X \times Y$ characterized by a membership function $\mu_R : X \times Y \longrightarrow [0,1]$ which associates with each pair (x,y) its grade of membership $\mu_R(x,y)$ in the intervale $[0,1]$.*

$$R = \{< (x,y), \mu_R(x,y) > |(x,y) \in X \times Y\}.$$

Particular cases:
If $X = Y$, then $R = \{< (x,y), \mu_R(x,y) > |(x,y) \in X \times X\}$.

Examples

Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$, the fuzzy relation R defined on $X \times Y$ by

$$R = \{< (x, y), \mu_R(x, y) > |(x, y) \in X \times Y\}.$$

R	x	y	z
a	0.49	0.68	0.17
b	0.53	1	0.77
c	0.25	1	0.85

1.6.1 Properties of fuzzy relations

(1) Reflexive
$\forall x \in X : R(x, x) = 1.$

(2) Symetrical
$\forall (x, y) \in X \times Y : R(x, y) = R(y, x).$

(3) Antisymerical
$R(x, y) \wedge R(y, x) \neq 0 \Rightarrow x = y.$

(4) Transitive
$R(x, y) \wedge R(y, z) \leq R(x, z), \forall x, y$ and $z \in X.$

(5) Antireflexive
$\forall x \in X : R(x, x) = 0.$

(6) No reflexive
$\exists x \in X : R(x, x) = 0$

(7) Asymetrical
$R(x, y) \wedge R(y, x) \neq 0 \Rightarrow x = y$

1.7 Operations on fuzzy relations

Below we will define the operations on fuzzy relations with some examples.

Definition 1.20 ***(Intersection).*** *Let X and Y be two non empty sets and let R and S be two fuzzy relations,the intersection defined by for all $(x, y) \in X \times Y$*

$$\mu_{R\cap S} = min\{\mu_R(x, y), \mu_S(x, y)\}.$$

Example 1.8 *Let R and S be tow fuzzy relation on $X \cap X$ such that $X = \{a, b, c, d\}$, represented by the following tables*

R	a	b	c	d
a	0.25	0.5	0.5	0.14
b	0.25	0.44	1	0.22
c	0.33	0.68	0.8	1
d	0.45	0.75	1	1

S	a	b	c	d
a	0.25	0.15	0.88	0.38
b	0.5	1	0.75	0.36
c	0.13	1	0.8	0.33
d	0.12	0.81	0.25	0.4

The intersection relations defined by

$R \cap S$	a	b	c	d
a	0.25	0.5	0.5	0.14
b	0.25	0.44	0.75	0.22
c	0.13	0.68	0.8	0.33
d	0.12	0.75	0.25	0.4

Definition 1.21 ***(union).*** *Let X and Y be two non empty sets and let R and S be two fuzzy relations,the union defined by for all $(x, y) \in X \times Y$*

$$\mu_{R \cup S}(x, y) = max\{\mu_R(x, y), \mu_S(x, y)\}.$$

Example 1.9 *Let R and S be tow fuzzy relation on $X \times X$ such that $X = \{a, b, c, d\}$, represented by the following tables*

R	a	b	c	d
a	0.55	0.41	0.57	1
b	0.25	1	0.77	0.76
c	0.45	0.36	0.89	0.95
d	0.67	1	0.34	1

S	a	b	c	d
a	0.25	1	0.45	0.99
b	1	0.76	0.5	0.97
c	0.79	0.86	1	0.7
d	0.46	0.86	1	0.09

The union relations defined by

$R \cup S$	a	b	c	d
a	0.55	1	0.57	1
b	1	1	0.77	0.97
c	0.79	0.86	1	0.95
d	0.67	1	1	1

Definition 1.22 ***(Containment)).****Let X and Y be two non empty sets and let R be a fuzzy relation, the Containment defined by for all $(x, y) \in X \times Y$*

$$\mu_R(x, y) \leq \mu_S(x, y).$$

Example 1.10 *Let R and S be tow fuzzy relations on $X \times X$ such that $X = \{a, b, c\}$,represented by the following table:*

R	a	b	c
a	0.01	0.35	0.46
b	0.25	1	0.58
c	0.55	0.8	0.6

and

S	a	b	c
a	0.12	0.44	0.58
b	0.75	1	0.88
c	0.77	0.9	0.65

$HenseR \subset S.$

Definition 1.23 ***(Complement).****Let X and Y be two non empty sets and let R be a fuzzy relation, the copmlement defined by for all $(x, y) \in X \times Y$*

$$\mu_{R^c}(x, y) = 1 - \mu_R(x, y).$$

Example 1.11 *Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$, the fuzzy relation R defined on $X \times Y$,represented by the following table*

R	x	y	z
a	0.5	0.18	1
b	0.45	0.75	0.9
c	0.25	0.55	0.95

The coplement relation defined by

R^c	x	y	z
a	0.5	0.82	0
b	0.55	0.25	0.1
c	0.75	0.45	0.05

Definition 1.24 ***(Inverse).****Let $R \subseteq X \times Y$ be a fuzzy relation, the inverse relation R^{-1} is defined by for all $(x, y) \subseteq X \times Y$:*

$$\mu_R^{-1}(x, y) = \mu_R(y, x)$$

.

Example 1.12 *Let R be a fuzzy relation on $X \times X$ such that $X = \{x, y, z\}$,represented by the following table:*

R	x	y	z
x	0.44	0.5	1
y	0	0.65	0.11
z	0.75	0.8	1

The inverse relation defined by

R^{-1}	x	y	z
x	0.44	0	0.75
y	0.5	0.65	0.8
z	1	0.11	1

1.8 α-cuts of fuzzy relation

Definition 1.25 ***(α-cuts of fuzzy relation).*** *Let $R \subseteq X \times Y$ be a fuzzy relation, and R_α is a α -cut relation . Then*

$$R_\alpha = \{(x, y) \in X \times Y | \mu_R(x, y) \geq \alpha\}.$$

Example 1.13 *Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$, and let R be a fuzzy relation represented by the following table*

R	y_1	y_2	y_3
x_1	0.41	0.8	0.34
x_2	0.64	0.32	0.55
x_3	0.13	1	1

$$R_{0.5} = \{(x, y) \in X \times Y | \mu_R(x, y) \geq 0.5\} = \{(x_1, y_2), (x_2, y_3), (x_3, y_2), (x_3, y_3)\}.$$

1.9 Composition of fuzzy relations

In this section, we will learn about the types of copmosition in fuzzy relations.

Definition 1.26 ***(Max-min composition).*** *Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two fuzzy relations the max-min composition of R and S denoted $R \circ S$ is then fuzzy set such that:*

$$\mu_{R \circ S}(x, z) = max_y \left[min \{\mu_R(x, y), \mu_S(y, z)\}\right].$$

Example 1.14 *Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two fuzzy relations as follows:*

R	y_1	y_2	y_3
x_1	1	0.45	0.47
x_2	0.25	0	0.44
x_3	0.1	0.33	1

and

S	y_1	y_2	y_3
x_1	0	1	0.5
x_2	0.13	1	0.75
x_3	0.25	0	1

The composition max-min is $R \circ S$:

$R \circ S$	y_1	y_2	y_3
x_1	0.25	0.45	0.47
x_2	0.25	0.25	0.44
x_3	0.33	0.33	1

Definition 1.27 ***(Max-product composition).****Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two fuzzy relations the max product composition of R and S denoted $R \circ S$ is then fuzzy set such that:*

$$\mu_{R \circ S}(x, z) = max \left\{ \mu_R(x, y) \times \mu_S(y, z) \right\}.$$

Example 1.15 *Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two fuzzy relations as follows:*

R	y_1	y_2	y_3
x_1	1	0	0.25
x_2	0.5	1	0.45
x_3	0.1	0	0.44

and

S	y_1	y_2	y_3
x_1	1	0.25	1
x_2	0	0.33	0.77
x_3	0.75	0	0.45

The composition max product is $R \circ S$

$R \circ S$	y_1	y_2	y_3
x_1	1	0.25	1
x_2	0.5	0	0.77
x_3	0.44	0.33	0.44

Definition 1.28 ***(max average composition).****Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two fuzzy relations the max average composition of R and S denoted $R \circ S$ is then fuzzy set such that*

$$\mu_{R \circ S}(x, z) = 1/2[max \left\{ \mu_R(x, y) + \mu_S(y, z) \right\}].$$

Example 1.16 *Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two fuzzy relations as follows:*

R	y_1	y_2	y_3
x_1	1	0.14	0.64
x_2	0.2	0	0.56
x_3	0.33	0.27	0.98

and

S	y_1	y_2	y_3
x_1	0	0.25	0.73
x_2	0.1	0.35	0.47
x_3	0.25	0.77	1

The composition max-average is $R \circ S$

$R \circ S$	y_1	y_2	y_3
x_1	0.25	0.64	0.73
x_2	0.25	0.56	0.56
x_3	0.25	0.77	0.98

1.10 Fuzzy order relations

Next, we recal the definition of the fuzzy order relation and we see some examples.

Definition 1.29 *If fuzzy relation R satisfies the followings $\forall x, y$ and $z \in X$, it is called fuzzy order relation.*

1. *Reflexive relation*
   $\forall x \in X \Rightarrow \mu_R(x,x) = 1.$
2. *Antisymmetric relation*
   $\forall (x,y) \in X \times X : \mu_R(x,y) \neq \mu_R(y,x) or \mu_R(x,y) = \mu_R(y,x) = 0.$
3. *Transitive relation*
   $R(x,y) \wedge R(y,z) \leq R(x,z), \forall x, y$ *and* $z \in X.$

Example 1.17 *Let $X = \{x, y, z\}$, then the fuzzy relation R defined on X by*

$$R = \left\{ \langle (a,b), \mu_R(a,b) \rangle \,|(a,b) \in X^2 \right\}$$

$\mu_R(a,b)$	x	y	z
x	1	0	0
y	0.1	1	0
z	0.25	0.77	1

Then R is a fuzzy order relation on X.

1.11 Fuzzy equivalent relations

In the followings we shows that the definition of fuzzy equivalent relation and example.

Definition 1.30 *If fuzzy relation R satisfies the followings $\forall x, y$ and $z \in X$, it is called fuzzy order relation*

1. *Reflexive relation*
   $\forall x \in X \Rightarrow \mu_R(x,x) = 1.$
2. *Symmetric relation*
   $\forall (x,y) \in X \times X : \mu_R(x,y) = \mu_R(y,x).$
3. *Transitive relation*
   $R(x,y) \wedge R(y,z) \leq R(x,z), \forall x, y$ *and* $z \in X.$

Example 1.18 *Let $X = \{x, y, z\}$, then the fuzzy relation R defined on X by*

$$R = \left\{ \langle (a,b), \mu_R(a,b) \rangle \,|(a,b) \in X^2 \right\}$$

$\mu_R(a,b)$	x	y	z
x	1	0.56	0.88
y	0.45	1	0
z	0.95	0	1

R is fuzzy equivalent relation on X.

CHAPTER 2

GENERALITIES ON INTUITIONISTIC FUZZY SETS

2.1 Intuitionistic fuzzy sets

In the following, we recall the definition of an intuitionistic fuzzy sets and example .

Definition 2.1 *Let X be a non empty set. An intuitionistic fuzzy set (IFS, for short) A on X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\}$$

Where function :

$$\mu_A : X \longrightarrow [0, 1]$$

$$\nu_A : X \longrightarrow [0, 1].$$

Define the degree of membership and the degree of non-membership of the element $x \in X$ to the set A, respectively, and for every $x \in X$

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

Obviously, every ordinary fuzzy set has the from $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \,|x \in X\}$.

Example 2.1 *Let $X = \{a, b, c\}$ be universal set.*

$$A = \{(a, 0.13, 0.43), (b, 0.45, 0.94), (c, 1, 0.66)\}$$

a intuitionistic fuzzy subset in X.

2.2 Operations on Intuitionistic fuzzy sets

In the section, we need following defintion of intersection, union, equality,containment,complement and sum of an intuitionistic fuzzy set with some examples.

Definition 2.2 ***(Intersection)*** *Let X be a non empty set and let A and B be two intuitionistic fuzzy subsets, the intersection defined by:*

$$A \cap B = \{\langle x, min\{\mu_A(x), \mu_B(x)\}, max\{\nu_A(x), \nu_B(x)\}\rangle \, | x \in X\}.$$

Example 2.2 *Let $X = \{x_1, x_2, x_3\}$ be universal set.*
Let $A = \{(1, 0.35, 0.33), (2, 0.19, 0.34), (3, 1, 0)\}$ and $B = \{(1, 0.16, 0.12), (2, 0.15, 0.53), (3, 1, 1)\}$
Then $A \cap B = \{(1, 0.16, 0.33), (2, 0.15, 0.53), (3, 1, 1)\}$.

Definition 2.3 ***(Union)*** *Let X be a non empty set and let A and B be two intuitionistic fuzzy subsets, the union defined by :*

$$A \cup B = \{\langle x, max\{\mu_A(x), \mu_B(x)\}, min\{\nu_A(x), \nu_B(x)\}\rangle \, | x \in X\}.$$

Example 2.3 *Let $X = \{x_1, x_2, x_3\}$ be universal set.*
Let $A = \{(1, 0.3, 0.2), (2, 0.9, 0.4), (3, 1, 0)\}$ and $B = \{(1, 0.6, 0.3), (2, 0.5, 0.3), (3, 1, 1)\}$.
Then $A \cup B = \{(1, 0.6, 0.2), (2, 0.9, 0.3), (3, 1, 0)\}$.

Definition 2.4 ***(Equality)*** *Let X be a non empty set and let A and B be two intuitionistic fuzzy subsets, we say that A=B, if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for all $x \in X$.*

Example 2.4 *Let $X = \{1, 2, 3\}$, and let $A = \{(1, 0.13, 0.11), (2, 0.3, 0.4), (3, 0.33, 0.55\})$*
and $B = \{(1, 0.13, 0.11), (2, 0.3, 0.4), (3, 0.33, 0.55\})$
Then $A = B$ for all 1,2 and 3 in X.

Definition 2.5 ***(Containment)*** *Let X be a non empty set and let A and B be two intuitionistic fuzzy subsets, we say that $A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$*
and $\nu_A(x) \geq \nu_B(x)$ for any $x \in X$.

Example 2.5 *Let $X = \{a, b, c\}$, and let $A = \{(a, 0.03, 0.17), (b, 0.27, 0.25), (c, 0.05, 0.4)\}$*
and $B = \{(a, 0.16, 0.15), (b, 0.33, 02.2), (c, 0.18, 0.38)\}$.
Then $A \subseteq B$ for all a,b and c in X.

Definition 2.6 ***(Complement)*** *The complement of an intuitionistic fuzzy set A is denoted by $C(A)$ and is defined by:*

$$C(A) = \{\langle x, \nu_A(x), \mu_A(x)\rangle \, | x \in X\}.$$

Example 2.6 *Let $X = \{1, 2, 3\}$.*
Let $A = \{(1, 0.03, 0.17), (2, 0.27, 0.25), (3, 0.05, 0.4)\}$.
Then $C(A) = \{(1, 0.17, 0.3), (2, 0.25, 0.27), (3, 0.04, 0.5)\}$.

Definition 2.7 ***(Product)*** *Let X be a non empty set and let A and B be two intuitionistic fuzzy subsets ,the product defined by:*

$$A \times B = \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x)\rangle \, | x \in X\}.$$

Example 2.7 *Let* $X = \{1, 2, 3\}$.
$A = \{(1, 0.2, 0.3), (2, 0.15, 0.2), (3, 1)\}$.
$B = \{(1, 0.2, 0.7), (2, 0.25, 0.22), (3, 0, 1)\}$.
Then $A \times B = \{(1, 0.04, 0.79), (2, 0.37, 0.38), (3, 0, 0)\}$.

Definition 2.8 ***(Sum)****Let* X *be a non empty set and let* A *and* B *be two intuitionistic fuzzy subsets ,the product defined by:*

$$A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \nu_A(x) \cdot \nu_B(x) \rangle \,| x \in X\}.$$

Example 2.8 *Let* $X = \{a, b, c\}$.
Let $A = \{(a, 0.1, 0.2), (b, 0.55, 0.75), (c, 0, 1)\}$ *and* $B = \{(a, 0.15, 0), (b, 0, 1), (c, 0.75, 0.15)\}$.
Then $A + B = \{(a, 0.23, 0), (b, 0.55, 0.75), (c, 0.75, 0.15)\}$.

2.2.1 Necessity and possibility operators

This section contains the necessity and possibility operators.

Definition 2.9 ***(Necessity)*** *Let* A *be an intuitionistic fuzzy set on* X*, the necessity of* A *denoted by* $\Box A$ *is defined by:*

$$\Box A(x) = \{\langle x, \mu_A(x), \mu_A^c(x) \rangle \,| \mu_A^c(x) = 1 - \mu_A(x)\}.$$

Example 2.9 *Let* $X = \{1, 2, 3\}$*, given by:*
$A = \{(1, 0.52, 0.5), (2, 0.15, 0.30), (3, 0.10, 0)\}$.
Then $\Box A(x) = \{(1, 0.52, 0.5), (2, 0.15, 0.85), (3, 0.10, 0.90)\}$.

Definition 2.10 ***(Possibility)*** *Let* A *be an intuitionistic fuzzy set on* X*, the possibility of* A *denoted by* $\diamond A$ *is defined by:*

$$\diamond A(x) = \{\langle x, \nu_A^c(x), \nu_A(x) \rangle \,| \nu_A^c(x) = 1 - \nu_A(x)\}.$$

Example 2.10 *Let* $X = \{a, b, c\}$*,and let* A *be intuitionistic fuzzy set given by :*

$$A = \{(a, 0.7, 0.5), (b, 0.44, 0.92), (c, 1, 1)\}.$$

Then $\diamond A(x) = \{(a, 0.5, 0.5), (b, 0.8, 0.92), (c, 0, 1)\}$.

2.3 Characteristics of intuitionistic fuzzy sets

In the section we defined the support, the kernel and the (α, β)-cut of an intuitionistic fuzzy set with examples.

Definition 2.11 ***(Support)****Let* A *be an intuitionistic fuzzy set on* X*,the support of denoted by Supp(A) is defined by:*

$$Supp(A) = \{x \in X \,| \mu_A(x) > 0 \;\; or \;\; (\mu_A(x) = 0 \;\; and \;\; \nu_A(x) < 1)\}.$$

Example 2.11 *Let* $X = \{x_1, x_2, x_3\}$,*and* $A = \{(x_1, 0, 0.3), (x_2, 1, 0), (x_3, 0, 0.42)\}$.

$$Supp(A) = \{x_1, x_2, x_3\}.$$

Definition 2.12 ***(Kernel)****Let* A *be an intuitionistic fuzzy set on* X*,the kernel of* A *denoted by* $Ker(A)$ *is defined by:*

$$Ker(A) = \{x \in X \ |\mu_A(x) = 1 \, and \, \nu_A(x) = 0\}.$$

Example 2.12 *Let* $X = \{x_1, x_2, x_3\}$,*and* $A = \{(x_1, 1, 0), (x_2, 1, 0), (x_3, 0.33, 0.88)\}$.

$$Ker(A) = \{x_1, x_2\}.$$

Definition 2.13 ***(***(α, β)***-cut)****Let* A *be an intuitionistic fuzzy set on* X*.The* (α, β)*-cut of* A *is a crisp subset*

$$A_(\alpha, \beta) = \{x \in X \ |\mu_A(x) \geq \alpha \, and \, \nu_A(x) \leq \beta\}.$$

where $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta \leq 1$.

Example 2.13 *Let* $X = \{x_1, x_2, x_3, x_4\}$,*and* $A = \{(x_1, 0.33, 0.15), (x_2, 0.22, 0.18), (x_3, 0.5, 0.2)\}$.
$A_{(0.2,0.1)} = \{x_1, x_2\}$.

2.3.1 Cartesian product on intuitionistic fuzzy set

The cartesian product of the intuitionistic fuzzy subsets is the minimum of these degrees of belonging and the maximum of these degrees of non-belonging.

Definition 2.14 *The cartesian product applied to n intuitonistic fuzzy sets can be defined as follows:*
Let $\mu_{A_1}, \mu_{A_2}, \ldots \mu_{A_n}$ *be membership functions of*$A_1, A_2, \ldots, A_n$*.Then, the membership de- gree of* $(x_1, x_2, \ldots x_n) \in X_1 \times X_2 \times \ldots \times X_n$ *on the intuitionistic fuzzy set* $A_1 \times A_2 \times \ldots \times A_n$ *is,*

$$\mu_{A_1 \times A_2 \times \ldots \times A_n}(x_1, x_2, \ldots, x_n) = min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)\}.$$

and the non-membership degree is,

$$\nu_{A_1 \times A_{2 \times \ldots \times A_n}}(x_1, x_2, \ldots, x_n) = max\{\nu_{A_1}(x1), \nu_{A_2}(x2), \ldots, \nu_{A_n}(x_n)\}.$$

Example 2.14 *Let* $X = \{x_1, x_2\}$,$Y = \{a, b\}$*and lets* A_1,A_2 *two intuitionistic fuzzy subsets respectively defined on* X *and* Y *given by:*
$A_1 = \{(x_1, 0.31, 0.2), (x_2, 0.54, 0.1)\}$.
$A_2 = \{(a, 0.02, 0.86), (b, 0.7, 0.53)\}$.
So, we get:
$A_1 \times A_2 = \{((x_1, a), 0.02, 0.86), ((x_1, b), 0.31, 0.53), ((x_2, a), 0.02, 0.86), ((x_2, b), 0.54, 0.53))\}$.

2.4 Intuitionistic fuzzy relations

In the following we give a definition and an example of intuitionistic fuzzy relations.

Definition 2.15 *Let X and Y be two non-empty sets.*
An intuitionistic fuzzy relation from X to Y (IFR, for short) is an intuitionistic fuzzy subset of $X \times Y$, i.e. is an expression R given by:

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \,|(x, y) \in X \times Y\}\,.$$

where

$$\mu_R : X \times Y \longrightarrow [0, 1].$$
$$\nu_R : X \times Y \longrightarrow [0, 1].$$

satisfy the condition $0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1$, for every $(x, y) \in X \times Y$. The value $\mu_R(x, y)$ is called the degree of membership of (x, y) in R and $\nu_R(x, y)$ is called the degree of non-membership of (x, y) in R.

Example 2.15 *Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$ be two nonempty sets, and let R be an intuitionistic fuzzy relation defined by:*

μ_R	x	y	z
a	0.49	0.68	0.17
b	0.53	1	0.77
c	0.25	1	0.85

ν_R	x	y	z
a	0.49	0.68	0.17
b	0.53	1	0.77
c	0.25	1	0.85

2.5 Operations on intuitionistic fuzzy relations

In the section we defined the intersection, union, containment and the complement of intuitionistic fuzzy relation plus some examples.

Definition 2.16 ***(Intersection)****Let X and Y be two non empty sets and let R and S be two intuitionistic fuzzy relations , the intersection defined by for all*
$(x,\ y) \in X \times Y$.

$$R \cap S = \{\langle (x, y), min\,\{\mu_R(x, y), \mu_S(x, y)\}\,, max\,\{\nu_R(x, y), \nu_S(x, y)\}\rangle\}\,.$$

Example 2.16 *Let R and S be tow intuitionistic fuzzy relations on $X \times X$ such that $X = \{x, y, z\}$, represented by the following tables:*

μ_R	x	y	z
x	0.59	0.8	0.87
y	0.63	0	0.37
z	0.15	0.55	0.65

ν_R	x	y	z
x	0.69	0.78	0.07
y	0.63	0.55	0.97
z	0.95	1	0.25

μ_S	x	y	z
x	0.49	0.68	0.17
y	0.53	1	0.77
z	0.25	1	0.85

ν_S	x	y	z
x	0.49	0.68	0.17
y	0.53	1	0.77
z	0.25	1	0.85

The intersection relations defined by

$\mu_{R\cap S}$	x	y	z
x	0.49	0.68	0.17
y	0.53	0	0.37
z	0.15	0.55	0.65

$\nu_{R\cap S}$	x	y	z
x	0.69	0.78	0.17
y	0.63	1	0.97
z	0.95	1	0.85

Definition 2.17 ***(Union)****Let X and Y be two non empty sets and let R and S be two intuitionistic fuzzy relations, the union defined by for all $(x, y) \in X \times Y$.*

$$R \cup S = \{\langle (x, y), max\,\{\mu_R(x, y), \mu_S(x, y)\}\,, min\,\{\nu_R(x, y), \nu_S(x, y)\}\rangle\}\,.$$

Example 2.17 *Let R and S be tow intuitionistic fuzzy relations on $X \times X$ such that $X = \{x, y, z\}$, represented by the following tables*

μ_R	x	y	z
x	0.9	0.18	0.47
y	0.3	1	0.87
z	0.85	0.05	0.6

ν_R	x	y	z
x	0.65	0.58	0.7
y	0.56	0.35	0.9
z	0.85	1	0.45

μ_S	x	y	z
x	0.44	0.08	0.47
y	0.53	1	0.17
z	0.86	1	0.80

ν_S	x	y	z
x	0.4	0.60	0.11
y	0.63	1	0.67
z	0.75	1	0.95

The union relations defined by

$\mu_{R\cup S}$	x	y	z
x	0.9	0.18	0.47
y	0.53	1	0.87
z	0.86	1	0.80

$\nu_{R\cup S}$	x	y	z
x	0.4	0.58	0.11
y	0.56	0.35	0.67
z	0.75	1	0.45

Definition 2.18 ***(Containment)****Let X and Y be two non empty sets and let R and S be two intuitionistic fuzzy relations, we say that $R \subseteq S$, if and only if $\mu_R(x, y) \leq \mu_S(x, y)$ and $\nu_R(x, y) \geq \nu_S(x, y)$ for any $(x, y) \in X \times Y$.*

Example 2.18 *Let R and S be tow intuitionistic fuzzy relations on $X \times X$ such that $X = \{x, y, z\}$, represented by the following tables:*

μ_R	x	y	z
x	0.55	0.88	0.88
y	0.25	0.25	0.45
z	0.16	0.4	0.77

ν_R	x	y	z
x	1	0.48	0.8
y	0.75	1	0.43
z	0.85	0.44	0.89

and

μ_S	x	y	z
x	1	0.95	1
y	0.8	0.33	0.66
z	0.9	0.75	0.88

ν_S	x	y	z
x	0.15	0.4	0.3
y	0.16	0.9	0.36
z	0.75	0.4	0.69

Then $\mu_R(x, y) \leq \mu_S(x, y)$ and $\nu_R(x, y) \geq \nu_S(x, y)$ $\forall x, y$ and $z \in X$.
Hense $R \subseteq S$.

Definition 2.19 *The complement of an intuitionistic fuzzy relation R is denoted by R^c and is defined by:*

$$R^c = \{\langle (x,y), \nu_R(x,y), \mu_R(x,y)\rangle \,|(x,y) \in X \times Y\}$$

Example 2.19 *Let R be an intuitionistic fuzzy relation on $X \times X$ such that $X = \{x, y, z\}$,represented by the following tables:*

μ_R	x	y	z
x	1	0.95	1
y	0.8	0.33	0.66
z	0.9	0.75	0.88

ν_R	x	y	z
x	0.15	0.4	0.3
y	0.16	0.9	0.36
z	0.75	0.4	0.69

The coplement relation defined by:

μ_{R^c}	x	y	z
x	0.15	0.4	0.3
y	0.16	0.9	0.36
z	0.75	0.4	0.69

ν_{R^c}	x	y	z
x	1	0.95	1
y	0.8	0.33	0.66
z	0.9	0.75	0.88

2.5.1 (α, β) cut of intuitionistic fuzzy relation

In the following, we extend the (α, β) cut of intuitionistic fuzzy relation.

Definition 2.20 *Let R be an intuitionistic fuzzy relation on a set X. The (α, β)-cut of R is a crisp subset*

$$R_{(\alpha,\beta)} = \left\{(x,y) \in X^2 | \mu_R(x,y) \geq \alpha \;\; and \;\; \nu_R(x,y) \leq \beta\right\}.$$

Example 2.20 *Let $X = \{x, y, z\}$, and let R be a intuitionistic fuzzy relation represented by the following table:*

μ_R	x	y	z
x	0.59	0.8	0.87
y	0.63	0	0.37
z	0.15	0.55	0.65

ν_R	x	y	z
x	0.69	0.78	0.07
y	0.63	0.55	0.97
z	0.95	1	0.25

$$R_{(0.5,0.7)} = \{(x,y) \in X^2 | \mu_R(x,y) \geq 0.5 and \nu_R(x,y) \leq 0.7\} = \{(x,x)\,,(x,y)\,,(y,y)\,,(z,z)\}.$$

2.6 Composition of intuitionistic fuzzy relations

In the section, we will study the composition of intuitionistic fuzzy relation.

Definition 2.21 *Let R and S be two intuitionistic fuzzy relations, one defines the composition "$\circ$" by $, y \in X$*

$$\mu_{R\circ S(x,y)} = max_z[min\,\{\mu_R(x,z), \mu_S(z,y)\}]$$

$$\nu_{R(x,y)} = min_z[max\,\{\nu_R(x,z), \nu_S(z,y)\}].$$

Example 2.21 *Let R and S be tow intuitionistic fuzzy relations on $X \times X$ such that $X = \{x, y, z\}$, represented by the following tables:*

μ_R	x	y	z
x	0.31	0.9	0.76
y	0.4	0.19	0.86
z	0.7	0.44	0.44

ν_R	x	y	z
x	0.15	0.75	0
y	0.33	1	0.13
z	0.45	0.88	0.11

and

μ_S	x	y	z
x	0.4	0.8	0.34
y	0.6	0.07	0.38
z	0.3	0.44	0.98

ν_S	x	y	z
x	0.45	0.43	0.5
y	0.41	0.13	0.07
z	0.34	0.1	0.09

The composition $R \circ S$ is defined by:

$\mu_{R\circ S}$	x	y	z
x	0.6	0.44	0.76
y	0.19	0.44	0.86
z	0.44	0.44	0.44

$\nu_{R\circ S}$	x	y	z
x	0.34	0.1	0.5
y	0.34	0.13	0.13
z	0.34	0.11	0.11

2.7 Intuitionistic fuzzy order relations

We will study now the intuitionistic fuzzy order relation.

Definition 2.22 *Let R be an intuitionistic fuzzy relation on the set X we say that R is an intuitionistic fuzzy order relation if anf only if:*

(1) *Reflexive:*
$\forall x \in X : \mu_R(x, x) = 1$ *and* $\nu_R(x, x) = 0$.

(2) *Antisymetrical:*
$\forall (x, y) \in X^2$ *and* $x \neq y : \mu_R(x, y) \neq \mu_R(y, x)$ *and* $\nu_R(x, y) \neq \nu(y, x)$.

(3) *Transitive:*
$\forall (x, y, z) \in X^3 : max[min\{\mu_R(x, y), \mu_R(y, z)\}] \leq \mu_R(x, z)$ *and*
$min[max\{\nu_R(x, y), \nu_R(y, z)\}] \geq \nu_R(x, z)$.

Example 2.22 *Let $X = \{x, y, z\}$. Then the intuitionistic fuzzy relation R defined on X by:*

$$R = \left\{ \langle (x, y), \mu_R(x, y), \nu_R(x, y) \rangle \,|(x, y) \in X^2 \right\}$$

where μ_R and ν_R given by the following tables :

μ_R	x	y	z
x	1	0.5	0.4
y	0.3	1	0
z	0	0.2	1

ν_R	x	y	z
x	0	0.1	0.5
y	0.7	0	0.2
z	0.4	0.7	0

is intuitionistic fuzzy ordering on X.

Definition 2.23 *An intuitionistic fuzzy order R on a universe X is called complete (or total) if for any* $(x, y) \in X^2$ *it holds that*

$$[\mu_R(x, y) > 0 \ \ or \ \ (\mu_R(x, y) = 0 \ \ and \ \ \nu_R(x, y) < 1)]$$

or

$$[\mu_R(y, x) > 0 \ \ or \ \ (\mu_R(y, x) = 0 \ \ and \ \ \nu_R(y, x) < 1)].$$

2.8 Intuitionistic fuzzy equivalent relations

Next, we will study now the intuitionistic fuzzy equivalent relation.

Definition 2.24 *Let R be an intuitionistic fuzzy relation on the set X we say that R is an intuitionistic fuzzy equivalent relation if anf only if:*

(1) *Reflexive:*
$\forall x \in X : \mu_R(x, x) = 1 \ \ and \ \ \nu_R(x, x) = 0.$

(2) *Symetrical:*
$\forall (x, y) \in X^2 : \mu_R(x, y) = \mu_R(y, x) \ \ and \ \ \nu_R(x, y) = \nu_R(y, x).$

(3) *Transitive:*
$\forall (x, y, z) \in X^3 : max[min\{\mu_R(x, y), \mu_R(y, z)\}] \leq \mu_R(x, z) \ \ and$
$min[max\{\nu_R(x, y), \nu_R(y, z)\}] \geq \nu_R(x, z).$

Example 2.23 *Let* $X = \{x, y, z\}$*. Then the intuitionistic fuzzy relation R defined on X by:*

$$R = \left\{ \langle (x, y), \mu_R(x, y), \nu_R(x, y) \rangle \, | (x, y) \in X^2 \right\}$$

where μ_R *and* ν_R *given by the following tables :*

μ_R	x	y	z
x	1	0.4	0.5
y	0.8	1	0.3
z	0.6	0.3	0

ν_R	x	y	z
x	0	0.4	0.5
y	0.4	0	0.3
z	0.5	0.3	0

is intuitionistic fuzzy equivalent on X.

CHAPTER 3

GENERALITIES ON NEUTROSOPHIC SETS

3.1 Neutrosophic set

In the following, we recall the definition of an neutrosophic set and example.

Definition 3.1 *Let X be a non empty set. An neutrosophic sets A on X is an object of the form :*

$$A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \, | x \in X\}.$$

Wher function :

$$\mu_A(x) : X \longmapsto [0,1].$$

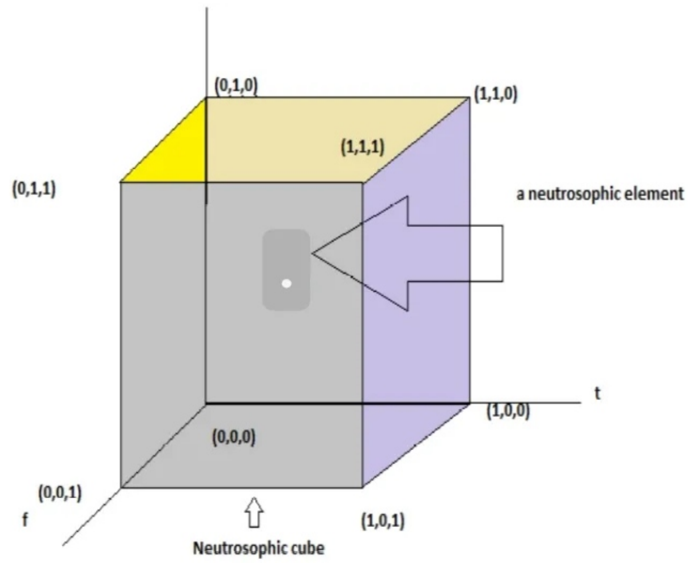
$$\sigma_A(x) : X \longmapsto [0,1].$$

$$\nu_A(x) : X \longmapsto [0,1].$$

Define the degree of membership , the degree of indeterminacy and degree non-membership of the element $x \in X$ to the set A , respectively, and for every $x \in X$.

$$0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3.$$

The following is a graphical representation of a single valued neutrosophic set. The elements of a single valued neutrosophic set, denoted henceforth by a neutrosophic element $x(\mu, \sigma, \nu)$, always remain inside and on a closed unit cube which henceforth will be called a neutrosophic cube. describes a neutrosophic cube.



Example 3.1 *Let* $X = \{a, b, c\}$ *be universal set.*

$$A = \{(a, 0.5, 0.7), (b, 0.25, 0.14), (c, 0.55, 0)\}.$$

A neutrosophic subset in X.

3.2 Operations on neutrosophic

In the section , we need following defintion of neutrosophic , union, equality , containment and complement of an neutrosophic fuzzy set with some examples .

Definition 3.2 (Intersection) *Let X non empty set and let A and B be two neutrosophic set , the intersection defined by :*

$$A \Cap B = \{\langle x, min\{\mu_A(x), \nu_B(x)\}, max\{\sigma_A(x), \sigma_B(x)\}, max\{\nu_A(x), \nu_A(x)\}\rangle \, | x \in X\}.$$

Example 3.2 *Let* $X = \{x_1, x_2, x_3\}$ *be universal set.*
Let $A = \{(x_1, 0.5, 0.4, 0), (x_2, 1, 0.33, 0.99), (x_3, 1, 0.77, 0, 9)\}$.
And let $B = \{(x_1, 1, 0.34, 0.77), (x_2, 0, 0.66, 0.78), (x_3, 0.35, 0.55, 0.90)\}$.
Then $A \Cap B = \{(x_1, 0.5, 0.4, 0.77), (x_2, 0, 0.66, 0.99), (x_3, 0.35, 0.77, 0.9)\}$.

Definition 3.3 (Union)*Let X non empty set, and let A and B be two neutrosophic set , the union defined by :*

$$A \Cup B = \{\langle x, max\{\mu_A(x), \mu_B(x)\}, min\{\sigma_A(x), \sigma_B(x)\}, min\{\nu_A(x), \nu_B(x)\}\rangle \, | x \in X\}.$$

Example 3.3 *Let* $X = \{x_1, x_2, x_3\}$ *be universal set.*
Let $A = \{(x_1, 0.15, 0.84, 0, 11), (x_2, 1, 0.73, 0.9), (x_3, 1, 0.47, 0, 19)\}$.
And let $B = \{(x_1, 1, 0.94, 0.07), (x_2, 0, 0.16, 0.68), (x_3, 0.39, 0.75, 0.19)\}$.
Then $A \Cup B = \{(x_1, 1, 0.84, 0.7), (x_2, 1, 0.16, 0.68), (x_3, 1, 0.47, 0.19)\}$.

Definition 3.4 (Containment) *Let X non empty set, and let A and B be two neutrosophic set ,we say that $A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$,$\sigma_A(x) \geq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for any $x \in X$.*

Example 3.4 *Let $X = \{x_1, x_2, x_3\}$ be universal set.*
Let $A = \{(x_1, 0.5, 0.7, 0, 13), (x_2, 0.4, 0.9, 1), (x_3, 0.13, 0.5, 1)\}$.
And let $B = \{(x_1, 0.75, 0.25, 0), (x_2, 0.65, 0.8, 0.9), (x_3, 0.2, 0.4, 0.5)\}$.
Then $A \subseteq B$ for all x_1, x_2 and x_3 in X.

Definition 3.5 (Complement) *The complement of an neutrosophic set A is denoted by $A^c$and is defined by:*

$$A^c = \{\langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x)\rangle \, | x \in X\}.$$

Example 3.5 *Let $X = \{x_1, x_2, x_3\}$ be universal set.*
Let $A = \{(x_1, 0.5, 0.75, 0, 40), (x_2, 0.64, 0.55, 0), (x_3, 0.45, 0.75, 1)\}$.
Then $A^c = \{(x_1, 0.4, 0.25, 0, 5), (x_2, 0, 0.45, 0.64), (x_3, 1, 0.25, 0.45)\}$.

Property 3.1 *Let A and B be two neutrosophic set.The fllowing results hold:*

(1) $A \subseteq A \Cup B$ and $B \subseteq A \Cup B$.

(2) $A \Cap B \subseteq A$ and $A \Cap B \subseteq B$.

(3) $(A^c)^c = A$.

(4) $(A \Cup B)^c = A^c \Cap B^c$.

(5) $(A \Cap B)^c = A^c \Cup B^c$.

3.2.1 Truth favourite and falsity favourite

This section contains the truth favourite and operators falsity favourite.For more see .

Definition 3.6 (Truth favourite) *The truth favourite of a neutrosophic set A, is denoted by ΔA which is defined as follows:*

$$\Delta A = \{\langle x, min\{\mu_A(x) + \sigma_A(x), 1\}, 0, \nu_A(x)\rangle \, | x \in X\}.$$

Example 3.6 *Let $X = \{x_1, x_2, x_3\}$ be universal set.*
Let $A = \{(x_1, 0.57, 0.65, 0, 14), (x_2, 0.94, 0.95, 1), (x_3, 0.85, 0.78, 0.13)\}$.
$\Delta A = \{(x_1, 0.71, 0, 0, 14), (x_2, 1, 0, 1), (x_3, 0.98, 0, 0.13)\}$.

Definition 3.7 (Falsity favourite) *The falsity favourite of a neutrosophic set A, is denoted by ∇A which is defined as follows:*

$$\nabla A = \{\langle x, \mu_A(x), 0, min\{\sigma_A(x) + \nu_A(x)\}\rangle \, | x \in X\}.$$

Example 3.7 *Let $X = \{x_1, x_2, x_3\}$ be universal set.*
Let $A = \{(x_1, 0.17, 0.65, 0, 45), (x_2, 0.19, 0.85, 1), (x_3, 0.55, 0.28, 0.93)\}$.
$\nabla A = \{(x_1, 0.17, 0, 1), (x_2, 0.19, 0, 1), (x_3, 0.55, 0, 1)\}$.

Remark 3.1 *∇A and ΔA are intuitionistic fuzzy set if and anly if :*

$$0 \leq \mu_{\nabla A(x)} + \nu_{\nabla A(x)} \leq 1.$$

$$0 \leq \mu_{\Delta A(x)} + \nu_{\Delta A(x)} \leq 1.$$

3.3 Characteristics of neutrosophic sets

In the section we defined the support, and (α,β,γ)-cuts the of an neutrosophic set with examples.

Definition 3.8 (Support)*Let A be a neutrosophic set on a X. The support of A is the crisp subset on X given by:*

$$Supp(A) = \{x \in X| \ \ \mu_A(x) \neq 0 \ \ and \ \ \sigma_A(x) \neq 0 \ \ and \ \ \nu_A(x) \neq 0\}.$$

Example 3.8 *Let $X = \{x_1, x_2, x_3\}$ be universal set.*
Let $A = \{(x_1, 0.17, 0, 0, 45), (x_2, 0.19, 0.85, 1), (x_3, 0.55, 0.28, 0.93)\}$.
$Supp(A) = \{x_2, x_3\}$.

Definition 3.9 (**(α,β,γ)**-cuts)*Let A be a neutrosophic set on X. The (α,β,γ)-cut of $A$is a crisp subset given by:*

$$A_{(\alpha,\beta,\gamma)} = \{x \in X|\mu_A(x) \geq \alpha \ \ and \ \ \sigma_A(x) \geq \beta \ \ and \ \ \nu_A(x) \leq \gamma\}.$$

3.3.1 Cartesien product on neutrosophic sets

Definition 3.10 *The cartesian product applied to n neutrosophic sets can be defined as follows:*
Let $\mu_{A_1}, \mu_{A_2}, \ldots \mu_{A_n}$ be membership functions of$A_1, A_2, \ldots, A_n$.Then, the membership de- gree of $(x_1, x_2, \ldots x_n) \in X_1 \times X_2 \times \ldots \times X_n$ on the intuitionistic fuzzy set $A_1 \times A_2 \times \ldots \times A_n$ is,

$$\mu_{A_1 \times A_2 \times \ldots \times A_n}(x_1, x_2, \ldots, x_n) = min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)\}.$$

indeterminacy degree is,

$$\sigma_{A_1 \times A_{2\times \ldots \times A_n}}(x_1, x_2, \ldots, x_n) = max\{\sigma_{A_1}(x1), \sigma_{A_2}(x2), \ldots, \sigma_{A_n}(x_n)\}.$$

and the non-membership degree is,

$$\nu_{A_1 \times A_{2\times \ldots \times A_n}}(x_1, x_2, \ldots, x_n) = max\{\nu_{A_1}(x1), \nu_{A_2}(x2), \ldots, \nu_{A_n}(x_n)\}.$$

Example 3.9 *Let $X = \{x_1, x_2\}$,$Y = \{a, b\}$and lets A_1,A_2 two neutrosophic subsets respectively defined on X and Y given by:*
$A_1 = \{(x_1, 0.31, 0.66, 0.2), (x_2, 0.54, 1, 0.1)\}$.
$A_2 = \{(a, 0.02, 0.34, 0.86), (b, 0.7, 0.77, 0.53)\}$.
So, we get:
$A_1 \times A_2 = \{((x_1, a), 0.02, 0.66, 0.86), ((x_1, b), 0.31, 0.77, 0.53), ((x_2, a), 0.02, 1, 0.86), ((x_2, b), 0.54, 1, 0.53))\}$

3.4 Neutrosophic relation

In the following we give a definition and an example of neutrosophic relations.

Definition 3.11 *A neutrosophic binary relation from a universe X to a universe$Y$ is a neutrosophic subset in $X \times Y$, i.e., is an expression Rgiven by:*

$$R = \{\langle (x,y), \mu_R(x,y), \sigma_R(x,y), \nu_R(x,y)\rangle \,|(x,y) \in X \times Y\}.$$

where for any $(x,y) \in X \times Y$

$$\mu_R : X \times Y \longrightarrow [0,1].$$
$$\sigma_R : X \times Y \longmapsto [0,1].$$
$$\nu_R : X \times Y \longrightarrow [0,1].$$

The value $\mu_R(x,y)$ is called the degree of a membership of (x,y) in R , $\sigma_R(x,y)$ is called the degree of indeterminacy of (x,y) in R and $\nu_R(x,y)$ is called the degree of non-membership of (x,y).

Example 3.10 *Let $X = \{a,b,c\}$ and $Y = \{x,y,z\}$ be two non empty sets, and let R be an neutrosophic relation defined by:*

μ_R	x	y	z
a	0.19	0.88	0.15
b	0.3	1	0.97
c	0.27	1	0.5

σ_R	x	y	z
a	0.79	0.48	0.7
b	0.54	1	0.07
c	0.65	1	0.8

ν_R	x	y	z
a	0.9	0.18	0.7
b	0.5	1	0.47
c	0.2	1	0.75

3.5 Operations on neutrosophic relations

In the section we defined the intersection, union, containment and the complement of neutrosophic relations plus some examples.

Definition 3.12 (Intersection) *Let X and Y be two non empty sets and let R and S be two neutrosophic relations , the intersection defined by for all $(x,y) \in X \times Y$.*
$R \Cap S = \{\langle (x,y), \mu_{R\Cap S}(x,y), \sigma_{R\Cap S}(x,y), \nu_{R\Cap S}(x,y)\rangle \,|(x,y) \in X \times Y\}.$
Where $\forall (x,y) \in X \times Y$, $\mu_{R\Cap S} = min\{\mu_R(x,y), \mu_S(x,y)\}, \sigma_{R\Cap S} = min\{\sigma_R(x,y), \sigma_S(x,y)\}$ and $\nu_{R\Cap S} = max\{\nu_R(x,y), \nu_S(x,y)\}$.

Example 3.11 *Let R and S be tow neutrosophic relations on $X \times X$ such that $X = \{x,y,z\}$, represented by the following tables:*

μ_R	x	y	z
x	0.09	0.78	0.27
y	0.46	1	0.67
z	0.54	0.65	0.6

σ_R	x	y	z
x	0.56	0.98	0.07
y	0.78	0.86	0.6
z	0.75	0.4	0.85

ν_R	x	y	z
x	0.89	0.08	0.37
y	0.64	0.5	0.57
z	0.85	1	0.5

and

μ_S	x	y	z
x	0.09	0.68	0.07
y	0.3	1	0.7
z	0.2	1	0.5

σ_S	x	y	z
x	0.59	0.08	0.87
y	0.63	0	0.37
z	0.15	0.55	0.65

ν_S	x	y	z
x	0.49	0.68	0.17
y	0.53	1	0.77
z	0.25	1	0.85

The intersection relations defined by

$\mu_{R\Cap S}$	x	y	z
x	0.09	0.68	0.07
y	0.3	1	0.67
z	0.2	0.65	0.5

$\sigma_{R\Cap S}$	x	y	z
x	0.56	0.08	0.07
y	0.63	0	0.37
z	0.15	0.4	0.65

$\nu_{R\Cap S}$	x	y	z
x	0.89	0.68	0.37
y	0.64	1	0.77
z	0.85	1	0.85

Definition 3.13 (Union)*Let X and Y be two non empty sets and let R and S be two neutrosophic relations , the union defined by for all $(x, y) \in X \times Y$.*
$R \Cup S = \{\langle (x, y), \mu_{R\Cup S}(x, y), \sigma_{R\Cup S}(x, y), \nu_{R\Cup S}(x, y)\rangle \,|(x, y) \in X \times Y\}$.
Where $\forall (x, y) \in X \times Y$, $\mu_{R\Cup S} = max\,\{\mu_R(x, y), \mu_S(x, y)\}$, $\sigma_{R\Cup S} = max\,\{\sigma_R(x, y), \sigma_S(x, y)\}$ *and* $\nu_{R\Cup S} = min\,\{\nu_R(x, y), \nu_S(x, y)\}$.

Example 3.12 *Let R and S be tow neutrosophic relations on $X \times X$ such that $X = \{x, y, z\}$, represented by the following tables:*

μ_R	x	y	z
x	0.9	0.48	0.47
y	0.6	0	0.7
z	0.15	0.65	0.6

σ_R	x	y	z
x	0.5	0.18	0.07
y	0.68	0.66	0.67
z	0.95	0.84	0.5

ν_R	x	y	z
x	0.69	0.78	0.07
y	0.63	0.55	0.97
z	0.95	1	0.25

and

μ_S	x	y	z
x	0.19	0.68	0.17
y	0.03	0	0.77
z	0.25	1	0.85

σ_S	x	y	z
x	0.59	0.8	0.87
y	0.63	0	0.37
z	0.15	0.55	0.65

ν_S	x	y	z
x	0.49	0.68	0.17
y	0.53	1	0.77
z	0.25	1	0.85

The union relations defined by

$\mu_{R\Cup S}$	x	y	z
x	0.9	0.68	0.47
y	0.6	0	0.77
z	0.25	1	0.85

$\sigma_{R\Cup S}$	x	y	z
x	0.59	0.8	0.87
y	0.68	0.66	0.67
z	0.95	0.84	0.65

$\nu_{R\Cup S}$	x	y	z
x	0.49	0.68	0.67
y	0.53	0.55	0.77
z	0.25	1	0.25

Definition 3.14 (Complement) *The complement of an neutrosophic relation R is denoted by R^c and is defined by:*

$$R^c = \{\langle (x, y), \mu_{R^c}(x, y), \sigma_{R^c}(x, y), \nu_{R^c}(x, y)\rangle \,|(x, y) \in X \times X\}\,.$$

Where $\forall (x, y) \in X$ $\mu_{R^c}(x, y) = \nu_R(x, y), \sigma_{R^c}(x, y) = 1 - \sigma_R(x, y)$
and $\nu_{R^c}(x, y) = \mu_R(x, y)$.

Example 3.13 *Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$ be two non empty sets, and let R be an neutrosophic relation defined by:*

μ_R	x	y	z
a	0.19	0.88	0.15
b	0.3	1	0.97
c	0.27	1	0.5

σ_R	x	y	z
a	0.79	0.48	0.7
b	0.54	1	0.07
c	0.65	1	0.8

ν_R	x	y	z
a	0.9	0.18	0.7
b	0.5	1	0.47
c	0.2	1	0.75

The coplement relation defined by:

μ_{R^c}	x	y	z
a	0.9	0.18	0.7
b	0.5	1	0.47
c	0.2	1	0.75

σ_{R^c}	x	y	z
a	0.21	0.52	0.3
b	0.46	0	0.93
c	0.35	0	0.1

μ_{R^c}	x	y	z
a	0.19	0.88	0.15
b	0.3	1	0.97
c	0.27	1	0.5

Definition 3.15 (Containment) *Let X and Y be two non empty sets and let R and S be two neutrosophic relations, we say that $R \subseteq S$, if and only if*
$\mu_R(x,y) \leq \mu_S(x,y)$, $\sigma_R(x,y) \geq \sigma_S(x,y)$ *and* $\nu_R(x,y) \geq \nu_R(x,y)$
for any $(x,y) \in X \times Y$.

Example 3.14 *Let R and S be tow neutrosophic relations on $X \times X$ such that $X = \{x, y, z\}$, represented by the following tables:*

μ_R	x	y	z
x	0.9	0.48	0.47
y	0.6	0	0.7
z	0.15	0.65	0.6

σ_R	x	y	z
x	0.5	0.18	0.07
y	0.68	0.66	0.67
z	0.95	0.84	0.5

ν_R	x	y	z
x	0.69	0.78	0.07
y	0.63	0.55	0.97
z	0.95	1	0.25

and

μ_S	x	y	z
x	0.91	0.68	0.9
y	0.75	1	0.8
z	0.5	1	0.75

σ_S	x	y	z
x	0.3	0.1	0.01
y	0.51	0.35	0.4
z	0.57	0.7	0.23

ν_S	x	y	z
x	0.49	0.68	0.01
y	0.53	0	0.77
z	0.25	0.5	0.12

Then $\mu_R(x,y) \leq \mu_S(x,y)$, $\sigma_R(x,y) \geq \sigma_S(x,y)$ *and* $\nu_R(x,y) \geq \nu_S(x,y)$.
Hense $R \subseteq S$.

Definition 3.16 (The inverse) *The inverse of an neutrosophic relation R is denoted by R^{-1} and is defined by:*

$$R^{-1} = \{\langle (x,y), \mu_{R^{-1}}(x,y), \sigma_{R^{-1}}(x,y), \nu_{R^{-1}}(x,y) \rangle \, | x \in X \times X\}.$$

Where $\forall (x,y) \in X \times X$, $\mu_{R^{-1}}(x,y) = \mu_R(y,x)$, $\sigma_{R^{-1}}(x,y) = \sigma_R(y,x)$
and $\nu_{R^{-1}}(x,y) = \nu_R(x,y)$.

Example 3.15 *Let R be neutrosophic relations on $X \times X$ such that $X = \{x, y, z\}$, represented by the following tables :*

μ_R	x	y	z
x	0.2	0.4	0.6
y	0.51	1	0.4
z	0	0.34	0.75

σ_R	x	y	z
x	0	0.7	0.13
y	0.51	1	0.48
z	0.4	0.26	0

ν_R	x	y	z
x	1	0.16	0.45
y	0.51	0	0.16
z	0.45	0.25	1

The inverse of R^{-1} are given by :

$\mu_{R^{-1}}$	x	y	z
x	0.2	0.51	0
y	0.4	1	0.34
z	0.6	0.4	0.75

$\sigma_{R^{-1}}$	x	y	z
x	0	0.51	0.4
y	0.7	1	0.26
z	0.13	0.48	0

$\nu_{R^{-1}}$	x	y	z
x	1	0.51	0.45
y	0.16	0	0.25
z	0.45	0.16	1

Theorem 3.1 *Let R, S and P be three neutrosophic relation in X. Then*

(1) $(R^c)^{-1} = (R^{-1})^c$.

(2) $(R^{-1})^{-1} = R$.

(3) $(R^c)^c = R$.

(4) $(R \Cup S) \supseteq R$, $(R \Cup S) \supseteq S$.

(5) $(R \Cap S) \subseteq R$, $(R \Cap S) \subseteq S$.

(6) *If* $R \subseteq S$,*then* $R^{-1} \subseteq S^{-1}$.

(7) *If* $P \supseteq S$ *and* $P \supseteq R$, *then* $P \supseteq (R \Cup S)$.

(8) *If* $P \subseteq S$ *and* $P \subseteq R$, *then* $P \subseteq (R \Cap S)$.

(9) *If* $R \subseteq S$, *then* $(R \Cup S) = S$ *and* $(R \Cup S) = R$.

(10) $(R \Cup S)^{-1} = (R^{-1}) \Cup (S^{-1})$, $(R \Cap S)^{-1} = R^{-1} \Cap S^{-1}$.

(11) $(R \Cup S)^c = R^c \Cap S^c$,$(R \Cap S)^c = R^c \Cup S^c$.

3.5.1 $(\alpha, \beta, \gamma)-$ cuts of neutrosophic relation

In the following, we extend the $(\alpha, \beta, \gamma)-$cut of neutrosophic relation.

Definition 3.17 *Let R be an neutrosophic relation on a set X . The $(\alpha, \beta, \gamma)-$ cuts of R is a crisp subset*

$$R_{(\alpha,\beta,\gamma)} = \left\{(x, y) \in X^2 | \mu_R(x, y) \geq \alpha \ \ and \ \ \sigma_R(x, y) \geq \beta \ \ and \ \ \nu_R(x, y) \leq \gamma\right\}.$$

Example 3.16 *Let $X = \{x, y, z\}$, and let R be a neutrosophic relation represented by the following table:*

μ_R	x	y	z
x	0.2	0.4	0.6
y	0.51	1	0.4
z	0	0.34	0.75

σ_R	x	y	z
x	0	0.7	0.13
y	0.51	1	0.48
z	0.4	0.26	0

ν_R	x	y	z
x	1	0.6	0.5
y	0.5	0	0.1
z	0.5	0.25	1

$R_{(0.2,0.1,0.5)} = \{(x, z), (y, x), (y, y), (y, z), (z, y)\}$.

3.6 Composition of neutrosophic relations

In the section, we will study the composition of neutrosophic relation.

Definition 3.18 *Let R neutrosophic relation on $X \times Y$ and let S neutrosophic relation on $Y \times Z$ .Then the composition of R and S denoted by $R \circ S$, is a neutrosophic relation on $X \times Z$ defined by :*

$$S \circ R = \{\langle x, \mu_{S \circ R}(x, z), \sigma_{S \circ R}(x, z), \nu_{S \circ R}(x, z)\rangle \,|(x, z) \in X \times Z\}.$$

Where for $(x,z) \in X \times Z$:

$$\mu_{S\circ R}(x,z) = max_{y\in X}\left[min\left\{\mu_R(x,y), \mu_S(x,y)\right\}\right].$$

$$\sigma_{S\circ R}(x,z) = min_{y\in X}\left[max\left\{\sigma_R(x,y), \sigma_S(x,y)\right\}\right].$$

$$\nu_{S\circ R}(x,z) = min_{y\in X}\left[max\left\{\nu_R(x,y), \nu_S(x,y)\right\}\right].$$

Example 3.17 *Let* R *and* S *be tow neutrosophic relations on* $X\times X$ *such that* $X = \{x,y,z\}$, *represented by the following tables:*

μ_R	x	y	z
x	0.79	0.38	1
y	0.69	0	0.87
z	0.19	0.65	0.46

σ_R	x	y	z
x	0.5	0.18	0.07
y	0.68	0.66	0.67
z	0.95	0.84	0.5

ν_R	x	y	z
x	0.09	0.7	0.97
y	0.3	0.5	0
z	0.9	1	0.25

and

μ_S	x	y	z
x	0.91	0.68	0.9
y	0.75	1	0.8
z	0.5	1	0.75

σ_S	x	y	z
x	0.53	0.17	0.91
y	0.45	0.5	0.46
z	0.7	0.71	0.23

ν_S	x	y	z
x	0.4	0.68	0.91
y	0.3	1	0
z	0.2	0.15	0.12

The composition $R \circ S$ *is defined by:*

$\mu_{S\circ R}$	x	y	z
x	0.79	0.65	0.91
y	0.75	0.65	0.75
z	0.69	0.65	0.87

$\sigma_{S\circ R}$	x	y	z
x	0.53	0.53	0.53
y	0.5	0.45	0.45
z	0.7	0.7	0.7

$\nu_{S\circ R}$	x	y	z
x	0.4	0.68	0.68
y	0.3	0.7	0.25
z	0.2	0.5	0.15

3.7 Neutrosophic order relations

We will study now the neutrosophic order relations.

Definition 3.19 *Let* R *be an neutrosophic relation on the set* X *we say that* R *is an neutrosophic order relation if anf only if:*

(1) *Reflexive*
$\mu_R(x,x) = 1, \sigma_R(x,x) = \nu_R(x,x) = 0$.

(2) *Antisymetrical*
$\forall (x,y) \in X\times X$ *and* $x \neq y$: $\mu_R(x,y) \neq \mu_R(y,x), \sigma_R(x,y) \neq \sigma_R(y,x)$ *and* $\nu_R(x,y) \neq \nu_R(y,x)$.

(3) *Transitive*
$R\circ R \subset R$ *i.e.,* $R^2 \subset R$.

Example 3.18 *Let* $X = \{x,y,z\}$. *Then the neutrosophic relation* R *defined on* X *by:*

$$R = \left\{\langle (x,y), \mu_R(x,y), \sigma_R(x,y), \nu_R(x,y)\rangle \,|(x,y) \in X\times X\right\}.$$

where μ_R , σ_R and ν_R given by the following tables :

μ_R	x	y	z
x	1	0.38	1
y	0.69	1	0.87
z	0.19	0.65	1

σ_R	x	y	z
x	0	0.18	0.07
y	0.68	0	0.67
z	0.95	0.84	0

ν_R	x	y	z
x	0	0.7	0.97
y	0.3	0	0
z	0.9	1	0

Then R neutrosophic order on X.

3.8 Neutrosophic equivalent relations

Next, we will study now the neutrosophic equivalent relations.

Definition 3.20 *Let R be an neutrosophic relation on the set X we say that R is an neutrosophic equivalent relation if anf only if:*

(1) *Reflexive*
$\mu_R(x,x) = 1, \sigma_R(x,x) = \nu_R(x,x) = 0.$

(2) *Symetrical*
$\forall(x,y) \in X \times X$ *and :* $\mu_R(x,y) = \mu_R(y,x), \sigma_R(x,y) = \sigma_R(y,x)$
and $\nu_R(x,y) = \nu_R(y,x)$.

(3) *Transitive*
$R \circ R \subset R$ *i.e.,* $R^2 \subset R$.

Example 3.19 *Let $X = \{x, y, z\}$. Then the neutrosophic relation R defined on X by:*

$$R = \{\langle (x,y), \mu_R(x,y), \sigma_R(x,y), \nu_R(x,y)\rangle \,|(x,y) \in X \times X\}\,.$$

where μ_R , σ_R and ν_R given by the following tables :

μ_R	x	y	z
x	1	0.69	1
y	0.69	1	0.87
z	1	0.87	1

σ_R	x	y	z
x	0	0.18	0.95
y	0.18	0	0.84
z	0.95	0.84	0

ν_R	x	y	z
x	0	0.7	0.97
y	0.7	0	1
z	0.97	1	0

R is neutrosophic equivalent on X.

3.9 Comparison between fuzzy sets and intuitionistic sets and neutrosophic sets

Definition *:*

Fuzzy sets *: Fuzzy sets is generaliztion of a benary logic wher the membership degree of a element is represented by a value between 0 and 1 .*
Intuitionistic sets *: Intuitionistic sets is a extension of a fuzzy sets that introduces a second value , called degree of a non membership .*
Neutrosophic sets *: Neutrosophic sets is a extension of a intuitionistic sets that introduces*

a third value , between membership degree and degree of a non membership , called degree of a indeterminacy .

Example 3.20 *Let $X = \{a, b, c\}$ be universal set . and A_1 , A_2 and A_3 be the three sets defined as :*

$$A_1 = \{\langle a, 0.4\rangle, \langle b, 0.7\rangle, \langle c, 1\rangle\}.$$

$$A_2 = \{\langle a, 0.4, 0.3\rangle, \langle b, 0.7, 0.1\rangle, \langle c, 1, 0.5\rangle\}.$$

$$A_1 = \{\langle a, 0.4, 0.6\rangle, \langle b, 0.7, 0, 0.34\rangle, \langle c, 1, 0.6, 09\rangle\}.$$

Then A_1 is a fuzzy sets, A_2 is a intuitionistic sets and A_3 neutrosophic sets .

Operation *:*

Fuzzy sets *: Fuzzy sets support various opration such as union , intersection , and complement . These operation are defind based fuzzy logic operators like min and max .*
Intuitionistic sets *: Intuitionistic sets also support operation like union , intersection , and complement . However , the oprations are defind differntly , tking into degree of a membership and degree non membership .*
Neutrosophic sets*: Neutrosophic sets also support operation like union , intersection , and complement . However , the oprations are defind differntly , tking into degree of a membership , degree of a indeterminacy , and degree non membership .*

Example 3.21 *.*
- At the union of two fuzzy sets in Definition 1.5 *we just calculated degree of membership.*

$$A \cup B = \{\langle x, max\{\mu_A, \mu_B\}\rangle | x \in X\}.$$

- At the union of two intuitionistic fuzzy sets in Definition 2.3 *we just calculated the degree of membership and the degree of non membership.*

$$A \cup B = \{\langle x, max\{\mu_A, \mu_B\}, min\{\nu_A, \nu_B\}\rangle | x \in X\}.$$

- At the union of two neutrosophic fuzzy sets in Definition 3.3 *we just calculated the degree of membership , degree of a indeterminacy and the degree of non membership.*

$$A \cup B = \{\langle x, max\{\mu_A, \mu_B\}, min\{\sigma_A, \sigma_B\}, min\{\nu_A, \nu_B\}\rangle | x \in X\}.$$

Applications *:*

Fuzzy sets *: Fuzzy sets find applications in fields such as decision-making, pattern recognition, control systems, image processing, and expert systems. They are particularly useful in situations where precise boundaries are difficult to define.*
Intuitionistic sets *: Intuitionistic fuzzy sets have applications in areas like information retrieval, knowledge representation, multi-criteria decisionmaking, and fuzzy clustering.*
Neutrosophic sets*: Neutrosophic sets find applications in fields such as decision-making , pattern recognition , image processing , and knowledge representation .*

Relations *:*

$\hookrightarrow$ ***Definition*** *.*

Fuzzy relation *:A fuzzy relation is a generalization of a crisp binary relation where the membership degree of an element in the relation is represented by a value between 0 and 1. It allows for degrees of membership and provides a quantitative measure of the relation between elements.*

Intuitionistic relation *: An intuitionistic fuzzy relation is an extension of a fuzzy relation that introduces a second value, called degree of non membership, that is, it also measures the degree of non membership and provides the non-existent quantity of the relation between the elements.*

Neutrosophic relation *: Neutrosophic relation is a extension of a intuitionistic relation that introduces a third value , between membership degree and degree of a non membership , called degree of a indeterminacy .*

Example 3.22 *.*

- At the fuzzy relation in Definition 1.19 *we just calculated degree of membership.*

$$R = \left\{ \left\langle x, \mu_(x, y) \right\rangle | x \in X \times Y \right\}.$$

- At the intuitionistic fuzzy relation in Definition 2.15 *we just calculated the degree of membership and the degree of non membership.*

$$R = \left\{ \left\langle x, \mu_(x, y), \nu(x, y) \right\rangle | x \in X \times Y \right\}.$$

- At the neutrosophic fuzzy relation in Definition 3.11 *we just calculated the degree of membership , degree of a indeterminacy and the degree of non membership.*

$$R = \left\{ \left\langle x, \mu_(x, y), \sigma(x, y), \nu(x, y) \right\rangle | x \in X \times Y \right\}.$$

$\hookrightarrow$***Representation*** *:*

Fuzzy relations *: Fuzzy relations are typically represented using matrices or graphs, where each entry represents the degree of membership or similarity between two elements. Fuzzy relations can also be represented using fuzzy logic operators.*

Intuitionistic fuzzy relations *: are often represented using two matrices or tables, where each entry contains two values: the degree of membership and degree of non membership, the sum of the two values is not necessarily 1.*

Neutrosophic relations *: are often represented using three matrices or tables, where each entry contains two values: the degree of membership, degree of a indeterminacy and degree of non membership, the sum of the two values is not necessarily 3.*

Example 3.23 *Let* $X = \{x, y\}$ *be a finite set. We represented the fuzzy relation* R_1 *by table :*

μ_{R_1}	x	y
x	0	0.7
y	0.7	0

And the intuitionistic fuzzy relation R_2 by the followings tables :

μ_{R_2}	x	y
x	1	0.3
y	0.4	1

ν_{R_2}	x	y
x	0	0.5
y	0.2	0

And the neutrosophic fuzzy relation R_3 by the followings tables :

μ_{R_3}	x	y
x	1	0.47
y	0.67	0

σ_{R_3}	x	y
x	0.59	0.7
y	0.7	0

ν_{R_3}	x	y
x	0.45	0.17
y	0.77	0

Remark 3.2 .

−Every fuzzy relations is intuitionistic fuzzy relations because we can give the degree of non-membership a value of $1-\mu(x)$. But not every intuitionistic fuzzy relation is a fuzzy relation .

− Any neutrosophic relations is intuitionistic fuzzy relations because we can give the degree of a indeterminacy a value of zero. But not every neutrosophic relation is a intuitionistic fuzzy relation .

Conclusion

In this memory, we have given a definition of fuzzy sets with some operations on them. Then, we have give definition of intuitionistic fuzzy sets and intuitionistic fuzzy relations ,and we have give definition of neutrosophic sets , neutrosohic relation and we concluded the study with a comparison between fuzzy sets intuitionistic fuzzy sets and neutrosophic sets. In summary, neutrosophic sets is a more general concept than fuzzy sets and intuitionistic fuzzy sets, and we think so that this result will open the door in front of many useful studies and researches about this scientific point in the future.

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ملخص

درسنا في هذا العمل إطاراً رياضياً يوسع مفهوم المجموعات الضبابية والمجموعات الحدسية من خلال تقديم المجموعات النيوتروسوفكسية والمعرفة بدرجة الانتماء ، درجة عدم الانتماء و درجة الحيادية بحيث مجموعهم يكون محصور بين ٠ و ٣ . ونظراً للغموض فيها فهي تستعمل في مجالات مختلفة كتحكم في الربوهات و الطائرات المسيرة ...الخ .

كلمات مفتاحية

المجموعات النيوتروسوفكية، العمليات في المجموعات النيوتروسوفكية، بعض التطبيقات المحددة للمجموعات النيوتروسوفكية.

Abstract

In this work, we studied a mathematical framework that expands the concept of fuzzy sets and intuitionistic sets by introducing neutrosophic sets, which are characterized by the degree of membership, the degree of non-membership, and the degree of indeterminacy, where the sum of the three is confined between 0 and 3. Due to the ambiguity in neutrosophic sets, they are used in various fields such as control of robots and unmanned aerial vehicles, etc.

Key words

Neutrosophic sets , Operation in neutrosophic sets , Some specific applications of neutrosophic sets.

Résumé

Dans ce travail, nous avons étudié un cadre mathématique qui élargit le concept des ensembles flous et des ensembles intuitionnistes en introduisant les ensembles neutrosophiques, caractérisés par le degré d'appartenance, le degré de non-appartenance et le degré d'indétermination, dont la somme est comprise entre 0 et 3. En raison de l'ambiguïté présente dans les ensembles neutrosophiques, ils sont utilisés dans divers domaines tels que le contrôle des robots et des véhicules aériens sans pilote, etc.

Mot-clés

Ensembles neutrosophiques , Fonctionnement dans les ensembles neutrosophiques , Quelques applications spécifiques des ensembles neutrosop.