Applied Mathematics E-Notes, 19(2019), ???-??? © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Initial value problem for nonlinear implicit fractional differential equations with Katugampola derivative*

Yacine Arioua[†], Bilal Basti[‡], Nouredine Benhamidouche[§]

Received 19 Mai 2018

Abstract

This work studies the existence and uniqueness of solutions for a class of nonlinear implicit fractional differential equations via the Katugampola fractional derivatives with an initial condition. The arguments for the study are based upon the Banach contraction principle, Schauders fixed point theorem and the nonlinear alternative of Leray-Schauder type.

1 Introduction

Fractional calculus is a mathematical branch which investigates the properties of derivatives and integrals of non-integer orders (also known as fractional derivatives and integrals, briefly differ-integrals). The interested readers in the subject should refer to the books (Samko et al. 1993 [17], Podlubny 1999 [16], Kilbas et al. 2006 [10], Diethelm 2010 [3]).

The differential equations of fractional order are generalizations of classical differential equations of integer order; they are increasingly used in such fields as fluid flow, control theory of dynamical systems, diffusive transport akin to diffusion, probability and statistics... etc. The boundary value problem of fractional differential equations is recently approached by various researchers ([1], [2], [8], [15], [18]).

In [1], Almeida, Malinowska and Odzijewicz, presented a new fractional operator, which generalizes the Caputo and the Caputo-Hadamard fractional derivative. Some fundamental properties of this operator are presented and proved. Then, an existence and uniqueness theorem for a fractional Cauchy type problem is given. In order to apply a numerical procedure for solving the mentioned fractional differential problem,

$$\begin{cases} {}^{C}\mathcal{D}_{a^{+}}^{\alpha;\rho}u\left(t\right)=f\left(t,u\left(t\right)\right), \quad \alpha\in\left(0,1\right),\ t\in\left[a,b\right],\ 0<a<b<\infty, \\ u\left(a\right)=u_{a}, \quad u_{a}\in\mathbb{R}, \end{cases}$$

${}^{C}\mathcal{D}_{a^{+}}^{\alpha;\rho}u$ is the left Caputo–Katugampola fractional derivative, $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

---

*Mathematics Subject Classifications: 34A08, 34A37.

[†]Laboratory for Pure and Applied Mathematics University of M'sila, Algeria.
[‡]Laboratory for Pure and Applied Mathematics University of M'sila, Algeria.
[§]Laboratory for Pure and Applied Mathematics University of M'sila, Algeria.

In [2], Benchohra and Lazreg, applied the Banach contraction principle, Schauder fixed-point theorem and Leray-Schauder type, to show the existence and uniqueness of solutions for an initial value problem of the nonlinear implicit fractional differential equation:

$$\begin{cases} {}^c\mathcal{D}_{0+}^{\alpha} u(t) = f\left(t, u(t), {}^c\mathcal{D}_{0+}^{\alpha} u(t)\right), & t \in [0,T],\ T > 0,\ 0 < \alpha \leq 1, \\ u(0) = u_0, \end{cases}$$

where ${}^c\mathcal{D}_{0+}^{\alpha} u$ is the Caputo fractional derivative, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, and $u_0 \in \mathbb{R}$.

In [15], Murad and Hadid, by means of Schauder fixed-point theorem and the Banach contraction principle, considered the boundary value problem of the fractional differential equation:

$$\begin{cases} \mathcal{D}_{0+}^{\alpha} u(t) = f\left(t, u(t), \mathcal{D}_{0+}^{\beta} u(t)\right), & t \in (0,1),\ 1 < \alpha \leq 2,\ 0 < \beta < 1,\ 0 < \gamma \leq 1, \\ u(0) = 0,\ u(1) = \mathcal{I}_{0+}^{\gamma} u(s), \end{cases}$$

where $\mathcal{D}_{0+}^{\alpha} u$ $\left[\text{resp. } \mathcal{I}_{0+}^{\alpha} u\right]$ is the Riemann-Liouville fractional derivative (resp. fractional integral), and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Our objective in this work is to study in a general manner, the existence and uniqueness of solutions of nonlinear fractional differential equations:

$$ {}^\rho\mathcal{D}_{0+}^{\alpha} u(t) = f\left(t, u(t), {}^\rho\mathcal{D}_{0+}^{\alpha} u(t)\right),\ t \in [0,T], \tag{1}$$

with the initial condition:

$$u(0) = 0. \tag{2}$$

Where $0 < \alpha \leq 1$, $\rho > 0$, and $T \leq (pc)^{\frac{1}{pc}}$ for any $1 \leq p \leq \infty$, $c > 0$, is a finite positive constant. The symbol ${}^\rho\mathcal{D}_{0+}^{\alpha}$ presents the Katugampola fractional derivative operator, and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

We obtain several existence and uniqueness results for the problem (1)-(2).

2 Preliminaries

In this section we present the necessary definitions from fractional calculus theory. As in [10], consider the space $X_c^p[0,T]$, $(c \in \mathbb{R},\ 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions y on $[0,T]$ for which $\|y\|_{X_c^p} < \infty$, where the norm is defined by:

$$\|y\|_{X_c^p} = \left(\int_0^T |s^c y(s)|^p \frac{ds}{s}\right)^{\frac{1}{p}} < \infty,$$

for $1 \leq p < \infty$, $c \in \mathbb{R}$. For the case $p = \infty$;

$$\|y\|_{X_c^\infty} = ess \sup_{0 \leq t \leq T} \left[t^c |y(t)|\right] \quad (c \in \mathbb{R}).$$

By $C[0,T]$ we denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm:

$$\|y\|_{\infty} = \sup_{0\leq t\leq T} |y(t)|.$$

REMARK 1. Let $p, c, T \in \mathbb{R}_{+}^{*}$, be such that $p \geq 1$, $c > 0$ and $T \leq (pc)^{\frac{1}{pc}}$.
It's clear that $\forall y \in C[0,T]$

$$\|y\|_{X_c^p} = \left(\int_0^T |s^c y(s)|^p \frac{ds}{s}\right)^{\frac{1}{p}} \leq \left(\|y\|_{\infty}^p \int_0^T s^{pc-1} ds\right)^{\frac{1}{p}} = \frac{T^c}{(pc)^{\frac{1}{p}}} \|y\|_{\infty}.$$

And

$$\|y\|_{X_c^{\infty}} = \text{ess} \sup_{0\leq t\leq T} [t^c |y(t)|] \leq T^c \|y\|_{\infty}.$$

Which implies that $C[0,T] \hookrightarrow X_c^p[0,T]$, and

$$\|y\|_{X_c^p} \leq \|y\|_{\infty}, \text{ for all } T \leq (pc)^{\frac{1}{pc}}.$$

We start with the definitions introduced in [10] with a slight modification in the notation.

DEFINITION 1 (*Riemann-Liouville* fractional integral [10]).
The left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $y : [0,T] \to \mathbb{R}$ is given by:

$${}^{RL}\mathcal{I}_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds, \ t \in [0,T].$$

DEFINITION 2 (*Riemann-Liouville* fractional derivative [10]).
The left-sided Riemann Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : [0,T] \to \mathbb{R}$ is given by:

$${}^{RL}\mathcal{D}_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} y(s)\, ds, \ t \in [0,T], \ n = [\alpha]+1.$$

DEFINITION 3 (*Hadamard* fractional integral [10]). The left-sided Hadamard fractional integral of order $\alpha > 0$ of a continuous function $y : [0,T] \to \mathbb{R}$ is given by:

$${}^{H}\mathcal{I}_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log\frac{t}{s}\right)^{\alpha-1} y(s) \frac{ds}{s}, \ t \in [0,T].$$

DEFINITION 4 (*Hadamard* fractional derivative [10]). The left-sided Hadamard fractional derivative of order $\alpha > 0$ of a continuous function $y : [0,T] \to \mathbb{R}$ is given by:

$${}^{H}\mathcal{D}_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_0^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} y(s) \frac{ds}{s}, \ t \in [0,T], \ n = [\alpha]+1.$$

A recent generalization, introduced by Udita Katugampola (2011) [9], generalizes the Riemann-Liouville fractional integral and the Hadamard fractional integral (see [10]).

The integral is now known as the Katugampola fractional integral, it is given in the following definition:

DEFINITION 5 (*Katugampola* fractional integral [9]). The *Katugampola* fractional integrals of order $\alpha > 0$ of a function $y \in X_c^p [0, T]$ is defined by:

$$^{\rho}\mathcal{I}_{0+}^{\alpha} y(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1} y(s)}{(t^{\rho} - s^{\rho})^{1-\alpha}} ds, \ t \in [0, T], \tag{3}$$

for $\rho > 0$. These integrals are called left-sided integrals.

Similarly we can define right-sided integrals [7]-[10]. In a similar way we have:

DEFINITION 6 (*Katugampola* fractional derivatives [7]).
The generalized fractional derivatives of order $\alpha > 0$, corresponding to the *Katugampola* fractional integrals (3) defined for any $t \in [0, T]$, by:

$$\begin{aligned} ^{\rho}\mathcal{D}_{0+}^{\alpha} y(t) &= \left(t^{1-\rho} \frac{d}{dt} \right)^n \left(^{\rho}\mathcal{I}_{0+}^{n-\alpha} y \right)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt} \right)^n \int_0^t \frac{s^{\rho-1} y(s)}{(t^{\rho} - s^{\rho})^{\alpha-n+1}} ds, \end{aligned} \tag{4}$$

where $n = [\alpha] + 1$, and $\rho > 0$. If the integrals exist.

REMARK 2 ([7]-[9]). As a basic example, we quote for $\alpha, \rho > 0$, and $\mu > -\rho$,

$$^{\rho}\mathcal{D}_{0+}^{\alpha} t^{\mu} = \frac{\rho^{\alpha-1}\Gamma\left(1 + \frac{\mu}{\rho}\right)}{\Gamma\left(1 - \alpha + \frac{\mu}{\rho}\right)} t^{\mu - \alpha\rho}.$$

Giving in particular:

$$^{\rho}\mathcal{D}_{0+}^{\alpha} t^{\rho(\alpha-m)} = 0, \text{ for each } m = 1, 2, \ldots, n.$$

In fact, for $\alpha, \rho > 0$, and $\mu > -\rho$, we have:

$$\begin{aligned} ^{\rho}\mathcal{D}_{0+}^{\alpha} t^{\mu} &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt} \right)^n \int_0^t s^{\rho+\mu-1} (t^{\rho} - s^{\rho})^{n-\alpha-1} ds \\ &= \frac{\rho^{\alpha-1}\Gamma\left(1 + \frac{\mu}{\rho}\right)}{\Gamma\left(1 + n - \alpha + \frac{\mu}{\rho}\right)} \left[n - \alpha + \frac{\mu}{\rho} \right] \cdots \left[1 - \alpha + \frac{\mu}{\rho} \right] t^{\mu - \alpha\rho} \qquad (5) \\ &= \frac{\rho^{\alpha-1}\Gamma\left(1 + \frac{\mu}{\rho}\right)}{\Gamma\left(1 - \alpha + \frac{\mu}{\rho}\right)} t^{\mu - \alpha\rho}. \qquad (6) \end{aligned}$$

If we put $m = \alpha - \frac{\mu}{\rho}$, we obtain from (5):

$$ {}^{\rho}\mathcal{D}_{0+}^{\alpha} t^{\rho(\alpha-m)} = \rho^{\alpha-1} \frac{\Gamma(\alpha - m + 1)}{\Gamma(n - m + 1)} (n-m)(n-m-1)\cdots(1-m) t^{-\rho m}. $$

So, for $m = 1, 2, \ldots, n$, we have ${}^{\rho}\mathcal{D}_{0+}^{\alpha} t^{\rho(\alpha-m)} = 0$, $\forall \alpha, \rho > 0$.

We present in the theorem below some properties of Katugampola fractional integrals and derivatives.

THEOREM 1 ([7]-[9]). Let $\alpha, \rho, c \in \mathbb{R}$, be such that $\alpha, \rho > 0$.
Then for any $f, g \in X_c^p [0, T]$, where $1 \leq p \leq \infty$, we have:
- Inverse property:

$$ {}^{\rho}\mathcal{D}_{0+}^{\alpha} \, {}^{\rho}\mathcal{I}_{0+}^{\alpha} f(t) = f(t), \text{ for all } \alpha \in (0, 1). \tag{7} $$

- Linearity property: for all $\alpha \in (0, 1)$, we have:

$$ \begin{cases} {}^{\rho}\mathcal{D}_{0+}^{\alpha} (f+g)(t) = {}^{\rho}\mathcal{D}_{0+}^{\alpha} f(t) + {}^{\rho}\mathcal{D}_{0+}^{\alpha} g(t), \\ {}^{\rho}\mathcal{I}_{0+}^{\alpha} (f+g)(t) = {}^{\rho}\mathcal{I}_{0+}^{\alpha} f(t) + {}^{\rho}\mathcal{I}_{0+}^{\alpha} g(t). \end{cases} \tag{8} $$

DEFINITION 7 (Equicontinuous).
Let E be a Banach space. Call a part P in $C(E)$ equicontinuous if

$$ \forall \varepsilon > 0, \ \exists \delta > 0, \ \forall u, v \in E, \ \ \forall \mathcal{A} \in P, \qquad \|u - v\| < \delta \Rightarrow \|\mathcal{A}(u) - \mathcal{A}(v)\| < \varepsilon. $$

THEOREM 2 (Ascoli-Arzela). Let E be a compact space.
If $\mathcal{A}$ is an equicontinuous, bounded subset of $C(E)$, then $\mathcal{A}$ is relatively compact.

DEFINITION 8 (Completely continuous). We say $\mathcal{A} : E \to E$ is completely continuous if for any bounded subset P of E, the set $\mathcal{A}(P)$ is relatively compact.

LEMMA 1 (Gronwall [6]). Let $u(t)$ and $g(t)$ be nonnegative, continuous functions on $0 \leq t \leq T$, for which the inequality:

$$ u(t) \leq \mu + \int_0^t g(s) u(s) \, ds, \ 0 \leq t \leq T, $$

holds, where μ is a nonnegative constant. Then:

$$ u(t) \leq \mu \exp\left(\int_0^t g(s) \, ds \right), \ 0 \leq t \leq T. $$

THEOREM 3 (Banach's fixed point [5]). Let P be a non-empty closed subset of a Banach space E, then any contraction mapping $\mathcal{A} : P \to P$ has a unique fixed point.

THEOREM 4 (Schauder's fixed point [5]). Let E be a Banach space, and P be a closed, convex and nonempty subset of E. Let $\mathcal{A} : P \to P$ be a continuous mapping such that $\mathcal{A}(P) \subset E$ is a relatively compact. Then $\mathcal{A}$ has at least one fixed point in P.

THEOREM 5 (Nonlinear Alternative of Leray-Schauder type [5]).
Let E be a Banach space with $P \subset E$ be a closed and convex. Assume U is a relatively open subset of P with $0 \in U$ and $\mathcal{A} : \bar{U} \to P$ is a compact map. Then either,
(i) $\mathcal{A}$ has a fixed point in $\bar{U}$; or
(ii) there is a point $u \in \partial U$ and $\mu \in (0,1)$ with $u = \mu \mathcal{A}(u)$.

3 Main results

Throughout the remaining of this paper T, p and c are real constants such that:

$$p \geq 1, \ c > 0, \text{ and } T \leq (pc)^{\frac{1}{pc}}.$$

In follows, we present some significant lemmas to show the principal theorems, we have:

LEMMA 2. Let $\alpha, \rho > 0$. If $u \in C[0,T]$, then:
(i) The fractional deferential equation ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) = 0$, has a unique solutions:

$$u(t) = C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)} + \cdots + C_n t^{\rho(\alpha-n)}, \text{ where } C_m \in \mathbb{R}, \text{ with } m = 1, 2, \ldots, n.$$

(ii) If ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u \in C[0,T]$ and $0 < \alpha \leq 1$, then:

$${}^{\rho}\mathcal{I}_{0+}^{\alpha} \ {}^{\rho}\mathcal{D}_{0+}^{\alpha} \ u(t) = u(t) + C t^{\rho(\alpha-1)}, \text{ for some constant } C \in \mathbb{R}. \tag{9}$$

PROOF. **(i)** Let $\alpha, \rho > 0$, from remark 2, we have:

$${}^{\rho}\mathcal{D}_{0+}^{\alpha} t^{\rho(\alpha-m)} = 0, \text{ for each } m = 1, 2, \ldots, n.$$

Then the fractional equation ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) = 0$, has a particular solution, as follows:

$$u(t) = C_m t^{\rho(\alpha-m)}, \ C_m \in \mathbb{R}, \text{ for each } m = 1, 2, \ldots, n. \tag{10}$$

Thus, the general solution of ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) = 0$, is a sum of particular solutions (10), i.e.

$$u(t) = C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)} + \cdots + C_n t^{\rho(\alpha-n)}, \ C_m \in \mathbb{R} \ (m = 1, 2, \ldots, n).$$

(ii) Let ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u \in C[0,T]$ be the fractional derivatives (4) of order $0 < \alpha \leq 1$.
If we apply the operator ${}^{\rho}\mathcal{D}_{0+}^{\alpha}$ to ${}^{\rho}\mathcal{I}_{0+}^{\alpha} \ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) - u(t)$, and use the properties (7), and (8) we have:

$$\begin{aligned} {}^{\rho}\mathcal{D}_{0+}^{\alpha} \left[{}^{\rho}\mathcal{I}_{0+}^{\alpha} \ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) - u(t) \right] &= {}^{\rho}\mathcal{D}_{0+}^{\alpha} \ {}^{\rho}\mathcal{I}_{0+}^{\alpha} \ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) - {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) \\ &= {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) - {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) = 0. \end{aligned}$$

After the step **(i)** we deduce there exists $C \in \mathbb{R}$, such that:

$${}^{\rho}\mathcal{I}_{0+}^{\alpha} \ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) - u(t) = C t^{\rho(\alpha-1)},$$

which implies the law of composition (9). The proof is complete.

Based on the previous lemma, we define the integral solution of the problem (1)-(2) .

LEMMA 3. Let $\alpha, \rho \in \mathbb{R}$, be such that $0 < \alpha \leq 1$, and $\rho > 0$.
We give $u, {}^{\rho}\mathcal{D}_{0+}^{\alpha} u \in C[0,T]$, and $f(t,u,v)$ is a continuous function. Then the problem (1)-(2) is equivalent to the fractional integral equation:

$$u(t) = \int_0^t G(t,s) f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(s)\right) ds, \tag{11}$$

where

$$G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1} \left(t^{\rho} - s^{\rho}\right)^{\alpha-1}. \tag{12}$$

PROOF. Let $0 < \alpha \leq 1$ and $\rho > 0$, we may apply lemma 2 to reduce the fractional equation (1) to an equivalent fractional integral equation.
By applying ${}^{\rho}\mathcal{I}_{0+}^{\alpha}$ to equation (1) we obtain:

$${}^{\rho}\mathcal{I}_{0+}^{\alpha}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) = {}^{\rho}\mathcal{I}_{0+}^{\alpha} f\left(t, u(t), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t)\right). \tag{13}$$

From lemma 2, we find easily:

$${}^{\rho}\mathcal{I}_{0+}^{\alpha}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) = u(t) + C t^{\rho(\alpha-1)},$$

for some $C \in \mathbb{R}$. Then, the fractional integral equation (13) , gives:

$$u(t) = {}^{\rho}\mathcal{I}_{0+}^{\alpha} f\left(t, u(t), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t)\right) - C t^{\rho(\alpha-1)}. \tag{14}$$

If we use the condition (2) in equation (14) we find:

$$u(0) = 0 = -C \lim_{t \to 0^+} t^{\rho(\alpha-1)} \Rightarrow C = 0.$$

Therefore, the problem (1)-(2) is equivalent to:

$$u(t) = \int_0^t G(t,s) f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(s)\right) ds, \tag{15}$$

where $G(t,s)$, which given by the equality (12) . The proof is complete.

Now, we will prove our first existence result for the problem (1)-(2) which is based on Banach's fixed point theorem.
We impose the following hypotheses:
(H1) $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.
(H2) For all $0 < \alpha \leq 1$, there exist two constants $\lambda, \beta > 0$, where $\beta < 1$ such that:

$$|f(t,u,v) - f(t,\tilde{u},\tilde{v})| \leq \lambda |u - \tilde{u}| + \beta |v - \tilde{v}|,$$

for any $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$ and $t \in [0,T]$.
(H3) There exists three positive functions $a, b, c \in C[0,T]$ such that:

$$|f(t,u,v)| \leq a(t) + b(t)|u| + c(t)|v| \text{ for all } t \in [0,T] \text{ and } u, v \in \mathbb{R}.$$

We denote:
$$M_0 = \frac{a^*}{1-c^*}, \text{ and } M_1 = \frac{b^*}{1-c^*},$$
where
$$a^* = \sup_{t\in[0,T]} a(t), \quad b^* = \sup_{t\in[0,T]} b(t), \quad c^* = \sup_{t\in[0,T]} c(t), \text{ with } c^* < 1.$$
In what follows, we present the principal theorems:

THEOREM 6. Assume the hypotheses (H1), (H2) hold. We give $0 < \alpha \leq 1$, and $\rho > 0$. If
$$\frac{\lambda T^{\rho\alpha}}{(1-\beta)\rho^{\alpha}\Gamma(\alpha+1)} < 1. \tag{16}$$
Then the problem (1)-(2) admits a unique solution on $[0,T]$.

PROOF. To begin the proof, we will transform the problem (1)-(2) into a fixed point problem. Define the operator $\mathcal{A} : C[0,T] \to C[0,T]$ by:
$$\mathcal{A}u(t) = \int_0^t G(t,s)\, f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\alpha}u(s)\right) ds. \tag{17}$$
Because the problem (1)-(2) is equivalent to the fractional integral equation (17), the fixed points of $\mathcal{A}$ are solutions of the problem (1)-(2).
Let $u, v \in C[0,T]$, be such that:
$${}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t) = f\left(t, u(t), {}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t)\right), \quad {}^{\rho}\mathcal{D}_{0+}^{\alpha}v(t) = f\left(t, v(t), {}^{\rho}\mathcal{D}_{0+}^{\alpha}v(t)\right).$$
Which implies that:
$$\mathcal{A}u(t) - \mathcal{A}v(t) = \int_0^t G(t,s)\left[f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\alpha}u(s)\right) - f\left(s, v(s), {}^{\rho}\mathcal{D}_{0+}^{\alpha}v(s)\right)\right] ds.$$
Then, for all $t \in [0,T]$
$$|\mathcal{A}u(t) - \mathcal{A}v(t)| \leq \int_0^t G(t,s) \left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u(s) - {}^{\rho}\mathcal{D}_{0+}^{\alpha}v(s)\right| ds. \tag{18}$$
By (H2) we have:
$$\begin{aligned} \left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t) - {}^{\rho}\mathcal{D}_{0+}^{\alpha}v(t)\right| &= \left|f\left(t, u(t), {}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t)\right) - f\left(t, v(t), {}^{\rho}\mathcal{D}_{0+}^{\alpha}v(t)\right)\right| \\ &\leq \lambda |u(t) - v(t)| + \beta \left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t) - {}^{\rho}\mathcal{D}_{0+}^{\alpha}v(t)\right|. \end{aligned}$$
Thus
$$\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t) - {}^{\rho}\mathcal{D}_{0+}^{\alpha}v(t)\right| \leq \frac{\lambda}{1-\beta}|u(t) - v(t)|.$$
From (18) we have:
$$|\mathcal{A}u(t) - \mathcal{A}v(t)| \leq \frac{\lambda}{1-\beta}\int_0^t G(t,s)\,|u(s) - v(s)|\, ds.$$

Then:

$$\left\| \mathcal{A}u-\mathcal{A}v\right\|_{\infty}\leq \frac{\lambda T^{\rho\alpha}}{\left(1-\beta\right)\rho^{\alpha}\Gamma\left(\alpha+1\right)}\left\| u-v\right\|_{\infty}.$$

This implies that by (16), $\mathcal{A}$ is a contraction operator.
As a consequence of theorem 3, using Banach's contraction principle [5], we deduce that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (1)-(2) on $[0,T]$.

THEOREM 7. Assume that hypotheses (H1)-(H3) hold. We give $0<\alpha\leq 1$, and $\rho>0$. If we put

$$\frac{M_1 T^{\rho\alpha}}{\rho^{\alpha}\Gamma\left(\alpha+1\right)}<1,$$

then the problem (1)-(2) has at least one solution on $[0,T]$.

PROOF. In the previous theorem, we already transform the problem (1)-(2) into a fixed point problem

$$\mathcal{A}u\left(t\right)=\int_0^t G\left(t,s\right)f\left(s,u\left(s\right),{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(s\right)\right)ds.$$

We demonstrate that $\mathcal{A}$ satisfies the assumption of Schauder's fixed point theorem 4. This could be proved through three steps:

Step 1. $\mathcal{A}$ is a continuous operator.
Let $\left(u_n\right)_{n\in\mathbb{N}}$ be a real sequence such that $\lim\limits_{n\to\infty}u_n=u$ in $C[0,T]$. Then for each $t\in[0,T]$,

$$\left|\mathcal{A}u_n\left(t\right)-\mathcal{A}u\left(t\right)\right|\leq\int_0^t G\left(t,s\right)\left|f\left(s,u_n\left(s\right),{}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(s\right)\right)-f\left(s,u\left(s\right),{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(s\right)\right)\right|ds, \tag{19}$$

where

$${}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(t\right)=f\left(t,u_n\left(t\right),{}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(t\right)\right),\ \text{and}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)=f\left(t,u\left(t\right),{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)\right).$$

As a consequence of (H2), we find easily ${}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\to{}^{\rho}\mathcal{D}_{0+}^{\alpha}u$ in $C[0,T]$. In fact we have:

$$\begin{aligned}\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(t\right)-{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)\right| &= \left|f\left(t,u_n\left(t\right),{}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(t\right)\right)-f\left(t,u\left(t\right),{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)\right)\right|\\ &\leq \lambda\left|u_n\left(t\right)-u\left(t\right)\right|+\beta\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(t\right)-{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)\right|.\end{aligned}$$

Thus:

$$\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(t\right)-{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)\right|\leq\frac{\lambda}{1-\beta}\left|u_n\left(t\right)-u\left(t\right)\right|$$

Since $u_n\to u$, then we get ${}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(t\right)\to{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)$ as $n\to\infty$ for each $t\in[0,T]$.
Now let $K_0>0$, be such that for each $t\in[0,T]$, we have:

$$\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(t\right)\right|\leq K_0,\quad\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)\right|\leq K_0.$$

Then, we have:

$$\begin{aligned}\left|\mathcal{A}u_n(t)-\mathcal{A}u(t)\right| &\leq \int_0^t G(t,s)\left|f\left(s,u_n(s),{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u_n(s)\right)-f\left(s,u(s),{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(s)\right)\right|ds \\ &\leq \int_0^t G(t,s)\left|{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u_n(s)-{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(s)\right|ds \\ &\leq \int_0^t G(t,s)\left[\left|{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u_n(s)\right|+\left|{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(s)\right|\right]ds \leq \int_0^t 2K_0G(t,s)\,ds.\end{aligned}$$

For each $t\in[0,T]$, the function $s\to 2K_0G(t,s)$ is integrable on $[0,t]$, then the Lebesgue dominated convergence theorem and (19) imply that:

$$\left|\mathcal{A}u_n(t)-\mathcal{A}u(t)\right|\to 0 \text{ as } n\to\infty,$$

and hence:

$$\lim_{n\to\infty}\left\|\mathcal{A}u_n-\mathcal{A}u\right\|_{\infty}=0.$$

Consequently, $\mathcal{A}$ is continuous.

Step 2. Let

$$r\geq \frac{M_0T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)-M_1T^{\rho\alpha}}.$$

We define:

$$P_r=\left\{u\in C[0,T]:\|u\|_{\infty}\leq r\right\}.$$

It is clear that P_r is a bounded, closed and convex subset of $C[0,T]$.
Let $u\in P_r$, and $\mathcal{A}:P_r\to C[0,T]$ be the integral operator defined in (17), then $\mathcal{A}(P_r)\subset P_r$.
In fact, for each $t\in[0,T]$, we have from (H3):

$$\left|{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(t)\right|=\left|f\left(t,u(t),{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(t)\right)\right|\leq a(t)+b(t)\left|u(t)\right|+c(t)\left|{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(t)\right|.$$

Then

$$\left|{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(t)\right|\leq \frac{a^*}{1-c^*}+\frac{b^*}{1-c^*}r=M_0+M_1r. \tag{20}$$

Thus

$$\begin{aligned}\left|\mathcal{A}u(t)\right| &\leq \int_0^t G(t,s)\left|f\left(s,u(s),{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(s)\right)\right|ds \leq \int_0^t G(t,s)\left[M_0+M_1r\right]ds \\ &\leq \frac{M_0T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}+\frac{M_1T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}r \\ &\leq \frac{\left[\rho^{\alpha}\Gamma(\alpha+1)-M_1T^{\rho\alpha}\right]\frac{M_0T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)-M_1T^{\rho\alpha}}+M_1T^{\rho\alpha}r}{\rho^{\alpha}\Gamma(\alpha+1)} \\ &\leq \frac{\left[\rho^{\alpha}\Gamma(\alpha+1)-M_1T^{\rho\alpha}\right]r+M_1T^{\rho\alpha}r}{\rho^{\alpha}\Gamma(\alpha+1)} \\ &\leq r.\end{aligned}$$

Then $\mathcal{A}(P_r) \subset P_r$.

Step 3. $\mathcal{A}(P_r)$ is relatively compact.
Let $t_1, t_2 \in [0,T]$, $t_1 < t_2$, and $u \in P_r$. Then

$$\begin{aligned} |\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| &= \left| \int_0^{t_2} G(t_2,s) f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(s)\right) ds \right. \\ &\quad \left. - \int_0^{t_1} G(t_1,s) f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(s)\right) ds \right| \\ &\le \int_0^{t_1} \left|\left[G(t_2,s) - G(t_1,s)\right] f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(s)\right)\right| ds \\ &\quad + \int_{t_1}^{t_2} G(t_2,s) \left|f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(s)\right)\right| ds \\ &\le (M_0 + M_1 r) \left[\int_0^{t_1} |(G(t_2,s) - G(t_1,s))| ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} G(t_2,s) ds \right]. \end{aligned} \tag{21}$$

We have:

$$\begin{aligned} G(t_2,s) - G(t_1,s) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1} \left[(t_2^{\rho} - s^{\rho})^{\alpha-1} - (t_1^{\rho} - s^{\rho})^{\alpha-1} \right] \\ &= \frac{-1}{\alpha\rho^{\alpha}\Gamma(\alpha)} \frac{d}{ds} \left[(t_2^{\rho} - s^{\rho})^{\alpha} - (t_1^{\rho} - s^{\rho})^{\alpha} \right] \end{aligned}$$

then

$$\int_0^{t_1} |(G(t_2,s) - G(t_1,s))| ds \le \frac{1}{\rho^{\alpha}\Gamma(\alpha+1)} \left[(t_2^{\rho} - t_1^{\rho})^{\alpha} + (t_2^{\rho\alpha} - t_1^{\rho\alpha}) \right]$$

we have also

$$\begin{aligned} \int_{t_1}^{t_2} G(t_2,s) ds &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} s^{\rho-1} (t_2^{\rho} - s^{\rho})^{\alpha-1} ds = \frac{-1}{\alpha\rho^{\alpha}\Gamma(\alpha)} \left[(t_2^{\rho} - s^{\rho})^{\alpha} \right]_{t_1}^{t_2} \\ &\le \frac{1}{\rho^{\alpha}\Gamma(\alpha+1)} (t_2^{\rho} - t_1^{\rho})^{\alpha}. \end{aligned}$$

Then (21) gives

$$|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| \le \frac{M_0 + M_1 r}{\rho^{\alpha}\Gamma(\alpha+1)} \left[2 (t_2^{\rho} - t_1^{\rho})^{\alpha} + (t_2^{\rho\alpha} - t_1^{\rho\alpha}) \right].$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.
As a consequence of steps 1 to 3 together, and by means of the Arzel-Ascoli theorem 2, we deduce that $\mathcal{A} : P_r \to P_r$ is continuous, compact and satisfies the assumption of Schauder's fixed point theorem 4. Then $\mathcal{A}$ has a fixed point which is a solution of the problem (1)-(2) on $[0,T]$.

Our next existence result is based on the nonlinear alternative of Leray-Schauder type.

THEOREM 8. Assume (H1)-(H3) holds. Then the problem (1)-(2) has at least one solution on $[0,T]$.

PROOF. Let $\alpha,\rho>0$, be such that $0<\alpha\leq 1$.
We shall show that the operator $\mathcal{A}$ defined in (17), satisfies the assumption of Leray-Schauder fixed point theorem 5. The proof will be given in several steps.

Step 1. Clearly $\mathcal{A}$ is continuous.

Step 2. $\mathcal{A}$ maps bounded sets into bounded sets in $C[0,T]$.
Indeed, it is enough to show that for any $\omega>0$ there exist a positive constant ℓ such that for each $u\in B_\omega=\{u\in C[0,T]:\|u\|_\infty\leq\omega\}$, we have $\|\mathcal{A}u\|_\infty\leq\ell$.
For $u\in B_\omega$, we have, for each $t\in[0,T]$,

$$|\mathcal{A}u(t)|\leq\int_0^t G(t,s)\left|f\left(s,u(s),{}^\rho\mathcal{D}_{0^+}^\alpha u(s)\right)\right|ds. \tag{22}$$

By $(H3)$, similarly of (20), for each $t\in[0,T]$, we have:

$$\left|f\left(t,u(t),{}^\rho\mathcal{D}_{0^+}^\alpha u(t)\right)\right|\leq M_0+M_1\omega.$$

Thus (22) implies that:

$$\|\mathcal{A}u\|_\infty\leq\frac{M_0T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}+\frac{M_1T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\omega=\ell.$$

Step 3. Clearly, $\mathcal{A}$ maps bounded sets into equicontinuous sets of $C[0,T]$.
We conclude that $\mathcal{A}:C[0,T]\to C[0,T]$ is continuous and completely continuous.

Step 4. A priori bounds.
We now show there exists an open set $U\subset C[0,T]$ with $u\neq\mu\mathcal{A}(u)$ for $\mu\in(0,1)$ and $u\in\partial U$. Let $u\in C[0,T]$ and $u=\mu\mathcal{A}(u)$ for some $0<\mu<1$.
Thus for each $t\in[0,T]$, we have:

$$u(t)\leq\mu\int_0^t G(t,s)\left|f\left(s,u(s),{}^\rho\mathcal{D}_{0^+}^\alpha u(s)\right)\right|ds.$$

By $(H3)$, for all solution $u\in C[0,T]$, of the problem (1)-(2), we have:

$$|u(t)|=\left|\int_0^t G(t,s)f\left(s,u(s),{}^\rho\mathcal{D}_{0^+}^\alpha u(s)\right)ds\right|\leq\int_0^t G(t,s)\left|{}^\rho\mathcal{D}_{0^+}^\alpha u(s)\right|ds.$$

Then for each $t\in[0,T]$, we have:

$$\left|{}^\rho\mathcal{D}_{0^+}^\alpha u(t)\right|=\left|f\left(t,u(t),{}^\rho\mathcal{D}_{0^+}^\alpha u(t)\right)\right|\leq a(t)+b(t)|u(t)|+c(t)\left|{}^\rho\mathcal{D}_{0^+}^\alpha u(t)\right|.$$

Then

$$\begin{aligned}\left|{}^\rho\mathcal{D}_{0^+}^\alpha u(t)\right| &\leq \frac{1}{1-c^*}\left(a^*+b^*|u(t)|\right)\\ &\leq M_0+M_1|u(t)|.\end{aligned}$$

Hence

$$|u(t)| \leq \frac{M_0 T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + \int_0^t M_1 G(t,s) |u(s)| \, ds.$$

After the Gronwall lemma [6], we have:

$$|u(t)| \leq \frac{M_0 T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \exp\left(\frac{M_1 T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)}\right).$$

Thus

$$\|u\|_\infty \leq \frac{M_0 T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \exp\left(\frac{M_1 T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)}\right) = M_2.$$

Let

$$U = \left\{u \in C[0,T] : \|u\|_\infty < M_2 + 1\right\}.$$

By choosing of U, there is no $u \in \partial U$, such that $u = \mu \mathcal{A}(u)$, for $\mu \in (0,1)$. As a consequence of Leray-Schauder's theorem 5, $\mathcal{A}$ has a fixed point u in U which is a solution to (1)-(2).

4 Examples

Example 1. Consider the following Cauchy problem:

$$\begin{cases} {}^1\mathcal{D}_{0^+}^{\frac{1}{2}} u(t) = \frac{\cos(t)}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\left[1+|u(t)|+\left|{}^1\mathcal{D}_{0^+}^{\frac{1}{2}} u(t)\right|\right]}, \ t \in \left[0, \frac{\pi}{4}\right], \\ u(0) = 0. \end{cases} \tag{23}$$

Set:

$$f(t,u,v) = \frac{\cos(t)}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\left[1+|u|+|v|\right]}, \ t \in \left[0, \frac{\pi}{4}\right], \ u, v \in \mathbb{R}.$$

Because $\sin(t)$, $\cos(t)$ are continuous positive functions $\forall t \in \left[0, \frac{\pi}{4}\right]$, the function f is jointly continuous. For any $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$ and $t \in \left[0, \frac{\pi}{4}\right]$, we have $\frac{\sqrt{2}}{2} \leq \cos(t) \leq 1$, and $0 \leq \sin(t) \leq \frac{\sqrt{2}}{2}$, then:

$$|f(t,u,v) - f(t,\tilde{u},\tilde{v})| \leq \frac{1}{\pi}\left(|u-\tilde{u}| + |v-\tilde{v}|\right).$$

Hence, the condition (H2) is satisfied with:

$$\lambda = \beta = \frac{1}{\pi} \simeq 0.3183 < 1.$$

It remains to show that the condition (16):

$$\frac{\lambda T^{\rho\alpha}}{(1-\beta)\rho^\alpha \Gamma(\alpha+1)} = \frac{\left(\frac{1}{\pi}\right)\left(\frac{\pi}{4}\right)^{\frac{1}{2}}}{\left(1-\frac{1}{\pi}\right)\Gamma\left(\frac{1}{2}+1\right)} = \frac{\sqrt{\pi}}{2(\pi-1)\Gamma\left(\frac{3}{2}\right)} \simeq 0.4669 < 1,$$

is satisfied. It follows from theorem 6 that the problem (23) has a unique solution.

Example 2. Consider the following Cauchy problem

$$\begin{cases} {}^1\mathcal{D}_{0^+}^{\frac{1}{2}} u(t) = \frac{\cos(t)\left[2+|u(t)|+\left|{}^1\mathcal{D}_{0^+}^{\frac{1}{2}} u(t)\right|\right]}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\left[1+|u(t)|+\left|{}^1\mathcal{D}_{0^+}^{\frac{1}{2}} u(t)\right|\right]}, \quad t \in \left[0, \frac{\pi}{4}\right], \\ u(0) = 0 \end{cases} \tag{24}$$

Set

$$f(t,u,v) = \frac{\cos(t)\left[2+|u|+|v|\right]}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\left[1+|u|+|v|\right]}, \; t \in \left[0, \frac{\pi}{4}\right], \; u, v \in \mathbb{R}.$$

Clearly, the function f is jointly continuous. For any $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$ and $t \in \left[0, \frac{\pi}{4}\right]$, we have

$$|f(t,u,v) - f(t,\tilde{u},\tilde{v})| \leq \frac{1}{\pi}\left(|u-\tilde{u}| + |v-\tilde{v}|\right).$$

Hence, the hypothese (H2) is satisfied with $\lambda = \beta = \frac{1}{\pi} < 1$. Also, we have:

$$|f(t,u,v)| \leq \frac{\cos(t)}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)}\left(2+|u|+|v|\right).$$

Thus, the hypothese (H3) is satisfied with

$$a(t) = \frac{2\cos(t)}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)}, \text{ and } b(t) = c(t) = \frac{\cos(t)}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)}.$$

We have also

$$a^* = \frac{2}{\pi}, \text{ and } b^* = c^* = \frac{1}{\pi} < 1, \text{ and } M_0 = \frac{2}{\pi-1}, \text{ and } M_1 = \frac{1}{\pi-1}.$$

And the condition

$$\frac{M_1 T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} = \frac{\left(\frac{1}{\pi-1}\right)\left(\frac{\pi}{4}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}+1\right)} = \frac{\sqrt{\pi}}{2(\pi-1)\Gamma\left(\frac{3}{2}\right)} \simeq 0.4669 < 1.$$

It follows from theorem 7 and theorem 8, that the problem (24) has at least one solution.

5 Conclusion

In this paper we have discussed the existence and uniqueness of solutions for a class of nonlinear implicit fractional differential equations with an initial condition, we made use of the Banach contraction principle, Schauder's fixed point theorem and the nonlinear alternative of Leray-Schauder type. The differential operator used is extended by Katugampola, which generalizes the Riemann-Liouville and the Hadamard fractional derivatives into a single form.

References

[1] R. Almeida, A. B. Malinowska, Odzijewicz, Fractional differential equations with dependence on the Caputo-Katugampola derivative, J. Comput Nonlinear Dynam. 11 (6) , 061017, (2016) .

[2] M. Benchohra, J. E. Lazreg, Existence results for nonlinear implicit fractional differential equations, Surveys in Mathematics and its Applications. 9 (2014), 79-92.

[3] K. Diethelm, The Analysis of Fractional Differential Equations, Springer (2010).

[4] A. M. A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, Nonlinear Anal. 33 (1998), 181-186.

[5] A. Granas, J. Dugundji, Fixed Point Theory, Springer-Verlag, New York (2003).

[6] T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Ann. Math. 20 (1918-1919), 292-296.

[7] U. N. Katugampola, A new approach to generalized fractional derivatives, Mathematical Analysis and Applications 6 (4) (2014), 1-15.

[8] U. N. Katugampola, Existence and uniqueness results for a class of generalized fractional differential equations, Department of Mathematics, University of Delaware, Newark DE 19716, USA. (2016).

[9] U. N. Katugampola, New approach to a generalized fractional integral, Applied Mathematics and Computation. 218 (3) (2011), 860-865.

[10] A. A. Kilbas, H. H. Srivastava , J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B.V, Amsterdam, 2006.

[11] A. A. Kilbas, J. J. Trujillo, Differential equations of fractional order: methods, results and problems I, Appl. Anal. 78 (2001), 153-192.

[12] A. A. Kilbas, J. J. Trujillo, Differential equations of fractional order: methods, results and problems II, Appl. Anal. 81 (2002), 435-493.

[13] K. S. Miller, Fractional differential equations, J. Fract. Calc. 3 (1993), 49-57.

[14] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[15] S. A. Murad, S. Hadid, Existence and uniqueness theorem for fractional differential equation with integral boundary condition, J. Frac. Calc. Appl. 3 (6) (2012), 1-9.

[16] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, Academic Press, New York, 1999.

[17] S. G. Samko, A. A. Kilbas, O.I. Marichev, Fractional Integral and Derivatives (Theory and Applications), Gordon and Breach, Switzerland, (1993).

[18] B. Zhanbing, L. Haishen, Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl. 311 (2005), 495-505.