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Semilinear Elliptic Equations with Singular Nonlinearities

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The journey was not short, nor was it meant to be easy.
The road was not always smooth with comforts, but I made itÃ‚Â… and here it ends.

To the one who carried my name, to my dear father, to the light that guided my path, and the lamp whose flame never faded in my heart, from whom I learned strength, determination, and pride.

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Today, I finally see the dream I once pursued, by the grace of Allah .

All praise is due to ***Allah*** *for all that He has granted me, and I ask Him to bless me wherever I may be.*

Marwa Bouafia

Contents

Notation

We introduce the necessary notations and definition which are used in the sequel.

$\mathbb{R}^N$	Euclidean space of dimension N, where N is a nonzero natural number.
Ω	open subset of $\mathbb{R}^N$.
X'	topological dual of X.
$(.,.)$	scalar product.
x	vector in $\mathbb{R}^N$, $x=(x_1,x_2,...,x_N)$, $x_i \in \mathbb{R}$, $1 \le i \le N$.
dx	Lebesgue measure in N-dimensional space.
$\partial\Omega = \Gamma$	boundary of Ω.
$\|E\|$	or mes (E) measure of the set E.
$B(x_0, r)$	open ball of radius r centered at $x0$.
$\|.\|$	Hilbert norm.
$\langle .,.\rangle$	duality bracket between X and its dual space.
$\int_\Omega f(x)dx$	integral of f in Ω with respect to the Lebesgue measure.
supp u	Support of the function u.
$\frac{\partial u}{\partial n}$	outward normal derivative.
div u	divergence of the vector u, $div u = \frac{\partial u}{dx_1} + \frac{\partial u}{dx_2} + ... + \frac{\partial u}{dx_N}$.
$\Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$	Laplacian of u.
$f_n \to f$	denotes that the sequence $\{f_n\}$ converge to f.
$f_n \rightharpoonup f$	denotes that the sequence $\{f_n\}$ converge weakly to f.
$L^p(\Omega)$	$= \{u : \Omega \to \mathbb{R}$ measurable and $\left(\int_\Omega \|u(x)\|^p dx\right)^{1/p} < +\infty$ such that $1 \le p < \infty\}$.
$\mathscr{H}$	Hilbert space.
$\|u\|_p$	$= \left[\int_\Omega \|u(x)\|^p dx\right]^{1/P} = \|u\|_{L^p}$.
$L^\infty(\Omega)$	$= \{u : \Omega \to \mathbb{R}$ measurable, $\exists M > 0 \mid \|u(x)\| \le M$a.e.$\}$.
$\\|u\\|_{L^\infty}$	$= \inf\{C : \|u(x)\| \le C$ a.e on $\Omega\}$.
q	Hölder conjugate of p : $q = \frac{p}{p-1}$ if $p > 1$ and $q = \infty$ if $p = 1$.
$L^q(\Omega) \subset L^p(\Omega), \quad \forall\, 1 \le p \le q < \infty$.	
$\\|u\\|_{C_c(\Omega)}$	$= \sup_{x\in\Omega} \|u(x)\| = \max_{x\in\Omega} \|u(x)\|$.
$L^p(\Omega)$	Lebesgue space.
$C_c(\Omega)$	space of continuous functions with compact.
$\mathscr{D}(\Omega)$	space of infinitely differentiable functions on Ω with compact support in Ω.
$W^{1,p}(\Omega)$	$= \left\{u \in L^p(\Omega) \mid \nabla u \in (L^p(\Omega))^N\right\}$.

$W_0^{1,p}(\Omega)$	$=\left\{u\in W^{1,p}(\Omega), \quad \text{with} \quad u=0 \quad \text{on} \quad \partial\Omega\right\}.$
a.e.	almost everywhere.

Introduction

Partial differential equations, which will be abbreviated as "PDE" in the following, constitute an important branch of applied mathematics and this field is becoming increasingly important in modern times. EDPs have great utility in mod- delusion of many phenomena of different natures such as physics, chemistry, biology and other sciences. In other words, EDPs intervene in many other fields: in chemistry to model reactions, in economics to study market behavior, in finance to study derivative products and in image processing to restore damage. These problems boil down to mathematical models usually written in the form

$$L(u) = f, \qquad (1)$$

where L is a definite operator from a function space E to a function space F, u is the unknown function and f a given function. PDEs probably first appeared during the 17th century. In- the field of PDEs has been enriched as the sciences have developed and especially physics. However, the systematic study of PDEs is much more recent and it is only during of the 20th century that mathematicians began to develop and indeed this theory has recently experienced a great theoretical and practical advancement. The mathematical analysis of these partial differential equations requires an appropriate choice of functional spaces and a clear definition of the notion of solution (the existence and sometimes uniqueness). One of the things to keep in mind about PDEs is to ask the question: can we get solutions explicitly?. So what mathematics can do on the other hand, is to say if one or more solutions exist, and to describe sometimes very precisely- ment some properties of these solutions. So usually an exact solution of problem (1) is not easy to find and sometimes we cannot even find the explicit solution. This leads to the introduction of the notion of the weak solution which appeared in 1934 in the work of Jean Leray. therefore, in most cases it is very difficult, if not impossible, to exhibit the solutions of a PDE. In some cases we manage to try to show that the problem admits a unique solution (it is said to be well-posed). But sometimes we can calculate numerical approximations of the solutions.

Moreover, the work presented in this dissertation concerns a particular case of the equations to partial derivatives of the elliptic type involving the divergent operator

$$Au = -\mathrm{div}(M(x)\nabla u).$$

We are interested in this thesis to study the following problem:

$$(p) \qquad \begin{cases} Au = \dfrac{f(x)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$), and $0 < \gamma \leq 1$ is a real number. We assume that f is a nonnegative function belonging to $L^1(\Omega)$, and $M(x)$ is a bounded elliptic matrix satisfying the following coercivity and boundedness conditions: there exist $0 < \alpha \leq \beta$ such that

$$\alpha|\xi|^2 \leq M(x)\xi \cdot \xi, \quad \text{and} \quad |M(x)| \leq \beta,$$

for every $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$.

We first illustrate the main difficulties that may arise when studying of the problem (P)

1) The irregularity in the case where $f \in L^1(\Omega)$; i.e. the second member of (P) does not belong to space $W^{-1,p'}(\Omega)$.

2) Singularity of the term $\frac{1}{u^\gamma}$

3) Ensuring positivity of the solution

4) Passing to the limit in nonlinear terms

Technique

We are going to prove the existence and regularity of the weak solutions of the problem (P). To do this, we will approximate the problem (P) by a sequence of approximate problems (P_n) given in L^∞ whose existence of the solution approximate is guaranteed by Theorem 2.1. Then we will prove some estimates uniform on the sequence of solutions of these problems (P_n) and their partial derivatives. Once this is done, the linearity of the operator with respect to the gradient as well as the boundedness and the continuity of a will make it possible to pass to the limit, thus finding the solution.

Thesis plan

This thesis is divided into three chapters as follows: The first chapter is dedicated to giving some basic definitions and results with functional analysis tools essential to the achievement of the objectives for the study of the (P) problem. For example we recall functional spaces (Lebesgue, Sobolev) , along with some of their properties , and we recall some definitions on the operators (himecontinuous, Monotone, Bounded and coercive).

In the second chapter, we take $\gamma = 0$, and we study the existence of weak solutions for a class of elliptic boundary value problems of elliptic operator with a regular and non-regular second member

- **Regular case:** In this case, we use Minty-Browder theorem to prove the existence of weak solutions $u \in W_0^{1,2}(\Omega)$, where $f \in L^\infty(\Omega)$
- **Irregular case:** In this case, we have $f \in L^1(\Omega)$, and to prove the existence of weak solutions $u \in W_0^{1,q}(\Omega)$ with $1 \leq q < \frac{N}{N-1}$,we follow the following steps:
  - Approximation of problem
  - Uniform Estimates
  - Passage to the limit

In the third chapter, we study the existence of weak solutions for a class of elliptic boundary value problems involving L^1 data and a singular nonlinearity of the type $\frac{f}{u^\gamma}$. By introducing an approximate problem, we first obtain the approximate solution $u_n \in H_0^1(\Omega)$ using the Leray–Schauder Fixed Point Theorem. Then, through truncation techniques, uniform estimates, and compactness arguments in the spirit of Boccardo–GallouÃ«t theory, we establish the existence of a weak solution. Finally, the strict positivity of the obtained solution is proved by applying the Strong Maximum Principle,The present work is inspired by the article of Lucio Boccardo [3].

Chapter 1

Preliminaries

This chapter presents some fundamental concepts related to the studied spaces, as well as some of their properties, based on the following references ([2, 4, 5, 9],and [10]).

1.1 Basic notions of functional analysis

1.1.1 Some definitions

Definition 1.1 (**Dual space**). *Let E be a normed space and let $f : E \to \mathbb{R}$ (or $\mathbb{C}$) be a linear functional. We say that f is* continuous *(or* bounded*) if there exists a constant $C > 0$ such that :*

$$|f(x)| \leq C\|x\|_E \quad \textit{for all } x \in E.$$

The dual space *of E, denoted by E', is the set of all linear and continuous functionals from E into $\mathbb{R}$ (or $\mathbb{C}$), that is,*

$$E' = \{f : E \to \mathbb{R}\ (or\ \mathbb{C}) \mid f \textit{ is linear and continuous}\}.$$

Moreover, for each $f \in E'$, we define the norm

$$\|f\|_{E'} = \sup_{\|x\|_E \leq 1} |f(x)|.$$

Remark 1.1.

- *The condition $|f(x)| \leq C\|x\|_E$ is equivalent to saying that f is continuous on E.*
- *The dual space E' collects all such continuous linear functionals defined on E.*
- *The norm $\|\cdot\|_{E'}$ is called the* operator norm.
- *Even when the space E is not complete, its dual space E' is always a Banach space when endowed with this norm.*

Definition 1.2 (**Bidual space**)**.** *Let E be a Banach space and let E' denote its dual space, i.e., the space of all continuous linear functionals on E.*

The bidual space of E, denoted by E'', is defined as the dual space of E', i.e., the space of all continuous linear functionals on E'.

For each element $F \in E''$, its norm is defined by

$$\|F\|_{E''} = \sup\{|F(f)| \,:\, f \in E',\ \|f\|_{E'} \leq 1\}.$$

There exists a natural (canonical) mapping

$$J : E \longrightarrow E'', \quad J(x)(f) = f(x) \quad \text{for all } x \in E, f \in E'.$$

This mapping is linear and isometric, i.e.,

$$\|J(x)\|_{E''} = \|x\|_E \quad \text{for all } x \in E.$$

Definition 1.3 (**Reflexive space**)**.** *Let E be a Banach space. We denote by E' its dual space and by E'' its bidual space. Consider the canonical mapping*

$$J : E \longrightarrow E'', \qquad J(x)(f) = f(x),\ \forall f \in E'.$$

The space E is said to be reflexive *if J is surjective, that is,*

$$J(E) = E'',\ (E = E'')$$

or equivalently, if E is isometrically isomorphic to E''.

Theorem 1.1. *[4] Let E be a reflexive Banach space. Then every bounded sequence in E admits a weakly convergent subsequence .*

Theorem 1.2. *[9] A Banach space E is reflexive if and only if its closed unit ball is weakly compact.*

Theorem 1.3. *[5] If H is a Hilbert space, then H is reflexive.*

Definition 1.4 (**Separable space**)**.** *Let E be a normed space. The space E is said to be* separable *if there exists a countable subset $D \subset E$ such that*

$$\overline{D} = E.$$

Theorem 1.4. *[5] Every separable normed space admits a countable dense subset.*

Theorem 1.5. *[9] If E' is separable, then E is separable .*

Theorem 1.6. *[4] If E is separable and reflexive, then the closed unit ball of E is weakly metrizable .*

1.1.2 Some notions of convergence

For more details on the notions of **strong convergence** and **weak convergence** (see [10])

Definition 1.5 (**Strong convergence**). *Convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ in a normed vector space $(E, \|\cdot\|_E)$ to an element $x \in E$, defined in the following way:*

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N} : n \geq n_0 \Rightarrow \|x_n - x\|_E \leq \varepsilon.$$

Or,

$$\lim_{n\to+\infty} \|x_n - x\|_E = 0.$$

We denote by $x_n \to x$ as $n \to +\infty$.

Definition 1.6 (**Weak convergence**). *We say that the sequence $(x_n)_{n\in\mathbb{N}} \subset E$ converges weakly to $x \in E$ if and only if :*

$$\langle f, x_n\rangle \to \langle f, x\rangle, \quad \forall f \in E'.$$

This weak convergence is written as $x_n \rightharpoonup x$, for $n \to +\infty$.

1.2 Functional spaces

In this section, we briefly mention some spaces and their most important properties and theories, which we will need in the following memory. For more details see [4].

1.2.1 Lebesgue spaces

Definition 1.7. *Let $p \in \mathbb{R}$ with $1 < p < \infty$; we set*

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R};\ f \text{ is measurable and } |f|^p \in L^1(\Omega)\}$$

with

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p\, dx\right]^{1/p}.$$

We shall check later on that $\|\ \|_p$ is a norm.

Definition 1.8. *We set*

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is measurable and there is a constant } C \\ \text{such that } |f(x)| \le C \text{ a.e. on } \Omega \end{array} \right\}$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf\{C;\ |f(x)| \le C \text{ a.e. on } \Omega\}.$$

The following remark implies that $\|\ \|_\infty$ is a norm:

Remark 1.2. *If $f \in L^\infty$ then we have:*

$$|f(x)| \le \|f\|_\infty \quad \text{a.e. on } \Omega.$$

For more (see [4])

1.2.2 Some Fundamental Inequalities

Notation: Let $1 \le p \le \infty$. We denote by p' the conjugate exponent of p, defined by the relation:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 1.7 (***Hölder's Inequality***)**.** *Let $1 \le p \le \infty$ and p' be its conjugate exponent. If $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, then $fg \in L^1(\Omega)$ and:*

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}.$$

Remark 1.3. *The case $p = 2$ corresponds to the **Cauchy-Schwarz inequality**:*

$$\int_\Omega |fg|\,dx \le \left(\int_\Omega |f|^2\,dx\right)^{1/2}\left(\int_\Omega |g|^2\,dx\right)^{1/2}.$$

Theorem 1.8 (***Minkowski Inequality***)**.** *Let $1 \le p < \infty$. If $f, g \in L^p(\Omega)$, then $f + g \in L^p(\Omega)$ and:*

$$\|f+g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

Definition 1.9 (***Young's Inequality***)**.** *Let $1 < p < \infty$ and p' be its conjugate exponent. For any non-negative real numbers a and b, we have:*

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \forall a, b \ge 0.$$

For more details, see [4, 2].

Theorem 1.9 (***Green's formulas***)**.** *Let $\Omega \subset \mathbb{R}^N$ be open, bounded and smooth. Let $u \in C^2(\Omega)$ and $v \in C^1(\Omega)$. Then,*

$$\int_\Omega (\Delta u) v\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \mathcal{V}} v\,d\sigma - \int_\Omega \nabla u \cdot \nabla v\,dx.$$

Where $\mathcal{V} = \mathcal{V}(x)$ is the outward normal to $\partial\Omega$ at x, $\frac{\partial u}{\partial \mathcal{V}}(x) = \nabla u(x) \cdot \mathcal{V}(x)$ and σ is the surface measure on $\partial\Omega$.

1.2.3 Some Results about Integration (Convergence Theorems)

For proofs and further details, see [4].

Theorem 1.10 (***Lebesgue's Dominated Convergence Theorem***)**.** *Let (f_n) be a sequence of functions in $L^1(\Omega)$ such that:*

- *$f_n(x) \to f(x)$ a.e. on Ω.*
- *There exists a function $g \in L^1(\Omega)$ such that for all $n \ge 1$:*

$$|f_n(x)| \le g(x) \quad \text{a.e. on } \Omega.$$

Then $f \in L^1(\Omega)$ and $\|f_n - f\|_{L^1(\Omega)} \to 0$ as $n \to \infty$.

Theorem 1.11 (***Converse of the Dominated Convergence Theorem***)**.** *Let (f_n) be a sequence in $L^p(\Omega)$ and let $f \in L^p(\Omega)$ such that:*

$$\|f_n - f\|_p \to 0 \quad \text{as } n \to \infty.$$

Then, there exists a subsequence (f_{n_k}) and a function $h \in L^p(\Omega)$ such that:

- $f_{n_k}(x) \to f(x)$ *a.e. on* Ω.

- $|f_{n_k}(x)| \leq h(x)$ *for all* k*, a.e. on* Ω.

Theorem 1.12 (**Giuseppe Vitali convergence theorem**)**.** *Let* $p \in [1, \infty[$ *and* (f_n) *be a sequence of functions in* $L^p(\Omega)$ *such that*

- $f_n \to f$ *a.e. in* Ω

- (f_n) *equal-integration in* Ω*, i.e.* $\forall \varepsilon > 0, \exists \delta > 0, \forall E \subset \Omega$ *with* $|E| \leq \delta$*, such that*

$$\int_E |f_n(x)|^p \, dx \leq \varepsilon, \quad \forall n \in \mathbb{N}$$

Or

$$\limsup_{|E| \to 0} \int_E |f_n(x)|^p \, dx = 0, \quad \forall n \in \mathbb{N}$$

Then

$$f \in L^p(\Omega) \quad \text{and} \quad f_n \to f \quad \text{in} \quad L^p(\Omega) \quad \text{strongly.}$$

Lemma 1.1 (***Fatou's Lemma***)**.** *Let* (f_n) *be a sequence of measurable functions in* $L^1(\Omega)$ *satisfying:*

- $f_n \geq 0$ *a.e. on* Ω *for all* n.

- $\liminf_{n \to \infty} \int_\Omega f_n \, dx < \infty$.

For almost all $x \in \Omega$*, we set* $f(x) = \liminf_{n \to \infty} f_n(x)$*. Then* $f \in L^1(\Omega)$ *and:*

$$\int_\Omega f \, dx \leq \liminf_{n \to \infty} \int_\Omega f_n \, dx.$$

Lemma 1.2. *[12] Let* f *be a strictly positive measurable function. Then, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every measurable* $A \subset \Omega$*, we have*

$$\int_A f \, dx \leq \delta \quad \Rightarrow \quad meas(A) \leq \varepsilon. \tag{1.1}$$

Lemma 1.3. *[11] Let* (u_n) *be a sequence of measurable functions of in* Ω *and* $\mathbb{R}^N$*. If* (u_n) *of Cauchy in measure then there exists a sub-sequence of* (u_n) *converge almost everywhere*

Proposition 1.1.

- *The Lebesgue spaces $L^p(\Omega)$ are reflexive for $1 < p < \infty$. (See Theorem 3.18 in [5]).*
- *The spaces $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive. (See [4]).*
- *The Lebesgue spaces $L^p(\Omega)$ are separable for $1 \le p < \infty$. (See [4]).*
- *The space $L^\infty(\Omega)$ is not separable. (See [9]).*

1.2.4 Sobolev spaces

Let $\Omega = (a, b)$ be an open interval, possibly unbounded, and let $p \in \mathbb{R}$ with $1 \le p \le \infty$.

Definition 1.10 ($W^{m,p}(\Omega)$ **space**)**.** *Given an integer $m \ge 2$ and a real number $1 \le p \le \infty$ we define by induction the space*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega)\,;\, D^\alpha u \in L^p(\Omega),\ \forall\, \alpha \le m\},$$

We also set

$$H^m(\Omega) = W^{m,2}(\Omega).$$

The space $W^{m,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{\alpha=1}^{m} \|D^\alpha u\|_{L^p(\Omega)}.$$

Definition 1.11 ($W^{1,p}(\Omega)$ **space**)**.** *The Sobolev space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \left\{u \in L^p(\Omega)\,;\, \exists g \in L^p(\Omega) \text{ such that } \int_\Omega u\varphi' = -\int_\Omega g\varphi \quad \forall \varphi \in C_c^1(\Omega)\right\}.$$

For $u \in W^{1,p}(\Omega)$ we denote $u' = g$.

The norm on $W^{1,p}(\Omega)$ is defined by:

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

Or,

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |u(x)|^p\, dx + \int_\Omega |\nabla u(x)|^p\, dx\right)^{\frac{1}{p}}.$$

and for $p = \infty$

$$\|u\|_{W^{1,\infty}(\Omega)} = \max\left(\|u\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}.\right)$$

In particular,

for $p = 2$*, we have* $H^1(\Omega) = W^{1,2}(\Omega)$ *and:*

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega |u(x)|^2\,dx + \int_\Omega |\nabla u(x)|^2\,dx\right)^{\frac{1}{2}}.$$

Theorem 1.13. *([4]) The space* $W^{1,p}(\Omega)$ *is a Banach space for* $1 \le p \le \infty$ *. It is reflexive for* $1 < p < \infty$ *and separable for* $1 \le p < \infty$ *.*

Definition 1.12 ($W_0^{1,p}(\Omega)$ **space).** *Let* $\Omega \subset \mathbb{R}^n$ *be an open bounded domain and* $1 \le p < \infty$*. The Sobolev space* $W_0^{1,p}(\Omega)$ *is defined as the closure of* $C_c^\infty(\Omega)$ *in* $W^{1,p}(\Omega)$*, that is,*

$$W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}}.$$

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) \,:\, u = 0 \text{ on } \partial\Omega\}.$$

The norm on $W_0^{1,p}(\Omega)$ *is given by*

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}.$$

1.2.5 Sobolev Embeddings

Notation: We define

$$p^* = \frac{Np}{N-p} \quad \text{if } p < N,$$

which is called the *critical Sobolev exponent.* When $p \ge N$, the exponent p^* is not defined.

Theorem 1.14 (Continuous Injection). *Let* $1 \le p \le \infty$*, we suppose that* Ω *is an open set of class* C^1*, a bounded frontier, and we take* $\Omega = \mathbb{R}_+^N$*.*

1) $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [1, p^*[\quad$ *if* $p < N$*.*

2) $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [p, \infty[\quad$ *if* $p = N$*.*

3) $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \quad$ *if* $p > N$*.*

Theorem 1.15 (**Compact Injection (Rellich Kondrachon)**). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^1.*

1) $W^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [1, p^*[\quad$ *if* $p < N$.

2) $W^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [p, \infty[\quad$ *if* $p = N$.

3) $W^{1,p}(\Omega) \subset C(\overline{\Omega}) \quad$ *if* $p > N$.

Definition 1.13 (**Poincare's inequality**). *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then there exists a constant $C > 0$ such that:*

$$\|u\|_{W^{1,p}(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega).$$

In other words, on $W_0^{1,p}(\Omega)$, the quantity $\|\nabla u\|_{L^p(\Omega)}$ is a norm equivalent to the $W^{1,p}$ norm.

1.3 Operators

Let V and W be two vector spaces. An operator is a mapping

$$A : V \to W$$

which associates to each element $x \in V$ a unique element $A(x) \in W$. If $V = W$, the mapping A is called an operator on V.

Exemple 1.1. *The operator*

$$A(u) = -\Delta u$$

can be viewed as an operator from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$.

1.3.1 Hemicontnuous Operators

Definition 1.14. *Let V is a reflexive and separable Banach space, and A is an application from $V \to V'$. We say that:*

A is hemicontinuous if for all $u, v, w \in V$, the application $\lambda \mapsto \langle A(u + tv), w\rangle$ is continuous

Exemple 1.2. *The operator* $A: H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ *defined by*

$Au = -\Delta u$ *is hemicontinuous. Indeed, let* $u, v, w \in H_0^1(\Omega)$ *and* $\lambda \in \mathbb{R}$ *:*

$$\begin{aligned}
\langle A(u+\lambda v), w\rangle &= \int_\Omega \nabla(u+\lambda v).\nabla w \\
&= \int_\Omega \nabla u.\nabla w + \lambda \int_\Omega \nabla v \nabla w \\
&= a + \lambda b.
\end{aligned}$$

This shows that $\lambda \longrightarrow \langle A(u+tv), w\rangle$ *is continuous.*

1.3.2 Monotone Operators

Definition 1.15. *Let* V *be a reflexive and separable Banach space, and* A *be an application of* $V \to V'$ *(generally non-linear). We say that:*

i *A is monotone :*

$$\langle Au - Av, u - v\rangle \geq 0, \ \forall u, v \in V.$$

ii *A is strictly monotone if moreover :*

$$\langle Au - Av, u - v\rangle > 0, \ \forall u, v \in V, \quad u \neq v$$

Exemple 1.3. *We define* $Au = -\Delta u$. *The operator* A *maps* $H_0^1(\Omega)$ *into its dual* $H^{-1}(\Omega)$. *It is monotone. Indeed, for all* $u, v \in H_0^1(\Omega)$.

we have:

$$\begin{aligned}
\langle Au - Av, u - v\rangle &= \int_\Omega \nabla(u-v).\nabla(u-v) \\
&= \|u-v\|^2_{H_0^1(\Omega)} \\
&\geq 0.
\end{aligned}$$

1.3.3 Bounded Operators

Definition 1.16. *Let* V *be a Banach space and* V' *its dual, let* $A: V \to V'$ *be an operator. We say that* A *is bounded if it maps every bounded set in* V *to a bounded set in* V'*; i.e*

$$\forall \rho > 0, \quad \exists C_\rho > 0: \quad A(B_V(0,\rho)) \subset B_{V'}(0, C_\rho).$$

where $B_V(0,\rho)$ *designates the open ball in* V *with center* 0 *and radius* $\rho > 0$ *and* $B_{V'}(0, C_\rho)$ *designates the open ball in* V' *with center* 0 *radius* $C_\rho > 0$.

Exemple 1.4. *The operator $Au = -\Delta_p$ is bounded from in $V' = H^{-1}(\Omega)$*

From the expression of the norm in a dual space, let $\rho > 0$, for $u \in B_V(0,\rho)$, we can write:

$$\| Au \|_{V'} = \sup_{\substack{v \in V \\ \|v\| \le 1}} |\langle Au, v\rangle| = \sup_{\substack{v \in V \\ \|v\| \le 1}} \left| \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla v \right|.$$

So,

$$\begin{aligned}
\left| \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla v \right| &\le \int_\Omega |\nabla u|^{p-1} . |\nabla v| \\
&\le \left(\int_\Omega |\nabla u||^p \right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla v|^p \right)^{\frac{1}{p}} \\
&= \| u \|_V^{p-1} \| v \|_V \\
&\le \rho^{p-1}
\end{aligned}$$

Hence $\| Au \|_{V'} \le \rho^{p-1}$. this shows that $A(B_V(0,\rho)) \subset B_{V'}(0,\rho^{p-1})$

1.3.4 Coercive Operators

Definition 1.17. *An operator $A : V \to V'$, we say that A is coercive if:*

$$\lim_{\|u\|_V \to +\infty} \frac{\langle Au, u\rangle}{\|u\|_V} = +\infty.$$

Exemple 1.5. *The operator $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ defined by*

$Au = -\Delta u$ is coercive, we have :

$$\frac{\langle Au, u\rangle}{\|u\|_{H_0^1}} = \|u\|_{H_0^1}^{p-1} \to +\infty. \ whene, \ \|u\|_{H_0^1} \to +\infty$$

Chapter 2

Existence of Solutions for Elliptic Equations with Irregular Data

In this chapter, we study the existence of weak solutions for a class of elliptic boundary value problems of elliptic operators with a regular and non-regular second member. i.e, not belonging to a dual space. we consider second members in L^1.

2.1 The problem (P_0)

We are interested in problems of this type.

$$(P_0) \qquad \begin{cases} -\mathrm{div}\,(M(x)\nabla u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$, where M is a bounded elliptic matrix which satisfies, for some positive constants α, β, for every $\xi \in \mathbb{R}$

$$\alpha|\xi|^2 \leq M(x)\xi\cdot\xi, \qquad |M(x)| \leq \beta. \tag{2.1}$$

we have two cases for f, if $f \in L^\infty(\Omega)$, and if $f \in L^1(\Omega)$.

2.2 Regular case

In this section, we study the existence of weak solutions to problem (P_0) with a regular second member, where $f \in L^\infty(\Omega)$

Theorem 2.1 (**Minty-Browder theorem**). *Let $A : V \to V'$ be an operator , and $f \in V'$ suppose that : A is Hemicontinuous, Monoton, Bounded and Coercive, then for every $f \in V'$ there exists a solution $u \in V$ such that : $A(u) = f$*

Lemma 2.1. *Let $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ be an operator defined by :*

$$Au = -div(M(x)\nabla u)\,,$$

The operator is Hemicontinuous, Monoton, Bounded, and Coercive. Then problem (P_0) has a unique solution $u \in W_0^{1,2}(\Omega)$

Proof.

1. **Hemicontinuity of** A:

   Let $u, v, w \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$:

$$\begin{aligned} \langle A(u+\lambda v), w\rangle &= \int_\Omega M(x)\nabla(u+\lambda v).\nabla w \\ &= \int_\Omega M(x)\nabla u.\nabla w + \lambda \int_\Omega M(x)\nabla v \nabla w \\ &= a + \lambda b. \end{aligned}$$

   This shows that $\lambda \longrightarrow \langle A(u+tv), w\rangle$ is continuous then.

   Therefore, A is hemicontinuous

2. **Monotoncity of** A:

   For all $u, v \in H_0^1(\Omega)$, we have:

$$\langle Au - Av, u - v\rangle = \int_\Omega M(x)\nabla(u-v).\nabla(u-v)$$

   by condition (2.1)

$$\begin{aligned} \langle Au - Av, u - v\rangle &\geq \alpha \| u - v\|^2_{H_0^1(\Omega)} \\ &\geq 0. \end{aligned}$$

   Therefore, A is monotone

3. **Boundedness of** A:

   Let $u, v \in W_0^{1,2}(\Omega)$, and $\| v\|_{W_0^{1,2}} \leq 1$, then:

$$\| u\|_{W_0^{1,2}} \leq \rho \Longrightarrow \| Au\|_{W^{-1,2}(\Omega)} \leq \rho'$$

We have

$$\begin{aligned}
|\langle Au, v\rangle| &= \left|\int_\Omega M(x)(\nabla u\cdot\nabla v)\,dx\right| \\
. &\leq \int_\Omega |M(x)||\nabla u||\nabla v|\,dx \\
&\leq \beta\int_\Omega |\nabla u|\,|\nabla v|\,dx.
\end{aligned}$$

By Hölder's inequality,

$$|\langle Au, v\rangle| \leq \beta\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}.$$

Since $\|v\|_{W_0^{1,2}} \leq 1$, it follows that

$$|\langle Au, v\rangle| \leq \beta\|\nabla u\|_{L^2(\Omega)}.$$

Hence

$$\begin{aligned}
\|Au\|_{W^{-1,2}(\Omega)} &= \sup_{\|v\|_{W_0^{1,2}}\leq 1} |\langle Au, v\rangle| \\
&\leq \beta\|u\|_{W_0^{1,2}(\Omega)} \\
&\leq \beta\rho = \rho'.
\end{aligned}$$

Therefore, A is bounded.

4. **Coercivity of** A:

   Let $u \in H_0^1(\Omega)$ we have :

$$\langle Au, u\rangle = \int_\Omega M(x)\nabla u.\nabla u\,dx$$

by condition (2.1)

$$\begin{aligned}
\langle Au, u\rangle &\geq \alpha\int_\Omega |\nabla u|^2\,dx \\
&\geq \alpha\,\|u\|^2_{H_0^1(\Omega)}
\end{aligned}$$

then

$$\lim_{\|u\|_{H_0^1(\Omega)}\to+\infty} \frac{\langle Au, u\rangle}{\|u\|_{H_0^1(\Omega)}} \geq \lim_{\|u\|_{H_0^1(\Omega)}\to+\infty} \alpha\,\|u\|_{H_0^1(\Omega)} = +\infty$$

Therefore, A is coercive

□

Remark 2.1. *In this case, we can also prove the existence of the weak solution by applying Lax-Milgram's theorem.*

2.3 Irregular case

In this case, we study the existence of a weak solution to problem (P_0) with a non-regular second member, where $f \in L^1(\Omega)$

2.3.1 Weak solutions

Definition 2.1. *says that u is a weak solution to problem* (P_0) *if:* $u \in W_0^{1,1}(\Omega)$ *and*

$$\int_\Omega M(x)\nabla u \nabla\varphi\, dx = \int_\Omega f\varphi\, dx,\ \forall\varphi \in W_0^{1,\infty}(\Omega). \tag{2.2}$$

The result proved in this chapter is the following theorem

Theorem 2.2. *Let* $f \in L^1(\Omega)$. *Then problem* (P_0) *has at least one weak solution u which is in* $W_0^{1,q}(\Omega)$*, for all*

$$q \in \left[1, q_c = \frac{N}{N-1}\right[.$$

2.3.2 Why the condition $1 \le q < \frac{N}{N-1}$.

Model Example

Let $\Omega = B(0,1)$, be the open Euclidean unit ball centered at zero with radius 1. find $u : \overline{\Omega} \to \mathbb{R}$ as a solution of

$$(P_{mod}) \qquad \begin{cases} \Delta u = -\mathrm{div}(\nabla u) &= \delta \text{ in } D'(\Omega); \\ u &= 0 \text{ on } \partial\Omega; \end{cases}$$

where $p = 2 > 1$ is a real number and δ is the Dirac distribution at the origin.

the problem (P_{mod}) has a solution (model) u_{mod} is given by (see [1]):

$$u_{mod} = \begin{cases} C(|x|^{1-N}) & \text{if } N \neq 2; \\ C\log|x| & \text{if } N = 2; \end{cases}$$

In which space -or set- should we look for a solution of (P_0)?

In the case $p \neq N$, we have

$$|\nabla u_{mod}| = C_1|x|^{1-N}$$

then

$$\begin{aligned} \int_\Omega |\nabla u_{mod}| dx &= C_1 \int_{B(0,1)} |x|^{1-N} dx \\ &= C_2 \int_0^1 r^{1-N} r^{N-1} dr \\ &= C_2 \int_0^1 r^0 dr. \end{aligned}$$

This integral is finite because $1-N+(N-1)=0>-1$. This calculation shows that:

$$u_{mod} \quad \text{is in} \quad W_0^{1,1}(B(0,1)).$$

The critical value of the exponent $q \geq 1$, so that $u_{mod} \in W_0^{1,q}(B(0,1))$ we have :

$$\int_\Omega |\nabla u_{mod}|^q dx = C_1 \int_0^1 r^{q(1-N)+N-1} dr dx.$$

the integral will be finite if $q(1-N)+N-1>-1$, which gives

$$u_{mod} \in W_0^{1,q}(B(0,1)) \quad \text{for all } q \text{ such that} \quad 1 \leq q < \frac{N}{N-1}.$$

Now, the proof of Theorem (2.2) needs several steps: First, we approximate the problem (P_0) with the sequence of problems (P_{0n}) having smooth solutions (u_n). Then, after deriving uniform estimates on u_n, we pass to the limit.

2.3.3 Approximation of (P_0)

Consider the following approximated problem :

$$(P_{0n}) \quad \begin{cases} -\operatorname{div}(M(x)\nabla u_n) = f_n(x), & \text{in } \Omega, \\ u_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

i.e., satisfying

$$\int_\Omega M(x)\nabla u_n \nabla\varphi\, dx = \int_\Omega f_n \varphi\, dx,\ \forall \varphi \in W_0^{1,\infty}(\Omega). \tag{2.4}$$

with f_n be a sequence of bounded functions defined in Ω defined by

$$f_n(x) = \frac{f(x)}{1+\frac{|f(x)|}{n}}. \tag{2.5}$$

such that

$$\begin{cases} \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, \\ |f_n| \leq n, \\ f_n \to f \ \text{ strongly in } \ L^1(\Omega) \quad . \end{cases} \tag{2.6}$$

The existence of a solution u_n in $W_0^{1,2}(\Omega)$ of the problem (P_{0n}) is classical, since $f_n \in W^{-1,2}(\Omega)$ and the fact that the operator $Au_n = -\operatorname{div}(M(x)\nabla u_n)$ Hemicontinuous, Monoton, Bounded and Coercive (see theorem 2.1) , i.e, there exists a function $u_n \in W_0^{1,2}(\Omega)$ solution of problem (P_{0n}).

2.3.4 Uniform Estimates

Since we only have $\|f_n\|_{L^1(\Omega)} \leq C, \quad \forall n \geq 1$ in order to to bound the term $\int_\Omega f_n \varphi$ uniformly with respect to n, we need to choose a test function

$$\varphi_n = \quad \text{a function of } u_n \text{ such that } \quad \|\varphi_n\|_{L^\infty(\Omega)} \leq C, \forall n \geq 1 \quad \text{with} \quad \varphi_n \in W_0^{1,\infty}(\Omega)$$

To choose the test functions, we use the following result due to Stampacchia :

Lemma 2.2 (**Stampacchia**)**.** *Let $T : \mathbb{R} \to \mathbb{R}$ be a globally lipschitz function,i.e.*

$$\exists C > 0 \quad \textit{such that} \quad |T(s) - T(t)| \leq C|s - t|, \quad \forall s, t \in \mathbb{R},$$

such that $T(0) = 0$. then, $\forall v \in W_0^{1,p}(\Omega)$ with $1 \leq p \leq \infty$ we have :

$$T(v) \in W_0^{1,p}(\Omega) \quad \textit{and} \quad \nabla T(v) = T'(v)\nabla v \quad \textit{in} \quad \mathscr{D}'(\Omega) \quad \textit{and} \quad almosteverywhere \quad \textit{in} \quad \Omega$$

Exemple 2.1. *Let $k > 0$. The truncation at levels $-k$ and k is defined by the function T_k of $\mathbb{R}$ from $\mathbb{R}$ given by*

$$T_k(r) = \begin{cases} k, & \textit{if } r \geq k, \\ r, & \textit{if } |r| < k, \\ -k, & \textit{if } r \leq -k. \end{cases}$$

It can be verified that the function T_k is a globally Lipschitz function,

We can verify that the function T_k is a globally Lipschitz function satisfying $|T_k(r)| \leq k$ and $|T_k(r)| \leq |r|$.

Lemma 2.3. *Let $\delta > 0$, there exists a constant C_δ independent of n such that*

$$\int_\Omega \frac{|\nabla u_n|^2}{(1 + |u_n|)^{1+\delta}} dx \leq C_\delta, \quad \forall n \geq 1.$$

Proof. Let $\delta > 0$. the function ψ_δ given by

$$\psi_\delta(t) = \int_0^t \frac{d\sigma}{(1 + |\sigma|)^{1+\delta}} = \frac{1}{\delta}\left(1 - \frac{1}{(1 + |t|)^\delta}\right) \text{sign}(t), \quad t \in \mathbb{R}$$

is globally Lipschitz $\psi_\delta(0) = 0$ and $|\psi_\delta(t)| \leq \frac{1}{\delta}, \quad \forall t \in \mathbb{R}$. Taking $\psi_\delta(u_n)$ as a test function in (P_{0n}), we obtain:

$$\langle Au_n, \psi_\delta(u_n)\rangle = \langle f_n, \psi_\delta(u_n)\rangle, \quad Au_n = -\text{div}(M(x)\nabla u_n)$$

so that

$$\int_\Omega M(x)\nabla u_n \nabla \psi_\delta(u_n) dx = \int_\Omega f_n \psi_\delta(u_n) dx$$

Using the fact that $\nabla\psi_\delta(u_n) = \nabla u_n \psi'_\delta(u_n)$ with $\psi'_\delta(u_n) = \frac{1}{(1+|u_n|)^{1+\delta}}$, we have

$$\begin{aligned}\int_\Omega M(x)\nabla u_n \nabla\psi_\delta(u_n)dx &= \int_\Omega M(x)\nabla u_n \nabla u_n \psi'_\delta(u_n)dx\\ &= \int_\Omega M(x)\nabla u_n \frac{\nabla u_n}{(1+|u_n|)^{1+\delta}}dx,\end{aligned}$$

Therefore,

$$\int_\Omega M(x)\nabla u_n \frac{\nabla u_n}{(1+|u_n|)^{1+\delta}}dx = \int_\Omega f_n \psi_\delta(u_n)dx \leq \frac{1}{\delta}\|f_n\|_{L^1(\Omega)},$$

and by (2.1) we have,

$$\int_\Omega M(x)\nabla u_n \frac{\nabla u_n}{(1+|u_n|)^{1+\delta}}dx \geq \alpha \int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{1+\delta}}dx$$

then, we obtain

$$\begin{aligned}\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{1+\delta}}dx &\leq \frac{1}{\alpha\delta}\|f_n\|_{L^1(\Omega)}, \quad \forall n \geq 1\\ &\leq \frac{1}{\alpha\delta}\|f\|_{L^1(\Omega)} = C.\end{aligned}$$

□

Lemma 2.4. *Let q be such that $q \in \left[1, \frac{N}{N-1}\right[$. Then, the sequence (∇u_n) is bounded in $L^q(\Omega)$, i.e., there exists a constant $C = C(q) > 0$ such that:*

$$\int_\Omega |\nabla u_n|^q\, dx \leq C, \quad \forall n \geq 1. \tag{2.7}$$

Proof. Note that for $N > 1$:

$$\frac{N}{N-1} - 2 = \frac{2-N}{N-1} \leq 0 \implies 1 \leq q < \frac{N}{N-1} \leq 2 \leq N \quad (\text{for } N \geq 2).$$

Let q^* be the Sobolev exponent associated with q, defined by $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{N}$, which gives $q^* = \frac{Nq}{N-q}$. By choosing δ such that:

$$\delta = q^* \frac{2-q}{q} - 1 \iff q^* = (1+\delta)\frac{q}{2-q},$$

Lemma 2.3 implies that there exists a constant $C = C(\delta) > 0$ such that:

$$\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{1+\delta}}\, dx \leq C, \quad \forall n \geq 1. \tag{2.8}$$

We can write the integral of interest as follows:

$$y_n = \int_\Omega |\nabla u_n|^q\, dx = \int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^{(1+\delta)q/2}}(1+|u_n|)^{(1+\delta)q/2}\, dx.$$

Applying Hölder's inequality with exponents $\frac{2}{q}$ and $\frac{2}{2-q}$, we obtain:

$$
\begin{aligned}
y_n &\le \left(\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{1+\delta}}\,dx\right)^{q/2} \left(\int_\Omega (1+|u_n|)^{(1+\delta)\frac{q}{2-q}}\,dx\right)^{1-q/2} \\
&\le C^{q/2}\left(\int_\Omega (1+|u_n|)^{q^*}\,dx\right)^{1-q/2}. \qquad (2.9)
\end{aligned}
$$

Using the elementary inequality $(a+b)^p \le 2^{p-1}(a^p+b^p)$ for $p\ge 1$ and $a,b\ge 0$, and the fact that $q^*>1$, we have:

$$
\begin{aligned}
y_n &\le C^{q/2}(2^{q^*-1})^{1-q/2}\left(\int_\Omega 1\,dx + \int_\Omega |u_n|^{q^*}\,dx\right)^{1-q/2} \\
&\le C'\left\{|\Omega|^{1-q/2} + \left(\int_\Omega |u_n|^{q^*}\,dx\right)^{1-q/2}\right\}.
\end{aligned}
$$

By the Sobolev-Poincaré inequality, we have:

$$
\|u_n\|_{L^{q^*}(\Omega)} = \left(\int_\Omega |u_n|^{q^*}\,dx\right)^{1/q^*} \le C_{q,N}\|\nabla u_n\|_{L^q(\Omega)} = C_{q,N}\, y_n^{1/q}.
$$

Substituting this back into the inequality for y_n, we get:

$$
y_n \le a + b y_n^{\frac{q^*}{q}(1-\frac{q}{2})}, \qquad (2.10)
$$

where a and b are constants depending on q, N and $|\Omega|$. In the case $N>2$, we analyze the exponent:

$$
\begin{aligned}
\frac{q^*}{q}\left(1-\frac{q}{2}\right)-1 &= \frac{Nq}{N-q}\,\frac{2-q}{2q}-1 \\
&= \frac{N(2-q)-2(N-q)}{2(N-q)} = \frac{q(2-N)}{2(N-q)} < 0.
\end{aligned}
$$

This implies that $\frac{q^*}{q}(1-\frac{q}{2})<1$. Consequently, the inequality (2.10) ensures that the sequence y_n is bounded.

Case $N=2$: We repeat the calculation but choose $\delta = \alpha\frac{N-q}{q}-1$ with $\max\{1,\frac{q}{N-q}\}<\alpha<q^*$. Using the continuity of the injection from $W_0^{1,q}(\Omega)$ into $L^\alpha(\Omega)$, we obtain:

$$
y_n \le a + b y_n^{\frac{\alpha}{q}(1-\frac{q}{N})}, \quad \text{with } \frac{\alpha}{q}\left(1-\frac{q}{N}\right)<1.
$$

This completes the proof that y_n is bounded in all cases. □

2.3.5 Passage to the limit

Proposition 2.1. *For all $q \in [1, q_c = \frac{N}{N-1}[$ and since $(u_n)_n$ is bounded in the space $W_0^{1,q}(\Omega)$, according to Rellich-Kondrachov theorem, we can extract from the sequence (u_n) a sub-sequence, also denote by (u_n) such that*

$$u_n \rightharpoonup u \quad \textit{weakly in} \quad W_0^{1,q}(\Omega), \tag{2.11}$$

$$u_n \longrightarrow u \quad \textit{strongly in} \quad L^q(\Omega). \tag{2.12}$$

Therefore

$$u_n \longrightarrow u \quad \textit{a.e in} \quad \Omega, \tag{2.13}$$

Proof. For the proof of this proposition, we can refer the reader to the book [4] □

Now, we have the operator $Au = -\text{div}(M(x)\nabla u)$ depends on the gradient ∇u, so we need more convergence on the gradients ∇u_n.

Lemma 2.5. *The sequence of gradients (∇u_n) converges almost everywhere to ∇u (possibly up to a subsequence).*

Proof. We show that (∇u_n) is **Cauchy in measure**, which implies that $\nabla u_n \to \nabla u$ almost everywhere for a subsequence. This consists of proving that:

$$\forall \delta > 0, \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall p, q \geq n_0, \quad \text{meas}\{x \in \Omega : |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \leq \varepsilon.$$

To do this, let us fix $\delta > 0$ and $\varepsilon > 0$. For any $\lambda > 0$ and $\eta > 0$, we have the following inclusion:

$$\{x \in \Omega : |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \subset E_1 \cup E_2 \cup E_3 \cup E_4,$$

where:

$$E_1 = \{x \in \Omega : |\nabla u_p| \geq \lambda\},$$

$$E_2 = \{x \in \Omega : |\nabla u_q| \geq \lambda\},$$

$$E_3 = \{x \in \Omega : |u_p - u_q| \geq \eta\},$$

$$E_4 = \{x \in \Omega : |u_p - u_q| \leq \eta, |\nabla u_p| \leq \lambda, |\nabla u_q| \leq \lambda, |\nabla u_p - \nabla u_q| \geq \delta\}.$$

Since the sequence (∇u_n) is bounded in $L^q(\Omega)$ (see Lemma 2.4), by choosing λ sufficiently large, we can make meas(E_1) and meas(E_2) arbitrarily small. Indeed:

$$\text{meas}(E_1) = \int_{E_1} 1\, dx \le \frac{1}{\lambda}\int_{E_1} |\nabla u_p|\, dx \le \frac{1}{\lambda}\|\nabla u_p\|_{L^1(\Omega)} \le \frac{C}{\lambda} \le \varepsilon.$$

Thus, meas$(E_1) \to 0$ as $\lambda \to +\infty$. Similarly for E_2. For E_3, we have:

$$\text{meas}(E_3) \le \frac{1}{\eta}\int_\Omega |u_p - u_q|\, dx.$$

Since (u_n) is a Cauchy sequence in $L^1(\Omega)$, for a fixed $\eta > 0$, we have meas$(E_3) \to 0$ as $p, q \to +\infty$.

It remains to control meas(E_4). By the monotonicity of the operator $a(x,\xi) = M(x)\xi$, we have $(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2) \ge 0$. Since the set

$$K = \{(\xi_1,\xi_2) : |\xi_1| \le \lambda, |\xi_2| \le \lambda, |\xi_1 - \xi_2| \ge \delta\}$$

is compact and the mapping is continuous, the function $(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2)$ attains a strictly positive minimum $\mu(x) > 0$ on K. Hence, by the Lemma 1.2 for any $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that:

$$\int_{E_4} \mu(x)\, dx < \varepsilon' \implies \text{meas}(E_4) < \varepsilon. \tag{2.14}$$

By the definition of $\mu(x)$, we have:

$$\begin{aligned}\int_{E_4} \mu(x)\, dx &\le \int_{E_4} (M(x)\nabla u_p - M(x)\nabla u_q)\nabla(u_p - u_q)\mathbf{1}_{\{|u_p - u_q| \le \eta\}}\, dx \\ &\le \int_\Omega (M(x)\nabla u_p - M(x)\nabla u_q)\nabla T_\eta(u_p - u_q)\, dx,\end{aligned} \tag{2.15}$$

where T_η is the truncation function. Using $T_\eta(u_p - u_q)$ as a test function for the problems satisfied by u_p and u_q and subtracting the results, we obtain:

$$\int_\Omega (M(x)\nabla u_p - M(x)\nabla u_q)\nabla T_\eta(u_p - u_q)\, dx = \int_\Omega (f_p - f_q) T_\eta(u_p - u_q)\, dx.$$

Since f_n converges in $L^1(\Omega)$ and $|T_\eta| \le \eta$, the integral on the right vanishes as $\eta \to 0$ uniformly in p, q. Specifically:

$$\int_\Omega (M(x)\nabla u_p - M(x)\nabla u_q)\nabla T_\eta(u_p - u_q)\, dx \le 2\eta \|f\|_{L^1}. \tag{2.16}$$

For a fixed ε', we can choose η small enough such that $\int_{E_4} \mu(x)\, dx < \varepsilon'$, which implies meas$(E_4) < \varepsilon$. Combining the estimates for E_1, E_2, E_3, and E_4, we conclude that ∇u_n is a Cauchy sequence in measure, and thus $\nabla u_n \to \nabla u$ a.e. in Ω up to a subsequence. □

Now let us fix $\varphi \in W_0^{1,\infty}(\Omega)$. We have

$$\int_\Omega M(x)\nabla u_n \nabla\varphi \quad = \quad \int_\Omega f_n \varphi.$$

1) Limit Passage in $\int_\Omega f_n\varphi$. We have :

$$\lim_{n\to+\infty}\int_\Omega f_n\varphi = \langle f,\varphi\rangle$$

2) Limit Passage in $\int_\Omega M(x)\nabla u_n \nabla\varphi\, dx$.

As $\quad \nabla u_n \rightharpoonup \nabla u \quad$ almost everywhere in Ω, we have:

$$M(x)\nabla u_n \to M(x)\nabla u \quad \text{almost everywhere in} \quad \Omega. \tag{2.17}$$

For any measurable subset for $E \subset \Omega$ we have :

$$\begin{aligned}
\int_E |M(x)\nabla u \nabla\varphi|\, dx &\leq \beta \|\nabla\varphi\|_{L^\infty(E)} \int_E |\nabla u_n| dx \\
&\leq \beta \|\nabla\varphi\|_{L^\infty(\Omega)} \int_E |\nabla u_n| dx \\
&\leq C\int_E |\nabla u_n| dx
\end{aligned}$$

Since $q > 1$ and according to Hölder inequality, we have :

$$\begin{aligned}
\int_E |M(x)\nabla u_n \nabla\varphi|\, dx &\leq C\left(\int_E |\nabla u_n|^q dx\right)^{1/q} . \left(\int_E 1 dx\right)^{1/q'} \\
&\leq C\left(\int_E |\nabla u_n|^q dx\right)^{1/q} . |E|^{1/q'}
\end{aligned}$$

Using lemma 2.4 we have (∇u_n) is bounded in $L^q(\Omega)$, we have :

$$\int_E |M(x)\nabla u_n \nabla\varphi|\, dx \leq C|E|^{1/q'} = C'$$

This proves the equi-integrability of the sequence $(M(x)\nabla u_n \nabla\varphi)_n$. It follows from Vitali's convergence theorem 1.12 that

$$\lim_{n\to+\infty}\int_\Omega M(x)\nabla u_n \nabla\varphi\, dx = \int_\Omega M(x)\nabla u \nabla\varphi\, dx$$

we conclude that

$$\int_\Omega M(x)\nabla u \nabla\varphi\, dx = \langle f,\varphi\rangle, \qquad \forall\varphi \in W_0^{1,\infty}(\Omega).$$

Then u is a solution of the problem

$$\begin{cases} Au &= f \quad \text{on} \quad \Omega; \\ u &\in W_0^{1,q}(\Omega), \quad q \in [1, \frac{N}{N-1}[\end{cases}$$

We therefore have to prove that u is a solution to problem (P_0). This finishes the proof of Theorem 2.2.

Chapter 3

Singular Elliptic Equations with L^1 Data

3.1 Singular Elliptic Problems with L^1 Data

In this chapter, we study the existence of weak solutions for a class of elliptic boundary value problems involving L^1 data and a singular nonlinearity of the type f/u^γ. Using approximation techniques, truncation methods, and compactness arguments in the spirit of Boccardo–GallouÃ«t theory, we establish the existence of a positive weak solution in $H_0^1(\Omega)$.

Our aim is to prove the existence of at least one positive solution u for the following semilinear elliptic problem:

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = \dfrac{f(x)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, and $0 < \gamma \leq 1$ is a real number. We assume that f is a nonnegative function belonging to $L^1(\Omega)$, and $M(x)$ is a bounded elliptic matrix satisfying the following coercivity and boundedness conditions: there exist $0 < \alpha \leq \beta$ such that

$$\alpha|\xi|^2 \leq M(x)\xi\cdot\xi, \quad \text{and} \quad |M(x)| \leq \beta, \tag{3.2}$$

for every $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$.

We establish existence results for problem (3.1) depending on the value of γ (with particular focus on the case $\gamma = 1$). For further details and background, we refer to [1, 3, 6, 14].

Theorem 3.1. *Let $\gamma = 1$ and let f be a nonnegative function in $L^1(\Omega)$. Then, there exists a positive weak solution $u \in H_0^1(\Omega)$ of problem (3.1), in the sense that:*

$$\int_\Omega M(x)\nabla u\cdot\nabla\varphi\,dx = \int_\Omega \frac{f\varphi}{u}\,dx, \tag{3.3}$$

for every test function $\varphi \in W_0^{1,\infty}(\Omega)$.

3.2 Approximating Problem

Let $n \in \mathbb{N}^*$. We consider the following approximating problem:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u_n) = \dfrac{f_n}{|u_n| + \frac{1}{n}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $f_n(x) = \frac{f(x)}{1+\frac{1}{n}f(x)}$ is a sequence of bounded functions that converge to f.

Lemma 3.1. *Let $f \in L^1(\Omega)$. Then there exists at least one nonnegative solution $u_n \in H_0^1(\Omega)$ to problem (3.4) in the sense that:*

$$\int_\Omega M(x)\nabla u_n \cdot \nabla\varphi \, dx = \int_\Omega \frac{f_n \varphi}{u_n + \frac{1}{n}} \, dx, \tag{3.5}$$

for every test function $\varphi \in W_0^{1,\infty}(\Omega)$.

To prove Lemma 3.1, we rely on the Leray–Schauder Fixed Point Theorem [14].

Theorem 3.2 (Leray–Schauder Fixed Point Theorem). *Let X be a Banach space and $T : X[0,1] \to X$ be a compact operator such that:*

$$T(x,0) = 0, \quad \forall x \in X.$$

Assume that there exists a constant $R > 0$ such that for all $(x,\sigma) \in X[0,1]$:

$$x = T(x,\sigma) \implies \|x\|_X < R. \tag{3.6}$$

Then, the operator $T_1 : X \to X$ defined by $T_1(x) = T(x,1)$ has at least one fixed point.

Proof the Lemma 3.1 Consider the following problem:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u_n) = \dfrac{f_n}{|u_n| + \frac{1}{n}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Fix $n \in \mathbb{N}^*$ and consider for $X = L^2(\Omega)$ the operator

$$T : X \times [0,1] \to X, \quad (v_n, \sigma) \mapsto u_n = T(v_n, \sigma).$$

where u_n is the only solution of the problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u_n) = \sigma \dfrac{f_n}{|v_n| + \frac{1}{n}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

It is clear that problem (3.8) has a unique solution whenever the right-hand side belongs to $L^{\infty}(\Omega)$, (see chapter 02).

- It is clear that $T(v_n, 0) = 0$ for all $v_n \in X$, since the only weak solution to the problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u_n) = 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

  is $u_n = 0 \in X$.

- Let us estimate the elements of X such that $v_n = T(v_n, \sigma)$.

For all $\varphi \in W_0^{1,\infty}(\Omega)$, we have

$$\int_{\Omega} M(x)\nabla v_n \nabla\varphi \, dx = \sigma \int_{\Omega} \frac{f_n \varphi}{|v_n| + \frac{1}{n}} \, dx. \tag{3.10}$$

Choosing $\varphi = v_n$ in (3.10), we get

$$\int_{\Omega} M(x)|\nabla v_n| \cdot |\nabla v_n| \, dx \le n^2 \int_{\Omega} |v_n| \, dx \tag{3.11}$$

and by (3.2), we get

$$\alpha \int_{\Omega} |\nabla v_n|^2 \, dx \le \int_{\Omega} M(x)|\nabla v_n| \cdot |\nabla v_n| \, dx \le n^2 \int_{\Omega} |v_n| \, dx$$

Using PoincarÃ©'s inequality, we obtain

$$\alpha \int_{\Omega} |\nabla v_n|^2 \, dx \le n^2 C \int_{\Omega} |\nabla v_n| \, dx,$$
$$\int_{\Omega} |\nabla v_n|^2 \, dx \le \frac{n^2 C}{\alpha} \int_{\Omega} |\nabla v_n| \, dx.$$

and by the Cauchy-Schwarz inequality we find

$$\begin{aligned} \int_{\Omega} |\nabla v_n|^2 \, dx &\le \frac{n^2 C}{\alpha} \left(\int_{\Omega} |\nabla v_n|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\Omega} 1 \, dx \right)^{\frac{1}{2}} \\ &\le \frac{n^2 C}{\alpha} |\Omega|^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v_n|^2 \, dx \right)^{\frac{1}{2}} \\ \|v_n\|^2_{W_0^{1,2}(\Omega)} &\le \frac{n^2 C}{\alpha} |\Omega|^{\frac{1}{2}} \|v_n\|_{W_0^{1,2}(\Omega)} \end{aligned}$$

then

$$\|v_n\|_{W_0^{1,2}(\Omega)} \le C', \quad \text{such that} \quad C' = \frac{n^2 C}{\alpha} |\Omega|^{\frac{1}{2}}.$$

Hence

$$\|v_n\|_{L^2(\Omega)} \le C'(n).$$

Then it follows from the Leray-Schauder theorem that the operator $T_1 : X \to X$ defined by $\forall x \in X : T_1(x) = T(x,1)$ has a fixed point. So, for every fixed n, there exists $u_n \in W_0^{1,2}(\Omega))$ such that $u_n = T_1(u_n)$.

Choosing $u_n^- = \min(u_n, 0)$ as a test function in (3.4) and using the assumption (3.2), we have the following.

$$\alpha \int_\Omega |\nabla u_n^-|^2 dx \leq \int_\Omega \frac{f_n u_n^-}{(|u_n| + \frac{1}{n})^\gamma} dx.$$

Using $\frac{f_n}{(|u_n|+\frac{1}{n})^\gamma} > 0$, we get

$$\alpha \| u_n^- \|_{W_0^{1,2}(\Omega)} \leq 0,$$

so that $u_n \geq 0$ almost everywhere in Ω. Therefore, u_n solves

$$\begin{cases} -\operatorname{div}(M(x)\nabla u_n) = \dfrac{f_n}{u_n + \frac{1}{n}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.12}$$

3.3 Uniform estimates

In this section, we state and prove a uniform estimate for the approximate solutions u_n of problem (3.4).

Following the same proof steps as in [3], we obtain the following two lemmas below.

Lemma 3.2. *The sequence (u_n) is increasing with respect to n.*

Proof. From (3.5) we have

$$\int_\Omega M(x)\nabla u_n \nabla \varphi \, dx = \int_\Omega \frac{f_n \varphi}{u_n + \frac{1}{n}} \, dx.$$

$$\int_\Omega M(x)\nabla u_{n+1} \nabla \varphi \, dx = \int_\Omega \frac{f_{n+1} \varphi}{u_{n+1} + \frac{1}{n+1}} \, dx.$$

then

$$\int_\Omega M(x)\nabla u_n \nabla \varphi \, dx - \int_\Omega M(x)\nabla u_{n+1} \nabla \varphi \, dx = \int_\Omega \frac{f_n \varphi}{u_n + \frac{1}{n}} \, dx - \int_\Omega \frac{f_{n+1} \varphi}{u_{n+1} + \frac{1}{n+1}} \, dx.$$

$$\int_\Omega M(x)(\nabla u_n - \nabla u_{n+1})\nabla \varphi \, dx = \int_\Omega \left(\frac{f_n}{u_n + \frac{1}{n}} - \frac{f_{n+1}}{u_{n+1} + \frac{1}{n+1}} \right) \varphi \, dx. \tag{3.13}$$

Taking $\varphi = max(u_n - u_{n+1}, 0) = (u_n - u_{n+1})^+$ as a test function in (3.13) and observing that if $0 \leq f_n \leq f_{n+1}$, then

$$\frac{f_n}{u_n + \frac{1}{n}} - \frac{f_{n+1}}{u_{n+1} + \frac{1}{n+1}} \leq f_{n+1} \left(\frac{1}{u_n + \frac{1}{n+1}} - \frac{1}{u_{n+1} + \frac{1}{n+1}} \right).$$

we find that the right-hand side gives

$$\int_\Omega M(x)(\nabla u_n - \nabla u_{n+1})\nabla(u_n - u_{n+1})^+ \, dx \le \int_\Omega f_{n+1}\left(\frac{1}{u_n + \frac{1}{n+1}} - \frac{1}{u_{n+1} + \frac{1}{n+1}}\right)(u_n - u_{n+1})^+ \, dx.$$

Since $f_{n+1} \ge 0$, $(u_n - u_{n+1})^+ \ge 0$ and

$$\frac{1}{u_n + \frac{1}{n+1}} - \frac{1}{u_{n+1} + \frac{1}{n+1}} \le 0 \quad in \quad \{x \in \Omega : u_n(x) \ge u_{n+1}(x)\},$$

we get

$$\int_\Omega M(x)(\nabla u_n - \nabla u_{n+1})\nabla(u_n - u_{n+1})^+ \, dx \le 0$$

by using (3.2) we get

$$\alpha \int_\Omega M(x)(\nabla(u_n - u_{n+1})^+)^2 \, dx \le \int_\Omega M(x)\nabla(u_n - u_{n+1}) \cdot \nabla(u_n - u_{n+1})^+ \, dx \le 0,$$

this implies

$$(u_n - u_{n+1})^+ = k \quad \text{a.e in}\Omega, \quad k \text{is a constant.}$$

so that $k = 0$. Hence

$$u_n \le u_{n+1} \tag{3.14}$$

The lemma is proved. $\square$

To prove the strict positivity of the solution, we recall the following strong maximum principle.

Theorem 3.3 (**Strong Maximum Principle with** $c \ge 0$ [7]). *Assume that*

$$u \in C^2(U) \cap C(\overline{U}),$$

and

$$c \ge 0 \quad in\, U.$$

Suppose also that U is connected.

1. *If*

$$Lu \le 0 \quad in\, U,$$

and u attains a nonnegative maximum over $\overline{U}$ at an interior point, then u is constant in U.

2. *Similarly, if*

$$Lu \geq 0 \quad \text{in } U,$$

and u attains a nonpositive minimum over $\overline{U}$ at an interior point, then u is constant in U.

Lemma 3.3. *For all $n \in \mathbb{N}^*$, $u_n \in L^\infty(\Omega)$ and for all $\Subset \Omega$, there exists $C > 0$ (independent of n) such that*

$$u_n \geq C > 0 \tag{3.15}$$

Proof. As the right-hand side of (3.4) belongs to $L^\infty(\Omega)$, therefore the $L^\infty(\Omega)$ estimate of $(u_n)_n$ is a direct consequence of Stampachia's result [13].

Now, since we have

$$-\text{div}(M(x)\nabla u_1) = \frac{f_1}{u_1 + 1} \geq \frac{f_1}{\|u_1\|_\infty + 1} \geq 0$$

the strong maximum principle (see Theorem 3.3) implies that if u_1 attains a nonpositive minimum over w at an interior point, then u_1 must be constant in w. Consequently, (3.14) gives (3.15). □

Lemma 3.4. *Let u_n be a nonnegative solution to problem (3.4), then u_n is bounded in $W_0^{1,2}(\Omega)$.*

Proof. Let us choose $\varphi = u_n$ we get

$$\int_\Omega M(x)\nabla u_n \cdot \nabla u_n = \frac{f_n u_n}{u_n + \frac{1}{n}}$$

then

$$\alpha \int_\Omega |\nabla u_n|^2 \, dx \leq \int_\Omega f_n \, dx \leq \int_\Omega f \, dx$$

Hence

$$\|u_n\|_{W_0^{1,2}(\Omega)} \leq C, \quad such\, that \quad C = \frac{\|f\|_{L^1(\Omega)}}{\alpha}.$$

□

3.4 Passage to the limit

By Lemma 3.4, (u_n) is bounded in $W_0^{1,2}(\Omega)$. Consequently, we can extract a subsequence (denoted again by (u_n)) such that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,2}(\Omega),$$

$$u_n \longrightarrow u \quad \text{strongly in} \quad L^2(\Omega).$$

Therefore

$$u_n \longrightarrow u \quad \text{a.e in} \quad \Omega,$$

Proof. For the proof of this proposition, we can refer the reader to the book [4] □

Now, we have the operator $Au = -\text{div}(M(x)\nabla u)$ depends on the gradient ∇u, so we need more convergence on the gradients ∇u_n.

Proof. From the *a priori* estimates obtained previously, we have

$$\|u_n\|_{H_0^1(\Omega)} \leq C,$$

for some positive constant C independent of n.

Hence, the sequence (u_n) is bounded in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is a reflexive Banach space, there exists a function

$$u \in H_0^1(\Omega)$$

such that, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega).$$

Consequently,

$$\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } (L^2(\Omega))^N.$$

□

So, for all $\varphi \in W_0^{1,\infty}(\Omega)$, we get

$$\int_\Omega M(x)\nabla u_n \nabla\varphi\, dx \to \int_\Omega M(x)\nabla u \nabla\varphi\, dx, \qquad as \quad n \to +\infty$$

On the other side we have, $u_n + \frac{1}{n} \geq u_n \geq C$, then

$$0 \leq \left|\frac{f_n\varphi}{u_n + \frac{1}{n}}\right| \leq \frac{f}{C}\|\varphi\|_{L^\infty}(\Omega).$$

Then the dominated Lebesgue's theorem permits us to conclude that

$$\int_\Omega \frac{f_n\varphi}{u_n + \frac{1}{n}}\, dx \to \int_\Omega \frac{f\varphi}{u}\, dx, \quad as \quad n \to +\infty.$$

for every $\varphi \in W^{1,\infty}$. Therefore. we have to prove that u is a solution to problem (3.1).This finishes the proof of theorem 3.1.

Conclusion

In this thesis, we have focused our attention on a class of nonlinear singular elliptic equations of the form:

$$(P) \qquad \begin{cases} -\operatorname{div}(M(x)\nabla u) = \dfrac{f(x)}{u} \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

where the function f belongs to $L^1(\Omega)$. The singularity is due to the term $\frac{f(x)}{u}$, which may become unbounded near $u = 0$.

The aim of this work is to apply the ideas and methods developed for nonlinear elliptic problems to the singular case. To prove the existence of weak solutions for problem (P), we approximate the original problem by a sequence of regularized problems (P_n), whose solvability is guaranteed by the Leray–Schauder Fixed Point Theorem. Then, we establish several uniform estimates on the sequence of approximate solutions and their derivatives. Once these estimates are obtained, compactness arguments together with suitable convergence results allow us to pass to the limit in the approximate problems and consequently obtain a weak solution of problem (P). Finally, the Strong Maximum Principle is used to prove that the obtained solution is strictly positive.

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Abstract

In this work, we prove the existence of a weak solutions of non-linear singular elliptic problems of the following type:

$$(P)\qquad \begin{cases} -\operatorname{div}(M(x)\nabla u) = \dfrac{f(x)}{u^{\gamma}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the function f belongs to $L^{(}\Omega)$, The singularity arises due to the term $\frac{f}{u^{\gamma}}$, which can become unbounded as $u \to 0$. To address this, we approximate the problem by introducing a regularized version, ensuring that the operator remains well-defined. We then derive a priori estimates for the sequence of approximate solutions and pass to the limit,proving the existence of a weak solution for problem (P)

keywords: Elliptic equations; $W_0^{1,1}$ solutions; singular equation.

Résumé

Dans ce travail, nous prouvons l'existence de solutions faibles pour des problÃ¨mes elliptiques singuliers non linÃ©aires du type :

$$(P)\qquad \begin{cases} -\operatorname{div}(M(x)\nabla u) = \dfrac{f(x)}{u^{\gamma}} & \text{dans } \Omega, \\ u > 0 & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

oÃ$^{1} lafonctionf$ appartient Ã $L^1(\Omega)$. La singularitÃ© provient du terme $\frac{f}{u^{\gamma}}$, qui peut devenir non bornÃ© lorsque u tend vers zÃ©ro. Pour surmonter cette difficultÃ©, nous approchons le problÃ¨me en introduisant une version rÃ©gularisÃ©e, garantissant ainsi que l'opÃ©rateur reste bien dÃ©fini. Ensuite, nous Ã©tablissons des estimations a priori pour la suite des solutions approchÃ©es, puis nous passons Ã la limite afin de prouver l'existence d'une solution faible du problÃ¨me (P).

Mots-clÃ©s: Ã‰quations elliptiques ; solutions $W_0^{1,1}$; Ã©quation singuliÃ¨re.

ملخّص

فِي هَذَا العَمل، نُثبِت وُجود حُلُول ضَعِيفَة لِمَسَائِل إِهلِيلِيجِيَّة غير خطية شَاذَّة مِن الشَكل:

$$(P)\begin{cases} -\operatorname{div}(M(x)\nabla u), = \dfrac{f(x)}{u^{\gamma}} & \text{في } \Omega, \\ u > 0 & \text{في } \Omega, \\ u = 0 & \text{علَى } \partial\Omega. \end{cases}$$

حَيثُ أَنَّ الدَالَّة f تَنتَمِي إِلَى $L^1(\Omega)$. تَنشَأ الخَاصِيَّة الشَاذَّة بِسَبَب الحَدّ $\frac{f}{u^{\gamma}}$ وَ الَذِي قَد يُصبِحُ غيرُ مَحدُود عندَمَا يَقتَرِبُ u مِنَ الصِفر. وَ لِمُعَالَجَةِ هذِهِ الصُعُوبَة، نَقُومُ بِتَقرِيبِ المَسأَلَة بِإِدخَالِ نُسخَة مُنَظَّمَة، مِمَّا يَضمَنُ بَقَاءَ المُؤَثِّر مُعرَّفًا جَيِّدًا. بَعدَ ذَلِك، نَضع التَقدِيرَات القَبلِيَّة لِمُتَتَالِيَة الحُلُول التَقرِيبِيَّة، ثُمَّ نَمُرُّ إِلَى النِهَايَة لِإِثبَاتِ وُجود حَلّ ضَعِيف لِلمَسأَلَة (P) .

اَلكَلِمَات المِفتَاحِيَّة: اَلمُعَادَلَات الإِهلِيلِيجِيَّة؛ اَلحُلُول الضَعِيفَة؛ اَلمُعَادَلَات الشَاذَّة.