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Master memory

Field : Mathematics and computer sciences
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Theme

On positive multiple summing operators

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اهداء

الى امي الله يرحمها

الى كل افراد العائلة

الى كل من مدّ لي يد العون من قريب أو بعيدّ

شكر وعرفان

أولا الحمد و الشكر لله عز و جل، كما أتوجه بالشكر الجزيل إلى المشرف على رسالتي الدكتور بلاعدة عبد العزيز على نصائحه الحكيمة وتوجيهاته لإتمام هذه الرسالة، كما أتقدم بشكري وامتناني لأعضاء لجنة التحكيم.

كما لا أنسى أن اشكر أساتذتي من المرحلة الابتدائية إلى الجامعة وجميع أفراد الأسرة وكل من ساعدوني من قريب أو بعيد في إعداد هذا العمل.

ملخص بالعربية :

يندرج عمل هذه المذكرة ضمن إطار نظرية المؤثرات المتعددة الخطية و في هذا الصدد تم دراسة المؤثرات الموجبة p multiple –جمعية و إعطاء بعض نظريات الاحتواء و المطابقة مع المؤثرات p multiple –جمعية و كذا المؤثرات p multiple –المقعرة

الكلمات المفتاحية :

المؤثرات الموجبة p multiple –جمعية، المؤثرات p multiple –جمعية

المؤثرات p multiple – المقعرة ، فضاءات بناخ لاتيص.

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Résumé: Le but de ce mémoire est l'étude de concept d'opérateurs multilinéaires positivement multi p-sommants, en donnant quelques proprietés à cette classe et leur relation avec les opérateurs multi p-concave.

Mots-clés: Banach réticulé, opérateurs multilinéaires positivement multi p-sommant, opérateurs p-concaves.

Abstract: The aim of this work is to study the class of positive multiple p-summing operators, we present some properties for this class and relationship by other classes of operators as multiple p-summing operators and multiple p-concave operators.

Keywords : Banach lattice, positive p-summing operator, positive multiple p-summing, p-concave operators, multiple p-summing operators

Notations

i_m	The canonical multilinear $i_m : X_1 \times ... \times X_m \to X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m$.
$\mathbb{k}$	The field of real or complex numbers
δ_m	The canonical polynomial $\delta_m : X \to \widehat{\otimes}^m_{\pi,s} X$.
$C(\Omega)$	The set of all continuous functions on the compact set Ω.
$\mathcal{L}_f(X_1, ..., X_m; Y)$	The space of all finite rank multilinear operators.
$\Pi_p(X;Y)$	The class of p-summing linear operators $(1 \leq p < \infty)$.
$\Pi_p^+(X;Y)$	The class of all positive p-summing linear operators $(1 \leq p < \infty)$.
$C_p(X;Y)$	The set of all p-concave operators $(1 \leq p < \infty)$.
p'	The conjugate of the number p $(1 \leq p \leq \infty)$, that is $1/p + 1/p' = 1$.
X^*	The topological dual of X
T^*	The adjoint linear operator of T
$\mathcal{B}(X;Y)$	The space of all bounded linear operators from X to Y
B_{X^*}	The unit ball of X^*.
$\Pi_p^{mult}(X_1, ..., X_n; Y)$	The space of all multiple p-summing operators.
$\Lambda_p(T)$	The space of all positive multiple p-summing operators.
$C_p^{mult}(X_1, ..., X_n; Y)$	The space of all multiple p-concave n-linear operators.

Introduction

The work of this memory is situated within the framework of the multilinear operators theory. The concept of positive p-summing linear operators has been introduced and studied in 1987 by O. Blasco [6]. "*Positive p-summing operators on L_p-spaces.*" (American Mathematical Society, 1987). A linear operator u between *Banach lattice* X *and Banach space* Y is called $(1 \leq p \leq \infty)$ *positive p-summing , if there exists a constant* $C > 0$ *such that for every* $x_1, ..., x_n$ positive elements in X, *we have*

$$(\sum_{i=1}^{n} \|T(x_i)\|^p)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{X^*}} (\sum_{i=1}^{n} |x^*(x_i)|^p)^{\frac{1}{p}}. \quad (0.0.1)$$

We denote by $\Pi_p^+(X;Y)$ *the class of all positive p-summing linear operators from* X *into* Y *and*

$$\pi_p^+(T) = \inf\{C \text{ verifying the inequality (1.3.2)}\}.$$

Later on, in 2015 Q. Bu and C. A. Labuschagne have been presented in [12] a generalization of some results given in [6] on the multilinear case.

The aim of this memory is to study the class of positive multiple p-summing operators, we present some properties for this class and relationship by other classes of operators as multiple p-summing operators and multilinear p-concave operators.

$$\Lambda_p^{mult}(X_1, ..., X_n; Y) \subseteq C_p^{mult}(X_1, ..., X_n; Y).$$

and

$$\Pi_p^{mult}(X_1, ..., X_n; Y) \subseteq \Lambda_p^{mult}(X_1, ..., X_n; Y)$$

Our work diveded into three chapters.

In the first chapter, we recalled by some concepts that we need it in the sequal of this memory, like Banach lattice, adjoint operator positive p-summing linear operators, linear operator, continuous n-linear mappings and tensor product.

The aim of the second chapter is to study the multiple p-summing operators and some of its properties and its relationship to the some class of opertors such as Hilbert-Schmidt operators.

The last chapter is devorted to present the class of positive multiple p-summing operators giving some properties. In the last of this chapter, we give some results of coincidence and inclusion about this class with replay some particular cases.

Chapter 1

Preliminaries

In this chapter, we present some basic concepts of Banach lattice and sequences.

1.1 Basic concepts

We begin by recaling briefly the abstract definition of Banach lattice

Banach space

we call banach space $(X; \|.\|)$ any normalized and complete vector space for the distance deduced from its norm $d(x; y) = \|x - y\|$

Any normed space $(X; \|.\|)$ of finite dimension is complete

Normed space $\mathcal{B}(X; Y)$

If $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are two normed vector spaces, we denote by $\mathcal{B}(X; Y)$ the vector space formed by all continuous linear maps from X to Y. When $X = Y$; we write $\mathcal{B}(X)$ instead of $\mathrm{B}(X; X)$

Definition 1.1.1 *(**Bounded operator**) A linear operator T defined on E in F is said to be bounded if there is a positive constant $C > 0$, such that*

$$\|T(x)\|_Y \leq C \|x\| X \ , \forall x \in X.$$

1.1.1 Banach lattice

Definition 1.1.2 (Banach lattice) *Let X be a Banach space. A real Banach lattice (resp, a real complete Banach lattice) X is equipped with a lattice (resp. a complete lattice) and for all x , y in X, then*

$$\begin{cases} (i) \ \||x|\| = \|x\| \\ (ii) \quad |x| \leq |y| \Longrightarrow \|x\| \ \leq \|y\| \end{cases}$$

Example 1.1.1 *The spaces L_p , $(1 \leq p \leq \infty)$ are complete Banach lattices.*

The $C(K)$ is a Banach lattice.

We denote by $X^+ = \{x \in X : x \geq 0\}$.

An element x of X is positive if $x \in X^+$.

The dual X of a Banach lattice X is a complete Banach lattice with the natural order,

$$x_1 \leq x_2 \iff \langle x_1^*, x\rangle \leq \langle x_2^*, x\rangle, \forall x \in X^+,$$

where $\langle ., .\rangle$ denotes the duality bracket.

Definition 1.1.3 *A Banach lattice is a Banach space with a lattice norm.*

Definition 1.1.4 (Linear operator) *Let T be a operator of a banach space X into a Banch space Y.We say that T is linear if,*

$$\begin{cases} \forall x, y \in X \ : T(x+y) = T(x) + T(y) \\ \forall x \in X, \ \ \forall \lambda \in \mathbb{R} : T(\lambda x) = \lambda T(y) \end{cases}$$

Definition 1.1.5 (Dual topological) *We call the topological dual of space X and denote by X^* the Banach space of continuous linear functionals $\mathcal{B}(X; \Bbbk)$*

Remark 1.1.1 *The set of all continuous linear operators $\mathcal{B}(X;Y)$ from X to Y, is a vector subspace of $\mathcal{B}(X;Y)$ the set of all linear operators over X to Y In particular the dual topological $X^* = \mathcal{B}(X; \Bbbk)$.*

Theorem 1.1.1 *A linear operator T is continuous, if and only if it is bounded.*

Proposition 1.1.1 *Let X and Y be two norm spaces and $T : X \to Y$ un linear operator, the following properties are equivalent:*

1. *The operator T is continuous on X.*
2. *The operator T is continuous at point 0_X.*
3. *The operator T is bounded.*

L_p **space**

A L_p space is a vector space of classes of functions whose power of exponent p is integrable in the sense of Lebesgue, where p is a strictly positive real number. Passing the limit of the exponent results in the construction of L_1 spaces of bounded functions.L_p spaces are called de Lebesgue. Two exponents $p; q \in [1; +\infty]$ will be said to be conjugate if $\frac{1}{p} + \frac{1}{q} = 1$ (dou be $p; q \in]1; +\infty[$; either one is and the other $+\infty$).

Theorem 1.1.2 *(Radon-Nikodym theorem) If $G : f \to X$ is a continuous vector measure of bounded variation, then there exists a Bochner integral g $(\in L_1(\mu; X))$ such that*

$$G(E) = \int_E g d\mu \text{ for everything } E \in f$$

1.1.2 Sequences

Definition 1.1.6 *For $1 \leq p < \infty$, let p' be its conjugate, that is, $1/p + 1/p' = 1$.*

For a Banach lattice X and a finite sequence $(x_i)_1^m \subseteq X$,

$$\begin{aligned} \|(x_i)_1^m\|_{\omega_p(X)} &= \sup\left\{\left(\sum_{i=1}^m |x^*(x_i)|^p\right)^{\frac{1}{p}} : x^* \in B_{X^*}\right\} \\ &= \sup\left\{\left\|\sum_{i=1}^m a_i x_i\right\|_X : (a_i)_1^m \in B_{\ell_{p'}}\right\}. \end{aligned}$$

Using the homogeneous functional calculus in a Banach lattice noted by Krivine see

We have that

$$\left(\sum_{i=1}^m |x_i|^p\right)^{\frac{1}{p}} = \sup\left\{\sum_{i=1}^m a_i x_i : (a_i)_1^m \in B_{\ell_{p'}}\right\} \tag{1.1.1}$$

from which it follows that

$$\|(x_i)_1^m\|_{\omega_p(X)} \leq \left\| \left(\sum_{i=1}^{m} |x_i|^p \right)^{\frac{1}{p}} \right\|_X \tag{1.1.2}$$

In the particular case where $X = C(\Omega)$, for a compact Hausdorff space Ω , it is known ($cf.$[1]) that

$$\|(x_i)_1^m\|_{\omega_p(C(\Omega))} = \left\| \left(\sum_{i=1}^{m} |x_i|^p \right)^{\frac{1}{p}} \right\|_{C(\Omega)} \tag{1.1.3}$$

Moreover, in the particular case where $p = 1$ and $(x_i)_1^m \subseteq X^+$, we have that

$$\|(x_i)\|_{\omega_1(X)} = \left\| \sum_{i=1}^{m} x_i \right\|_X \tag{1.1.4}$$

1.2 Multililinear operator

Definition 1.2.1 *(n-linear operator) Let $n \in \mathbb{N}$ and $X_1, ..., X_n; Y$ be Banach spaces. An operator $T : X_1, ..., X_n; Y$;*

we say that operators ou application n-linear if

$$T(x^1, ..., \alpha x^j + \beta y^j, ..., x^n) = \alpha T(x^1, ..., x^j, ..., x^n) + \beta T(x^1, ..., y^j, ..., x^n);$$

for all j $(1 \leq j \leq n)$ et $x^j, y^j \in X_j, \alpha, \beta \in \mathbb{k}$ ($\mathbb{k} = \mathbb{R}$ *ou* $\mathbb{C}$).

If $Y = \mathbb{k}; T$ is said multilinear form

Definition 1.2.2 *A n-linear operator $U : X_1 \times ... \times X_n \to Y$ is continuous if it is continuous as a function between two normed spaces.*

As a consequence of this definition, following the linear case, we have a result that gives the characterization of the continuous n-linear mappings.

Theorem 1.2.1 *Let $U : X_1 \times .. \times X_n \to Y$ be a n-linear mapping. Then the following assertions are equivalent:*

$(i)U$ is continuous.

(ii) U is continuous in $(0,0,...,0)$.

(iii) there exists a constant $C > 0$ such that for every choice of elements $x_i \in X_i$, $1 \leq i \leq n$, we have

$$\|U(x_1,...,x_n)\| \leq C\,\|x_1\| \times .. \times \|x_n\|\,.$$

In this case, we set

$$\begin{aligned}\|U\| &= \sup_{\|x_j\|_{X_j} \leq 1, j=i,...,n} \|U(x_1,...,x_n)\| \\ &= \inf\{C; C \quad \text{verifying the above equality}:\}\end{aligned}$$

and we can say that U is bounded.

Remark 1.2.1 *The space of continuous (or bounded) n-linear operators of $X_1 \times ... \times X_n$ in Y is a Banach space, we denote it $\mathcal{L}(X_1;...;X_n;Y)$.*

If $Y = \mathbb{k}$, we write $\mathcal{L}(X_1;...;X_n)$.

If $X_1 = ... = X_n = X$, we simply denote by $\mathcal{L}(^nX;Y)$.

Definition 1.2.3 *(adjoint operator) Let $T \in \mathcal{B}(X,Y)$, where X and Y are banach spaces. An adjoint operator $T^* \in \mathcal{B}(X,Y)$ is one such that*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \ \ ,\forall x \in X \ \ , y \in Y.$$

Definition 1.2.4 *The definition of adjoint of an m-linear mapping is due to M. S. Ramanujan and E. Schock* [64]. *Recall, if $T \in \mathcal{L}(X_1,...,X_m;Y)$ we define the adjoint of T by*

$$T^* : Y^* \rightarrow \mathcal{L}(X_1,...,X_n), y^* \rightarrow T^*(y^*) : X_1 \times ... \times X_n \rightarrow \mathbb{k},$$

with

$$T^*(y^*)(x_1,...,x_n) = y^*(T(x_1,...,x_n)),$$

and has the property that T^* is linear and $\|T^*\| = \|T\|$.

It is easy to see that, if $T \in \mathcal{L}(X_1,...,X_m;Y)$ and $u \in \mathcal{L}(Y,Z)$ we have

$$(u \circ T)^* = T^* \circ u^*.$$

As applications of the composition method we give the following important result

Definition 1.2.5 *(adjoint operator) the adjoint of an n-linear operator is defined as follows, if $T \in \mathcal{L}(X_1, ..., X_n; Y)$, we define the adjoint of T by*

$$T^* : Y^* \to \mathcal{L}(X_1, ..., X_m), y^* \to T^*(y^*) : X_1 \times ... \times X_m \to \Bbbk$$

with

$$T^*\left(y^*\right)\left(x^1, ..., x^m\right) = y^*(T(x^1, ..., x^m))$$

and $\Bbbk = \mathbb{R}$ or $\mathbb{C}$.

Definition 1.2.6 (Tensor product) *the tensor product of x and y is the vector space generated by the symbols $x \otimes y$ with $x \in X$ and $y \in Y$*

we can build a tensor product $X_1 \otimes ... \otimes X_n$ the spaces $X_1, ..., X_n$ by elements from space $(L(X_1, ..., X_n; Y))$: for $x^j \in X_j$ $(j = 1, ..., n)$;we define the linear operator

$$x^1 \otimes ... \otimes x^n : L(X_1, ..., X_n) \longrightarrow \Bbbk$$

by

$$x^1 \otimes ... \otimes x^n(\phi) := \phi(x^1, ..., x^n),$$

for any linear cheese ϕ sure $X_1 \times ... \times X_n$: the functional $x^1 \otimes ... \otimes x^n$ is an elementary tensor.

The tensor product $X_1 \otimes ... \otimes X_m$ of the vector spaces $X_1, ..., X_n$ can be constructed from the elements of the space $(L(X_1, ..., X_n; Y))$. For $x^j \in X_j$ $(j = 1, ..., m)$ we define the linear mapping n

$$x^1 \otimes ... \otimes x^n : L(X_1, ..., X_m) \longrightarrow \Bbbk$$

by

$$x^1 \otimes ... \otimes x^n(\phi) := \phi(x^1, ..., x^n),$$

for each n-linear form ϕ on $X_1 \times ... \times X_n$.

The functional $x^1 \otimes ... \otimes x^n$ is called an elementary tensor.

1.3 Positive p-summing operators

1.3.1 Positive p-summing linear operators

In this paragrah we present the definition of positive p-summing linear operators. And some properties that we need later. For more information, see [6].

We start with the definition p-summing linear operators introduced and studied in 1967 by Pietsch [30].

Definition 1.3.1 *Let $1 \leq p \leq \infty$. A linear operator $u : X \longrightarrow Y$ is p-summing if there exists a constant $C \geq 0$ such that, for any $x_1, ..., x_n \in X$, we have*

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{X^*} \leq 1} \left(\sum_{i=1}^{n} |\xi(x_i)|^p \right)^{\frac{1}{p}}. \tag{1.3.1}$$

The class of p-summing linear operators from X into Y, which is denoted by $\Pi_p(X;Y)$. is a Banach space for the norm $\pi_p(T)$, i.e., the smallest constant C such that the inequality (1.3.1) holds.

Definition 1.3.2 *Let X be a Banach lattice and Y be a Banach space. An operator $T : X \longrightarrow Y$ is positive p-summing for $1 \leq p \leq \infty$, if there exists a constant $C > 0$ such that for every $x_1, ..., x_n$ positive elements in X, we have*

$$(\sum_{i=1}^{n} \|T(x_i)\|^p)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{X^*}} (\sum_{i=1}^{n} |x^*(x_i)|^p)^{\frac{1}{p}}. \tag{1.3.2}$$

We denote by $\Pi_p^+(X;Y)$ the class of all positive p-summing linear operators from X into Y and

$$\pi_p^+(T) = \inf\{C \text{ verifying the inequality } (1.3.2)\}.$$

Linear rueslts (Balasco 1987, see [6])

(*a*) *If* $1 \leq p \leq \infty$, *then*

$$\Pi_p(X;Y) \subset \Pi_p^+(X;Y).$$

(*b*) *If* $1 \leq p \leq \infty$, *then.*

$$\Pi_p^+(L_1(\mu), Y) = \mathcal{B}(L_1(\mu), Y).$$

(c) *If* $1 \leq p \leq q \leq \infty$, *then* .

$$\Pi_p^+ (X; Y) \subset \Pi_q^+ (X; Y) .$$

(d) For all $1 \leq p \leq \infty$, we have

$$\Pi_p^+ (L_{p'} (\mu) ; Y) = \Pi_p^+ (L_{p'} (\mu) ; Y) .$$

(e) *If* $1 \leq p \leq \infty$., *then*

$$\Pi_p(X, Y) \subsetneqq \Pi_p^+(X, Y) \subsetneqq C_P(X, Y).$$

(f) *If* $1 \leq p \leq \infty$., *then*

$$\Pi_p^+(C(\Omega), Y) = \Pi_p^+(C(\Omega), Y) = C_p(C(\Omega), Y).$$

(e) *If* $X_1 \subseteq X_2$, $\overline{X_1} = X_2$, *then*

$$\Pi_p^+ (X_2; Y) \subseteq \Pi_p^+ (X_1; Y)$$

(g) Ideal property.

Let $\in \Pi_p^+(X, Y)$ $v : E \to X$ continuous linear positive and $W : Y \to F$ linear continuous positive (E and F are any two spaces where F is lattice). So wTv is positively p-summing and

$$\pi_p^+(wTv) \leq \|w\| \, \pi_p^+(T) \, \|v\| .$$

(h) For all $1 \leq p \leq \infty$, we have

$$\Pi_p^+ (L_{p'} (\mu) ; Y) = \Pi_1^+ (L_{p'} (\mu) ; Y) .$$

Proof. We prove (g) Ideal property.

We have the operator wTv is linear,$wTv \in \Pi_p^+(X; Y)$ We also have

$$\|wTv(x)\| \leq \|w\| \, \|T(v(x))\| \, ; \forall x \in E,$$

let $n \in N$ and $\{x_1, ..., x_n\}$in E^+then$\{v(x_1), ..., v(x_n)\} \subset X^+$, so

$$\left(\sum_1^n \|T(v(x_i))\|^p\right)^{\frac{1}{p}} \leq \pi_p^+(T) \sup_{\xi \in B_{X^*}} \left(\sum_1^n |\langle v(x_i), \xi\rangle|^p\right)^{\frac{1}{p}}$$
$$\leq \pi_p^+(T) \sup_{\xi \in B_{X^*}} \left(\sum_1^n |\langle x_i, v^\star(\xi)\rangle|^p\right)^{\frac{1}{p}}$$

We pose $\eta = \frac{v^\star(\xi)}{\|v\|} \in B_{E^\star}$,so

$$\left(\sum_1^n \|T(v(x_i))\|^p\right)^{\frac{1}{p}} \leq \pi_p^+(T)\|v\| \sup_{\eta \in B_{E^*}} \left(\sum_1^n |\langle x_i, \eta\rangle|^p\right)^{\frac{1}{p}}$$

■

Remark 1.3.1 ***Theorem 1.3.1 Proposition 1.3.1 Proof.*** *from where*

$$\left(\sum_1^n \|wT(v(x_i))\|^p\right)^{\frac{1}{p}} \leq \|w\| \pi_p^+(T)\|v\| \sup_{\eta \in B_{E^*}} \left(\sum_1^n |\langle x_i, \eta\rangle|^p\right)^{\frac{1}{p}}$$

Consequently $wTv \in \Pi_p^+(E; Z)$ and

$$\pi_p^+(wTv) \leq \|w\| \pi_p^+(T)\|v\|$$

■

1.3.2 Ideals of multilinear mappings

Definition 1.3.3 *[3] (The m-linear mappings of finite type) A multilinear mapping $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is of finite type if it is a finite sum of operators of the form*

$$T_{y \otimes_{j=1}^m x_j^*} = x_1^* \otimes ... \otimes x_m^* \otimes y : \left(x^1, ..., x^m\right) \to x_1^*\left(x^1\right) ... x_m^*\left(x^m\right) y, \tag{4.1}$$

where $x_j^* \in X_j^*$ $(1 \leq j \leq m)$ and $y \in Y$. We denote by $\mathcal{L}_f(X_1, ..., X_m; Y)$ the space of all finite type multilinear operators.

Definition 1.3.4 *[3] (Ideal of multilinear mapping) An ideal of multilinear mappings (or multi-ideal) $\mathcal{M}$ is a subclass of the class for all continuous multilinear mapping such that for all $m \in \mathbb{N}$ and Banach spaces $X_1, ..., X_m$ and Y, the componente*

$$\mathcal{M}(X_1, ..., X_m; Y) = \mathcal{L}(X_1, ..., X_m; Y) \cap \mathcal{M}$$

satisfy:

(1) $\mathcal{M}(X_1, ..., X_m; Y)$ *is a linear subspace of* $\mathcal{L}(X_1, ..., X_m; Y)$ *which contains the* m*-linear mappings of finite type.*

(2) *The ideal property: If* $T \in \mathcal{M}(X_1, ..., X_m; Y)$, $u_j \in \mathcal{B}(E_j; X_j)$ *and* $v \in \mathcal{B}(Y; F)$, *then* $v \circ T \circ (u_1, ..., u_m)$ *is in* $\mathcal{M}(E_1, ..., E_m; F)$.

If $\|.\|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^+$ *satisfies*

(1′) $(\mathcal{M}(X_1, ..., X_m; Y), \|.\|_{\mathcal{M}})$ *is a normed (Banach) for all Banach spaces* $X_1, ..., X_m$ *and* Y *and for all* $m \in \mathbb{N}$.

(2′) $\|A^n : \mathbb{K}^n \to \mathbb{K};\ A^n(x_1, ..., x_m) = x_1 ... x_m\|_{\mathcal{M}} = 1$.

(3′) *If* $T \in \mathcal{M}(X_1, ..., X_m; Y), u_j \in \mathcal{B}(E_j; X_j), v \in \mathcal{B}(Y; F)$,

$$\|v \circ T \circ (u_1, ..., u_j)\|_{\mathcal{M}} \leq \|v\| \|T\|_{\mathcal{M}} \|u_1\| ... \|u_m\|,$$

then $(\mathcal{M}; \|.\|_{\mathcal{M}})$ *is called an ormed (Banach) multi-idea*

Chapter 2

Multiple summing operators

In this chapter, we present the class of multiple p-summing, positive multiple p-summing, and multiple p-concave and we also offer prove of some results concerning of positive multiple p-summing.

2.1 Multiple p-summing operators

Let us recall the definitions of concept of multiple p-summing and the multiple p-concave [12].

Definition 2.1.1 (Multiple p-summing) *Let $X_1, ..., X_n$, Y Banach spaces. An n-linear operator $T : X_1 \times ... \times X_n \to Y$ is called multiple p-summing if there exists a constant $C > 0$ such that for every choice of finite sequences $(x_i^j)_{i=1}^{mj} \subseteq X_j, 1 \leq j \leq n$,*

$$\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\| T(x_{i_1}^1, ..., x_{i_n}^n) \right\|^p \right)^{\frac{1}{p}} \leq C \prod_{j=1}^{n} \cdot \left\| (x_i^j)_{i=1}^{m_j} \right\|_{\omega_p(X_j)} . \tag{2.1.1}$$

In this case, we define the positive multiple p-summing norm of T by

$$\pi_p^{mult}(T) = inf\{C : C \text{ verifies the inequality(2.1.1)}\}.$$

In other words, we give a definition

Definition 2.1.2 *[2]Let $1 \leq p < \infty$. A bounded multilinear operator $T : X_1 \times ... \times X_n \to Y$ is multiple p-summing if there exists a constant $C > 0$ such that, for every choice of finite systems $(x^j_{i_j})_{1 \leq i_j \leq m_j} \subset X_j$ $(1 \leq j \leq n)$, the following relation holds*

$$\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\| T(x^1_{i_1}, ..., x^n_{i_n}) \right\|^p \right)^{\frac{1}{p}} \leq C w_p \left(x^1_{i_1} \mid 1 \leq i_1 \leq m_1 \right) ... w_p(x^n_{i_n} \mid 1 \leq i_n \leq m_n) \quad (2.1.2)$$

In this case, we define the multiple p-summing norm of T by

$$\pi_p^{mult}(T) = min\{C : C \text{ verifies } (2.1.2)\}.$$

and we denote by $\Pi_p^{mult}(X_1, ..., X_n; Y)$.

Remark 2.1.1 (1) *In the case of $n = 1$, the previous definition coincides with the definition p-summing linear operators*

(2) *The class $\Pi_p^{mult}(X_1, ..., X_n; Y)$ of multiple p-summing n-linear operators is a Banach space with the norm $\pi_p^{mult}(.)$.*

Now by giving the class multiple p-concave operators, for more details see [12].

Definition 2.1.3 *Let $X_1, ..., X_n$ be Banach lattices and Y a Banach space. An n-linear operator $T : X_1 \times ... \times X_n \to Y$ is called multiple p-concave if there exists a constant $C > 0$ such that for every choice of finite sequences $(x^j_i)_{i=1}^{m_j} \subseteq X_j$, $1 \leq j \leq n$,*

$$\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\| T(x^1_{i_1}, ..., x^n_{i_n}) \right\|^p \right)^{\frac{1}{p}} \leq C. \prod_{j=1}^{n} \left\| \left(\sum_{i=1}^{m} |x^j_i|^p \right)^{\frac{1}{p}} \right\|_{E_j} \quad (2.1.3)$$

In this case, we define the multiple p-concave norm of T by

$$C_p(T) = inf\{C : C\text{verifies the inequality}(2.1.3)\}.$$

It is easily verified that the class $C_p^{mult}(X_1, ..., X_n; Y)$ of multiple p-concave n-linear operators, with its associated norm C_p, is a Banach space.

Proposition 2.1.1 *[2, Remark 2.5] Let $T \in \Pi_p^{mult}(X_1, ..., X_n; Y)$. Then for every choice of finite systems*

$$\left(x^j_{i_j} \right)_{1 \leq i_j \leq m_j} \subset X_j \ (1 \leq j \leq n),$$

$$\left(\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} \left\|T(x_{i_1}^1,\dots,x_{i_n}^n)\right\|^p\right)^{\frac{1}{p}} \le \pi_p^{mult}(T) w_p\left(x_{i_1}^1 \mid 1 \le i_1 \le m_1\right) .. w_p(x_{i_n}^n \mid 1 \le i_n \le m_n) \tag{2.1.4}$$

and $\|T\| \le \pi_p^{mult}(T)$.

Proof. Since, by the definition, we have $\pi_p^{mult}(T) = min\{C : C \text{ verifies}(\mathit{2.1.2})\ \}$,then there exists a sequence $(C_k)_{k\in N} \subset \{C > 0 \mid C \text{ verifies}(\mathit{2.1.2})\}$ such that $lim_{k\to\infty} C_k = \pi_p^{mult}(T)$. Rewriting the definition for each $C_k, k \in \mathbb{N}$ we obtain

$$\left(\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} \left\|T(x_{i_1}^1,\dots,x_{i_n}^n)\right\|^p\right)^{\frac{1}{p}} \le C_K w_p\left(x_{i_1}^1 \mid 1 \le i_1 \le m_1\right) \dots w_p(x_{i_n}^n \mid 1 \le i_n \le m_n).$$

Now, passing to the limit by k, it results the desired relationship. To obtain the second relation, we shall write the definition for $m_1 = 1, \dots, m_n = 1$. Thus, for $x^j \in X_j, (1 \le j \le n)$, we obtain

$$\left\|T(x^1,\dots,x^n)\right\| \le C \sup_{\|x^*\|\le 1}\left(\left|x^*(x^1)\right|\right) \dots \sup_{\|x^*\|\le 1}\left(\left|x^*(x^n)\right|\right)$$

which by the consequence of the Hahn-Banach theorem, means that for each $x^j \in X_j, 1 \le j \le n$,

$$\left\|T(x^1,\dots,x^n)\right\| \le C \left\|x^1\right\| \dots \|x^n\|$$

hence

$$\|T\| \le C.$$

Going to infimum, which is the greatest lower bound, we get

$$\|T\| \le \pi_p^{mult}(T).$$

■

2.1.1 The ideal of multiple p-summing operators

In this section we prove that $\left(\Pi_p^{mult}, \pi_p^{mult}\right)$ is a Banach ideal of multilinear operators, which is a well known fact,

but we do not know an exact reference for its proof.

Proposition 2.1.2 *The class* $\left(\Pi_p^{mult}, \pi_p^{mult}\right)$ *is a Banach ideal of multilinear operators.*

Proof. In order to prove this, we shall apply the definition for ideals of multilinear operators,

that was given following A. Pietsch [34], by K. Floret and

D. Garcia in [13]. Thus, we have to show that

(**a**) for $X_1, ..., X_n, Y$ be Banach spaces

$$\Pi_p^{mult}(X_1, ..., X_n; Y) = \mathcal{L}(X_1, ..., X_n; Y) \cap \Pi_p^{mult}$$

is a linear subspace of $\mathcal{L}(X_1, ..., X_n; Y)$;

(**b**)(Π_p^{mult}, $\pi_p^{mult}(.)$) has the ideal property i.e for $U_i \in \mathcal{L}(X_i, Y_i), S \in \mathcal{L}(Z, W)$ and $T \in \Pi_p^{mult}(Y_1, ..., Y_n; Z)$, we have

$$S \circ T \circ (U_1, ..., U_n) \in \Pi_p^{mult}(Y_1, ..., Y_n; Z)$$

and

$$\pi_p^{mult}(S \circ T \circ (U_1, ..., U_n)) \leq \|S\| \, \pi_p^{mult}(T) \, \|U_1\| \, ... \, \|U_n\|$$

(**c**) The mapping

$$I_{\mathbb{k}} : \mathbb{k}^n \to \mathbb{k}, I_{\mathbb{k}}(\lambda_1, ..., \lambda_n) = \lambda_1 ... \lambda_n, I_{\mathbb{k}} \in \Pi_p^{mult}$$

and

$$\pi_p^{mult}(I_{\mathbb{k}}) = 1;$$

(**d**) $\left(\Pi_p^{mult}(X_1, ..., X_n; Y), \pi_p^{mult}(.)\right)$ is a Banach space.

(**i**) To start, let $U, V \in \Pi_p^{mult}\ (X_1, ..., X_n; Y)$ and ($x_{i_j}^j)_{1 \leq i_j \leq m_j} \subset X_j$ $(1 \leq j \leq n)$.

Then

$$\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(U+V)(x_{i_1}^1, ..., x_{i_n}^n)\right\|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|U(x_{i_1}^1, ..., x_{i_n}^n)\right\|^p\right)^{\frac{1}{p}} + \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|V(x_{i_1}^1, ..., x_{i_n}^n)\right\|^p\right)^{\frac{1}{p}}$$

which by $(2.1.2)$ gives

$$\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(U+V)(x^1_{i_1},...,x^n_{i_n})\right\|^p\right)^{\frac{1}{p}} \leq (\pi_p^{mult}(U)+\pi_p^{mult}(V)).w_p(x^1_{i_1} \mid 1 \leq i_1 \leq m_1)... .w_p(x^n_{i_n} \quad | \quad 1 \leq i_n \leq m_n)$$

,so $U+V$ is p-summing and $\pi_p^{mult}(U+V) \leq \pi_p^{mult}(U)+\pi_p^{mult}(V)$.

By a similar argument, we have that

$$\begin{aligned}\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(\alpha U)(x^1_{i_1},...,x^n_{i_n})\right\|^p\right)^{\frac{1}{p}} &= |\alpha| \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|U(x^1_{i_1},...,x^n_{i_n})\right\|^p\right)^{\frac{1}{p}} \\ &\leq |\alpha|\, \pi_p^{mult}(U).w_p(x^1_{i_1} \mid 1 \leq i_1 \leq m_1)...w_p(x^n_{i_n} \mid 1 \leq i_n \leq m_n)\end{aligned}$$

hence, αU is p-summing and $\pi_p^{mult}(\alpha U) \leq |\alpha| \pi_p^{mult}(U)$.

For the reverse inequality,

we consider $\alpha \to 1\alpha, U \to \alpha U$ and then $\pi_p^{mult}(U) \leq 1\alpha\ \pi_p^{mult}(\alpha U)$

i.e.$\pi_p^{mult}(\alpha U) \geq |\alpha| \pi_p^{mult}(U)$

And so we showed that $\Pi_p^{mult}\ (X_1,...,X_n;Y)$ is a linear subspace of $\mathcal{L}(X_1,...,X_n;Y)$ and that $\pi_p^{mult}(.)$ is a norm on this space.

(**ii**) Further, let $U_j \in \mathcal{L}(X_j,Y_j), 1 \leq j \leq n, T \in \Pi_p^{mult}(Y_1,...,Y_n;Z)$ and

$S \in \mathcal{L}(Z,W)$ and let $(x^j_{i_j})_{1 \leq i_j \leq m_j}$ be some finite systems of elements in $X_j, 1 \leq j \leq n$. Then

$$\begin{aligned}\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(S \circ T \circ (U_1,...,U_n))(x^1_{i_1},...,x^n_{i_n})\right\|^p\right)^{\frac{1}{p}} &= \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(S(T(U_1,...,U_n)))(x^1_{i_1},...,x^n_{i_n})\right\|^p\right)^{\frac{1}{p}} \\ &\leq \|S\| \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(T(U_1,...,U_n))(x^1_{i_1},...,x^n_{i_n})\right\|^p\right)^{\frac{1}{p}} \\ &= \|S\| \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|T(U_1(x^1_{i_1}),...,U_n(x^n_{i_n}))\right\|^p\right)^{\frac{1}{p}}\end{aligned}$$

Since T is p-summing, it follows that

$$\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(S \circ T \circ (U_1,...,U_n))(x^1_{i_1},...,x^n_{i_n})\right\|^p\right)^{\frac{1}{p}} \leq \|S\|\, \pi_p^{mult}(T) w_p(U_1\left(x^1_{i_1}\right) \mid 1 \leq i_1 \leq m_1)\ ... .w_p(U_n(x^n_{i_n}) \quad | \quad 1 \leq i_n \leq m_n)$$

Now, we have to evaluate $w_p\left(U_j(x^j_{i_j})|1 \leq i_j \leq m_j\right), 1 \leq j \leq n$.We have

$$\begin{aligned} w_p(U_j(x^j_{i_j})|1 &\leq i_j \leq m_j) = \sup_{\varphi_j \in Y_j^*}\left(\sum_{i_j=1}^{m_j}\left|\varphi_j\left(U_j(x^j_{i_j})\right)\right|^p\right)^{\frac{1}{p}} \\ &= \|U_j\| \sup_{\varphi_j \in Y_j^*}\left(\sum_{i_j=1}^{m_j}\left|\varphi_j\left(\frac{U_j}{\|U_j\|}(x^j_{i_j})\right)\right|^p\right)^{\frac{1}{p}}, \\ &\leq \|U_j\| \sup_{\psi_j \in X_j^*}\left(\sum_{i_j=1}^{m_j}\left|\psi_j(x^j_{i_j})\right|^p\right)^{\frac{1}{p}}, \end{aligned}$$

where $\psi_j = \varphi_j \circ \frac{U_j}{\|U_j\|}, 1 \leq j \leq n$. Since ,

$$\|\psi_j\| = \left\|\varphi_j \circ \frac{U_j}{\|U_j\|}\right\| = \|\varphi_j\| \left\|\frac{U_j}{\|U_j\|}\right\| = \|\varphi_j\| \leq 1 1 \leq j \leq n$$

it follows that

$$w_p(U_j(x^j_{i_j})|1 \leq i_j \leq m_j) \leq \|U_j\| \, w_p(x^j_{i_j}|1 \leq i_j \leq m_j)(1 \leq j \leq n)$$

Hence

$$\begin{aligned} \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n}\left\|(S \circ T \circ (U_1, ..., U_n))(x^1_{i_1}, ..., x^n_{i_n})\right\|^p\right)^{\frac{1}{p}} &\leq \|S\| \, \pi_p^{mult}(T). \\ \|U_1\| ... \|U_n\| \, w_p(x^1_{i_1} &\mid 1 \leq i_1 \leq m_1)...w_p(x^n_{i_n} \mid 1 \leq i_n \leq m_n) \end{aligned}$$

This means that $S \circ T \circ (U_1, ..., U_n)$ is p-summing and

$$\pi_p^{mult}(S \circ T \circ (U_1, ..., U_n)) \leq \|S\| \, \pi_p^{mult}(T) \|U_1\| ... \|U_n\|$$

(iii) Let $(\lambda^j_{i_j})$ $1 \leq i_j \leq m_j$ be some finite systems of elements in $\Bbbk$.

Then

$$\begin{aligned}
\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left| I_{\mathbb{k}}(\lambda^1_{i_1},...,\lambda^n_{i_n})\right|^p\right)^{\frac{1}{p}} &= \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left|(\lambda^1_{i_1},...,\lambda^n_{i_n})\right|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i_1=1}^{m_1} \left|\lambda^1_{i_1}\right|^{\frac{1}{p}}\right)^{\frac{1}{p}} ... \left(\sum_{i_n=1}^{m_n} \left|\lambda^n_{i_n})\right|^p\right)^{\frac{1}{p}} \\
&\leq \sup_{\left\|x_1^*\right\|\leq 1} \left(\sum_{i_1=1}^{m_1} \left|x_1^*\left(\lambda^1_{i_1}\right)\right|^p\right)^{\frac{1}{p}} ... \sup_{\|x_n^*\|\leq 1} \left(\sum_{i_n=1}^{m_n} \left|x_n^*\left(\lambda^n_{i_n}\right)\right|^p\right)^{\frac{1}{p}} \\
&= w_p(\lambda^1_{i_1} \mid 1\leq i_1\leq m_1)...w_p(\lambda^n_{i_n} \mid 1\leq i_n\leq m_n) \quad .
\end{aligned}$$

Hence, $I_{\mathbb{k}}$ is p-summing and $\pi_p^{mult}(I_{\mathbb{k}})\leq 1$. In order to reverse inequality, we have that $\|I_{\mathbb{k}}\|\leq \pi_p^{mult}(I_{\mathbb{k}})$ and since$\|I_{\mathbb{k}}\|=1$,

we obtain $\pi_p^{mult}(I_{\mathbb{k}})=1$.

(**iv**') In order to prove that $\left(\Pi_p^{mult}(X_1,...,X_n;Y),\pi_p^{mult}(.)\right)$ is a Banach space, we shall consider a Cauchy sequence $(U_n)_{n\in\mathbb{N}}\subset \pi_p^{mult}(.)$.

Hence $\forall\varepsilon>0,\exists n_\varepsilon\in\mathbb{N}$ such that $\pi_p^{mult}(U_n-U_m)<\varepsilon,\forall\ n,m\geq n_\varepsilon$.

By Remark 2.5,we have $\|U_n-U_m\|\leq\pi_p^{mult}(U_n-U_m)<\varepsilon$,which means that $(U_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{L}(X_1,...,X_n;Y)$.

Thus, it exists $U\in\mathcal{L}(X_1,...,X_n;Y)$ such that $\|Un-U\|\to 0$.

Now, let $(x^j_{i_j})_{1\leq i_j\leq m_j}\subset X_j,\ 1\leq j\leq n$. Since U_n-U_m is a p-summing operator, it follows that for every $n,m\geq n_\varepsilon$, we have

$$\begin{aligned}
\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left|(U_n-U_m)(x^1_{i_1},...,x^n_{i_n})\right|^p\right)^{\frac{1}{p}} &\leq \pi_p^{mult}(U_n-U_m). \\
w_p(x^1_{i_1} &\mid 1\leq i_1\leq m_1)...w_p(x^n_{i_n}\mid 1\leq i_n\leq m_n) \\
&\leq \varepsilon w_p(x^1_{i_1}\mid 1\leq i_1\leq m_1)...w_p(x^n_{i_n}\mid 1\leq i_n\leq m_n)
\end{aligned}$$

Since $U_n(x_j)\to U(x_j),\forall x_j\in X_j$, after passing to the limit for $m\to\infty$, we obtain that for every $n\geq n_\varepsilon$

$$\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left|(U_n-U)(x^1_{i_1},...,x^n_{i_n})\right|^p\right)^{\frac{1}{p}} \leq \varepsilon w_p(x^1_{i_1}\mid 1\leq i_1\leq m_1)...w_p(x^n_{i_n}\mid 1\leq i_n\leq m_n)$$

which means that $U_n - U$ is p-summing and hence, U is p-summing. Inaddition, π_p^{mult} $(U_n - U) \leq \varepsilon, \forall n \geq n_\varepsilon$ i.e.

the sequence $(U_n)_{n\in N}$ is convergent to $U \in \pi_p^{mult}(.)$ with respect to the p-summing norm,$\pi_p^{mult}(.)$. ■

The definition of the Hilbert-Schmidt multilinear mappings was introduced by Dwyer [19].

Definition 2.1.4 (Hilbert-Schmidt multilinear) *Let $H_1, ..., H_m, H$ be Hilbert spaces. A mapping $T \in \mathcal{L}(H_1, ..., H_m; H)$ is said to be Hilbert -Schmidt if there is an orthonormal basis $(e_{i_j})_{i_j \in I_j}$ for H_j, for each $j = 1, .., m$, such that*

$$\|T\|_{\mathcal{HS}} = \sum_{i_1\in I_1,...,i_m\in I_m} \|T(e_{i_1}, ..., e_{i_m})\|^2 < \infty.$$

We denote by $\mathcal{L}_{\mathcal{HS}}(H_1, ..., H_m; H)$ the space of all Hilbert-Schmidt multilinear mappings.

It is easy to show that it is a Hilbert space under the norm $\|.\|_{\mathcal{HS}}$ defined by the inner product

$$\langle T_1, T_2 \rangle = \sum_{i_1\in I_1,...,i_m\in I_m} \left\langle T_1(e_{i_1}, ..., e_{i_m}), \overline{T_2(e_{i_1}, ..., e_{i_m})} \right\rangle.$$

2.1.2 Relation with Hilbert-Schmidt multilinear mappings

The relationship between the class of multiple p-summing operators and the class of Hilbert-Schmidt ones has been studied by M. Matos in [26, Proposition 5.7] and by D. Peerez-Garcia in [26, Theorem 4.2]. Thus, they have showed the following resul

Theorem 2.1.1 *Let $1 \leq p < \infty$ and $H_1, ..., H_n; H$ be Hilbert spaces ,we have*

$$\Pi_p^{mult}(H_1, ..., H_n; H) = \mathcal{L}_{\mathcal{HS}}(H_1, ..., H_n; H).$$

Multiple rueslts (**Matos 1987,** see)

Chapter 3

Positive multiple p-summing operators

In this chapter, we student the class of positive multiple p-summing operators with some properties

3.1 Positive multiple p-summing operators

In this paragraph, we present class the positive multiple p-summing operators with some properties. The definition was introduced by Q. Bu and Labuschagne in [12].

Definition 3.1.1 (Positive multiple p-summing) *[12] Let $X_1, ..., X_n$, Y, be Banach lattices and Y a Banach space. An n-linear operator $T : X_1 \times ... \times X_n \to Y$ is called positive multiple p-summing if there exists a constant $C > 0$ such that for every choice of finite sequences $(x_i^j)_{i=1}^{m_j} \subseteq X_j^+, 1 \leq j \leq n$,*

$$\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\| T(x_{i_1}^1, ..., x_{i_n}^n) \right\|^p \right)^{\frac{1}{p}} \leq C \prod_{j=1}^{n} \cdot \left\| (x_i^j)_{\ i=1}^{m_j} \right\|_{\omega_p(X_j)} . \tag{3.1.1}$$

The class of the positive multiple p-summing norm of T by

$$\Lambda_p(T) = inf\{C : C \text{ verifies the inequality} \textit{(3.1.1)} \ \}.$$

Which is denoted by $\Lambda_p^{mult}(X_1, ..., X_n; Y)$ is a Banach space for the norm $\Lambda_p(T)$.

3.1.1 Properties of the class $\Lambda_p^{mult}(X_1,...,X_n;Y)$

By the relation (3.1.1), we have (1) and (3) .

(1) *All* multiple p-summing operator *is* positive multiple p-summing *operator.*

(i.e., If $1 \leq p < \infty$, then $\Pi_p^{mult}(X_1,...,X_n;Y) \subseteq \Lambda_p^{mult}(X_1,...,X_n;Y)$).

Furthermore, we have

$$\Lambda_p(.) \leq \pi_p^{mult}(.).$$

(2) *If* $1 \leq p \leq q \leq \infty$, *then*

$$\Pi_p^{mult}(X_1,...,X_n;Y) \subset \Pi_q^{mult}(X_1,...,X_n;Y).$$

(3) *If* $1 \leq p \leq q \leq \infty$, *then*

$$\Lambda_p^{mult}(X_1,...,X_n;Y) \subset \Lambda_q^{mult}(X_1,...,X_n;Y).$$

Proposition 3.1.1 *The class* $\Lambda_p^{mult}(X_1,...,X_n;Y)$ *is a Banach space.*

Proof. On verify that the class $\Lambda_p^{mult}(X_1,...,X_n;Y)$ of positive multiple p-summing n-linear operators, with its associated norm Λ_p, is a Banach space.

• $(\left(\Lambda_p^{mult}(X_1,...,X_n;Y), \Lambda_p^{mult}(.)\right)$ is a Banach space.

(i)To start, *let* $X_1,...,X_n$, Y, *be Banach lattices and* Y *a Banach space.*

Let $U,V \in \Lambda_p^{mult}$ $(X_1,...,X_n;Y)$ and ($x_{i_j}^j)_{1\leq i_j\leq m_j} \subset X_j$ $(1 \leq j \leq n)$.

Then

$$\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(U+V)(x_{i_1}^1,...,x_{i_n}^n)\right\|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|U(x_{i_1}^1,...,x_{i_n}^n)\right\|^p\right)^{\frac{1}{p}} + \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|V(x_{i_1}^1,...,x_{i_n}^n)\right\|^p\right)^{\frac{1}{p}}$$

which by (2.3) gives

$$\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(U+V)(x_{i_1}^1,...,x_{i_n}^n)\right\|^p\right)^{\frac{1}{p}} \leq (\Lambda_p^{mult}(U)+\Lambda_p^{mult}(V)).w_p(x_{i_1}^1 \mid 1 \leq i_1 \leq m_1)..$$
$$.w_p(x_{i_n}^n \quad | \quad 1 \leq i_n \leq m_n)$$

,which means that $U+V$ is p-summing and $\Lambda_p^{mult}(U+V) \leq \Lambda_p^{mult}(U) + \Lambda_p^{mult}(V)$.

By a similar argument, we have that

$$\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(\alpha U)(x^1_{i_1},...,x^n_{i_n})\right\|^p\right)^{\frac{1}{p}} = |\alpha| \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|U(x^1_{i_1},...,x^n_{i_n})\right\|^p\right)^{\frac{1}{p}}$$
$$\leq |\alpha|\, \Lambda_p^{mult}(U).w_p(x^1_{i_1} \mid 1 \leq i_1 \leq m_1)...w_p(x^n_{i_n} \mid 1 \leq i_n \leq m_n)$$

hence, αU is p-summing and $\Lambda_p^{mult}(\alpha U) \leq |\alpha|\Lambda_p^{mult}(U)$. For the reverse inequality,

we consider $\alpha \to 1\alpha, U \to \alpha U$ and then $\Lambda_p^{mult}(U) \leq 1\alpha\ \Lambda_p^{mult}(\alpha U)$

i.e.$\Lambda_p^{mult}(\alpha U) \geq |\alpha|\Lambda_p^{mult}(U)$

Thus, we have shown that $\Lambda_p^{mult}\ (X_1,...,X_n;Y)$ is a linear subspace of $L(X_1,...,X_n;Y)$ and

that $\Lambda_p^{mult}(.)$ is a norm on this space.

(ii) Further, let $U_j \in L(X_j, Y_j), 1 \leq j \leq n, T \in \Lambda_p^{mult}(Y_1,...,Y_n;Z)$ and $S \in L(Z,W)$ and let $(\ x^j_{i_j})_{1\leq i_j \leq m_j}$ be some finite systems of elements in $X_j, 1 \leq j \leq n$. Then

$$\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(S\circ T\circ(U_1,...,U_n))(x^1_{i_1},...,x^n_{i_n})\right\|^p\right)^{\frac{1}{p}} = \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(S(T(U_1,...,U_n)))(x^1_{i_1},...,x^n_{i_n})\right\|^p\right)^{\frac{1}{p}}$$
$$\leq \|S\| \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(T(U_1,...,U_n))(x^1_{i_1},...,x^n_{i_n})\right\|^p\right)^{\frac{1}{p}}$$
$$= \|S\| \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|T(U_1(x^1_{i_1}),...,U_n(x^n_{i_n}))\right\|^p\right)^{\frac{1}{p}}$$

Since T is p-summing, it follows that

$$\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\|(S\circ T\circ(U_1,...,U_n))(x^1_{i_1},...,x^n_{i_n})\right\|^p\right)^{\frac{1}{p}} \leq \|S\|\, \Lambda_p^{mult}(T) w_p(U_1\left(x^1_{i_1}\right) \mid 1 \leq i_1 \leq m_1\big)\ ...$$
$$.w_p(U_n(x^n_{i_n}) \quad | \quad 1 \leq i_n \leq m_n)$$

Now, we have to evaluate $w_p(U_j(x^j_{i_j})|1 \leq i_j \leq m_j), 1 \leq j \leq n$.We have

$$\begin{aligned} w_p(U_j(x^j_{i_j})|1 &\leq i_j \leq m_j) = \sup_{\varphi_j \in Y_j^*} \left(\sum_{i_j=1}^{m_j} \left| \varphi_j \left(U_j(x^j_{i_j}) \right) \right|^p \right)^{\frac{1}{p}} \\ &= \|U_j\| \sup_{\varphi_j \in Y_j^*} \left(\sum_{i_j=1}^{m_j} \left| \varphi_j \left(\frac{U_j}{\|U_j\|}(x^j_{i_j}) \right) \right|^p \right)^{\frac{1}{p}}, \\ &\leq \|U_j\| \sup_{\psi_j \in X_j^*} \left(\sum_{i_j=1}^{m_j} \left| \psi_j(x^j_{i_j}) \right|^p \right)^{\frac{1}{p}}, \end{aligned}$$

where $\psi_j = \varphi_j \circ \frac{U_j}{\|U_j\|}, 1 \leq j \leq n$. Since, ,

$$\|\psi_j\| = \left\| \varphi_j \circ \frac{U_j}{\|U_j\|} \right\| = \|\varphi_j\| \left\| \frac{U_j}{\|U_j\|} \right\| = \|\varphi_j\| \leq 1 1 \leq j \leq n$$

it follows that

$$w_p(U_j(x^j_{i_j})|1 \leq i_j \leq m_j) \leq \|U_j\| \, w_p\left(x^j_{i_j}|1 \leq i_j \leq m_j \right) (1 \leq j \leq n)$$

Hence

$$\begin{aligned} \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left\| (S \circ T \circ (U_1,...,U_n))(x^1_{i_1},...,x^n_{i_n}) \right\|^p \right)^{\frac{1}{p}} &\leq \|S\| \, \Lambda_p^{mult}(T). \, \|U_1\| ... \|U_n\| \, . \\ w_p(x^1_{i_1} &\mid 1 \leq i_1 \leq m_1)...w_p(x^n_{i_n} \mid 1 \leq i_n \leq m_n) \end{aligned}$$

which means that $S \circ T \circ (U_1, ..., U_n)$ is p-summing and

$$\Lambda_p^{mult}(S \circ T \circ (U_1, ..., U_n)) \leq \|S\| \, \Lambda_p^{mult}(T) \, \|U_1\| ... \|U_n\|$$

(iii) Let $(\lambda^j_{i_j}), 1 \leq i_j \leq m_j$ be some finite systems of elements in K. Then

$$\begin{aligned}
\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left|I_K(\lambda^1_{i_1},...,\lambda^n_{i_n})\right|^p\right)^{\frac{1}{p}} &= \left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left|(\lambda^1_{i_1},...,\lambda^n_{i_n})\right|^p\right)^{\frac{1}{p}} \\
&\le \left(\sum_{i_1=1}^{m_1} \left|\lambda^1_{i_1}\right|^{\frac{1}{p}}\right)^{\frac{1}{p}} ... \left(\sum_{i_n=1}^{m_n} \left|\lambda^n_{i_n})\right|^p\right)^{\frac{1}{p}} \\
&\le \sup_{\|x_1^*\|\le 1} \left(\sum_{i_1=1}^{m_1} \left|x_1^*\left(\lambda^1_{i_1}\right)\right|^p\right)^{\frac{1}{p}} ... \sup_{\|x_n^*\|\le 1} \left(\sum_{i_n=1}^{m_n} \left|x_n^*\left(\lambda^n_{i_n}\right)\right|^p\right)^{\frac{1}{p}} \\
&= w_p(\lambda^1_{i_1} \mid 1 \le i_1 \le m_1)...w_p(\lambda^n_{i_n} \mid 1 \le i_n \le m_n) \quad .
\end{aligned}$$

Hence, I_K is p-summing and $\Lambda_p^{mult}(I_K) \le 1$. For the reverse inequality, we have that $\|I_K\| \le \Lambda_p^{mult}(I_K)$ and since$\|I_K\| = 1$, we obtain $\Lambda_p^{mult}(I_K) = 1$.

(iv) In order to prove that $\left(\Lambda_p^{mult}(X_1,...,X_n;Y), \Lambda_p^{mult}(.)\right)$ is a Banach space, we shall consider a Cauchy sequence $(U_n)_{n\in N} \subset \Lambda_p^{mult}(.)$. Hence $\forall \varepsilon > 0, \exists n_\varepsilon \in N$ such that $\Lambda_p^{mult}(U_n - U_m) < \varepsilon, \forall n,m \ge n_\varepsilon$. By Remark 2.5,we have $\|U_n - U_m\| \le \Lambda_p^{mult}(U_n - U_m) < \varepsilon$,which means that $(U_n)_{n\in N}$ is a Cauchy sequence in the Banach space $L(X_1,...,X_n;Y)$.Thus, it exists $U \in L(X_1,...,X_n;Y)$ such that $\|Un - U\| \to 0$.

Now, let $(x^j_{i_j})_{1\le i_j \le m_j} \subset X_j$, $1 \le j \le n$. Since $U_n - U_m$ is a p-summing operator, it follows that for every $n, m \ge n_\varepsilon$, we have

$$\begin{aligned}
\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left|(U_n - U_m)(x^1_{i_1},...,x^n_{i_n})\right|^p\right)^{\frac{1}{p}} &\le \Lambda_p^{mult}(U_n - U_m).w_p(x^1_{i_1} \mid 1 \le i_1 \le m_1).. \\
.w_p(x^n_{i_n} &\mid 1 \le i_n \le m_n) \\
&\le \varepsilon w_p(x^1_{i_1} \mid 1 \le i_1 \le m_1)...w_p(x^n_{i_n} \mid 1 \le i_n \le m_n)
\end{aligned}$$

Since $U_n(x_j) \to U(x_j), \forall x_j \in X_j$, after passing to the limit for $m \to \infty$, we obtain that for every $n \ge n_\varepsilon$

$$\begin{aligned}
\left(\sum_{i_1,...,i_n=1}^{m_1,...,m_n} \left|(U_n - U)(x^1_{i_1},...,x^n_{i_n})\right|^p\right)^{\frac{1}{p}} &\le \varepsilon w_p(x^1_{i_1} \mid 1 \le i_1 \le m_1)... \\
.w_p(x^n_{i_n} &\mid 1 \le i_n \le m_n)
\end{aligned}$$

which means that $U_n - U$ is p-summing and hence, U is p-summing. Inaddition, Λ_p^{mult} $(U_n - U) \leq \varepsilon, \forall n \geq n_\varepsilon$

(i.e. the sequence $(U_n)_{n\in N}$ is convergentto $U \in \Lambda_p^{mult}(.)$ with respect to the p-summing norm, $\Lambda_p^{mult}(.)$.) ■

3.2 Some inclusions and coincidence results

In this section, we give some properties and coincidence results for a positive multiple p-summing operators

Theorem 3.2.1 *If $1 \leq p < \infty$, then*

$$\Lambda_p^{mult}(X_1, ..., X_n; Y) \subseteq C_p^{mult}(X_1, ..., X_n; Y).$$

Proof. For $n = 2$, let $T \in \Lambda_p^{mult}(X_1, X_2; Y)$, and take $(x_i)_1^m \subseteq X_1, (y_j)_1^k \subseteq X_2$.By*(1.1.2)*

$$\begin{aligned}
\left(\sum_{i,j=1}^{m,k} \|T(x_i, y_j)\|^p \right)^{\frac{1}{P}} &= \left(\sum_{i,j=1}^{m,k} \left\|T(x_i^+ - x_i^-, y_j^+ - y_j^-)\right\|^p \right)^{\frac{1}{P}} \\
&\leq \left(\sum_{i,j=1}^{m,k} \left\|T(x_i^+, y_j^+)\right\|^p \right)^{\frac{1}{P}} + \left(\sum_{i,j=1}^{m,k} \left\|T(x_i^+, y_j^-)\right\|^p \right)^{\frac{1}{P}} \\
&\quad + \left(\sum_{i,j=1}^{m,k} \left\|T(x_i^-, y_j^+)\right\|^p \right)^{\frac{1}{P}} + \left(\sum_{i,j=1}^{m,k} \left\|T(x_i^-, y_j^-)\right\|^p \right)^{\frac{1}{P}}.
\end{aligned}$$

So

$$\begin{aligned}\left(\sum_{i,j=1}^{m,k}\|T(x_i,y_j)\|^p\right)^{\frac{1}{P}} &\leq \Lambda_p(T).\left[\left\|\left(\sum_{i=1}^{m}|x_i^+|^p\right)^{\frac{1}{p}}\right\|_{X_1}.\left\|\left(\sum_{j=1}^{k}|y_j^+|^p\right)^{\frac{1}{p}}\right\|_{X_2}\right.\\ &\quad+\left\|\left(\sum_{i=1}^{m}|x_i^+|^p\right)^{\frac{1}{p}}\right\|_{X_1}.\left\|\left(\sum_{j=1}^{k}|y_j^-|^p\right)^{\frac{1}{p}}\right\|_{X_2}\\ &\quad+\left\|\left(\sum_{i=1}^{m}|x_i^-|^p\right)^{\frac{1}{p}}\right\|_{X_1}.\left\|\left(\sum_{j=1}^{k}|y_j^+|^p\right)^{\frac{1}{p}}\right\|_{X_2}\\ &\quad\left.+\left\|\left(\sum_{i=1}^{m}|x_i^-|^p\right)^{\frac{1}{p}}\right\|_{X_1}.\left\|\left(\sum_{j=1}^{k}|y_j^-|^p\right)^{\frac{1}{p}}\right\|_{X_2}\right]\\ &\leq 4\Lambda_p(T).\left\|\left(\sum_{i=1}^{m}|x_i|^p\right)^{\frac{1}{p}}\right\|_{X_1}.\left\|\left(\sum_{j=1}^{k}|y_j|^p\right)^{\frac{1}{p}}\right\|_{X_2}\end{aligned}$$

Therefore, T is in $C_p^{mult}(X_1, X_2; Y)$. ■

Corollary 3.2.1 *If $1 \leq p < \infty$, then*

$$\Pi_p^{mult}(X_1, ..., X_n; Y) \subset \Lambda_p^{mult}(X_1, ..., X_n; Y) \subseteq C_p^{mult}(X_1, ..., X_n; Y).$$

3.2.1 Particular cases

The relations *(1.1.4)* and *(1.1.3)* yield the following results

1) For $p = 1$, then the following propostion

Proposition 3.2.1 *We have*

$$\Lambda_1^{mult}(X_1, ..., X_n; Y) = C_1^{mult}(X_1, ..., X_n; Y).$$

2) If $X_1, ..., X_n$ are $C(\Omega)$-spaces, then the following propostion gives the coincidence between $\Pi_p^{mult}(X_1, ..., X_n; Y)$, $\Lambda_p^{mult}(X_1, ..., X_n; Y)$ and $C_p^{mult}(X_1, ..., X_n; Y)$.

Proposition 3.2.2 *If $1 \leq p < \infty$*

$$\Pi_p^{mult}(X_1, ..., X_n; Y) = \Lambda_p^{mult}(X_1, ..., X_n; Y) = C_p^{mult}(X_1, ..., X_n; Y).$$

Positive multiple rueslts (Q. Bu and C. A. Labuschagne 2015 , see [12])

(i) Let $1 \leq p, q < \infty$ and Y be a Banach space.If

$$\Pi_p^{mult}(C(\Omega_1), ..., C(\Omega_n); Y) \subseteq \Pi_q^{mult}(C(\Omega_1), ..., C(\Omega_n); Y)$$

for any continuous function spaces $C(\Omega_1), ..., C(\Omega_n)$, then

$$C_p^{mult}(X_1, ..., X_n; Y) \subseteq C_q^{mult}(X_1, ..., X_n; Y)$$

for any Banach lattices $X_1, ..., X_n$

(**ii**) If $1 \leq p < \infty$, then

$$C_1^{mult}(X_1, ..., X_n; Y) \subseteq C_p^{mult}(X_1, ..., X_n; Y).$$

(**3i**) If $1 \leq p \leq q < 2$, then

$$C_p^{mult}(X_1, ..., X_n; Y) \subseteq C_q^{mult}(X_1, ..., X_n; Y).$$

(**4i**) If $1 \leq p < \infty$. Then

$$\Lambda_1^{mult}(X_1, ..., X_n; Y) \subseteq \Lambda_p^{mult}(X_1, ..., X_n; Y).$$

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