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On fuzzy rings and their ideal

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in front of the jury :

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Dedicaces

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To all my family and all my friends, to my supervisor, Dr. Amroune Abdelaziz . For everyone My dear teachers, and to everyone who loves good deeds and does not back down from life Barriers. To you, dear reader.

CONTENTS

INTRODUCTION

The concept of a fuzzy set was introduced by Zadeh in his article "Fuzzy Sets" [17] as a generalisation of the notion of the classical set. This theory was then developed by many authors[12, 13] .
Fuzzy set theory, a branch of mathematics dealing with uncertainty and vagueness, has diverse applications across various fields. Here are some of its key application areas: Control Systems, pattern recognition and image processing, data mining...etc.
The concept of fuzzy ideals and fuzzy rings emerged from fuzzy set theory, which extends classical set theory to handle uncertainty. In the late 1960 s and early 1970s, Lotfi Zadeh introduced fuzzy set theory as a way to deal with imprecise and vague information. Fuzzy ideals and fuzzy rings are generalizations of classical ideals and rings, allowing for degrees of membership rather than strict membership. In fuzzy set theory, each element can belong to a set with a degree of membership between 0 and 1, representing the degree to which the element possesses the characteristics of the set. Fuzzy ideals and fuzzy rings extend this notion to algebraic structures, such as rings and ideals, by allowing elements to belong to these structures with varying degrees of membership. These concepts have found applications in various areas, including control systems, decision making, and pattern recognition, where uncertainty is inherent. Fuzzy ideals and fuzzy rings provide a framework for reasoning about uncertainly in algebraic structures, enabling more flexible and robust analysis and decision-making processes. The notions of fuzzy rings and fuzzy ideals on a ring, which are fuzzy subsets are just a part of fuzzy logic and they were introduced by Liu[11] and they were developed by Mukherjee.
The aim of this memory is to investigate fuzzy subrings and fuzzy ideals concepts on rings and their fundamental properties.
The memory is divided into three chapters.
The first chapter, we give some fundamental concepts of fuzzy sets, operations of a fuzzy sets, characteristics sets of a fuzzy set. Also, we recall the fundamental concepts of groups, rings and ideals of a ring, morphism of a ring.
The second chapter, we have defined Fuzzy subring on a crisp ring, properties of a fuzzy subrings, operations of fuzzy subrings, characteristic sets of a fuzzy subring, and ring morphisms.
In the third chapter, we have defined by fuzzy ideals of a ring, operations on fuzzy ideals on a ring, characterisation of fuzzy ideals of a ring, and direct image and converse image

of an ideal via morphisms.

CHAPTER 1

PRELIMINARIES ON FUZZY SETS AND RINGS

Fuzzy sets constitute a generalisation of the notion of classical sets proposed, by Lotfi Zadeh in 1965. In this chapter, we present some basic concepts of fuzzy sets and some of their fundamental properties. Also, we give some rules for algebraic calculations in fuzzy sets. We will give definitions for each one of the following vocabularies: group, ring, ideal of a ring, and morphism in both classical and fuzzy logic.

1.1 Fuzzy set

1.1.1 Basic definitions and examples

Definition 1.1 (Classic set, intuitive definition) *A classical set is a collection of distinct objects. The objects that make up a set (also known as set elements or members) can be anything: numbers, people, letters of the alphabet, other sets, and so on.According to **Georg Cantor**, one of the founders of set theory).*
A set can be written,

1. ***In extension**, we give the list of its elements. For example, if $a_1, a_2, ..., a_n$ are the elements of the set A , we write:*
$$A = \{a_1, a_2, ..., a_n\}.$$
2. ***In understanding**, the properties that characterize its elements. For example, if the elements of the set B satisfying the conditions $p_1, p_2, ..., p_n$then the set B is defined by:*
$$B = \{b/b \text{ satisfied } p_1, p_2, ..., p_n\}.$$
3. ***In characteristic function**, characteristic function of A is a function on X ,is defined by:*

$$\chi_A : X \longrightarrow \{0,1\}$$

$$x \longmapsto \begin{cases} 0 \;\; if \;\; x \notin A, \\ 1 \;\; if \;\; x \in A. \end{cases}$$

Example 1.1 *Let $X = [a,b]$ such that $a, b \in R$ and let A be a classical subset of X defined by:*

$$\chi_A \longmapsto \begin{cases} 0 \;\; if \;\; x \leq a, \\ 1 \;\; if \;\; x \;\; a < x \leq b. \end{cases}$$

Definition 1.2 (fuzzy set) *[17] A fuzzy set A is characterised by a generalised characteristic function. $\mu_A : X \longrightarrow [0,1]$, called the membership function of A and defined over a universe of discourse X.*

$$A = \{\langle x, \mu_A(x)\rangle \,|x \in X\}$$

Example 1.2 $X = \{motorbike, car, train\}$ *means of transport,*
Let A be a subset of X, the means of fast transport

$$A = \{(motorbike, 0.7), (car, 0.5), (train, 1)\}$$

Example 1.3 *(i) Let $X = \{a,b,c\}$ be universal set. $A = \{(a,0.2),(b,0.8),(c,1)\}$ a fuzzy subset in X ;*

(ii) Let $X = [0,7]$, and A fuzzy subset in X, defined by :

$$\mu_A(x) = \frac{1}{(1+x)^2}$$

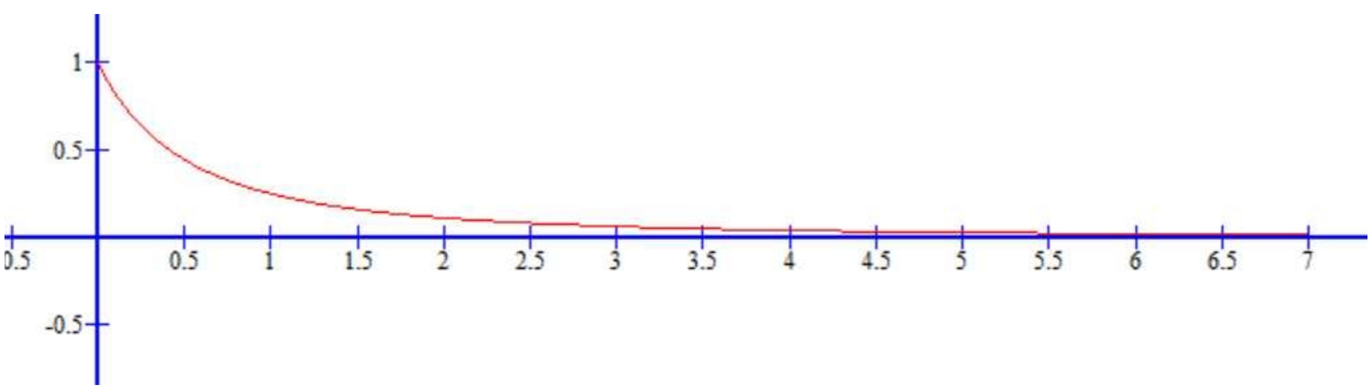

Figure 1.1: *
graph of μ_A

1.1.2 Operations of fuzzy sets

In this section we will see some operations in this fuzzy sets, some of which we will mention: equality, inclusion, union, intersection, sum, product, complementation, and cartesian product.

Definition 1.3 (Equality) *[9] Let A, B two fuzzy sets of X. We say that: $A = B$, if and only if $\mu_A(x) = \mu_B(x)$, for all $x \in X$.*

Definition 1.4 (Inclusion) *[9] Let A, B two fuzzy sets of X. We say that the fuzzy set A is included in B if:*

$$\mu_A(x) \leq \mu_B(x), \forall x \in X.$$

Definition 1.5 (Intersection) *[9] Let A, B two fuzzy sets of X. The intersection of A and B is the fuzzy set $A \cap B$ with,*

$$\mu_{A\cap B}(x) = min\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x), \forall x \in X.$$

Definition 1.6 (Union) *[9] Let X be a non-empty set and let A and B two fuzzy subsets of X. The union of A and B is the fuzzy set $A \cup B$, where*

$$\mu_{A\cup B}(x) = max\,\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \vee \mu_B(x), \forall x \in X.$$

Definition 1.7 (Complement) *[9] Let A be a fuzzy subset. The complement of A is the fuzzy subset A^c where*

$$\mu_{A^c}(x) = 1 - \mu_A(x), \forall x \in X.$$

Definition 1.8 (Sum) *[9] Let A, B be two fuzzy subset in X . The sum of A and B is the fuzzy subset, where*

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \forall x \in X.$$

Example 1.4 *Let $X = \{a, b, c\}$, and let $A = \{(a, 0.2), (b, 0.4), (c, 0.7)\}$, and*

$$B = \{(a, 0), (b, 0.3), (c, 1)\},$$

we have :

1. $A \cap B = \{(a, 0), (b, 0.3), (c, 0.7)\}$,
2. $A \cup B = \{(a, 0.2), (b, 0.4), (c, 1)\}$,
3. $A + B = \{(a, 0.2), (b, 0.58), (c, 1)\}$,
4. $A^c = \{(a, 0.8), (b, 0.6), (c, 0.3)\}$.

• Among these degrees of membership, the minimum is the cartesian product of the fuzzy subsets.

Definition 1.9 (Cartesian product of fuzzy subsets) *[12]*
The cartesian product applied to n fuzzy subsets can be defined as follows: let $\mu_{A_1}, \mu_{A_2}, ...\mu_{A_n}$, be the membership functions of $A_1, A_2, ..., A_n$. Then, the membership degree of $(x_1, x_2, ...x_n) \in X_1 \times X_2 ... X_n$ on the fuzzy subset $A_1 \times A_2 \times ... \times A_n$ is given by:
$\mu_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) = min \{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\}$.

Example 1.5 *Lets $X_1 = \{a, b\}, X_2 = \{\alpha, \beta\}$, and let A_1, A_2 be two fuzzy subset, respectively defined on X_1 and X_2 by:*
$A_1 = (a, 0.1), (b, 0.4)$,
$A_2 = (\alpha, 0.2), (\beta, 0.6)$.
Then,
$A_1 \times A_2 = \{((a, \alpha), 0.1), ((a, \beta), 0.1), ((b, \alpha), 0.2), ((b, \beta), 0.4)\}$.

1.1.3 Characteristics sets of fuzzy subsets

This section gives definitions of concepts like "height", "cardinality of a fuzzy subset", "ker" and "support," with an example.

Definition 1.10 (Support) *[17, 18] Let X be a non-empty set and let A be a fuzzy subset of X. The support of A is the crisp subset of X given by:*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

Definition 1.11 (Kernel) *[17, 18] Let X be a non-empty set and let A be a fuzzy subset of X. The kernel of A is the crisp subset of X given by:*

$$ker(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

Definition 1.12 (Height) *[17, 18] Let X be a non-empty set and let A be a fuzzy subset of X. The height of A is the highest value taken by its membership function, given by:*

$$H(A) = sup\{\mu_A(x) | x \in X\}.$$

Definition 1.13 (Cardinality) *[17, 18] The cardinality of a finite fuzzy subset A denoted $\mid A \mid$, is defined as:*

$$\mid A \mid = \sum_{x \in X} \mu_A(x).$$

Example 1.6 *Let $X = \{1, 2, ..., 6\}$, and let A be a fuzzy subset of X given by:*

$$A = \{\langle x, \mu_A(x) \rangle\} = \{\langle 1, 0.2 \rangle, \langle 2, 0 \rangle, \langle 3, 0.8 \rangle, \langle 4, 1.0 \rangle, \langle 5, 0.5 \rangle, \langle 6, 1.0 \rangle\}.$$

Then:
$supp(A) = \{1, 3, 4, 5, 6\}$, $Ker(A) = \{4, 6\}$,
$H(A) = \{1\}$, $|A| = 3.5$.

Proposition 1.1 *The kernel and support of a fuzzy subset satisfy the following properties:*

(i) $Supp(A^c) = X - ker(A)$.

(ii) $ker(A^c) = X - Supp(A)$.

Proof
(i)

$$\begin{aligned} Supp(A^c) &= \{x \in X \mid \mu_{A^c}(x) \neq 0\}, \\ &= \{x \in X \mid 1 - \mu_A(x) \neq 0\}, \\ &= \{x \in X \mid \mu_A(x) \neq 1\}, \\ &= \{x \in X \mid x \notin ker(A)\}, \\ &= X - ker(A). \end{aligned}$$

(ii)

$$\begin{aligned} ker(A^c) &= \{x \in X \mid \mu_{A^c}(x) = 1\}, \\ &= \{x \in X \mid 1 - \mu_A(x) = 1\}, \\ &= \{x \in X \mid \mu_A(x) = 0\}, \\ &= \{x \in X \mid x \notin supp(A)\}, \\ &= X - supp(A). \end{aligned}$$

Definition 1.14 ($\alpha - cuts$) *[17, 18] For a given fuzzy subset A on a universe X, the $\alpha - cuts$ of A, written A_α, is defined as:*
$A_\alpha = \{x \in X, \mu_A(x) \geq \alpha\}$*, for* $\alpha \in]0, 1]$ *particular cases:*

1. *if* $\alpha = 0, then \quad A_0 = X$,
2. *if* $\alpha = 1, then \quad A_1 = ker(A)$.

Example 1.7 *let* $X = \{1, 2, 3, ..., 10\}$*, and A be a fuzzy subset of* X *given by:*

$$A = \{\langle 1, 0.2\rangle, \langle 2, 0.5\rangle, \langle 3, 0.8\rangle, \langle 4, 1\rangle, \langle 5, 0.7\rangle, \langle 6, 0.3\rangle, \langle 7, 0\rangle, \langle 8, 0\rangle, \langle 9, 0\rangle, \langle 10, 0\rangle\},$$

the $\alpha - cuts$ *of A.*
$A_0 = X$,
$A_{0.2} = \{x \in X, A(x) \geq 0.2\} = \{1, 2, 3, 4, 5, 6\}$,
$A_{0.3} = \{x \in X, A(x) \geq 0.3\} = \{2, 3, 4, 5, 6\}$,
$A_{0.5} = \{x \in X, A(x) \geq 0.5\} = \{2, 3, 4, 5\}$,
$A_{0.7} = \{x \in X, A(x) \geq 0.2\} = \{3, 4, 5\}$,
$A_{0.8} = \{x \in X, A(x) \geq 0.2\} = \{3, 4\}$,
$A_1 = \{x \in X, A(x) \geq 1\} = \{4\}$.

Proposition 1.2 (properties of $\alpha - cuts$) *: Let A, B be two fuzzy subsets on a universe X and* $\alpha, \beta \in]0, 1]$*. It holds that:*

(i) $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$,

(ii) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$,

(iii) if $\alpha \leq \beta$, *then* $A_\beta \subseteq A_\alpha$,

(iv) $A_0 = X$,

(v) $A_1 = ker(A)$.

Proof.
(i)

$$\begin{aligned}(A \cup B)_\alpha &= \{x \in X : \mu_A(x) \cup \mu_B(x) \geq \alpha\}, \\ &= \{x \in X : max(\mu_A(x), \mu_B(x)) \geq \alpha\}, \\ &= \{x \in X : \mu_A(x) \geq \alpha \vee \mu_B(x) \geq \alpha\}, \\ &= \{x \in X : \mu_A(x) \geq \alpha\} \cup \{x \in X : \mu_B(x) \geq \alpha\}, \\ &= A_\alpha \cup B_\alpha.\end{aligned}$$

(ii)

$$\begin{aligned}(A \cap B)_\alpha &= \{x \in X : \mu_A(x) \cap \mu_B(x) \geq \alpha\}, \\ &= \{x \in X : min(\mu_A(x), \mu_B(x)) \geq \alpha\}, \\ &= \{x \in X : \mu_A(x) \geq \alpha \wedge \mu_B(x) \geq \alpha\}, \\ &= \{x \in X : \mu_A(x) \geq \alpha\} \cap \{x \in X : \mu_B(x) \geq \alpha\}, \\ &= A_\alpha \cap B_\alpha.\end{aligned}$$

(iii)
Let $x \in A_\beta, i.e., \mu_A(x) \geq \beta$. Since $\alpha \leq \beta$, it holds that $\mu_A(x) \geq \alpha$.
Hence, $x \in A_\alpha$, and thus $A_\beta \subseteq A_\alpha$.
(iv)

$$A_0 = \{x \in X : \mu_A(x) \geq 0\} = X.$$

(v)

$$\begin{aligned}A_1 &= \{x \in X : \mu_A(x) \geq 1\} \\ &= \{x \in X : \mu_A(x) = 1\} \\ &= ker(A).\end{aligned}$$

Definition 1.15 (The strong $\alpha - cuts$**)** *[17] For any* α *of* $]0, 1]$, *we define the strong* $\alpha - cut$ *of the fuzzy subset A as the subset:*

$$A_\alpha^+ = \{x \in X, \mu_A(x) > \alpha\}.$$

Example 1.8 *Let* $X = \{1, 2, 3, 4\}$ *and let* $A = \{(1, 0.1), (2, 0.3)(3, 0)(4, 0.23)\}$*,*
Then
$A_{0.1}^+ = \{2, 4\}$,
$A_{0.3}^+ = \{\emptyset\}$,
$A_0^+ = \{1, 2, 4\}$,
$A_{0,23}^+ = \{2\}$.

Proposition 1.3 *Let A,B are two fuzzy subset .For any $\alpha, \beta \in]0.1]$, we have:*

(i) $A^+_\alpha \subseteq A_\alpha$.

(ii) $\alpha \leq \beta$ *implies* $A^+_\alpha \supseteq A^+_\beta$*, for all* $\alpha, \beta \in [0.1[$.

(iii) $\alpha \leq \beta$ *implies* $A_\alpha \supseteq A_\beta$*, for all* $\alpha, \beta \in]0.1]$.

(iv) $A \subseteq B$ *if only if* $A_\alpha \subseteq B_\alpha$*, for all* $\alpha \in]0.1]$.

(v) $A \subseteq B$ *if only if* $A^+_\alpha \subseteq B^+_\alpha$*, for all* $\alpha \in [0.1[$.

Proof

(i) Let $x \in A^+_\alpha$ implies $x \in A_\alpha$,
$A_\alpha = \{x \in X, \mu_A(x) \geq \alpha\}$,
$A^+_\alpha = \{x \in X, \mu_A(x) > \alpha\}$,
If $x \in A^+_\alpha$ implies $\mu_A(x) > \alpha$.
So, $x \in A_\alpha$ then $A^+_\alpha \subseteq A_\alpha$.

(ii) Let $\alpha \leq \beta$ implies $A^+_\beta \subseteq A^+_\alpha$,
If $x \in A^+_\beta$ implies $x \in A^+_\alpha$,
Now,
let $x \in A^+_\beta$ implies $\mu_A(x) > \beta > \alpha$,
implies $\mu_A(x) > \alpha$. So, $x \in A^+_\alpha$.
Then $A^+_\beta \subseteq A^+_\alpha$.

(iii) Assume $\alpha \leq \beta$ and suppose that $x \in A_\beta$.
Then $\mu_A(x) \geq \beta \geq \alpha$.
Thus $x \in A_\alpha$.

(iv) Assume $A \subseteq B$, and $\mu_A(x) \leq \mu_B(x)$, for all $x \in X$.
To prove that: $A_\alpha \subseteq B_\alpha$, suppose that $x \in A_\alpha$,
then $\mu_A(x) \geq \alpha$, for all $\alpha \in]0.1]$.
Since $\mu_B(x) \geq \mu_A(x) \geq \alpha$, for all $x \in X$.
It follows that, $\mu_B(x) \geq \alpha$.
Thus, $x \in B_\alpha$.
Conversely, Let $A_\alpha \subseteq B_\alpha$,
$x \in A_\alpha$ implies $x \in B_\alpha$, $\mu_A(x) \geq \alpha$ implies $\mu_B(x) \geq \alpha$,
$\mu_A(x) \leq \mu_B(x)$.
Finally $A \subseteq B$.

(v) First let us assume that $A \subseteq B$, then $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$.
To prove that $A^+_\alpha \subseteq B^+_\alpha$,
Let $x \in A^+_\alpha$ implies $\mu_A(x) > x$, implies $\mu_B(x) \geq \mu_A(x) > \alpha$,
then $\mu_B(x) > \alpha$ implies $x \in B^+_\alpha$.

Conversely, let assume that $A_\alpha^+ \subseteq B_\alpha^+$.
To prove that : $A \subseteq B$,
$A_\alpha^+ \subseteq B_\alpha^+$then $x \in A_\alpha^+$ implies $x \in B_\alpha^+$.
So, $\mu_A(x) > \alpha$ implies $\mu_B(x) > \alpha$, then $\mu_B(x) \geq \mu_A(x)$.

Definition 1.16 ($\alpha - line$) *[17] Let X be a non-empty set and let A be a fuzzy subset of X. For $\alpha \in]0,1]$, the $\alpha - line$ of A denoted L. We mean all elements of X that belong to A to a degree equal. That is the L classical set defined by:*

$$L_\alpha(A) = \{x \in X | \mu_A(x) = \alpha\}.$$

Example 1.9 *Let $X = \{1,2,3,4,5,6\}$ and let $A = \{(1,0.2)(2,0.5)(3,0.8)(4,1)(5,0.7)(6,0.3)\}$.*
$L_{0.2}(A) = 1$,
$L_{0.5}(A) = 2$,
$L_{0.8}(A) = 3$,
$L_1(A) = 4$,
$L_{0.7}(A) = 5$,
$L_{0.3}(A) = 6$.

1.2 The direct image and the reciprocal image of a fuzzy subset

In this section, we study the direct image and the reciprocal image of a fuzzy subset.

Definition 1.17 (The direct image) *Let X and Y be two non-empty sets, and let $f : X \longrightarrow Y$ be a function. For a fuzzy subset A in X, if $f[A]$ is a fuzzy subset in Y, the membership function is given by:*

$$x \longmapsto \begin{cases} sup_{x \in f^{-1}[y]} \ \ if \quad f^{-1}[y] \ is \ non - empty, \\ 0 \ \ if \quad f^{-1}[y] \ is \ empty. \end{cases}$$

For all $y \in Y$.

Definition 1.18 (the reciprocal image) *Let f be a function from X to Y, the reciprocal image. If a fuzzy subset B of Y is written as $f^{-1}(B)$, is a fuzzy subset in X, the function Membership is defined by: $f^{-1}[B](x) = B(f(x))$, for all $x \in X$.*
Or $f^{-1}(B) = \{\langle x, f^{-1}[B](x)\rangle, x \in X\}$, with $f^1[B](x) = B[f(x)]$.

Example 1.10 *Let $X = \{a,b,c\}, Y = \{l,m,n\}$. Let A,B be two fuzzy subsets in Y respectively.*
$A = \{\langle a, 0.4\rangle, \langle b, 0.1\rangle, \langle c, 0.6\rangle\}$, $B = \{\langle d, 0.5\rangle, \langle e, 0.8\rangle, \langle g, 0.9\rangle\}$.
Let $f : X \longrightarrow Y$ is a function defined by: $f(a) = d$, $f(b) = d$ and $f(c) = e$, then :
(i)
For $f^{-1}(d) = \{a,b\} \neq \emptyset$.
So,
$f[A](d) = sup\{A(x) : x \in f^{-1}(d)\}$
$= sup\{\mu_A(a), \mu_A(b)\}$
$= sup\{0.4, 0.1\} = 0.4$.

For $f^{-1}(e) = c \neq \emptyset$.
Then, $f[A](e) = sup\{A(x) : x \in f^{-1}(e)\} = sup\{\mu_A(c)\} = 0.6$.
For $f^{-1}(g) = \emptyset$, *so :* $f[A](g) = 0$.
Then, $f[A] = \{(d, 0.4), (e, 0.6), (g, 0)\}$.
(ii)
For $f^{-1}[B](a) = B(f(a)) = B(d) = 0.5$.
$f^{-1}[B](b) = B(f(b)) = B(d) = 0.5$.
$f^{-1}[B](c) = B(f(c)) = B(e) = 0.8$.
Then, $f^{-1}[B] = \{(a, 0.2), (b, 0.2), (c, 0.7)\}$.

1.3 Rings and ideals

In this section, we will give definitions for each one of the following vocabularies: group strict, ring, ideal on a ring, and morphism

1.3.1 structure group

Here, we provide a definition and examples of groups, abelian groups and subgroups.

Definition 1.19 (Group) *[10] Let G be a non-empty set, and let the internal composition law be* $" \cdot "$,

$$" \cdot " : G \times G \longrightarrow G$$
$$(x, y) \longmapsto x \cdot y$$

We say that $(G, \cdot)$ *is a group if it satisfies the following conditions:*

(i) *G is closed under the operation* $" \cdot "$*, i.e.,* $x \cdot y \in G$*, for all* $x, y \in G$.

(ii) *The operation* $" \cdot "$ *is associative, i.e.,* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$*, for all* $x, y, z \in G$.

(iii) *There is an identity element* $e \in G$ *such that* $x \cdot e = e \cdot x = x$*, for all* $x \in G$.

(iv) *Each element* $x \in G$ *has an inverse element* $x^{-1} \in G$ *such that* $x \cdot x^{-1} = x^{-1} \cdot x = e$.

Definition 1.20 (Abelian groupe) *[10] Let* $(G, \cdot)$ *be a group . Then* $(G, \cdot)$ *is called a abelian group, or commutative group, if* $x \cdot y = y \cdot x$ *for all* $x, y \in G$.

Example 1.11 $(\mathbb{Z}, +), (\mathbb{R}^*, \times)$ *are abelian groups.*

Definition 1.21 (subgroup) *[10] Let* A *be a non-empty subset of a group* G. A *is called a subgroup of G if* H *is itself a group with respect to the operation on* G.

Theorem 1.1 *Let* A *be a subset* H *of the group* G. A *is a subgroup of* G *if and only if these conditions are satisfied:*

(ii) *:H is nonempty,*

(ii) :$x \in Handy \in Himplyx - y \in H$.

Example 1.12 $(\mathbb{Z}, +)$ *is a subgroup of* $(\mathbb{R}, +)$.

1.3.2 Rings

In the following,we provide definitions and examples of ring,abelian ring, and subring.

Definition 1.22 (Ring) *[10] Let A be a ring.* $(R, +, \cdot)$ *is a set* R*, together with two binary operations "+" and "* $\cdot$ *" on* R*, satisfying the following axioms":*

(i) $(R, +)$ *is an abelian group,*

(ii) associativity of multiplication, i.e., $(x \cdot y) \cdot z = x \cdot (y \cdot z)$*,*
for all $x, y, z \in R$*,*

(iii) two distributive laws hold in R *such that* $x.(y + z) = x.y + x.z$*,*
for all $x, y, z \in R$*.*

Remark 1.1 *Let* $(R, +, \cdot)$ *be a crisp ring. The multiplicative identity of* R *will be denoted* e*, and we say that the ring* R *is with identity.*

Definition 1.23 (Abelian ring) *[10] Let* $(R, +, \cdot)$ *be a ring. Then* $(R, +, \cdot)$ *is called a commutative ring, if* $x \cdot y = y \cdot x$ *for all* $x, y \in R$

Example 1.13 *The structures* $(\mathbb{Z}, +, \times)$*,* $(\mathbb{Q}, +, \times)$*,* $(\mathbb{R}, +, \times)$ *and* $(\mathbb{C}, +, \times)$ *are commutative ring with identity.*

Definition 1.24 (Subring) *[10] If* $(R, +, \cdot)$ *is a ring, A non-empty subset* S *of* R *is a subring of* R *if* S *itself is a ring with respect to the operations on* R*.*

Next, we provide an equivalent definition of the subring.

Theorem 1.2 *A subset* S *of the ring* R *is a subring of* R *if and only if these conditions are satisfied :*

(i) S *is non-empty ,*

(ii) $(S, +)$ *is a subgroup de* R *,*

(iii) for all $x, y \in S \Longrightarrow x \cdot y \in S$*.*

Example 1.14 $(\mathbb{Z}, +, \times)$ *is a subring of* $(\mathbb{R}, +, \times)$*.*

1.3.3 Morphism of a rings

A morphism (or homomorphism) between two rings is a function that preserves the operations of addition and multiplication as well as the identity.

Definition 1.25 *[10] Let* $(R, +, \cdot)$ *and* $(S, \circ, *)$ *be two rings.*
The function $f : R \longrightarrow S$ *is called a ring morphism if for all* $x, y \in R$ *we have :*

(i) $f(x + y) = f(x) \circ f(y)$ *,*

(ii) $f(x \cdot y) = f(x) * f(y)$ *,*

(iii) $f(1_R) = 1_S$*.*

A ring isomorphism is a bijective ring morphism. We say R and S are isomorphic rings and we write $R \cong S$.

1.3.4 Ideal of a rings

Now, we give definition and example to ideal on a ring.

Definition 1.26 *[10] A non-empty set I of a ring R is called an ideal on R if*

(i) $(I, +)$ *subgroup of a group* $(R, +)$ *,*

(ii) $\forall a \in I, \forall x \in R \Rightarrow a \cdot x \in I, x \cdot a \in I.$

Example 1.15 *Consider the ring* $(\mathbb{Z}, +, \times)$*. Let* $n \in \mathbb{N}$*. Then* $I = \{qn \mid q \in \mathbb{Z}\}$ *is an ideal of* $\mathbb{Z}$*.*

CHAPTER 2

FUZZY SUBRING AND THEIR PROPERTIES

In this chapter we will give the definition of the fuzzy ring as well as some of its related fundamental properties.

2.1 Fuzzy subring of a crisp ring

Definition 2.1 *[14] Let* $(\mathbb{R}, +, \cdot)$ *be a crisp ring, and let A be a fuzzy subset of* $\mathbb{R}$*. Then A is said to be a fuzzy subring of* $\mathbb{R}$ *if the following conditions are satisfied:*

1. $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y) ... (I_1)$,
2. $\mu_A(x \cdot y) \geq \mu_A(x) \wedge \mu_A(y) ... (I_2)$.

Example 2.1 *Let us consider the crisp ring* $(\mathbb{Z}, +, \cdot)$*, and let A be a fuzzy subset of* $\mathbb{Z}$ *defined by:*

$$\mu_A(x) = \begin{cases} 0.4 \textit{ if } x \in 2\mathbb{Z}, \\ 0.2 \textit{ if } x \in 2\mathbb{Z} + 1. \end{cases} \tag{2.1}$$

x	y	$\mu_A(x)$	$\mu_A(y)$	$x - y$	$x \cdot y$	$\mu_A(x-y)$	$\mu_A(x \cdot y)$	$\mu_A(x) \wedge \mu_A(y)$	I_1	I_2
$2\mathbb{Z}$	$2\mathbb{Z}$	0.4	0.4	$2\mathbb{Z}$	$2\mathbb{Z}$	0.4	0.4	0.4	T	T
$2\mathbb{Z}$	$2\mathbb{Z}+1$	0.4	0.2	$2\mathbb{Z}+1$	$2\mathbb{Z}$	0.2	0.4	0.2	T	T
$2\mathbb{Z}+1$	$2\mathbb{Z}$	0.2	0.4	$2\mathbb{Z}+1$	$2\mathbb{Z}$	0.2	0.4	0.2	T	T
$2\mathbb{Z}+1$	$2\mathbb{Z}+1$	0.2	0.2	$2\mathbb{Z}$	$2\mathbb{Z}$+1	0.4	0.2	0.2	T	T

Then A is a fuzzy subring of $\mathbb{Z}$.

2.2 Properties of fuzzy subrings

Proposition 2.1 *Let $(\mathbb{R}, +, \cdot)$ be a ring, and let A be a fuzzy subring of $\mathbb{R}$. Then*

(i) $\mu_A(0) \geq \mu_A(x)$ *for all* $x \in R$,

(ii) $\mu_A(x) = \mu_A(-x)$ *for all* $x \in R$.

Proof

(i) Since A is a fuzzy subring of $\mathbb{R}$, then for all $x \in R$,
$\mu_A(0) = \mu_A(x - x) \geq \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$.

(ii) For all $x \in \mathbb{R}$, we have,
$\mu_A(-x) = \mu_A(0 - x) \geq \mu_A(0) \wedge \mu_A(x) = \mu_A(x)$,
then $\mu_A(-x) \geq \mu_A(x)$.
Now, replacing x by $-x$ we get, $\mu_A(x) \geq \mu_A(-x)$.
Hence $\mu_A(-x) = \mu_A(x)$.

Proposition 2.2 *Let $(\mathbb{R}, +, \cdot)$ be a crisp ring, and let A be a fuzzy subring of $\mathbb{R}$. Then $\mu_A(r.x) \geq \mu_A(x)$, for all $x \in \mathbb{R}$ and for all integers r.*

Proof

Case 1 : Let r be a positive integer.
for all $x \in \mathbb{R}$. Here, P(1) is trivially true. Since A is a fuzzy subring of $\mathbb{R}$,
therefore, Let us assume that $P(r)$ is true and we will prove $P(r + 1)$.
Then $\mu_A((r + 1)x) = \mu_A(rx + x) \geq \mu_A(rx) \wedge \mu_A(x) \geq \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$,
then, by recurrence, $\mu_A(r.x) \geq \mu_A(x)$ for r positive.

Case 2 : Let r be a negative integer.
For all $x \in \mathbb{R}$, we pose $r = -s$,
then, $\mu_A(r.x) = \mu_A(-s.x) \geq \mu_A(s.x) \geq \mu_A(x)$.

2.3 Operations of fuzzy subring

Proposition 2.3 *Let $(\mathbb{R}, +, \cdot)$ be a crisp ring, and let A and B be two fuzzy subring of $\mathbb{R}$. Then $A \cap B$ is a fuzzy subring of $\mathbb{R}$.*

Proof Let A and B be two fuzzy subrings of a ring $\mathbb{R}$.

$$\begin{aligned}
\mu_{A\cap B}(x - y) &= \mu_A(x - y) \wedge \mu_B(x - y), \\
&\geq (\mu_A(x) \wedge \mu_A(y)) \wedge (\mu_B(x) \wedge \mu_B(y)), \\
&= (\mu_A(x) \wedge \mu_B(x)) \wedge (\mu_A(y) \wedge \mu_B(y)), \\
&= \mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y).
\end{aligned}$$

Then $\mu_{A\cap B}(x-y) \geq \mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y)$.

$$\begin{aligned} \mu_{A\cap B}(x\cdot y) &= \mu_A(x\cdot y)\wedge\mu_B(x\cdot y), \\ &\geq (\mu_A(x)\wedge\mu_A(y))\wedge(\mu_B(x)\wedge\mu_B(y)), \\ &= (\mu_A(x)\wedge\mu_B(x))\wedge(\mu_A(y)\wedge\mu_B(y)), \\ &= \mu_{A\cap B}(x)\wedge\mu_{A\cap B}(y). \end{aligned}$$

Then, $\mu_{A\cap B}(x\cdot y) \geq \mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y)$.

So, $A\cap B$ is a fuzzy subring of $\mathbb{R}$.

Remark 2.1 *Let $(\mathbb{R},+,\cdot)$ be a crisp ring, and let A and B be two fuzzy subrings of $\mathbb{R}$. The union $A\cup B$ is not necessarily a fuzzy subring of $\mathbb{R}$. Indeed,*

Example 2.2 *Let A and B be two fuzzy subrings of a ring $(\mathbb{Z},+,\cdot)$ defined by:*

$$\mu_A(x) = \begin{cases} 0.6 \textit{ if } x\in 2\mathbb{Z}, \\ 0.2 \textit{ if } x\in 2\mathbb{Z}+1. \end{cases}$$

And

$$\mu_B(x) = \begin{cases} 0.5 \textit{ if } x\in 3\mathbb{Z}, \\ 0.3 \textit{ otherwise}. \end{cases}$$

Then,

$$\mu_{A\cup B}(x) = \begin{cases} 0.6 \textit{ if } x\in 6\mathbb{Z}, \\ 0.6 \textit{ if } x\in 2\mathbb{Z}, \\ 0.5 \textit{ if } x\in 3\mathbb{Z}, \\ 0.3 \textit{ otherwise}. \end{cases}$$

We have $\mu_{A\cup B}(3-2) = \mu_{A\cup B}(1) = 0.3$,

$\mu_{A\cup B}(3)\wedge\mu_{A\cup B}(2) = 0.5\wedge 0.6 = 0.5$,

$\mu_{A\cup B}(3-2) \not\geq \mu_{A\cup B}(3)\wedge\mu_{A\cup B}(2)$,

then $A\cup B$ is not fuzzy subring.

Proposition 2.4 *Let A and B be two fuzzy subrings of a ring $(\mathbb{R},+,\cdot)$. Then $A\times B$ is a fuzzy subring of $R\times R$.*

proof Let A and B be two fuzzy subrings of a ring $\mathbb{R}$. Then:

$$\begin{aligned} \mu_{A\times B}(x-y) &= \mu_A(x-y)\wedge\mu_B(x-y), \\ &\geq (\mu_A(x)\wedge\mu_A(y))\wedge(\mu_B(x)\wedge\mu_B(y)), \\ &= (\mu_A(x)\wedge\mu_B(x))\wedge(\mu_A(y)\wedge\mu_B(y)), \\ &= \mu_{A\times B}(x)\geq\mu_{A\times B}(y). \end{aligned}$$

Then, $\mu_{A\times B}(x-y) \geq \mu_{A\times B}(x) \wedge \mu_{A\times B}(y)$.

$$\begin{aligned}
\mu_{A\times B}(x\cdot y) &= \mu_A(x\cdot y) \wedge \mu_B(x\cdot y),\\
&\geq (\mu_A(x) \wedge \mu_A(y)) \wedge (\mu_B(x) \wedge \mu_B(y)),\\
&= (\mu_A(x) \wedge \mu_B(x)) \wedge (\mu_A(y) \wedge \mu_B(y)),\\
&= \mu_{A\times B}(x) \wedge \mu_{A\times B}(y).
\end{aligned}$$

Then, $\mu_{A\times B}(x\cdot y) \geq \mu_{A\times B}(x) \wedge \mu_{A\times B}(y)$.

Hence, $A \times B$ is a fuzzy subring of $R \times R$.

Remark 2.2 *The fuzzy complement of a fuzzy subring is not necessarily a fuzzy subring. Indeed,*

Example 2.3 *In example 2.2 we have*
Let $(\mathbb{Z}, +, \cdot)$ be a crisp ring, and let A be fuzzy subring of a ring $\mathbb{R}$, defined by:

$$\mu_A(x) = \begin{cases} 0.6 \textit{ if } x \in 2\mathbb{Z}, \\ 0.2 \textit{ if } x \in 2\mathbb{Z}+1. \end{cases}$$

Then,

$$\mu_{\bar{A}}(x) = \begin{cases} 0.4 \textit{ if } x \in 2\mathbb{Z}, \\ 0.8 \textit{ if } x \in 2\mathbb{Z}+1. \end{cases}$$

$$\begin{aligned}
\mu_{\bar{A}}(3+1) &\geq \mu_{\bar{A}}(3) \wedge \mu_{\bar{A}}(1),\\
\mu_{\bar{A}}(4) &\geq \mu_{\bar{A}}(3) \wedge \mu_{\bar{A}}(1),\\
0.4 &\geq 0.8 \wedge 0.8,\\
0.4 &\not\geq 0.8.
\end{aligned}$$

Hence, $\bar{A}$ is not fuzzy subring.

2.4 Characterisations of a fuzzy subring

Proposition 2.5 *Let $(\mathbb{R}, +, .)$ be a crisp ring and let A be a fuzzy subring of $\mathbb{R}$. Then $supp(A)$ is a crisp subring of $\mathbb{R}$.*

Proof Let A be a fuzzy subring of $\mathbb{R}$.

1. Let $x, y \in supp(A)$, then $\mu_A(x) > 0$ and $\mu_A(y) > 0$, and we have $\mu_A(x-y) > \mu_A(x) \wedge \mu_A(y) > 0 \wedge 0 = 0$, then $x - y \in supp(A)$.
2. Let $x, y \in supp(A)$ then, $\mu_A(x) > 0$ and $\mu_A(y) > 0$ and we have $\mu_A(x.y) > \mu_A(x) \wedge \mu_A(y) > 0 \wedge 0 = 0$, then $x.y \in supp(A)$.

Hence, $supp(A)$ is a crisp subring of $\mathbb{R}$.

Proposition 2.6 *Let $(\mathbb{R}, +, .)$ be a crisp ring, and let A be a fuzzy subring of $\mathbb{R}$. Then $ker(A)$ is a crisp subring of $\mathbb{R}$.*

Proof Let A be a fuzzy subring of $\mathbb{R}$.

1. Let $x, y \in ker(A)$, then $\mu_A(x) = 1$ and $\mu_A(y) = 1$, and we have $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y) = 1 \wedge 1 = 1$, then $\mu_A(x - y) = 1$, so $x - y \in ker(A)$.
2. Let $x, y \in ker(A)$,then $\mu_A(x) = 1$ and $\mu_A(y) = 1$, and we have $\mu_A(x.y) \geq \mu_A(x) \wedge \mu_A(y) = 1 \wedge 1 = 1$, then $\mu_A(x.y) = 1$, hence $x.y \in ker(A)$.

Hence, $ker(A)$ is a crisp subring of $\mathbb{R}$.

Proposition 2.7 *Let $(\mathbb{R}, +, .)$ be a crisp ring and let A be a fuzzy subset of R, then A is a fuzzy subring of $\mathbb{R}$. If and only if A_α is a crisp subring of $\mathbb{R}$, for all $\alpha \in]0.1]$.*

Proof Let A be a fuzzy subring of $\mathbb{R}$.

1. Let $x, y \in A_\alpha$, then $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$, and we have $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y) \geq \alpha \wedge \alpha = \alpha$, then $x - y \in A_\alpha$.
2. Let $x, y \in A_\alpha$, then $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$, and we have $\mu_A(x.y) \geq \mu_A(x) \wedge \mu_A(y) \geq \alpha \wedge \alpha = \alpha$, then $x.y \in A_\alpha$.

Hence, A_α is a crisp subring of $\mathbb{R}$.
Conversely,
Suppose that A_α is a cris subring, for every $\alpha \in]0, 1]$.
We put $\mu_A(x) \wedge \mu_A(y) = \alpha$.
(1)
If $\alpha = 0$, it is easy to see that
$\mu_A(x - y) \geq 0 = \alpha$, and $\mu_A(x \cdot y) \geq 0 = \alpha$,
$\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y) = \alpha$,
$\mu_A(x \cdot y) \geq \mu_A(x) \wedge \mu_A(y) = \alpha$.
(2)
If $\alpha > 0$, $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$,
then, $x \in A_\alpha$ and $y \in A_\alpha$,
so, $x - y \in A_\alpha$, and $x \cdot y \in A_\alpha$.
therefore,
$\mu_A(x - y) \geq \alpha = \mu_A(x) \wedge \mu_A(y)$,
$\mu_A(x \cdot y) \geq \alpha = \mu_A(x) \wedge \mu_A(y)$.
Consequently, A is a fuzzy subring of R.

Proposition 2.8 *Let $(\mathbb{R}, +, .)$ be a crisp ring and let A be a fuzzy subset of R, then A is a fuzzy subring of $\mathbb{R}$, then A is a fuzzy subring of $\mathbb{R}$ if and only if A^+_α is a crisp subring of $\mathbb{R}$, for all $\alpha \in [0, 1[$.*

Proof
Let A be a fuzzy subring of R.

1. Let $x, y \in A^+_\alpha$ then $\mu_A(x) > \alpha$ and $\mu_A(y) > \alpha$ and we have $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y) > \alpha \wedge \alpha = \alpha$, then $x - y \in A^+_\alpha$

2. Let $x, y \in A_\alpha^+$ then $\mu_A(x) > \alpha$. and $\mu_A(y) > \alpha$ and we have $\mu_A(x.y) \geq \mu_A(x) \wedge \mu_A(y) > \alpha \wedge \alpha = \alpha$, then $x.y \in A_\alpha^+$.

Hence, A_α^+ is a crisp subring of R.
Conversely,
Suppose that A_α is a cris subring, for every $\alpha \in [0, 1[$.
We put $\mu_A(x) \wedge \mu_A(y) = \beta$.
(1)
If $\beta = 0$, it is easy to see
$\mu_A(x - y) \geq 0 = \beta, \ \ \mu_A(x \cdot y) \geq 0 = \beta.$
$\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y) = \beta.$
(2)
If $\beta > 0, \ \ \mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \beta$,
then,
$x, y \in A_\beta, \ \ x - y \in A_\beta$, and $x \cdot y \in A_\beta$.
Therefore,
$\mu_A(x - y) \geq \alpha = \mu_A(x) \wedge \mu_A(y)$, and $\mu_A(x \cdot y) \geq \alpha = \mu_A(x) \wedge \mu_A(y)$.
Consequently,A is a fuzzy subring of $\mathbb{R}$.

Remark 2.3 *The L_α does not necessarily be a fuzzy subring.*

Example 2.4 *Let A be fuzzy subrings of a ring $(\mathbb{Z}, +, \cdot)$ defined by*

$$\mu_A(x) = \begin{cases} 0.6 \textit{ if } x \in 2\mathbb{Z}, \\ 0.2 \textit{ if } x \in 2\mathbb{Z} + 1. \end{cases}$$

$L_{0.2}(A) = 2\mathbb{Z} + 1.$
Let 1 and 3 $\in 2\mathbb{Z} + 1$, it is easy to see $3 + 1 = 4$ does not belong in $2\mathbb{Z} + 1$.
So, $L_{0.2}(A)$ is not fuzzy subring.

2.5 Ring morphisms

Proposition 2.9 *Let $(R, +, .)$ and $(S, *, \times)$ be two rings, and let A be a fuzzy subring on R, then for a bijective ring homomorphism $f : R \to S$, $f(A)$ is a fuzzy subring on S.*

Proof let A be a fuzzy subring on R
$\mu_{f(A)}(x) = \bigvee_{a \in f^{-1}(x)} \mu_A(a)$ Since f is bijective therefore $f^{-1}(x)$ must be a singleton set, there exists an unique $a \in R$ such that $f(a) = x$, then

1.
$$\begin{aligned} \mu_{f(A)}(x - y) &= \mu_A(f^{-1}(x - y)) = \mu_A(f^{-1}(x) - f^{-1}(y)), \\ &\geq \mu_A(f^{-1}(x)) \wedge \mu_A(f^{-1}(y)) \text{because A is fuzzy subring}, \\ &= \mu_{f(A)}(x) \wedge \mu_{f(A)}(y). \end{aligned}$$

   Then $\mu_{f(A)}(x - y) \geq \mu_{f(A)}(x) \wedge \mu_{f(A)}(y)$.

2.

$$\begin{aligned}\mu_{f(A)}(x \times y) &= \mu_A(f^{-1}(x \times y)),\\ &= \mu_A(f^{-1}(x) \times f^{-1}(y)),\\ &\geq \mu_A(f^{-1}(x)) \wedge \mu_A(f^{-1}(x))\text{because A is fuzzy subring},\\ &= \mu_{f(A)}(x) \wedge \mu_{f(A)}(y).\end{aligned}$$

Then $\mu_{f(A)}(x \times y) \geq \mu_{f(A)}(x) \wedge \mu_{f(A)}(y)$.

Proposition 2.10 *Let $(R, +, .)$ and $(S, *, \times)$ be two rings, and let A be a fuzzy subring on S, then for a bijective ring homomorphism $f : R \to S$, $f^{-1}(A)$ is a fuzzy subring on R.*

proof let A be a fuzzy subring on S
$\mu_{f^{-1}(A)}(x) = \mu_A(f(x))$, then

1.

$$\begin{aligned}\mu_{f^{-1}(A)}(x - y) &= \mu_A(f(x - y)),\\ &= \mu_A(f(x) - f(y)),\\ &\geq \mu_A(f(x)) \wedge \mu_A(f(x)) \text{ because A is fuzzy subring },\\ &= \mu_{f^{-1}(A)}(x) \wedge \mu_{f^{-1}(A)}(y).\end{aligned}$$

Then $\mu_{f^{-1}(A)}(x - y) \geq \mu_{f^{-1}(A)}(x) \wedge \mu_{f^{-1}(A)}(y)$.

2.

$$\begin{aligned}\mu_{f^{-1}(A)}(x \times y) &= \mu_A(f(x \times y)),\\ &= \mu_A(f(x) \times f(y)),\\ &\geq \mu_A(f(x)) \wedge \mu_A(f(x)) \text{ because A is fuzzy subring },\\ &= \mu_{f^{-1}(A)}(x) \wedge \mu_{f^{-1}(A)}(y).\end{aligned}$$

Then $\mu_{f^{-1}(A)}(x \times y) \geq \mu_{f^{-1}(A)}(x) \wedge \mu_{f^{-1}(A)}(y)$.

CHAPTER 3

FUZZY IDEALS OF A RINGS

In this chapter, we will give the definition of the fuzzy ideal and some of its basic properties. We will also study direct and indirect images of these ideals.

3.1 Fuzzy ideals of a crisp ring

Definition 3.1 *[14] Let R be a ring, and let I be a fuzzy subset of R. Then I is called a fuzzy ideal if, for any $x, y \in R$, the following conditions are satisfied:*

1. $\mu_I(x - y) \geq \mu_I(x) \wedge \mu_I(y) ... (I_1)$,
2. $\mu_I(x \cdot y) \geq \mu_I(x) \vee \mu_I(y) ... (I_2)$.

Example 3.1 *Let $(\mathbb{Z}_4, +, \cdot)$ be a ring. The fuzzy subset I on the ring $\mathbb{Z}_4$ defined by:*

$$\mu_I(x) = \begin{cases} 0.8 \ \ if \ \ x \ = \ 0, x = 2, \\ 0.1 \ \ if \ \ x = 1, x = 3. \end{cases} \tag{3.1}$$

x	y	$x-y$	$\mu_A(x)$	$\mu_A(y)$	$x \cdot y$	$\mu_A(x-y)$	$\mu_A(x \cdot y)$	$\mu_A(x) \wedge \mu_A(y)$	$\mu_A(x) \vee \mu_A(y)$	I_1	I_2
P	P	P	0.8	0.8	P	0.8	0.8	0.8	0.8	T	T
P	I	I	0.8	0.1	P	0.1	0.8	0.1	0.8	T	T
I	P	I	0.1	0.8	P	0.1	0.8	0.1	0.8	T	T
I	I	P	0.1	0.1	I	0.8	0.1	0.1	0.1	T	T

Then I is a fuzzy ideal of $(\mathbb{Z}_4)$.

3.2 Operations of fuzzy ideals of a ring

In this section, we will give some basic properties of fuzzy ideals of a ring.
In the following proposition, we study the intersection of two fuzzy ideals of a ring.

Proposition 3.1 *Let I, J be two fuzzy ideals of the ring R. Then $I \cap J$ is a fuzzy ideal on R.*

Proof Let I, J be two fuzzy ideals of R. We will show that $I \cap J$ is a fuzzy ideal on R. Since I is an ideal, it follows that:

$$\begin{cases} \mu_I(x-y) \geq \mu_I(x) \wedge \mu_I(y), \\ \mu_I(x \cdot y) \geq \mu_I(x) \vee \mu_I(y). \end{cases}$$

and J is an ideal, then:

$$\begin{cases} \mu_J(x-y) \geq \mu_J(x) \wedge \mu_J(y), \\ \mu_J(x \cdot y) \geq \mu_J(x) \vee \mu_J(y). \end{cases}$$

Then,

$$\begin{aligned} \mu_{I\cap J} &= \mu_I(x-y) \wedge \mu_J(x-y), \\ &\geq [\mu_I(x) \wedge \mu_I(y)] \wedge [\mu_J(x) \wedge \mu_G(y)], \\ &\geq [\mu_I(x) \wedge \mu_J(x)] \wedge [\mu_I(y) \wedge \mu_G(y)], \\ &\geq \mu_{I\cap J}(x) \wedge \mu_{I\cap J}(y). \end{aligned}$$

And,

$$\begin{aligned} \mu_{I\cap J} &= \mu_I(x \cdot y) \wedge \mu_J(x \cdot y), \\ &\geq [\mu_I(x) \vee \mu_I(y)] \wedge [\mu_J(x) \vee \mu_J(y)], \\ &\geq [\mu_I(x) \wedge \mu_J(x)] \vee [\mu_I(y) \wedge \mu_J(y)], \\ &\geq \mu_{I\cap J}(x) \vee \mu_{I\cap J}(y). \end{aligned}$$

Then, $I \cap J$ is a fuzzy ideal on the ring $\mathbb{R}$.

Remark 3.1 *The union of two fuzzy ideals does not necessarily be a fuzzy ideal.*

Example 3.2 *Let I and J be two fuzzy ideals of a ring $(\mathbb{Z}, +, \cdot)$. defined by:*

$$\mu_I(x) = \begin{cases} 0.8 \textit{ if } x \in 2\mathbb{Z}, \\ 0.1 \textit{ if } x \in 2\mathbb{Z}+1. \end{cases}$$

And,

$$\mu_G(x) = \begin{cases} 0.7 \textit{ if } x \in 3\mathbb{Z}, \\ 0.5 \textit{ otherwise}. \end{cases}$$

Then,

$$\mu_{I\cup J}\left(x\right)=\begin{cases}0.8 \text{ if } x\in 6\mathbb{Z},\\ 0.8 \text{ if } x\in 2\mathbb{Z},\\ 0.7 \text{ if } x\in 3\mathbb{Z},\\ 0.5 \text{ otherwise}.\end{cases}$$

We have $\mu_{I\cup J}(3-2)=\mu_{I\cup J}(1)=0.5$*,*
$\mu_{I\cup J}(3)\wedge\mu_{I\cup J}(2)=0.7\wedge 0.8=0.7$*,*
$\mu_{I\cup J}(3-2)\not\geq\mu_{I\cup J}(3)\wedge\mu_{I\cup J}(2)$*,*
then $I\cup J$ *is not fuzzy ideals.*

Proposition 3.2 *Let I and J be two fuzzy ideals of a ring* $(\mathbb{R},+,\cdot)$*. Then* $I\times J$ *is a fuzzy ideals of* $\mathbb{R}\times\mathbb{R}$*.*

Proof Let I and J be two fuzzy ideals of a ring R. Then:

$$\begin{aligned}\mu_{I\times J}(x-y) &= \mu_I(x-y)\wedge\mu_J(x-y),\\ &\geq (\mu_I(x)\wedge\mu_I(y))\wedge(\mu_J(x)\wedge\mu_J(y)),\\ &= (\mu_I(x)\wedge\mu_J(x))\wedge(\mu_I(y)\wedge\mu_J(y)),\\ &= \mu_{I\times J}(x)\wedge\mu_{I\times J}(y).\end{aligned}$$

Then, $\mu_{I\times J}(x-y)\geq\mu_{I\times J}(x)\wedge\mu_{I\times J}(y)$.

$$\begin{aligned}\mu_{I\times J}(x\cdot y) &= \mu_I(x\cdot y)\vee\mu_J(x\cdot y),\\ &\geq (\mu_I(x)\vee\mu_I(y))\vee(\mu_J(x)\vee\mu_J(y)),\\ &= (\mu_I(x)\vee\mu_J(x))\vee(\mu_I(y)\vee\mu_J(y)),\\ &= \mu_{I\times J}(x)\vee\mu_{I\times J}(y).\end{aligned}$$

Then, $\mu_{I\times J}(x\cdot y)\geq\mu_{I\times J}(x)\vee\mu_{I\times J}(y)$.

Hence, $I\times J$ is a fuzzy ideals of $R\times R$.

Remark 3.2 *The fuzzy complement of a fuzzy ideal is not necessarily a fuzzy ideal.*

Example 3.3 *From the previous example 3.1 we find: Let* $(\mathbb{Z}_4,+,\cdot)$ *be a ring. The fuzzy subset* I *on the ring* $\mathbb{Z}$ *defined by:*

$$\mu_I(x)=\begin{cases}0.8 \;\; if \;\; x \; = \; 0, x=2,\\ 0.1 \;\; if \;\; x=1, x=3.\end{cases}$$

Then,
Let $(\mathbb{Z}_4,+,\cdot)$ *be a ring. The fuzzy subset* I *on the ring* $\mathbb{Z}$ *defined by:*

$$\mu_{\bar{I}}(x)=\begin{cases}0.2 \;\; if \;\; x \; = \; 0, x=2,\\ 0.9 \;\; if \;\; x=1, x=3.\end{cases}$$

We have $\mu_{\bar{I}}(3-1)=\mu_{\bar{I}}(2)=0.2$*,*
$\mu_{\bar{I}}(3)\wedge\mu_{\bar{I}}(1)=0.9\wedge 0.9=0.9$*.*
$\mu_{\bar{I}}(3-1)\not\geq\mu_{\bar{I}}(3)\wedge\mu_{\bar{I}}(1)$*,*
then $\bar{I}$ *is not fuzzy ideals.*

3.3 Characterization of fuzzy ideals of a ring

In this section, we offer intriguing descriptions of fuzzy ideals of a ring concerning their support and $\alpha - level$ sets.

Proposition 3.3 *Let $(R, +, \cdot)$ be a ring. If I is a fuzzy ideal of R, then its support is a crisp ideal of R.*

Proof Let I is a fuzzy ideal on a ring R.
(i)
Let $x, y \in Supp(I)$. Since $x \in Supp(I)$, it follows that $\mu_I(x) > 0$. Since, $y \in Supp(I)$, it follows that $\mu_I(y) > 0$, we have that I is a fuzzy ideal on a ring R, then $\mu_I(x - y) \geq \mu_I(x) \wedge \mu_I(y) > 0$. Hence, $x - y \in Supp(I)$.
(ii)
Let $x \in Supp(I)$ and $a \in A$. We prove that $x.a \in Supp(I)$ and $a.x \in Supp(I)$. Since $x \in Supp(I)$ it holds that $\mu_I(x) > 0$, we have that I is a fuzzy ideal on R, then $\mu_I(x.a) \geq \mu_I(x) \vee \mu_I(a) > 0$. Hence, $x.a \in Supp(I)$. By using the same method as above we get that $a.x \in Supp(I)$.
Thus, $Supp(I)$ is a crisp ideal of R.

Proposition 3.4 *Let $(R, +, \cdot)$ be a crisp ring, and let I be a fuzzy ideal of R. Then $ker(I)$ is a crisp ideal of R.*

proof

1. Let $x, y \in ker(I)$, then $\mu_I(x) = 1$ and $\mu_I(y) = 1$, and we have
   $\mu_I(x - y) \geq \mu_I(x) \wedge \mu_I(y) = 1 \wedge 1 = 1$, then $\mu_I(x - y) = 1$, then $x - y \in ker(I)$.

2. Let $x, y \in ker(I)$, then $\mu_I(x) = 1$ and $\mu_I(y) = 1$, and we have
   $\mu_I(x.y) \geq \mu_I(x) \vee \mu_I(y) = 1 \vee 1 = 1$, then $\mu_I(x.y) = 1$, then $x.y \in ker(I)$.

Hence, $ker(I)$ is a crisp ideal of R.

Proposition 3.5 *Let $(R, +, \cdot)$ be a ring and let I be a fuzzy subset of R. Then, I is a fuzzy ideal on R if and only if all it α-cuts are ideals of R.*

Proof
Suppose that I is a fuzzy ideal. We show that I_α is an ideal on R for $\alpha \in]0, 1]$:

1. Let $x, y \in I_\alpha$, then $\mu_I(x) \geq \alpha$ and $\mu_I(y) \geq \alpha$. Since I is a fuzzy ideal on R, hence $\mu_I(x - y) \geq \mu_I(x) \wedge \mu_I(y) \geq \alpha$. Then, $x - y \in I_\alpha$.

2. Let $x \in I_\alpha$ and $a \in A$, hence $\mu_I(x) \geq \alpha$ and $a \in A$. Since I is a fuzzy ideal on R, hence $\mu_I(x \cdot a) \geq \mu_I(x) \vee \mu_I(a) \geq \alpha$. Then, $x \cdot a \in I_\alpha$. By using the same method as above we get that $a \cdot x \in I_\alpha$.

Hence, I_α are ideals on R.
Conversely,
Suppose that I_α are ideals on R for all $\alpha \in]0, 1]$. We show that I is a fuzzy ideal on R.

1. Let $x, y \in R$. Setting $\alpha = \mu_I(x) \wedge \mu_I(y)$, then it follows that $\mu_I(x) \geq \alpha$ and $\mu_I(y) \geq \alpha$. The case $\alpha = 0$ is obvious. Let $\alpha \in]0, 1]$ and $x, y \in I_\alpha$. Since I_α are ideals on R, then $x - y \in I_\alpha$ for all $\alpha \in]0, 1]$ this implies that $\mu_I(x - y) \geq \alpha$. Then, $\mu_I(x - y) \geq \mu_I(x) \wedge \mu_I(y)$.

2. Let $x, y \in I_\alpha$ then $x \cdot y \in I_\alpha$ because I_α is an ideal. Then $\mu_I(x) \geq \alpha$ and it follows that $\mu_I(x \cdot y) \geq \alpha$. Hence $\mu_I(x \cdot y) \geq \mu_I(x) and \mu_I(x \cdot y) \geq \mu_I(y)$. Thus, $\mu_I(x \cdot y) \geq \mu_I(x) \vee \mu_I(y)$.

We conclude that I is a fuzzy ideal on R.

3.4 Direct and converse images of ideals

Proposition 3.6 *Let $(R_1, +, \cdot)$ and $(R_2, +, \cdot)$ denote two rings, and let $f : R_1 \to R_2$ represent a ring homomorphism. If I is a fuzzy ideal in R_1, then its image under f, denoted by $f(I)$, forms a fuzzy ideal in R_2.*

Proof
Suppose I is a fuzzy ideal in R_1. Our goal is to demonstrate that $f(I)$ is also a fuzzy ideal in R_2. That is:

$$f(I)(x - y) \geq f(I)(x) \wedge f(I)(y),$$
$$f(I)(x \cdot y) \geq f(I)(x) \vee f(I)(y).$$

We have

$$\begin{aligned} f(I)(x - y) &= \mu_I(f(x - y)), \\ &= \mu_I(f(x) - f(y)), \\ &\geq \mu_I(f(x)) \wedge \mu_I(f(y)) \\ &= f(I)(x) \wedge f(I)(y). \end{aligned}$$

And

$$\begin{aligned} f(I)(x \cdot y) &= \mu_I(f(x \cdot y)), \\ &= \mu_I(f(x) \cdot f(y)), \\ &\geq \mu_I(f(x)) \vee \mu_I(f(y)) \\ &= f(I)(x) \vee f(I)(y). \end{aligned}$$

Then, $f(I)$ is a fuzzy ideal on R_2.

Proposition 3.7 *Let $(R_1, +, \cdot)$ and, $(R_2, +, \cdot)$ be two rings and $f : R_1 \longrightarrow R_2$ be a homomorphism of rings. If I is a fuzzy ideal on R_2, then $f^-1(I)$ is a fuzzy ideal on R_1.*

Proof Let I be a fuzzy ideal on R_2. We show that $f^-1(I)$ is a fuzzy ideal on R_1 i.e

$$f^-1(I)(x - y) \geq f^-1(I)(x) \wedge f^-1(I)(y),$$
$$f^-1(I)(x \cdot y) \geq f^-1(I)(x) \vee f^-1(I)(y).$$

We have

$$\begin{aligned} f^{-}1(I)(x-y) &= \mu_I(f^{-}1(x-y)), \\ &= \mu_I(f^{-}1(x)-f^{-}1(y)), \\ &\geq \mu_I(f^{-}1(x)) \wedge \mu_I(f^{-}1(y)), \\ &= f^{-}1(I)(x) \wedge f^{-}1(I)(y). \end{aligned}$$

And

$$\begin{aligned} f^{-}1(I)(x \cdot y) &= \mu_I(f^{-}1(x \cdot y)), \\ &= \mu_I(f^{-}1(x) \cdot f^{-}1(y)), \\ &\geq \mu_I(f^{-}1(x)) \vee \mu_I(f^{-}1(y)), \\ &= f^{-}1(I)(x) \vee f^{-}1(I)(y). \end{aligned}$$

Then, $f^{-}1(I)$ is a fuzzy ideal on R_1.

CONCLUSION

In this work, we have studied some algebraic structures, such as crisp rings and their ideals. Then we explored these structures in fuzzy cases as a generalisation of classical cases. Also, we have studied some fundamental properties of fuzzy rings and their ideals. Moreover, we have explored the direct and converse images of fuzzy ideal via a ring homomorphism.

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ملخص

قدمنا في هذه المذكرة بعض المفاهيم الاساسية حول المجموعات الضبابية والحلقات. كما قدمنا لمحة عامة عن الحلقات الفرعية الضبابية وخصائصها. ثم درسنا المثل الضبابية المرتبطة بحلقة فرعية ضبابية. وانهينا هذا العمل بخلاصة عامة.

كلمات مفتاحية

مجموعات غامضة، مثالية، حلقة، حلقة فرعية غامضة، مثالية غامضة.

Abstract

In this dissertation, we have presented some basic notions about fuzzy sets and rings. Also, we gave an overview of fuzzy subrings and their properties. Then, we studied the fuzzy ideals associated with a fuzzy subring. We, ended this work with a general conclusion.

Key words

Fuzzy sets ,ideal,ring, Fuzzy subring , fuzzy ideal.

Résumé

Dans ce mémoire, nous avons présenté certaines notions de base sur les ensembles flous et les anneaux. Aussi, nous avons donné un apercu sur les sous anneau flou et leurs propriétés.Ensuite, nous avons étudié les idéaux flous associés à un sous anneau flous. Nous avons terminé ce travail par une conclusion générale.

Mot-clés

Ensembles flous, idéal, anneau, sous-anneau flou, idéal flou.