RÉPUBLIQUE DÉMOCRATIQUE POPULAIRE ALGÉRIENNE
MINISTÈRE D'ENSEIGNEMENT SUPÉRIEUR ET DE RECHERCHE SCIENTIFIQUE
UNIVERSITÉ DE M'SILA
FACULTÉ DE MATHÉMATIQUES ET DE L'INFORMATIQUE
DÉPARTEMENT DE MATHÉMATIQUES

N° d'ordre: .....

# T H È S E

*Présentée pour l'obtention de grade de docteur en sciences*

Spécialité:

*Mathématiques pur et application*

Option:

*EDPs et optimisation*

Par:

Titraoui Maria

*Intitulée:*

# Étude d'existence et d'unicité de certaines classes d'équations différentielles fractionnaires

Soutenu publiquement le : 05 / 11 / 2020, devant le jury :

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Année universitaire: 2019 / 2020

Thesis presented to obtain the doctoral degree in sciences
Speciality: Pure Mathematics and application
Option: PDEs and optimization

By:

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Titled:

# Study the existence and uniqueness of certain classes of fractional differential equations

Supervisors: Arioua Yacine & Benhamidouche Nourddine

University of M'sila, 2020.

*To my family and all my friends,*

*especially my husband*
*and my mother*

# ACKNOWLEDGEMENTS

First of all, I thank my God who gave me the strength to finish this thesis.
I would like to thank my promoter: Dr. Arioua Yacine for the confidence he has shown in me by offering me this subject, his encouragement and his patience. The scientific discussions he was able to generate, his remarks and his suggestions which allowed me to finalize this thesis. I wish to convey my gratitude and my deepest gratitude to him.

I wish to express my sincere thanks to Professors A. Gasmi, N. Benhamidouche, M. Benbachir and to Doctors A. Merzougui, B. Gagui who kindly agreed to accept on my thesis committee.

I cannot close my thanks without turning to the people who are most dear to me; my family who had an essential and continuous role in my success.

This thesis is typeset using LaTeX2$_\varepsilon$ and is based on a version of the *Classic Thesis* template, with many modifications.

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**Title :** **Study the existence and uniqueness of certain classes of fractional differential equations**

**Specialty**: Pure Mathematics and applications

**Option**: PDEs and optimization

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ABSTRACT

In this thesis, we focus on the existence and uniqueness of positive, bounded and unbounded solutions to certain boundary value problems on the half-line for nonlinear fractional differential equations in the sense of Erdélyi–Kober and Caputo Erdélyi–Kober derivatives in Banach space. The results were proven analytically using the Banach contraction principle, Schauder and GuoKrasnosel'skii fixed point theorems, the technique of the nonlinear alternative of Leray-Schauder type and the diagonalization argument method.

*Keywords:*

Fractional differential equations, Boundary value problems, Erdélyi–Kober operators, Caputo Erdélyi–Kober derivative, Fixed point theorems, Diagonalization argument, Existence, uniqueness, Unbounded solution, Positive solution, Bounded solution, Fixed point theorems.

***العنوان:*** دراسة وجود ووحدانية الحلول لفئات معينة من المعادلات التفاضلية الكسرية.

***ملخص:*** في هذه الأطروحة، نركز على وجود ووحدانية الحلول الموجبة والمحدودة وغير المحدودة لبعض مسائل القيمة الحدية على نصف المجال الموجب للمعادلات التفاضلية الكسرية غير الخطية بمفهوم مشتق اردیلي كوبر وكبوتو اردیلي كوبر في فضاء بناخ. وقد تم إثبات النتائج بشكل تحليلي باستخدام مبدأ انكماش بناخ، ونظريات شودر وجيو كراسنوسلسكي للنقطة الثابتة، وتقنية البديل غير الخطي لنوع لاري شودر ونضرية التقطير.

***كلمات مفتاحية:*** المعادلات التفاضلية الكسرية، مسائل القيمة الحدية، مأثرات اردیلي كوبر، مشتق كابوتو اردیلي كوبر، نظريات النقطة الثابتة، نظرية التقطير، ووجود، ووحدانية، حل غير محدود، حل موجب، حل محدود.

Titre: **Étude d'existence et d'unicité de certaines classes d'équations différentielles fractionnaires**

RÉSUMÉ:

Dans cette thèse, nous nous intéressons a l'existence et l'unicité des solutions positives, bornées et non bornées des certains problèmes aux limites sur la droit réal d' équations différentielles fractionnaires non linéaires au sens de dérivé Erdélyi–Kober et Caputo Erdélyi–Kober dans un espace de Banach. Les résultats ont été prouvés analytiquement en utilisant le principe de contraction de Banach, les théorèmes de point fixe de Schauder, GuoKrasnosel'skii, l'alternative non linéaire de type Leray-Schauder et la méthode d'argument de diagonalisation.

*Mots Clés:*

Équations différentielles fractionnaires, Problèmes aux limites, Les opérateurs d Erdélyi–Kober, La dérive de Caputo Erdélyi–Kober, Argument de diagonalisation, Existence, Unicité, Solution non bornée, Solution positive, Solution bornée, Théorèmes de point fixe.

# NOTATIONS

| | |
|---|---|
| $\mathbb{N}$ | Natural numbers $\{0,1,2,3,\ldots\}$. |
| $\mathbb{N}^*$ | Nonzero natural numbers $\{1,2,3,\ldots\}$. |
| $\mathbb{R}$ | Real numbers $(-\infty,\infty)$. |
| $\mathbb{R}_+$ | Positive real numbers $(0,\infty)$. |
| $\mathbb{R}^*$ | Nonzero real numbers $(-\infty,0)\cup(0,\infty)$. |
| $\mathbb{C}$ | Complex numbers, $z\in\mathbb{C}$, then $z=x+iy$, where $x,y\in\mathbb{R}$, and $i^2=-1$. |
| $\mathrm{Re}\,(\alpha)$ | Real part of complex $\alpha$. |
| $\Omega$ | Finite closed interval of the real axis $\mathbb{R}$. |
| $L^1(\Omega)$ | Space of Lebesgue complex-valued measurable functions $u$ on $\Omega$, for which $\|u\|_{L^1}=\int_\Omega \|u(s)\|\,ds<\infty$. |
| $L^p(\Omega)$ | Space of all measurable functions $u$, for which $\|u\|^p\in L^1(\Omega)$, for any $1<p<\infty$. |
| $C(\Omega)$ | The space of all continuous functions from $\Omega$ into $\mathbb{R}$. |
| $C^n_\alpha, \alpha\in\mathbb{R}, n\in\mathbb{N}$ | $\{f(t), t>0:\ f(t)=t^p f_1(t)$ with $p>\alpha$ and $f_1\in C^n([0,\infty))\}$. |
| $\mathcal{M}u$ | Mellin transform of a function $u$. |
| $\Gamma(\cdot)$ | Euler gamma function. |
| $(\alpha)_n$ | Pochhammer symbol, where $(\alpha)_0=1,\ (\alpha)_n=\prod_{i=0}^{n-1}(\alpha+i)$, $\alpha\in\mathbb{C}$, $n\in\mathbb{N}^*$. |
| $B(\cdot,\cdot)$ | Beta function. |

| | |
|---|---|
| $E-K$ | Erdélyi-Kober. |
| FDE | Fractional Differential Equation. |
| BVP | Boundary Value Problem. |
| $\mathcal{I}^1 u$ | Primitive of Lebesgue summable function $u$. |
| $\mathcal{I}^n u$ | Cauchy formula for the $n^{th}$ integrals, $n \in \mathbb{N}$. |
| $\mathcal{I}^\alpha u$ | Riemann-Liouville's fractional integral of order $\alpha$. |
| $\mathcal{D}^\alpha u$ | Riemann-Liouville's fractional derivative of order $\alpha$. |
| ${}^C\mathcal{D}^\alpha u$ | Caputo's fractional derivative of order $\alpha$. |
| $\mathcal{I}_\beta^{\gamma,\delta} u$ | Erdélyi-Kober fractional integral of order $\delta$. |
| $\mathcal{D}_\beta^{\gamma,\delta} u$ | Erdélyi-Kober fractional derivative of order $\delta$. |
| ${}^c\mathcal{D}_\beta^{\gamma,\delta} u$ | Caputo Erdélyi-Kober fractional derivative of order $\delta$. |

# CONTENTS

# 0

# INTRODUCTION

Fractional-order differential equations have been used in the study of models of many phenomena in various fields of science and engineering, such as viscoelasticity, fluid mechanics, electrochemistry, control, porous media, mathematical biology, diffusive transport akin to diffusion, probability, statistics and electromagnetic bioengineering. Recent investigations have shown that sometimes physical systems can be modeled more accurately using fractional derivative formulations. For the basic theory on fractional differential equations, the reader can refer to the books Samko et al. 1993 [51], Podlubny 1999 [48], Kilbas et al. 2006 [29], Sabatier et al. 2007 [50], Das 2008 [15], Diethelm 2010 [16], and Mathai and Haubold 2018 [40], Kiryakova [30, 31].

The classical fractional calculus is based on several definitions for the operators of integration and differentiation of arbitrary order [30]. Among the various definitions of fractional differentiation, the Riemann–Liouville and Caputo fractional derivatives are widely used in the literature. The most useful classical fractional integrals, however, seem to be the Erdélyi–Kober operators. These were introduced by Sneddon (see, for example, [53–55]), who studied their basic properties and emphasized their useful applications to generalized axially symmetric potential theory and other physical problems, such as in electrostatics and elasticity. For details and applications of the Erdélyi–Kober fractional integrals, we refer the reader to a series of papers and texts [18, 27, 29, 31, 54, 60].

In 1998 Yuri Luchko and Rudolf Gorenflo [35] determine a symmetry group of scaling transformations for a time- and space-fractional (of orders $\alpha$ and $\beta$, respectively) partial differential equation ($\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = d\frac{\partial^\beta u(x,t)}{\partial x^\beta}$). For its group-invariant solutions an ordinary differential equation of fractional order with the new independent variable $z = xt^{-\frac{\alpha}{\beta}}$ is derived. The derivatives then are the well known right- and left-hand sided Erdélyi Kober derivatives depending on $\alpha$, $\beta$, and on a parameter $\gamma$ of its scaling group $\left(P_{\frac{\beta}{\alpha}}^{1+\gamma-\alpha,\alpha}u\right)(z) = dz^{-\beta}\left(D_1^{-\beta,\beta}u\right)(z)\,,\ z > 0.$

The Caputo-type Erdélyi Kober fractional derivatives introduced for the first time by Gorenflo and all. [21] in connection with their investigation of the scale-invariant solutions of the diffusion-wave equation.

With the results presented in [46], the E–K operators and their Caputo-type modifications were pushed to a class of fractional operators important for models in applications and thus worth to be investigated in-depth from the mathematical viewpoint.

The theory of boundary value problems on infinite intervals arises quite naturally and has many applications [8]; it is important and several authors have done much work on this topic [1–4, 9, 22, 32, 36, 37, 39, 43, 45, 46, 52, 57, 61, 62, 64–66] and the references therein.

In this thesis we provide conditions assuring the existence and uniqueness of the solutions to certain boundary value problems on the positive half-line $(0, +\infty)$ for nonlinear fractional differential equations involving the Erdélyi Kober (or Caputo Erdélyi Kober) derivatives, in special Banach space.

This thesis is organized as follows:

The first chapter will be devoted to the basic elements of fractional calculus, a historical reminder and some preliminary concepts will be introduced such as the gamma and the Beta function which plays an important role in the theory of fractional differential equations, we gives the definitions of the approaches fractional integrals and derivatives (Riemann-Liouville and Caputo) and their most important properties, as well as the generalized Erdélyi Kober operators and their Caputo-type modifications. At the end we give some Studies and results of FDE's solutions.

In the second chapter, we discuss the existence and uniqueness of a positive solutions to the boundary value problems of nonlinear fractional differential equations involving the Erdélyi Kober operators on the positive half line:

$$\begin{cases} \mathcal{D}_{\beta}^{\gamma,\delta} u(t) + f(t, u(t)) = 0, \ t > 0, \\ \lim_{t\to 0} t^{\beta(2+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} u(t) = 0, \\ \lim_{t\to\infty} t^{\beta(1+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} u(t) = 0, \end{cases}$$

where $\mathcal{D}_{\beta}^{\gamma,\delta}$ denotes the Erdélyi–Kober fractional derivative operator of order $\delta$ and $\mathcal{I}^{\delta+\gamma,2-\delta}$ is the Erdélyi–Kober fractional integral of order $2-\delta$, with $1 < \delta \leq 2$, $-2 < \gamma < -1$, $\beta > 0$, and $f$ is a given function. First we define an integral solution of this problem and special Banach space $C_\infty$. We use the leray schauder non linear alternative to prove the existence of solution in this space, then we prove the existence of a positive solution when the proof technique is the Guo–Krasnosel'skii fixed point theorem. on the other hand, we apply the Banach contraction

principle fixed point theorem to prove the uniqueness of the positive solution. Finally, the main result is strengthened through examples.

In the third chapter, the main focus is the existence and uniqueness of unbounded solutions to the following nonlinear fractional differential equations involving the Caputo Erdélyi Kober derivatives:

$$\begin{cases} *\mathcal{D}_{\beta}^{\gamma,\delta} u(t) + f(t, u(t)) = 0, \ t > 0, \\ \lim_{t\to 0} t^{\beta(1+\gamma)} u^{(k)}(t) = 0, \ \text{with } k = \overline{0, n-2}, \\ \lim_{t\to\infty} t^{\beta(1+\gamma)} u(t) = \rho u(1), \end{cases}$$

where ${}_*\mathcal{D}_{\beta}^{\gamma,\delta}$ is the Caputo-type Erdélyi Kober fractional derivative operator of order $\delta$, such that $n-1 < \delta \leq n$, $-n < \gamma < 1-n$, $n \in \mathbb{N}$, $n \geq 2$, $\beta > 0$, $0 < \rho < 1$, and $f$ is a given function. The results in this chapter are based on Schauder's, nonlinear alternative Leray–Schauder's and contraction mapping principle fixed point theorems technique to prove the existence and uniqueness of unbounded solutions in a special Banach space $C_\infty$. Also, we present two examples to explain the application of our main results.

In the last chapter, we will discuss the existence and uniqueness of bounded solutions to certain boundary value problems on the positive half-line $(0, +\infty)$ for nonlinear fractional differential equations involving the Erdélyi Kober derivatives. Specifically, we will consider the BVP

$$\begin{cases} \mathcal{D}_{\beta}^{\gamma,\delta} u(t) + f(t, u(t)) = 0, \ t \in J = (0, \infty) \\ \lim_{t\to 0} t^{\beta(1+\gamma)} \frac{d^k}{dt^{(k)}} I^{\gamma+\delta, m-\delta} u(t) = 0, \ \text{with } k = \overline{0, \ m-2}, \ u(t) \ \text{bounded on } J, \end{cases}$$

where $\mathcal{D}_{\beta}^{\gamma,\delta}$ denotes the Erdélyi–Kober fractional derivative operator of order $\delta$ and $\mathcal{I}^{\delta+\gamma, m-\delta}$ is the Erdélyi–Kober fractional integral of order $m-\delta$, with $m-1 < \delta \leq m$, $-m < \gamma < 1-m, m \in \mathbb{N} \setminus \{0, 1\}$, $\beta > 0$ and $f$ is a function. First we solve similar problems on finite intervals of type $(0, T_n]$, $n \in \mathbb{N}^*$ and prove (theorem 3.1) the existence of bounded solutions $u_n$, $n \in \mathbb{N}^*$. The proof technique of this result is mainly based on the Schauder's fixed point theorem. Then, starting with the sequence $\{u_n\}_{n\in\mathbb{N}^*}$ , we use a diagonalization method, which is based on the ArzelA-Ascoli criterion, to conclude that this problem admits bounded solutions on $J$. Also, the main result is strengthened through an example.

Finally, general conclusion and future research lines are given.

Most parts of results presented in this thesis have already been published or submitted for publication in peer-reviewed international journals. Results included in chapter 2 have been described in [12]. and those included in chapter 4 have been accepted in [59].

# 1

# PRELIMINARIES

This chapter will be devoted to the basic definitions and basics of fractional computing such as: fractional derivation, fractional integration, fractional order operator definitions and other notions that we will need in the rest of our work. We will begin by giving a historical of the development of fractional derivation theory.

## 1.1 HISTORICAL DEVELOPMENT OF FRACTIONAL CALCULUS

The concept of differentiation and integration to non-integer order is by no means new. A lots of authors give a very detailed histories in reference works [41, 44]. Our goal in this section is to present the main historical steps in the development of fractional computing, up to its growth in application development in the 1970s. We rely on works [26, 34, 41, 44, 49] to cover the period from 1695 to 2000.

In 30 September 1695 Leibniz invented the notation $d^n y/dx^n$. Perhaps it was a naive play with symbols that prompted L'Hopital in 1695 to ask Leibniz: "What if $n$ be $\frac{1}{2}$?" Leibniz replied: "Thus it follows that $d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx : x}$. This is an apparent paradox from which, one day, useful consequences will be drawn."

In another letter to J. Wallis and J. Bernulli (in 28 May 1697) , Leibniz mentioned the possible approach to fractional-order differentiation in that sense, that for non-integer values of n the definition could be the following: $\frac{d^n e^{mx}}{dx^n} = m^n e^{mx}$.

In1730 Euler is the second great mathematician to address the issue. In his article [20] where he introduces his famous Gamma function which generalizes the factorial ($\Gamma(n+1) = n!$). He obtained the basic fractional derivative

$$\frac{d^n}{dx^n} x^m = \frac{m!}{(m-n)!} x^{m-n}.$$

He suggested to use this relationship also for negative or non-integer (rational) values of n and m.

In 1822 Joseph B. J. Fourier was the next to mention derivatives of arbitrary order. His definition of fractional operations was obtained from his integral representation of f(x):

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\alpha) \cos(p(x-\alpha))\, d\alpha dp.$$

Fourier made a remark that

$$\frac{d^n}{dx^n} f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\alpha)\, p^n \cos\left[p(x-\alpha) + \frac{n\pi}{2}\right] \alpha dp,$$

and this relationship could serve as a definition of the n-th order derivative for non-integer n.

In 1823 Abel was probably the first applied the fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone. He found that the solution could be accomplished via an integral transform, which could be written as a semi-derivative. More precisely, the integral transform considered by Abel was $k = \int_0^x (x-t)^{-\frac{1}{2}} f(t)\, dt$.

After all, Liouville is the first to study in detail the fractional calculation, as the eight articles he published between 1832 and 1837 seem to attest. his first definition of a derivative of arbitrary order $v$ involved an infinite series. This had the disadvantage that $v$ must be restricted to those values for which the series converges. Liouville seemed aware of the restrictive nature of his first definition, therefore, Liouville tried to put his effort to define fractional derivative again of $x^a$ whenever $x$ and $a$ are positive. After these for any $v > 0$ Liouville was able to obtain the result known as his second definition:

$$\frac{d^v}{dx^v} x^{-a} = \frac{(-1)^v \Gamma(a+v)}{\Gamma(a)} x^{-a-v}.$$

Indeed, Liouville was able to extend this definition to include complex values for $a$ and $v$. By piecing together the somewhat disjointed accomplishments of many notable mathematicians, especially Liouville and Riemann, modern analysts can now define the integral of arbitrary order. The fractional integral of order $v$ is defined as follows:

$$D^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t)\, dt.$$

In 1847 Riemann sought a generalization of a Taylor's series expansion and derived the following definition for fractional integration:

$$\frac{d^{-r}}{dx^{-r}}u(x) = \frac{1}{\Gamma(r)}\int_c^x (x-k)^{r-1}\, u(k)\, dk.$$

However, he saw fit to add a complementary function to the above definition. Today, this definition is in common use as a definition for fractional integration but with the complementary function taken to be identically zero, and the lower limit of integration c is usually zero.

The earliest work that ultimately led to what is now called the Riemann-Liouville definition appears to be the paper by N. Ya. Sonin in 1869 [56], where he used Cauchy's integral formula as a starting point to reach differentiation with arbitrary index.

In 1940 H. Kober [33] and A. Erdélyi [19], introduced a complex parameter $\gamma$ and deal with the operator

$$\begin{aligned} I_+^{\gamma,\delta} f &= \frac{t^{-\gamma-\delta}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} s^{\gamma} f(s)\, ds, \\ I_-^{\gamma,\delta} f &= \frac{t^{-\gamma-\delta}}{\Gamma(\delta)}\int_t^{\infty} (s-t)^{\delta-1} s^{\gamma} f(s)\, ds. \end{aligned}$$

In 1960 IN. Sneddon [53],[54],[55] introduce what is now called the Erdélyi–Kober operators, he studied their basic properties and emphasized their useful applications to generalized axially symmetric potential theory and other physical problems, such as in electrostatics and elasticity.

The earliest work that ultimately led to what is now called the The Caputo Erdélyi–Kober fractional derivative appears to be the paper by Gorenflo, Luchko and Mainardi in 2000 [21] in connection with their investigation of the scale-invariant solutions of the diffusion-wave equation.

## 1.2 BASIC FRACTIONAL CALCULUS

In this section, we will present the fractional integration operators and the most used definitions of fractional derivatives and giving the most important properties of these notions. We will begin by giving some special functions and functional spaces.

### 1.2.1 *Functional spaces*

Let $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ be a finite closed interval of the real axis $\mathbb{R} = (-\infty, \infty)$. We denote by $L^p(\Omega)$ $(1 \leq p \leq \infty)$ the space of those LEBESGUE complex-valued measurable functions $u$ on $\Omega$ for which $\|u\|_{L^p} < \infty$, where

$$\|u\|_{L^p} = \left( \int_\Omega |u(s)|^p \, ds \right)^{\frac{1}{p}}. \quad (1 \leq p < \infty)$$

and

$$\|u\|_{L^\infty} = ess \sup_{a \leq t \leq b} |u(t)|.$$

Here $ess \sup_{a \leq t \leq b} |u(t)|$ is the essential maximum of the function $|u(t)|$ [see, for example, NIKOL'SKII [42], p. 12-13].

**Definition 1.1 ([38])** *The space of functions $C_\alpha^n$, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, consists of all functions $f(t)$, $t > 0$, that can be represented in the form $f(t) = t^p f_1(t)$ with $p > \alpha$ and $f_1 \in C^n([0, \infty))$.*

### 1.2.2 *Special functions of fractional calculus*

Here, we give some information on the EULER gamma and the Beta functions which play the most important role in the theory of the differentiation of arbitrary order.

*Euler Gama function*

One of the basic function of the fractional calculus is EULER gamma function $\Gamma(z)$, which generalizes the factorial $n!$ and allows $n$ to take also non-integer and even complex values. first, we give the MELLIN transform and the POCHHAMMER symbol $(\alpha)_n$ definitions, which have an important role in the definition of EULER gamma function.

**Definition 1.2 (Mellin transform[29])** *The MELLIN transform of a function $u(t)$ of a real variable $t \in \mathbb{R}^+ = (0, \infty)$ is defined by*

$$(\mathcal{M}u)(s) = \mathcal{M}[u(t)](s) = u^*(s) = \int_0^\infty t^{s-1} u(t) \, dt, Re(s) > 0,$$

*and the inverse of MELLIN transform is given for $t \in \mathbb{R}^+$ by the formula:*

$$\left(\mathcal{M}^{-1} u^*\right)(t) = \mathcal{M}^{-1}[u^*(s)](t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} t^{-s} u^*(s) \, ds, \gamma = Re(s).$$

*The direct and inverse of MELLIN transforms are inverse to each other for "sufficiently good" functions $u$ and $v$, $\mathcal{M}^{-1}\mathcal{M}u = u$, and $\mathcal{M}\mathcal{M}^{-1}v = v$.*

**Definition 1.3 ([29])** *The* POCHHAMMER *symbol* $(\alpha)_n$ *is defined for complex* $\alpha \in \mathbb{C}$ *and non-negative integer* $n \in \mathbb{N}$ *by*

$$(\alpha)_0 = 1, \text{ and } (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \; n \in \mathbb{N}^*. \tag{1.1}$$

*Note that* $(1)_n = n!$.

The definition of the EULER gamma function is given as follows

**Definition 1.4 (Euler Gamma function[29])** *The* EULER *gamma function* $\Gamma(\alpha)$ *is defined by the so-called* EULER *integral of the second kind*

$$\Gamma(\alpha) = \mathcal{M}\left[e^{-s}\right](\alpha) = \int_0^\infty s^{\alpha-1}e^{-s}ds, (Re(\alpha) > 0), \tag{1.2}$$

*which converges for all complex* $\alpha \in \mathbb{C}$ $(Re(\alpha) > 0)$,. *It follows from Definition 1.2 that the gamma function is the* MELLIN *transform of the exponential function*

$$\mathcal{M}\left[e^{-s}\right](\alpha) = \Gamma(\alpha)(Re(\alpha) > 0). \tag{1.3}$$

We also indicate some properties of the EULER gamma function:

1. 
   - We have $\Gamma(1) = 1$, $\Gamma(0^+) = +\infty$. The EULER gamma function $\Gamma(\alpha)$ is a monotonous and strictly decreasing for $0 < \alpha \leq 1$.
   - Gamma is a a monotonous and strictly increasing function for $\alpha \geq 2$ so it is convex for $\alpha \in ]0, +\infty[$, with point of minimum equal to $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$.
   - $\Gamma(\alpha)$ is infinite for all the negative integer values of $\alpha$, ie., $\Gamma(-1)$, $\Gamma(-2)$, ..., $\Gamma(-n)$,... are infinite.
2. One of the basic properties of the gamma function is that it satisfies the following function equation

   $$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), Re(\alpha) > 0, \tag{1.4}$$

   which can be easily proved by integrating by parts

   $$\Gamma(\alpha+1) = \int_0^\infty s^\alpha e^{-s}ds = \left[-s^\alpha e^{-s}\right]_0^\infty + \alpha\int_0^\infty s^{\alpha-1}e^{-s}ds = \alpha\Gamma(\alpha).$$

3. It is natural to expect a connection between the Gamma function and the factorial. This is provided by the formula (1.4) and by the fact that $\Gamma(1) = 1$:

   $$\Gamma(n+1) = n\Gamma(n) = n!, \forall n \in \mathbb{N}.$$

4. Using the formula (1.1), and the relation (1.4), the EULER gamma function is extended to the half-plane $Re(\alpha) \leq 0$ by

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+n)}{(\alpha)_n}, Re(\alpha) > -n, \text{ for } n \in \mathbb{N}^*, Re(\alpha) \notin \mathbb{Z}_0^- = \{\ldots, -3, -2, -1, 0\}. \quad (1.5)$$

*Beta function*

The Beta function is very important for the computation of the fractional derivatives of the power function.

**Definition 1.5 (Beta function[29])** *The Beta function is a type of* EULER *integral. For a positive values of the two parameters, $p$ and $q$ ($Re(p) > 0$ and $Re(q) > 0$), the function is defined by*

$$B(p,q) = \int_0^1 s^{p-1}(1-s)^{q-1}\, ds. \quad (1.6)$$

In addition, $B(p,q)$ is used sometimes for convenience to replace a combination of Gamma functions. This relation between the gamma and beta function

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad (1.7)$$

is used letter on.

Equation (1.7) provides the analytical continuation of the Beta function to the entire complex plane via the analytical continuation of the EULER gamma function. It should also be mentioned that Beta function is symmetric, i.e.,

$$B(p,q) = B(q,p).$$

### 1.2.3 *The Riemann-Liouville fractional integrals*

One of the common way to motivate the definition of fractional integrals is to start with an n-tuple iterated integral and show that it can be expressed as a single integral involving the parameter n. The fractional integral of order $\alpha > 0$ is then defined just replacing the integer n by the positive real .

Let $[a,b]$ be a finite closed interval of the real axis $\mathbb{R} = (-\infty, \infty)$, and let $u$ be a measurable continuous function on $[a,b]$ in $\mathbb{R}$. Let's start by noting $\mathcal{I}_{a^+}^1$ the primitive of $u$, and we give

$$\mathcal{I}_{a^+}^1 u(t) = \int_a^t u(s)\, ds. \quad (1.8)$$

The iteration of $\mathcal{I}_{a^+}^1$ allows to obtain the primitive second of $u$. Moreover, according to the theorem of FUBINI,

$$\begin{aligned}\mathcal{I}_{a^+}^2 u(t) &= \mathcal{I}_{a^+}^1 \circ \mathcal{I}_{a^+}^1 u(t) = \int_a^t \left(\int_a^s u(\tau)\, d\tau\right) ds = \int_a^t u(\tau) \left(\int_\tau^t ds\right) d\tau \\ &= \int_a^t (t-\tau)\, u(\tau)\, d\tau.\end{aligned}$$

The RIEMANN-LIOUVILLE's approach is based on the CAUCHY formula (1.8) for the $n^{th}$ integral which uses only a simple integration so as to provide a good basis for generalization

$$\mathcal{I}_{a^+}^n u(t) = \int_a^t \int_a^{t_1} \dots \int_a^{t_{n-1}} u(t_n)\, dt_n dt_{n-1} \dots dt_1 = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} u(s)\, ds. \tag{1.9}$$

Now it is clear how to get an integral of arbitrary order. We simply generalize the CAUCHY formula (1.9), the integer $n$ is substituted by a positive real number $\alpha$ and the EULER gamma function is used instead of the factorial.

If we go back to the starting relationship (1.8) for a function $u : [a,b] \to \mathbb{R}$, we can notice that the integral

$$\mathcal{I}_{b^-}^1 u(t) = \int_b^t u(s)\, ds = -\int_t^b u(s)\, ds,$$

is also a primitive of $u$, which this time involves the values to the right of $u$.

From the relationship

$$\int_b^t (t-s)^{n-1} u(s)\, ds = (-1)^n \int_t^b (s-t)^{n-1} u(s)\, ds,$$

we could define in the same way the right-sided integral of order $n$ of $u$ by

$$\forall t \in [a,b],\ \mathcal{I}_{b^-}^n u(t) = \frac{(-1)^n}{(n-1)!} \int_t^b (s-t)^{n-1} u(s)\, ds.$$

by replacing the integer n by positive real number $\alpha$, we obtain the following definition.

**Definition 1.6 ( [29])** *Let $u \in L^1([a,b], \mathbb{R})$ and $\alpha > 0$. Then, for any $t \in [a,b]$, the integrals*

$$\mathcal{I}_{a^+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s)\, ds,\ t \in [a,b], \tag{1.10}$$

$$\mathcal{I}_{b^-}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s)\, ds,\ t \in [a,b], \tag{1.11}$$

*are called left-sided and right-sided* RIEMANN-LIOUVILLE'S *fractional integral of order $\alpha > 0$.*

*Properties of fractional integrals*

Let $\alpha > 0$, then the fractional integrals, both the left-side and right-side, $\mathcal{I}^{\alpha}$ have the following properties (see [kilbas 1993])

1. Suppose $u \in L^1([a,b])$, then for any $t \in (a,b)$ ,the fractional integrals $I_{a^+}^{\alpha}$ and $I_{b^-}^{\alpha}$ are well defined.

2. For any $t \in (a,b)$, $\gamma > -1$, one can easily show that:

$$\begin{aligned}\left(I_{a^+}^{\alpha}(t-a)^{\gamma}\right)(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(t-a)^{\alpha+\gamma},\\ \left(I_{b^-}^{\alpha}(b-t)^{\gamma}\right)(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(b-t)^{\alpha+\gamma}.\end{aligned}$$

3. By convention is set: $I^0 u(t) = u(t)$.

4. Semigroup property: for any $u \in L^1([a,b])$ and $\beta > 0$

$$I_{a^+}^{\alpha} I_{a^+}^{\beta} u(t) = I_{a^+}^{\beta} I_{a^+}^{\alpha} u(t) = I_{a^+}^{\alpha+\beta} u(t).$$

5. $I_{a^+}^{\alpha} u(t) = 0$ implies that $u = 0$ almost everywhere.

6. Fractional integration by parts formula: let $u \in L^p([a,b])$ and $v \in L^q([a,b])$ either with $\alpha \geq 1$, $p = q = 1$, or with $0 < \alpha < 1$, $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, $p, q > 1$. Then,

$$\int_a^b u(s)\, I_{a^+}^{\alpha} v(s)\, ds = \int_a^b I_{a^+}^{\alpha} u(s)\, v(s)\, ds.$$

*Fractional integrals on the real line*

It is natural to extend the definitions (1.10) and (1.11) to the axes $\mathbb{R}$ and $\mathbb{R}^+$.

**Definition 1.7 ( [29])** *The left-sided* Riemann-Liouville*'s fractional integral of order $\alpha > 0$ of a continuous function $u : \mathbb{R} \to \mathbb{R}$ is given by*

$$\mathcal{I}_{+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} u(s)\, ds,\ t \in \mathbb{R},$$

*The left-sided* Riemann-Liouville*'s fractional integral of order $\alpha > 0$ of a continuous function $u : \mathbb{R}^+ \to \mathbb{R}$ is given by*

$$\mathcal{I}_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} u(s)\, ds,\ t > 0, \tag{1.12}$$

*The Right-sided* Riemann-Liouville*'s fractional integral of order $\alpha > 0$ of a continuous function on $\mathbb{R}^+$ and $\mathbb{R}$is given by*

$$\mathcal{I}_{-}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{+\infty} (s-t)^{\alpha-1} u(s)\, ds.$$

### 1.2.4 *The Riemann-Liouville fractional derivatives*

The definition of fractional integrals is very straightforward and there are no complications. A more difficult question to deal with is how to define a fractional derivative. There is no formula for the $n^{th}$ derivative analogous to (1.8) so we have to generalize the derivatives through a fractional integral.

If $\alpha > 0$, we denote $[\alpha]$ the integer part of $\alpha$, $[\alpha]$ is the unique integer satisfying

$$[\alpha] \leq \alpha < [\alpha] + 1.$$

Let $u : [a, b] \to \mathbb{R}$. From the classic relationship $\frac{d}{dt} = \frac{d^2}{dt^2} \circ \mathcal{I}^1_{a^+}$ we can define a fractional derivative of order $0 \leq \alpha < 1$ by

$$\frac{d^\alpha}{dt^\alpha} = \frac{d}{dt} \circ \mathcal{I}^{1-\alpha}_{a^+}.$$

More generally, if $\alpha > 0$ and if $n = [\alpha] + 1$, we can put

$$\frac{d^\alpha}{dt^\alpha} = \left(\frac{d}{dt}\right)^n \circ \mathcal{I}^{n-\alpha}_{a^+}. \tag{1.13}$$

**Definition 1.8 ( [29])** *Let $\alpha > 0$, and $n = [\alpha] + 1$.The left-sided* Riemann-Liouville's *fractional derivative of order $\alpha$ of a continuous function $u : [a, b] \to \mathbb{R}$ is given by*

$$\mathcal{D}^\alpha_{a^+} u(t) = \left(\frac{d}{dt}\right)^n \circ \mathcal{I}^{n-\alpha}_{a^+} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} u(s)\, ds.$$

*The right-sided* Riemann Liouville *fractional derivative of order $\alpha$ of a continuous function $u : [a, b] \to \mathbb{R}$ is given by*

$$\mathcal{D}^\alpha_{b^-} u(t) = \left(-\frac{d}{dt}\right)^n \circ \mathcal{I}^{n-\alpha}_{a^+} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^b (s-t)^{n-\alpha-1} u(s)\, ds.$$

*Properties of fractional derivatives*

Let $0 < \alpha < 1$. The fractional derivative $D^\alpha_{a^+}$ has the following properties (see [kilbas 1993])

1. For any $u \in L^1([a, b])$, we have that $D^\alpha_{a^+} I^\alpha_{a^+} u = u$
2. The latter can be generalized. In fact, if the function $I^{1-\alpha}_{a^+} u$ is absolutely continuous on $[a, b]$, then

$$I^\alpha_{a^+} D^\alpha_{a^+} u(t) = u(t) - \frac{I^{1-\alpha}_{a^+} u(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1}, \; t \in (a, b),$$

where $I^{1-\alpha}_{a^+} u(a) = \lim_{s \to a^+} \left(I^{1-\alpha}_{a^+} u\right)(s)$, which is in general non zero.

3. According to (1.13), $\forall n \in \mathbb{N}$, all these derivatives coincide with the usual derivatives for integer orders

$$\mathcal{D}_{a^+}^n u(t) = \left(\tfrac{d}{dt}\right)^n \circ \mathcal{I}_{a^+}^0 = \left(\tfrac{d}{dt}\right)^n [u(t) - u(a)] = u^{(n)},$$
$$\mathcal{D}_{b^-}^n u = \left(-\tfrac{d}{dt}\right)^n \circ \mathcal{I}_{b^-}^0 = (-1)^n \left(\tfrac{d}{dt}\right)^n [u(b) - u(t)] = (-1)^n u^{(n)}.$$

4. For any $\alpha, \beta > 0$, we have

$$\mathcal{D}_{a^+}^\alpha (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}. \tag{1.14}$$

*Fractional derivatives on the real line*

Now, if $u : \mathbb{R} \to \mathbb{R}$, then the preceding definitions are generalized directly and are called Liouville derivatives.

**Definition 1.9 ( [29])** *Let $\alpha > 0$, and $n = [\alpha] + 1$. The left-sided* Liouville *fractional derivative of order $\alpha$ of a continuous function $u : \mathbb{R} \to \mathbb{R}$ is given by*

$$\mathcal{D}_+^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_{-\infty}^t (t-s)^{n-\alpha-1} u(s)\, ds.$$

*The right-sided* Liouville *fractional derivative of order $\alpha$ of a continuous function $u : \mathbb{R} \to \mathbb{R}$ is given by*

$$\mathcal{D}_-^\alpha u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^{+\infty} (s-t)^{n-\alpha-1} u(s)\, ds.$$

### 1.2.5 *Caputo-type fractional derivatives*

The Caputo fractional derivative is one of the most used definitions of a fractional derivative along with the Riemann-Liouville. Whereas the Riemann-Liouville definition of a fractional derivative is usually employed in mathematical texts and not so frequently in applications, the Caputo approach appears often while modeling applied problems by means of fractional derivatives and fractional order differential equations.

Moreover, if we go back to $[a, b]$, the inversion of the compositions in the right side of (1.13) seems also reasonable to define a fractional derivative

$$\frac{d^\alpha}{dt^\alpha} = \mathcal{I}_{a^+}^{n-\alpha} \circ \left(\frac{d}{dt}\right)^n. \tag{1.15}$$

It should be noted, however, that this definition is less natural than the previous one, since $\frac{d}{dt} \circ \mathcal{I}_{a^+}^1 u(t) = u(t)$, while $\mathcal{I}_{a^+}^1 \circ \frac{d}{dt} u(t) = u(t) - u(a)$.

This problem of terms of border (here $u(a)$) is in fact very often found in the fractional calculation. The definition given by (1.15) is called the Caputo's derivative.

**Definition 1.10 ( [29])** *Let $\alpha > 0$, and $n = [\alpha] + 1$. The left-sided* CAPUTO*'s fractional derivative of order $\alpha$ of a function $u \in C^n([a,b], \mathbb{R})$, is given by*

$$ {}^{C}\mathcal{D}^{\alpha}_{a^+} u(t) = \mathcal{I}^{n-\alpha}_{a^+} \circ \left(\frac{d}{dt}\right)^n u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s)\, ds, $$

*The right-sided* CAPUTO*'s fractional derivative of order $\alpha$ of a function $u \in C^n([a,b], \mathbb{R})$, is given by*

$$ {}^{C}\mathcal{D}^{\alpha}_{b^-} u(t) = \mathcal{I}^{n-\alpha}_{b^-} \circ \left(-\frac{d}{dt}\right)^n u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s)\, ds, $$

*Properties of Caputo fractional derivatives*

we present some properties of the caputo fractional derivative.

1. Let $\alpha \in \mathbb{R}^+ \backslash \mathbb{N}$, be such that $n = [\alpha] + 1$. If $u^{(n)}$ is a continuous function, so almost everywhere

$$ \lim_{\alpha \to n^-} {}^{C}\mathcal{D}^{\alpha}_{a^+} u(t) = u^{(n)}(t), $$
$$ \lim_{\alpha \to n^-} {}^{C}\mathcal{D}^{\alpha}_{b^-} u(t) = (-1)^n u^{(n)}(t). $$

2. If $\alpha \notin \mathbb{N}$ and $u(t)$ is a function for which the CAPUTO's fractional derivatives ${}^{C}\mathcal{D}^{\alpha}_{a^+} u(t)$ and ${}^{C}\mathcal{D}^{\alpha}_{b^-} u(t)$ of order $\alpha > 0$ exist together with the RIEMANN-LIOUVILLE's fractional derivatives $\mathcal{D}^{\alpha}_{a^+} u(t)$ and $\mathcal{D}^{\alpha}_{b^-} u(t)$, then we have

$$ {}^{C}\mathcal{D}^{\alpha}_{a^+} u(t) = \mathcal{D}^{\alpha}_{a^+} u(t) - \textstyle\sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}, \quad n = [\alpha] + 1, $$
$$ {}^{C}\mathcal{D}^{\alpha}_{a^+} u(t) = \mathcal{D}^{\alpha}_{a^+} u(t), \qquad u(a) = u'(a) = \cdots = u^{(n-1)}(a) = 0. $$

Also

$$ {}^{C}\mathcal{D}^{\alpha}_{b^-} u(t) = \mathcal{D}^{\alpha}_{b^-} u(t) - \textstyle\sum_{k=0}^{n-1} \frac{u^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-t)^{k-\alpha}, \quad n = [\alpha] + 1, $$
$$ {}^{C}\mathcal{D}^{\alpha}_{b^-} u(t) = \mathcal{D}^{\alpha}_{b^-} u(t), \qquad u(b) = u'(b) = \cdots = u^{(n-1)}(b) = 0. $$

3. If $\alpha, \beta > 0$, then

$$ {}^{C}\mathcal{D}^{\alpha}_{a^+} (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}. $$

*Caputo fractional derivatives on the real line*

Fractional derivatives can be defined also in the real line.

**Definition 1.11 ( [29])** *Let $\alpha > 0$, and $n = [\alpha] + 1$. The let-sided* CAPUTO*'s fractional derivative of order $\alpha$ of a function $u \in C^n(\mathbb{R}, \mathbb{R})$, is given by*

$$ {}^{C}\mathcal{D}^{\alpha}_{+} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^t (t-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s)\, ds. $$

*The right-sided* CAPUTO*'s fractional derivative of order $\alpha$ of a function $u \in C^n(\mathbb{R}, \mathbb{R})$, is given by*

$$ {}^{C}\mathcal{D}^{\alpha}_{-} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^{+\infty} (s-t)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s)\, ds. $$

### 1.2.6 *Erdélyi–Kober fractional integral and derivative*

In the mathematical texts and applications, the so called Erdélyi–Kober fractional derivative, as a generalization of the Riemann-Liouville's fractional derivative. In this part we present the definitions and some properties of the Erdélyi–Kober fractional integrals and derivatives.

**Definition 1.12 (*Erdélyi–Kober* fractional integrals [38])** *The right- and left-hand Erdélyi–Kober fractional integrals of the orders $\delta$ and $\alpha$, respectively, of the function $u \in C_\alpha$ are defined by*

$$\left(\mathcal{I}_\beta^{\gamma,\delta} u\right)(t) = \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+\delta)} \int_0^t \left(t^\beta - s^\beta\right)^{\delta-1} s^{\beta(\gamma+1)-1} u(s)\, ds,\ \delta, \beta > 0,\ \gamma \in \mathbb{R}, \tag{1.16}$$

*and*

$$\left(\mathcal{J}_\beta^{\tau,\alpha} u\right)(t) = \frac{\beta}{\Gamma(\alpha)} t^{\beta\tau} \int_t^\infty \left(s^\beta - t^\beta\right)^{\alpha-1} s^{-\beta(\tau+\alpha-1)-1} u(s)\, ds, \alpha, \beta > 0,\ \tau \in \mathbb{R}, \tag{1.17}$$

*where $\Gamma$ is the Euler gamma function.*

- In the special case $\gamma = 0$ and $\beta = 1$, the Erdelyi–Kober fractional integral operator (1.16) is reduced to the Riemann-Liouville fractional integral (1.12) of order $\delta$, here noted by $I^\delta$, with a power weight

$$I_1^{0,\delta} f(t) = \frac{t^{-\delta}}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s)\, ds = t^{-\delta} I^\delta f(t).$$

Similarly, we define generalized fractional derivatives that correspond to the generalized fractional integrals (*1.16*) and (*1.17*).

**Definition 1.13 (*Erdélyi–Kober* fractional derivatives [38])** *Let $n-1 < \delta \leq n,\ n \in \mathbb{N}$ and $m-1 < \alpha \leq m,\ m \in \mathbb{N}$. The right-hand Erdélyi–Kober fractional derivative of the order $\delta$ of the function $u \in C_\alpha^n$ is defined by*

$$\left(\mathcal{D}_\beta^{\gamma,\delta} u\right)(t) = \prod_{j=1}^{n} \left(\gamma + j + \frac{1}{\beta} t \frac{d}{dt}\right) \left(\mathcal{I}_\beta^{\gamma+\delta,n-\delta} u\right)(t). \tag{1.18}$$

*The left-hand Erdélyi–Kober fractional derivative of the order $\alpha$ of the function $u \in C_\alpha^m$ is defined by*

$$\left(\mathcal{P}_\beta^{\tau,\alpha} u\right)(t) = \prod_{j=0}^{m-1} \left(\tau + j - \frac{1}{\beta} t \frac{d}{dt}\right) \left(\mathcal{J}_\beta^{\tau+\alpha,m-\alpha} u\right)(t), \tag{1.19}$$

*where*

$$\prod_{j=1}^{n} \left(\gamma + j + \frac{1}{\beta} t \frac{d}{dt}\right) \left(\mathcal{I}_\beta^{\gamma+\delta,n-\delta} u\right) = \left(\gamma + 1 + \frac{1}{\beta} t \frac{d}{dt}\right) \ldots \left(\gamma + n + \frac{1}{\beta} t \frac{d}{dt}\right) \left(\mathcal{I}_\beta^{\gamma+\delta,n-\delta} u\right).$$

- It emerges that the Erdelyi–Kober and the Riemann–Liouville fractional derivatives are related through the formula

$$D_1^{-\delta,\delta} f(t) = t^{\delta} D^{\delta} f(t).$$

- A further important property of the Erdelyi–Kober fractional derivative is its reduction to the identity operator when $\delta = 0$, i.e.,

$$D_{\beta}^{\gamma,0} f(t) = f(t).$$

*Properties of the Erdélyi-Kober operators*

Here, we give some lemmas and remarks that present the main properties of the generalized fractional operators , we mainly restrict our attention to the right-hand sided operators (1.16), (1.18) (of course, the left-hand sided Erdélyi-Kober operators (1.17), (1.19) possesses similar properties).

**Lemma 1.1 ([38])** *Let $\delta, \beta > 0$, $\gamma \in \mathbb{R}$, and $u \in C_{\alpha}$. The Erdélyi–Kober fractional integrals defined by (1.16) has the following properties:*

$$\begin{aligned} \left(\mathcal{I}_{\beta}^{\gamma,\delta} x^{\lambda\beta} u\right)(t) &= x^{\lambda\beta}\left(\mathcal{I}_{\beta}^{\gamma+\lambda,\delta} u\right)(t), \\ \left(\mathcal{I}_{\beta}^{\gamma,\delta} \mathcal{I}_{\beta}^{\gamma+\delta,\alpha} u\right)(t) &= \left(\mathcal{I}_{\beta}^{\gamma,\delta+\alpha} u\right)(t), \qquad (1.20) \\ \left(\mathcal{I}_{\beta}^{\gamma,\delta} \mathcal{I}_{\beta}^{\alpha,\eta} u\right)(t) &= \left(\mathcal{I}_{\beta}^{\alpha,\eta} \mathcal{I}_{\beta}^{\gamma,\delta} u\right)(t). \qquad (1.21) \end{aligned}$$

**Remark 1.1** *Let $\delta, \beta > 0$ and $\gamma \in \mathbb{R}$. Then we have*

$$\mathcal{D}_{\beta}^{\gamma,\delta} t^{p} = \frac{\Gamma\left(\gamma + \delta + \frac{p}{\beta} + 1\right)}{\Gamma\left(\gamma + \frac{p}{\beta} + 1\right)} t^{p}, \; p + \beta(\gamma + 1) > 0.$$

*In particular,*

$$\mathcal{D}_{\beta}^{\gamma,\delta} t^{-\beta(\gamma - i)} = 0, \text{ for each } i = \{1, 2, ..., n\}.$$

*In fact, from Definition 1.13, for $\delta, \beta > 0$, $\gamma \in \mathbb{R}$, and $p + \beta(\gamma + 1) > 0$, we obtain*

$$\mathcal{D}_{\beta}^{\gamma,\delta} t^{p} = \prod_{j=1}^{n} \left(\gamma + j + \frac{1}{\beta} t \frac{d}{dt}\right) \mathcal{I}_{\beta}^{\gamma+\delta, n-\delta} t^{p},$$

*also from Definition (1.12), we find*

$$\mathcal{I}_{\beta}^{\gamma+\delta, n-\delta} t^{p} = \frac{\beta}{\Gamma(n - \delta)} t^{-\beta(\gamma+n)} \int_0^t \left(t^{\beta} - s^{\beta}\right)^{n-\delta-1} s^{p+\beta(\gamma+\delta+1)-1} ds.$$

*If we put* $x = \frac{s^\beta}{t^\beta}$, *then we get*

$$\mathcal{I}_\beta^{\gamma+\delta,n-\delta} t^p = \frac{1}{\Gamma(n-\delta)} t^p \int_0^1 (1-x)^{n-\delta-1} x^{\gamma+\delta+\frac{p}{\beta}} dx,$$

*By the definition of the beta function, we obtain*

$$\mathcal{I}_\beta^{\gamma+\delta,n-\delta} t^p = \frac{\Gamma\left(\gamma+\delta+\frac{p}{\beta}+1\right)}{\Gamma\left(n+\gamma+\frac{p}{\beta}+1\right)} t^p,\ p+\beta(\gamma+\delta+1) > 0. \tag{1.22}$$

*Now, if we choose* $h = \frac{\Gamma\left(\gamma+\delta+\frac{p}{\beta}+1\right)}{\Gamma\left(n+\gamma+\frac{p}{\beta}+1\right)}$, *then it holds that*

$$\begin{aligned}
\mathcal{D}_\beta^{\gamma,\delta} t^p &= \prod_{j=1}^{n} \left(\gamma + j + \frac{1}{\beta} t \frac{d}{dt}\right) h t^p \\
&= \left(\gamma+1+\frac{1}{\beta}t\frac{d}{dt}\right)\left(\gamma+2+\frac{1}{\beta}t\frac{d}{dt}\right)\dots\left(\gamma+n+\frac{1}{\beta}t\frac{d}{dt}\right) h t^p \\
&= \left(\gamma+1+\frac{1}{\beta}t\frac{d}{dt}\right)\dots\left(\gamma+n-1+\frac{1}{\beta}t\frac{d}{dt}\right)\left(\gamma+n+\frac{p}{\beta}\right) h t^p \\
&= \left(\gamma+1+\frac{1}{\beta}t\frac{d}{dt}\right)\dots\left(\gamma+n-1+\frac{1}{\beta}t\frac{d}{dt}\right) h_1 t^p,\ h_1 = \left(\gamma+n+\frac{p}{\beta}\right) h \\
&\ \ \vdots \\
&= \left(\gamma+1+\frac{p}{\beta}\right)\left(\gamma+2+\frac{p}{\beta}\right)\dots\left(\gamma+n-1+\frac{p}{\beta}\right)\left(\gamma+n+\frac{p}{\beta}\right) h t^p.
\end{aligned}$$

*Thus,*

$$\mathcal{D}_\beta^{\gamma,\delta} t^p = \frac{\left(\gamma+1+\frac{p}{\beta}\right)\left(\gamma+2+\frac{p}{\beta}\right)\dots\left(\gamma+n+\frac{p}{\beta}\right)\Gamma\left(\gamma+\delta+\frac{p}{\beta}+1\right)}{\Gamma\left(n+\gamma+\frac{p}{\beta}+1\right)} t^p; \tag{1.23}$$

*furthermore,* $\Gamma\left(n+\gamma+\frac{p}{\beta}+1\right) = \left(n+\gamma+\frac{p}{\beta}\right)\dots\left(1+\gamma+\frac{p}{\beta}\right)\Gamma\left(1+\gamma+\frac{p}{\beta}\right)$; *it follows that*

$$\mathcal{D}_\beta^{\gamma,\delta} t^p = \frac{\Gamma\left(\gamma+\delta+\frac{p}{\beta}+1\right)}{\Gamma\left(1+\gamma+\frac{p}{\beta}\right)} t^p,\ p+\beta(1+\gamma) > 0.$$

*In particular, if we put* $i = -\left(\gamma+\frac{p}{\beta}\right)$, *then from (1.23), we obtain that*

$$\mathcal{D}_\beta^{\gamma,\delta} t^{-\beta(\gamma+i)} = (1-i)(2-i)\dots(n-1-i)(n-i)\frac{\Gamma(\delta-i+1)}{\Gamma(n-i+1)} t^p.$$

*Therefore, for* $i = 1, 2, \dots, n$, *we can conclude that*

$$\mathcal{D}_\beta^{\gamma,\delta} t^{-\beta(\gamma+i)} = 0,\ \forall \delta, \beta > 0,\ \gamma \in \mathbb{R}.$$

**Lemma 1.2** *Given* $u \in C_\alpha^n$, $n \in \mathbb{N}$, $\delta, \beta > 0$, *and* $\gamma \in \mathbb{R}$ , *such that* $\alpha \geq -\beta(\gamma+1)$ , *the fractional deferential equation* $\mathcal{D}_\beta^{\gamma,\delta} u(t) = 0$ *has the following solutions:*

$$u(t) = C_1 t^{-\beta(\gamma+1)} + C_2 t^{-\beta(\gamma+2)} + \dots + C_n t^{-\beta(\gamma+n)}, C_i \in \mathbb{R} \text{ and } i = 1, 2, \dots, n. \tag{1.24}$$

In fact, from Remark 1.1, we have $\mathcal{D}_{\beta}^{\gamma,\delta} t^{-\beta(\gamma+i)} = 0$, for each $i = \{1, 2, ..., n\}$. Then, the fractional differential equation $\mathcal{D}_{\beta}^{\gamma,\delta} u(t) = 0$ admits a solution as follows

$$u(t) = C_1 t^{-\beta(\gamma+1)} + C_2 t^{-\beta(\gamma+2)} + ... + C_n t^{-\beta(\gamma+n)},\ C_i \in \mathbb{R} \text{ and } i = \{1, 2, .., n\}.$$

**Lemma 1.3 ([38])** *Let $n-1 < \delta < n, n \in \mathbb{N}$, $\alpha \geq -\beta(\gamma+1)$, and $u \in C_{\alpha}^{n}$. Then, the following relationship between the Erdélyi–Kober fractional derivative and the Erdélyi–Kober fractional integral of order $\delta$ is given by*

$$\left(\mathcal{I}_{\beta}^{\gamma,\delta} \mathcal{D}_{\beta}^{\gamma,\delta} u\right)(t) = u(t) - \sum_{k=0}^{n-1} c_k t^{-\beta(1+\gamma+k)},$$

*where* $c_k = \frac{\Gamma(n-k)}{\Gamma(\delta-k)} \lim_{t\to 0} t^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1} \left(1+\gamma+i+\frac{1}{\beta} t \frac{d}{dt}\right) \left(\mathcal{I}_{\beta}^{\gamma+\delta,n-\delta} u\right)(t).$

**Lemma 1.4 ([38])** *For the functions from the space $C_{\alpha}$, $\alpha \geq -\beta(\gamma+1)$, the right-hand sided Erdélyi–Kober fractional derivative is a left-inverse operator to the right-hand sided Erdélyi–Kober fractional integration operator, i.e.,*

$$\left(D_{\beta}^{\gamma,\delta} I_{\beta}^{\gamma,\delta} u\right)(x) = u(x),\ u \in C_{\alpha}.$$

*Caputo type of the Erdélyi–Kober fractional derivative*

In this section we define and give some basic properties of the Caputo type modification of the Erdélyi–Kober fractional derivatives.

**Definition 1.14 (See [38])** *Let $n-1 < \delta \leq n$, $n \in \mathbb{N}$. The right hand sided Caputo Erdélyi–Kober fractional derivative of the order $\delta$, of the function $u \in C_{\alpha}^{n}$ is defined by*

$$\left({}_{*}\mathcal{D}_{\beta}^{\gamma,\delta} u\right)(t) = \left(\mathcal{I}_{\beta}^{\gamma+\delta,n-\delta} \prod_{j=0}^{n-1} \left(1+\gamma+j+\frac{1}{\beta} t \frac{d}{dt}\right) u\right)(t). \tag{1.25}$$

**Lemma 1.5 (See [38])** *Let $n-1 < \delta < n$, $n \in \mathbb{N}$, $\alpha \geq -\beta(\gamma+1)$ and $u \in C_{\alpha}^{n}$, then the following relationship between the Caputo Erdélyi–Kober fractional derivative and the Erdélyi–Kober fractional integral of order $\delta$ is given by*

$$\left(\mathcal{I}_{\beta}^{\gamma,\delta}\ {}_{*}\mathcal{D}_{\beta}^{\gamma,\delta} u\right)(t) = u(t) - \sum_{k=0}^{n-1} c_k t^{-\beta(1+\gamma+k)},$$

*where*

$$c_k = \frac{\Gamma(n-k)}{\Gamma(\delta-k)} \lim_{t\to 0} t^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1} \left(1+\gamma+i+\frac{1}{\beta} t \frac{d}{dt}\right) u(t). \tag{1.26}$$

**Lemma 1.6 (See [38])** *The Caputo-type modification* ${}^{*}D_{\beta}^{\gamma,\delta}$ *of the Erdélyi–Kober fractional derivative is a left-inverse operator to the Erdélyi–Kober fractional integral for the functions from the functional space* $C_{\alpha}$, $\alpha \geq -\beta(\gamma+1)$, *i.e.,*

$$\left({}^{*}D_{\beta}^{\gamma,\delta} I_{\beta}^{\gamma,\delta} u\right)(x) = u(x), \; u \in C_{\alpha}.$$

## 1.3 FIXED POINT THEOREMS

In this section we recall the famous fixed point theorems that we will use to obtain various existence results. first , we give some definitions and theorems, which have an important role in defining these fixed point theorems.

**Definition 1.15** *Let $E$ be a real Banach space; a nonempty closed convex set $P \subset E$ is called a cone of $E$ if it satisfies the following conditions:*
**(i)** $u \in P$, $\lambda \geq 0$, *implies* $\lambda u \in P$,
**(ii)** $u \in P$, $-u \in P$, *implies* $u = 0$.

**Definition 1.16 (Equicontinuous)** *Let $E$ be a Banach space; a subset $P$ in $C(E)$ is called equicontinuous if*

$$\forall \varepsilon > 0, \; \exists \delta > 0, \; \forall u, v \in E, \; \forall \mathcal{A} \in P, \; \|u - v\| < \delta \Rightarrow |\mathcal{A}(u) - \mathcal{A}(v)| < \varepsilon.$$

**Theorem 1.1 (Ascoli–Arzela)** *Let $E$ be a compact space. If $\mathcal{P}$ is an equicontinuous, bounded subset of $C(E, \mathcal{R})$, then $\mathcal{P}$ is relatively compact.*

**Definition 1.17 (Relatively compact subset)** *A subset $\mathcal{P}$ of a topological space $E$ is a relatively compact if and only if any sequence in $\mathcal{P}$ has sub-sequence convergent in $E$.*

**Definition 1.18 (Completely continuous)** *Let $E$ be a Banach space; we say that $\mathcal{A} : E \to E$ is completely continuous if $\mathcal{A} : E \to E$ is continuous and for any bounded subset $P$ of $E$, the set $\mathcal{A}(P)$ is relatively compact.*

**Theorem 1.2 (Schauder's fixed point [1])** *Let $E$ be a Banach space and let $P$ be a closed, convex and nonempty subset of $E$. Let $\mathcal{A} : P \to P$ be a continuous mapping such that:*
*$\mathcal{A}(P)$ is a relatively compact subset of $E$. Then $\mathcal{A}$ has at least one fixed point in $P$.*

**Theorem 1.3 (Leray–Schauder Nonlinear Alternative theorem [31])** *Let $E$ be a Banach space, and $\Omega$ a bounded open subset of $E$ with $0 \in \Omega$. Then, every completely continuous map $\mathcal{A} : \overline{\Omega} \to E$ has at*

*least one of the following two properties:*

**(i)** $\mathcal{A}$ *has a fixed point in* $\overline{\Omega}$,

**(ii)** *There is an* $x \in \partial\Omega$ *and* $\lambda \in (0,1)$ *with* $x = \lambda \mathcal{A}x$.

**Theorem 1.4 (Guo–Krasnosel'skii fixed point theorem [1])** *Let* $E$ *be a Banach space, and let* $P \subseteq E$ *be a cone. Assume that* $\Omega_1, \Omega_2$ *are open subsets of* $E$ *with* $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega$, *and let* $\mathcal{A} : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ *be a completely continuous operator such that:*

**(i)** $\|\mathcal{A}x\| \leq \|x\|, x \in P \cap \partial\Omega_1$ *and* $\|\mathcal{A}x\| \geq \|x\|, x \in P \cap \partial\Omega_2$, *or*

**(ii)** $\|\mathcal{A}x\| \geq \|x\|, x \in P \cap \partial\Omega_1$ *and* $\|\mathcal{A}x\| \leq \|x\|, x \in P \cap \partial\Omega_2$,

*holds. Then,* $\mathcal{A}$ *has at least one positive solution in* $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

**Theorem 1.5 (Banach's fixed point theorem [1])** *Let* $E$ *be a Banach space,* $D$ *be closed subset of* $E$, *and* $\mathcal{A} : D \to D$ *be a strict contraction, i.e.,*

$$\|\mathcal{A}u - \mathcal{A}v\| \leq k \|u - v\| \text{ for some } k \in (0,1) \text{ and all } u, v \in D.$$

*Then,* $\mathcal{A}$ *has a unique fixed point.*

## 1.4 STUDIES AND RESULTS OF FDE'S SOLUTIONS

In recent years, some results dealing with the existence, uniqueness and multiplicity of real or positive solutions of nonlinear fractional boundary value problems on infinite intervals have appeared, by using a lot of operators, (see.eg, [1–4, 8, 9, 22, 32, 36, 37, 39, 45, 46, 52, 57, 61, 62, 64, 66]). For instance, In [52] an explicit solutions are given. In order to apply the transmutation method for solving the mentioned fractional integral and differential equations involving Erdé1yi–Kober operators

$$\begin{aligned} &t^{-\beta\delta}\mathcal{D}_{\beta}^{\alpha,\delta}u(t) - \lambda u(t) = f(t) = 0, \ t > 0, \\ &u(t) - \lambda t^{\beta\delta}\mathcal{I}_{\beta}^{\alpha,\delta}u(t) = f(t) = 0, \ t > 0, \end{aligned}$$

where $\alpha, \lambda \in \mathbb{R}$, $\beta, \delta > 0$, $f$ is a given function, and $\mathcal{I}_{\beta}^{\alpha,\delta}, \mathcal{D}_{\beta}^{\alpha,\delta}$ denote the Erdélyi–Kober fractional integral and derivative, respectively.

The author in [43] considered the following nonlinear fractional differential problem on the half-line $\mathbb{R}_+ = (0, +\infty)$,

$$\begin{cases} \mathcal{D}^{\alpha}u(t) + f(t, u(t)) = 0, \ \ u > 0, \\ \lim_{t\to 0} u(t) = 0, \end{cases} \tag{1.27}$$

where $1 < \alpha \leq 2$ and $f$ is a measurable function in $\mathbb{R}_+ \times \mathbb{R}_+$ that satisfies an appropriate condition. Then, he established the existence of infinitely many solutions of (1.27).

In [65], Zhao and Ge were the first to prove the existence of unbounded solutions for the following nonlinear fractional boundary value problem on an infinite interval

$$\begin{cases} \mathcal{D}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t > 0, \\ u(0) = 0, \ \lim_{t \to +\infty} \mathcal{D}^{\alpha-1} u(t) = \alpha u(\xi), \end{cases}$$

by using the Leray–Schauder nonlinear alternative. Here, $1 < \alpha \leq 2$ and $\mathcal{D}^{\alpha}$ denotes the Riemann–Liouville fractional derivative.

In [63], Zhang and all studied the following fractional nonlinear integral differential equation of mixed type on a semi-infinite interval in a Banach space $E$

$$\begin{cases} \mathcal{D}^{\alpha} u(t) + f(t, u(t), Tu(t), Su(t)) = \theta, \ t \in [0, \infty), \\ u(0) = u'(0) = u''(0) = \ldots = u^{(n-2)}(0) = \theta, \ \mathcal{D}^{\alpha-1} u(\infty) = u_{\infty}, \end{cases}$$

applying the fixed point theorem and the monotone iterative technique, they proved the existence of a unique solution. Where $n-1 < \alpha \leq n, \ n \in \mathbb{N}, \ n \geq 2, f \in C([0, \infty) \times E^3, E), u_{\infty} \in E$ and $D^{\alpha}$ is the Riemann–Liouville fractional derivative, $\theta$ refers to the zero in the space $E$, and $T, \ S$ are defined.

In [23], Guezane-Lakoud and Kılıçman proved the existence of unbounded positive solution for the following nonlinear fractional boundary value problem on infinite interval

$$\begin{cases} {}^{c}\mathcal{D}^{q} u(t) = f(t, u(t), u'(t)), \quad t > 0, \\ u(0) = u''(0) = 0, \ \lim_{t \to +\infty} {}^{c}\mathcal{D}^{q-1} u(t) = \alpha u(1), \end{cases}$$

by using Leray-Schauder nonlinear alternative and Guo-Krasnoselskii fixed point theorems, such that $2 < q \leq 3$, $\alpha > 0$, $f : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a given function, and ${}^{c}\mathcal{D}^{\alpha}$ denotes the caputo fractional derivative.

In [10], Arara et al studied the existence solutions of fractional differential problem of the form:

$$\begin{cases} {}^{c}D^{\alpha} y(t) = f(t, y(t)), \ t \in [0, \infty), \\ y(0) = 0, \ y \text{ is bounded on } [0, \infty), \end{cases}$$

by using the fixed point theorem of Schauder's combined with the diagonalization method. Where, $1 < \alpha \leq 2$ and ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative, $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $y_0 \in \mathbb{R}$.

# 2

## NEW CLASS OF BVP FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING ERDÉLYI–KOBER DERIVATIVE

This chapter is an article published in: Communications in Mathematics, (see [12]).

This chapter deals with the existence and uniqueness of a positive solution to boundary value problem of a nonlinear fractional differential equation involving Erdélyi–Kober differential operators on an infinite interval

$$\mathcal{D}_{\beta}^{\gamma,\delta} u(t) + f(t, u(t)) = 0, \ t > 0, \tag{2.1}$$

with the boundary conditions

$$\lim_{t\to 0} t^{\beta(2+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} u(t) = 0, \ \lim_{t\to\infty} t^{\beta(1+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} u(t) = 0, \tag{2.2}$$

where $\mathcal{D}_{\beta}^{\gamma,\delta}$ denotes the Erdélyi–Kober fractional derivative operator of order $\delta$ and $\mathcal{I}^{\delta+\gamma,2-\delta}$ is the Erdélyi–Kober fractional integral of order $2-\delta$, with $1 < \delta \leq 2$, $-2 < \gamma < -1$, $\beta > 0$, and $f$ is a given function satisfying certain of the following hypotheses:

**(H1)** $f : (0,\infty) \times \mathbb{R} \longrightarrow (0,\infty)$ is continuous.

**(H2)** For all $(t,u) \in (0,\infty) \times \mathbb{R}$,

$$F(t,u) = t^{\beta(1+\gamma)-1} f\left(t, \left(1 + t^{-\beta(1+\gamma)}\right) u\right), \text{ such that } |F(t,u)| \leq \psi(t)\omega(|u|),$$

with $\omega \in C((0,\infty),(0,\infty))$ nondecreasing and $\psi \in L^1(0,\infty)$.

**(H3)** $f : (0,\infty) \times (0,\infty) \longrightarrow (0,\infty)$ is continuous, such that

$$t^{\beta(1+\gamma)-1} f(t,u) = a(t)\, g(t,u),$$

where $a \in L^1(\mathbb{R}_+, \mathbb{R}_+)$, $g \in C(\mathbb{R}_+^2, \mathbb{R}_+)$, and $0 < \int_{\frac{\tau}{\lambda}}^{\tau} a(t)\, dt < \infty$, with $\tau > 0$, $\lambda > 1$.

**(H4)** There exists a positive function $q(t)$ with

$$q^* = \int_0^{+\infty} \left(1 + t^{-\beta(1+\gamma)}\right) q(t)\, dt < \infty,$$

such that

$$t^{\beta(\gamma+1)-1} |f(t,u) - f(t,v)| \leq q(t)\, |u - v|, \ t \in (0,\infty), u, v \in \mathbb{R}.$$

## 2.1 DEFINITION OF INTEGRAL SOLUTION

In this section, we prove a preparatory lemma for the boundary value problem of nonlinear fractional differential equations with Erdélyi–Kober derivative and we give some properties of the green function

**Lemma 2.1** *Let $y \in C^2_\alpha$, with $\int_0^\infty s^{\beta(\gamma+m)-1}y(s)ds < \infty$, $m = \{1,2\}$. Then, the fractional differential equation*

$$\mathcal{D}^{\gamma,\delta}_\beta u(t) + y(t) = 0,\ t > 0,\ 1 < \delta \leq 2,\ -2 < \gamma < -1,\ \beta > 0, \tag{2.3}$$

*with the conditions*

$$\lim_{t\to 0} t^{\beta(2+\gamma)}\mathcal{I}^{\delta+\gamma,2-\delta}u(t) = 0, \tag{2.4}$$

$$\lim_{t\to \infty} t^{\beta(1+\gamma)}\mathcal{I}^{\delta+\gamma,2-\delta}u(t) = 0, \tag{2.5}$$

*has a unique solution given by*

$$u(t) = \int_0^\infty G(t,s)\, s^{\beta(\gamma+1)-1}y(s)\, ds, \tag{2.6}$$

*where*

$$G(t,s) = \begin{cases} \frac{\beta}{\Gamma(\delta)}\left[t^{-\beta(\gamma+1)} - t^{-\beta(\delta+\gamma)}\left(t^\beta - s^\beta\right)^{\delta-1}\right], & 0 < s \leq t < \infty, \\ \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+1)}, & 0 < t \leq s < \infty, \end{cases} \tag{2.7}$$

*is called the Green function of the boundary value problem (2.3)-(2.4)-(2.5).*

**Proof.** Let $1 < \delta \leq 2$, with $-2 < \gamma < -1$ and $\beta > 0$; it is easy to prove that the operator $\mathcal{I}^{\gamma,\delta}_\beta$ has the linearity property for all $\delta > 0$. Now, by applying $\mathcal{I}^{\gamma,\delta}_\beta$ to equation (2.3) we obtain

$$\mathcal{I}^{\gamma,\delta}_\beta \mathcal{D}^{\gamma,\delta}_\beta u(t) + \mathcal{I}^{\gamma,\delta}_\beta y(t) = 0. \tag{2.8}$$

By using Lemma 1.5, for $1 < \delta \leq 2$, we can easily find that

$$\mathcal{I}^{\gamma,\delta}_\beta \mathcal{D}^{\gamma,\delta}_\beta u(t) = u(t) - c_0 t^{-\beta(1+\gamma)} + c_1 t^{-\beta(2+\gamma)},$$

for some constants $c_0, c_1 \in \mathbb{R}$. Thus, (2.8) gives

$$u(t) - c_0 t^{-\beta(1+\gamma)} - c_1 t^{-\beta(2+\gamma)} + \mathcal{I}^{\gamma,\delta}_\beta y(t) = 0,$$

which means that

$$u(t) = c_0 t^{-\beta(1+\gamma)} + c_1 t^{-\beta(2+\gamma)} - \mathcal{I}_\beta^{\gamma,\delta} y(t). \tag{2.9}$$

The boundary condition (2.4) implies that

$$\begin{aligned}
&c_0 \lim_{t\to 0} t^{\beta(2+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} t^{-\beta(1+\gamma)} + c_1 \lim_{t\to 0} t^{\beta(2+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} t^{-\beta(2+\gamma)} \\
&- \lim_{t\to 0} t^{\beta(2+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} \mathcal{I}_\beta^{\gamma,\delta} y(t) = 0;
\end{aligned}$$

consequently, from (1.20), (1.21), (1.22), we obtain

$$\begin{aligned}
\lim_{t\to 0} t^{\beta(2+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} t^{-\beta(1+\gamma)} &= \lim_{t\to 0} t^{\beta(2+\gamma)} \frac{\Gamma(\delta)}{\Gamma(2)} t^{-\beta(1+\gamma)} = 0, \\
\lim_{t\to 0} t^{\beta(2+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} t^{-\beta(2+\gamma)} &= \lim_{t\to 0} t^{\beta(2+\gamma)} \frac{\Gamma(\delta-1)}{\Gamma(1)} t^{-\beta(2+\gamma)} = \Gamma(\delta-1), \\
\lim_{t\to 0} t^{\beta(2+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} \mathcal{I}_\beta^{\gamma,\delta} y(t) &= \lim_{t\to 0} \frac{\beta}{\Gamma(2)} \int_0^t \left(t^\beta - s^\beta\right) s^{\beta(\gamma+1)-1} y(s) ds \\
&= \lim_{t\to 0} \frac{\beta}{\Gamma(2)} t^\beta \int_0^t s^{\beta(\gamma+1)-1} y(s) ds \\
&\quad - \lim_{t\to 0} \frac{\beta}{\Gamma(2)} \int_0^t s^{\beta(\gamma+2)-1} y(s) ds \\
&= 0,
\end{aligned}$$

and therefore, $c_1 = 0$.

In view of the boundary condition (2.5), we conclude that

$$c_0 \lim_{t\to\infty} t^{\beta(1+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} t^{-\beta(1+\gamma)} - \lim_{t\to\infty} t^{\beta(1+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} \mathcal{I}_\beta^{\gamma,\delta} y(t) = 0,$$

consequently, from (1.20), (1.21), (1.22), we find that

$$\begin{aligned}
\lim_{t\to\infty} t^{\beta(1+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} t^{-\beta(1+\gamma)} &= \lim_{t\to\infty} t^{\beta(1+\gamma)} \frac{\Gamma(\delta)}{\Gamma(2)} t^{-\beta(1+\gamma)} = \frac{\Gamma(\delta)}{\Gamma(2)}, \\
\lim_{t\to\infty} t^{\beta(1+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} \mathcal{I}_\beta^{\gamma,\delta} y(t) &= \lim_{t\to\infty} \frac{\beta}{\Gamma(2)} t^{-\beta} \int_0^t \left(t^\beta - s^\beta\right) s^{\beta(\gamma+1)-1} y(s) ds \\
&= \lim_{t\to\infty} \frac{\beta}{\Gamma(2)} \int_0^t s^{\beta(\gamma+1)-1} y(s) ds \\
&\quad - \lim_{t\to\infty} \frac{\beta}{\Gamma(2)} t^{-\beta} \int_0^t s^{\beta(\gamma+2)-1} y(s) ds \\
&= \frac{\beta}{\Gamma(2)} \int_0^\infty s^{\beta(\gamma+1)-1} y(s) ds,
\end{aligned}$$

and therefore, $c_0 = \frac{\beta}{\Gamma(\delta)} \int_0^\infty s^{\beta(\gamma+1)-1} y(s) ds$.

Hence, the unique solution of the problem (2.3)-(2.4)-(2.5) is given by

$$\begin{aligned} u(t) &= \frac{\beta}{\Gamma(\delta)} t^{-\beta(1+\gamma)} \int_0^\infty s^{\beta(\gamma+1)-1} y(s)\, ds \\ &\quad - \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+\delta)} \int_0^t \left(t^\beta - s^\beta\right)^{\delta-1} s^{\beta(\gamma+1)-1} y(s)\, ds \\ &= \frac{\beta}{\Gamma(\delta)} \int_0^t \left[t^{-\beta(1+\gamma)} - t^{-\beta(\gamma+\delta)} \left(t^\beta - s^\beta\right)^{\delta-1}\right] s^{\beta(\gamma+1)-1} y(s)\, ds \\ &\quad + \frac{\beta}{\Gamma(\delta)} \int_t^\infty t^{-\beta(1+\gamma)} s^{\beta(\gamma+1)-1} y(s)\, ds \\ &= \int_0^{+\infty} G(t,s)\, s^{\beta(\gamma+1)-1} y(s) ds. \end{aligned}$$

■

Now, we present some properties of Green's function that form the basis of our main work.

**Lemma 2.2** *For $1 < \delta \leq 2$, $-2 < \gamma < -1$, and $\beta > 0$, the function $G(t,s)$ in Lemma 2.1 satisfies the following conditions:*

1. $\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} > 0, \forall t, s \in (0,\infty)$,
2. $\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} \leq \frac{\beta}{\Gamma(\delta)}, \forall t, s \in (0,\infty)$,
3. *For all $0 < \frac{\tau}{\lambda} \leq t \leq \tau$ and $\forall s > \frac{\tau}{\lambda^2}$, where $\lambda > 1$, $\tau > 0$, we have*

$$\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} \geq \frac{\beta(\delta-1)\tau^{-\beta(1+\gamma)}}{\Gamma(\delta)\lambda^{\beta(1-\gamma)}\left(1+\tau^{-\beta(1+\gamma)}\right)} = \frac{\beta}{\Gamma(\delta)} p(\tau).$$

**Proof.**

1. For $t \leq s$. It is easy to check that $\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} > 0$.

   For $s \leq t$, it holds that

$$\begin{aligned} \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} &= \frac{\beta}{\Gamma(\delta)\left(1+t^{-\beta(1+\gamma)}\right)} \left[t^{-\beta(\gamma+1)} - t^{-\beta(\delta+\gamma)} \left(t^\beta - s^\beta\right)^{\delta-1}\right] \\ &= \frac{\beta t^{-\beta(\gamma+1)}}{\Gamma(\delta)\left(1+t^{-\beta(1+\gamma)}\right)} \left[1 - \left[1 - \left(\frac{s}{t}\right)^\beta\right]^{\delta-1}\right] \\ &= \frac{\beta t^{-\beta(\gamma+1)}}{\Gamma(\delta)\left(1+t^{-\beta(1+\gamma)}\right)} (\delta-1) \int_{1-\frac{s^\beta}{t^\beta}}^1 u^{\delta-2} du \\ &\geq \frac{\beta t^{-\beta(\gamma+1)}}{\Gamma(\delta-1)\left(1+t^{-\beta(1+\gamma)}\right)} (1)^{\delta-2} \left[1 - \left(1 - \frac{s^\beta}{t^\beta}\right)\right] \\ &\geq \frac{\beta t^{-\beta(\gamma+1)}}{\Gamma(\delta-1)\left(1+t^{-\beta(1+\gamma)}\right)} \frac{s^\beta}{t^\beta} > 0. \end{aligned} \tag{2.10}$$

2. For each $t, s \in (0, \infty)$,

$$G(t,s) \leq \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+1)} \text{ implies that } \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} \leq \frac{\beta}{\Gamma(\delta)}.$$

3. Let $0 < \frac{\tau}{\lambda} \leq t \leq \tau$, where $\lambda > 1$, $\tau > 0$. For $t \leq s$, we obtain

$$\begin{aligned}
\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} &= \frac{\beta t^{-\beta(1+\gamma)}}{\Gamma(\delta)\left(1+t^{-\beta(1+\gamma)}\right)} \\
&\geq \frac{\beta \tau^{-\beta(1+\gamma)}}{\Gamma(\delta)\lambda^{-\beta(1+\gamma)}\left(1+\tau^{-\beta(1+\gamma)}\right)} \\
&\geq \frac{(\delta-1)\beta \tau^{-\beta(1+\gamma)}}{\Gamma(\delta)\lambda^{\beta(1-\gamma)}\left(1+\tau^{-\beta(1+\gamma)}\right)} = \frac{\beta}{\Gamma(\delta)} p(\tau).
\end{aligned}$$

For $s \leq t$, from (2.10), we obtain

$$\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} \geq \frac{\beta t^{-\beta(\gamma+1)}}{\Gamma(\delta-1)\left(1+t^{-\beta(1+\gamma)}\right)} \frac{s^{\beta}}{t^{\beta}}.$$

We notice that there are two cases to consider.

FIRST CASE: suppose that $0 < s \leq \frac{\tau}{\lambda}$; if we choose $s \in \left[\frac{\tau}{\lambda^2}, \frac{\tau}{\lambda}\right]$, then

$$\begin{aligned}
\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} &\geq \frac{\beta \tau^{-\beta(\gamma+1)}}{\lambda^{-\beta(1+\gamma)}\Gamma(\delta-1)\left(1+\tau^{-\beta(1+\gamma)}\right)} \frac{\tau^{\beta}}{\lambda^{2\beta}\tau^{\beta}} \\
&\geq \frac{\beta \tau^{-\beta(\gamma+1)}}{\lambda^{\beta(1-\gamma)}\Gamma(\delta-1)\left(1+\tau^{-\beta(1+\gamma)}\right)} = \frac{\beta}{\Gamma(\delta)} p(\tau).
\end{aligned}$$

SECOND CASE: suppose that $\frac{\tau}{\lambda} \leq s \leq t$; then, it follows that

$$\begin{aligned}
\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} &\geq \frac{\beta \tau^{-\beta(\gamma+1)}}{\lambda^{-\beta(1+\gamma)}\Gamma(\delta-1)\left(1+\tau^{-\beta(1+\gamma)}\right)} \frac{\tau^{\beta}}{\lambda^{\beta}\tau^{\beta}} \\
&\geq \frac{\tau^{-\beta(\gamma+1)}}{\lambda^{-\beta\gamma}\Gamma(\delta-1)\left(1+\tau^{-\beta(1+\gamma)}\right)} \\
&\geq \frac{\beta \tau^{-\beta(\gamma+1)}}{\lambda^{\beta(1-\gamma)}\Gamma(\delta-1)\left(1+\tau^{-\beta(1+\gamma)}\right)} = \frac{\beta}{\Gamma(\delta)} p(\tau).
\end{aligned}$$

■

## 2.2 FUNDAMENTAL LEMMAS

We now turn to the question of existence for the boundary value problem (2.1)–(2.2).

Let

$$C([0,\infty]) = \left\{ u \;\middle|\; \begin{array}{l} u \text{ is a continuous function on } (0,+\infty) \text{ such that} \\ \lim_{t\to 0} u(t) \text{ and } \lim_{t\to+\infty} u(t) \text{ exist} \end{array} \right\}.$$

It is easy to see that $C([0,\infty])$ is a Banach space with the norm

$$\|u\|_* = \sup_{t>0} |u(t)|,$$

for instance, see [14, 64].

In this work, we use the space $C_\infty$ to study the problem (2.1)–(2.2), which is denoted by

$$C_\infty((0,\infty),\mathbb{R}) = \left\{ u \in C((0,\infty),\mathbb{R}) \;\middle|\; \begin{array}{l} \lim_{t\to 0} \frac{u(t)}{1+t^{-\beta(1+\gamma)}} \text{ and} \\ \lim_{t\to+\infty} \frac{u(t)}{1+t^{-\beta(1+\gamma)}} \text{ exist} \end{array} \right\}; \tag{2.11}$$

$C_\infty$ is a Banach space with the norm

$$\|u\|_\infty = \sup_{t>0} \left| \frac{u(t)}{1+t^{-\beta(1+\gamma)}} \right|.$$

In fact, it is easy to see that $C_\infty((0,\infty),\mathbb{R})$ is a normed linear space.

Let $\{x_m\}$ be a Cauchy sequence in $C_\infty$; then, $\left\{ y_m \,\middle|\, y_m = \frac{x_m}{1+t^{-\beta(1+\gamma)}} \right\} \subset C([0,\infty])$ is also a Cauchy sequence. Therefore, there exists $y_0 \subset C([0,\infty])$ such that

$$\lim_{m\to+\infty} \|y_m - y_0\|_* = 0.$$

Let $x_0(t) = \left(1+t^{-\beta(1+\gamma)}\right) y_0(t)$. Then, $x_0 \in C_\infty((0,\infty),\mathbb{R})$ and

$$\begin{aligned} \lim_{m\to+\infty} \|x_m - x_0\|_\infty &= \lim_{m\to+\infty} \sup_{t>0} \left| \frac{x_m(t) - x_0(t)}{1+t^{-\beta(1+\gamma)}} \right| \\ &= \lim_{m\to+\infty} \|y_m - y_0\|_* = 0. \end{aligned}$$

Hence, $C_\infty$ is Banach space.

Define an integral operator $\mathcal{A} : C_\infty \to C_\infty$ by

$$\mathcal{A}u(t) = \int_0^\infty G(t,s)\, s^{\beta(1+\gamma)-1} f(s,u(s))ds, \; t \in (0,\infty), \tag{2.12}$$

where $G(t,s)$ is defined by (2.7).

Clearly, from Lemma 2.1, the fixed points of the operator $\mathcal{A}$ coincide with the solutions of the problem (2.1)–(2.2).

To obtain the complete continuity of $\mathcal{A}$, the following lemma is still needed.

**Lemma 2.3 ([65])** *Let* $V = \{u \in C_\infty \,|\|u\|_\infty < l,\ l > 0\}$, $V_1 = \left\{\frac{u(t)}{1+t^{-\beta(1+\gamma)}} \,|u \in V\right\}$. *If* $V_1$ *is equicontinous on any compact intervals of* $(0,\infty)$ *and equiconvergent at infinity, then* $V$ *is relatively compact on* $C_\infty$.

**Remark 2.1 ([65])** $V_1$ *is called equiconvergent at infinity if and only if for all* $\varepsilon > 0$, *there exists* $v(\varepsilon) > 0$ *such that for all* $u \in V_1$, $t_1, t_2 \geq v$, *it holds that*

$$\left|\frac{u(t_1)}{1+t_1^{-\beta(1+\gamma)}} - \frac{u(t_2)}{1+t_2^{-\beta(1+\gamma)}}\right| < \varepsilon.$$

**Lemma 2.4** *If (H1)–(H2) hold, then* $\mathcal{A} : C_\infty \to C_\infty$ *is completely continuous.*

**Proof.** First, for all $u \in C_\infty$, we have

$$\begin{aligned}
\|\mathcal{A}u(t)\|_\infty &= \sup_{t>0} \frac{|\mathcal{A}u(t)|}{1+t^{-\beta(1+\gamma)}} = \sup_{t>0} \left|\int_0^\infty \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f(s,u(s))ds\right| \\
&\leq \frac{\beta}{\Gamma(\delta)} \int_0^\infty \left|s^{\beta(\gamma+1)-1} f(s,u(s))\right| ds,
\end{aligned}$$

together with conditions (H1) and (H2), it then follows that

$$\begin{aligned}
\int_0^\infty \left|s^{\beta(\gamma+1)-1} f(s,u(s))\right| ds &= \int_0^\infty \left|s^{\beta(\gamma+1)-1} f\left(s, \frac{\left(1+s^{-\beta(1+\gamma)}\right)u(s)}{1+s^{-\beta(1+\gamma)}}\right)\right| ds \\
&= \int_0^\infty \left|F\left(s, \frac{u(s)}{1+s^{-\beta(1+\gamma)}}\right)\right| ds \\
&\leq \int_0^\infty \psi(s)\omega\left(\frac{|u(s)|}{1+s^{-\beta(1+\gamma)}}\right) ds \\
&\leq \omega(\|u\|_\infty) \int_0^\infty \psi(s)ds < \infty.
\end{aligned}$$

Hence, $\mathcal{A} : C_\infty \to C_\infty$ is well-defined.

Let $\Omega = \{u \in C_\infty, \|u\|_\infty \leq k,\ k > 0\}$ be a bounded subset of $C_\infty$.

In the following, we divide the proof into several steps.

STEP 1: $\mathcal{A}$ is continuous.

Let $(u_n)_{n\in\mathbb{N}} \in C_\infty$ be a convergent sequence to $u$ in $C_\infty$ such that $\|u\|_\infty \leq k$; from Lemma 2.2, we obtain that

$$\begin{aligned}
\|\mathcal{A}u_n - \mathcal{A}u\|_\infty &= \sup_{t\in(0,\infty)} \left|\frac{\mathcal{A}u_n(t) - \mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}}\right| \\
&\leq \frac{\beta}{\Gamma(\delta)} \sup_{t\in(0,\infty)} \left|\int_0^\infty s^{\beta(\gamma+1)-1} f(s,u_n(s))ds \right. \\
&\quad \left. - \int_0^\infty s^{\beta(\gamma+1)-1} f(s,u(s))ds\right|.
\end{aligned}$$

By the condition (H2), we obtain

$$\begin{aligned}
\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right| &= \left|s^{\beta(\gamma+1)-1}f\left(s,\frac{\left(1+s^{-\beta(1+\gamma)}\right)u(s)}{1+s^{-\beta(1+\gamma)}}\right)\right| \\
&= \left|F\left(s,\frac{u(s)}{1+s^{-\beta(1+\gamma)}}\right)\right| \\
&\leq \psi(s)\,\omega\left(\frac{|u(s)|}{1+s^{-\beta(1+\gamma)}}\right) \\
&\leq \psi(s)\,\omega\left(\|u\|_\infty\right) \\
&\leq \omega(k)\psi(s) \in L^1(0,\infty).
\end{aligned}$$

Together with the continuity of the function $s^{\beta(\gamma+1)-1}f(s,u(s))$, the Lebesgue dominated convergence theorem (theorem 12.12, page 199 in [13]) yields $u \to \int_0^\infty s^{\beta(\gamma+1)-1}f(s,u(s))ds$ is continuous, and it follows that

$$\int_0^\infty s^{\beta(\gamma+1)-1}f(s,u_n(s))ds \to \int_0^\infty s^{\beta(\gamma+1)-1}f(s,u(s))ds \text{ as } n\to\infty.$$

Therefore,

$$\|\mathcal{A}u_n - \mathcal{A}u\|_\infty \to 0, \text{ as } n\to\infty.$$

STEP2: $\mathcal{A}(\Omega)$ is relatively compact.

First, we show that $\mathcal{A}(\Omega)$ is uniformly bounded. Let $u\in\Omega$; by the condition (H2), we obtain

$$\begin{aligned}
\frac{|\mathcal{A}u(t)|}{1+t^{-\beta(1+\gamma)}} &= \left|\int_0^\infty \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds\right| \\
&\leq \frac{\beta}{\Gamma(\delta)}\int_0^\infty \left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&\leq \frac{\beta\omega(k)}{\Gamma(\delta)}\int_0^\infty \psi(s)ds < \infty;
\end{aligned}$$

consequently,

$$\|\mathcal{A}u\|_\infty \leq \frac{\beta\omega(k)}{\Gamma(\delta)}\int_0^\infty \psi(s)ds < \infty, \text{ for all } u\in\Omega, \tag{2.13}$$

and hence, $\mathcal{A}(\Omega)$ is uniformly bounded.

Next, letting $V = \left\{ \frac{\mathcal{A}u}{1+t^{-\beta(1+\gamma)}} \,|u \in \Omega \right\}$, we show that $V$ is equicontinuous on any compact interval of $\mathbb{R}_+$.

For all $u \in \Omega$, $t_1, t_2 \in [a, b]$, $0 < a < b < \infty$, and $t_1 \leq t_2$, we can find

$$\begin{aligned}
& \left| \frac{\mathcal{A}u(t_2)}{1+t_2^{-\beta(1+\gamma)}} - \frac{\mathcal{A}u(t_1)}{1+t_1^{-\beta(1+\gamma)}} \right| \\
\leq & \int_0^\infty \left| \frac{G(t_2,s)}{1+t_2^{-\beta(1+\gamma)}} - \frac{G(t_1,s)}{1+t_1^{-\beta(1+\gamma)}} \right| \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\
\leq & \int_0^\infty \left| \frac{G(t_2,s)}{1+t_2^{-\beta(1+\gamma)}} - \frac{G(t_1,s)}{1+t_2^{-\beta(1+\gamma)}} + \frac{G(t_1,s)}{1+t_2^{-\beta(1+\gamma)}} \right. \\
& \left. - \frac{G(t_1,s)}{1+t_1^{-\beta(1+\gamma)}} \right| \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\
\leq & \int_0^\infty \left| \frac{G(t_2,s) - G(t_1,s)}{1+t_2^{-\beta(1+\gamma)}} - \frac{G(t_1,s)\left(t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}\right)}{\left(1+t_2^{-\beta(1+\gamma)}\right)\left(1+t_1^{-\beta(1+\gamma)}\right)} \right| \\
& \times \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\
\leq & \int_0^\infty \frac{|G(t_2,s) - G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\
& + \int_0^\infty \frac{G(t_1,s)\left(t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}\right)}{\left(1+t_2^{-\beta(1+\gamma)}\right)\left(1+t_1^{-\beta(1+\gamma)}\right)} \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds.
\end{aligned}$$

It remains to show that the right-hand side of the above inequality tends to zero. It is easy to see that

$$\begin{aligned}
& \int_0^\infty \frac{|G(t_2,s) - G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\
\leq & \int_0^{t_1} \frac{|G(t_2,s) - G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\
& + \int_{t_1}^{t_2} \frac{|G(t_2,s) - G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\
& + \int_{t_2}^\infty \frac{|G(t_2,s) - G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds
\end{aligned}$$

$$
\begin{aligned}
\leq \quad & \frac{\beta\omega(k)}{\Gamma(\delta)}\int_0^{t_1}\frac{1}{1+t_2^{-\beta(1+\gamma)}}\left|t_2^{-\beta(1+\gamma)}-t_1^{-\beta(1+\gamma)}\right. \\
& \left.-t_2^{-\beta(\delta+\gamma)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}+t_1^{-\beta(\delta+\gamma)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta-1}\right|\psi(s)ds \\
& +\frac{\beta\omega(k)}{\Gamma(\delta)}\int_{t_1}^{t_2}\frac{1}{1+t_2^{-\beta(1+\gamma)}}\left|t_2^{-\beta(1+\gamma)}-t_1^{-\beta(1+\gamma)}\right. \\
& \left.-t_2^{-\beta(\delta+\gamma)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}\right|\psi(s)ds \\
& +\frac{\beta\omega(k)}{\Gamma(\delta)}\int_{t_2}^{\infty}\frac{t_2^{-\beta(1+\gamma)}-t_1^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}\psi(s)ds, \\
\to \quad & 0, \text{ uniformly as } t_1\to t_2 \text{ for all } u\in\Omega.
\end{aligned}
$$

Analogously, we can obtain

$$
\int_0^{\infty}\frac{G(t_1,s)\left(t_2^{-\beta(1+\gamma)}-t_1^{-\beta(1+\gamma)}\right)}{\left(1+t_2^{-\beta(1+\gamma)}\right)\left(1+t_1^{-\beta(1+\gamma)}\right)}s^{\beta(\gamma+1)-1}\left|f(s,u(s))\right|ds\to 0,
$$

uniformly as $t_1\to t_2$ for all $u\in\Omega$.

Hence, $V$ is locally equicontinuous on $(0,\infty)$.

Finally, we show that $V$ is equiconvergent at $\infty$.

We know that

$$
\begin{aligned}
\mathcal{A}u(t) \quad = \quad & \frac{\beta}{\Gamma(\delta)}t^{-\beta(1+\gamma)}\int_0^{\infty}s^{\beta(\gamma+1)-1}f(s,u(s))ds \\
& -\frac{\beta}{\Gamma(\delta)}t^{-\beta(\gamma+\delta)}\int_0^{t}\left(t^{\beta}-s^{\beta}\right)^{\delta-1}s^{\beta(\gamma+1)-1}f(s,u(s))ds;
\end{aligned}
\tag{2.14}
$$

observing that for any $u\in\Omega$, the condition (H2) gives

$$
\int_0^{\infty}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds\leq\omega\left(\|u\|_{\infty}\right)\int_0^{\infty}\psi(s)ds<\infty, \tag{2.15}
$$

for a given $\varepsilon>0$, there exists a constant $L>0$, such that

$$
\int_L^{\infty}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds<\varepsilon. \tag{2.16}
$$

However, because $\lim\limits_{t\to+\infty}\frac{t^{-\beta(1+\gamma)}}{1+t^{-\beta(1+\gamma)}}=1$, there exists a constant $T_1>0$, such that for any $t_1,t_2\geq T_1$, and we obtain

$$
\left|\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|\leq\left|1-\frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|+\left|1-\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}\right|<\varepsilon. \tag{2.17}
$$

Similarly, $\lim\limits_{t\to+\infty}\frac{t^{-\beta(\delta+\gamma)}\left(t^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t^{-\beta(1+\gamma)}}=1$, and thus there exists a constant $T_2>L>0$, such that for any $t_1,t_2\geq T_2$ and $0<s\leq L$, it holds that

$$\begin{aligned}
&\left|\frac{t_2^{-\beta(\delta+\gamma)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(\delta+\gamma)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\right| \\
\leq\ &\left|1-\frac{t_1^{-\beta(\delta+\gamma)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\right|+\left|1-\frac{t_2^{-\beta(\delta+\gamma)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}\right| \\
\leq\ &\left|1-\frac{t_1^{-\beta(\delta+\gamma)}\left(t_1^{\beta}-L^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\right|+\left|1-\frac{t_2^{-\beta(\delta+\gamma)}\left(t_2^{\beta}-L^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}\right| \\
<\ &\varepsilon. \qquad (2.18)
\end{aligned}$$

Now, we choose $T>\max\{T_1,T_2\}$, for all $t_1,t_2\geq T$; by (2.14), we can obtain

$$\begin{aligned}
&\left|\frac{\mathcal{A}u(t_2)}{1+t_2^{-\beta(1+\gamma)}}-\frac{\mathcal{A}u(t_1)}{1+t_1^{-\beta(1+\gamma)}}\right| \\
\leq\ &\frac{\beta}{\Gamma(\delta)}\left|\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|\int_0^{\infty}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\left|\int_0^{t_2}\frac{t_2^{-\beta(\gamma+\delta)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds\right. \\
&\left.-\int_0^{t_1}\frac{t_1^{-\beta(\gamma+\delta)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds\right| \\
\leq\ &\frac{\beta}{\Gamma(\delta)}\left|\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|\int_0^{\infty}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\left|\int_0^{L}\frac{t_2^{-\beta(\gamma+\delta)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds\right. \\
&\left.-\int_0^{L}\frac{t_1^{-\beta(\gamma+\delta)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds\right| \\
&+\frac{\beta}{\Gamma(\delta)}\left|\int_L^{t_2}\frac{t_2^{-\beta(\gamma+\delta)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds\right. \\
&\left.-\int_L^{t_1}\frac{t_1^{-\beta(\gamma+\delta)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds\right|;
\end{aligned}$$

a direct calculation yields

$$\begin{aligned}
&\left|\frac{\mathcal{A}u(t_2)}{1+t_2^{-\beta(1+\gamma)}}-\frac{\mathcal{A}u(t_1)}{1+t_1^{-\beta(1+\gamma)}}\right| \\
\leq\ & \frac{\beta}{\Gamma(\delta)}\left|\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|\int_0^{\infty}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\int_0^{L}\left|\frac{t_2^{-\beta(\gamma+\delta)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}\right. \\
&\left.-\frac{t_1^{-\beta(\gamma+\delta)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\right|\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\int_L^{t_2}\frac{t_2^{-\beta(\gamma+\delta)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\int_L^{t_1}\frac{t_1^{-\beta(\gamma+\delta)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds.
\end{aligned}$$

From (2.15),(2.16),(2.17),(2.18) and for $t_1, t_2 \to \infty$, we obtain

$$\begin{aligned}
&\left|\frac{\mathcal{A}u(t_2)}{1+t_2^{-\beta(1+\gamma)}}-\frac{\mathcal{A}u(t_1)}{1+t_1^{-\beta(1+\gamma)}}\right| \\
<\ & \frac{\beta}{\Gamma(\delta)}\varepsilon\int_0^{\infty}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\varepsilon\int_0^{L}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds+\frac{2\beta}{\Gamma(\delta)}\varepsilon.
\end{aligned}$$

Hence, $V$ is equiconvergent at $\infty$. Consequently, Lemma 2.3 yields that $V$ is relatively compact.

Therefore, $\mathcal{A} : C_\infty \to C_\infty$ is completely continuous.

■

## 2.3 EXISTENCE OF AT LEAST ONE SOLUTION

In this section, we will prove the existence of at least one solution. We will use the Leray–Schauder nonlinear alternative fixed point theorem.

**Theorem 2.1** *Assume that hypotheses (H1)–(H2) hold, and that there exists $k > 0$, such that*

$$\frac{\beta\omega(k)\int_0^\infty \psi(s)ds}{k\Gamma(\delta)} < 1; \tag{2.19}$$

*then, the fractional boundary value problem (2.1)–(2.2) has at least one solution $u \in \Omega$.*

**Proof.** From the proof of Lemma 2.4, we know that $\mathcal{A}$ is a completely continuous operator. We apply the nonlinear alternative of Leray–Schauder to prove that $\mathcal{A}$ has at least one nontrivial solution in $\Omega$. Let $u \in \partial\Omega$, such that $u = \lambda\mathcal{A}u$, $\lambda \in (0,1)$; from (2.13), we obtain

$$\|u\|_\infty = \lambda\|\mathcal{A}u\|_\infty \leq \|\mathcal{A}u\|_\infty \leq \frac{\beta\omega(k)}{\Gamma(\delta)}\int_0^\infty \psi(s)ds,$$

and thus

$$k \leq \frac{\beta\omega(k)}{\Gamma(\delta)}\int_0^\infty \psi(s)ds;$$

hence,

$$\frac{\beta\omega(k)\int_0^\infty \psi(s)ds}{k\Gamma(\delta)} \geq 1,$$

which contradicts (2.19). By Theorem 1.3 and Lemma 2.4, the boundary value problem (2.1)–(2.2) has at least one solution $u \in \Omega$. ■

## 2.4 EXISTENCE OF POSITIVE SOLUTION

In this section, we establish the existence of a positive solution for the boundary value problem (2.1)–(2.2). First, we introduce the following results.

**Remark 2.2** *Let $\delta, \beta, \gamma, \lambda, \varrho, l, \tau \in \mathbb{R}$, such that $1 < \delta \leq 2$, $\beta > 0$, $-2 < \gamma \leq -1$, $\lambda > 1$, and $l, \tau, \varrho > 0$. If the conditions (H2)–(H3) hold, then*

$$\int_0^{+\infty} s^{\beta(1+\gamma)-1} f(s, u(s))\, ds \leq \eta \int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1} f(s, u(s))\, ds, \tag{2.20}$$

*where $\eta = \frac{l}{\varrho(\lambda^2-1)} + 1 > 1$.*

In fact, for all $t \in \left[\frac{\tau}{\lambda^2}, \tau\right]$, there exists a finite constant $\varrho > 0$, such that $t^{\beta(1+\gamma)-1} f(t,u) \geq \varrho$. Thus,

$$\begin{aligned}\int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1} f(s,u(s))\,ds &\geq \int_{\frac{\tau}{\lambda^2}}^{\tau} s^{\beta(1+\gamma)-1} f(s,u(s))\,ds \\ &\geq \frac{\tau\left(\lambda^2-1\right)}{\lambda^2}\varrho,\end{aligned}$$

hence,

$$\frac{\lambda^2}{\tau\left(\lambda^2-1\right)\varrho}\int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1} f(s,u(s))\,ds \geq 1. \tag{2.21}$$

If $\sup\limits_{t\in\left(0,\frac{\tau}{\lambda^2}\right]} t^{\beta(1+\gamma)-1} f(t,u(t))$ is bounded for $u \in (0,\infty)$, then there exists some $l_0 > 0$, such that

$$\left|t^{\beta(1+\gamma)-1} f(t,u(t))\right| \leq l_0, \quad \forall t \in \left(0, \frac{\tau}{\lambda^2}\right].$$

Similarly, if $\sup\limits_{t\in\left(0,\frac{\tau}{\lambda^2}\right]} t^{\beta(1+\gamma)-1} f(t,u(t))$ is unbounded for $u \in (0,\infty)$, then there exists $M_0 > 0$, such that

$$\sup_{0<u\leq M_0}\ \sup_{t\in\left(0,\frac{\tau}{\lambda^2}\right]} t^{\beta(1+\gamma)-1} f(t,u(t)) \leq l_1, \text{ for some } l_1 > 0.$$

In all cases, if we choose $l = \max\{l_0, l_1\}$, for all $t \in \left(0, \frac{\tau}{\lambda^2}\right]$, then we obtain $t^{\beta(1+\gamma)-1} f(t,u(t)) \leq l$, and thus

$$\int_0^{\frac{\tau}{\lambda^2}} s^{\beta(1+\gamma)-1} f(s,u(s))\,ds \leq \frac{l\tau}{\lambda^2}. \tag{2.22}$$

From (2.21) and (2.22), we can find that

$$\begin{aligned}\int_0^{+\infty} s^{\beta(1+\gamma)-1} f(s,u(s))\,ds &= \int_0^{\frac{\tau}{\lambda^2}} s^{\beta(1+\gamma)-1} f(s,u(s))\,ds \\ &\quad + \int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1} f(s,u(s))\,ds \\ &\leq \frac{l\tau}{\lambda^2} + \int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1} f(s,u(s))\,ds \\ &\leq \frac{l}{\varrho\left(\lambda^2-1\right)}\int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1} f(s,u(s))\,ds \\ &\quad + \int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1} f(s,u(s))\,ds \\ &\leq \left(\frac{l}{\varrho\left(\lambda^2-1\right)} + 1\right)\int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1} f(s,u(s))\,ds \\ &\leq \eta\int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1} f(s,u(s))\,ds.\end{aligned}$$

Let us define the cone $K$ by

$$K=\left\{u\in C_{\infty}\ \middle|\ u(t)>0,\ \text{and}\ \forall t>0;\ \min_{t\in\left[\frac{\tau}{\lambda},\tau\right]}\frac{u\left(t\right)}{1+t^{-\beta(1+\gamma)}}\geq\frac{p\left(\tau\right)}{\eta}\left\|u\right\|_{\infty}\right\}.$$

**Lemma 2.5** *We have $\mathcal{A}\left(K\right)\subset K$.*

**Proof.** We know from Lemma 2.2 that

$$\begin{aligned}\left\|\mathcal{A}u\left(t\right)\right\|_{\infty} &= \sup_{t>0}\frac{\left|\mathcal{A}u\left(t\right)\right|}{1+t^{-\beta(1+\gamma)}}=\sup_{t>0}\left|\int_{0}^{\infty}\frac{G\left(t,s\right)}{1+t^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds\right| \\ &\leq \frac{\beta}{\Gamma\left(\delta\right)}\int_{0}^{\infty}s^{\beta(\gamma+1)-1}f(s,u(s))ds; \end{aligned}\tag{2.23}$$

also from (2.20) and Lemma 2.2, for all $t\in\left[\frac{\tau}{\lambda},\tau\right]$, $\tau>0$, and $\lambda>1$, we obtain

$$\begin{aligned}\frac{\mathcal{A}u\left(t\right)}{1+t^{-\beta(1+\gamma)}} &= \int_{0}^{\infty}\frac{G\left(t,s\right)}{1+t^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds \\ &= \int_{0}^{\frac{\tau}{\lambda^{2}}}\frac{G\left(t,s\right)}{1+t^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds \\ &\quad+\int_{\frac{\tau}{\lambda^{2}}}^{\infty}\frac{G\left(t,s\right)}{1+t^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds \\ &\geq \int_{\frac{\tau}{\lambda^{2}}}^{\infty}\frac{G\left(t,s\right)}{1+t^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds \\ &\geq \frac{\beta}{\Gamma\left(\delta\right)}p\left(\tau\right)\int_{\frac{\tau}{\lambda^{2}}}^{\infty}s^{\beta(\gamma+1)-1}f(s,u(s))ds \\ &\geq \frac{p\left(\tau\right)}{\eta}\frac{\beta}{\Gamma\left(\delta\right)}\int_{0}^{+\infty}s^{\beta(1+\gamma)-1}f\left(s,u\left(s\right)\right)ds. \end{aligned}\tag{2.24}$$

From (2.23), we obtain

$$\frac{\mathcal{A}u\left(t\right)}{1+t^{-\beta(1+\gamma)}}\geq\frac{p\left(\tau\right)}{\eta}\left\|\mathcal{A}u\left(t\right)\right\|_{\infty};$$

therefore, $\min_{t\in\left[\frac{\tau}{\lambda},\tau\right]}\frac{\mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}}\geq\frac{p(\tau)}{\eta}\left\|\mathcal{A}\left(u\right)\right\|_{\infty}$, which proves that $\mathcal{A}\left(K\right)\subset K$. ■

For convenience, we denote some important constants:

$$F_{0}=\lim_{u\to0}\sup_{t>0}\frac{g\left(t,\left(1+t^{-\beta(1+\gamma)}\right)u\right)}{u},\qquad f_{\infty}=\lim_{u\to+\infty}\inf_{t>0}\frac{g\left(t,\left(1+t^{-\beta(1+\gamma)}\right)u\right)}{u},$$
$$f_{0}=\lim_{u\to0}\inf_{t>0}\frac{g\left(t,\left(1+t^{-\beta(1+\gamma)}\right)u\right)}{u},\qquad F_{\infty}=\lim_{u\to+\infty}\sup_{t>0}\frac{g\left(t,\left(1+t^{-\beta(1+\gamma)}\right)u\right)}{u}.$$

**Theorem 2.2** *Assume that hypotheses (H2)–(H3) hold. If the following condition is satisfied:*

$$F_{0}=0,\quad f_{\infty}=\infty,$$

*then the boundary value problem (2.1)–(2.2) has at least one positive solution.*

**Proof.** From Lemma 2.4, $\mathcal{A}$ is a completely continuous operator.

Now, because $F_0 = 0$, we may choose $r_1 > 0$, such that

$$g\left(t, \left(1 + t^{-\beta(1+\gamma)}\right) u\right) \leq \varepsilon u, \text{ for } 0 < u \leq r_1, \ t > 0, \tag{2.25}$$

where $\varepsilon > 0$ satisfies

$$\varepsilon \leq \frac{\Gamma(\delta)}{\beta \int_0^{+\infty} a(s)\, ds}. \tag{2.26}$$

Therefore, for all $u \in K$ and $\|u\|_\infty = r_1$, from Lemma 2.2, we have

$$\begin{aligned} \frac{\mathcal{A}u(t)}{1 + t^{-\beta(1+\gamma)}} &= \int_0^\infty \frac{G(t,s)}{1 + t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f(s, u(s)) ds \\ &\leq \frac{\beta}{\Gamma(\delta)} \int_0^\infty s^{\beta(1+\gamma)-1} f(s, u(s)) ds. \end{aligned}$$

By the condition (H3) and from (2.25), we obtain

$$\begin{aligned} \frac{\mathcal{A}u(t)}{1 + t^{-\beta(1+\gamma)}} &\leq \frac{\beta}{\Gamma(\delta)} \int_0^\infty a(s)\, g(s, u(s)) ds \\ &\leq \frac{\beta}{\Gamma(\delta)} \int_0^\infty a(s)\, g(s, \left(1 + s^{-\beta(1+\gamma)}\right) \frac{u(s)}{1 + s^{-\beta(1+\gamma)}}) ds \\ &\leq \frac{\beta}{\Gamma(\delta)} \int_0^\infty a(s)\, \varepsilon \frac{u(s)}{1 + s^{-\beta(1+\gamma)}}) ds \\ &\leq \frac{\beta}{\Gamma(\delta)} \varepsilon \|u\|_\infty \int_0^\infty a(s)\, ds. \end{aligned}$$

Therefore, from (2.26), we obtain

$$\|\mathcal{A}u\|_\infty \leq \|u\|_\infty, u \in K \text{ and } \|u\|_\infty = r_1, \ \forall t > 0. \tag{2.27}$$

If we choose

$$\Omega_1 = \{u \in C_\infty, \|u\|_\infty < r_1\},$$

then (2.27) shows that

$$\|\mathcal{A}u\|_\infty \leq \|u\|_\infty, \text{ for } u \in K \cap \partial\Omega_1.$$

Furthermore, because $f_\infty = \infty$, there exists $r > 0$ such that

$$g\left(t, \left(1 + t^{-\beta(1+\gamma)}\right) u\right) \geq mu, \text{ for } u \geq r, \ t > 0, \tag{2.28}$$

where $m > 0$ is chosen so that

$$m \geq \frac{\eta^2 \Gamma(\delta)}{\beta p^2(\tau) \int_{\frac{\tau}{\lambda}}^{\tau} a(s)\, ds}. \tag{2.29}$$

Let $r_2 \geq \max\left\{r_1, \frac{\eta r}{p(\tau)}\right\}$ and $\Omega_2 = \{u \in C_\infty, \|u\|_\infty < r_2\}$; then, $\Omega_1 \subset \Omega_2$.
Thus, for all $u \in K$ and $\|u\|_\infty = r_2$, we have that

$$\frac{u(t)}{1+t^{-\beta(1+\gamma)}} \geq \min_{\frac{\tau}{\lambda} \leq t \leq \tau} \frac{u(t)}{1+t^{-\beta(1+\gamma)}} \geq \frac{p(\tau)}{\eta}\|u\|_\infty = \frac{p(\tau)}{\eta} r_2 \geq r;$$

from (2.24), (2.28), (2.29) and by the condition (H3), for all $\frac{\tau}{\lambda} \leq t \leq \tau$, we have

$$\begin{aligned}
\frac{\mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}} &\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} \int_0^\infty s^{\beta(1+\gamma)-1} f(s,u(s))ds \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} \int_0^\infty a(s)\, g(s,u(s))ds \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} \int_0^\infty a(s)\, g(s, \left(1+s^{-\beta(1+\gamma)}\right) \frac{u(s)}{1+s^{-\beta(1+\gamma)}})ds \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} m \int_0^\infty a(s) \frac{u(s)}{1+s^{-\beta(1+\gamma)}} ds \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} m \min_{\frac{\tau}{\lambda} \leq t \leq \tau} \frac{u(t)}{1+t^{-\beta(1+\gamma)}} \int_{\frac{\tau}{\lambda}}^\tau a(s)\, ds \\
&\geq \frac{\beta}{\eta^2\Gamma(\delta)} p^2(\tau)\, m \|u\|_\infty \int_{\frac{\tau}{\lambda}}^\tau a(s)\, ds \geq \|u\|_\infty .
\end{aligned}$$

Hence, $\|\mathcal{A}u\|_\infty \geq \|u\|_\infty$, for $u \in K \cap \partial\Omega_2$. Therefore, by the first part of Theorem 1.4, it follows that $\mathcal{A}$ has a fixed point in $K \cap \partial\left(\overline{\Omega}_2 \backslash \Omega_1\right)$. ■

Similarly to the previous theorem, we can prove the following theorem.

**Theorem 2.3** *Assume that hypotheses (H2)–(H3) hold. If the following condition is satisfied:*

$$f_0 = \infty, \quad F_\infty = 0,$$

*then the boundary value problem (2.1)–(2.2) has at least one positive solution.*

**Proof.** From Lemma 2.4, $\mathcal{A}$ is a completely continuous operator.
Now, because $f_0 = \infty$, there exists $R_1 > 0$ such that

$$g\left(t, \left(1+t^{-\beta(1+\gamma)}\right) u\right) \geq Mu, \text{ for } 0 < u \leq R_1,\ t > 0, \tag{2.30}$$

where $M > 0$ is chosen so that

$$M \geq \frac{\eta^2\Gamma(\delta)}{\beta p^2(\tau) \int_0^\infty a(s)\, ds}. \tag{2.31}$$

Let $\Omega_1 = \{u \in C_\infty, \|u\|_\infty < R_1\}$. Thus, if $u \in K$ and $\|u\|_\infty = R_1$ $(u \in \partial\Omega_1)$, then for all $t \in \left[\frac{\tau}{\lambda}, \tau\right]$, from (2.24), (2.30), (2.31) and by the condition (H3), we obtain

$$\begin{aligned}
\frac{\mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}} &\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} \int_0^\infty s^{\beta(1+\gamma)-1} f(s,u(s))ds \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} \int_0^\infty a(s)\, g(s,u(s))ds \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} \int_0^\infty a(s)\, g(s, \left(1+s^{-\beta(1+\gamma)}\right) \frac{u(s)}{1+s^{-\beta(1+\gamma)}})ds \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} M \int_{\frac{\tau}{\lambda}}^\tau a(s) \frac{u(s)}{1+s^{-\beta(1+\gamma)}} ds \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} M \int_{\frac{\tau}{\lambda}}^\tau a(s) \frac{p(\tau)}{\eta} \|u\|_\infty ds \\
&\geq \frac{\beta p^2(\tau)}{\eta^2\Gamma(\delta)} M \|u\|_\infty \int_{\frac{\tau}{\lambda}}^\tau a(s)\, ds \geq \|u\|_\infty .
\end{aligned}$$

Hence,

$$\|\mathcal{A}u\|_\infty \geq \|u\|_\infty, \text{ for } u \in K \cap \partial\Omega_1.$$

Furthermore, from $F_\infty = 0$, there exists $R > 0$, such that

$$g\left(t, \left(1+t^{-\beta(1+\gamma)}\right) u\right) \leq \varepsilon u, \text{ for } u > R, \text{ and } t > 0, \tag{2.32}$$

where $\varepsilon > 0$ satisfies

$$\varepsilon \leq \frac{\Gamma(\delta)}{\beta \int_0^\infty a(s)\, ds}. \tag{2.33}$$

Let $\Omega_2 = \{u \in C_\infty, \|u\|_\infty < R_2\}$, where $R_2 > \max\{R_1, R\}$. Then, $\Omega_1 \subset \Omega_2$.
Now, we define the function $J$ as

$$\begin{aligned}
&J : \mathbb{R}_+ \to \mathbb{R}_+, \\
&J(a) = \sup_{0<u\leq a} \sup_{t>0} g\left(t, \left(1+t^{-\beta(1+\gamma)}\right) u\right).
\end{aligned}$$

Suppose $u \in K$ and $\|u\|_\infty = R_2$ $(u \in \partial\Omega_2)$. Then, from (2.32), we obtain

$$\begin{aligned}
\sup_{0<u\leq R_2} \sup_{t>0} g\left(t, \left(1+t^{-\beta(1+\gamma)}\right) u\right) &\leq \varepsilon \sup_{0<u\leq R_2} u = \varepsilon R_2 \\
&\Longrightarrow J(R_2) \leq \varepsilon R_2 = \varepsilon \|u\|_\infty .
\end{aligned} \tag{2.34}$$

Given Lemma 2.2 and the condition (H3), (2.33) and (2.34) yield that

$$\begin{aligned}
\frac{\mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}} &\leq \frac{\beta}{\Gamma(\delta)}\int_0^\infty s^{\beta(1+\gamma)-1} f(s,u(s))ds \\
&\leq \frac{\beta}{\Gamma(\delta)}\int_0^\infty a(s)\, g(s,u(s))ds \\
&\leq \frac{\beta}{\Gamma(\delta)}\int_0^\infty a(s)\, g(s,\left(1+s^{-\beta(1+\gamma)}\right)\frac{u(s)}{1+s^{-\beta(1+\gamma)}})ds \\
&\leq \frac{\beta}{\Gamma(\delta)}\int_0^\infty a(s) \sup_{0<u\leq R_2}\sup_{s>0} g(s,\left(1+s^{-\beta(1+\gamma)}\right)\frac{u(s)}{1+s^{-\beta(1+\gamma)}})ds \\
&= \frac{\beta}{\Gamma(\delta)}\int_0^\infty a(s)\, J(R_2)\, ds \\
&\leq \frac{\beta}{\Gamma(\delta)}\varepsilon \|u\|_\infty \int_0^\infty a(s)\, ds.
\end{aligned}$$

Hence, $\|\mathcal{A}u\|_\infty \leq \|u\|_\infty$, for $u \in K \cap \partial\Omega_2$.

Therefore, by the second part of Theorem 1.4, it follows that $\mathcal{A}$ has a fixed point in $K \cap \partial\left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

The proof is complete. ■

## 2.5 UNIQUENESS OF SOLUTION

The last result of the existence is based on the Banach contraction principle theorem.

**Theorem 2.4** *Assume that hypotheses (H1), (H2), and (H4) hold. If*

$$\frac{q^*\beta}{\Gamma(\delta)} < 1, \tag{2.35}$$

*then the boundary value problem (2.1)–(2.2) has a unique solution $u \in C_\infty$.*

**Proof.** We shall show that the operator $\mathcal{A}$ defined by (2.12) is a contraction mapping.

Let $u, v \in C_\infty$, from Lemma 2.2 and by the condition (H4), we can obtain that

$$\begin{aligned}
\left|\frac{\mathcal{A}u(t)-\mathcal{A}v(t)}{1+t^{-\beta(1+\gamma)}}\right| &= \int_0^\infty \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1}\left|f(s,u(s))-f(s,v(s))\right| ds \\
&\leq \frac{\beta}{\Gamma(\delta)}\int_0^\infty s^{\beta(\gamma+1)-1}\left|f(s,u(s))-f(s,v(s))\right| ds \\
&\leq \frac{\beta}{\Gamma(\delta)}\int_0^\infty q(s)\left|u-v\right| ds \\
&\leq \frac{\beta}{\Gamma(\delta)}\int_0^\infty q(s)\left(1+s^{-\beta(1+\gamma)}\right)\left|\frac{u-v}{1+s^{-\beta(1+\gamma)}}\right| ds,
\end{aligned}$$

this implies that

$$\begin{aligned}
\|\mathcal{A}(u)-\mathcal{A}(v)\|_\infty &\leq \frac{\beta}{\Gamma(\delta)}\|u-v\|_\infty \int_0^\infty q(s)\left(1+s^{-\beta(1+\gamma)}\right) ds \\
&\leq \frac{\beta q^*}{\Gamma(\delta)}\|u-v\|_\infty .
\end{aligned}$$

It follows from the assumption (2.35) and the preceding estimate that $\mathcal{A}$ is a contraction mapping. Applying Banach's fixed point Theorem 1.5, the operator $\mathcal{A}$ has a fixed point that corresponds to the unique solution of problem (2.1)–(2.2). ■

## 2.6 EXAMPLES

In this section, we present some examples to illustrate the usefulness of our main results.

**Example 2.1** *Consider the following boundary value problem:*

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2},\frac{3}{2}} u(t) + t^{\frac{3}{2}} \sqrt{\left|\frac{u}{1+t^{\frac{1}{2}}}\right|} e^{-t} = 0, \ t > 0, \\ \lim_{t\to 0} t^{\frac{1}{2}} I_1^{0,\frac{1}{2}} u(t) = 0, \ \lim_{t\to\infty} t^{-\frac{1}{2}} I_1^{0,\frac{1}{2}} u(t) = 0. \end{cases} \tag{2.36}$$

*Here,* $f(t,u) = t^{\frac{3}{2}} \sqrt{\left|\frac{u}{1+t^{\frac{1}{2}}}\right|} e^{-t}$, $\delta = \frac{3}{2}$, $\gamma = -\frac{3}{2}$, *and* $\beta = 1$.

***(H1)*** *It is easy to show that the function* $f$ *is continuous for any* $(t,u) \in (0,\infty) \times \mathbb{R}$.

***(H2)*** *From the expression of the function* $f$, *it follows that*

$$F(t,u) = t^{\beta(1+\gamma)-1} f\left(t, \left(1 + t^{-\beta(1+\gamma)}\right) u\right) = \sqrt{|u|} e^{-t}.$$

*If we choose* $\omega(u) = \sqrt{u}$, $\psi(t) = e^{-t}$, *then we obtain*

$$|F(t,u)| \le \psi(t)\omega(|u|), \ on \ (0,\infty) \times \mathbb{R},$$

*with* $\omega \in C((0,\infty),(0,\infty))$ *nondecreasing and* $\psi \in L^1(0,\infty)$. *Then, the condition (H2) holds.*

*If we choose* $k > \frac{4}{\pi}$, *we show that*

$$\frac{\beta\omega(k) \int_0^\infty \psi(s)ds}{k\Gamma(\delta)} = \frac{2}{k\sqrt{\pi}} < 1,$$

*therefore, (2.19) is satisfied. Hence, all the conditions of Theorem 2.1 hold, and problem (2.36) has at least one solution.*

**Example 2.2** *Consider the following problem:*

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2},\frac{5}{3}} u(t) + \frac{t^{\frac{3}{2}} \exp(-t) u^2}{(1+\sqrt{t})} = 0, \ t > 0, \\ \lim_{t\to 0} t^{\frac{1}{2}} I_1^{\frac{1}{6},\frac{1}{3}} u(t) = 0, \ \lim_{t\to\infty} t^{-\frac{1}{2}} I_1^{\frac{1}{6},\frac{1}{3}} u(t) = 0. \end{cases} \tag{2.37}$$

*Here,* $f(t,u) = \frac{t^{\frac{3}{2}} \exp(-t) u^2}{(1+\sqrt{t})}$, $\beta = 1$, $\gamma = -\frac{3}{2}$, *and* $\delta = \frac{5}{3}$.

***(H2)*** $F(t,u) = t^{\beta(1+\gamma)-1} f\left(t, \left(1 + t^{-\beta(1+\gamma)}\right) u\right) = \exp(-t) u^2$, *verify*

$$|F(t,u)| \le \psi(t)\omega(|u|), \ on \ (0,+\infty) \times \mathbb{R},$$

*where* $\psi(t) = \exp(-t) \in L^1(0,+\infty)$ *and* $\omega(u) = u^2 \in C((0,+\infty),(0,+\infty))$. *Then, the condition (H2) holds.*

***(H3)*** *It is clear that the function* $f$ *is continuous on* $(0,+\infty)\times(0,+\infty)$, *and*

$$t^{-\frac{3}{2}} f(t,u) = \frac{\exp(-t)\, u^2}{\left(1+\sqrt{t}\right)} = a(t)\, g(t,u),$$

*where* $a(t) = \exp(-t)$, *and* $g(t,u) = \frac{u^2}{1+\sqrt{t}} \in C((0,+\infty)\times(0,+\infty))$. *Hence, the condition (H3) is satisfied.*

*We have* $g\left(t, \left(1+\sqrt{t}\right)u\right) = u^2$, *which implies that*

$$\begin{aligned} F_0 &= \lim_{u\to 0}\sup_{t>0} \frac{g\left(t,\left(1+\sqrt{t}\right)u\right)}{u} = 0, \\ f_\infty &= \lim_{u\to\infty}\inf_{t>0} \frac{g\left(t,\left(1+\sqrt{t}\right)u\right)}{u} = \infty. \end{aligned}$$

*It follows from Theorem 2.2 that problem (2.37) has at least one positive solution.*

**Example 2.3** *We take* $\beta = 1$, $\gamma = -\frac{3}{2}$, *and* $\delta = \frac{7}{6}$. *Consider the following problem:*

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2},\frac{7}{6}} u(t) + f(t,u) = 0, \ t>0, \\ \lim_{t\to 0} t^{\frac{1}{2}} I_1^{\frac{16}{6},\frac{5}{6}} u(t) = 0, \ \lim_{t\to\infty} t^{-\frac{1}{2}} I_1^{\frac{16}{6},\frac{5}{6}} u(t) = 0, \end{cases} \tag{2.38}$$

*where* $f(t,u) = t^{\frac{3}{2}}\left[\arctan\left(\frac{|u|}{1+\sqrt{t}}\right)+1\right]\exp(-t)$.

***(H2)*** *It is clear that* $F(t,u) = [\arctan(|u|)+1]\exp(-t)$, *verify*

$$|F(t,u)| \le \psi(t)\omega(|u|),$$

*where* $\psi(t) = \exp(-t) \in L^1(0,+\infty)$, $\omega(u) = \arctan(|u|)+1 \in C((0,+\infty),(0,+\infty))$. *Then, the condition (H2) holds.*

***(H3)*** $f(t,u)$ *is continuous on* $((0,+\infty)\times(0,+\infty),(0,+\infty))$, *and*

$$t^{-\frac{3}{2}} f(t,u) = \left[\arctan\left(\frac{|u|}{1+\sqrt{t}}\right)+1\right]\exp(-t) = a(t)\, g(t,u),$$

*where* $a(t) = \exp(-t)$, $g(t,u) = \left[\arctan\left(\frac{|u|}{1+\sqrt{t}}\right)+1\right] \in C((0,+\infty)\times(0,+\infty))$. *Hence, the condition (H3) is satisfied.*

*We have* $g\left(t,\left(1+\sqrt{t}\right)u\right) = \arctan(|u|)+1$, *which implies that*

$$\begin{aligned} F_0 &= \lim_{u\to 0}\sup_{t>0} \frac{g\left(t,\left(1+\sqrt{t}\right)u\right)}{u} = \lim_{u\to 0}\frac{\arctan(|u|)+1}{u} = \infty, \\ f_\infty &= \lim_{u\to\infty}\inf_{t>0} \frac{g\left(t,\left(1+\sqrt{t}\right)u\right)}{u} = \lim_{u\to\infty}\frac{\arctan(|u|)+1}{u} = 0. \end{aligned}$$

*It follows from Theorem 2.3 that problem (2.38) has at least one positive solution.*

**Example 2.4** *We take $\beta = 1$, $\gamma = -\frac{3}{2}$, and $\delta = \frac{3}{2}$. Consider the following problem:*

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2},\frac{3}{2}} u(t) + f(t,u) = 0, \ t > 0, \\ \lim_{t\to 0} t^{\frac{1}{2}} I_1^{0,\frac{1}{2}} u(t) = 0, \ \lim_{t\to\infty} t^{-\frac{1}{2}} I_1^{0,\frac{1}{2}} u(t) = 0, \end{cases} \tag{2.39}$$

*where* $f(t,u) = \frac{t^{\frac{3}{2}} \exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} \arctan(|u|)$.

***(H1)*** *$f(t,u)$ is continuous on $((0,+\infty) \times \mathbb{R}, (0,+\infty))$.*

***(H2)*** *It is clear that $F(t,u) = \frac{\exp(-t)}{2\sqrt{\pi}} \arctan(|u|)$, verify*

$$|F(t,u)| \leq \psi(t)\omega(|u|),$$

*where $\psi(t) = \exp(-t) \in L^1(0,+\infty)$, $\omega(u) = \frac{\arctan(|u|)}{2\sqrt{\pi}} \in C((0,+\infty),(0,+\infty))$. Then, (H2) holds.*

***(H4)*** *We have*

$$\begin{aligned} \left|t^{-\frac{3}{2}} f(t,u) - t^{-\frac{3}{2}} f(t,v)\right| &= \left| \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} \arctan(|u|) \right. \\ &\quad \left. - \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} \arctan(|v|) \right| \\ &= \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} |\arctan(|u|) - \arctan(|v|)| \\ &\leq \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} ||u| - |v|| \\ &\leq \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} |u - v|. \end{aligned}$$

*If we put $q(t) = \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)}$, then we obtain*

$$q^* = \int_0^{+\infty} \left(1+t^{\frac{1}{2}}\right) \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} dt = \frac{1}{2\sqrt{\pi}} < \infty.$$

*Hence, the condition (H4) is satisfied.*

*Moreover, we have $\frac{\beta q^*}{\Gamma(\delta)} = \frac{1}{2\Gamma\left(\frac{3}{2}\right)} = \frac{1}{\pi} < 1$, and the condition (2.35) is satisfied. It follows from Theorem (2.4) that the boundary value problem (2.39) has a unique solution $u \in C_\infty$.*

# 3

## UNBOUNDED SOLUTION OF BVP FOR NONLINEAR CAPUTO E–K FRACTIONAL DIFFERENTIAL EQUATION ON THE HALF-LINE

In this chapter, we discuss the existence and uniqueness of unbounded solution for boundary value problem of nonlinear fractional differential equation involving Caputo-type Erdélyi–Kober fractional differential operator on infinite interval

$$ {}_{*}\mathcal{D}_{\beta}^{\gamma,\delta} u(t) + f(t, u(t)) = 0, \ t > 0, \tag{3.1} $$

with the boundary conditions

$$ \lim_{t\to 0} t^{\beta(1+\gamma)} u^{(k)}(t) = 0, \text{ and } \lim_{t\to\infty} t^{\beta(1+\gamma)} u(t) = \rho u(1), \text{ with } k = \overline{0, n-2}, \tag{3.2} $$

where ${}_{*}\mathcal{D}_{\beta}^{\gamma,\delta}$ is the Caputo-type Erdélyi Kober fractional derivative operator of order $\delta$, such that $n-1 < \delta \leq n$, $-n < \gamma < 1-n$, $n \in \mathbb{N}$, $n \geq 2$, $\beta > 0$, $0 < \rho < 1$, and $f$ is a given function.

Throughout this chapter, we assume that the following conditions hold:

**(C1)** $f : (0,\infty) \times \mathbb{R} \longrightarrow (0,\infty)$ is continuous.

**(C2)** There exist non negative functions $a(t)$, $b(t) \in L_1(0,\infty)$ such that

$$ \left| t^{\beta(1+\gamma)-1} f(t,u) \right| \leq a(t)|u| + b(t), \text{ for any } u \in \mathbb{R}, \ t \in (0,\infty), $$

with

$$ \int_0^\infty \left(1 + t^{-\beta(1+\gamma)}\right) a(t)\, dt < \frac{(1-\rho)\Gamma(\delta)}{\beta}, \text{ and } \int_0^\infty b(t)\, dt < \infty. $$

**(C3)** There exists a positive function $q(t)$ with

$$ q^* = \int_0^{+\infty} \left(1 + t^{-\beta(1+\gamma)}\right) q(t)\, dt < \infty, $$

such that

$$ t^{\beta(\gamma+1)-1} |f(t,u) - f(t,v)| \leq q(t)|u-v|, $$

for all $t \in (0,\infty)$ and $u, v \in \mathbb{R}$.

## 3.1 DEFINITION OF INTEGRAL SOLUTION

In this section, we established some important lemmas which play a key role in the proofs of the existence results.

**Lemma 3.1** *Let $n-1<\delta\leq n,\ -n<\gamma<1-n,\ n\in\mathbb{N},\ n\geq 2,\ \beta>0$. We give $y\in C^2_\alpha$, with $\int_0^\infty s^{\beta(\gamma+1)-1}y(s)ds<\infty$, then the fractional differential equation:*

$$ {}_*\mathcal{D}_\beta^{\gamma,\delta}u\left(t\right)+y(t)=0,\ t>0, \tag{3.3}$$

*with the boundary conditions*

$$\lim_{t\to 0}t^{\beta(1+\gamma)}u^{(k)}(t)=0,\ k=\overline{0,n-2}, \tag{3.4}$$

$$\lim_{t\to\infty}t^{\beta(1+\gamma)}u(t)=\rho u\left(1\right),\ 0<\rho<1, \tag{3.5}$$

*has a unique solution given by*

$$u\left(t\right)=\int_0^\infty G\left(t,s\right)s^{\beta(\gamma+1)-1}y\left(s\right)ds, \tag{3.6}$$

*where*

$$G\left(t,s\right)=\begin{cases}\frac{\beta t^{-\beta(\gamma+1)}}{(1-\rho)\Gamma(\delta)}\left[1-\rho\left(1-s^\beta\right)^{\delta-1}\right]-\frac{\beta t^{-\beta(\delta+\gamma)}}{\Gamma(\delta)}\left(t^\beta-s^\beta\right)^{\delta-1}, & 0<s\leq\min\left\{t,1\right\},\\ \frac{\beta}{(1-\rho)\Gamma(\delta)}t^{-\beta(\gamma+1)}\left[1-\rho\left(1-s^\beta\right)^{\delta-1}\right], & t\leq s\leq 1,\\ \frac{\beta}{(1-\rho)\Gamma(\delta)}t^{-\beta(\gamma+1)}-\frac{\beta}{\Gamma(\delta)}t^{-\beta(\delta+\gamma)}\left(t^\beta-s^\beta\right)^{\delta-1}, & 1\leq s\leq t,\\ \frac{\beta}{(1-\rho)\Gamma(\delta)}t^{-\beta(\gamma+1)}, & s\geq\max\left\{t,1\right\},\end{cases} \tag{3.7}$$

*is called the Green function of boundary value problem (3.3)–(3.4)–(3.5).*

**Proof.** Let $n-1<\delta\leq n$, with $-n<\gamma<1-n$, $n\in\mathbb{N}$, $n\geq 2$, and $\beta>0$, it is easy to prove that the operator $\mathcal{I}_\beta^{\gamma,\delta}$ has the linearity property for all $\delta>0$. By applying $\mathcal{I}_\beta^{\gamma,\delta}$ to equation (3.3) we obtain

$$\mathcal{I}_\beta^{\gamma,\delta}\left[{}_*\mathcal{D}_\beta^{\gamma,\delta}u\left(t\right)+y(t)\right]=0\iff\mathcal{I}_\beta^{\gamma,\delta}\,{}_*\mathcal{D}_\beta^{\gamma,\delta}u\left(t\right)+\mathcal{I}_\beta^{\gamma,\delta}y(t)=0. \tag{3.8}$$

By using Lemma 1.5, for $n-1<\delta\leq n$, we can easily find that

$$\mathcal{I}_\beta^{\gamma,\delta}\,{}_*\mathcal{D}_\beta^{\gamma,\delta}u\left(t\right)=u(t)-c_0t^{-\beta(1+\gamma)}+c_1t^{-\beta(2+\gamma)}+\cdots+c_{n-1}t^{-\beta(n+\gamma)},$$

for some $c_0,c_1,\cdots c_{n-1}\in\mathbb{R}$. Thus (3.8) gives

$$u(t)-c_0t^{-\beta(1+\gamma)}-c_1t^{-\beta(2+\gamma)}+\cdots+c_{n-1}t^{-\beta(n+\gamma)}+\mathcal{I}_\beta^{\gamma,\delta}y(t)=0.$$

Therefore, the unique solution of the problem (3.3)–(3.4)–(3.5) is given by

$$u(t) = c_0 t^{-\beta(1+\gamma)} + c_1 t^{-\beta(2+\gamma)} + \cdots + c_{n-1} t^{-\beta(n+\gamma)} - \mathcal{I}_\beta^{\gamma,\delta} y(t). \tag{3.9}$$

From the formula (1.26) of Lemma 1.5, it follows that

$$\begin{aligned}
c_0 &= \lim_{t\to 0} t^{\beta(1+\gamma)} \prod_{i=1}^{n-1} \left(1+\gamma+i+\frac{1}{\beta} t \frac{d}{dt}\right) u(t) \\
&= \lim_{t\to 0} t^{\beta(1+\gamma)} \left[ \begin{array}{l} (2+\gamma)(3+\gamma)\cdots(n+\gamma)\, u(t) + \xi_1(n,\gamma)\, \frac{1}{\beta} t u'(t) + \cdots \\ +\xi_{n-2}(n,\gamma)\, \frac{1}{\beta^{n-2}} t^{n-2} u^{(n-2)}(t) + \frac{1}{\beta^{n-1}} t^{n-1} u^{(n-1)}(t) \end{array} \right]. \\
c_1 &= \lim_{t\to 0} t^{\beta(2+\gamma)} \prod_{i=2}^{n-1} \left(1+\gamma+i+\frac{1}{\beta} t \frac{d}{dt}\right) u(t) \\
&= \lim_{t\to 0} t^{\beta(2+\gamma)} \left[ \begin{array}{l} (3+\gamma)\cdots(n+\gamma)\, u(t) + \xi_1(n,\gamma)\, \frac{1}{\beta} t u'(t) + \cdots \\ +\xi_{n-3}(n,\gamma)\, \frac{1}{\beta^{n-3}} t^{n-3} u^{(n-3)}(t) + \frac{1}{\beta^{n-2}} t^{n-2} u^{(n-2)}(t) \end{array} \right]. \\
&\vdots \\
c_{n-2} &= \lim_{t\to 0} t^{\beta(n-1+\gamma)} \prod_{i=n-1}^{n-1} \left(1+\gamma+i+\frac{1}{\beta} t \frac{d}{dt}\right) u(t) \\
&= \lim_{t\to 0} t^{\beta(n-1+\gamma)} \left[ (n+\gamma)\, u(t) + \frac{1}{\beta} t u'(t) \right]. \\
c_{n-1} &= \lim_{t\to 0} t^{\beta(n+\gamma)} \prod_{i=n}^{n-1} \left(1+\gamma+i+\frac{1}{\beta} t \frac{d}{dt}\right) u(t) = \lim_{t\to 0} t^{\beta(n+\gamma)} u(t).
\end{aligned}$$

with $\xi_1, \cdots, \xi_{n-2} \in \mathbb{R}$.

On one hand, the condition (3.4) implies that $c_{n-1} = c_{n-2} = \cdots = c_1 = 0$, which means that we can rewrite (3.9) as

$$u(t) = c_0 t^{-\beta(1+\gamma)} - \mathcal{I}_\beta^{\gamma,\delta} y(t). \tag{3.10}$$

On the other hand, (3.10) with the condition (3.5) implies that

$$c_0 - \lim_{t\to\infty} t^{\beta(1+\gamma)} \mathcal{I}_\beta^{\gamma,\delta} y(t) = \rho u(1),$$

in particular, from (3.3) we get

$$\begin{aligned}
\lim_{t\to\infty} t^{\beta(1+\gamma)} \mathcal{I}_\beta^{\gamma,\delta} y(t) &= \lim_{t\to\infty} \frac{\beta}{\Gamma(\delta)} t^{\beta(1-\delta)} \int_0^t \left(t^\beta - s^\beta\right) s^{\beta(\gamma+1)-1} y(s) ds \\
&= \lim_{t\to\infty} \frac{\beta}{\Gamma(\delta)} \int_0^t \left(1 - \left(\frac{s}{t}\right)^\beta\right) s^{\beta(\gamma+1)-1} y(s) ds \\
&= \frac{\beta}{\Gamma(\delta)} \int_0^\infty s^{\beta(\gamma+1)-1} y(s) ds,
\end{aligned}$$

from (1.16) and (3.10), we can easily see that

$$\rho u(1) = \rho c_0 - \frac{\rho\beta}{\Gamma(\delta)} \int_0^1 \left(1 - s^\beta\right) s^{\beta(\gamma+1)-1} y(s) ds,$$

therefore

$$c_0 = \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_0^\infty s^{\beta(\gamma+1)-1} y(s) ds - \frac{\rho\beta}{(1-\rho)\Gamma(\delta)} \int_0^1 \left(1 - s^\beta\right) s^{\beta(\gamma+1)-1} y(s) ds.$$

Hence, we can conclude that the unique solution of problem (3.3)–(3.4)–(3.5) is given by

$$\begin{aligned} u(t) &= \frac{\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(1+\gamma)} \int_0^\infty s^{\beta(\gamma+1)-1} y(s) ds \\ &\quad - \frac{\rho\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(1+\gamma)} \int_0^1 \left(1 - s^\beta\right) s^{\beta(\gamma+1)-1} y(s) ds \\ &\quad - \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+\delta)} \int_0^t \left(t^\beta - s^\beta\right)^{\delta-1} s^{\beta(\gamma+1)-1} y(s) ds \\ &= \int_0^{+\infty} G(t,s) s^{\beta(\gamma+1)-1} y(s) ds. \end{aligned}$$

■

Now, we give some properties of Green's function (3.7).

**Lemma 3.2** *Let $n-1 < \delta \leq n$, $-n < \gamma < 1-n$, $n \in \mathbb{N}$, $n \geq 2$, and $\beta > 0$. The function $G(t,s)$ in Lemma 3.1 satisfies the following conditions*

1. $G(t,s) \geq 0$, $\forall t, s \in (0,\infty)$.
2. *For all $t \in (0,\infty)$ and $s \in \left(\frac{1}{10}, \infty\right)$, $G(t,s) \geq \chi t^{-\beta(\gamma+1)}$, with*

$$\chi = \begin{cases} \frac{\rho\beta}{10^\beta \Gamma(\delta-1)}, & n = 2, \\ \frac{\rho\beta}{10^\beta \Gamma(\delta)}, & n > 2, \end{cases}$$

3. $\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} \leq \frac{\beta}{(1-\rho)\Gamma(\delta)}$, $\forall t, s \in (0,\infty)$.

**Proof.** Let $n-1 < \delta \leq n$, $-n < \gamma < 1-n$, $n \in \mathbb{N}$, $n \geq 2$, and $\beta > 0$.

1. For $s \geq \max\{t, 1\}$. It is easy to check that $G(t,s) \geq 0$.

   For $s \leq \min\{t, 1\}$, it holds that

$$\begin{aligned} G(t,s) &= \frac{\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} - \frac{\beta}{\Gamma(\delta)} t^{-\beta(\delta+\gamma)} \left(t^\beta - s^\beta\right)^{\delta-1} \\ &\quad - \frac{\rho\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} \left(1 - s^\beta\right)^{\delta-1} \\ &= \frac{\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} - \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+1)} \left(1 - \frac{s^\beta}{t^\beta}\right)^{\delta-1} \\ &\quad - \frac{\rho\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} \left(1 - s^\beta\right)^{\delta-1} \\ &\geq \frac{\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} - \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+1)} - \frac{\rho\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} \geq 0. \end{aligned}$$

For $t \leq s \leq 1$, it follows that

$$\begin{aligned} G(t,s) &= \frac{\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} - \frac{\rho\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} \left(1-s^{\beta}\right)^{\delta-1} \\ &\geq \frac{\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} - \frac{\rho\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} \\ &\geq \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+1)} \geq 0. \end{aligned}$$

For $1 \leq s \leq t$, we have

$$\begin{aligned} G(t,s) &= \frac{\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} - \frac{\beta}{\Gamma(\delta)} t^{-\beta(\delta+\gamma)} \left(t^{\beta}-s^{\beta}\right)^{\delta-1} \\ &\geq \frac{\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} - \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+1)} \\ &\geq \frac{\rho\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} \geq 0. \end{aligned}$$

2. For $s \geq \max\{t,1\}$, we can easily see that

$$G(t,s) = \frac{\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} \geq \frac{\rho\beta}{10^{\beta}\Gamma(\delta)} t^{-\beta(\gamma+1)}.$$

For $s \leq \min\{t,1\}$, it holds that

$$\begin{aligned} G(t,s) &\geq \frac{\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} - \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+1)} \\ &\quad - \frac{\rho\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} \left(1-s^{\beta}\right)^{\delta-1} \\ &\geq \frac{\rho\beta t^{-\beta(\gamma+1)}}{(1-\rho)\Gamma(\delta)} \left[1-\left(1-s^{\beta}\right)^{\delta-1}\right] \\ &\geq \frac{\rho\beta t^{-\beta(\gamma+1)}}{(1-\rho)\Gamma(\delta)} (\delta-1) \int_{1-s^{\beta}}^{1} u^{\delta-2} du \\ &\geq \frac{\rho\beta t^{-\beta(\gamma+1)}}{(1-\rho)\Gamma(\delta-1)} s^{\beta}. \end{aligned}$$

If we choose $s \in \left[\frac{1}{10}, \min\{t,1\}\right]$, we observe that $G(t,s) \geq \frac{\rho\beta t^{-\beta(\gamma+1)}}{10^{\beta}\Gamma(\delta-1)}$.
For $t \leq s \leq 1$, then we will show that

$$G(t,s) \geq \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+1)} \geq \frac{\rho\beta}{10^{\beta}\Gamma(\delta)} t^{-\beta(\gamma+1)}.$$

For $1 \leq s \leq t$, we get

$$G(t,s) \geq \frac{\rho\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)} \geq \frac{\rho\beta}{10^{\beta}\Gamma(\delta)} t^{-\beta(\gamma+1)}.$$

We notice that there are two cases to consider:

- First case: for $n = 2$, we obtain $\Gamma(\delta - 1) \geq \Gamma(\delta)$, then it follows that

$$G(t,s) \geq \frac{\rho\beta}{10^{\beta}\Gamma(\delta - 1)} t^{-\beta(\gamma+1)}.$$

- Second case: since $n > 2$, we get $\Gamma(\delta) \geq \Gamma(\delta - 1)$, then it holds that

$$G(t,s) \geq \frac{\rho\beta}{10^{\beta}\Gamma(\delta)} t^{-\beta(\gamma+1)}.$$

Finally, we conclude that

$$G(t,s) \geq \chi t^{-\beta(\gamma+1)}.$$

3. For every $t, s \in (0,\infty)$, we know that $G(t,s) \leq \frac{\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+1)}$, this implies that $\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} \leq \frac{\beta}{(1-\rho)\Gamma(\delta)}$.

■

## 3.2 FUNDAMENTAL LEMMA

In this section, we give a significant lemmas that will be needed later in the existence and uniqueness results.

Now, we define an integral operator $\mathcal{A} : C_\infty \to C_\infty$ by

$$\mathcal{A}u(t) = \int_0^\infty G(t,s)\, s^{\beta(1+\gamma)-1} f(s,u(s))ds, \tag{3.11}$$

where $G(t,s)$ defined by (3.7) and $C_\infty$ defined by (2.11) .

Clearly, from Lemma 3.1, the fixed points of the operator $\mathcal{A}$ coincide with the solution of the problem (3.1)–(3.2).

**Lemma 3.3** *If (C1)–(C2) hold, then $\mathcal{A} : C_\infty \to C_\infty$ is completely continuous.*

**Proof.** First, for all $u \in C_\infty$, we have

$$\begin{aligned} \|\mathcal{A}u(t)\|_\infty &= \sup_{t\in(0,\infty)} \left| \int_0^\infty \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f(s,u(s))ds \right| \\ &\leq \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_0^\infty \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds, \end{aligned}$$

together with conditions (C1) and (C2), then it follows that

$$\begin{aligned}
& \frac{\beta}{(1-\rho)\Gamma(\delta)}\int_0^\infty \left|s^{\beta(\gamma+1)-1}f(s,u(s))\right| ds \\
\leq\ & \frac{\beta}{(1-\rho)\Gamma(\delta)}\int_0^\infty [a(s)|u(s)|+b(s)]\, ds \\
\leq\ & \frac{\beta}{(1-\rho)\Gamma(\delta)}\int_0^\infty \left[a(s)\left(1+s^{-\beta(1+\gamma)}\right)\frac{|u(s)|}{1+s^{-\beta(1+\gamma)}}+b(s)\right] ds \\
\leq\ & \frac{\beta}{(1-\rho)\Gamma(\delta)}\left[\|u\|_\infty\int_0^\infty \left(1+s^{-\beta(1+\gamma)}\right)a(s)\, ds+\int_0^\infty b(s)\, ds\right] \\
\leq\ & \|u\|_\infty+\frac{\beta}{(1-\rho)\Gamma(\delta)}\int_0^\infty b(s)\, ds<\infty.
\end{aligned}$$

Hence, $\mathcal{A}: C_\infty \to C_\infty$ is well-defined.

Choose

$$k \geq \frac{\beta\int_0^\infty b(s)\, ds}{(1-\rho)\Gamma(\delta)-\beta\int_0^\infty \left(1+s^{-\beta(1+\gamma)}\right)a(s)\, ds}, \tag{3.12}$$

and $\Omega=\{u\in C_\infty,\ \|u\|_\infty\leq k,\ k>0\}$.

In what follows we divide the proof into several steps.

STEP 1: $\mathcal{A}:\Omega\to C_\infty$ is continuous.

Let $(u_n)_{n\in\mathbb{N}}\in\Omega$ be a convergent sequence to $u$ in $\Omega$. Lemma 3.2, we ensure that

$$\begin{aligned}
\|\mathcal{A}u_n-\mathcal{A}u\|_\infty &= \sup_{t\in[0,\infty)}\left|\int_0^\infty \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}\left[f(s,u_n(s))-f(s,u(s))\right] ds\right| \\
&\leq \frac{\beta}{(1-\rho)\Gamma(\delta)}\sup_{t\in[0,\infty)}\left|\int_0^\infty s^{\beta(\gamma+1)-1}f(s,u_n(s))ds\right. \\
&\quad \left.-\int_0^\infty s^{\beta(\gamma+1)-1}f(s,u(s))ds\right|.
\end{aligned}$$

Due to the condition (C2), we can write

$$\begin{aligned}
\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right| &\leq a(s)|u(s)|+b(s) \\
&\leq a(s)\left(1+s^{-\beta(1+\gamma)}\right)\frac{|u(s)|}{1+s^{-\beta(1+\gamma)}}+b(s) \\
&\leq \left(1+s^{-\beta(1+\gamma)}\right)a(s)\|u\|_\infty+b(s) \\
&\leq ka(s)\left(1+s^{-\beta(1+\gamma)}\right)+b(s).
\end{aligned}$$

Since the right hand side of the above inequality is in $L^1(0,\infty)$, and the function $s^{\beta(\gamma+1)-1}f(s,u(s))$ is continuous, it follows that the Lebesgue dominated convergence Theorem (Theorem 12.12, page 199 in [13]) yields $u\to\int_0^\infty s^{\beta(\gamma+1)-1}f(s,u(s))ds$ is continuous. Hence, it holds

$$\int_0^\infty s^{\beta(\gamma+1)-1}f(s,u_n(s))ds\to\int_0^\infty s^{\beta(\gamma+1)-1}f(s,u(s))ds \text{ as } n\to\infty.$$

Therefore $\|\mathcal{A}u_n-\mathcal{A}u\|_\infty\to 0$, as $n\to\infty$.

STEP2: $\mathcal{A}(\Omega)$ is relatively compact.

First, we show that $\mathcal{A}(\Omega)$ is uniformly bounded. Let $u \in \Omega$, from Lemma 3.2 and by the condition (C2) and (3.12), it holds that

$$\begin{aligned}
\frac{|\mathcal{A}u(t)|}{1+t^{-\beta(1+\gamma)}} &= \left| \int_0^\infty \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f(s,u(s)) ds \right| \\
&\leq \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_0^\infty [a(s)|u(s)| + b(s)] ds \\
&\leq \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_0^\infty \left[ a(s)\left(1+s^{-\beta(1+\gamma)}\right) \frac{|u(s)|}{1+s^{-\beta(1+\gamma)}} + b(s) \right] ds \\
&\leq \frac{\beta}{(1-\rho)\Gamma(\delta)} \left[ \|u\|_\infty \int_0^\infty \left(1+s^{-\beta(1+\gamma)}\right) a(s) ds + \int_0^\infty b(s) ds \right] \leq k,
\end{aligned}$$

consequently

$$\|\mathcal{A}u\|_\infty \leq k, \text{ for all } u \in \Omega. \tag{3.13}$$

Hence, $\mathcal{A}(\Omega)$ is uniformly bounded.

Let $V = \left\{ \frac{\mathcal{A}u}{1+t^{-\beta(1+\gamma)}} | u \in \Omega \right\}$.

Next, we show that $V$ is equicontinuous on any compact interval of $\mathbb{R}^+$.

For all $u \in \Omega$, $t_1, t_2 \in [a,b]$, $0 < a < b < \infty$, and $t_1 \leq t_2$, we obtain

$$\begin{aligned}
&\left| \frac{\mathcal{A}u(t_2)}{1+t_2^{-\beta(1+\gamma)}} - \frac{\mathcal{A}u(t_1)}{1+t_1^{-\beta(1+\gamma)}} \right| \\
\leq\ & \int_0^\infty \left| \frac{G(t_2,s)}{1+t_2^{-\beta(1+\gamma)}} - \frac{G(t_1,s)}{1+t_1^{-\beta(1+\gamma)}} \right| \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\
\leq\ & \int_0^\infty \left| \frac{G(t_2,s)}{1+t_2^{-\beta(1+\gamma)}} - \frac{G(t_1,s)}{1+t_2^{-\beta(1+\gamma)}} + \frac{G(t_1,s)}{1+t_2^{-\beta(1+\gamma)}} - \frac{G(t_1,s)}{1+t_1^{-\beta(1+\gamma)}} \right| \\
& \times \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\
\leq\ & \int_0^\infty \frac{|G(t_2,s) - G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\
& + \int_0^\infty \frac{G(t_1,s)\left(t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}\right)}{\left(1+t_2^{-\beta(1+\gamma)}\right)\left(1+t_1^{-\beta(1+\gamma)}\right)} \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds.
\end{aligned}$$

It remain to show that the right hand side of the above inequality tends to zero, there are three cases to consider

CASE 1: $t_1 \leq t_2 \leq 1$

$$\begin{aligned}
&\int_0^\infty \frac{|G(t_2,s)-G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left|s^{\beta(\gamma+1)-1} f(s,u(s))\right| ds \\
\leq\ & \int_0^{t_1} \frac{|G(t_2,s)-G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left|s^{\beta(\gamma+1)-1} f(s,u(s))\right| ds \\
&+ \int_{t_1}^{t_2} \frac{|G(t_2,s)-G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left|s^{\beta(\gamma+1)-1} f(s,u(s))\right| ds \\
&+ \int_{t_2}^{1} \frac{|G(t_2,s)-G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left|s^{\beta(\gamma+1)-1} f(s,u(s))\right| ds \\
&+ \int_{1}^{\infty} \frac{|G(t_2,s)-G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left|s^{\beta(\gamma+1)-1} f(s,u(s))\right| ds.
\end{aligned}$$

In addition, from (3.7) we can also rewrite the last one as

$$\begin{aligned}
&\int_0^\infty \frac{|G(t_2,s)-G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left|s^{\beta(\gamma+1)-1} f(s,u(s))\right| ds \\
\leq\ & \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_0^{t_1} \frac{1}{1+t_2^{-\beta(1+\gamma)}} \left| \left[t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}\right] \left[1-\rho\left(1-s^\beta\right)^{\delta-1}\right] \right. \\
&\left. - (1-\rho)\left[t_2^{-\beta(\delta+\gamma)}\left(t_2^\beta - s^\beta\right)^{\delta-1} - t_1^{-\beta(\delta+\gamma)}\left(t_1^\beta - s^\beta\right)^{\delta-1}\right] \right| \\
&\times \left[a(s)\left(1+s^{-\beta(1+\gamma)}\right)k + b(s)\right] ds \\
&+ \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_{t_1}^{t_2} \frac{1}{1+t_2^{-\beta(1+\gamma)}} \left| \left[t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}\right] \left[1-\rho\left(1-s^\beta\right)^{\delta-1}\right] \right. \\
&\left. - (1-\rho)\, t_2^{-\beta(\delta+\gamma)}\left(t_2^\beta - s^\beta\right)^{\delta-1} \right| \times \left[a(s)\left(1+s^{-\beta(1+\gamma)}\right)k + b(s)\right] ds \\
&+ \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_{t_2}^{1} \frac{\left| \left[t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}\right] \left[1-\rho\left(1-s^\beta\right)^{\delta-1}\right] \right|}{1+t_2^{-\beta(1+\gamma)}} \\
&\times \left[a(s)\left(1+s^{-\beta(1+\gamma)}\right)k + b(s)\right] ds \\
&+ \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_{1}^{\infty} \frac{t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}} \left[a(s)\left(1+s^{-\beta(1+\gamma)}\right)k + b(s)\right] ds,
\end{aligned}$$

as $t_1 \to t_2$, for all $u \in \Omega$ the right hand side of the above inequality tends to zero.

CASE 2: $t_1 \leq 1 \leq t_2$

Similarly, from (3.7) and with a simple computation we find

$$\begin{aligned}
&\int_0^\infty \frac{|G(t_2,s) - G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\
\leq\ & \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_0^{t_1} \frac{1}{1+t_2^{-\beta(1+\gamma)}} \left| \left[ t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)} \right] \left[ 1 - \rho \left( 1 - s^\beta \right)^{\delta-1} \right] \right. \\
& \left. - (1-\rho) \left[ t_2^{-\beta(\delta+\gamma)} \left( t_2^\beta - s^\beta \right)^{\delta-1} - t_1^{-\beta(\delta+\gamma)} \left( t_1^\beta - s^\beta \right)^{\delta-1} \right] \right| \\
& \times \left[ a(s) \left( 1 + s^{-\beta(1+\gamma)} \right) k + b(s) \right] ds \\
& + \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_{t_1}^1 \frac{1}{1+t_2^{-\beta(1+\gamma)}} \left| \left[ t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)} \right] \left[ 1 - \rho \left( 1 - s^\beta \right)^{\delta-1} \right] \right. \\
& \left. - (1-\rho) t_2^{-\beta(\delta+\gamma)} \left( t_2^\beta - s^\beta \right)^{\delta-1} \right| \times \left[ a(s) \left( 1 + s^{-\beta(1+\gamma)} \right) k + b(s) \right] ds \\
& + \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_1^{t_2} \frac{\left| \left[ t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)} \right] - (1-\rho) t_2^{-\beta(\delta+\gamma)} \left( t_2^\beta - s^\beta \right)^{\delta-1} \right|}{1+t_2^{-\beta(1+\gamma)}} \\
& \times \left[ a(s) \left( 1 + s^{-\beta(1+\gamma)} \right) k + b(s) \right] ds \\
& + \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_{t_2}^\infty \frac{t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}} \left[ a(s) \left( 1 + s^{-\beta(1+\gamma)} \right) k + b(s) \right] ds,
\end{aligned}$$

as $t_1 \to t_2$, for all $u \in \Omega$ the right hand side of the above inequality tends to zero.

CASE 3: $1 \leq t_1 \leq t_2$

In this case due to (3.7), we will show that

$$\begin{aligned}
&\int_0^\infty \frac{|G(t_2,s)-G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left|s^{\beta(\gamma+1)-1} f(s,u(s))\right| ds \\
\leq\ & \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_0^1 \frac{1}{1+t_2^{-\beta(1+\gamma)}} \left| \left[t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}\right] \left[1-\rho\left(1-s^\beta\right)^{\delta-1}\right] \right. \\
& \left. -(1-\rho)\left[t_2^{-\beta(\delta+\gamma)}\left(t_2^\beta - s^\beta\right)^{\delta-1} - t_1^{-\beta(\delta+\gamma)}\left(t_1^\beta - s^\beta\right)^{\delta-1}\right]\right| \\
& \times \left[a(s)\left(1+s^{-\beta(1+\gamma)}\right)k + b(s)\right] ds \\
& + \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_1^{t_1} \frac{1}{1+t_2^{-\beta(1+\gamma)}} \left| \left[t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}\right] \right. \\
& \left. -(1-\rho)\left[t_2^{-\beta(\delta+\gamma)}\left(t_2^\beta - s^\beta\right)^{\delta-1} - t_1^{-\beta(\delta+\gamma)}\left(t_1^\beta - s^\beta\right)^{\delta-1}\right]\right| \\
& \times \left[a(s)\left(1+s^{-\beta(1+\gamma)}\right)k + b(s)\right] ds \\
& + \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_{t_1}^{t_2} \frac{\left|\left[t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}\right] - (1-\rho)\, t_2^{-\beta(\delta+\gamma)}\left(t_2^\beta - s^\beta\right)^{\delta-1}\right|}{1+t_2^{-\beta(1+\gamma)}} \\
& \times \left[a(s)\left(1+s^{-\beta(1+\gamma)}\right)k + b(s)\right] ds \\
& + \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_{t_2}^\infty \frac{t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}} \left[a(s)\left(1+s^{-\beta(1+\gamma)}\right)k + b(s)\right] ds,
\end{aligned}$$

as $t_1 \to t_2$, for all $u \in \Omega$ the right hand side of the above inequality tends to zero.

Analogously, we can obtain

$$\int_0^\infty \frac{G(t_1,s)\left(t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}\right)}{\left(1+t_2^{-\beta(1+\gamma)}\right)\left(1+t_1^{-\beta(1+\gamma)}\right)} s^{\beta(\gamma+1)-1} \left|f(s,u(s))\right| ds \to 0,$$

uniformly as $t_1 \to t_2$ for all $u \in \Omega$.

Hence, $V$ is locally equicontinuous on $(0,\infty)$.

Finally, we show that $V$ is equiconvergent at $\infty$. We know that

$$\begin{aligned}
Au(t) \ =\ & \frac{\beta t^{-\beta(1+\gamma)}}{(1-\rho)\Gamma(\delta)} \int_0^\infty s^{\beta(\gamma+1)-1} f(s,u(s))ds \\
& - \frac{\beta t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^t \left(t^\beta - s^\beta\right)^{\delta-1} s^{\beta(\gamma+1)-1} f(s,u(s))ds \\
& - \frac{\rho\beta}{(1-\rho)\Gamma(\delta)} t^{-\beta(\gamma+\delta)} \int_0^1 \left(1-s^\beta\right)^{\delta-1} s^{\beta(\gamma+1)-1} f(s,u(s))ds. \qquad (3.14)
\end{aligned}$$

Observing that for any $u \in \Omega$, the condition (C2) gives

$$\int_0^\infty \left|s^{\beta(\gamma+1)-1} f(s,u(s))\right| ds \leq k \int_0^\infty \left(1+s^{-\beta(1+\gamma)}\right) a(s)ds + \int_0^\infty b(s)ds < \infty,$$

and for a given $\varepsilon > 0$. There exists a constant $L > 0$ such that

$$\int_L^{\infty} \left| s^{\beta(\gamma+1)-1} f(s, u(s)) \right| ds < \varepsilon. \tag{3.15}$$

On one hand, since

$$\lim_{t \to +\infty} \frac{t^{-\beta(1+\gamma)}}{1 + t^{-\beta(1+\gamma)}} = 1.$$

There exists a constant $T_1 > 0$ such that for any $t_1, t_2 \geq T_1$, we get

$$\left| \frac{t_2^{-\beta(1+\gamma)}}{1 + t_2^{-\beta(1+\gamma)}} - \frac{t_1^{-\beta(1+\gamma)}}{1 + t_1^{-\beta(1+\gamma)}} \right| \leq \left| 1 - \frac{t_1^{-\beta(1+\gamma)}}{1 + t_1^{-\beta(1+\gamma)}} \right| + \left| 1 - \frac{t_2^{-\beta(1+\gamma)}}{1 + t_2^{-\beta(1+\gamma)}} \right| < \varepsilon, \tag{3.16}$$

similarly

$$\lim_{t \to +\infty} \frac{t^{-\beta(\delta+\gamma)} \left(t^{\beta} - s^{\beta}\right)^{\delta-1}}{1 + t^{-\beta(1+\gamma)}} = 1,$$

so there exists a constant $T_2 > L > 0$, such that for any $t_1, t_2 \geq T_2$, and $0 < s \leq L$, it holds that

$$\begin{aligned}
& \left| \frac{t_2^{-\beta(\delta+\gamma)} \left(t_2^{\beta} - s^{\beta}\right)^{\delta-1}}{1 + t_2^{-\beta(1+\gamma)}} - \frac{t_1^{-\beta(\delta+\gamma)} \left(t_1^{\beta} - s^{\beta}\right)^{\delta-1}}{1 + t_1^{-\beta(1+\gamma)}} \right| \\
\leq & \left| 1 - \frac{t_1^{-\beta(\delta+\gamma)} \left(t_1^{\beta} - s^{\beta}\right)^{\delta-1}}{1 + t_1^{-\beta(1+\gamma)}} \right| + \left| 1 - \frac{t_2^{-\beta(\delta+\gamma)} \left(t_2^{\beta} - s^{\beta}\right)^{\delta-1}}{1 + t_2^{-\beta(1+\gamma)}} \right| \\
\leq & \left| 1 - \frac{t_1^{-\beta(\delta+\gamma)} \left(t_1^{\beta} - L^{\beta}\right)^{\delta-1}}{1 + t_1^{-\beta(1+\gamma)}} \right| + \left| 1 - \frac{t_2^{-\beta(\delta+\gamma)} \left(t_2^{\beta} - L^{\beta}\right)^{\delta-1}}{1 + t_2^{-\beta(1+\gamma)}} \right| < \varepsilon.
\end{aligned} \tag{3.17}$$

On the other hand

$$\lim_{t \to +\infty} \frac{t^{-\beta(\delta+\gamma)} \left(1 - s^{\beta}\right)^{\delta-1}}{1 + t^{-\beta(1+\gamma)}} = 0.$$

There exists a constant $T_3 > 1$, such that for any $t_1, t_2 \geq T_3$, and $0 < s \leq 1$, it follows that

$$\left| \frac{t_2^{-\beta(\delta+\gamma)} \left(1 - s^{\beta}\right)^{\delta-1}}{1 + t_2^{-\beta(1+\gamma)}} - \frac{t_1^{-\beta(\delta+\gamma)} \left(1 - s^{\beta}\right)^{\delta-1}}{1 + t_1^{-\beta(1+\gamma)}} \right| < \varepsilon. \tag{3.18}$$

Now, if we choose $T > \max\{T_1, T_2, T_3\}$, for all $t_1, t_2 \geq T$, and due to (3.14), we obtain

$$\begin{aligned}
&\left|\frac{\mathcal{A}u(t_2)}{1+t_2^{-\beta(1+\gamma)}} - \frac{\mathcal{A}u(t_1)}{1+t_1^{-\beta(1+\gamma)}}\right| \\
\leq\ & \frac{\beta}{(1-\rho)\Gamma(\delta)}\left|\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}} - \frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|\int_0^{\infty}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\left|\int_0^{t_2}\frac{t_2^{-\beta(\gamma+\delta)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds\right. \\
&\left.-\int_0^{t_1}\frac{t_1^{-\beta(\gamma+\delta)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds\right| \\
&+\frac{\rho\beta}{(1-\rho)\Gamma(\delta)}\left|\int_0^{1}\left[\frac{t_2^{-\beta(\gamma+\delta)}\left(1-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}} - \frac{t_1^{-\beta(\gamma+\delta)}\left(1-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\right]\right. \\
&\left.\times s^{\beta(\gamma+1)-1}f(s,u(s))ds\right|.
\end{aligned}$$

In addition, we recall that

$$\begin{aligned}
&\frac{\beta}{\Gamma(\delta)}\left|\int_0^{t_2}\frac{t_2^{-\beta(\gamma+\delta)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds\right. \\
&\left.-\int_0^{t_1}\frac{t_1^{-\beta(\gamma+\delta)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f(s,u(s))ds\right| \\
\leq\ & \frac{\beta}{\Gamma(\delta)}\int_0^{L}\left|\frac{t_2^{-\beta(\gamma+\delta)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}} - \frac{t_1^{-\beta(\gamma+\delta)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\right|\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\int_L^{t_2}\frac{t_2^{-\beta(\gamma+\delta)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\int_L^{t_1}\frac{t_1^{-\beta(\gamma+\delta)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds,
\end{aligned}$$

this means that

$$
\begin{aligned}
&\left|\frac{\mathcal{A}u\left(t_2\right)}{1+t_2^{-\beta(1+\gamma)}}-\frac{\mathcal{A}u\left(t_1\right)}{1+t_1^{-\beta(1+\gamma)}}\right| \\
\leq\ & \frac{\beta}{(1-\rho)\Gamma(\delta)}\left|\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|\int_0^{\infty}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\int_0^{L}\left|\frac{t_2^{-\beta(\gamma+\delta)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(\gamma+\delta)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\right|\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\int_L^{t_2}\frac{t_2^{-\beta(\gamma+\delta)}\left(t_2^{\beta}-L^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\int_L^{t_1}\frac{t_1^{-\beta(\gamma+\delta)}\left(t_1^{\beta}-L^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\rho\beta}{(1-\rho)\Gamma(\delta)}\int_0^{1}\left|\frac{t_2^{-\beta(\gamma+\delta)}\left(1-s^{\beta}\right)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(\gamma+\delta)}\left(1-s^{\beta}\right)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\right|\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds.
\end{aligned}
$$

From (3.15), (3.16), (3.17), (3.18), and for $t_1, t_2 \to \infty$, we get

$$
\begin{aligned}
&\left|\frac{\mathcal{A}u\left(t_2\right)}{1+t_2^{-\beta(1+\gamma)}}-\frac{\mathcal{A}u\left(t_1\right)}{1+t_1^{-\beta(1+\gamma)}}\right| \\
\leq\ & \frac{\beta}{(1-\rho)\Gamma(\delta)}\varepsilon\int_0^{\infty}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\varepsilon\int_0^{L}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\beta}{\Gamma(\delta)}\int_L^{\infty}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds+\frac{\beta}{\Gamma(\delta)}\int_L^{\infty}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\rho\beta}{(1-\rho)\Gamma(\delta)}\varepsilon\int_0^{1}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
<\ & \frac{\beta}{(1-\rho)\Gamma(\delta)}\varepsilon\int_0^{\infty}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds+\frac{\beta}{\Gamma(\delta)}\varepsilon\int_0^{L}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds \\
&+\frac{\rho\beta}{(1-\rho)\Gamma(\delta)}\varepsilon\int_0^{1}\left|s^{\beta(\gamma+1)-1}f(s,u(s))\right|ds+\frac{2\beta}{\Gamma(\delta)}\varepsilon.
\end{aligned}
$$

Hence, $V$ is equiconvergente at $\infty$, consequently, Lemma 2.3 yields that $\mathcal{A}(\Omega)$ is relatively compact. Therefore, $\mathcal{A} : C_\infty \to C_\infty$ is completely continuous.

■

## 3.3 EXISTENCE OF UNBOUNDED SOLUTION

First, we will prove the first following existence result, we use Schauder's fixed point theorem.

**Theorem 3.1** *Assume that the hypotheses (C1)–(C2) and (3.12) hold, then the fractional boundary value problem (3.1)–(3.2) has at least one solution $u \in \Omega$.*

**Proof.** Let $u \in \Omega$. From Lemma 3.3 we have $\mathcal{A} : \Omega \to C_\infty$ is continuous, and (3.13) implies that $\mathcal{A}(\Omega) \subset \Omega$, which means that $\mathcal{A} : \Omega \to \Omega$ is continuous. Moreover, Lemma 3.3 gives $\mathcal{A}(\Omega)$ is relatively compact.
Hence, all conditions of Theorem 1.4 hold. Thus, the problem (3.1)–(3.2) has at least one solution $u \in \Omega$. ■
Our next existence result is based on the Leray–Schauder nonlinear alternative fixed point theorem.

**Theorem 3.2** *Assume that the hypotheses (C1)–(C2) and the relation (3.12) hold, then the fractional boundary value problem (3.1)–(3.2) has at least one solution $u \in \Omega$.*

**Proof.** From Lemma 3.3, we know that $\mathcal{A}$ is a completely continuous operator. Now we apply the nonlinear alternative of leray–Schauder to prove that $\mathcal{A}$ has at least one nontrivial solution in $\Omega$.
Let $u \in \partial\Omega$, such that $u = \lambda \mathcal{A}u$, $\lambda \in (0,1)$, from (3.13) we obtain

$$\begin{aligned} \|u\|_\infty &= \lambda \|\mathcal{A}u\|_\infty \leq \|\mathcal{A}u\|_\infty \leq \frac{\beta k}{(1-\rho)\Gamma(\delta)} \int_0^\infty \left(1 + s^{-\beta(1+\gamma)}\right) a(s)ds \\ &\quad + \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_0^\infty b(s)ds, \end{aligned}$$

so

$$k \leq \frac{\beta k}{(1-\rho)\Gamma(\delta)} \int_0^\infty \left(1 + s^{-\beta(1+\gamma)}\right) a(s)ds + \frac{\beta}{(1-\rho)\Gamma(\delta)} \int_0^\infty b(s)ds,$$

hence

$$k \leq \frac{\beta \int_0^\infty b(s)\,ds}{(1-\rho)\Gamma(\delta) - \beta \int_0^\infty \left(1 + s^{-\beta(1+\gamma)}\right) a(s)\,ds},$$

which contradicts with (3.12). By theorem 1.3 and lemma 3.3, the boundary value problem (3.1)–(3.2) has at least one solution $u \in \Omega$. ■

## 3.4 UNIQUENESS OF UNBOUNDED SOLUTION

The last result of the existence are based on Banach contraction principle theorem.

**Theorem 3.3** *Assume that the hypotheses (C1)–(C2)–(C3) hold. Then the problem (3.1)–(3.2) has a unique solution $u \in C_\infty$ if*

$$\frac{q^* \beta}{(1-\rho)\,\Gamma(\delta)} < 1. \tag{3.19}$$

**Proof.** Assume that $\frac{q^*\beta}{(1-\rho)\Gamma(\delta)} < 1$. We shall show that the operator $\mathcal{A}$ defined by (3.11) is a contraction mapping.

Let $u, v \in C_\infty$, from Lemma 3.2 with the condition (C3), we can obtain that

$$\begin{aligned}
\left|\frac{\mathcal{A}u(t) - \mathcal{A}v(t)}{1+t^{-\beta(1+\gamma)}}\right| &= \int_0^\infty \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} \left|f(s,u(s)) - f(s,v(s))\right| ds \\
&\leq \frac{\beta}{(1-\rho)\,\Gamma(\delta)} \int_0^\infty s^{\beta(\gamma+1)-1} \left|f(s,u(s)) - f(s,v(s))\right| ds \\
&\leq \frac{\beta}{(1-\rho)\,\Gamma(\delta)} \int_0^\infty q(s) \left|u - v\right| ds \\
&\leq \frac{\beta}{(1-\rho)\,\Gamma(\delta)} \int_0^\infty q(s) \left(1 + s^{-\beta(1+\gamma)}\right) \left|\frac{u-v}{1+s^{-\beta(1+\gamma)}}\right| ds,
\end{aligned}$$

this implies that

$$\begin{aligned}
\|\mathcal{A}(u) - \mathcal{A}(v)\|_\infty &\leq \frac{\beta}{(1-\rho)\,\Gamma(\delta)} \|u-v\|_\infty \int_0^\infty q(s) \left(1 + s^{-\beta(1+\gamma)}\right) ds \\
&\leq \frac{\beta q^*}{(1-\rho)\,\Gamma(\delta)} \|u-v\|_\infty .
\end{aligned}$$

It follows from the assumption (3.19) and the preceding estimate that $\mathcal{A}$ is a contraction mapping. Applying Banach's fixed point theorem 1.5, the operator $\mathcal{A}$ has a fixed point which corresponds to the unique solution of the problem (3.1)–(3.2). ■

**Remark 3.1** *If $\int_0^\infty s^{\beta(\gamma+1)-1} f(s,u(s))ds < \infty$, then with lemma 3.2 every solution of the problem (3.1)–(3.2) is unbounded. Indeed*

$$\begin{aligned}
u(t) &= \int_0^\infty G(t,s)\, s^{\beta(\gamma+1)-1} f(s,u(s))ds \\
&\geq \int_{\frac{1}{10}}^\infty G(t,s)\, s^{\beta(\gamma+1)-1} f(s,u(s))ds \\
&\geq \chi t^{-\beta(\gamma+1)} \int_{\frac{1}{10}}^\infty s^{\beta(\gamma+1)-1} f(s,u(s))ds.
\end{aligned}$$

## 3.5 ILLUSTRATIVE EXAMPLES

In this section, we present some examples to illustrate the usefulness of our main results.

**Example 3.1** *Consider the following boundary value problem of nonlinear Caputo-type Erdélyi–Kober fractional differential equations*

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2},\frac{3}{2}} u(t) + \frac{t^{\frac{3}{2}} \ln(1+|u(t)|)}{30(1+t^2)} + \frac{t^{\frac{3}{2}}|u(t)|}{60\exp\left(\sqrt{t}\right)} + t^{\frac{3}{2}} \exp(-t) = 0, \ t > 0, \\ \lim_{t\to 0} t^{-\frac{1}{2}} u(t) = 0, \ \lim_{t\to\infty} t^{-\frac{1}{2}} u(t) = \frac{1}{2} u(1), \end{cases} \tag{3.20}$$

*here,*

$$f(t,u) = \frac{t^{\frac{3}{2}} \ln(1+|u(t)|)}{30(1+t^2)} + \frac{t^{\frac{3}{2}}|u|}{60\exp\left(\sqrt{t}\right)} + t^{\frac{3}{2}} \exp(-t),$$

*and* $\delta = \frac{3}{2}$, $\gamma = -\frac{3}{2}$, $\beta = 1$, *and* $\rho = \frac{1}{2}$, $n = 2$.

*(C1) Clearly* $f \in C((0,\infty) \times \mathbb{R}, (0,\infty))$, *for any* $(t,u) \in (0,\infty) \times \mathbb{R}$.

*(C2) Obviously*

$$\begin{aligned} t^{-\frac{3}{2}} f(t,u) &= \frac{\ln(1+|u|)}{30(1+t^2)} + \frac{|u|}{60\exp\left(\sqrt{t}\right)} + \exp(-t) \\ &\leq \left( \frac{1}{30(1+t^2)} + \frac{1}{60\exp\left(\sqrt{t}\right)} \right) |u| + \exp(-t). \end{aligned}$$

*With a simple computation we find that*

$$\begin{aligned} &\int_0^\infty \left(1+\sqrt{t}\right) \left( \frac{1}{30(1+t^2)} + \frac{1}{60\exp\left(\sqrt{t}\right)} \right) dt \\ = &\frac{\left(1+\sqrt{2}\right)\pi}{60} \approx 0.22 < \frac{1}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1}{4}\sqrt{\pi} \approx 0.44, \end{aligned}$$

*and*

$$\int_0^\infty \exp(-t)\, dt = 1 < \infty.$$

*Thus the hypothesis (C2) is satisfied.*

*If we choose* $k = 5$, *we show* $k \geq \frac{1}{\frac{1}{4}\sqrt{\pi} - \frac{1}{60}\pi\left(\sqrt{2}+1\right)} \approx 3.15$. *Therefore, (3.12) is satisfied. Hence, all conditions of theorem 3.1 and theorem 3.2 hold. Thus, the problem (3.20) has at least one solution in* $\Omega$.

**Example 3.2** *Consider the following boundary value problem of nonlinear Caputo-type Erdélyi–Kober fractional differential equations*

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2},\frac{3}{2}} u(t) + \frac{t^{\frac{3}{2}} \exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} \arctan(|u|) = 0, \ t > 0, \\ \lim_{t\to 0} t^{-\frac{1}{2}} u(t) = 0, \ \lim_{t\to\infty} t^{-\frac{1}{2}} u(t) = \frac{1}{2} u(1), \end{cases} \tag{3.21}$$

*here,*

$$f(t,u) = \frac{t^{\frac{3}{2}} \exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} \arctan(|u|),$$

*and* $n = 2,\ \beta = 1, \rho = \frac{1}{2},\ \gamma = -\frac{3}{2}$, *and* $\delta = \frac{3}{2}$.

*(C1) Clearly* $f \in C((0,\infty) \times \mathbb{R}, (0,\infty))$, *for any* $(t,u) \in (0,\infty) \times \mathbb{R}$.

*(C2) Obviously*

$$\begin{aligned} t^{-\frac{3}{2}} f(t,u) &= \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} \arctan(|u|) \\ &\leq \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} |u|. \end{aligned}$$

*with the aid of simple computation we find that*

$$\int_0^\infty \left(1+\sqrt{t}\right) \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} dt = \frac{1}{2\sqrt{\pi}} \approx 0.282,$$

*and* $\frac{1}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1}{4}\sqrt{\pi} \approx 0.44$. *Then (C2) holds.*

*(C3) We have*

$$\begin{aligned} \left|t^{-\frac{3}{2}} f(t,u) - t^{-\frac{3}{2}} f(t,v)\right| &= \left|\frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} \arctan(|u|) - \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} \arctan(|v|)\right| \\ &= \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} |\arctan(|u|) - \arctan(|v|)| \\ &\leq \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} ||u| - |v|| \\ &\leq \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} |u-v|. \end{aligned}$$

*If we put* $q(t) = \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)}$, *then we obtain*

$$q^* = \int_0^{+\infty} \left(1+t^{\frac{1}{2}}\right) \frac{\exp(-t)}{2\sqrt{\pi}\left(1+t^{\frac{1}{2}}\right)} dt = \frac{1}{2\sqrt{\pi}} < \infty.$$

*Hence, the condition (C3) is satisfied.*

*Moreover, we have*

$$\frac{\beta q^*}{(1-\rho)\Gamma(\delta)} = \frac{1}{\Gamma\left(\frac{3}{2}\right)\sqrt{\pi}} = \frac{2}{\pi} < 1,$$

*then the condition (3.19) is satisfied. It follows from Theorem 3.3 that the boundary value problem (3.21) has a unique solution* $u \in C_\infty$.

# 4

## BOUNDED SOLUTION FOR BVP OF DIFFERENTIAL EQUATIONS INVOLVING E–K FRACTIONAL DERIVATIVE ON UNBOUNDED DOMAIN

This chapter is an article accepted in: Annals of the University of Craiova–Mathematics and Computer Science series, (see [59]).

The aim of this chapter is to study the existence of bounded solution for the boundary value problem of nonlinear fractional differential equation involving Erdélyi–Kober differential operator on unbounded domain

$$\mathcal{D}_{\beta}^{\gamma,\delta} u\left(t\right) + f\left(t, u\left(t\right)\right) = 0, \ t \in J = \left(0, \infty\right), \tag{4.1}$$

with the boundary conditions

$$\lim_{t \to 0} t^{\beta(1+\gamma)} \frac{d^k}{dt^{(k)}} I^{\gamma+\delta, m-\delta} u(t) = 0, \text{ with } k = \overline{0, \ m-2}, \ u\left(t\right) \text{ bounded on } J, \tag{4.2}$$

where $\mathcal{D}_{\beta}^{\gamma,\delta}$ denotes the Erdélyi–Kober fractional derivative operator of order $\delta$ and $\mathcal{I}^{\delta+\gamma, m-\delta}$ is the Erdélyi–Kober fractional integral of order $m-\delta$, with $m-1 < \delta \leq m$, $-m < \gamma < 1-m, m \in \mathbb{N}, m \geq 2$, $\beta > 0$ and $f$ is a given function required to satisfy the following conditions:

**(D1)** $f : J \times \mathbb{R} \longrightarrow [0, \infty)$ is continuous.

**(D2)** There exist $\psi(t) \in C\left(\left(0, \infty\right), \left(0, \infty\right)\right)$, and $\omega(t) : (0, \infty) \to (0, \infty)$ continuous and non-decreasing such that

$$\left| t^{\beta(1+\gamma)-1} f\left(t, u\right) \right| \leq \psi(t) \omega(|u|) \text{ on } (0, \infty) \times \mathbb{R}.$$

In this chapter, we assume that $T_n \in J$, $n \in \mathbb{N}^*$, such that $0 < T_1 < T_2 < ... < T_n < ...$ with $T_n \to \infty$ as $n \to \infty$.

In the sequel we set $J_n = (0, T_n]$.

In this chapter, we use the space $C_l(0,\infty)$ to study the problem (4.1)-(4.2), which is denoted by

$$C_l(0,\infty) = \left\{ u \;\middle|\; \begin{array}{l} u \text{ is a continuous function on } (0,+\infty) \text{ such that} \\ \lim_{t\to 0} u(t) \text{ and } \lim_{t\to+\infty} u(t) \text{ exist} \end{array} \right\},$$

from [14],[64], $C_l(0,\infty)$ is a Banach space with the norm

$$\|u\|_{C_l(0,\infty)} = \sup_{t\in(0,\infty)} |u(t)|,$$

furthermore

$$C_l(0,T_n] = \left\{ u \;\middle|\; \begin{array}{l} u \text{ is a continuous function on } (0,T_n] \text{ such that} \\ \lim_{t\to 0} u(t) \text{ exists} \end{array} \right\}.$$

It is easily seen that $C_l(0,T_n]$ is a Banach space with the norm

$$\|u\|_{C_l(0,T_n]} = \sup_{0<t\leq T_n} |u(t)|.$$

## 4.1 FINITE INTERVAL PROBLEM

To present existence theory for the problem (4.1) -(4.2) we begin with the following existence principle for the problem on the finite interval.

Fix $n \in \mathbb{N}^*$, for $m = \{2,3,\ldots\}$, with $m-1 < \delta \leq m$, $-m < \gamma < 1-m$, $\beta > 0$, we consider the following boundary value problem

$$\begin{cases} \mathcal{D}_\beta^{\gamma,\delta} u(t) + f(t,u(t)) = 0, \; t \in J_n, \\ \lim_{t\to 0} t^{\beta(1+\gamma)} \frac{d^k}{dt^{(k)}} I^{\gamma+\delta,m-\delta} u(t) = 0, \text{ with } k = \overline{0,m-2}, \\ u(T_n) = 0. \end{cases} \tag{4.3}$$

Based on the previous lemma, we will define the integral solution of the finite interval problem (4.3).

**Lemma 4.1** *Let $m-1<\delta\leq m,\ -m<\gamma<1-m,\ m\geq 2,\ \beta>0$ and $y\in C^2_\alpha$, then the fractional differential equation*

$$\mathcal{D}^{\gamma,\delta}_{\beta}u\left(t\right)+y(t)=0,\ t\in J_n, \tag{4.4}$$

*with the conditions*

$$\lim_{t\to 0}t^{\beta(1+\gamma)}\frac{d^k}{dt^{(k)}}I^{\gamma+\delta,m-\delta}u(t)=0,\ \textit{with } k=\overline{0,m-2}, \tag{4.5}$$

$$u(T_n)=0,\ n\in\mathbb{N}^*\ \textit{is fixed.} \tag{4.6}$$

*has a unique solution given by*

$$u\left(t\right)=\int_0^{T_n}G_n\left(t,s\right)s^{\beta(\gamma+1)-1}y\left(s\right)ds, \tag{4.7}$$

*where*

$$G_n\left(t,s\right)=\begin{cases}\frac{\beta}{\Gamma(\delta)}\left[t^{-\beta(\gamma+1)}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}-t^{-\beta(\delta+\gamma)}\left(t^{\beta}-s^{\beta}\right)^{\delta-1}\right], & 0<s\leq t\leq T_n,\\ \frac{\beta}{\Gamma(\delta)}t^{-\beta(\gamma+1)}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}, & 0<t\leq s\leq T_n,\end{cases} \tag{4.8}$$

*is called the Green function of boundary value problem (4.4)-(4.5)-(4.6).*

**Proof.** Let $m-1<\delta\leq m$, with $-m<\gamma<1-m$, and $\beta>0$; it is easy to prove that the operator $\mathcal{I}^{\gamma,\delta}_{\beta}$ has the linearity property for all $\delta>0$. By applying $\mathcal{I}^{\gamma,\delta}_{\beta}$ to equation (4.4) we obtain

$$\mathcal{I}^{\gamma,\delta}_{\beta}\ \mathcal{D}^{\gamma,\delta}_{\beta}u\left(t\right)+\mathcal{I}^{\gamma,\delta}_{\beta}y(t)=0. \tag{4.9}$$

By using Lemma 1.5, for $m-1<\delta\leq m$, we can easily find that

$$\mathcal{I}^{\gamma,\delta}_{\beta}\ \mathcal{D}^{\gamma,\delta}_{\beta}u\left(t\right)=u(t)-c_0t^{-\beta(1+\gamma)}+c_1t^{-\beta(2+\gamma)}+\cdots+c_{m-1}t^{-\beta(m+\gamma)},$$

for some constants $c_0,c_1,\cdots c_{m-1}\in\mathbb{R}$. Thus, (4.9) gives

$$u(t)-c_0t^{-\beta(1+\gamma)}-c_1t^{-\beta(2+\gamma)}+\cdots+c_{m-1}t^{-\beta(m+\gamma)}+\mathcal{I}^{\gamma,\delta}_{\beta}y(t)=0,$$

which means that

$$u\left(t\right)=c_0t^{-\beta(1+\gamma)}+c_1t^{-\beta(2+\gamma)}+\cdots+c_{m-1}t^{-\beta(m+\gamma)}-\mathcal{I}^{\gamma,\delta}_{\beta}y(t). \tag{4.10}$$

From the formula (1.26) of Lemma 1.5, it follows that

$$
\begin{aligned}
c_0 &= \lim_{t\to 0} t^{\beta(1+\gamma)} \prod_{i=1}^{m-1} \left(1+\gamma+i+\frac{1}{\beta}t\frac{d}{dt}\right)\left(I^{\delta+\gamma,m-\delta}u\right)(t) \\
&= \lim_{t\to 0} t^{\beta(1+\gamma)} \left[\begin{array}{l}
(2+\gamma)(3+\gamma)\cdots(m+\gamma)\left(I^{\delta+\gamma,m-\delta}u\right)(t) \\
+\xi_1(n,\gamma)\frac{1}{\beta}t\frac{d}{dt}\left(I^{\delta+\gamma,m-\delta}u\right)(t)+\cdots \\
+\xi_{m-2}(m,\gamma)\frac{1}{\beta^{m-2}}t^{m-2}\frac{d^{(m-2)}}{dt^{(m-2)}}\left(I^{\delta+\gamma,m-\delta}u\right)(t) \\
+\frac{1}{\beta^{m-1}}t^{m-1}\frac{d^{(m-1)}}{dt^{(m-1)}}\left(I^{\delta+\gamma,m-\delta}u\right)(t), \\
\text{with, } \xi_1,\cdots,\xi_{m-2}\in\mathbb{R}
\end{array}\right] \\
c_1 &= \lim_{t\to 0} t^{\beta(2+\gamma)} \prod_{i=2}^{m-1} \left(1+\gamma+i+\frac{1}{\beta}t\frac{d}{dt}\right)\left(I^{\delta+\gamma,m-\delta}u\right)(t) \\
&= \lim_{t\to 0} t^{\beta(2+\gamma)} \left[\begin{array}{l}
(3+\gamma)\cdots(m+\gamma)\left(I^{\delta+\gamma,m-\delta}u\right)(t) \\
+\xi_1(m,\gamma)\frac{1}{\beta}t\frac{d}{dt}\left(I^{\delta+\gamma,m-\delta}u\right)(t)+\cdots \\
+\xi_{m-3}(m,\gamma)\frac{1}{\beta^{m-3}}t^{m-3}\frac{d^{(m-3)}}{dt^{(m-3)}}\left(I^{\delta+\gamma,m-\delta}u\right)(t) \\
+\frac{1}{\beta^{m-2}}t^{m-2}\frac{d^{(m-2)}}{dt^{(m-2)}}\left(I^{\delta+\gamma,m-\delta}u\right)(t), \\
\text{with, } \xi_1,\cdots,\xi_{m-3}\in\mathbb{R}
\end{array}\right]; \\
&\vdots \\
c_{m-2} &= \lim_{t\to 0} t^{\beta(m-1+\gamma)} \prod_{i=m-1}^{m-1} \left(1+\gamma+i+\frac{1}{\beta}t\frac{d}{dt}\right)\left(I^{\delta+\gamma,m-\delta}u\right)(t) \\
&= \lim_{t\to 0} t^{\beta(m-1+\gamma)} \left[(m+\gamma)\left(I^{\delta+\gamma,m-\delta}u\right)(t)+\frac{1}{\beta}t\frac{d}{dt}\left(I^{\delta+\gamma,m-\delta}u\right)(t)\right]; \\
c_{m-1} &= \lim_{t\to 0} t^{\beta(m+\gamma)} \prod_{i=m}^{m-1} \left(1+\gamma+i+\frac{1}{\beta}t\frac{d}{dt}\right)\left(I^{\delta+\gamma,m-\delta}u\right)(t) \\
&= \lim_{t\to 0} t^{\beta(m+\gamma)} \left(I^{\delta+\gamma,m-\delta}u\right)(t).
\end{aligned}
$$

The boundary condition (4.5) implies that $c_{m-1} = c_{m-2} = \cdots = c_1 = 0$, which means that we can rewrite the integral equation (4.10) as

$$u(t) = c_0 t^{-\beta(1+\gamma)} - \mathcal{I}_\beta^{\gamma,\delta} y(t),$$

in view of the boundary condition (4.6) we conclude that

$$c_0 T_n^{-\beta(1+\gamma)} - \mathcal{I}_\beta^{\gamma,\delta} y(T_n) = 0.$$

Consequently, we find that

$$\begin{aligned} c_0 &= \frac{\beta T_n^{-\beta(\gamma+\delta)}}{\Gamma(\delta)\, T_n^{-\beta(1+\gamma)}} \int_0^{T_n} \left(T_n^\beta - s^\beta\right)^{\delta-1} s^{\beta(\gamma+1)-1} y(s) ds \\ &= \frac{\beta}{\Gamma(\delta)} \int_0^{T_n} T_n^{-\beta(\delta-1)} \left(T_n^\beta - s^\beta\right)^{\delta-1} s^{\beta(\gamma+1)-1} y(s) ds \\ &= \frac{\beta}{\Gamma(\delta)} \int_0^{T_n} \left(1 - \left(\frac{s}{T_n}\right)^\beta\right)^{\delta-1} s^{\beta(\gamma+1)-1} y(s) ds, \end{aligned}$$

and therefore, the unique solution of the problem (4.4)-(4.5)-(4.6) is given by

$$\begin{aligned} u(t) &= \frac{\beta}{\Gamma(\delta)} \left[ \begin{array}{l} t^{-\beta(1+\gamma)} \int_0^{T_n} \left(1 - \left(\frac{s}{T_n}\right)^\beta\right)^{\delta-1} s^{\beta(\gamma+1)-1} y(s) ds \\ -t^{-\beta(\gamma+\delta)} \int_0^t \left(t^\beta - s^\beta\right)^{\delta-1} s^{\beta(\gamma+1)-1} y(s)\, ds \end{array} \right] \\ &= \int_0^{T_n} G_n(t,s)\, s^{\beta(\gamma+1)-1} y(s) ds. \end{aligned}$$

■

Now, we present some notations and some properties of Green's function that form the basis of our main work.

**Remark 4.1** *For $m-1 < \delta \leq m$, $-m < \gamma < 1-m$, and $\beta > 0$ the following conditions is hold*

1. *For all $\forall t, s \in (0, \infty)$, the function $G_n(t,s) \geq 0$.*
2. *For each $n > 0$, the function $t \in J_n \to \int_0^{T_n} |G_n(t,s)|\, ds$ is continuous and bounded on $J_n$.*

We now turn to the question of existence for the problem (4.3).
Define an integral operator $\mathcal{A} : C_l(0, T_n] \to C_l(0, T_n]$ by

$$\mathcal{A}u(t) = \int_0^{T_n} G_n(t,s)\, s^{\beta(1+\gamma)-1} f(s, u(s)) ds, \ t \in J_n, \tag{4.11}$$

where $G_n(t,s)$ defined by (4.8).
Clearly, from Lemma 4.1, the fixed points of the operator $\mathcal{A}$ coincide with the solutions of the problem (4.3).
We put

$$\widetilde{G}_n = \sup \left\{ \int_0^{T_n} |G_n(t,s)|\, ds, \ t \in J_n \right\}. \ \psi_n^* = \sup \{\psi(s), \ s \in J_n\}, n > 0.$$

**Lemma 4.2** *If (D1)-(D2) hold, Then $\mathcal{A} : C_l(0, T_n] \to C_l(0, T_n]$ is completely continuous.*

**Proof.** First, for $u \in C_l(0, T_n]$ we have

$$\begin{aligned}\|\mathcal{A}u(t)\|_{C_l(0,T_n]} &= \sup_{0<t\leq T_n} |\mathcal{A}u(t)| = \sup_{0<t\leq T_n} \left| \int_0^{T_n} G_n(t,s)\, s^{\beta(\gamma+1)-1} f(s,u(s))ds \right| \\ &\leq \sup_{0<t\leq T_n} \int_0^{T_n} |G_n(t,s)| \left| s^{\beta(\gamma+1)-1} f(s,u(s))ds \right|\end{aligned}$$

together with conditions (D1) and (D2), it then follows that

$$\begin{aligned}&\sup_{0<t\leq T_n} \int_0^{T_n} |G_n(t,s)| \left| s^{\beta(\gamma+1)-1} f(s,u(s))ds \right| \\ \leq\ & \sup_{0<t\leq T_n} \int_0^{T_n} |G_n(t,s)|\, \psi(s)\omega(|u(s)|)\, ds \\ \leq\ & \psi_n^* \omega\left(\|u\|_{C_l(0,T_n]}\right) \int_0^{T_n} |G_n(t,s)|\, ds \\ \leq\ & \psi_n^* \omega\left(\|u\|_{C_l(0,T_n]}\right) \widetilde{G}_n < \infty.\end{aligned}$$

Hence, $\mathcal{A} : C_l(0, T_n] \to C_l(0, T_n]$ is well-defined.

Choose

$$M \geq \psi_n^* \omega(M)\, \widetilde{G}_n, \tag{4.12}$$

and let

$$\Omega = \left\{ u \in C_l(0, T_n], \|u\|_{C_l(0,T_n]} \leq M,\ M > 0 \right\}.$$

In what follows we divide the proof into several steps.

STEP 1: $\mathcal{A} : \Omega \to C_l(0, T_n]$ is continuous.

Let $(u_q)_{q\in\mathbb{N}} \in \Omega$ be a convergent sequence to $u$ in $\Omega$; from Lemma 4.1 we obtain that

$$\begin{aligned}\|\mathcal{A}u_q - \mathcal{A}u\|_{C_l(0,T_n]} &= \sup_{0<t\leq T_n} \left| \int_0^{T_n} G_n(t,s)\, s^{\beta(\gamma+1)-1} \left[ f(s,u_q(s)) - f(s,u(s)) \right] ds \right| \\ &\leq \sup_{0<t\leq T_n} \left| \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+1)} \int_0^{T_n} \left(1 - \left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} s^{\beta(\gamma+1)-1} \right. \\ &\quad \left. \times \left[ f(s,u_q(s)) - f(s,u(s)) \right] ds \right| \\ &\leq \sup_{0<t\leq T_n} \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+1)} \left| \int_0^{T_n} \left(1 - \left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} s^{\beta(\gamma+1)-1} \right. \\ &\quad \left. \times f(s,u_q(s))ds - \int_0^{T_n} \left(1 - \left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} s^{\beta(\gamma+1)-1} f(s,u(s))ds \right|.\end{aligned}$$

Due to the condition (D2), we get

$$\begin{aligned}\left|\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} s^{\beta(\gamma+1)-1} f(s,u(s))\right| &\leq \left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} \psi(s)\omega\left(|u(s)|\right) \\ &\leq \left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} \psi(s)\omega\left(\|u\|\right) \\ &\leq \omega\left(M\right)\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} \psi(s).\end{aligned}$$

Since the right hand side of the above inequality is in $L^1(0,\infty)$ and the function $\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} s^{\beta(\gamma+1)-1} f(s,u(s))$, is continuous, it follows that the Lebesgue dominated convergence theorem (theorem 12.12, page 199 in [13]) yields $u \to \int_0^{T_n}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} s^{\beta(\gamma+1)-1} f(s,u(s))ds$ is continuous. Hence, it holds that

$$\begin{aligned}&\int_0^{T_n}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} s^{\beta(\gamma+1)-1} f(s,u_q(s))ds \\ \to\ &\int_0^{T_n}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} s^{\beta(\gamma+1)-1} f(s,u(s))ds \text{ as } q\to\infty.\end{aligned}$$

Therefore, $\left\|\mathcal{A}u_q-\mathcal{A}u\right\|_{C_l(0,T_n]}\to 0$, as $q\to\infty$.

STEP 2: $\mathcal{A}(\Omega)$ is relatively compact.

First, we show that $\mathcal{A}(\Omega)$ is uniformly bounded. Let $u\in\Omega$; by the condition (D2), we obtain

$$\begin{aligned}\left\|\mathcal{A}u(t)\right\|_{C_l(0,T_n]} &= \sup_{0<t\leq T_n}\left|\mathcal{A}u(t)\right| = \sup_{0<t\leq T_n}\left|\int_0^{T_n} G_n(t,s)\, s^{\beta(\gamma+1)-1} f(s,u(s))ds\right| \\ &\leq \sup_{0<t\leq T_n}\int_0^{T_n}\left|G_n(t,s)\right|\left|s^{\beta(\gamma+1)-1} f(s,u(s))ds\right| \\ &\leq \psi_n^*\omega\left(\|u\|_{C_l(0,T_n]}\right)\int_0^{T_n}\left|G_n(t,s)\right|ds \\ &\leq \psi_n^*\omega\left(M\right)\widetilde{G}_n\end{aligned}$$

hence, $\mathcal{A}(\Omega)$ is uniformly bounded.

Next, we show that $\mathcal{A}(\Omega)$ is equicontinuous on $J_n$.

For all $u\in\Omega$, $t_1,t_2\in J_n$, and $t_1\leq t_2$, we can find

$$\begin{aligned}\left|\mathcal{A}u(t_2)-\mathcal{A}u(t_1)\right| &\leq \int_0^{T_n}\left|G_n(t_2,s)-G_n(t_1,s)\right|\left|s^{\beta(\gamma+1)-1} f(s,u(s))\right|ds \\ &\leq \psi_n^*\omega\left(M\right)\int_0^{T_n}\left|G_n(t_2,s)-G_n(t_1,s)\right|ds \to 0, \text{uniformly as } t_1\to t_2.\end{aligned}$$

Hence, $\mathcal{A}(\Omega)$ is locally equicontinuous on $J_n$. Consequently, $\mathcal{A}(\Omega)$ is relatively compact. Therefore, $\mathcal{A} : C_l(0, T_n] \to C_l(0, T_n]$ is completely continuous.

■

Now, to prove the existence result for the problem (4.3), we use the fixed point theorem of schauder's

**Theorem 4.1** *Assume that the hypotheses (D1)-(D2) hold, and that there exicts $M \in \mathbb{R}$ satisfied (4.12), then the fractional boundary value problem (4.3) has at least one solution $u \in \Omega$.*

**Proof.** From the proof of Lemma 4.2, we know that $\mathcal{A}$ is a completely continuous operator. Also we have $\mathcal{A}(\Omega) \subset \Omega$ because of

$$\begin{aligned}
\|\mathcal{A}u(t)\|_{C_l(0,T_n]} &= \sup_{0<t\leq T_n} |\mathcal{A}u(t)| = \sup_{0<t\leq T_n} \left| \int_0^{T_n} G_n(t,s)\, s^{\beta(\gamma+1)-1} f(s,u(s))ds \right| \\
&\leq \sup_{0<t\leq T_n} \int_0^{T_n} |G_n(t,s)| \left| s^{\beta(\gamma+1)-1} f(s,u(s))ds \right| \\
&\leq \psi_n^* \omega\left(\|u\|_{C_l(0,T_n]}\right) \int_0^{T_n} |G_n(t,s)|\, ds \\
&\leq \psi_n^* \omega(M)\, \widetilde{G}_n \leq M.
\end{aligned}$$

Hence, by Theorem 1.4 the boundary value problem (4.3) has at least one solution $u$ in $\Omega$ such that

$$|u(t)| \leq M, \text{ for each } t \in J_n.$$

■

## 4.2 SEMI-INFINITE INTERVAL PROBLEM

The ideas in the previous part together with a diagonalization argument enable us to treat the problem (4.1)-(4.2) defined on semi-infinite interval.

**Theorem 4.2** *Assume that the hypotheses (D1)-(D2) hold and that there exits $M \in \mathbb{R}$ satisfying (4.12). Then the fractional boundary value problem (4.1)-(4.2) has at least one solution $u$ on $J$*

**Proof.** The proof will be given in tow parts

PART 1: From Theorem 4.1, for all $n \in \mathbb{N}^*$ we show that the following boundary value problems

$$\begin{cases}
\mathcal{D}_\beta^{\gamma,\delta} u(t) + f(t, u(t)) = 0, \ t \in J_n, \\
\lim_{t\to 0} t^{\beta(1+\gamma)} \frac{d^k}{dt^{(k)}} I^{\gamma+\delta, m-\delta} u(t) = 0, \text{ with } k = \overline{0, m-2}, \\
u(T_n) = 0.
\end{cases}$$

have a solution $u_n \in C_l(J_n, \mathbb{R})$ verifies that for each $t \in J_n$, $|u_n(t)| \leq M$, with

$$u_n(t) = \int_0^{T_n} G_n(t,s)\, s^{\beta(\gamma+1)-1} y(s)\, ds,$$

where, $G_n(t,s)$ defined by (4.8).

PART 2: Diagonalization argument

Define

$$v_n(t) = \begin{cases} u_n(t), \ t \in (0, T_n], \\ 0, \ t \in [T_n, \infty). \end{cases}$$

Then $v_n$ is in $C_l(0,\infty)$ with $\|v_n(t)\|_{C_l(0,\infty)} \leq M$, $t \in (0,\infty)$.
Let $S = \{(v_n)_{n\in\mathbb{N}^*}\}$. For $t \in (0, T_1]$, we have

$$|v_n(t)| \leq |u_n(t)| \leq M, \ \forall n \in \mathbb{N}^*,$$

which means that, for all $t \in (0, T_1]$; $(v_n)_{n\in\mathbb{N}^*}$ is bounded. Furthermore for all $t_1, t_2 \in (0, T_1]$, $\forall n \in \mathbb{N}^*$, we have

$$\begin{aligned} |v_n(t_2) - v_n(t_1)| &\leq |u_n(t_2) - u_n(t_1)| \\ &\leq \int_0^{T_1} |G_1(t_2,s) - G_1(t_1,s)| \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\ &\leq \psi_1^* \omega(M) \int_0^{T_1} |G_1(t_2,s) - G_1(t_1,s)|\, ds \to 0 \text{ for } t_1 \to t_2. \end{aligned}$$

It holds that, for all $t \in (0, T_1]$; $(v_n)_{n\in\mathbb{N}^*}$ is equicontinous. Thus, $S = \{(v_n)_{n\in\mathbb{N}^*}\}$ is relatively compact on $(0, T_1]$. Let $\mathbb{N}^1 = \mathbb{N}^* - \{1\}$. For all $(v_n)_{n\in\mathbb{N}^*} \in S$, the Arzela-Ascoli Theorem 1.1, guarantees that there is a sub-sequence $(v_n)_{n\in\mathbb{N}^1}$ and a function $z_1$ in $C_l(0, T_1]$ such that $(v_n)_{n\in\mathbb{N}^1} \to z_1$ uniformly on $(0, T_1]$ as $n \to \infty$.
Let $S_1 = \{(v_n)_{n\in\mathbb{N}^1}\}$. For $t \in (0, T_2]$, we have

$$|v_n(t)| \leq |u_n(t)| \leq M, \ \forall n \in \mathbb{N}^1,$$

which means that for all $t \in (0, T_2]$, $(v_n)_{n\in\mathbb{N}^1}$ is bounded. Furthermore for all $t_1, t_2 \in (0, T_2]$, $n \in \mathbb{N}^*$, we have

$$\begin{aligned} |v_n(t_2) - v_n(t_1)| &\leq |u_n(t_2) - u_n(t_1)| \\ &\leq \int_0^{T_2} |G_2(t_2,s) - G_2(t_1,s)| \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\ &\leq \psi_2^* \omega(M) \int_0^{T_2} |G_2(t_2,s) - G_2(t_1,s)|\, ds \to 0 \text{ for } t_1 \to t_2. \end{aligned}$$

It holds that, for all $t \in (0, T_2]$, $(v_n)_{n\in\mathbb{N}^1}$ is equicontinous. Thus, $S_1 = \{(v_n)_{n\in\mathbb{N}^1}\}$ is relatively compact on $(0, T_2]$.

Let $\mathbb{N}^2$=$\mathbb{N}^1 - \{2\}$. For all $(v_n)_{n\in\mathbb{N}^1} \in S$, the Arzela-Ascoli Theorem 1.1, guarantees that there is a sub-sequence $(v_n)_{n\in\mathbb{N}^2}$ and a function $z_2$ in $C_l(0, T_2]$ such that $(v_n)_{n\in\mathbb{N}^2} \to z_2$ uniformly on $(0, T_2]$ as $n \to \infty$. Note that $z_2 = z_1$ on $(0, T_1]$ since $\mathbb{N}^2 \subset \mathbb{N}^1$. Proceed inductively to obtain for $q \in \{3, 4, \cdots\}$ there is a sub-sequence $(v_n)_{n\in\mathbb{N}^q}$ with $\mathbb{N}^q \subset \mathbb{N}^*$ and $\mathbb{N}^q \subset \mathbb{N}^{q-1}$ and a function $z_q$ in $C_l\left(0, T_q\right]$ such that $(v_n)_{n\in\mathbb{N}^q} \to z_q$ as $n \to \infty$. Also $z_q = z_{q-1}$ on $\left(0, T_{q-1}\right]$.

Define a function $u$ as follows

$$u(t) = \begin{cases} z_q(t), \ t \in \left(0, T_q\right], \\ 0, \ t \in \left[T_q, \infty\right). \end{cases}$$

Then $u \in C_l(0, \infty)$, $\lim_{t\to 0} t^{\beta(1+\gamma)} \frac{d^k}{dt^{(k)}} I^{\gamma+\delta, n-\delta} u(t) = 0$, with $k = \overline{0, m-2}$, $m \geq 2$, and $|u(t)| \leq M$, for $t \in (0, \infty)$. Again fix $t \in (0, \infty)$ and let $q \in \mathbb{N}^*$ with $t \leq T_q$. Then for $n \in \mathbb{N}^q$ we have

$$v_n(t) = \int_0^{T_q} G_q(t, s) s^{\beta(\gamma+1)-1} f(s, v_n(s)) ds.$$

Passing to limit, as $n \to \infty$ through $\mathbb{N}^q$ (using [25], p. 38 or [28], p. 35) we obtain

$$z_q(t) = \int_0^{T_q} G_q(t, s) s^{\beta(\gamma+1)-1} f(s, z_q(s)) ds.$$

Thus

$$u(t) = \int_0^{T_q} G_q(t, s) s^{\beta(\gamma+1)-1} f(s, u(s)) ds,$$

which implies $\lim_{t\to 0} t^{\beta(1+\gamma)} \frac{d^k}{dt^{(k)}} I^{\gamma+\delta, n-\delta} u(t) = 0$, with $k = \overline{0, m-2}$, $u \in C_l(0, \infty)$ and

$$\mathcal{D}_\beta^{\gamma,\delta} u(t) + f(t, u(t)) = 0, \ t \in J = (0, \infty).$$

■

## 4.3 AN EXAMPLE

Consider the following boundary value problem

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2},\frac{3}{2}} u(t) + t^{\frac{3}{2}} \sqrt{\left|\frac{u}{1+t^{\frac{1}{2}}}\right|} e^{-t} = 0, \ t > 0, \\ \lim_{t\to 0} t^{-\frac{1}{2}} \frac{d^k}{dt^{(k)}} I_1^{0,m-\frac{1}{2}} u(t) = 0, \text{ with } k = \overline{0, m-2}, \ u(t) \text{ bounded in } J. \end{cases} \tag{4.13}$$

Here, $f(t,u) = t^{\frac{3}{2}}\sqrt{\left|\frac{u}{1+t^{\frac{1}{2}}}\right|}e^{-t}$, $\delta = \frac{3}{2}$, $\gamma = -\frac{3}{2}$, and $\beta = 1$.

**(D1)** It is easy to show that the function $f$ is continuous for any $(t,u) \in (0,\infty) \times \mathbb{R}$.

**(D2)** From the expression of the function $f$, it follows that

$$F(t,u) = t^{\beta(1+\gamma)-1} f\left(t, \left(1 + t^{-\beta(1+\gamma)}\right) u\right) = \sqrt{|u|}e^{-t}.$$

If we choose $\omega(u) = \sqrt{u}$, $\psi(t) = e^{-t}$, then we obtain

$$|F(t,u)| \leq \psi(t)\omega(|u|), \text{ on } (0,\infty) \times \mathbb{R},$$

with $\omega \in C((0,\infty),(0,\infty))$ non-decreasing and $\psi \in L^1(0,\infty)$. Then, the condition (D2) holds.

On the other hand, we show that

i)$\psi_n^* = \sup_{t\in(0,T_n]} \psi(t) = 1$.

ii)$\widetilde{G}_n = \sup_{t\in(0,T_n]} \int_0^{T_n} |G_n(t,s)|\, ds$; we have to consider tow cases

case 1: for $s \leq t$, we have

$$\begin{aligned}
\widetilde{G}_n &= \sup_{t\in(0,T_n]} \frac{\beta}{\Gamma(\delta)} \int_0^{T_n} \left| t^{-\beta(\gamma+1)} \left(1 - \left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} - t^{-\beta(\delta+\gamma)} \left(t^{\beta} - s^{\beta}\right)^{\delta-1} \right| ds \\
&= \sup_{t\in(0,T_n]} \int_0^{T_n} \left| t^{\frac{1}{2}} \left(1 - \frac{s}{T_n}\right)^{\frac{1}{2}} - (t-s)^{\frac{1}{2}} \right| ds \\
&\leq \sup_{t\in(0,T_n]} t^{\frac{1}{2}} \int_0^{T_n} \left| \left(1 - \frac{s}{T_n}\right)^{\frac{1}{2}} \right| ds + \sup_{t\in(0,T_n]} \int_0^{T_n} \left|(t-s)^{\frac{1}{2}}\right| ds \\
&\leq \sup_{t\in(0,T_n]} t^{\frac{1}{2}} \int_0^{T_n} \left(1 - \frac{s}{T_n}\right)^{\frac{1}{2}} ds + \sup_{t\in(0,T_n]} \int_0^{T_n} (t-s)^{\frac{1}{2}} ds \\
&\leq \sup_{t\in(0,T_n]} \left[\frac{2}{3} t^{\frac{1}{2}} T_n - \frac{2}{3}(t-T_n)^{\frac{3}{2}} + \frac{2}{3} t^{\frac{3}{2}}\right] \\
&\leq \sup_{t\in(0,T_n]} \frac{2}{3} t^{\frac{1}{2}} T_n - \inf_{t\in(0,T_n]} \frac{2}{3}(t-T_n)^{\frac{3}{2}} + \sup_{t\in(0,T_n]} \frac{2}{3} t^{\frac{3}{2}} \\
&\leq 2T_n^{\frac{3}{2}};
\end{aligned}$$

case 2: for $t \leq s$, we have

$$\begin{aligned}
\widetilde{G}_n &= \sup_{t\in(0,T_n]} \frac{\beta}{\Gamma(\delta)} \int_0^{T_n} \left| t^{-\beta(\gamma+1)} \left(1 - \left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} \right| ds \\
&= \sup_{t\in(0,T_n]} t^{\frac{1}{2}} \int_0^{T_n} \left(1 - \frac{s}{T_n}\right)^{\frac{1}{2}} ds \\
&= \sup_{t\in(0,T_n]} \frac{2}{3} t^{\frac{1}{2}} T_n \\
&= \frac{2}{3} T_n^{\frac{3}{2}} \\
&\leq 2T_n^{\frac{3}{2}}.
\end{aligned}$$

Now, if we choose $M \geq 4T_n^3$, then we get

$$\psi_n^* \omega(M)\, \widetilde{G}_n = 2T_n^{\frac{3}{2}} \sqrt{M} \leq M;$$

therefore, (4.12) is satisfied. Hence, all the conditions of Theorem 4.2 hold, witch means that the boundary value problem (4.13) has at least one solution.

# 5

## GENERAL CONCLUSIONS

This thesis devoted to study the existence and uniqueness solutions for certain classes of fractional order differential equations.

In the first chapter, we recall the history of fractional calculus and we present some functional spaces. Then, we expose the theory of fractional derivation: different types of fractional derivation (Riemann–Liouville, Caputo, Erdélyi–Kober, Caputo Erdélyi–Kober), Mellin transformation, Euler and Beta functions.

The second and the third chapter of this thesis are dedicated to the question of existence and uniqueness of a positive (respectively unbounded) solution of nonlinear fractional differential equation in the sense of Erdélyi–Kober derivative (respectively Caputo Erdélyi–Kober derivative) with boundary conditions on infinite interval in special Banach spaces . These results were obtained by using the Banach contraction principle, Guo–Krasnoselskii, Schauders and nonlinear alternative Leray–Schauders fixed point theorem. We gave in each chapter some examples to illustrate the applicability of our results.

In the last chapter, a different criterion presented for the study of the existence and uniqueness of bounded solution of nonlinear fractional differential equation on the half line. This criterion based on the Schauders fixed point theorem combined with diagonalization argument method. Also, the main result strengthened through an example.

This work opens the way for other developments on fractional differential equations. In particular, we can offer the following perspectives:

- The search for numerical methods of resolution of fractional differential equations involving Erdélyi–Kober derivative.
- The search for methods of resolution of partial differential equations of fractional order in the sense of Erdélyi–Kober and caputo Erdélyi–Kober derivative.

These perspectives are possible directions for future works.

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