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Master memory

Field : Mathematics and computer sciences
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Theme

Extrapolation theorems for (p, q)-factorable operators

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In front of the jury composed of :

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University year 2020/2021

Dedications

In the name of ALLAH the most gracious the most merciful

I dedicate this work to :

The deare

whom can not calculate their virtue

My dear sister Anfel and my brothers: Ramzi, Ammare and Amine.

To my middle's school teacher "Mo

All my friends and family

All the student

during academic year 2020/2021.

Acknowledgments

First, I am so grateful to " **Allah** " for giving us protection and ability to start and finish this work.

I would like to express my most sincere thanks to my supervisor doctor **Rachid YAHI** for his precious advice and guidance.

I am also grateful to the president of jury Professor **Abdelmoumen TIAIBA** and the examiner doctor **Maatougui BELAALA** .

I would like to thank Professor **Orlando GALDAMES-BRAVO** whom helped me on my work.

I do not forget to thank my teachers of our department of mathematics at M'sila University for their unceasing encouragement, support and help.

Contents

Notaions

q'	The conjugate of the number q $(1 \leq q \leq \infty)$, that is $\frac{1}{q} + \frac{1}{q'} = 1$
X^*	The topological dual of X.
B_X	The closed unit ball of X.
χ_A	The characteristic function.
$Z(\mu)$	The Banach function space (BFS) over μ.
$Z(\mu)^*$	The topological dual of $Z(\mu)$.
$Z(\mu)'$	The Köth dual of $Z(\mu)$.
$Z(\mu)_{[p]}$	The p-th power space of $Z(\mu)$.
L^0	The space of classes of measurable function.
$L^1(m)$	The space of m-integrable functions.
$\mathcal{L}(X;Y)$	The sets of all continuous linear operators.
$\mathcal{L}_{p,q}(X;Y)$	The operator ideal of (p,q)-factorable operators.
$\mathcal{F}_p(X;Y)$	The p-th power factorable operators.
$\mathcal{F}_{p,q}(X;Y)$	The (p,q)-th power factorable operators.
$\mathcal{I}(X,Y)$	The ideal of all linear operators.
$\Pi_p(X,Y)$	The ideal of p-summing operators.

$\mathcal{K}_Y$	The isometric embedding defined between Y and its bidual Y^{**}.
$i_{[p]}$	The natural (and continuous) injection.
$T_{[p]}$	The linear extension of T.
$[i]$	The inclusion/ quotient map.
$\mathcal{I}_p$	The ideal of p-integral operators.
$\|m\|$	The variation a vector measure m .
$\|\|m\|\|$	The semivariation of the vector measure m.

Abstract

Abstract

In this research, we studied some classes of linear operators ideals such as the class of p-factorable operators, p-th power factorable operators, and the class of (p, q)-factorable operators. The main aim of our work is to present the ideal of $\mathcal{F}_{p,q}$-factorable operators which introduced by **Orlando GALDAMES-BRAVO**. Some extrapolation theorems related to this class are shown.

Keywords : (p, q)- factorable operator, p-th power factorable operator, vector measure, p-summing operator.

Résumé

Dans ce mémoire, nous avons présenté quelques classes d'idéaux d'opérateurs linéaires tels que la classe des opérateurs p-factorables, les opérateurs p-th power factorables, les opérateures (p, q) factorables, L'objectif principal de notre travail est de présenter l'idéal des opérateurs $\mathcal{F}_{p,q}$-factorables introduite par **Orlando GALDAMES-BRAVO**. Certains théorèmes d'extrapolation liés à cette classe sont présentés.

Mots clés: opératuer (p, q)- factorable , opératuer p-th power , espace mésuré vector, opératuer p-summing sommant.

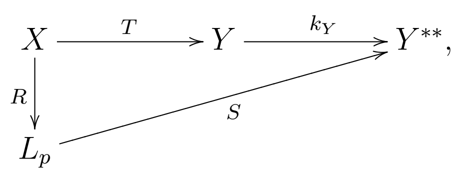
ملخص

لقد قمنا في هذه المذكرة بدراسة بعض المثاليات الخطية نذكر منها المؤثرات p قابلة للتفكيك، المؤثرات p اسية قابلة للتفكيك. المثاليات (p, q) قابلة للتفكيك، وقد ركزنا عملنا على نوع جديد من المؤثرات الخطية الذي نشر بحث لها من قبل العالم الرياضي الاسباني اورلندو سنة 2018 حيث عرضنا فيها بعض نظريات الاستقطاب الخارجي الخاصة بهذا الصنف من المثاليات الخطية.

الكلمات المفتاحية: مثاليات المؤثرات (p, q) قابلة للتفكيك ، مثاليات المؤثرات p أسية قابلة للتفكيك، المؤثرات p جمعية

Introduction

The ideal of p-factorable operators (denoted by $\mathcal{L}_p$) was introduced by Kwapień [7, page 217]. A linear operator between two Banach spaces X and Y is called $p-$factorable if there are a finite measure space (Ω, Σ, μ) and operators $R \in \mathcal{L}(X \to L_p(\mu))$, $S \in \mathcal{L}(L_p(\mu) \to Y^{**})$, such that



where k_Y is the cannonical injection of Y into Y^{**}. We write

$$\gamma_{p(u)} := \inf \|R\|\|S\|,$$

where the infimum extends over all conceivable factorization of the form we have indicated. The collection of all p-factorable operators from X into Y denoted by $\mathcal{L}_p(X, Y)$

From [3, Theorem 18.11], let X and Y be Banach spaces, and let $p, q \in [1, \infty)$ be such that $1/p + 1/q > 1, T \in \mathcal{L}(X, Y)$ is (p, q)-factorable if and only if there are a finite measure μ, operators

$R \in \mathcal{L}(X, L^{q'}(\mu))$ and $S \in \mathcal{L}(L^p(\mu), Y^{**})$ such that $k_Y \circ T = S \circ I \circ R$

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \xrightarrow{K_Y} & Y^{**} \\ {\scriptstyle R}\downarrow & & & & \uparrow{\scriptstyle S} \\ L^{q'}(\mu) & \hookrightarrow & \xrightarrow{I} & & L^p(\mu) \end{array}$$

where I is the natural inclusion. The norm is given by $\alpha_{p,q}(T) := inf \, \|R\| \, \|I\| \, \|S\|$, where the infimum is taken over all such factorizations . Recall that when $\frac{1}{p} + \frac{1}{q} = 1$ we have $\mathcal{L}_{p,q}(X,Y) = \mathcal{L}_p(X,Y)$

In [8] Maurey studied the class of operators that factor through L^p spaces of a finite measure, providing an extrapolation theorem for p-summing operators which establishes, in one of its versions, that $\Pi_p(X,Y) = \Pi_r(X,Y)$ for every $r \in [1,p]$, provided that $\Pi_p(X,l^p) = \Pi_r(X,l^p)$ for some $1 \leqslant r < p < \infty$ and that X is a Banach space (see [4, Theorem 3.17]).

The aim of this memory is to present an extrapolation theorems for (p,q)-power factorable operators, so our work based in the article of **Orlando Galdames-Bravo** (see[5]), this memory organized as follow

In the first chapter, we recall by some basic definitions concerning Banach function space, linear operator ideals, p-factorable operators and (p,q)-factorable operators. The second chapter is devoted to present the classes of p-th power factorable operator presented in the [9], definitions, inclusion theorem for this class of operators are given.

The ideal of $\mathcal{F}_{p,q}$-factorable operators presented in the last chaptere. Some characterizations and extrapolation theorems related to this class of operators are given.

Chapter 1

Preliminaries

In this chapter, we present some concepts which will use in the sequel of this Memory, as measure spaces, Banach Lattice, Banach function spaces

1.1 Preliminaries on Measure space

Definition 1.1.1. *[1, page 89] (**Measure space** (Ω, Σ, μ)) Let (Ω, Σ, μ) denote a measure space, i.e., Ω is a set and*

(i) Σ is a Σ-algebra in Ω, i.e., Σ is a collection of subsets of Ω such that:

(a) $\phi \in \Sigma$,

(b) $A \in \Sigma \implies A^c \in \Sigma$,

(c) $\bigcup_{n=1}^{\infty} A_n \in \Sigma$ whenever $A_n \in \Sigma \qquad \forall n$,

(ii) μ is a measure, i.e.,$\mu : \Sigma \longrightarrow [0, \infty[$ satisfies

(a) $\mu(\phi) = 0$,

(b) $\mu(\bigcup_{n=1}^{\infty}) = \bigcup_{n=1}^{\infty} \mu(A_n)$ whenever (A_n) is a disjoint countable family of members of Σ. (The members of Σ are colled the measurable sets. Sometimes we shall write$|A|$ instead of $\mu(A)$.

(iii) Ω is Σ-finite, i.e., there exists a countable family (Ω_n) in Σ such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$and $\mu(\Omega_n) < \infty, \forall n$.

Remark 1.1.2. *The sets $E \in \Sigma$ with the property that $\mu(E) = 0$ are called the null sets. We say that a property holds a.e. (or for almost all $x \in \Omega$) if it holds everywhere on Ω except on a null set.*

1.2 Banach lattices

Let recall some notations and results from [11]. A Riesz space is a partially ordered real vector space E which in addition is a lattice, i.e., any two elements $x; y \in E$ have a least upper bound, denoted by $x \vee y = sup\{x, y\}$ and a greatest lower bound, denoted by $x \wedge y = inf\{x, y\}$. For every $x \in E$, let $x^+ = x \vee 0, x^- = x \wedge 0$ and $|x| = x^+ \vee x^-$ be the positive part, the negative part and the absolute value of x, respectively. We have the identities $x = x^+ + x^-$ and $x = x^+ + x$. The set $E = \{x; x > 0\}$, is called the positive cone of E and its elements are called positive .

Example 1.2.1. *Let (Ω, Σ, μ) be a measure space. Define $\boldsymbol{C}$ to be the set of all realvalued Σ-measurable functions on Ω and order $\boldsymbol{C}$ by $f \leq g$ if $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$ and $f, g \in \boldsymbol{C}$. Let $\boldsymbol{B}$ be the space of all equivalence classes of functions in $\boldsymbol{C}$ for the equivalence relation $f \approxeq g$ if $f - g = 0$ μ-a.e. Order $\boldsymbol{B}$ by $[f] \leq [g]$ if $f \leq g$*

Definition 1.2.2. ***(Banach Lattice)*** *Let E be a Riesz space, A norm $\|\cdot\|$ on E is a lattice norm whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ $\forall x, y \in E$. A Riesz space equipped with a lattice norm is known as a normed Riesz space. If a normed Riesz space is also norm complete, then it is referred to as a Banach lattice.*

1.3 Banach function space

• The space of classes of measurable function equal almost everywhere with respect to μ is denoted by $L^0(\mu)$. Let $A \in \Sigma$; hence $\chi_A \in L^0(\mu)$ denotes the characteristic function. A Banach function space (BFS for short) over μ is a Banach space $Z(\mu)$ continuously embedded into $L^0(\mu)$ and satisfying the following:

(i) (Ideal property). If $g \in Z(\mu)$ and $|f| \leqslant |g|$ $(f \in L^0(\mu))$, then $f \in Z(\mu)$ and $\|f\|_{Z(\mu)} \leqslant \|g\|_{Z(\mu)}$.

(ii) For every $A \in \Sigma, \chi_A \in Z(\mu)$.

1.4 The topological dual of Banach space $Z(\mu)^*$

Definition 1.4.1. *[9] The duality between a B.f.s.$Z(\mu)$ and its (topological) dual space $Z(\mu)^* = \mathcal{L}(Z(\mu), \mathbb{C})$ is denoted by*

$$\langle f, \xi \rangle := \xi(f), \qquad f \in Z(\mu), \quad \xi \in Z(\mu)^*.$$

Then $Z(\mu)^*$ is equipped with the dual norm

$$\|\cdot\|_{Z(\mu)^*} : \xi \mapsto \sup \left\{ |\langle f, \xi \rangle| : f \in B_{Z(\mu)} \right\}, \qquad \xi \in Z(\mu)^*.$$

Note that $\|\cdot\|_{Z(\mu)^*}$ is indeed a norm for which $Z(\mu)^*$ is complete because $\mathbb{C}$ is a Banach space and because of the inequality

$$|\langle f, \xi \rangle| \leqslant \|f\|_{Z(\mu)} \|\xi\|_{Z(\mu)^*}, \qquad f \in Z(\mu), \quad \xi \in Z(\mu)^*.$$

1.5 The Köthe dual space $Z(\mu)'$

Definition 1.5.1. *[9] The Köthe dual of $Z(\mu)$, also called the associate space of $Z(\mu)$, is the order ideal of $L^0(\mu)$ defined by*

$$Z(\mu)' := \left\{ g \in L^0(\mu) : g \cdot Z(\mu) \subseteq L^1(\mu) \right\}.$$

Since both $Z(\mu)$ and $L^1(\mu)$ are order ideals with $\chi_\Omega \in Z(\mu)$, it follows that $Z(\mu)' \subseteq L^1(\mu)$. Given $g \in Z(\mu)'$, the linear functional

$$\xi_g : f \mapsto \int_\Omega g f d\mu, \qquad f \in Z(\mu)$$

is necessarily continuous, that is, $\xi_g \in Z(\mu)^*$. Indeed, the multiplication operator $M_g : f \mapsto gf$, for $f \in Z(\mu)$, is a closed operator from $Z(\mu)$ into $L^1(\mu)$ and hence, is continuous via the Closed Graph Theorem. So, ξ_g is continuous because it is the composition of M_g with the continuous linear functional $h \mapsto \int_\Omega h d\mu$ on $L^1(\mu)$. Next, observe that the linear map

$$g \mapsto \xi_g, \qquad g \in Z(\mu)'.$$

Clearly, $\|\cdot\|_{Z(\mu)'}$ is a lattice norm. Accordingly, $(Z(\mu)', \|\cdot\|_{Z(\mu)'})$ is a lattice normed function space over (Ω, Σ, μ).

Definition 1.5.2. • *A BFS $Z(\mu)$ is called σ-order continuous or simply order continuous (OC for short) if, for every sequence of functions $(f_i)_i \subseteq Z(\mu)^+$ such that $f_i \downarrow 0$, We have $\|f_i\|_{z(\mu)} \downarrow 0$ (note that σ-OC and OC coincide in BFS).*

• *A BFS $Z(\mu)$ is called σ-Fatou or simply Fatou if, for every sequence of functions $(f_i)_i \subseteq Z(\mu)^+$ such that $f_i \uparrow f \in L^0(\mu)$ and $sup_i \|f_i\|_{Z(\mu)} < \infty$, we have $f \in Z(\mu)$ and $\|f_i\|_{Z(\mu)} \uparrow \|f\|_{Z(\mu)}$. The main characterizations of these properties are that a BFS $Z(\mu)$ is OC if and only if $Z(\mu)^* = Z(\mu)'$, and it is Fatou if and only if $Z(\mu)'' = Z(\mu)$.*

1.6 Linear operator ideals

Recall that a linear operator $T \in \mathcal{L}(X, Y)$ is said to have finite rank if $T(X)$ is a finite dimensional subspace of Y. The class of all finite rank linear operators between Banach spaces is denoted by

$\mathcal{L}_f(X,Y)$. An operator has rank one if and only if has the form

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle \, y$$

i.e. if $u \in \mathcal{L}_f(X,Y)$ we have

$$u = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset X^*$ and $(y_i)_{i=1}^n \subset Y$ (see [10, Page 25]).

Definition 1.6.1. *An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces X and Y its components $\mathcal{I}(X,Y) := \mathcal{L}(X,Y) \cap \mathcal{I}$ satisfy:*

(i) $\mathcal{I}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$ which contains the finite rank operators.

(ii) The ideal property: if $v \in \mathcal{L}(G,X)$, $u \in \mathcal{I}(X,Y)$ and $w \in \mathcal{L}(Y,H)$, then the composition $w \circ v \circ u$ is in $\mathcal{I}(G,H)$.

If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies: (i') $(\mathcal{I}(X,Y), \|.\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces E and F.

(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$.

(iii') If $v \in \mathcal{L}(G,X)$, $u \in \mathcal{I}(E,F)$ and $w \in \mathcal{L}(Y,H)$.

$$\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \, \|v\|_{\mathcal{I}} \, \|u\| \, ,$$

then $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal.

The operator ideal $\mathcal{I}$ is said to be *closed* if each $\mathcal{I}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$ for the sup norm.

Definition 1.6.2. *(injective operator ideal)*

A normed operator ideal $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is said to be injective if for every metric injection $i : Y \hookrightarrow G$

and every $u \in \mathcal{L}(X,Y)$ *it follows from* $i \circ u \in \mathcal{I}(X,G)$ *that* $u \in \mathcal{I}(X,Y)$. *Moreover*

$$\|i \circ u\|_{\mathcal{I}} = \|u\|_{\mathcal{I}},$$

The ideal $\mathcal{L}_f$ of finite rank linear operators is the smallest operator ideal and $\mathcal{L}$ the largest one [10, Theorem 1.2.2].

1.7 Ideal of p-summing linear operators.

The theory of p-summing operators is based on a crucial criterion due to Pietsch [10]. We mention that the linear p-summing operators are the starting point in the study of Lipschitz p-summing mappings.

Let $1 \leq p < \infty$. A linear operator $T : X \longrightarrow Y$ is said to be p-summing if there exists a constant $C \geq 0$ such that for all finite sequence $(x_i)_{1\leq i\leq n}$ in X

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{E^*}\leq 1} \left(\sum_{i=1}^{n} |\xi(x_i)|^p\right)^{\frac{1}{p}}. \tag{1.1}$$

The infimum of all such constants $C \geq 0$ is denoted by $\pi_p(T)$. The collection of all p-summing operators between X and Y is denoted by $\Pi_p(X,Y)$.

Theorem 1.7.1. *[4, Page 39] If* $1 \leq p \leq q < \infty$, *then* $\Pi_p(X,Y) \subset \Pi_q(X,Y)$. *Moreover,* $\pi_q(T) \leq \pi_p(T)$ *for every* $u \in \Pi_p(X,Y)$.

The following basic result about p-summing operators is due to A. Pietsch, and it characterizes the p-summability by means of a domination theorem.

Theorem 1.7.2. *(Pietsch Domination Theorem) [4, page 44]*

Let $1 \leq p < \infty$ *and* $T \in \mathcal{L}(X,Y)$. *Then* T *is p-summing if and only if there exist a constant* C

and a regular Borel probability measure μ on B_{E^*} (with the weak star topology) so that

$$\|T(x)\| \leq C \int_{B_{X^*}} |\langle x, x^* \rangle|^p \, d\mu(x^*), \qquad x \in E. \tag{1.2}$$

In this case, $\pi_p(T)$ is the least of all the constants C such that (1.2) holds.

In order to adapt the previous result into a factorization theorem, we present basic examples of p-summing linear operators.

Example 1.7.3. *see [4, Example 2.9 (b),(d)]*

(1) Let K be a compact Hausdorff space, let μ be a positive regular Borel measure on K, and let $1 \leq p < \infty$. The canonical inclusion

$$J_p : C(K) \longrightarrow L_p(\mu),$$

is p-summing with $\pi_p(J_p) = \|J_p\| = \mu(K)^{\frac{1}{p}}$.

(2) Let (Ω, Σ, μ) be a finite measure space and let $1 \leq p < \infty$. The formal inclusion map

$$I_{\infty,p} : L_\infty(\mu) \longrightarrow L_p(\mu),$$

is p-summing, with $\pi_p(I_{\infty,p}) = \mu(\Omega)^{\frac{1}{p}}$.

We denote by i_X the isometric embedding $X \longrightarrow C(B_{X^*})$ given by $i_X(x) = \langle x, . \rangle$.

Corollary 1.7.4. *[4, page 45] (Pietsch Factorization Theorem)*

Let $1 \leq p < \infty$ and $T \in \mathcal{L}(X, Y)$. The following are equivalent

(i) T is p-summing.

(ii) There exist a regular Borel probability measure μ on B_{X^*} (with the weak star topology), a closed subspace X_p of $L_p(\mu)$ and a linear continuous operator $\widetilde{u} : X_p \longrightarrow Y$ such that $J_p \circ i_X(X) \subset X_p$ and $\widetilde{u} \circ J_p \circ i_X(x) = T(x)$ for all $x \in E$.

In other words, if $\overline{J_p}$ is the map $i_X(X) \longrightarrow X_p$ induced by J_p, then the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ i_X \downarrow & & \uparrow \widetilde{T} \\ i_X(X) & \xrightarrow{\overline{J_p}} & X_p \\ \cap & & \cap \\ C\left(B_{X^*}\right) & \xrightarrow{J_p} & L_p(\mu). \end{array}$$

In addition, we may choose μ and $\widetilde{T}$ so that $\left\|\widetilde{T}\right\| = \pi_p(T)$.

1.8 Extrapolation theorem for p-summing operators

Theorem 1.8.1. *[4, page70] Let $1 < r < p < \infty$, and let X be a Banach spaces such that*

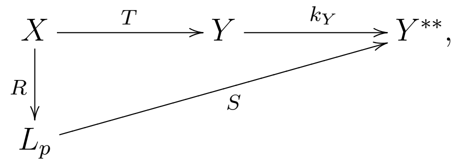
$$\Pi_p\left(X, l_p\right) = \Pi_r\left(X, l_p\right).$$

Then, for every Banach space Y,

$$\Pi_p\left(X, Y\right) = \Pi_1\left(X, Y\right).$$

1.9 p-factorable operators

Definition 1.9.1. *([7, page 217]) A linear operator between two Banach spaces X and Y is called $p-$factorable if there are a finite measure space (Ω, Σ, μ) and operators $R \in \mathcal{L}(X \to L_p(\mu))$, $S \in \mathcal{L}(L_p(\mu) \to Y^{**})$, such that*



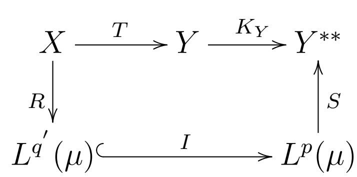
where k_Y is the cannonical injection of Y into Y^{**}. We write

$$\gamma_{p(u)} := \inf \|R\| \|S\|,$$

where the infimum extends over all conceivable factorization of the form we have indicated. The collection of all p-factorable operators from X into Y denoted by $\mathcal{L}_p(X, Y)$

1.10 (p, q)-factorable operators

Definition 1.10.1. *([3, Theorem 18.11]) Let X and Y be Banach spaces, and let $p, q \in [1, \infty)$ be such that $1/p + 1/q > 1, T \in \mathcal{L}(X, Y)$ is (p, q)-factorable if and only if there are a finite measure μ, operators $R \in \mathcal{L}(X, L^{q'}(\mu))$ and $S \in \mathcal{L}(L^p(\mu), Y^{**})$ such that $k_Y \circ T = S \circ I \circ R$*



where I is the natural inclusion. The norm is given by

$$\alpha_{p,q}(T) := inf \|R\| \, \|I\| \, \|S\| \,,$$

where the infimum is taken over all such factorizations . Recall that when $\dfrac{1}{p} + \dfrac{1}{q} = 1$ we have $\mathcal{L}_{p,q}(X, Y) = \mathcal{L}_p(X, Y)$

Remark 1.10.2. *We have that $\mathcal{L}_{r,s} \subseteq \mathcal{L}_{p,q}$ is satisfied when $1 \leqslant r \leqslant p < \infty$ and $1 \leqslant s \leqslant q <$*

∞. *The ideal of p-integral operators, denoted by* $[\mathcal{I}_p, i_p]$, *is characterized by the* $(p, 1)$*-factorable operators, that is,* $\mathcal{I}_p = \mathcal{L}_{p,1}$. *Recall that every p-integrable operator is p-summing.*

Chapter 2

p-th power factorable operators

The aim of this chapter is to relate the factorization results for p-th power factorable operators.

2.1 p-th power of a Banach function space:

Definition 2.1.1. *[9, Page 39] Let $0 < p < \infty$, and let $Z(\mu)$ be a BFS. The p-th power space (sometimes called the $(1/p)$-th power space) of $Z(\mu)$ is the space*

$$Z(\mu)_{[p]} := \left\{ f \in L^0(\mu) : |f|^{1/p} \in Z(\mu) \right\},$$

equipped with the norm $\|f\|_{Z(\mu)_{[p]}} := \left\| |f|^{1/p} \right\|^p_{Z(\mu)}, \quad f \in Z(\mu)_{[p]}$.

The term "p-th power" is derived from the fact that

$$|f| \in Z(\mu) \Longleftrightarrow |f|^p \in Z(\mu)_{[p]} \quad \textit{provided} \quad f \in L^0(\mu),$$

Lemma 2.1.2. *[9, Lemma 2.21] Let $(Z(\mu), \|\cdot\|_{Z(\mu)})$ be a BFS based on (Ω, Σ, μ) and $K \geqslant 1$ be a constant satisfying*

$$\|f_1 + f_2\|_{Z(\mu)} \leqslant K(\|f_1\|_{Z(\mu)} + \|f_2\|_{Z(\mu)}), \qquad f_1, f_2 \in Z(\mu).$$

(i) If the norm of $Z(\mu)$ is σ-OC, then the norm of $Z(\mu)_{[p]}$ is also σ-OC for every $0 < p < \infty$.

(ii) If $0 < p \leqslant q < \infty$, then $Z(\mu)_{[p]} \subseteq Z(\mu)_{[q]}$ and the natural inclusion map is continuous. Indeed,

$$\|f\|_{Z(\mu)_{[q]}} \leqslant K^q \, \|\chi_\Omega\|^{q-p}_{Z(\mu)} \, \|f\|_{Z(\mu)_{[p]}}, \qquad f \in Z(\mu)_{[p]}.$$

In particular, $Z(\mu) \subseteq Z(\mu)_{[p]}$ for $1 \leqslant p < \infty$ and $Z(\mu)_{[p]} \subseteq Z(\mu)$ for $0 < p \leqslant 1$.

Remark 2.1.3. *[9] Even when $Z(\mu)$ is a BFS, the space $Z(\mu)_{[p]}$ may only be a normed function space (for some p). For instance, if $Z(\mu) := L^1([0,,1])$ and $1 < p < \infty$, then $Z(\mu)_{[p]} = L^{1/p}([0,1])$ with $0 < (1/p) < 1$ and so is a non-normable BFS, whereas for $0 < p \leqslant 1$ we see that $Z(\mu)_{[p]} = L^{1/p}([0,1])$ with $0 < (1/p) < \infty$ is actually a BFS.*

2.2 p-th power factorable operators

Definition 2.2.1. *[9] Let $Z(\mu)$ be an OC BFS, $1 \leqslant p < \infty$ and E be a Banach space. A continuous linear operator $T : Z(\mu) \longrightarrow E$ is called p-th power factorable if there exists a continuous linear operator $T_{[p]} : Z(\mu)_{[p]} \longrightarrow E$ such that*

$$T = T_{[p]} \circ i_{[p]}$$

In other words, $T_{[p]}$ is an E-valued continuous linear extension of T to the Banach function space $Z(\mu)_{[p]}$ (in which $Z(\mu)$ is always dense). Hence, the following diagram commutes:

$$\begin{array}{ccc} Z(\mu) & \xrightarrow{\quad T \quad} & E, \\ & {\scriptstyle i_{[p]}} \searrow \quad \nearrow {\scriptstyle T_{[p]}} & \\ & Z(\mu)_{[p]}\,, & \end{array} \tag{2.1}$$

where $i_{[p]} : Z(\mu) \longrightarrow Z(\mu)_{[p]}$ denotes the natural (and continuous) injection.

Lemma 2.2.2. *[9] Let $Z(\mu)$ be a σ-order continuous BFS and E be a Banach space. Suppose that $1 \leqslant p < \infty$. Then a continuous linear operator $T : Z(\mu) \longrightarrow E$ is p-th power factorable if and only* *if there is a constant $C > 0$ such that*

$$\|T(f)\|_E \leqslant C\,\|f\|_{Z(\mu)_{[p]}} = C\left\|\,|f|^{1/p}\right\|^p_{Z(\mu)}, \qquad f \in Z(\mu) \subseteq Z(\mu)_{[p]}. \tag{2.2}$$

Proof. Let T be p-th power factorable. Let $f \in Z(\mu)$, in which case $i_{[p]}(f) = f$. Since $T_{[p]} : Z(\mu)_{[p]} \longrightarrow E$ is continuous:

$$\begin{aligned}\|T(f)\|_E &= \left\|(T_{[p]} \circ i_{[p]})(f)\right\|_E \leqslant \left\|T_{[p]}\right\| \cdot \left\|i_{[p]}(f)\right\|_{Z(\mu)_{[p]}} \\ &= \left\|T_{[p]}\right\| \cdot \|f\|_{Z(\mu)_{[p]}} = \left\|T_{[p]}\right\| \cdot \left\|\,|f|^{1/p}\right\|^p_{Z(\mu)}.\end{aligned}$$

In other words, (2.2) holds with $C := \left\|T_{[p]}\right\|$. For the converse implication see ([9, page 39]). □

Remark 2.2.3. *[9, Page 211] In the above definition, the extension $T_{[p]}$ of a pth power factorable operator $T : Z(\mu) \longrightarrow E$ to $Z(\mu)_{[p]}$ is unique because $i_{[p]}$ is continuous and $i_{[p]}(Z(\mu))$ is dense in $Z(\mu)_{[p]}$. It is clear that the collection of all E-valued, pth power factorable operators on $Z(\mu)$ equals the set*

$$\mathcal{L}(Z(\mu)_{[p]}, E) \circ i_{[p]} := \left\{S \circ i_{[p]} : S \in \mathcal{L}(Z(\mu)_{[p]}, E)\right\}.$$

In particular, for $p = 1$, the collection of all E-valued, 1-th power factorable operators on $Z(\mu)$ is exactly $\mathcal{L}(Z(\mu), E)$.

Lemma 2.2.4. *[9] Let $Z(\mu)$ be a σ-order continuous BFS and E be a Banach space. Suppose that $1 \leqslant p < \infty$ and $T : Z(\mu) \longrightarrow E$ is a p-th power factorable operator. Given any Banach space G and $S \in \mathcal{L}(E, G)$, the composition $S \circ T : Z(\mu) \longrightarrow G$ is also a p-th power factorable operator and $(S \circ T)_{[p]} = S \circ T_{[p]}$.*

Proof. Since T admits the unique continuous linear extension $T_{[p]} : Z(\mu)_{[p]} \longrightarrow E$, the operator $S \circ T_{[p]} : Z(\mu)_{[p]} \longrightarrow R$ is also a continuous linear extension of $S \circ T$ to $Z(\mu)_{[p]}$. So, $S \circ T$ is p-th power factorable and $(S \circ T)_{[p]} = S \circ T_{[p]}$

□

Let X be a Banach space, the class of all p-th power factorable operators in $\mathcal{L}(Z(\mu), X)$ is denoted by $\mathcal{F}_p(Z(\mu), Y)$, and we denote by $\mathcal{F}_q^{dual}(X, Z(\mu))$ the class of all operators $R \in \mathcal{L}(X, Z(\mu))$ such that $R^* \in \mathcal{F}_q(Z(\mu)^*, X^*)$.

Remark 2.2.5. *We say that $S \in \mathcal{L}(Z(\mu), Y^{**})$ is p-th power factorable if and only if there is some $K > 0$ such that $\|Sf\|_{Y^{**}} \leqslant K \|f\|_{Z(\mu)_{[p]}}$ for all $f \in Z(\mu)$ or, equivalently, $|\langle f, S^* y^{**}\rangle| \leqslant K \|y^{***}\|_{(Y^{**})^*} \|f\|_{Z(\mu)_{[p]}}$ for every $y^{***} \in (Y^{**})^*$ and every $f \in Z(\mu)$. Now assume that $Z(\mu)$ has an OC dual space. Then given $R \in \mathcal{L}(X, Z(\mu))$, we say that R^* is q-th power factorable if and only if there is some $K > 0$ such that $|\langle Rx, g\rangle| \leqslant K \|x\|_X \|g\|_{(Z(\mu)')_{[q]}}$ for every $x \in X$ and every $g \in Z(\mu)' = Z(\mu)^*$.*

Remark 2.2.6. *[9] Given $1 \leqslant p < \infty$, the collection of all p-th power factorable operators from a σ-order continuous BFS. $Z(\mu)$ into a Banach space E is a linear subspace of $\mathcal{L}(Z(\mu), E)$. In fact, if $T, S \in \mathcal{L}(Z(\mu), E)$ are p-th power factorable and $a, b \in \mathbb{C}$, then $(aT + bS)$ is p-th power factorable and $(aT + bS)_{[p]} = aT_{[p]} + bS_{[p]}$.*

2.3 Vector Measures and Integration Operators

Definition 2.3.1. *[9] Let (Ω, Σ) be a Measurable space, and let $m : \Sigma \longrightarrow E$ be a vector measure, that is, it is a σ-additive set function. The variation measure of m, denoted by $|m| : \Sigma \longrightarrow [0, \infty]$, is defined as for scalar measures. Given $x^* \in E^*$, let $\langle m, x^*\rangle : \Sigma \longrightarrow \mathbb{C}$ denote the scalar measure*

$$\langle m, x^*\rangle : A \mapsto \langle m, x^*\rangle (A) = \langle m(A), x^*\rangle, \qquad A \in \Sigma, x^* \in E^*;$$

its variation measure $|\langle m, x^*\rangle| : \Sigma \longrightarrow [0, \infty)$ is necessarily finite. The semivariation $\|m\|$ of m is the set function defined by

$$\|m\| (A) := \sup_{x^* \in B_{E^*}} |\langle m, x^*\rangle| (A), \qquad A \in \Sigma$$

Definition 2.3.2. ***(The space of*** m***-integrable functions:)*** *The space* $L^0(m)$ *is the space of* $\|m\|$ *almost everywhere classes of real function, where* $\|m\|$ *is the semi-variation of* m*. We say that a function* $f \in L^0(m)$ *is* m*-integrable (or* $f \in L^1(m)$ *) if it satisfies the following.*

(i) For each $x^* \in X^*, f \in L^1(|\langle m, x^*\rangle|)$*.*

(ii) There exists $x_0 \in E$ *such that*

$$\langle x_0, x^* \rangle = \int_{\Omega} |f| \, d \langle m, x^* \rangle$$

for every $x^* \in E^*$*.*

Remark 2.3.3. $L^1(m)$ *is Banach space and its norm define by*

$$\|f\|_{L^1(m)} := \sup_{x^* \in B_{E^*}} \int_{\Omega} |f| \, d \langle m, x^* \rangle .$$

Definition 2.3.4. *[9] (**Integration operator**) Let* (Ω, Σ) *be a measurable space and* E *be a complex Banach space. Let* $m : \Sigma \longrightarrow E$ *be a vector measure. Associated with* m *is the integration operator* $I_m : L^1(m) \longrightarrow E$ *defined by*

$$I_m(f) := \int_{\Omega} f dm, \qquad f \in L^1(m);$$

it is clearly linear and continuous. Moreover, its operator norm $\|I_m\| = 1$*.*

Remark 2.3.5. *Let* μ *finite scalar measure and* $0 < p < \infty$*.* $Z(\mu)$ *be a Banach function space and* $Z(\mu)_{[p]}$ *its* p*-th power space of* $Z(\mu)$*.* $L^1(m)$ *is a BFS over* $\langle m, x^* \rangle$ *and* $L^p(m) := L^1(m)_{[1/p]}$*.* $T : Z(\mu) \longrightarrow Y$ *linear operator,* $Z(\mu)$ *order continuous,* T *always factors through* $L^1(m_T)$*, where* $m_T(A) := T(\chi_A)$ *and sometimes* T *factors through* $L^P(m_T)$*.*

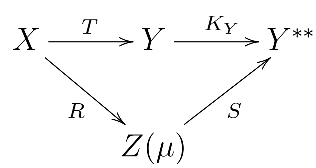
Chapter 3

Extraction theorems for (p,q)-factorable operators

In this chapter, we present the class of $\mathcal{F}_{p,q}$-factorable operators, some characterizations and properties of this class are given. This part based on the article of of **Orlando Galdames-Bravo** (see[5])

3.1 Factorization through p-th power factorable operators

Definition 3.1.1. *Let $1 \leqslant p, q < \infty$, Let X and Y be Banach spaces, and let $T \in \mathcal{L}(X,Y)$. We say that T is $\mathcal{F}_{p,q}$-factorable if there exist a finite measure μ, an OC and Fatou BFS $Z(\mu)$ with OC dual space, and two operators $R \in \mathcal{F}_q^{dual}(X, Z(\mu))$ and $S \in \mathcal{F}_p(Z(\mu), Y^{**})$ such that $\mathcal{K}_Y \circ T = S \circ R$.*



We denote by $\mathcal{F}_{p,q}(X,Y)$ the class of all $\mathcal{F}_{p,q}$-factorable operators in $\mathcal{L}(X,Y)$, endowed with the norm defined as $\varphi_{p,q}(T) := inf\, \|S\|\, \|R\|$, where the infimum is taken over all operators R and S as in the definition above.

Example 3.1.2. *[6] [Hardy type operators] Let $s \geqslant 0$ and consider the Kernel operator H_s with*

Kernel function $K(x,y) := \frac{1}{x^s}\chi_{[0,x]}(y)dy$, *i.e.*

$$(H_s f)(x) = \int_0^1 K(x,y)f(y)dy = \int_0^1 \frac{1}{x^s} f(y)\chi_{[0,x]}(y)dy = \frac{1}{x^s}\int_0^x f(y)dy.$$

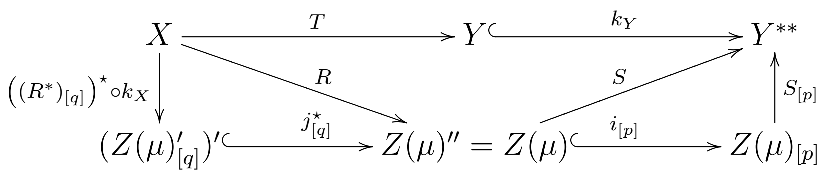
Note that by Hölder's inequality the operator $H_s : L^u[0,1] \longrightarrow L^v[0,1]$ *is always well defined and continuous for* $1 \leqslant v < u$ *when* $s < 1/v - 1/u$ *(in fact, it is continuous in more cases, see for instance [2, Theorem3.10]). Under these restrictions for* u, v *and* s *we can consider the following factorization. For* $f(x) = x_{-s}$ *and the volterra operator* $V : L^u[0,1] \longrightarrow L^u[0,1]$*, we can write*

$$H_s = M_f \circ V : L^u[0,1] \xrightarrow{\quad V \quad} L^u[0,1] \xrightarrow{\quad M_f \quad} L^v[0,1].$$

It is known that V *is p-th power factorable for all* $1 \leqslant p \leqslant u$ *(see [9, Example 5.9]). On the other hand, notice that* $(M_f)^{'} = M_f : L^{v^{'}}[0,1] \longrightarrow L^{u^{'}}[0,1]$ *and for* $g \in L^u[0,1]$ *we have that* $M_f(g) \in L^v[0,1]$*. Take then an index* $1 < t \leqslant v^{'}$ *such that* $s < 1/u^{'} - 1/v^{'}$ *(note that these requirements are compatible with the restrictions on the indexes written above). Then a direct computation using Hölder's inequality gives the continuity of the map* $M_f : L^{v^{'}/t} \longrightarrow L^{u^{'}}$*,i.e.* $(M_f)^{'}$ *is t-th power factorable. Consequently,* H_s *is* (p,t)*-th factorable for all* $1 \leqslant p \leqslant u$*.*

Remark 3.1.3. *Suppose that* $T \in \mathcal{L}(X,Y)$ *factors through a BFS. Then* T *is* $\mathcal{F}_{1,1}$*-factorable since every continuous operator (between the suitable spaces) is 1st power factorable. Observe that we do not need an extension to the 1-st power space, so we do not need the order continuity or Fatou condition. For example, the class of the* $\mathcal{F}_{1,1}$*-factorable operators includes all the p-factorable and p-integrable operators for every* $p \in [1,\infty)$*. The Fatou condition is required to obtain a commutative diagram as in the following remark. For instance,* $\mathcal{F}_{p,1}$*-factorization does not require the BFS in the factorization to be Fatou.*

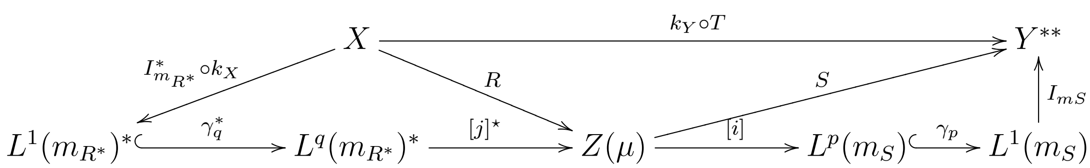
Remark 3.1.4. *From the canonical factorizations of p-th power factorable operators(see[**?**, Section 5.2]) , and taking into account the fact that* $Z(\mu)$ *and* $Z(\mu)^*$ *are OC and* $Z(\mu)$ *is Fatou, we deduce* *the following two factorization schemes:*



(3.1)

where $j_{[q]}$ and $i_{[p]}$ are continuous inclusions. Observe that the order continuity of $Z(\mu)$ and $Z(\mu)^*$ implies that $Z(\mu)^* = Z(\mu)'$ and also $Z(\mu)^{**} = (Z(\mu)^*)^* = (Z(\mu)')^* = Z(\mu)''$. Thus, Fatou and reflexive are the same for $Z(\mu)$. This allows us to identify the operator R with the composition $R^{**} \circ \mathcal{K}_X$. In consequence, the diagram makes sense. Moreover, if $Z(\mu)$ is OC, then so is $Z(\mu)_q$ (see[9, Lemma 2.21]). Thus $((Z(\mu)^*)_{[q]})^* = ((Z(\mu)')_{[q]})'$.

The factorization scheme above is equivalent to the following one



(3.2)

where γ_r denotes the canonical inclusion. Recall that every p-th power factorable operator is r-th power factorable for every $r \in [1, p]$. As a result, T factors through $L^r(m_T)$ for $r \in [1, p]$.

Theorem 3.1.5. *[5] Let $1 \leqslant p, q < \infty$, and let X_0, X, Y_0, and Y be Banach spaces. Then we have the following:*

(i) $\mathcal{F}_{p,q}(X, Y)$ is a linear subspace of $\mathcal{L}(X, Y)$ containing all the finite-rank operators of $\mathcal{L}(X, Y)$. Moreover, $\varphi_{p,q}$ is a Banach space norm on $\mathcal{F}_{p,q}(X, Y)$, and $\|T\| \leqslant \varphi_{p,q}(T)$ for all $T \in \mathcal{F}_{p,q}(X, Y)$.

(ii) The composition of an $\mathcal{F}_{p,q}$-factorable operator with any operator is $\mathcal{F}_{p,q}$-factorable. More formally, if $T \in \mathcal{F}_{p,q}(X, Y), G \in \mathcal{L}(X_0, X)$ and $F \in \mathcal{L}(Y, Y_0)$, then $FTG \in \mathcal{F}_{p,q}(X, Y)$ and $\varphi_{p,q}(FTG) \leqslant \|F\| \, \varphi_{p,q}(T) \, \|G\|$.

The following proposition is an immediate consequence of the definition. It establishes the first inclusion property for $\mathcal{F}_{p,q}$-factorable operators.

Proposition 3.1.6. *Let $1 \leqslant r \leqslant p < \infty$ and $1 \leqslant s \leqslant q < \infty$. Let X and Y be Banach spaces. Then $\mathcal{F}_{p,q}(X,Y) \subseteq \mathcal{F}_{r,s}(X,Y)$, and $\varphi_{r,s}(T) \leqslant \varphi_{p,q}(T)$ for every $T \in \mathcal{F}_{p,q}(X,Y)$.*

Proof. We just prove the norm equality. Let $\varepsilon > 0$ and $T \in \mathcal{F}_{p,q}(X,Y)$. Hence there exist a finite measure μ, a Fatou and OC BFS with OC dual space $Z(\mu)$, and two operators $R \in \mathcal{F}_q^{dual}(X, Z(\mu))$ and $S \in \mathcal{F}_p(Z(\mu), Y^{**})$ such that $\mathcal{K}_Y \circ T = S \circ R$. On the one hand, we know that $S = S_{[r]} \circ i_{[r]} = S_{[p]} \circ i_{[p]}$ and also that $R^* = R^*_{[s]} \circ j_{[s]} = R^*_{[q]} \circ j_{[q]}$. On the other hand, from the diagram abov, we can choose R and S so that

$$
\begin{aligned}
\varphi_{p,q}(T) + \varepsilon \geqslant \|R\| \, \|S\| &= \left\|R^*_{[q]} \circ j_{[q]}\right\| \left\|S_{[q]} \circ i_{[p]}\right\| \\
&= \left\|R^*_{[s]} \circ j_{[s]}\right\| \left\|R_{[r]} \circ i_{[r]}\right\| \geqslant \varphi_{r,s}(T),
\end{aligned}
$$

obtaining the inequality from the arbitrary choice of $\varepsilon > 0$. □

As we have seen in the preliminary section, several characterizations of p-th power factorable operators exist. We present below some of these characterizations in order to provide some new ones for $\mathcal{F}_{p,q}$-factorable operators.

Proposition 3.1.7. *Let $1 \leqslant p, q < \infty$, let X and Y be Banach spaces, and let $T \in \mathcal{L}(X,Y)$. Then the following statements are equivalent.*

(i) $T \in \mathcal{F}_{p,q}(X,Y)$.

(ii) There exist a finite measure μ, an OC and Fatou BFS $Z(\mu)$, with OC dual space, and two operators $F \in \mathcal{L}(X, ((Z(\mu)')_{[q]})$ and $G \in \mathcal{L}(Z(\mu)_{[p]}, Y^{**})$ such that $\mathcal{K}_Y \circ T = G \circ i_{[p]} \circ (j_{[q]})^* \circ F$, where $i_{[p]}$ and $j_{[q]}$ are the inclusions into the p-th and q-th power spaces of $Z(\mu)$ and $Z(\mu)'$, respectively.

(iii) *There exist a finite measure μ, an OC and Fatou BFS $Z(\mu)$, with OC dual space, and two operators $R \in \mathcal{L}(X, Z(\mu))$ and $S \in \mathcal{L}(Z(\mu), Y^{**})$ such that $\mathcal{K}_Y \circ T = S \circ R$ and the diagram*

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \xhookrightarrow{\mathcal{K}_Y} & Y^{**} \\ {\scriptstyle F}\downarrow & & & & \uparrow{\scriptstyle G} \\ L^q(m_{R^*})^* & & \xrightarrow{\quad H \quad} & & L^p(m_S) \end{array}$$

commutes, where F and G are bounded operators, $H = [i] \circ [j]^*$, and $[i]$ and $[j]$ denote the inclusion/quotient maps.

Proof. $(i) \Longrightarrow (ii)$: This is clear from diagram (3.1).

$(ii) \Longrightarrow (iii)$: We use the characterization of p-th power factorable operator given in [6, Lemma 3.3] to obtain the factorization

$$T : X \xrightarrow{I^*_{m_{R^*}} \circ \mathcal{K}_X} L^q(m_{R*})^* \xrightarrow{[i]} Z(\mu) \xrightarrow{I_{m_S}} Y^{**}$$

where $[j]$ and $[i]$ are not inclusions necessarily.

$(iii) \Longrightarrow (i)$: By hypothesis, it follows that $Z(\mu) \hookrightarrow^{[i]} L^p(m_S)$ and $Z(\mu)' \hookrightarrow^{[i]} L^q(m_{R^*})$. Hence, the characterization in [6, Lemma 3.3] again implies (i) □

Note that characterization (ii) of the above proposition coincides with the characterization of (p, q)-factorable operators when we take $Z(\mu) := L^c(\mu)$ for a suitable $c \in [1, \infty)$ (see[3, Theorem18.11]).

Theorem 3.1.8. *Let $1 \leqslant p, q < \infty$, let X and Y be Banach spaces such that Y and Y^* are reflexive, and let $T \in \mathcal{F}(X, Y)$. Then the following two statements are equivalent.*

(i) *$T \in \mathcal{F}_{p,q}(X, Y)$.*

(ii) *There exist a finite measure μ, an OC and Fatou BFS $Z(\mu)$, with OC dual space, a constant $K > 0$, and operators $R \in \mathcal{L}(X, Z(\mu))$ ans $S \in \mathcal{L}(Z(\mu), Y^{**})$ such that $\mathcal{K}_Y \circ T = S \circ R$ and*

$$|\langle Tx, y^* \rangle| \leqslant K(\|x\|_X \, \|(S^* \circ \mathcal{K}_{Y^*})y^*\|_{(Z(\mu)')_{[q]}})^\theta (\|y^*\|_{Y^*} \, \|Rx\|_{Z(\mu)_{[p]}})^{1-\theta},$$

for every $x \in X, y^* \in Y^*$ and every $\theta \in (0,1)$. The (i)$\Longrightarrow$ (ii).

If in (ii) we have that $R(X)$ is dense in $Z(\mu)$ and $S^* \circ \mathcal{K}_{Y^*}(Y^*)$ is dence in $Z(\mu)'$, then (ii)$\Longrightarrow$ (i).

Proof. (i)$\Longrightarrow$(ii): Let $x \in X$ and $y^* \in Y^*$. From the factorization $\mathcal{K}_Y \circ T = S \circ R$, the characterization of s-th power factorable operators given in Remark (2.2.5), and taking into account the fact that $y^* = (\mathcal{K}^*{}_Y \circ \mathcal{K}_{y^*})y^*$, by reflexivity of Y and Y^*, we get:

$$\begin{aligned}
|\langle Tx, y^*\rangle| &= |\langle Tx, (\mathcal{K}^*{}_Y \circ \mathcal{K}_{y^*})y^*\rangle| = |\langle \mathcal{K}_Y \circ Tx, \mathcal{K}_{y^*}y^*\rangle| \\
&= |\langle Rx, (S^* \circ \mathcal{K}_{Y^*})y^*\rangle|^{\theta} \, |\langle Rx, S^*(\mathcal{K}_{Y^*}y^*)\rangle|^{1-\theta} \\
&\leqslant K \left(\|x\|_X \, \|(S^* \circ \mathcal{K}_{Y^*})y^*\|_{(Z(\mu)')_{[q]}}\right)^{\theta} \left(\|y^*\|_{Y^*} \, \|Rx\|_{z(\mu)_{[p]}}\right)^{1-\theta},
\end{aligned}$$

for some $K \geqslant \left(\left\|R^*_{[q]}\right\|\right)^{\theta} \left(\left\|S_{[p]}\right\| \|\mathcal{K}_{Y^*}\|\right)^{1-\theta}$ and for every $\theta \in (0,1)$.

(ii)$\Longrightarrow$ (i):This follows from the next property of the exponential map. We claim that $0 \leqslant a \leqslant b^{\theta} \cdot c^{1-\theta}$ for every $\theta \in (0,1)$ implies that $a \leqslant b$ and $a \leqslant c$.

Assume that $b \leqslant c$, thus $a \leqslant c$. Moreover, $b^{\theta} \cdot c^{1-\theta}$ is a decreasing function of θ. Hence, by continuity of such a function, we have that $a \leqslant \inf\left\{b^{\theta} \cdot c^{1-\theta} : \theta \in (0,1)\right\} = b$. Analgously, we get the same result assuming that $c \leqslant b$. By applying this property, the characterization given in Remark (2.2.5), and the density hypothesis, we obtain our result. Let $\varepsilon > 0$, and fix $f \in Z(\mu)'$ and $x^* \in X^*$. By density of $(S^* \circ \mathcal{K}_{Y^*})(Y^*)$, there exists some $y_0^* \in Y^*$ such that $\|f - (S^*(\mathcal{K}_{Y^*}))y_0^*\|_{Z(\mu)'} < \varepsilon/2$. Thus, there exists some $K_0 > 0$ such that :

$$\begin{aligned}
|\langle Rx, f\rangle| &\leqslant |\langle Rx, (S^* \circ \mathcal{K}_{Y^*})y_0^*\rangle| + |\langle Rx, f - (S^* \circ \mathcal{K}_{Y^*})y_0^*\rangle| \\
&\leqslant K_0 \|x\|_X \left(\varepsilon/2 + \|(S^* \circ \mathcal{K}_{Y^*})y_0^*\|_{(Z(\mu)')_{[q]}}\right) \\
&\leqslant K_0 \|x\|_X \left(\varepsilon/2 + \|(S^* \circ \mathcal{K}_{Y^*})y_0^* - f\|_{(Z(\mu)')_{[q]}} + \|f\|_{(Z(\mu)')_{[q]}}\right) \\
&\leqslant K_0 \|x\|_X \left(\varepsilon + \|f\|_{(Z(\mu)')_{[q]}}\right).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary for a fixed $f \in Z(\mu)'$, we have that R^* is q-th power factorable. To show that S is p-th power factorable, we just take into account the fact that $(\mathcal{K}^*{}_Y \circ \mathcal{K}_{Y^*})y^* = y^* \in Y^* = Y^{**}$ and proceed in an analogous way. □

Proposition 3.1.9. *Let $1 \leqslant p, q < \infty$. Let X and Y be Banach spaces such that Y is reflexive. Then $\mathcal{F}_{p,q}(X,Y) = \mathcal{F}^{dual}_{q,p}(X,Y)$.*

Proof. Let X and Y be Banach spaces, and let $T \in \mathcal{F}_{p,q}(X,Y)$. Then there exist a finite measure μ and an OC and Fatou BFS $Z(\mu)$, with OC dual space, such that

$$\mathcal{K}_{X^*} \circ T^* : Y^* = (Y^{**})^* \xrightarrow{S^*} Z(\mu)' \xrightarrow{R^*} X^* \xrightarrow{\mathcal{K}_{X^*}} (X^*)^{**}$$

for some operators R and S, where $\mathcal{K}_{X^*} \circ R^*$ is q-th power factorable. Now, taking into account the fact that Y is reflexive and $Z(\mu)$ is so (thanks to the order continuity and Fatou properties), we have that $S^{**} = S$, hence it is p-th power factorable. This means that $T^* \in \mathcal{F}_{q,p}(Y^*, X^*)$. The conditions on $Z(\mu)'$ also hold, because it is OC and the dual $Z(\mu)'' = Z(\mu)$ is so by hypothesis. Moreover, $Z(\mu)'$ is also Fatou since $(Z(\mu)')'' = (Z(\mu)'')' = Z(\mu)'$ and $Z(\mu)''$ is also OC since it coincides with $Z(\mu)$. With this and by the same process, we obtain the other inclusion. The equality of norms is trivially fulfilled. □

Remark 3.1.10. *As we have seen, there are several situations where we need reflexivity properties of the range or its dual. This is due to the definition of $\mathcal{F}_{p,q}$-factorable operator, in which we have included a factorization through an operator $S \in \mathcal{F}_p(Z(\mu), Y^{**})$. This bidual as a range space has a main role in the following section, in which we relate this class with the class of (p,q)-factorable operators.*

3.2 Extrapolation theorems

In the beginning of this Chapter, we presented the ideal of $\mathcal{F}_{p,q}$-factorable operators. We now show that every $\mathcal{F}_{p,q}$-factorable operator between finite-dimensional Banach spaces is 1-summing. From the characterization of (p,q)-factorable operators by means of the embedding $L^{q'}(\mu) \hookrightarrow L^p(\mu)$, it is not hard to show the first inclusion relation between such operators and the $\mathcal{F}_{p,q}$-factorable operators.

Proposition 3.2.1. *Let* $1 \leqslant p,q,r,s < \infty$ *such that* $1/(pr)+1/(qs) \geqslant 1$*. Let* X *and* Y *be Banach spaces. Then* $\mathcal{L}_{p,q}(X,Y) \subseteq \mathcal{F}_{r,s}(X,Y)$ *and* $\varphi_{r,s} \leqslant \alpha_{p,q}$*. In particular, if* $1/p^2+1/q^2 \geqslant 1$*, then* $\mathcal{L}_{p,q}(X,Y) \subseteq \mathcal{F}_{r,s}(X,Y)$*.*

Proof. Let $T \in \mathcal{L}_{p,q}(X,Y)$. Then there exists a probability measure μ such that:

$$T : X \xrightarrow{R_0} L^{q'}(\mu) \xhookrightarrow{I} L^p(\mu) \xrightarrow{S_0} Y^{**},$$

where I is the canonical inclusion. Choosing any $c \in [pr,(qs)']$, we have the factorization for such an inclusion map:

$$I : L^{q'}(\mu) \xhookrightarrow{I_1} L^{(c'/s)'}(\mu) \xhookrightarrow{I_2} L^c(\mu) \xhookrightarrow{I_3} L^{c/r}(\mu) \xhookrightarrow{I_4} L^p(\mu).$$

Since $L^{u/v}(\mu) = L^u(\mu)_{[v]}$, we have that $R := I_2 \circ I_1 \circ R_0 \in \mathcal{F}_s^{dual}(X, L^c(\mu))$ and $S := S_0 \circ I_4 \circ I_3 \in \mathcal{F}_r(L^c(\mu), Y^{**})$ (see[9, Lemma5.4]). Let $\varepsilon > 0$, and choose R_0 and S_0 so that

$$\begin{aligned} \alpha_{p,q}(T) + \varepsilon &\geqslant \|S_0\|\,\|I\|\,\|R_0\| \\ &= \|S_0\|\,\|I_4 \circ I_3\|\,\|I_2 \circ I_1\|\,\|R_0\| \\ &\geqslant \varphi_{r,s}(T). \end{aligned}$$

The equality of the norms of the inclusion maps is given by the choice of μ as a probability measure. ☐

With this inclusion property and conditions for compactness of the operators involved, we can obtain some extrapolation results. For example, when the operators is over finite-dimensional Banach spaces, such operators are actually absolutely summing.

Proposition 3.2.2. *Let* $1 \leqslant p, q < \infty$ *be such that* $1/p + 1/q \geqslant 1$*, and let* E *and* F *be finite-dimensional spaces. Then* $\mathcal{F}_{p,q}(E,F) \subseteq \Pi_1(E,F)$ *and* $\pi_1 \leqslant \varphi_{p,q}$.

Proof. Let $T \in \mathcal{F}_{p,q}(E,F)$. Hance there exist a BFS $Z(\mu)$ and two operators $R \in \mathcal{F}_q^{dual}(E, Z(\mu))$ and $S \in \mathcal{F}_p(Z(\mu), F)$ such that $T = S \circ R$. Observe that the vector measures m_{R^*} and m_S take values in the finite-dimensional spaces E^* and F, respectively. This implies that $L^q(m_{R^*})^* = L^{q'}(|m_{R^*}|)$ and $L^p(m_S) = L^p(|m_S|)$ order isomorphically (see, e.g.,[9, Remark 3.17]). In consequence, taking into account diagram (3.2) and Maurey's factorization theorem(see, e.g.,[3, theorem 18.9]), there exists a probability measure μ_0 such that T factors through the embedding $L^\infty(\mu_0) \hookrightarrow L^1(\mu_0)$; that is, $T \in \mathcal{L}_{1,1}(E,F) = \mathcal{I}_1(E,F) \subseteq \Pi_1(E,F)$. Let $\varepsilon > 0$, and take R and S as above, that satisfies (3.2) and also

$$
\begin{aligned}
\varphi_{p,q}(T) + \varepsilon \geqslant \varphi_{1,1}(T) + \varepsilon \geqslant \|S\| \, \|R\| &= \|[i]\| \, \|[j^*]\| \\
&= \|I_{m_s}\| \, \|[i]\| \, \|[j^*]\| \left\| I_{m^*_{R^*}} \right\| = \|I_{m_S}\| \, \|F\| \, \|I\| \, \|G\| \left\| I^*_{m_{R^*}} \right\| \\
&\geqslant \|I_{m_S} \circ F\| \, \|I\| \left\| G \circ I^*_{m_{R^*}} \right\| \geqslant \alpha_{1,1}(T) = i_1(T) \geqslant \pi_1(T),
\end{aligned}
$$

Where $L^\infty(|m_{R^*}|) \longrightarrow^G L^\infty(\mu_0) \hookrightarrow^I L^1(\mu_0) \longrightarrow^F L^1(|m_S|)$. Recall (see Proposition 3.1.6, [3, Theorem 18.9]) that $\|[i]\| = \|S\| \, \|[j]\| = \|R\|$, and $\|I_{m_S}\| = \|I_{m_{R^*}}\| = 1$. □

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