PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESERACH

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Faculty of Mathematics and Informatics
Departement of Mathematics

Domain : Mathematics and Informatics
Field: Mathematics
Specialty: Partial Differential Equations and Applications

Existence results for a class of fractional p-Kirchhoff problems

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University years 2024/2025

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Dedication

All praise is due to Allah, Lord of the Worlds. May peace and blessings be upon our Prophet Muhammad, his family, and his companions.

I dedicate this work to my noble parents, whose unwavering support and sincere sacrifices have had the greatest impact on my academic journey. To them, I offer my highest expressions of gratitude and appreciation.

I also dedicate it to my beloved family, who have always been a source of strength through their generosity, patience, and unconditional love.

This work is likewise dedicated to my esteemed professors, whose academic guidance and example in diligence and dedication have profoundly shaped my scholarly path.

I sincerely ask Allah, the Almighty, to accept this effort, make it purely for His sake, and grant benefit through it to all who read it. I also pray that He blesses the people of knowledge and guides them to all that is good—for he alone is the All-Powerful, the All-Wise.

Introduction

Fractional calculus, whose origins trace back to a 1695 correspondence between Leibniz and L'Hôpital, has evolved into a rich and dynamic area of research. While initially perceived as paradoxical, it now finds applications across various fields such as physics, biology, and engineering. Despite its widespread use, the precise definition of a fractional derivative remains under debate among researchers [1].

Although the foundational tools of fractional calculus have existed for centuries, the rigorous study of fractional differential equations began more recently [3], [10], [9], [19]. In particular, equations involving Riemann–Liouville operators play a significant role in modeling physical phenomena [7], [8], [16], [20], [23]. One of the earliest systematic studies was presented by Joseph Liouville in 1832, laying essential groundwork for the field.

The Caputo derivative [3] has become especially prominent due to its physically meaningful initial conditions and its property that the derivative of a constant is zero.

Fractional derivatives are applied in various domains, including viscoelasticity, neuroscience, electrochemistry, control theory, porous media, and electromagnetism.

In parallel, **Critical Point Theory and Variational Methods** offer powerful tools for analyzing nonlinear problems through functional minimization, intersecting with areas like differential equations, calculus of variations, and mathematical physics.

Additionally, in 1845, **Gustav Kirchhoff** [2] formulated his famous circuit laws, forming a bridge between electrical engineering and applied mathematics. These laws rely on principles of differential equations, linear algebra, and complex analysis, and play a fundamental role in analyzing both DC and AC electrical networks.

My mémoire consists of three chapters. In Chapter One, we provided an overview of fractional integrals and fractional derivatives, highlighting their mathematical foundations. The chapter included essential definitions and key properties that form the basis for the subsequent theoretical developments and applications discussed in later chapters. In Chapter Two, we considered the p-Laplacian problem as an example to illustrate these concepts and explore their practical applications. In Chapter Three, we generalized the results to the p-Kirchhoff case.

Chapter 1

Preliminaries

Introduction 1.1 *In this chapter, we introduce the fundamental concepts relevant to our study. In particular, we provide the definitions of the fractional integral and derivative operators-namely, the Riemann–Liouville and Caputo operators—along with their main properties. In addition, we define the space of absolutely continuous functions in which our problems are studied, and present its fundamental properties.*

1.1 Fractional calculus

In this section we introduce the definitions of the Riemann-Liouville fractional integrals and fractional derivatives.

1.1.1 Riemann-Liouville fractional integrals

Let $[a, b]$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I^{\alpha}_{a+}f$ and $I^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{R}$ $(\alpha>0)$ are defined by:

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{f(t)dt}{(x-t)^{1-\alpha}}, \quad (x > a, \alpha > 0). \tag{1.1}$$

and

$$(I^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{b}\frac{f(t)dt}{(t-x)^{1-\alpha}}, \quad (x < b, \alpha > 0). \tag{1.2}$$

Here $\Gamma(\alpha)$ is the Gamma function, defined by:

$$\Gamma(\alpha) = \int_{0}^{+\infty} t^{\alpha-1}e^{-t}dt, \quad (\alpha \in \mathbb{C}, Re(\alpha) > 0). \tag{1.3}$$

with: $t^{\alpha-1}{=}e^{(\alpha-1)\ln t}$.

Remark 1.1 *When* $\alpha= n \in \mathbb{N}$, *the definitions* (1.1) *and* (1.2) *coincide with the nth integrals of the form:*

$$\begin{aligned}(I^{n}_{a+}f)(x) &= \int_{a}^{x}dt_1\int_{a}^{t_1}dt_2 ... \int_{a}^{t_{n-1}}f(t_n)dt_n \\ &= \frac{1}{(n-1)!}\int_{a}^{x}(x-t)^{n-1}f(t)dt \ (n \in \mathbb{N}) \end{aligned}\tag{1.4}$$

and

$$\begin{aligned}(I^n_{b-}f)(x) &= \int_x^b dt_1 \int_{t_1}^b dt_2 ... \int_{t_{n-1}}^b f(t_n)dt_n \\ &= \frac{1}{(n-1)!}\int_x^b (t-x)^{n-1} f(t)dt \ (n \in \mathbb{N}). \qquad (1.5)\end{aligned}$$

1.1.2 Riemann-Liouville Fractional derivatives

The Riemann-Louville fractional derivative $D^\alpha_{a+}y$ and $D^\alpha_{b-}y$ of order $\alpha \in \mathbb{R}$ ($\alpha \geq 0$) are defined by:

$$\begin{aligned}(D^\alpha_{a+}y)(x) &= \left(\frac{d}{dx}\right)^n \circ (I^{n-\alpha}_{a+}y)(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{y(t)dt}{(x-t)^{\alpha-n+1}}, (n = [\alpha]+1; x > a). \qquad (1.6)\end{aligned}$$

and

$$\begin{aligned}(D^\alpha_{b-}y)(x) &= \left(\frac{-d}{dx}\right)^n \circ (I^{n-\alpha}_{b-}y)(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{-d}{dx}\right)^n \int_x^b \frac{y(t)dt}{(t-x)^{\alpha-n+1}}, (n = [\alpha]+1; x < b). \qquad (1.7)\end{aligned}$$

In particulare case, when α = n $\in \mathbb{N}_0$, we have:

$$(D^0_{a+}y)(x) = (D^0_{b-}y)(x) = y(x), \quad (D^n_{a+}y)(x) = y^{(n)}(x),$$

$$(D^n_{a+}y)(x) = y^{(n)}(x), \quad (D^n_{b-}y)(x) = (-1)^n y^{(n)}(x), n \in \mathbb{N}.$$

Lemma 1.1 (*a*) *The fractional integration operators* I^α_{a+} *and* I^α_{b-} *with* $\alpha > 0$ *are bounded in* L_p*(a,b)* ($1 \leq p \leq \infty$)*, and we have:*

$$\|I^\alpha_{a+}f\|_p \leq K\|f\|_p, \quad \|I^\alpha_{b-}f\|_p \leq K\|f\|_p \ \ with \ \left(K = \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right) \qquad (1.8)$$

(*b*) *If* $0 < \alpha < 1$ *and* $1 < p < \frac{1}{\alpha}$*, then the operators* I^α_{a+} *and* I^α_{b-} *are bounded from* L_p*(a,b) into* L_q*(a,b), where* $q = \frac{p}{1-\alpha p}$.

Proof. the assertion (a) was poved in Samko et al. [[22] (theorem 2.6)], while the assertion is known as the Hardy-Littlewood theorem, see [[22] et al, (theorem 3.5)]. ■

1.1.3 Space of absolutely continuous functions:

Definition 1.1 *Let* $\Omega = [a, b], (-\infty \leq a < b \leq \infty)$ *be a finite interval of* $\mathbb{R}$ *and* $n \in \mathbb{N}$.
AC(Ω) *be the space of functions f which are absolutely continuous on* Ω*(space of primitive functions f); i.e.*

$$f \in AC(\Omega) \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt \ (\varphi \in L(\Omega)). \qquad (1.9)$$

With: $c = f(a)$.

Definition 1.2 *For $n \in \mathbb{N}^*$, we denot by $AC^n[a,b]$ the space of complex valued functions f(x) which have continuous derivatives up to order $(n-1)$ and such that $f^{(n-1)}(x) \in AC(\Omega)$:*
$AC(\Omega) = \{\ f : [a,b] \longrightarrow \mathbb{R}, D^{(n-1)}f(x) \in AC[a,b], D = \frac{d}{dx}\}$
In particular, if $n = 1$; we have $AC^1(\Omega) = AC(\Omega)$.

Lemma 1.2 *Let $\alpha \geq 0$, and $n = [\alpha]+1$. If $y \in AC^n[a,b]$, then the fractional derivatives $D^\alpha_{a+}y$ and $D^\alpha_{b-}y$ exist almost everywhere on $[a,b]$ and can be represented in the forms:*

$$(D^\alpha_{a+}y)(x) = \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(1+k-\alpha)}(x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} \tag{1.10}$$

$$(D^\alpha_{b-}y)(x) = \sum_{k=0}^{n-1} \frac{(-1)^k y^{(k)}(b)}{\Gamma(1+k-\alpha)}(b-x)^{k-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{y^{(n)}(t)dt}{(t-x)^{\alpha-n+1}}, \tag{1.11}$$

respectively.

Corollary 1.1 *If $0 < \alpha < 1$ and $y \in AC[a,b]$, then:*

$$(D^\alpha_{a+}y)(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{y(a)}{(x-a)^\alpha} + \int_a^x \frac{y'(t)dt}{(x-t)^\alpha} \right] \tag{1.12}$$

and

$$(D^\alpha_{b-}y)(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{y(b)}{(b-x)^\alpha} - \int_x^b \frac{y'(t)dt}{(t-x)^\alpha} \right] \tag{1.13}$$

Remark 1.2 *The relations (1.12) and (1.13) were proved in Samko et al. [[22] (Lemma 2.2 and 2.3)].*

Lemma 1.3 *If $\alpha > 0$, and $f \in L_p(a,b)$ $(1 \leq p \leq \infty)$, then the following equalities:*

$$(D^\alpha_{a+}I^\alpha_{a+})(x) = f(x) \text{ and } (D^\alpha_{b-}I^\alpha_{b-})(x) = f(x),\ (\alpha > 0) \tag{1.14}$$

hold almost every where on $[a,b]$.

Property 1.1 *Let $\alpha \geq 0$, $m \in \mathbb{N}$ and $D = d/dx$.*
(a)If the fractional derivatives $(D^\alpha_{a+})(x)$ and $(D^{\alpha+m}_{a+})(x)$ exist, then:

$$(D^m D^\alpha_{a+}y)(x) = (D^{\alpha+m}_{a+}y)(x). \tag{1.15}$$

(b)If the fractional derivatives $(D^\alpha_{b-})(x)$ and $(D^{\alpha+m}_{b-})(x)$ exist, then:

$$(D^m D^\alpha_{b-}y)(x) = (-1)^m (D^{\alpha+m}_{b-}y)(x). \tag{1.16}$$

To present the next property, we use the spaces of functions $I^\alpha_{a+}(L^p)$ and $I^\alpha_{b-}(L^p)$ defined for $\alpha > 0$ and $1 \leq p \leq \infty$ by:

$$I^\alpha_{a+} := f : f = I^\alpha_{a+}\varphi, \varphi \in L^p(a,b) \tag{1.17}$$

and

$$I^\alpha_{b-} := f : f = I^\alpha_{b-}\phi, \phi \in L^p(a,b) \tag{1.18}$$

respectively.

Lemma 1.4 *Let $\alpha > 0$, $n = [\alpha] + 1$ and let $f_{n-\alpha}(x) = (I_{a+}^{n-\alpha} f)(x)$ be the fractional integral (1.1) of order $n - \alpha$.*
(a) If $1 \leq p \leq \infty$ and $f \in I_{a+}^{\alpha}(L^p)$, then:

$$(I_{a+}^{\alpha} D_{a+}^{\alpha} f)(x) = f(x). \tag{1.19}$$

(b) If $f \in L^1(a, b)$ and $f_{n-\alpha} \in AC^n[a, b]$, then the equality:

$$(I_{a+}^{\alpha} D_{a+}^{\alpha} f)(x) = f(x) - \sum_{j=1}^{n} \frac{f_{n-\alpha}^{n-j}(a)}{\Gamma(\alpha - j + 1)} (x - a)^{\alpha - j}, \tag{1.20}$$

hold almost everywhere on $[a, b]$.
In particular, If $0 < \alpha < 1$, then:

$$(I_{a+}^{\alpha} D_{a+}^{\alpha} f)(x) = f(x) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)} (x - a)^{\alpha - 1}, \tag{1.21}$$

Where $f_{1-\alpha}(x) = (I_{a+}^{1-\alpha} f)(x)$, while for $\alpha = n \in \mathbb{N}$, the following equality holds:

$$(I_{a+}^{n} D_{a+}^{n} f)(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x - a)^{k}. \tag{1.22}$$

Property 1.2 *Let $\alpha > 0$ and $\beta > 0$ be such that $n - 1 < \alpha \leq n$, $m - 1 < \beta \leq m$. $(n, m \in \mathbb{N})$ and $\alpha + \beta < n$, and let $f \in L^1(a, b)$ and $f_{m-\alpha} \in AC^m[a, b]$. Then we have the following index rule:*

$$(D_{a+}^{\alpha} D_{a+}^{\beta} f)(x) = (D^{\alpha+\beta} f)(x) - \sum_{j=1}^{m} (D_{a+}^{\beta - j} f)(a+) \frac{(x - a)^{-j-\alpha}}{\Gamma(1 - j - \alpha)}. \tag{1.23}$$

Lemma 1.5 *Let $\alpha > 0$, $p \geq 1$, $q \geq 1 \geq$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$)*
(a)

$$If : \varphi(x) \in L^p(a, b) \text{ and } \psi(x) \in L^q[a, b]$$

then:

$$\int_a^b \varphi(x)(I_{a+}^{\alpha} \psi)(x) dx = \int_a^b \psi(x)(I_{b-}^{\alpha} \varphi)(x) dx \tag{1.24}$$

(b)

$$If : f(x) \in I_{b-}^{\alpha}(L^p) \text{ and } g(x) \in I_{a+}^{\alpha}(L^q)$$

then:

$$\int_a^b f(x)(D_{a+}^{\alpha} g)(x) dx = \int_a^b g(x)(D_{b-}^{\alpha} f)(x) dx \tag{1.25}$$

Definition 1.3 *Let $0 < \alpha \leq 1$ and $1 < p < \infty$.The fractional derivative the space $E_0^{\alpha,p}$ is defined by the closure of $C_0^{\infty}([0, T], R^N)$ with respect to the norm*

$$\|u\|_{\alpha,p} = \left(\int_0^T |u(t)|^p dt + \int_0^T |{}_0D_t^{\alpha} u(t)|^p dt \right)^{\frac{1}{p}}, \forall u \in E_0^{\alpha,p}. \tag{1.26}$$

It is obvious that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u \in L^p([0, T], R^N)$ having an α-order fractional derivative $D_t^{\alpha} u \in L^P([0, T], R^N)$ and $u(0) = u(T) = 0$, Furthermore, it is easy to verify that $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

Proposition 1.1 *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

Proof. In fact , since $L^p([0,T], R^N)$ is reflexive and separable, The Cartesian product space $L_2^p([0,T], R^N) = L^p([0,T], R^N) \times L^p([0,T], R^N)$
is also a reflexive and separable Banach space with respect to the norm:

$$\|v\|_{L_2^p} = \left(\sum_{i=1}^{2} \| v_i \|_{L^p}^p \right)^{\frac{1}{p}}, \tag{1.27}$$

where $v = (v_1, v_2) \in L_2^p([0,T], R^N)$.
Consider the space $\Omega = (u, D_{0+}^\alpha u) : \forall u \in E_0^{\alpha,p}$, which is a closed subset of $L_2^p([0,T], R^N)$ as $E_0^{\alpha,p}$ is closed. There for, Ω is also a reflexive and separable Banach space with the norm (1.27) for $v = (v_1, v_2) \in \Omega$.
We form the operator $A : E_0^{\alpha,p} \longrightarrow \Omega$ as follows:

$$A : u \longrightarrow (u, D_{a+}^\alpha u), \forall u \in E_0^{\alpha,p}.$$

It is obvious that

$$\|u\|_{\alpha,p} = \|Au\|_{L_2^p},$$

Which means that the operator $A : u \longrightarrow (u, D_{0+}^\alpha)$ is a isometric isomorphic mapping and the space $E_0^{\alpha,p}$ is isometric isomorphic to the space Ω. Thus $E_0^{\alpha,p}$ is a reflexive and separable Banach space, and this completes the proof. ■

Lemma 1.6 (Kilbas et al [14]., 2006;Somko et al [23], 1993;Jiao & Zhou [12], 2011) . *Let $0 < \alpha \leq 1$ and $1 \leq p \leq \infty$.*

$$\|I_{0+}^\alpha f\|_{L^p([0,t])} \leq \frac{t^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p([0,t])}, \quad for\ t \in [0,T] \tag{1.28}$$

Proposition 1.2 *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, if $\alpha > \frac{1}{p}$, we have $I_{0+}^\alpha(D_{0+}^\alpha u(t)) = u(t)$. Moreover, we can get that $E_0^{\alpha,p} \subset C_0([0,T], R^N)$.*

Remark 1.3 *In the case that $1 - \alpha \geq \frac{1}{p}$, for any $u \in E_0^{\alpha,p}$, we also have $D_{0+}^{-\alpha}(D_{0+}^\alpha u(t)) = u(t)$. In fact, set $f(t) = D_{0+}^{\alpha-1} u(t)$. According to property, we only need to prove that $f(0) = [D_{0+}^{\alpha-1} u(t)]_{t=0} = 0$. Noting that $1 - \alpha \geq \frac{1}{p}$, by using Hölder inequality, Lemma 6 and the similar method in the proof of $[\ [11], 2004, Lemma 7]$, we can obtain the desired result, i.e. $f(0) = 0$. We skip the proof since it is similar to $[\ [11], 2004, Lemma 7]$*

Proposition 1.3 *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$,if $1 - \alpha \geq \frac{1}{p}$ or $\alpha > \frac{1}{p}$ we have:*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|D_{0+}^\alpha u\|_{L^p}, \tag{1.29}$$

If $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then:

$$\|u\|_\infty \leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{p}}} \left\| D_{0+}^\alpha u \right\|_{L^p}. \tag{1.30}$$

Proof. In order to prove inequalities (1.29) and (1.30) we only need to prove that:

$$\|I^{\alpha}_{0+}(D^{\alpha}_{0+}u)\|_{L^p} \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)}\|D^{\alpha}_{0+}u\|_{L^p}, \tag{1.31}$$

for $1-\alpha > \frac{1}{p}$ or $\alpha > \frac{1}{p}$, and

$$\|I^{\alpha}_{0+}(D^{\alpha}_{0+}u)\|_{\infty} \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\|D^{\alpha}_{0+}u\|_{L^p}. \tag{1.32}$$

for $\alpha > \frac{1}{p}$ and $\frac{1}{p}+\frac{1}{q}=1$ according to Proposition (1.2) and Remark (1.3). Firstly, we not that $D^{\alpha}_{0+}u \in L^p([0,T], R^N)$, the inequality (1.31) follows from (1.28) directly.
We ar now a position to prove (1.32). For $\alpha > \frac{1}{p}$, choose q such that $\frac{1}{p}+\frac{1}{q}=1$.
$\forall u \in E^{\alpha,p}_0$, we have:

$$\begin{aligned}
|I^{\alpha}_{0+}(D^{\alpha}_{0+}u(t))| &= \frac{1}{\Gamma(\alpha)}|\int_0^t (t-s)^{\alpha-1}D^{\alpha}_{0+}u(s)ds| \\
&\leq \frac{1}{\Gamma(\alpha)}(\int_0^t (t-s)^{(\alpha-1)q}ds)^{\frac{1}{q}}\|D^{\alpha}_{0+}u\|_{L^p} \\
&\leq \frac{T^{\frac{1}{q}+\alpha-1}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\|D^{\alpha}_{0+}u\|_{L^p},
\end{aligned}$$

and this completes the poof. ■

1.1.4 Caputo Fractional derivatives

Definition 1.4 *The Caputo fractional derivatives* ${}^cD^{\alpha}_{a+}y$ *and* ${}^cD^{\alpha}_{b-}y$ *of order* $\alpha \in \mathbb{R}(\alpha \geq 0)$ *are defined by:*

$$\begin{aligned}
({}^cD^{\alpha}_{a+}y)(x) &= (I^{n-\alpha}_{a+}) \circ \left(\frac{d}{dx}\right)^n y(x) \\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{y^{(n)}(t)dt}{(x-t)^{\alpha-n+1}}, (n=[\alpha]+1; x>a).
\end{aligned} \tag{1.33}$$

and

$$\begin{aligned}
({}^cD^{\alpha}_{b-}y)(x) &= (I^{n-\alpha}_{b-}) \circ \left(-\frac{d}{dx}\right)^n y(x) \\
&= \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{y^{(n)}(t)dt}{(t-x)^{\alpha-n+1}}, (n=[\alpha]+1; x<b).
\end{aligned} \tag{1.34}$$

The Riemann-Liouville fractional derivatives and the Caputo fractional derivatives are connected with each other by the following relations.

Property 1.3 ([15] [20])

Let $n \in \mathbb{N}$ *and* $n-1 < \alpha < n$. *If* x *is a function defined on* $[a,b]$ *for which the Caputo fractional derivatives* ${}^cD^{\alpha}_{a+}y(x)$ *and* ${}^cD^{\alpha}_{b-}y(x)$ *of order* α *exist together with The Riemann-Liouville fractional derivatives* $D^{\alpha}_{a+}y(x)$ *and* $D^{\alpha}_{b-}y(x)$, *then:*

$${}^cD^{\alpha}_{a+}y(x) = D^{\alpha}_{a+}y(x) - \sum_{j=0}^{n-1}\frac{y^{(j)}(a)}{\Gamma(j-\alpha+1)}(x-a)^{j-\alpha},$$

and

$$^{c}D^{\alpha}_{b-}y(x) = D^{\alpha}_{b-}y(x) - \sum_{j=0}^{n-1} \frac{y^{(j)}(b)}{\Gamma(j-\alpha+1)}(b-x)^{j-\alpha}.$$

In particular, when $0 < \alpha < 1$ *we get:*

$$^{c}D^{\alpha}_{a+}y(x) = D^{\alpha}_{a+}y(x) - \frac{y(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha}, \tag{1.35}$$

$$^{c}D^{\alpha}_{b-}y(x) = D^{\alpha}_{b-}y(x) - \frac{y(b)}{\Gamma(1-\alpha)}(b-x)^{-\alpha}. \tag{1.36}$$

1.2 Differentiability on Banach space

Definition 1.5 *Let E be a Banach space, $\Omega \subset E$ an open set, and let $I : \Omega \longrightarrow \mathbb{R}$ be a functional, we say that I is Gâteau differentiable (G-differentiable) at $u \in \Omega$, if there exists $A \in E'$ (A linear and continuous), denoted by $I'_G(u)$ such that, for all $v \in E$,*

$$\lim_{t\to 0} \frac{I(u+tv) - I(u)}{t} = \langle A, v \rangle, \tag{1.37}$$

If I is Gâteaux differentiable at u, there exists only one linear functional $A \in E'$ satisfying (1.37). It is called the Gâteaux differential of I at u and is denoted by $I'_G(u)$.

Definition 1.6 *If the functional I is Gâteaux differentiable at every u of an open set $U \subset E$, we say that I is Gâteaux differentiable on U. The map*

$$\begin{aligned} I'_G : E &\to E' \\ u &\mapsto I'_G(u) \end{aligned}$$

is called the Gâteaux derivative of I.

Example 1.1 *Let $1 < p < \infty$, the functional*

$$\begin{aligned} I : L^p(\Omega) &\to \mathbb{R} \\ u &\mapsto I(u) = \int_\Omega |u|^p \, dx \end{aligned}$$

is Gâteaux differentiable and we have

$$\langle I'_G(u), v\rangle = p\int_\Omega |u|^{p-2}uvdx.$$

Example 1.2 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be an open set. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. suppose that there exists $\sigma \in [1, 2^*]$, $2^* = \frac{2N}{N-2}$ is be the critical Sobolev exponent, and there exist $a, b \in \mathbb{R}$ such that*

$$|f(t)| \leq a + b|t|^{\sigma-1}, \quad \forall t \in \mathbb{R}. \tag{1.38}$$

We set $F(t) = \int_0^t f(s)\, ds$ and we consider $J : H^1_0(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega F(u(x))\, dx.$$

Then J is is Gâteaux differentiable and

$$\langle J'_G(u), v\rangle = \int_\Omega f(u(x))v(x)\, dx.$$

1.3 Fréchet differentiability

Definition 1.7 *Let E be a Banach space, $\Omega \subset E$ an open set and let $I : \Omega \longrightarrow \mathbb{R}$ be a functional, we say that I is Fréchet differentiable at $u \in \Omega$, if there exists $A_u \in E'$ such that*

$$\lim_{\|v\| \to 0} \frac{I(u+v) - I(v) - A_u v}{\|v\|} = 0 \tag{1.39}$$

Thus, for a Fréchet differentiable functional I, we have

$$I(u+v) - I(u) = A_u(v) + o(\|v\|),$$

where $\lim_{\|v\| \to 0} \frac{o(\|v\|)}{\|v\|} = 0.$

If a functional I is differentiable at u, then A_u is a unique.

Definition 1.8 *Let $I : \Omega \to \mathbb{R}$ be differentiable at $u \in \Omega$. The unique element A_u such that (1.39) holds is called the Fréchet differential of I at u, and is denoted by $I'(u)$. Then we can write*

$$I(u+v) - I(u) = \langle I'(u), v \rangle + o(\|v\|)$$

as $\|v\| \to 0.$

Definition 1.9 *Let $\Omega \subset E$ be an open set.*

1. *If the functional I is differentiable at every $u \in \Omega$, we say that I is differentiable on Ω.*
2. *The map $I' : \Omega \to E'$ that sends $u \in \Omega$ to $I'(u) \in E'$ is called the Fréchet derivative of I. Note that I' is in general a nonlinear map.*
3. *If the derivative I' is continuous from Ω to E' we say that I is of class C^1 on Ω and we write $I \in C^1(\Omega)$.*

Remark 1.4

- *Notice that $I'(u)$ is defined on E, even if I is defined only in Ω.*
- *If I is Fréchet differentiable at $u \in \Omega$ then I is continuous at u.*
- *If I is Fréchet differentiable at $u \in \Omega$ then I is Gâteaux differentiable at $u \in \Omega$ and $I'(u) = I'_G(u)$.*

The converse of the third point of Remark 1.4 is not always true but we have the following result:

Proposition 1.4 *Suppose that $\Omega \subseteq E$ is on open set, such that I G-differentiable in Ω and that $I'_G : \Omega \in E \to E'$ is continuous, then I is also Fréchet differentiable at u, and we have $I'_G = I'(u)$.*

Remark 1.5 *Calculating the Gâteaux derivative and then proving its continuity is often technically easier than directly proving Fréchet differentiability. From this, we deduce that the functional is of class C^1.*

Example 1.3 *We prove that the functional* $J : L^p(\Omega) \to \mathbb{R}$ $(p > 1)$ *defined by*

$$u \to J(u) = \int_\Omega |u|^p dx.$$

is a Fréchet differentiable on $L^p(\Omega)$*. Indeed; we already prove that* J *is G-differentiable, with*

$$\langle J'_G(u), v\rangle = p \int_\Omega |u|^{p-1} uv\ dx, \quad \forall u, v \in L^p(\Omega). \tag{1.40}$$

Example 1.4 *Let* $\Omega \subset \mathbb{R}^N$*,* $N \geq 3$*, be an open set. Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfies the condition (1.38). We set* $F(t) = \int_0^t f(s)\ ds$ *and we consider* $J : H_0^1(\Omega) \to \mathbb{R}$ *defined by*

$$J(u) = \int_\Omega F(u(x))\ dx.$$

Then J *is is Fréchet differentiable and*

$$\langle J'(u), v\rangle = \int_\Omega f(u(x))v(x)\ dx.$$

Chapter 2

Existence of weak solution for a fractional $p-$Laplacian problem

Introduction 2.1 *In this chapter, we present a fractional boundary value problem involving left and right fractional derivative operators and the p-Laplacian. By applying critical point theory, several results are obtained concerning the existence and characterization of the weak solution to such a fractional boundary value problem.*

2.1 Presentation of problem

we consider the following problem:

$$(P_1)\begin{cases} D^{\alpha}_{T-}\phi_p\left(D^{\alpha}_{0+}u(t)\right) = f(t,u(t)), \quad t\in[0,T], \\ u(0)=u(T)=0, \end{cases} \tag{2.1}$$

Where D^{α}_{T-} and D^{α}_{0+} are the left and right Riemann-Liouville fractional derivative of order $\alpha \in (0,1]$ respectively, $\phi_p : \mathbb{R} \longrightarrow \mathbb{R}$ is the p-laplacian defined by:

$$\phi_p(s) = |s|^{p-2}s \ if \ s \neq 0, \ \phi_p(0)=0, \ p>1,$$

and $f:[0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following condition:
(H_1) $f \in C([0,T]\times\mathbb{R},\mathbb{R})$.
Note that p-laplacian ϕ_p introduce by Leibenson [18] often occurs in non-Newtonian fluid theory, nonlinear elastic mechanics, and so forth. Moreover, when $p=2$, then nonlinear operator $D^{\alpha}_{T-}\phi_p\left(D^{\alpha}_{0+}\right)$ reduces to the linear operator $D^{\alpha}_{T-}D^{\alpha}_{0+}$.

Now we present the rule for fractionl integration by parts, see lemma(1.5) .

2.2 Fractional derivative space and variational structure

Definition 2.1 ([13]) *Let* $0 < \alpha \leq 1$ *and* $1 < p < \infty$*. The fractional derivative space* $E_0^{\alpha,p}$ *is defined by:*

$$E_0^{\alpha,p} = \left\{ u \in L^p\left([0,T],\mathbb{R}\right) |^c D_{0+}^{\alpha} u \in L^p\left([0,T],\mathbb{R}\right), u(0) = u(T) = 0 \right\}.$$

with the norm:

$$\|u\|_{\alpha,p} = \left(\|u\|_{L^p}^p + \|^c D_{0+}^{\alpha}\|_{L^p}^p \right)^{\frac{1}{p}}, \ \forall u \in E_0^{\alpha,p}, \tag{2.2}$$

Where: $\|u\|_{L^p} = (\int_0^T |u(t)|^p dt)^{\frac{1}{p}}$ *is the norm of* $L^p([0,T],\mathbb{R})$*.*

Remark 2.1 *For any* $u \in E_0^{\alpha,p}$*, according to* (1.35) *and* (1.36) *in view of* $u(0) = u(T) = 0$*, we have:* ${}^c D_{0+}^{\alpha} u(t) = D_{0+}^{\alpha} u(t)$*,* ${}^c D_{T-}^{\alpha} u(t) = D_{T-}^{\alpha} u(t)$ *for* $t \in [0,T]$*.*

Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$is a reflexive and separable Banach space, see lemma proposition (1.1).

Recall that Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For $u \in E_0^{\alpha,p}$, we have:

$$\|u\|_{L^p} \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|^c D_{0+}^{\alpha} u\|_{L^p}. \tag{2.3}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then:

$$\|u\|_{\infty} \leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \|^c D_{0+}^{\alpha} u\|_{L^p}, \tag{2.4}$$

Where: $\|u\|_{\infty} = \max_{t \in [0,T]} |u(t)|$ is the norm of $C([0,T],\mathbb{R})$.
Here, we also conclude that: $E_0^{\alpha,p} \hookrightarrow C([0,T],\mathbb{R})$. (*see proposition*(1.3))

Remark 2.2 *According to* (2.3)*, we know that the norm* (2.2) *is equivalent to the norm of the form:*

$$\|u\|_{\alpha,p} = \|^c D_{0+}^{\alpha} u\|_{L^p}. \tag{2.5}$$

Hence, in what follows, we can consider $E_0^{\alpha,p}$ *whith the norm* (2.5)*.*

Proof.
We have:

$$\begin{aligned} \|u\|_{\alpha,p}^p &= \|^c D_{0+}^{\alpha} u\|_{L^p}^p + \|u\|_{L^p}^p \\ &\geq \|^c D_{0+}^{\alpha} u\|_{L^p}^p \\ &\Rightarrow \|u\|_{\alpha,p} \geq \|^c D_{0+}^{\alpha} u\|_{L^p}. \end{aligned}$$

Conversely, we obtain:

$$\begin{aligned} \|u\|_{\alpha,p}^p &= \|^c D_{0+}^{\alpha} u\|_{L^p}^p + \|u\|_{L^p}^p \\ &\leq (C+1) \ \|^c D_{0+}^{\alpha} u\|_{L^p}^p \\ \Rightarrow \|u\|_{\alpha,p} &\leq (C+1)^{\frac{1}{p}} \ \|^c D_{0+}^{\alpha} u\|_{L^p} \\ \Rightarrow \|^c D_{0+}^{\alpha} u\|_{L^p} \leq \|u\|_{\alpha,p} &\leq (C+1)^{\frac{1}{p}} \ \|^c D_{0+}^{\alpha} u\|_{L^p}. \end{aligned}$$

It follows that: the two norms are equivalent. ■

Lemma 2.1 ([13]) *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and the sequence $\{u_k\}$ converges weakly to u in $E_0^{\alpha,p}$, that is, $u_k \rightharpoonup u$. Then $u_k \longrightarrow u$ in $C([0,T],\mathbb{R})$, that is:*

$$\|u_k - u\|_\infty \longrightarrow 0,\ k \longrightarrow \infty.$$

Lemma 2.2 *for $v \in E_0^{\alpha,p}$ by Remak (2.1) and Definition of the Riemann-Louville fractional derivative we have:*

$$\int_0^T [D_{T-}^{\alpha}\phi_p\left(D_{0+}^{\alpha}u(t)\right)]v(t)dt = \int_0^T \phi_p\left({}^cD_{0+}^{\alpha}u(t)\right)\ {}^cD_{0+}^{\alpha}v(t)dt.$$

Proof.
There is:

$$\begin{aligned}\int_0^T [D_{T-}^{\alpha}\phi_p\left(D_{0+}^{\alpha}u(t)\right)]v(t)dt &= \int_0^T \left[D_{T-}^{\alpha}\phi_p\left({}^cD_{0+}^{\alpha}u(t)\right)\right]v(t)dt \\ &= -\int_0^T v(t)d\left[D_{T-}^{\alpha-1}\phi_p\left({}^cD_{0+}^{\alpha}u(t)\right)\right]dt \\ &= \int_0^T \left[D_{T-}^{\alpha-1}\phi_p\left({}^cD_{0+}^{\alpha}u(t)\right)\right]v'(t)dt.\end{aligned}$$

Thus, from Lemma (1.5) and Definition of Caputo fractional derivatives we have:

$$\begin{aligned}\int_0^T \left[D_{T-}^{\alpha}\phi_p\left(D_{0+}^{\alpha}u(t)\right)\right]v(t)dt &= \int_0^T \phi_p({}^cD_{0+}^{\alpha}u(t))D_{0+}^{\alpha-1}v'(t)dt \\ &= \int_0^T \phi_p\left({}^cD_{0+}^{\alpha}u(t)\right)\ {}^cD_{0+}^{\alpha}v(t)dt.\end{aligned}$$

■

So we can define the weak solutions of Problem (2.1). as follows.

Definition 2.2 *By weak solution to Problem (2.1) we mean a function $u \in E_0^{\alpha,p}$ such that, $f(.,u(.)) \in L^1([0,T],\mathbb{R})$ and the following equation holds:*

$$\int_0^T \phi_p({}^cD_{0+}^{\alpha}u(t)){}^cD_{0+}^{\alpha}v(t)dt = \int_0^T f(t,u(t))v(t)dt,\ \forall v \in E_0^{\alpha,p}.$$

Next, we shall introduce a functional for Problem (2.1) on $E_0^{\alpha,p}$. Also, we will show that the critical points of that functional are weak solutions of Problem (2.1).
Define the functional $I : E_0^{\alpha,p} \longrightarrow \mathbb{R}$ by:

$$I(u) = \frac{1}{p}\int_0^T |{}^cD_{0+}^{\alpha}u(t)|^p dt - \int_0^T F(t,u(t))dt, \tag{2.6}$$

Where $F(t,s) = \int_0^s f(t,\tau)d\tau$.

Remark 2.3 *By Lemma (2.1) we get that the functional $u \longrightarrow \int_0^T F(t,u(t))dt$ is weakly continuous on $E_0^{\alpha,p}$. Hence, as the sum of a convex continuous functional and a weakly continuous one, I is a weakly lower semicontinuous functional on $E_0^{\alpha,p}$ with $\alpha > \frac{1}{p}$. Moreover, following [21], we can show that $I \in C^1(E_0^{\alpha,p},\mathbb{R})$, and we have:*

$$\begin{aligned}\langle I'(u),v\rangle &= \int_0^T \phi_p({}^cD_{0+}^{\alpha}u(t)){}^cD_{0+}^{\alpha}v(t)dt \\ &\quad -\int_0^T f(t,u(t))v(t)dt,\ \forall v \in E_0^{\alpha,p}.\end{aligned} \tag{2.7}$$

Remark 2.4 *By Definition (2.2) and (2.7), if $u \in E_0^{\alpha,p}$ is a solution of the Euler equation $I'(u) = 0$, then u is a weak solution of Problem (2.1).*

2.3 Existence of weak solutions of problem (2.1)

Definition 2.3 *Let $\varphi \in C^1(X,\mathbb{R})$. If any sequence $\{u_k\} \subset X$ for which $\{\varphi(u_k)\}$ is bounded and $\varphi'(u_k) \longrightarrow 0$ as $k \longrightarrow \infty$ possesses a convergent subsequence , then we say that φ satisfies the Palais-Smale condition (P.S.condition for short).*

Theorem 2.1 *, Let X be a real reflexive Banach space. If the functional $\varphi : X \longrightarrow \mathbb{R}$ is weakly lower semicontinuous and coercive, that is, $\lim\limits_{\|z\|\longrightarrow\infty} \varphi(z) = +\infty$, then there exists $z_0 \in X$ such that $\varphi(z_0) = \inf\limits_{z\in X} \varphi(z)$. Moreover, if φ is also Fréchet differentiable on X, then $\varphi'(z_0) = 0$.*

Theorem 2.2 *(Mountain pass theorem [21]) Let X be a real Banach space, and $\varphi \in C^1(X,\mathbb{R})$ satisfy the P.S condition. Suppose that:*
(C_1) $\varphi(0) = 0$,
(C_2) there exist $\rho > 0$ and $\sigma > 0$ such that $\varphi(z) \geq \sigma$ for all $z \in X$ with $\|z\| = \rho$,
(C_3) there exists $z_1 \in X$ with $\|z_1\| \geq \rho$ such that $\varphi(z_1) < \sigma$.
Then φ possesses a critical value $c \geq \sigma$. Moreover, c can be characterized as

$$c = \inf_{g\in\Omega,}\max_{z\in g([0,1])} \varphi(z),$$

Where $\Omega = \{g \in C([0,1],X) | g(0) = 0, g(1) = z_1\}$.
First, we use Theorem (2.1) to prove the existence of weak solutions of problem (2.1)

Theorem 2.3 *Let $\frac{1}{p} < \alpha \leq 1$ and (H_1) be satisfied. Assume that:*
(H_2) there exist $a \in (0,(\Gamma(\alpha+1))^p/pT^{\alpha p})$ and $b \in L^1([0,T],\mathbb{R}^+)$, such that:

$$|F(t,x)| \leq a|x|^p + b(t),\ \forall t \in [0,T],\ x \in \mathbb{R}.$$

Then (2.1) has at least one weak solution that minimized I on $E_0^{\alpha,p}$.

Proof. According to Proposition (1.1), Remark (2.3), and Theorem (2.1), we only need to prove that the functional I is **coercive**.

For any $u \in E_0^{\alpha,p}$, it follows from (H2) that:

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|_{\alpha,p}^p - \int_0^T F(t,u(t))\,dt \\ &\geq \frac{1}{p}\|u\|_{\alpha,p}^p - a\int_0^T |u(t)|^p\,dt - \int_0^T b(t)\,dt. \end{aligned}$$

This can be rewritten as:

$$I(u) \geq \frac{1}{p}\|u\|_{\alpha,p}^p - a\|u\|_{L^p}^p - \|b\|_{L^1}.$$

which, together with (2.3), implies:

$$I(u) \geq \left[\frac{1}{p} - \left(\frac{aT^{\alpha p}}{(\Gamma(\alpha+1))^p}\right)\right]\|u\|_{\alpha,p}^p - \|b\|_{L^1}.$$

thus, noting that: $\alpha \in (0, (\Gamma(\alpha+1))^p/pT^{\alpha p})$,we have: $\lim\limits_{\|u\|_{\alpha,p}\to\infty} I(u) = +\infty$, which means that I is **coercive**.
I is **weakly lower semicontinuous** ($w.l.s.c$), that is:

$$\forall u_n \in E_0^{\alpha,p} : u_n \rightharpoonup u \Rightarrow I(u) \leq \lim_{n\longrightarrow+\infty} \inf I(u_n)$$

let $\forall (u_n) \subset E_0^{\alpha,p}$ such that $u_n \rightharpoonup u \in E_0^{\alpha,p}$.We have: $\|u\|_{\alpha,p} \leq \lim\limits_{n\longrightarrow+\infty} \inf \|u\|_{\alpha,p}$, since for any norm is w.l.s.c in Banach space.
Moreover, we have:

$$\int_0^T F(t, u_n(t))dt \longrightarrow \int_0^T F(t, u(t))dt,$$

Indeed by Lemma (2.1) we have:
$u_n \longrightarrow u \;\; in \; C([0,T]\mathbb{R})$, and (u_n) is uniformly bounded. The continuity of F implies that:

$$F(t, u_n(t)) \longrightarrow F(t, u(t))$$

On other hand:

$$|F(t, u_n(t))| \leq a|u_n|^p + b(t) \leq aM^p + b(t) \in L^1,$$

By dominated convergence theorem, we get:

$$\int_0^T F(t, u_n(t))dt \longrightarrow \int_0^T F(t, u(t))dt.$$

■

Hence: we found, $I(u) \leq \lim\limits_{n\longrightarrow+\infty} \inf I(u_n)$ which means that I is w.l.s.c. In conclusion, I is coercive, w.l.s.c and differentiable, then I has a least critical point which is a weak solution of Problem (2.1).

By **Theorem** (2.2) in order to discuss the existence of mountain pass solutions of the fractional boundary value Problem (2.1).

Theorem 2.4 *Let* $\frac{1}{p} < \alpha \leq 1$ *and* (H_1) *be satisfied. Assume that*
(H_3) *There exist constants* $\mu \in (0, 1/p)$ *and* $M > 0$ *such that:*

$$0 < F(t, x) \leq \mu x f(t, x), \;\; \forall t \in [0, T], \;\; x \in \mathbb{R} \; with \; |x| \geq M,$$

(H_4) *For* $t \in [0, T]$ *and* $x \in \mathbb{R}$, *we have:*

$$\lim_{|x|\longrightarrow 0} \sup \frac{F(t,x)}{|x|^p} < \frac{(\Gamma(\alpha+1))^p}{2pT^{\alpha p}}.$$

Then the fractional boundary value Problem (2.1) *has at least one nontrivial weak solution on* $E_0^{\alpha,p}$.

Remark 2.5 *Since* $(t, x) \longmapsto F(t, x) - \mu x f(t, x)$ *is continuous on* $[0, T] \times \mathbb{R}$, *In particular, on* $[0, T] \times [-M, M]$ *for any* $M > 0$ *fixed, this function is bounded, that is, there exist* $C > 0$ *such that:*

$$|F(t, x) - \mu x f(t, x)| \leq C, \;\; \forall (t, x) \in [0, T] \times [-M, M].$$

Then,

$$F(t,x) \leq \mu x f(t,x) + C, \ \forall (t,x) \in [0,T] \times [-M,M]. \tag{2.8}$$

Thus, from (H_3) *we get:*

$$F(t,x) \leq \mu x f(t,x) + C, \ t \in [0,T], \ x \in \mathbb{R}. \tag{2.9}$$

Remark 2.6 *By* (H_4)*, we have:*

$$\begin{aligned} \forall \xi > 0, \exists \ \delta_\xi : \left| \frac{F(t,x)}{|x|^p} - l \right| \leq \xi, \ \forall |x| \leq \delta, \\ \Rightarrow l - \xi \leq \frac{F(t,x)}{|x|^p} \leq l + \xi < \frac{(\Gamma(\alpha+1))^p}{2pT^{\alpha p}} |x|^p, \\ t \in [0,T], \ x \in \mathbb{R}, \ \forall |x| \leq \delta_\xi. \end{aligned} \tag{2.10}$$

Proof. We will verify that I satisfies all the conditions of Theorem(2.2).
First, we show that I satisfies the P.S condition. Let $\{u_k\} \subset E_0^{\alpha,p}$ be a sequence such that:

$$|I(u_k)| \leq K, \quad I'(u_k) \longrightarrow 0 \ \ as \ k \longrightarrow \infty. \tag{2.11}$$

Accoding to (2.7), we have:

$$\begin{aligned} \langle I'(u_k), u_k \rangle &= \int_0^T \phi_p \left({}^c D_{0+}^\alpha u_k(t) \right) \ {}^c D_{0+}^\alpha u_k(t) dt - \int_0^T f(t, u_k(t)) u_k(t) dt, \\ &= \int_0^T |{}^c D_{0+}^\alpha u_k(t)|^{p-2} ({}^c D_{0+}^\alpha u_k(t))^2 dt - \int_0^T f(t, u_k(t)) u_k(t) dt \\ &= \int_0^T |{}^c D_{0+}^\alpha u_k(t)|^p dt - \int_0^T f(t, u_k(t)) u_k(t) dt \\ &= \|u_k\|_{\alpha,p}^p - \int_0^T f(t, u_k(t)) u_k(t) dt. \end{aligned}$$

which, together with (2.8), yields:

$$\begin{aligned} K &\geq I(u_k) \\ &= \frac{1}{p} \|u_k\|_{\alpha,p}^p - \int_0^T F(t, u_k(t)) dt \\ &\geq \frac{1}{p} \|u_k\|_{\alpha,p}^p - \mu \int_0^T f(t, u_k(t)) u_k(t) dt - CT \\ &= \left(\frac{1}{p} - \mu \right) \|u_k\|_{\alpha,p}^p + \mu \langle I'(u_k), u_k \rangle - CT. \end{aligned}$$

Since:

$$\begin{aligned} &|\langle I'(u_k), u_k \rangle| \leq \|I'(u_k)\|_{-\alpha,q} \ . \ \|u_k\|_{\alpha,p} \\ &\qquad \Rightarrow -\|I'(u_k)\|_{-\alpha,q} \ . \ \|u_k\|_{\alpha,p} \leq \langle I'(u_k), u_k \rangle \leq \|I'(u_k)\|_{(E_0^{\alpha,p})'} \ . \ \|u_k\|_{\alpha,p}. \end{aligned}$$

We have:

$$\mu < \frac{1}{p} \Rightarrow \frac{1}{p} - \mu > 0.$$

Then:

$$K \geq \left(\frac{1}{p} - \mu\right) \|u_k\|_{\alpha,p}^p - \mu \|I'(u_k)\|_{(E_0^{\alpha,p})'} \|u_k\|_{\alpha,p} - CT.$$

Since: $I'(u_k) \longrightarrow 0$, there exists $N_0 \in \mathbb{N}$ such that:

$$K \geq \left(\frac{1}{p} - \mu\right) \|u_k\|_{\alpha,p}^p - \|u_k\|_{\alpha,p} - CT, \quad K > N_0.$$

By (2.11) we have:
$\mu \|I'(u_k)\|_{(E_0^{\alpha,p})'} \leq 1$, for k is big enough.

It follows from $\mu \in (0, 1/p)$ that $\{u_k\}$ is bounded in $E_0^{\alpha,p}$. Since $E_0^{\alpha,p}$ is a reflexive space, up to a subsequence, we can assume that $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$. Hence, we have:

$$\begin{cases} u_k \longrightarrow u \ in \ C([0,T], \mathbb{R}) \\ \|u_k - u\|_\infty \longrightarrow 0 \ as \ k \longrightarrow +\infty. \end{cases}$$

Consequently:

$$\begin{aligned} \langle I'(u_k) - I'(u), u_k - u \rangle &= \langle I'(u_k), u_k - u \rangle - \langle I'(u), u_k - u \rangle \\ &\leq \|I'(u_k)\|_{-\alpha,q} \|u_k - u\|_{\alpha,p} - \langle I'(u), u_k - u \rangle \\ &\longrightarrow 0, \ k \longrightarrow \infty. \end{aligned}$$

Indeed;

$$I'(u) \in (E_0^{\alpha,p})' \Rightarrow \langle I'(u), u_k \rangle \longrightarrow \langle I'(u), u \rangle \Leftrightarrow \langle I'(u), u_k - u \rangle \longrightarrow 0.$$

and

$$\|I'(u_k)\|_{(E_0^{\alpha,p})'} \|u_k - u\|_{\alpha,p} \leq \|I'(u_k)\|_{-\alpha,q} \left(\|u_k\|_{\alpha,p} + \|u\|_{\alpha,p}\right) \longrightarrow 0.$$

because: $\|I'(u_k)\|_{-\alpha,q} \longrightarrow 0$ and $(\|u_k\|_{\alpha,p} + \|u\|_{\alpha,p}) \leq C \ (constant)$.
Moreover, Lemma (2.1) we obtain that u_k is bounded in $C([0,T], \mathbb{R})$ and $\|u_k - u\|_\infty \longrightarrow 0$ as $k \longrightarrow \infty$. Then we get:

$$\int_0^T (f(t, u_k(t)) - f(t, u(t)))(u_k(t) - u(t))dt \longrightarrow 0, \ k \longrightarrow \infty. \tag{2.12}$$

Because:

$$\int_0^T (f(t, u_k(t)) - f(t, u(t)))(u_k(t) - u(t))dt \leq \|u_k - u\|_\infty \int_0^T f(t, u_k(t)) - f(t, u(t))dt.$$

Where: $\|u_k - u\|_\infty \longrightarrow 0$, and $\int_0^T f(t, u_k(t)) - f(t, u(t))dt$ is bounded, that is:

$$f(t, u_k) - f(t, u) \longrightarrow 0$$

Then:

$$|f(t,u_k)-f(t,u)|\leq M$$

So:

$$\int_0^T f(t,u_k)-f(t,u)dt\leq MT\leq C\ (constant).$$

Note that:

$$\begin{aligned}\langle I'(u_k)-I'(u),u_k-u\rangle=&\int_0^T\left(\phi_p({}^cD^\alpha_{0+}u_k(t))-\phi_p({}^cD^\alpha_{0+}u(t))\right)\left({}^cD^\alpha_{0+}u_k(t)-{}^cD^\alpha_{0+}u(t)\right)dt\\&-\int_0^T\left(f(t,u_k(t))-f(t,u(t))\right)\left(u_k(t)-u(t)\right)dt.\end{aligned}$$

Thus, from (2.9) and (2.12) we have:

$$\int_0^T\left(\phi_p({}^cD^\alpha_{0+}u_k(t))-\phi_p({}^cD^\alpha_{0+}u(t))\right)({}^cD^\alpha_{0+}u_k(t)-{}^cD^\alpha_{0+}u(t))dt\longrightarrow 0 \tag{2.13}$$

as $k\longrightarrow\infty$.

Using the following elementary inequalities (see[22]):

Let $x,y\in\mathbb{R}^N$ and $\langle .,.\rangle$ the standard scalar prodact in $\mathbb{R}$. Then:

$$\langle |x|^{p-2}x-|y|^{p-2}y,x-y\rangle\geq\begin{cases}c_p|x-y|^p, & \text{if } p\geq 2,\\ c_p\frac{|x-y|^2}{(|x|+|y|)^{2-p}}, & \text{if } 1<p<2.\end{cases}$$

With:

$$\begin{cases}x= {}^cD^\alpha_{0+}u_k(t),\\ y= {}^cD^\alpha_{0+}u(t).\end{cases}$$

We obtain that there exist $c_1,c_2>0$ such that:

$$\begin{aligned}&\int_0^T\left(\phi_p({}^cD^\alpha_{0+}u_k(t))-\phi_p({}^cD^\alpha_{0+}u(t))\right)({}^cD^\alpha_{0+}u_k(t)-{}^cD^\alpha_{0+}u(t))dt\\&\geq\begin{cases}c_1\int_0^T\left|{}^cD^\alpha_{0+}u_k(t)-{}^cD^\alpha_{0+}u(t)\right|^p dt, & \text{if } p\geq 2,\\ c_2\int_0^T\frac{\left|{}^cD^\alpha_{0+}u_k(t)-{}^cD^\alpha_{0+}u(t)\right|^2}{(|{}^cD^\alpha_{0+}u_k(t)|+|{}^cD^\alpha_{0+}u(t)|)^{2-p}}dt, & \text{if } 1<p<2.\end{cases}\end{aligned} \tag{2.14}$$

When $1<p<2$, using Hölder inequality, we get:

$$\begin{aligned}&\int_0^T\left|{}^cD^\alpha_{0+}u_k(t)-{}^cD^\alpha_{0+}u(t)\right|^p dt\\&\leq\left(\int_0^T\frac{\left|{}^cD^\alpha_{0+}u_k(t)-{}^cD^\alpha_{0+}u(t)\right|^2}{(|{}^cD^\alpha_{0+}u_k(t)|+|{}^cD^\alpha_{0+}u(t)|)^{2-p}}\right)^{\frac{p}{2}}\\&.\left(\int_0^T\left(\left|{}^cD^\alpha_{0+}u_k(t)\right|+\left|{}^cD^\alpha_{0+}u(t)\right|\right)^p dt\right)^{\frac{2-p}{2}}.\end{aligned}$$

Thus, noting that $(s_1+s_2)^\gamma \leq 2^{\gamma-1}(s_1^\gamma+s_2^\gamma)$ where $s_1, s_2 \geq 0$, $\gamma \geq 1$, we have:

$$\begin{aligned}\int_0^T \left|{}^cD_{0+}^\alpha u_k(t) -{}^c D_{0+}^\alpha u(t)\right|^p dt \\ \leq c_3(\|u_k\|_{\alpha,p}^p + \|u\|_{\alpha,p}^p)^{\frac{2-p}{2}} \\ \cdot\left(\int_0^T \frac{\left|{}^cD_{0+}^\alpha u_k(t) -{}^c D_{0+}^\alpha u(t)\right|^2}{(|{}^cD_{0+}^\alpha u_k(t)| + |{}^cD_{0+}^\alpha u(t)|)^{2-p}} dt\right)^{\frac{p}{2}}.\end{aligned}$$

Where: $c_3 = 2^{(p-1)(2-p)/2}$, which, together with (2.14), implies:

$$\begin{aligned}\int_0^T \left(\phi_p({}^cD_{0+}^\alpha u_k(t)) - \phi_p({}^cD_{0+}^\alpha u(t))\right)({}^cD_{0+}^\alpha u_k(t) -{}^c D_{0+}^\alpha u(t))dt \\ \geq c_2 c_3^{-\frac{2}{p}}(\|u_k\|_{\alpha,p}^p + \|u\|_{\alpha,p}^p)^{\frac{p-2}{p}}\|u_k-u\|_{\alpha,p}^2, \quad 1<p<2.\end{aligned} \tag{2.15}$$

When $p \geq 2$, by (2.14) we get:

$$\begin{aligned}\int_0^T \left(\phi_p({}^cD_{0+}^\alpha u_k(t)) - \phi_p({}^cD_{0+}^\alpha u(t))\right)({}^cD_{0+}^\alpha u_k(t) -{}^c D_{0+}^\alpha u(t))dt \\ \geq c_1\|u_k-u\|_{\alpha,p}^p, \quad p\geq 2.\end{aligned} \tag{2.16}$$

It follows from (2.13), (2.15), and (2.16) that:

$$\|u_k-u\|_{\alpha,p} \longrightarrow 0, \ k \longrightarrow \infty,$$

that is, $\{u_k\}$ converges strongly to $u \in E_0^{\alpha,p}$. It follows that I satisfies the C_1 condition.
Now we show that I satisfies the geometry conditions of mountain pass theorem.
Let $\rho = \frac{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}{T^{\alpha-1/p}}\delta > 0$ and $\sigma = \rho^p/2p > 0$. Then, by (2.4) we have:

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\|u\|_{\alpha,p} = \delta, \ u \in E_0^{\alpha,p} \ with \ \|u\|_{\alpha,p} = \rho,$$

Which, together with (2.3) and (2.10), implies:

$$\begin{aligned}I(u) &= \frac{1}{p}\|u\|_{\alpha,p}^p - \int_0^T F(t,u(t))dt \\ &\geq \frac{1}{p}\|u\|_{\alpha,p}^p - \frac{(\Gamma(\alpha+1))^p}{2pT^{\alpha p}}\int_0^T |u(t)|^p dt \\ &\geq \frac{1}{p}\|u\|_{\alpha,p}^p - \frac{1}{2p}\|u\|_{\alpha,p}^p \\ &= \frac{1}{2p}\|u\|_{\alpha,p}^p \\ &= \sigma, \ \forall u \in E_0^{\alpha,p} \ with: \ \|u\|_{\alpha,p} = \rho.\end{aligned}$$

Hence, condition C_2 in Theorem (2.2) is satisfied.
By (H_3) there exists $c_4, c_5 > 0 : F(t,x) \geq c_4|x|^{\frac{1}{\mu}} - c_5, \forall t \in [0,T], \forall x \in \mathbb{R}$. Indeed;we have:
Proof.

$$0 < F(t,x) < \mu x f(t,x) \ \forall t \in [0,T], \ x \in \mathbb{R} \ with: \ |x| \geq M,$$

$$\int_M^x \frac{f(t,x)}{F(t,x)}dt \geq \int_M^x \frac{1}{\mu x}$$

So:

$$\ln\left|\frac{F(t,x)}{F(t,M)}\right| \geq \ln\frac{|x|^{\frac{1}{\mu}}}{M^{\frac{1}{\mu}}}$$

Then:

$$F(t,x) \geq \frac{F(t,M)}{M^{\frac{1}{\mu}}}|x|^{\frac{1}{\mu}}\ \forall t \in [0,T]$$

$$F(t,x) \geq \frac{c}{M^{\frac{1}{\mu}}}|x|^{\frac{1}{\mu}}$$

$$F(t,x) \geq c_4|x|^{\frac{1}{\mu}}$$

where:

$$c_4 = \frac{c}{M^{\frac{1}{\mu}}}.$$

The same result holds over the domain $[x,-M]$.
and we have:

$$\begin{aligned} |F(t,x)| &\leq c \\ \Rightarrow -c &\leq F(t,x) \\ |F(t,x) - c_4|x|^{\frac{1}{\mu}}| &\geq -c_5 \\ \Rightarrow F(t,x) &\geq c_4|x|^{\frac{1}{\mu}} - c_5,\ t \in [0,T], x \in \mathbb{R}. \end{aligned} \tag{2.17}$$

■

For any $u \in E_0^{\alpha,p}\backslash\{0\}$, $\xi \in \mathbb{R}^+$, noting that $\mu \in (0, 1/p)$, using (2.17) we get:

$$\begin{aligned} I(\xi u) &= \frac{1}{p}\|\xi u\|_{\alpha,p}^p - \int_0^T F(t,\xi u(t))\,dt \\ &\leq \frac{\xi^p}{p}\|u\|_{\alpha,p}^p - c_4\int_0^T |\xi u(t)|^{\frac{1}{\mu}}\,dt + c_5T \\ &= \frac{\xi^p}{p}\|u\|_{\alpha,p}^p - c_4\xi^{\frac{1}{\mu}}\|u\|_{L^{\frac{1}{\mu}}}^{\frac{1}{\mu}} + c_5T \longrightarrow -\infty \end{aligned}$$

$,\xi \longrightarrow \infty.$

Because:

$$\frac{1}{\mu} > p.$$

■

Thus, for ξ_0 large enough and taking $e = \xi_0 u$, we have $I(e) < 0$. Therefore, condition (C_3) in Theorem(2.2) is also satisfied.
finally, noting that $I(0) = 0$, we get the existence of a critical point u such that $I(u) \geq \sigma > 0$. Hence, u is a nontrivial weak solution of Problem (2.1).

Chapter 3

Existence results for:a class of fractional p-kirchhoff problem

Introduction 3.1 *In this chapter, we investigate a fractional boundary value problem that involves both left- and right-sided fractional derivative operators, in addition to a p-Kirchhoff-type operator, subject to various boundary conditions. By employing critical point theory, we establish results concerning the existence of weak solutions to the proposed fractional boundary value problem.*

3.1 Fractional p-kirchhoff problem

3.1.1 Presentation of problem

We consider the following problem:

$$(P_2)\begin{cases} M\left(\|D^{\alpha}_{0+}u\|^{p}_{L^p}\right) D^{\alpha}_{T-}\phi_p\left(D^{\alpha}_{0+}u(t)\right) = f(t,u(t)), \quad t\in[0,T], \\ u(0)=u(T)=0, \end{cases} \tag{3.1}$$

Where: $M : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is given by: $M(t) = a + bt^{\theta-1}$ with: $a, b > 0$, $\theta \geq 1$. and $f : [0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following condition:
(H_1) $f \in C([0,T]\times\mathbb{R},\mathbb{R})$.
Here, the integral by part version with the presence of kirchhoff term:

Lemma 3.1 *For any $u, v \in E^{\alpha,p}_0$, we have:*

$$\begin{aligned}&\int_0^T M\left(\|D^{\alpha}_{0+}u\|^{p}_{L^p}\right) D^{\alpha}_{T-}\phi_p\left(D^{\alpha}_{0+}u(t)\right)v(t)dt\\ &= M\left(\|D^{\alpha}_{0+}u\|^{p}_{L^p}\right)\int_0^T \phi_p\left({}^cD^{\alpha}_{0+}u(t)\right)^c D^{\alpha}_{0+}v(t)dt.\end{aligned}$$

So we can define the weak solution of Problem (3.1).

Definition 3.1 *We say that $u \in E^{\alpha,p}_0$ is a weak solution of Problem* (3.1) *if and only if:*

$$\left(a + b\|^cD^{\alpha}_{0+}u\|^{p(\theta-1)}_{L^p}\right)\int_0^T \phi_p\left({}^cD^{\alpha}_{0+}u(t)\right)^c D^{\alpha}_{0+}v(t)dt = \int_0^T f(t,u(t))v(t)dt.$$

for any $v \in E_0^{\alpha,p}$.
We consider the associated functional $J : E_0^{\alpha,p} \longrightarrow \mathbb{R}$ defined by::

$$J(u) = \left(\frac{a}{p}\|^c D_{0+}^{\alpha} u\|_{L^p}^p + \frac{b}{p\theta}\|^c D_{0+}^{\alpha} u\|_{L^p}^{p\theta}\right) - \int_0^T F(t,u(t))dt.$$

Clearly J is well defined and differentiable on $E_0^{\alpha,p}$ and we have:

$$\langle J'(u), v\rangle = \left(a + b\|^c D_{0+}^{\alpha} u\|_{L^p}^{p(\theta-1)}\right)\int_0^T \phi_p\left({}^c D_{0+}^{\alpha} u(t)\right)^c D_{0+}^{\alpha} v(t)dt - \int_0^T f(t,u(t))v(t)dt, \quad (3.2)$$

$\forall v \in E_0^{\alpha,p}$.

3.1.2 Existence of weak solutions of problem (3.1)

Theorem 3.1 *Let* $\frac{1}{p} < \alpha \leq 1$ *and* (H_1) *be satisfied. Assume that:*
(H_2') *there exist* $\xi \in (0, b\Gamma(\alpha+1)^{p\theta}/p\theta T^{\alpha p\theta})$ *and* $\lambda \in L^1([0,T],\mathbb{R}^+)$ *such that:*

$$|F(t,x)| \leq \xi|x|^{p\theta} + \lambda(t), \;\; \forall t \in [0,T], \;\; x \in \mathbb{R}.$$

Proof. We apply Theorem (2.1):
J is **coercive**:
For $u \in E_0^{\alpha,p}$, it follows from (H_2') that:

$$\begin{aligned} J(u) &= \frac{a}{p}\|u\|_{\alpha,p}^p + \frac{b}{p\theta}\|u\|_{\alpha,p}^{p\theta} - \int_0^T F(t,u(t))dt \\ &\geq \frac{a}{p}\|u\|_{\alpha,p}^p + \frac{b}{p\theta}\|u\|_{\alpha,p}^{p\theta} - \xi\|u\|_{L^p}^{p\theta} - \|\lambda\|_{L^1}. \end{aligned}$$

By (2.3) we have:

$$J(u) \geq \frac{a}{p}\|u\|_{\alpha,p}^p + \frac{b}{p\theta}\|u\|_{\alpha,p}^{p\theta} - \frac{\xi T^{\alpha p\theta}}{\Gamma(\alpha+1)^{p\theta}}\|u\|_{\alpha,p}^{p\theta} - \|\lambda\|_{L^1}$$

$$= \left(\frac{b}{p\theta} - \frac{\xi T^{\alpha p\theta}}{(\Gamma(\alpha+1))^{p\theta}}\right)\|u\|_{\alpha,p}^{p\theta} + \frac{a}{p}\|u\|_{\alpha,p}^p - \|\lambda\|_{L^1}$$

Since: $\xi \in \left(0, \frac{b(\Gamma(\alpha+1))^{p\theta}}{T^{\alpha p\theta}p\theta}\right)$, we have:

$$\frac{b}{p\theta} - \frac{\xi T^{\alpha p\theta}}{(\Gamma(\alpha+1))^{p\theta}} > 0.$$

Then:

$$\lim_{\|u\|_{\alpha,p}\longrightarrow\infty} J(u) = \lim_{\|u\|_{\alpha,p}\longrightarrow\infty}\left(\frac{b}{p\theta} - \frac{\xi T^{\alpha p\theta}}{\Gamma(\alpha+1)^{p\theta}}\right)\|u\|_{\alpha,p}^{p\theta} = +\infty.$$

Thus, J is **coercive**.
We now proceed to the next step, since J is **w.l.s.c** .
Assume that $u_n \rightharpoonup u \in E_0^{\alpha,p}$. We have: $\|u\|_{\alpha,p} \leq \lim_{n\longrightarrow+\infty}\inf\|u\|_{\alpha,p}$, since any norm is **w.l.s.c** in Banach space.

Moreover we have: $\int_0^T F(t,u_n(t))dt \longrightarrow \int_0^T F(t,u(t))dt$ and by Lemma (2.1) we have: $u_n \rightharpoonup u \in C([0,T],\mathbb{R})$ and (u_n) is uniformly bounded, we get: $F(t,u_n(t)) \longrightarrow F(t,u(t))$

$$|F(t,u_n)| \leq \xi |u_n|^{p\theta} + \lambda(t) \leq \xi M^{p\theta} + \lambda(t) \in L^1,$$

With $M^{p\theta} = C$.
By **D.C.T**, we get:

$$\int_0^T F(t,u_n(t))dt \longrightarrow \int_0^T F(t,u(t))dt$$

Hence, we deduce that: $J(u) \leq \lim_{n\longrightarrow+\infty} \inf J(u_n)$, which means that J is **w.l.s.c**.
In conclusion: J is coercive, w.l.s.c and diff, then J has at least critical point which is a weak solution of Problem (3.1) ■
Using **Theorem** (2.2) we discuss the existence of mountain pass solutions of the fractional boundary value Problem (3.1).

Theorem 3.2 *Let* $\frac{1}{p} < \alpha \leq 1$ *and* (H_1) *be satisfied. Assume that*
(H_3') *there exist constants* $\mu \in (0, 1/p\theta)$ *and* $M > 0$ *such that:*

$$0 < F(t,x) \leq \mu x f(t,x), \ \forall t \in [0,T], \ x \in \mathbb{R} \ with \ |x| \geq M,$$

(H_4') *for* $t \in [0,T]$ *and* $x \in \mathbb{R}$*, we have:*

$$\lim_{|x|\longrightarrow 0} \sup \frac{F(t,x)}{|x|^{p\theta}} < \frac{(\Gamma(\alpha+1))^{p\theta}}{2p\theta T^{\alpha p\theta}}.$$

Then the fractional boundary value Problem (3.1) has at least one nontrivial weak solution on $E_0^{\alpha,p}$.

Proof. We will verify that J satisfies all the conditions of Theorem(2.2).
Firstly, we show that J satisfies the P.S condition. Since $F(t,x) - \mu x f(t,x)$ is continuous, there exists $C \in \mathbb{R}^+$ (*see Remark* (2.5)) such that:

$$F(t,x) \leq \mu x f(t,x) + C, \ t \in [0,T], \ x \in \mathbb{R}. \tag{3.3}$$

Let $\{u_K\} \subset E_0^{\alpha,p}$ be such that:

$$|J(u_k)| \leq K, \ J'(u_k) \longrightarrow 0 \ \ as \ k \longrightarrow \infty.$$

According to (3.2), we have:

$$\begin{aligned}
\langle J'(u_k), u_k \rangle &= \left(a + b\|u_k\|_{\alpha,p}^{p(\theta-1)}\right) \int_0^T \phi_p\left({}^cD_{0+}^\alpha u_k(t)\right) {}^cD_{0+}^\alpha u_k(t)dt - \int_0^T f(t,u_k(t))u_k(t)dt, \\
&= \left(a + b\|u_k\|_{\alpha,p}^{p(\theta-1)}\right) \int_0^T |{}^cD_{0+}^\alpha u_k(t)|^p dt - \int_0^T f(t,u_k(t))u_k(t)dt, \\
&= \left(a + b\|u_k\|_{\alpha,p}^{p(\theta-1)}\right) \|u\|_{\alpha,p}^p - \int_0^T f(t,u_k(t))u_k(t)dt, \\
&= a\|u_k\|_{\alpha,p}^p + b\|u_k\|_{\alpha,p}^{p\theta} - \int_0^T f(t,u_k(t))u_k(t)dt.
\end{aligned} \tag{3.4}$$

which, together with (2.8), yields:

$$\begin{aligned} K &\geq J(u_k) \\ &= \frac{a}{p}\|u_k\|^p_{\alpha,p} + \frac{b}{p\theta}\|u_k\|^{p\theta}_{\alpha,p} - \int_0^T F(t,u_k(t))dt \\ &\geq \frac{a}{p}\|u_k\|^p_{\alpha,p} + \frac{b}{p\theta}\|u_k\|^{p\theta}_{\alpha,p} - \mu\int_0^T f(t,u_k(t))u_k(t)dt - CT \end{aligned}$$

From the equation (3.4), we find that:

$$-\mu\int_0^T f(t,u_k(t))u_k(t)dt = \mu\langle J'(u_k),u_k\rangle - \mu a\|u_k\|^p_{\alpha,p} - \mu b\|u_k\|^{p\theta}_{\alpha,p}$$

Hence:

$$\begin{aligned} K &\geq \frac{a}{p}\|u_k\|^p_{\alpha,p} + \frac{b}{p\theta}\|u_k\|^{p\theta}_{\alpha,p} + \mu\langle J'(u_k),u_k\rangle - \mu a\|u_k\|^p_{\alpha,p} - \mu b\|u_k\|^{p\theta}_{\alpha,p} - CT \\ &= a\left(\frac{1}{p}-\mu\right)\|u_k\|^p_{\alpha,p} + b\left(\frac{1}{p\theta}-\mu\right)\|u_k\|^{p\theta}_{\alpha,p} + \mu\langle J'(u_k),u_k\rangle - CT \\ &\geq a\left(\frac{1}{p}-\mu\right)\|u_k\|^p_{\alpha,p} + b\left(\frac{1}{p\theta}-\mu\right)\|u_k\|^{p\theta}_{\alpha,p} - \mu\|J'(u_k)\|_{(E_0^{\alpha,p})'}\|u_k\|_{\alpha,p} - CT \\ &\geq a\left(\frac{1}{p}-\mu\right)\|u_k\|^p_{\alpha,p} + b\left(\frac{1}{p\theta}-\mu\right)\|u_k\|^{p\theta}_{\alpha,p} - \|u_k\|_{\alpha,p} - CT \end{aligned}$$

Because:

$$\mu\|J'(u_k)\|_{(E_0^{\alpha,p})'} \longrightarrow 0, \quad as\ k \longrightarrow +\infty.$$

Since:
$0 < \mu < \frac{1}{p\theta}$, we deduce that (u_n) is bounded in $E_0^{\alpha,p}$, because if (u_n) do not bounded, for a subsequence (u_{n_k}) such that $\|u_{n_k}\| \longrightarrow +\infty$, so we get:

$$\lim_{\|u_k\|_{\alpha,p}\longrightarrow+\infty} a\left(\frac{1}{p}-\mu\right)\|u_k\|^p_{\alpha,p} + b\left(\frac{1}{p\theta}-\mu\right)\|u_k\|^{p\theta}_{\alpha,p} - \|u_k\|_{\alpha,p} - CT = +\infty$$

Which is contradiction, therefore (u_n) is bounded.

It follows from $\mu \in (0, 1/p\theta)$ that $\{u_k\}$ is bounded in $E_0^{\alpha,p}$. Since $E_0^{\alpha,p}$ is a reflexive space, up to a subsequence, we can assume that $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$. Hence, we have:

$$\begin{cases} u_k \longrightarrow u\ in\ C([0,T],\mathbb{R}) \\ \|u_k - u\|_\infty \longrightarrow 0\ as\ k \longrightarrow +\infty. \end{cases}$$

and

$$\begin{aligned} \langle J'(u_k) - J'(u), u_k - u\rangle &= \langle J'(u_k), u_k - u\rangle - \langle J'(u), u_k - u\rangle \\ &\leq \|J'(u_k)\|_{-\alpha,q}\|u_k - u\|_{\alpha,p} - \langle J'(u), u_k - u\rangle \longrightarrow 0, \\ &as\ k \longrightarrow \infty. \end{aligned}$$

Because:

$$J'(u) \in E_0^{-\alpha,q} \Rightarrow \langle J'(u), u_k \rangle \longrightarrow \langle J'(u), u \rangle \Leftrightarrow \langle J'(u), u_k - u \rangle \longrightarrow 0.$$

and

$$\|J'(u_k)\|_{-\alpha,q} \|u_k - u\|_{\alpha,p} \leq \|J'(u_k)\|_{-\alpha,q} (\|u_k\|_{\alpha,p} + \|u\|_{\alpha,p}) \longrightarrow 0$$

Because: $\|J'(u_k)\|_{-\alpha,q} \longrightarrow 0$ and $(\|u_k\|_{\alpha,p} + \|u\|_{\alpha,p}) \leq C \ \ ((u_k) \ is \ bounded \ in \ E_0^{\alpha,p})$.
Moreover, Lemma (2.1) leads to (u_k) is bounded in $C([0,T],\mathbb{R})$ and $\|u_k - u\|_\infty \longrightarrow 0$ as $k \longrightarrow \infty$. Let us consider the sequence:

$$P_n = \langle J'(u_n), u_n \rangle + \int_0^T f(t, u_n(t)) u_n(t) dt - \langle J'(u_n), u \rangle - \int_0^T f(t, u_n(t)) u(t) dt, \tag{3.5}$$

we have:

$$\lim_{n \longrightarrow +\infty} P_n = 0. \tag{3.6}$$

Indeed:

$$\begin{aligned} \langle J'(u_n), u_n \rangle &\leq \|J'(u_n)\| \|u_n\| \longrightarrow 0, \ because : J'(u_n) \longrightarrow 0 \ in \ E_0^{\alpha,p}, \ and \ \|u_n\| \leq C, \\ \langle J'(u_n), u \rangle &\leq \|J'(u_n)\| \|u\| \longrightarrow 0, \ because : J'(u_n) \longrightarrow 0. \end{aligned}$$

and:

$$\begin{aligned} \int_0^T f(t, u_n(t)) u_n(t) dt &\longrightarrow \int_0^T f(t, u(t)) u(t) dt \\ \int_0^T f(t, u_n(t)) u(t) dt &\longrightarrow \int_0^T f(t, u(t)) u(t) dt \end{aligned}$$

as $n \longrightarrow \infty$.
Indeed, since $u_n \rightharpoonup u$ in $E_0^{\alpha,p}$, by using the compact embbeding $E_0^{\alpha,p} \hookrightarrow\hookrightarrow C([0,T],\mathbb{R})$ we have:

$$u_n \longrightarrow u \ in \ C([0,T],\mathbb{R}). \tag{3.7}$$

By using (H_1) we deduce that:

$$f(t, u_n(t)) \longrightarrow f(t, u(t))$$

Hence:

$$f(t, u_n(t)) u_n(t) \longrightarrow f(t, u(t)) u(t), \ for \ any \ t \in [0,T]. \tag{3.8}$$

on other hand, from (3.7), we can see that:

$$|u_n(t)| \leq M, \ \forall t \in [0,T].$$

Then:

$$|f(t, u_n(t))| \leq \sup_{t \in [0,T], \ y \in [-M,M]} |f(t,y)| \leq C.$$

Thanks the continuous of $f(.,.)$, we deduce that:

$$|f(t,u_n(t))|.|u_n(t)| \leq C.M \ \in L^1(0,T) \tag{3.9}$$

from (3.8) and (3.9) and applying D.C.T, we get:

$$\int_0^T f(t,u_n(t))u_n(t)dt \longrightarrow \int_0^T f(t,u(t))u(t)dt$$

By the almost the same method we can prove that:

$$\int_0^T f(t,u_n(t))u \ dt \longrightarrow \int_0^T f(t,u(t))u(t)dt$$

finally we get: (3.6).

On other hand, by direct calculation, we get:

$$\begin{aligned} P_n &= \left(a+b\|u_n\|_{\alpha,p}^{p(\theta-1)}\right)\int_0^T \phi_p({}^cD_{0+}^{\alpha}u_n(t)){}^cD_{0+}^{\alpha}u_n(t)dt - \left(a+b\|u_n\|_{\alpha,p}^{p(\theta-1)}\right)\int_0^T \phi_p({}^cD_{0+}^{\alpha}u_n(t)){}^cD_{0+}^{\alpha}u(t)dt \\ &= \left(a+b\|u_n\|_{\alpha,p}^{p(\theta-1)}\right)\int_0^T \phi_p({}^cD_{0+}^{\alpha}u_n(t))\left({}^cD_{0+}^{\alpha}u_n(t) -{}^c D_{0+}^{\alpha}u(t)\right)dt. \end{aligned}$$

Let us consider the sequence:

$$K_n = -\left(a+b\|u_n\|_{\alpha,p}^{p(\theta-1)}\right)\int_0^T \phi_p({}^cD_{0+}^{\alpha}u(t)){}^cD_{0+}^{\alpha}u_n(t)dt + \left(a+b\|u_n\|_{\alpha,p}^{p(\theta-1)}\right)\|u\|_{\alpha,p}^p \tag{3.10}$$

we know that, $u_n \rightharpoonup u$ in $E_0^{\alpha,p}$, then $\|u_n\|$ is bounded, so, passing to a subsequence if : necessary, we have:

$$\|u_n\|^p \longrightarrow t_0 \in \mathbb{R}$$

Since: $M:\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous, we get:

$$M\left(\|u_n\|^p\right) \longrightarrow M(t_0)$$

Since: $\phi_p({}^cD_{0+}^{\alpha}u(t)) \in L^{p'}(0,T)$, and $u_n \rightharpoonup u$, we have:

$$\int_0^T \phi_p({}^cD_{0+}^{\alpha}u(t)){}^cD_{0+}^{\alpha}u_n(t)dt \longrightarrow \int_0^T \phi_p({}^cD_{0+}^{\alpha}u(t)){}^cD_{0+}^{\alpha}u(t)dt = \|u_n\|_{\alpha,p}^p.$$

So:

$$\begin{cases} \left(a+b\|u_n\|_{\alpha,p}^{p(\theta-1)}\right)\int_0^T \phi_p({}^cD_{0+}^{\alpha}u(t)){}^cD_{0+}^{\alpha}u_n(t)dt \longrightarrow M(t_0)\|u_n\|_{\alpha,p}^p, \\ \left(a+b\|u_n\|_{\alpha,p}^{p(\theta-1)}\right)\|u\|_{\alpha,p}^p \longrightarrow M(t_0)\|u_n\|_{\alpha,p}^p. \end{cases}$$

We conclude that:

$$\lim_{n\longrightarrow+\infty} K_n = 0. \tag{3.11}$$

From (3.6) and (3.11) we have:

$$\lim_{n\longrightarrow+\infty} P_n + K_n = 0.$$

But, by simple calculating, we get:

$$\begin{aligned}
P_n + K_n &= \left(a + b\|u_n\|_{\alpha,p}^{p(\theta-1)}\right) \int_0^T \phi_p({}^cD_{0+}^{\alpha} u_n(t)) \left({}^cD_{0+}^{\alpha} u_n(t) -{}^c D_{0+}^{\alpha} u(t)\right) dt \\
&- \left(a + b\|u_n\|_{\alpha,p}^{p(\theta-1)}\right) \int_0^T \phi_p({}^cD_{0+}^{\alpha} u(t)){}^cD_{0+}^{\alpha} u_n(t) dt \\
&+ \left(a + b\|u_n\|_{\alpha,p}^{p(\theta-1)}\right) \int_0^T \phi_p({}^cD_{0+}^{\alpha} u(t)){}^cD_{0+}^{\alpha} u(t) dt \\
&= \left(a + b\|u_n\|_{\alpha,p}^{p(\theta-1)}\right) \int_0^T \phi_p\left({}^cD_{0+}^{\alpha} u_n(t) - \phi_p({}^cD_{0+}^{\alpha} u(t))\right) \left({}^cD_{0+}^{\alpha} u_n(t) -{}^c D_{0+}^{\alpha} u(t)\right) dt \\
&\geq a + \int_0^T \phi_p\left({}^cD_{0+}^{\alpha} u_n(t) - \phi_p({}^cD_{0+}^{\alpha} u(t))\right) \left({}^cD_{0+}^{\alpha} u_n(t) -{}^c D_{0+}^{\alpha} u(t)\right) dt
\end{aligned}$$

$$\geq a \begin{cases} c_1 \int_0^T \left|{}^cD_{0+}^{\alpha} u_n(t) -{}^c D_{0+}^{\alpha} u(t)\right|^p dt, & \text{if } p \geq 2, \\ c_2 \int_0^T \dfrac{\left|{}^cD_{0+}^{\alpha} u_n(t) -{}^c D_{0+}^{\alpha} u(t)\right|^2}{(|{}^cD_{0+}^{\alpha} u_n(t)| + |{}^cD_{0+}^{\alpha} u(t)|)^{2-p}} dt, & \text{if } 1 < p < 2. \end{cases} \tag{3.12}$$

The same analytical steps previously applied to the relation (2.14), It follows that:

$$\|u_n - u\|_{\alpha,p} \longrightarrow 0, \; n \longrightarrow \infty,$$

that is, u_n converges strongly to u in $E_0^{\alpha,p}$. Consequently, J satisfies the cerami condition.
Now we show that J satisfies the geometry conditions of mountain pass theorem.
By applying the same argument as in the condition (2.10) can be expressed in this form:

$$F(t,x) \leq \frac{(\Gamma(\alpha+1))^{p\theta}}{2p\theta T^{\alpha p\theta}} |x|^{p\theta}, \; for \; t \in [0,T] \; and \; x \in \mathbb{R}, \; \forall |x| \leq \delta'_{\xi}. \tag{3.13}$$

Let $\rho' = \frac{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}{T^{\alpha-1/p}}\delta' > 0$ and $\sigma' = (2b-1)\rho'^{p\theta}/p\theta > 0$. Then, by(2.4) we have:

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \|u\|_{\alpha,p} = \delta', \; u \in E_0^{\alpha,p} \; with \; |u\|_{\alpha,p} = \rho',$$

Which, together with (2.3) and (3.13), implies:

$$\begin{aligned}
J(u) &= \frac{a}{p}\|u\|_{\alpha,p}^p + \frac{b}{p\theta}\|u\|_{\alpha,p}^{p\theta} - \int_0^T F(t,u(t))dt \\
&\geq \frac{a}{p}\|u\|_{\alpha,p}^p + \frac{b}{p\theta}\|u\|_{\alpha,p}^{p\theta} - \int_0^T \frac{(\Gamma(\alpha+1))^{p\theta}}{2p\theta T^{\alpha p\theta}} |u(t)|^{p\theta} dt \\
&\geq \frac{a}{p}\|u\|_{\alpha,p}^p + \frac{b}{p\theta}\|u\|_{\alpha,p}^{p\theta} - \frac{(\Gamma(\alpha+1))^{p\theta}}{2p\theta T^{\alpha p\theta}} \int_0^T |u(t)|^{p\theta} dt \\
&\geq \frac{a}{p}\|u\|_{\alpha,p}^p + \frac{b}{p\theta}\|u\|_{\alpha,p}^{p\theta} - \frac{(\Gamma(\alpha+1))^{p\theta}}{2p\theta T^{\alpha p\theta}} \|u\|_{L^p}^{p\theta}
\end{aligned}$$

We know:

$$\|u\|_{L^p}^{p\theta} \leq \frac{T^{\alpha p\theta}}{(\Gamma(\alpha+1))^{p\theta}} \|u\|_{\alpha,p}^{p\theta}$$

It follows that:

$$\begin{aligned} J(u) &\geq \frac{a}{p}\|u\|_{\alpha,p}^{p} + \frac{b}{p\theta}\|u\|_{\alpha,p}^{p\theta} - \frac{1}{2p\theta}\|u\|_{\alpha,p}^{p\theta} \\ &= \frac{a}{p}\|u\|_{\alpha,p}^{p} + \left(\frac{b}{p\theta} - \frac{1}{2p\theta}\right)\|u\|_{\alpha,p}^{p\theta} \\ &\geq \left(\frac{2b-1}{p\theta}\right)\|u\|_{\alpha,p}^{p\theta} \\ &= \sigma' > 0, \quad \forall u \in E_0^{\alpha,p} \ with \ \|u\|_{\alpha,p} = \rho'. \end{aligned}$$

Since: $b > \frac{1}{2}$.
Hence, condition C_2 in Theorem (2.2) is satisfied.
By(H_3') and following the same steps in (2.3) we can prove the following inequality:

$$F(t,x) \geq c_4|x|^{\frac{1}{\mu}} - c_5, \ t \in [0,T], x \in \mathbb{R}.$$

For any $u \in E_0^{\alpha,p}\backslash\{0\}$, $\xi' \in \mathbb{R}^+$, noting that $\mu \in (0, 1/p\theta)$, we get:

$$\begin{aligned} J(\xi' u) &= \frac{a}{p}\|\xi' u\|_{\alpha,p}^{p} + \frac{b}{p\theta}\|\xi' u\|_{\alpha,p}^{p\theta} - \int_0^T F(t,\xi' u(t))dt \\ &\leq \frac{a}{p}\|\xi' u\|_{\alpha,p}^{p} + \frac{b}{p\theta}\|\xi' u\|_{\alpha,p}^{p\theta} - c_4\int_0^T |\xi' u(t)|^{\frac{1}{\mu}}\, dt + c_5 T \\ &= \frac{a\xi'^{p}}{p}\|u\|_{\alpha,p}^{p} + \frac{b\xi'^{p\theta}}{p\theta}\|u\|_{\alpha,p}^{p\theta} - c_4\xi'^{\frac{1}{\mu}}\|u\|_{L^{\frac{1}{\mu}}}^{\frac{1}{\mu}} + c_5 T \longrightarrow -\infty \end{aligned}$$

$, \ as \ \xi' \longrightarrow \infty.$

Because:

$$\frac{1}{\mu} > p\theta$$

■

Thus, for ξ_0' large enough and taking $e = \xi_0' u$, we have $J(e) < 0$. Therefore, condition (C_3) in Theorem(2.2) is also satisfied.
Finally, noting that $J(0) = 0$, we get the existence of critical point u such that $J(u) \geq \sigma' > 0$. Hence, u is a nontrivial weak solution of Problem (3.1).

3.2 Fractional p-kirchhoff problem with R.B condition

3.2.1 Presentation of problem

We consider the following nonlinear problem:

$$(P_3)\begin{cases} M\left(\|D_{0+}^{\alpha}u\|_{L^p}^{p}\right) D_{T-}^{\alpha}\phi_p\left(D_{0+}^{\alpha}u(t)\right) = f(t,u(t)), \quad t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{3.14}$$

Where: $M : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function satisfies the following condition:
(M_1) There exists $\eta > 0$ such that:

$$M(T) \geq \eta, \ for \ all \ t \geq 0,$$

(M_2) There exists $\mu_1 > 0$ such that:

$$\widehat{M}(t) \geq \mu_1 t M(t), \ \forall t \geq 0.$$

Where: $\widehat{M}(t) = \int_0^T M(s)ds$.

Definition 3.2 *we say that $u \in E_0^{\alpha,p}$ is a weak solution of problem (P_3) if and only if:*

$$M\left(\|u\|_{\alpha,p}^p\right) \int_0^T \phi_p\left({}^cD_{0+}^{\alpha}u(t)\right) \ {}^cD_{0+}^{\alpha}v(t)dt = \int_0^T f(t,u(t))v(t)dt,$$

$\forall v \in E_0^{\alpha,p}$.
We consider the associated functional: $J_1 : E_0^{\alpha,p} \longrightarrow \mathbb{R}$ defined by:

$$J_1(u) = \frac{1}{p}\widehat{M}\left(\|u\|_{\alpha,p}^p\right) - \int_0^T F(t,u(t))dt.$$

With: $\widehat{M}(\sigma) = \int_0^\sigma M(s)ds$, $F(t,\sigma) = \int_0^\sigma f(t,s)ds$.
Clearly J_1 is well defined and differentiable on $E_0^{\alpha,p}$ and we have:

$$\begin{aligned}\langle J_1'(u), v\rangle &= M\left(\|u\|_{\alpha,p}^p\right)\left[\int_0^T |{}^cD_{0+}^{\alpha}u(t)|^{p-2}.\ {}^cD_{0+}^{\alpha}u(t).{}^cD_{0+}^{\alpha}v(t)dt\right] - \int_0^T f(t,u(t))v(t)dt \\ &= M\left(\|u\|_{\alpha,p}^p\right)\int_0^T \phi_p\left({}^cD_{0+}^{\alpha}u(t)\right).{}^cD_{0+}^{\alpha}v(t)dt - \int_0^T f(t,u(t))v(t)dt \qquad (3.15)\end{aligned}$$

u is a weak solution of (P_3), so u is a critical point of J_1.

3.2.2 Existence of weak solutions of problem (3.14)

The main result of this part is:

Theorem 3.3 *Let $\frac{1}{p} < \alpha \leq 1$ and (H_1) be satisfied. Assume that:*
(H_2'') there exist $a' \in \left(0, \frac{\eta\mu_1}{p}\frac{\Gamma(\alpha+1)^p}{T^{\alpha p}}\right)$ and $b' \in L^1[(0,T),\mathbb{R}^+]$ such that:

$$|F(t,x)| \leq a'|x|^p + b'(t), \ \forall t \in [0,T], \ x \in \mathbb{R}.$$

Then, Problem (P_3) has at least a nontrivial solution.
Proof. J_1 is **coercive**:
Using (H_2'') and (2.3) we have:

$$\begin{aligned}J_1(u) &= \frac{1}{p}\widehat{M}\left(\|u\|_{\alpha,p}^p\right) - \int_0^T F(t,u(t))dt. \\ &\geq \frac{\mu_1}{p}\|u\|_{\alpha,p}^p M(\|u\|_{\alpha,p}^p) - a'\int_0^T |u(t)|^p dt - \int_0^T b'(t)dt \\ &\geq \frac{\eta\mu_1}{p}\|u\|_{\alpha,p}^p - a'\|u\|_{L^p}^p - \|b'\|_{L^1} \\ &\geq \frac{\eta\mu_1}{p}\|u\|_{\alpha,p}^p - a'\frac{T^{\alpha p}}{(\Gamma(\alpha+1))^p}\|u\|_{\alpha,p}^p - \|b'\|_{L^1} \\ &= \left[\frac{\eta\mu_1}{p} - a'\frac{T^{\alpha p}}{(\Gamma(\alpha+1))^p}\right]\|u\|_{\alpha,p}^p - \|b'\|_{L^1}\end{aligned}$$

as:

$$a' < \frac{\eta\mu_1}{p}\frac{\Gamma(\alpha+1)^p}{T^{\alpha p}}$$

Now passing to the limit:

$$\lim_{\|u\|_{\alpha,p}\longrightarrow\infty} J_1(u) = \lim_{\|u\|_{\alpha,p}\longrightarrow\infty}\left[\frac{\eta\mu_1}{p} - a'\frac{T^{\alpha p}}{\Gamma(\alpha+1)^p}\right]\|u\|^p_{\alpha,p} = +\infty.$$

Thus, J_1 is **coercive**.
Let $(u_n) \subset E_0^{\alpha,p}$ such that $u_n \rightharpoonup n$ in $E_0^{\alpha,p}$. By dominated convergence theorem, we can show that

$$\int_0^T F(t,u_n(t))dt \longrightarrow \int_0^T F(t,u(t))dt$$

we know that:

$$\|u\|_{\alpha,p} \leq \lim_{n\longrightarrow+\infty} \inf \|u\|_{\alpha,p}$$

Since: $\widehat{M}$ is increasing:

$$\widehat{M}(\|u\|_{\alpha,p}) \leq \lim_{n\longrightarrow+\infty} \inf \hat{M}(\|u\|_{\alpha,p})$$

Hence:

$$J_1(u) \leq \lim_{n\longrightarrow+\infty} \inf J_1(u)$$

In conclusion: J_1 is coercive, weakly lower semicontinuous, differentiable, there J_1 has a least critical point which is a weak solution of Problem (3.14). ■
New, we use **Theorem** (2.2) to discuss the existence of mountain pass solutions of the fractional boundary value Problem (3.14).

Theorem 3.4 *Assume that* $\frac{1}{p} < \alpha \leq 1, \quad 0 < \mu_1 < p$ *and* (H_1), (M_1), (M_2), *satisfied. Assume that:*
(H_3'') $\exists\mu_2 \in \left(0, \frac{\mu_1}{p}\right)$ *and* $M > 0$ *such that:*

$$F(t,x) \leq \mu_2 x f(t,x), \ \forall t \in [0,T], \ x \in \mathbb{R} \ \textit{with} \ |x| \geq M,$$

(H_4'') *for* $t \in [0,T]$ *and* $x \in \mathbb{R}$ *we have:*

$$\lim_{|u|\longrightarrow 0} \sup \frac{F(t,x)}{|x|^p} \leq \frac{\eta(\Gamma(\alpha+1))^p}{2pT^{\alpha p}}.$$

Proof. We will verify that J_1 satisfies all the conditions of Theorem(2.2).
Firstly, we show that J_1 satisfies the P.S condition. Since $F(t,x) - \mu x f(t,x)$ is continuous, there exists $C \in \mathbb{R}^+$ see Remark (2.5) to show (H_3'') it's the same work. and we conclude that:

$$F(t,x) \leq \mu_2 x f(t,x) + C, \ t \in [0,T], \ x \in \mathbb{R}. \tag{3.16}$$

Let $\{u_K\} \subset E_0^{\alpha,p}$ be such that:

$$|J_1(u_k)| \leq K, \ J_1'(u_k) \longrightarrow 0 \ \ as \ k \longrightarrow \infty.$$

Using (M_1), (M_2) and accoding to (3.15), we get:

$$\langle J_1'(u_k), u_k\rangle = M\left(\|u_k\|^p_{\alpha,p}\right)\|u_k\|^p_{\alpha,p} - \int_0^T f(t,u_k(t))u_k(t)dt$$

which, together with (3.16), yields:

$$\begin{aligned}
K &\geq J_1(u_k) \\
&= \frac{1}{p}\hat{M}\left(\|u_k\|_{\alpha,p}^p\right) - \int_0^T F(t,u_k(t))dt \\
&\geq \frac{\mu_1}{p}M\left(\|u_k\|_{\alpha,p}^p\right)\|u_k\|_{\alpha,p}^p - \mu_2\int_0^T f(t,u_k(t))u_k(t)dt - CT \\
&\geq \frac{\mu_1}{p}M\left(\|u_k\|_{\alpha,p}^p\right)\|u_k\|_{\alpha,p}^p + \mu_2\langle J_1'(u_k),u_k\rangle - \mu_2 M\left(\|u_k\|_{\alpha,p}^p\right)\|u_k\|_{\alpha,p}^p - CT \\
&= M\left(\|u_k\|_{\alpha,p}^p\right)\|u_k\|_{\alpha,p}^p\left(\frac{\mu_1}{p}-\mu_2\right) + \mu_2\langle J_1'(u_k),u_k\rangle - CT
\end{aligned}$$

we have:

$$\mu_2 < \frac{\mu_1}{p}, \ \left(\mu_2 \in \left(0,\frac{\mu_1}{p}\right)\right)$$

then:

$$K \geq M\left(\|u_k\|_{\alpha,p}^p\right)\|u_k\|_{\alpha,p}^p\left(\frac{\mu_1}{p}-\mu_2\right) - \mu_2\|J_1'(u_k)\|_{(E_0^{\alpha,p})'}\|u_k\|_{\alpha,p} - CT$$

Since: $J_1'(u_k) \longrightarrow 0$, there exists $N_0 \in \mathbb{N}$ such that:

$$\begin{aligned}
K &\geq M\left(\|u_k\|_{\alpha,p}^p\right)\|u_k\|_{\alpha,p}^p\left(\frac{\mu_1}{p}-\mu_2\right) - \|u_k\|_{\alpha,p} - CT \\
&\geq \eta\left(\frac{\mu_1}{p}-\mu_2\right)\|u_k\|_{\alpha,p}^p - \|u_k\|_{\alpha,p} - CT, \quad \forall k \geq N_0
\end{aligned}$$

It follows from $\mu_2 \in (0,\mu_1/p)$ that $\{u_k\}$ is bounded in $E_0^{\alpha,p}$. Since $E_0^{\alpha,p}$ is a reflexive space, up to a subsequence, we can assume that $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$. Hence, we have:

$$\begin{cases} u_k \longrightarrow u \ in \ C([0,T],\mathbb{R}) \\ \|u_k - u\|_\infty \longrightarrow 0 \ as \ k \longrightarrow +\infty. \end{cases}$$

and

$$\begin{aligned}
\langle J_1'(u_k) - J'(u), u_k - u\rangle &= \langle J_1'(u_k), u_k - u\rangle - \langle J_1'(u), u_k - u\rangle \\
&\leq \|J_1'(u_k)\|_{-\alpha,q}\|u_k - u\|_{\alpha,p} - \langle J_1'(u), u_k - u\rangle \\
&\longrightarrow 0, \ as \ k \longrightarrow \infty.
\end{aligned}$$

Moreover, Lemma (2.1) we leads that (u_k) is bounded in $C([0,T],\mathbb{R})$ and $\|u_k - u\|_\infty \longrightarrow 0$ as $k \longrightarrow \infty$. Then, let us consider the sequence:

$$P_n' = \langle J_1'(u_n),u_n\rangle + \int_0^T f(t,u_n(t))u_n(t)dt - \langle J_1'(u_n),u\rangle - \int_0^T f(t,u_n(t))u(t)dt, \tag{3.17}$$

We have:

$$\lim_{n\longrightarrow+\infty} P_n' = 0.$$

Because:

$$\langle J_1'(u_n),u_n\rangle \leq \|J_1'(u_n)\|\|u_n\| \longrightarrow 0, \ because: J_1'(u_n) \longrightarrow 0 \ in \ E_0^{\alpha,p}, and \ \|u_n\| \leq C,$$

$$\langle J_1'(u_n), u\rangle \leq \|J_1'(u_n)\|\|u\| \longrightarrow 0,\ because : J_1'(u_n) \longrightarrow 0.$$

and:

$$\int_0^T f(t,u_n(t))u_n(t)dt \longrightarrow \int_0^T f(t,u(t))u(t)dt.$$
$$\int_0^T f(t,u_n(t))u(t)dt \longrightarrow \int_0^T f(t,u(t))u(t)dt.$$

on other hand , by simple calculating we get:

$$P_n' = M\left(\|u_n\|_{\alpha,p}^p\right)\|u_n\|_{\alpha,p}^p - M\left(\|u_n\|_{\alpha,p}^p\right)\int_0^T \phi_p({}^cD_{0+}^\alpha u_n(t))^cD_{0+}^\alpha u(t)dt$$

Let us consider:

$$K_n' = -M\left(\|u_n\|_{\alpha,p}^p\right)\int_0^T \phi_p({}^cD_{0+}^\alpha u(t))^cD_{0+}^\alpha u_n(t)dt + M\left(\|u_n\|_{\alpha,p}^p\right)\|u_n\|_{\alpha,p}^p \tag{3.18}$$

We know that $M\left(\|u_n\|_{\alpha,p}^p\right) \longrightarrow M(t_0) > 0$ since $u_n \rightharpoonup u \in E_0^{\alpha,p}$ and $\phi_p({}^cD_{0+}^\alpha u(t)) \in L^{\frac{p}{p-1}}(0,T)$, we have:

$$\int_0^T \phi_p({}^cD_{0+}^\alpha u(t))^cD_{0+}^\alpha u_n(t)dt \longrightarrow \int_0^T \phi_p({}^cD_{0+}^\alpha u(t))^cD_{0+}^\alpha u(t)dt = \|u_n\|_{\alpha,p}^p.$$

So,

$$\lim_{n\longrightarrow+\infty} K_n' = 0. \tag{3.19}$$

Hence:

$$\begin{aligned}
P_n' + K_n' &= M\left(\|u_n\|_{\alpha,p}^p\right)\|u_n\|_{\alpha,p}^p - M\left(\|u_n\|_{\alpha,p}^p\right)\int_0^T \phi_p({}^cD_{0+}^\alpha u_n(t))^cD_{0+}^\alpha u(t)dt \\
&- M\left(\|u_n\|_{\alpha,p}^p\right)\int_0^T \phi_p({}^cD_{0+}^\alpha u(t))^cD_{0+}^\alpha u_n(t)dt + M\left(\|u_n\|_{\alpha,p}^p\right)\int_0^T \phi_p({}^cD_{0+}^\alpha u(t))^cD_{0+}^\alpha u(t)dt \\
&= M\left(\|u_n\|_{\alpha,p}^p\right)\int_0^T \phi_p({}^cD_{0+}^\alpha u_n(t))\left({}^cD_{0+}^\alpha u_n(t) -{}^c D_{0+}^\alpha u(t)\right)dt \\
&- M\left(\|u_n\|_{\alpha,p}^p\right)\int_0^T \phi_p({}^cD_{0+}^\alpha u(t))\left({}^cD_{0+}^\alpha u_n(t) -{}^c D_{0+}^\alpha u(t)\right)dt \\
&= M\left(\|u_n\|_{\alpha,p}^p\right)\int_0^T \phi_p\left({}^cD_{0+}^\alpha u_n(t) - {}^cD_{0+}^\alpha u(t)\right)\left({}^cD_{0+}^\alpha u_n(t) -{}^c D_{0+}^\alpha u(t)\right)dt \\
&\geq \eta\int_0^T \phi_p\left({}^cD_{0+}^\alpha u_n(t) - {}^cD_{0+}^\alpha u(t)\right)\left({}^cD_{0+}^\alpha u_n(t) -{}^c D_{0+}^\alpha u(t)\right)dt
\end{aligned}$$

$$\geq \eta\begin{cases} c_1\int_0^T \left|{}^cD_{0+}^\alpha u_n(t) -{}^c D_{0+}^\alpha u(t)\right|^p dt, & \text{if } p \geq 2, \\ c_2\int_0^T \dfrac{\left|{}^cD_{0+}^\alpha u_n(t) -{}^c D_{0+}^\alpha u(t)\right|^2}{\left(|{}^cD_{0+}^\alpha u_n(t)| + |{}^cD_{0+}^\alpha u(t)|\right)^{2-p}}dt, & \text{if } 1 < p < 2. \end{cases} \tag{3.20}$$

we follow the same steps previously applied to the relation (2.14), we have:

$$\|u_n - u\|_{\alpha,p} \longrightarrow 0,\ n \longrightarrow \infty,$$

that is, u_n converges strongly to u in $E_0^{\alpha,p}$, and J_1 satisfies cerami condition.
Now we show that J_1 satisfies the geometry conditions of mountain pass theorem.
By applying the same argument as in the Remark (2.6), it follows that the equation (2.10) can be expressed in this form:

$$F(t,x) \leq \eta \frac{(\Gamma(\alpha+1))^p}{2pT^{\alpha p}}|x|^p,\ for\ t \in [0,T]\ and\ x \in \mathbb{R},\ \forall |x| \leq \delta''_\xi. \tag{3.21}$$

Let $\rho'' = \frac{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}{T^{\alpha-1/p}}\delta'' > 0$ and $\sigma'' = \rho''^p\eta/2p > 0$. Then, by(2.4) we have:

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\|u\|_{\alpha,p} = \delta'',\ u \in E_0^{\alpha,p}\ with\ |u\|_{\alpha,p} = \rho'',$$

Which, together with (2.3) and (3.21), implies:

$$J_1(u) = \frac{1}{p}\widehat{M}\left(\|u\|^p_{\alpha,p}\right) - \int_0^T F(t,u(t))dt$$

By (M_2), (M_1) we have:

$$\begin{aligned}
J_1(u) &\geq \frac{1}{p}M\left(\|u\|^p_{\alpha,p}\right)\|u\|^p_{\alpha,p} - \eta\frac{(\Gamma(\alpha+1))^p}{2pT^{\alpha p}}|u|^p_{L^p} \\
&\geq \frac{\eta}{p}\|u\|^p_{\alpha,p} - \frac{\eta}{2p}\|u\|^p_{\alpha,p} \\
&= \frac{1}{p}\left(\eta - \frac{\eta}{2}\right)\|u\|^p_{\alpha,p} \\
&= \frac{\eta}{2p}\|u\|^p_{\alpha,p} \\
&= \sigma'' > 0,\ \forall u \in E_0^{\alpha,p}\ with\ \|u\|_{\alpha,p} = \rho''
\end{aligned}$$

Hence, condition C_2 in Theorem (2.2) is satisfied.
By (H_3'') the same steps as before are followed $(see$ (2.3)) to prove the following relation:

$$F(t,x) \geq c_4|x|^{\frac{1}{\mu_2}} - c_5,\ t \in [0,T], x \in \mathbb{R}.$$

For any $u \in E_0^{\alpha,p}\backslash\{0\}$, $\xi' \in \mathbb{R}^+$, noting that $\mu_2 \in (0,\mu_1/p)$, we get:

$$J_1(\xi''u) = \frac{1}{p}\hat{M}\left(\|\xi''u\|^p_{\alpha,p}\right) - \int_0^T F(t,\xi''u(t))dt$$

Notice that (M_2) implies that for any $t > \varepsilon_0$, we have:

$$\int_{\varepsilon_0}^t \frac{M(s)}{\widehat{M}(s)}ds \leq \frac{1}{\mu_1}\int_{\varepsilon_0}^t \frac{1}{s}ds$$

$$\ln\frac{\widehat{M}(t)}{\hat{M}(\varepsilon_0)} \leq \frac{1}{\mu_1}\ln\frac{t}{\varepsilon_0}$$

$$\widehat{M}(t) \leq \frac{\widehat{M}(\varepsilon_0)}{\varepsilon_0^{\frac{1}{\mu_1}}}\ t^{\frac{1}{\mu_1}}$$

$$\widehat{M}(t) \leq Ct^{\frac{1}{\mu_1}}, \quad with : C = \frac{\widehat{M}(\varepsilon_0)}{(\varepsilon_0)^{\frac{1}{\mu_1}}}$$

Then, we get:

$$\begin{aligned} J_1(\xi'' u) &\leq \frac{1}{p}\hat{M}\left(\|\xi'' u\|^p_{\alpha,p}\right) - c_4 \int_0^T |\xi'' u(t)|^{\frac{1}{\mu}}\, dt + c_5 T \\ &\leq \frac{C}{p}\|\xi'' u\|^{\frac{\mu_1}{p}}_{\alpha,p} - c_4 \xi''^{\frac{1}{\mu_2}} \|u\|^{\frac{1}{\mu_2}}_{L^{\frac{1}{\mu_2}}} + c_5 T \\ &= \frac{\xi'' C}{p}\|u\|^{\frac{\mu_1}{p}}_{\alpha,p} - c_4 \xi''^{\frac{1}{\mu_2}} \|u\|^{\frac{1}{\mu_2}}_{L^{\frac{1}{\mu_2}}} + c_5 T \longrightarrow -\infty \end{aligned}$$

$, \xi' \longrightarrow \infty.$

■

Thus, taking ξ''_0 large enough and letting $e = \xi''_0 u$, we have $J(e) < 0$. Therefore, condition (C_3) in Theorem (2.2) is also satisfied.
Noting that $J_1(0) = 0$, we get the existence of a critical point u such that $J_1(u) \geq \sigma'' > 0$. Hence, u is a nontrivial weak solution of Problem (3.14).

3.3 Multiplicity of solutions

3.3.1 Presentation of problem

We consider the following problem:

$$(P_4)\begin{cases} M\left(\|D^\alpha_{0+}u\|^p_{L^p}\right) D^\alpha_{T-}\phi_p\left(D^\alpha_{0+}u(t)\right) = f(t,u(t)), & t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{3.22}$$

Where: $M : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function satisfies the following hypotheses from M and f:
(1) There exist positive constants A, B *and* $\theta > 0$ such that, for all $t > 0$,

$$At^\theta \leq M(t) \leq Bt^\theta.$$

(2) There exist positive constants $Q_1, Q_2,$ *and* q such that:

$$Q_1 t^{q-1} \leq f(x,t) \leq Q_2 t^{q-1}.$$

(3) for all $u \in E^{\alpha,p}_0$ positive, and for all $t \in [0,T]$ where: $q \in \left(p, p* = \frac{Np}{N-p}\right)$ and $\theta > q/p$,

$$f(t,u(t)) = -f(t,-u(t))$$

Definition 3.3 *we say that $u \in E^{\alpha,p}_0$ is a weak solution of problem (P_4) if and only if:*

$$M\left(\|u\|^p_{\alpha,p}\right)\int_0^T \phi_p\left({}^cD^\alpha_{0+}u(t)\right)\, {}^cD^\alpha_{0+}v(t)dt = \int_0^T f(t,u(t))v(t)dt,$$

$\forall v \in E_0^{\alpha,p}$.

Let us go back to the problem (P_4). For this, we consider the functional $J_2 : E_0^{\alpha,p} \to \mathbb{R}$, defined by:

$$J_2(u) = \frac{1}{p}\widehat{M}(\|u\|^p) - \int_0^T F(t, u(t))\, dt,$$

With: $\widehat{M}(\sigma) = \int_0^\sigma M(s)ds$, $F(t,\sigma) = \int_0^\sigma f(t,s)ds$.

Clearly J_2 is well defined and differentiable on $E_0^{\alpha,p}$ and we have:

$$\begin{aligned}\langle J_2'(u), v\rangle &= M\left(\|u\|_{\alpha,p}^p\right)\left[\int_0^T |{}^cD_{0+}^\alpha u(t)|^{p-2}.\ {}^cD_{0+}^\alpha u(t).{}^cD_{0+}^\alpha v(t)dt\right] - \int_0^T f(t,u(t))v(t)dt \\ &= M\left(\|u\|_{\alpha,p}^p\right)\int_0^T \phi_p\left({}^cD_{0+}^\alpha u(t)\right).{}^cD_{0+}^\alpha v(t)dt - \int_0^T f(t,u(t))v(t)dt. \qquad (3.23)\end{aligned}$$

u is a weak solution of (P_4), so u is a critical point of J_2.

Theorem 3.5 *Assume that* (1), (2) *and* (3) *then* (P_4) *has infinitely many solutions.*

3.3.2 Krasnoselskii genus

We will start by considering some basic notions on the Krasnoselskii genus that we will use in the proof of our main result.

Definition 3.4 *Let $A \in \mathcal{A}$. The **Krasnoselskii genus** $\gamma(A)$ of A is defined as being the least positive integer k such that there is an odd mapping $\varphi \in C(A, \mathbb{R}^k)$ satisfying $\varphi(x) \neq 0$ for all $x \in A$. If such a k does not exist, we set $\gamma(A) = \infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.*

Theorem 3.6 *Let $E = \mathbb{R}^N$, and let $\partial\Omega$ be the boundary of an open, symmetric, and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then,*

$$\gamma(\partial\Omega) = N.$$

Corollary 3.1

$$\gamma(S^{N-1}) = N.$$

As a consequence of this , if E is an infinite-dimension and separable , and S is the unit sphere in E, then

$$\gamma(S) = \infty.$$

Theorem 3.7 *Let $J \in C^1(X, \mathbb{R})$ be a functional satisfying the Palais–Smale condition. Furthermore let us Suppose that:*

(i) J is bounded from below and even;

(ii) there is a compact set $K \in \mathcal{A}$ such that $\gamma(K) = k$ and

$$\sup_{x\in K} J(x) < J(0).$$

Then J possesses at least k pairs of distinct critical points, and their corresponding critical values are less than $J(0)$.

In the proof of theorem (3.5) we shall need the following technical results:

Lemma 3.2 *J_2 is bounded from below.*

Proof. Using (1) and (2) we get:

$$\begin{aligned} J_2(u) &= \frac{1}{p}\widehat{M}(\|u\|^p) - \int_0^T F(t,u(t))\,dt \\ &\geq \frac{1}{p}\int_0^{\|u\|^p} A(t)^\theta dt - \int_0^T Q_2|u|^{q-1}dt \\ &\geq \frac{A}{p(\theta+1)}\|u\|_{\alpha,p}^{p(\theta+1)} - C\int_0^T |u|^q dt \\ &\geq \frac{A}{p(\theta+1)}\|u\|_{\alpha,p}^{p(\theta+1)} - C\frac{T^{\alpha q}}{(\Gamma(\alpha+1))^q}|u\|_{\alpha,p}^q \end{aligned}$$

We know that:

$$\theta+1 > \theta > \frac{q}{p}.$$

So, J_2 is bounded from below. ■

Lemma 3.3 *J_2 satisfies the (P.S) condition.*

We can prove that J_2 is satisfies the (P.S) condition, by the same method with previouse section. Moreover, J_2 is even because f is odd ($F(x,.)$ *is even*), and $J_2 \in C^1$.
Proof. of Theorem (3.5).
Let us consider $\{e_1, e_2, \ldots\}$, a Schauder basis of $E_0^{\alpha,p}$, and for each $k \in \mathbb{N}$ consider:

$$X_k = \text{span}\{e_1, e_2, \ldots, e_k\},$$

the subspace of $E_0^{\alpha,p}$ generated by the k vectors $e_1, e_2, \ldots, e_k$.

Note that $X_k \hookrightarrow L^q(\Omega)$, for all $1 \leq q \leq p^*$, with continuous embeddings. Thus, the norms of $E_0^{\alpha,p}$ and $L^q(\Omega)$ are equivalent on X_k. Hence, there exists a positive constant $C(k)$, which depends on k, such that:

$$\|.\|_{L^q(\Omega)} \equiv \|.\|_{E_0^{\alpha,p}} \quad in\ X_k.$$

Hence:

$$-\|u\|_{L^q(\Omega)} \leq -C(k)\|u\|_{E_0^{\alpha,p}},$$

for all $u \in X_k$. We now use (1) and (2) to conclude that:

$$\begin{aligned} J_2 &= \frac{1}{p}\widehat{M}(\|u\|^p) - \int_0^T F(t,u(t))\,dt \\ &\leq \frac{B}{p(\theta+1)}\|u\|^{p(\theta+1)} - \frac{Q_1}{q}\int_0^T |u(t)|^q dt \\ &\leq \frac{B}{p(\theta+1)}\|u\|^{p(\theta+1)} - \frac{C(k)Q_1}{q}\|u\|^q \\ &= \|u\|^q\left[\frac{B}{p(\theta+1)}\|u\|^{p(\theta+1)-q} - \frac{C(k)Q_1}{q}\right]. \end{aligned}$$

Let R be a positive constant such that:

$$\frac{B}{p(\theta+1)-q} R^{p(\theta+1)} < \frac{C(k)Q_1}{q}$$

Thus, for all $0 < r < R$, and considering $K = u \in \ X_k : \|u\| = r$, we get:

$$J_2(u) \leq r^q \left(\frac{B}{p(\theta+1)} \|u\|^{p(\theta+1)-q} - \frac{C(k)Q_1}{q} \right) < R^q \left(\frac{B}{p(\theta+1)} \|u\|^{p(\theta+1)-q} - \frac{C(k)Q_1}{q} \right) < 0 = J(0).$$

which implies:

$$\sup_K J_2(u) < 0 = J(0).$$

Since X_k and $\mathbb{R}^k$ are isomorphic, and K and S^{k-1} are homeomorphic, we conclude that $\gamma(K) = k$.

Moreover, from condition **(3)**, the functional J_2 is even. By applying Clarke's theorem, we deduce that J_2 has at least k pairs of different critical points.

Since k is arbitrary, we obtain infinitely many critical points of J_2. ■

Conclusion

This mémoire has investigated the existence of weak solutions for a class of fractional p-Kirchhoff type differential equations. Employing variational methods and critical point theory within the framework of fractional calculus, we have rigorously established existence results for fractional p-Laplacian problems.

These findings contribute significantly to the theoretical understanding of nonlinear and nonlocal fractional models, which are increasingly instrumental in describing complex physical and engineering systems exhibiting anomalous diffusion and memory effects.

Future research directions include exploring the uniqueness and regularity of solutions, extending the analysis to more general nonlinear fractional equations, and developing efficient numerical methods to facilitate practical applications.

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Abstract

This mémoire investigates the existence and multiplicity of solutions for a class of nonlinear fractional differential equations of Kirchhoff type involving the fractional p-Laplacian operator. The study is conducted within a variational framework, relying on tools from critical point theory and nonlinear analysis.