جامعة محمد بوضياف - المسيلة
Université Mohamed Boudiaf - M'sila

PEOPLE's DEMOCRATIC REPUBLIC OF ALGERIA

**MINISTRY OF HIGHER EDUCATION
AND SCIENTIFIC RESEARCH**

Mohamed Boudiaf University of M'sila
Faculty of Mathematics and Computer Sciences
Department of Mathematics

Master's Thesis

Field : Mathematics and Computer Science
Branch : Mathematics
Option : Partial Differential Equations and Applications

Theme

Insights into Nonlinear Diffusion Problems and Implications for Biological Systems

---

Presented by :

BOUZID FARIDA

Supported on : June 14^{th}, 2025.

In front of the jury composed of :

NOUREDDINE BENHAMIDOUCHE	Prof.,	Mohamed Boudiaf University of M'sila	**President**.
RABAH DJEMIAT	MCB,	Higher National School of Hydro-Blida	**Examiner**.
BILAL BASTI	MCA,	Mohamed Boudiaf University of M'sila	**Supervisor**.

University Year 2024/2025

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمْ

"وَقُلِ اعْمَلُواْ فَسَيَرَى اللّهُ عَمَلَكُمْ وَرَسُولُهُ
وَالْمُؤْمِنُونَ وَسَتُرَدُّونَ إِلَى عَالِمِ الْغَيْبِ وَالشَّهَادَةِ
فَيُنَبِّئُكُم بِمَا كُنتُمْ تَعْمَلُونَ" [التوبة: 105]

Acknowledgments

All praise is due to Allah, Lord of the worlds, who granted me the capability and perseverance to complete this research. I extend my sincere gratitude to all who contributed to the realization of this work. I am deeply indebted to *Dr.* Bilal Basti for his esteemed supervision, insightful guidance, and meticulous academic oversight. My respectful thanks also go to the examination committee members, *Prof.* Noureddinne Benhamidouche and *Prof.* Rabah Djemiat, for their time and effort in evaluating this research. I also gratefully acknowledge the support of my professors and colleagues throughout my academic journey. Special recognition goes to my class of 2025 peers, whose collaboration and encouragement were indispensable.

Notation

PDEs	Partial Differential Equations.
ODEs	Ordinary Differential Equations.
$\mathbb{R}$	Real numbers $(-\infty, \infty)$.
$\mathbb{R}_+$	Positive real numbers $(0, \infty)$.
$\mathbb{R}^n$	Euclidean space of real numbers with dimension n.
$C_0^1(0, \infty)$	The space of continuously differentiable functions on the open interval $(0, \infty)$ that vanish at the boundary.
$\mathfrak{Dfe}$	Disease-free equilibrium point.
$\mathfrak{Eqp}$	Endemic equilibrium point.
$\mathfrak{Rb}$	Basic reproduction number.
$\mathbb{N}_0$	Zero natural numbers $\{0, 1, 2, 3, \ldots\}$.

Contents

Introduction

Diffusion phenomena are fundamental processes observed in a wide range of scientific domains. They govern the spread of substances, energy, or even information through space and time. Examples include heat conduction in physics, mixing of chemical species, gas transport in porous media, and the transmission of signals or nutrients in biological systems. These processes also extend beyond the natural sciences, such as modeling the spread of information in social networks. Due to their ubiquity and importance, diffusion phenomena have inspired the development of sophisticated mathematical models, particularly those based on nonlinear partial differential equations (PDEs), to better understand their dynamics.

This thesis presents a detailed and in-depth study of a set of nonlinear partial differential equations (PDEs) used to model diffusion phenomena in diverse physical and biological contexts. The thesis is structured into four interrelated chapters, beginning with foundational concepts and progressively moving toward more complex and realistic models.

In the first chapter, we review the fundamental definitions and concepts related to nonlinear diffusion equations, with a focus on important mathematical properties such as Banach's fixed point theorem, as discussed in several references, notably [12] and [**?**]. We also highlight the role of self-similarity, a key analytical tool in studying the solutions of these equations, which facilitates the analysis by transforming PDEs into ordinary differential equations, as illustrated in key references such as [5] and [6].

The second chapter is devoted to the study of the porous medium equation, which models the diffusion of gas through porous materials (such as soil and rocks). This equation is widely used in vital applications including environmental pollution control, oil recovery, and drug delivery systems. It has attracted significant attention in previous studies, particularly concerning the existence and uniqueness of compactly supported solutions, as investigated by Gilding and Peletier in [11].

The third chapter focuses on a broader class of nonlinear diffusion equations, specifically non-divergence form PDEs. These equations generalize the porous medium equation and encompass the non-divergence formulations studied in reference [2] within a unified framework. We discuss these models in terms of their structure, mathematical analysis, and solution behavior.

Finally, in the fourth chapter, we shift from physical to biological models by employing reaction-diffusion systems to simulate the spatiotemporal dynamics of disease transmission. These models account for spatial variations in population density, geographical barriers, and environmental factors, making them more realistic and accurate in describing epidemic spread in heterogeneous environments.

CHAPTER 1

PRELIMINARIES ON DIFFUSION PROBLEMS

This chapter presents the key definitions related to diffusion phenomena, whether from a physical or biological perspective, in addition to the definitions relevant to the content presented in this thesis. These definitions are essential for understanding the theoretical foundations and mathematical models that will be used in the subsequent chapters to analyze diffusion phenomena.

1.1 Introduction to Partial Differential Equations (PDEs)

A PDE describes the relationship between a function and its partial derivatives. Such equations are used to model various phenomena in physics, chemistry, and other sciences. A PDE typically involves an unknown function u that depends on variables $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and can be expressed in the general form:

$$F\left(x_1, x_2, \ldots, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}, \ldots, \frac{\partial^n u}{\partial x_n^n}\right) = 0,$$

where

- $u = u(x)$ is the unknown function.
- $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots$ represent the first-order partial derivatives of u.
- $\frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}, \ldots$ represent the second-order partial derivatives of u.
- F is a function that defines the relationship between these terms.

Generalities on PDEs

The following concepts are fundamental to the study of PDEs and can be found in various references such as [14] and [15].

Definition 1.1 (Linear PDE)**.** *A linear PDE involves partial derivatives of an unknown function, where the relationship between these derivatives is linear. This means that the derivatives appear only to the first degree, and there are no products or powers of the derivatives or the function itself.*

Definition 1.2 (Nonlinear PDE)**.** *A nonlinear PDE is an equation that contains partial derivatives, where the relationship between these derivatives is nonlinear. This includes equations with products or powers of the derivatives or the function itself.*

Example 1.1. *Consider the following equations:*

$$\begin{aligned} \tfrac{\partial^2 u}{\partial x^2} - \tfrac{\partial^2 u}{\partial y^2} &= 0, && \text{(linear PDE)}, \\ \left(\tfrac{\partial u}{\partial x}\right)^2 + \left(\tfrac{\partial u}{\partial y}\right)^2 &= 1, && \text{(nonlinear PDE)}. \end{aligned}$$

First-Order PDEs

The most general form of a first-order linear PDE with two variables is:

$$A\,(x, y)\,\frac{\partial u}{\partial x} + B\,(x, y)\,\frac{\partial u}{\partial y} + C\,(x, y)\,u = D\,(x, y)\,,$$

where $A\,(x, y)$, $B\,(x, y)$, $C\,(x, y)$, and $D\,(x, y)$ are functions.

Second-Order PDEs

A second-order linear PDE with constant coefficients can be written as:

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g, \tag{1.1}$$

where:

- The first three terms constitute the principal part of the equation.
- a, b, c, d, e, f, and g are either constants or functions of x and y.
- The type of PDE (1.1) depends on the sign of the discriminant $b^2 - 4ac$.

Classification of Second-Order PDEs

The equation (1.1) is classified as

- **Elliptic** if $b^2 - 4ac < 0$.
- **Hyperbolic** if $b^2 - 4ac > 0$.
- **Parabolic** if $b^2 - 4ac = 0$.

Fixed Point Theorems

Fixed point theorems study mappings that take a set into itself and, under certain conditions, guarantee the existence of a fixed point, an element x, such that $f(x) = x$. These theorems have wide applications in analysis and topology [16]. For example, the fixed-point theorem of Banach[1] is a fundamental result in this area.

Theorem 1.1 (Banach's Fixed-Point Theorem [12]). *Let X be a non-empty closed subset of a Banach space E, and let $M : X \to X$ be a contraction mapping. Then M has a unique fixed point in X.*

1.2 Self-Similar Solutions in PDEs

Self-similar solutions are a special class of solutions to PDEs that maintain their shape under appropriate rescaling of spatial and temporal variables. Such solutions can often be expressed in the form

$$u(x,t) = t^{\alpha}\varphi\left(\frac{x}{t^{\beta}}\right),$$

where α and β are constants that determine the scaling behavior of the solution with respect to time t, and $\varphi(\xi)$ is the self-similar profile depending on the similarity variable $\xi = xt^{-\beta}$.

Definition 1.3. *A solution of a PDE is said to be self-similar if it is scale-invariant, meaning that it satisfies the transformation*

$$u(x,t) \to a^{\lambda}u(a^{s}x, a^{\gamma}t), \qquad a > 0.$$

Scale Invariance Condition

Definition 1.4. *Let $P(x,t,u,\ldots) = 0$ be a PDE. Then, P admits a self-similar solution if and only if it is* ***scale-invariant*** *under the action of dilation. That is, if we apply the following change of variables:*

$$u = a^{\lambda}u, \quad x = a^{s}x, \quad t = a^{\gamma}t,$$

where $a > 0$ and s, λ, γ are positive parameters, the equation transforms as:

$$P\left(a^{s}x, a^{\gamma}t, a^{\lambda}u, \ldots\right) = 0.$$

Here s, γ, and λ are the scaling exponents.

[1]Stefan Banach, born in 1892 in Poland, was a mathematician and a founder of functional analysis.

Differentiation Rules

To verify similarity, we apply the following differentiation rule:

$$\frac{\partial\left(a^{\lambda}u\right)}{\partial\left(a^{\gamma}t\right)}=a^{\lambda-\gamma}\frac{\partial u}{\partial t},\qquad \frac{\partial\left(a^{\lambda}u\right)}{\partial\left(a^{s}x\right)}=a^{\lambda-s}\frac{\partial u}{\partial x}.$$

If we set $a^{\gamma}t=1$, then

$$a^{\lambda}u\left(a^{s}x,1\right)\rightarrow a^{\lambda}\varphi\left(a^{s}x\right),$$

from which we deduce

$$a=t^{\frac{-1}{\gamma}}.$$

Thus, the self-similar solution takes the form

$$u\left(x,t\right)=t^{\frac{-\lambda}{\gamma}}\varphi\left(t^{\frac{-s}{\gamma}}x\right).$$

This form of self-similar solution is called **highly singular**.

Example 1.2. *Consider the heat equation:*

$$\begin{cases}\frac{\partial u}{\partial t}=\frac{\partial^{2}u}{\partial x^{2}}, & (x,t)\in\mathbb{R}\times\mathbb{R}^{+},\\ u\left(x,0\right)=u_{0}, & x,u_{0}\in\mathbb{R}.\end{cases}$$

Applying the scaling transformation $u\left(x,t\right)\rightarrow a^{\lambda}u\left(a^{s}x,a^{\gamma}t\right)$, we obtain

$$\frac{\partial\left(a^{\lambda}u\right)}{\partial\left(a^{\gamma}t\right)}=a^{\lambda-\gamma}\frac{\partial u}{\partial t}\quad\text{and}\quad\frac{\partial^{2}\left(a^{\lambda}u\right)}{\partial\left(a^{s}x\right)^{2}}=a^{\lambda-2s}\frac{\partial^{2}u}{\partial x^{2}}.$$

Thus

$$a^{\lambda-\gamma}\frac{\partial u}{\partial t}=a^{\lambda-2s}\frac{\partial^{2}u}{\partial x^{2}},$$

which implies

$$\lambda-\gamma=\lambda-2s\quad\Rightarrow\quad\gamma=2s.$$

And this represents the scale invariance condition. For similarity, we assume $a^{\gamma}t=1$, which gives $a=t^{\frac{-1}{\gamma}}$. Substituting, we find

$$u\left(x,t\right)=t^{\frac{-\lambda}{\gamma}}\varphi\left(t^{\frac{-s}{\gamma}}x\right).$$

For the heat equation, $\beta=\frac{1}{2}$ transforms the PDE into an ordinary differential equation (ODE) in terms of the similarity variable $\xi=\frac{x}{\sqrt{t}}$. The goal is to determine the profile φ that is the solution of the ODE and the exponents α and β.

Types of Self-Similarity

In this thesis, we study the following three types of self-similar solutions:

- **Type 1:** Solutions that evolve over time with a self-similar scaling behavior:

$$u_1\,(x,t) = (t+\tau)^{\alpha}\, f_1(\eta), \quad \text{for} \quad \eta = (t+\tau)^{\beta}\, x, \quad \tau > 0. \tag{1.2}$$

- **Type 2:** Solutions that terminate at a finite time $t = \tau$ (if $\alpha < 0$, τ represents the blow-up time):

$$u_2\,(x,t) = (\tau - t)^{\alpha}\, f_2\,(\eta)\,, \quad \text{for} \quad \eta = (\tau - t)^{\beta}\, x, \quad \tau > 0. \tag{1.3}$$

- **Type 3:** Solutions that exhibit exponential growth or propagation, often in the context of unstable systems:

$$u_3\,(x,t) = e^{\alpha(t+\tau)} f_3\,(\eta)\,, \quad \text{for} \quad \eta = xe^{\beta(t+\tau)}, \quad \forall\tau. \tag{1.4}$$

1.3 Fundamentals in Mathematical Modeling

Mathematical modeling transforms complex biological phenomena into mathematical frameworks that facilitate their analysis and understanding. These models are used to study disease spread, population growth, and biological interactions, contributing to prediction and informed scientific decision-making. This topic has been extensively covered in many references and studies, including [7] and [10].

Dynamical system

Dynamical system it is a mathematical framework used to describe the evolution of a system over time according to a fixed set of rules. These rules are usually expressed through various types of models.

Types of models

There are several types of models, but the most common ones are:

1. **Deterministic Dynamical System:** Where the system's evolution is completely determined by fixed laws and initial conditions, with no randomness involved.

2. **Stochastic Dynamical System:** Contains random elements or noise that affect the system's evolution.

3. **Continuous Dynamical System:** Described by differential equations where time changes continuously.

4. **Discrete Dynamical System:** Studied in discrete time steps using difference equations.

Applications of Mathematical Modeling in Biology

Mathematical modeling plays a vital role in biology by providing a structured way to understand, analyze, and predict complex biological processes. Some key applications include:

1. **Epidemiology:** Modeling the spread of infectious diseases (e.g., SIR, SEIR models) to predict outbreaks, evaluate control strategies, and inform public health policies.

2. **Population Dynamics:** Studying growth, interactions, and competition between species to understand ecosystem stability and biodiversity.

3. **Genetics and Evolution:** Modeling gene frequency changes, natural selection, and evolutionary dynamics over generations.

4. **Cell Biology:** Analyzing cellular processes such as signaling pathways, metabolic networks, and cell cycle regulation.

5. **Neuroscience:** Modeling neural networks and brain activity to understand information processing and behavior.

6. **Physiology:** Simulating organ functions, such as heart rhythms or respiratory processes, to assist in medical diagnosis and treatment planning.

Key Analytical Tools in Mathematical Biology

Mathematical biology relies on several essential analytical tools that help interpret and predict the behavior of biological systems. Among the most fundamental are:

- **Basic Reproduction Number ($\mathfrak{R}_0$):** A critical threshold quantity used in epidemiological modeling. It represents the average number of secondary infections produced by a single infected individual in a fully susceptible population. If $\mathfrak{Rb} > 1$, the disease tends to spread; if $\mathfrak{Rb} < 1$, the infection will likely die out.

- **Equilibrium Points:** These are states of the system where variables no longer change over time. In biological models, equilibrium points often correspond to disease-free or endemic steady states, depending on the nature of the model.
- **Stability Analysis:** Stability is determined by analyzing the system's Jacobian matrix at the equilibrium point. If all eigenvalues have negative real parts, the equilibrium point is considered (asymptotically) stable. However, if at least one eigenvalue has a positive real part, the equilibrium is unstable.

CHAPTER 2

NONLINEAR DIFFUSION OF POROUS MEDIUM EQUATION

In this chapter, we study the porous medium equation. This nonlinear partial differential equation models the diffusion of physical quantities, such as the one-dimensional gas flow in a porous medium. The equation is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2}, \tag{2.1}$$

where $u > 0$, x is the spatial variable, and t represents time, while $m > 1$.

Remark 2.1. *When $m = 1$, the equation reduces to the heat equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

From (1.2), we find the solution for u_1 :

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= \frac{\partial}{\partial t}\left((t+\tau)^{\alpha} f_1((t+\tau)^{\beta} x)\right) \\
&= \alpha (t+\tau)^{\alpha-1} f_1(\eta) + \beta x (t+\tau)^{\beta-1+\alpha} f_1'(\eta) \\
&= (t+\tau)^{\alpha-1} \left[\alpha f_1(\eta) + \beta\eta f_1'(\eta)\right].
\end{aligned} \tag{2.2}$$

Also

$$\begin{aligned}
\frac{\partial^2 u_1^m}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u_1^m}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(t+\tau)^{\alpha m} f_1^m\left(x (t+\tau)^{\beta}\right)\right) \\
&= \frac{\partial}{\partial x}\left(m (t+\tau)^{\alpha m} (t+\tau)^{\beta} f_1^{m-1} f_1'\right) \\
&= (t+\tau)^{\alpha m+2\beta}\left(m(m-1) f_1^{m-2} (f_1')^2 + m f_1^{m-1} f_1''\right) \\
&= (t+\tau)^{\alpha m+2\beta} (f_1^m)''.
\end{aligned} \tag{2.3}$$

From the equations (2.2) and (2.3), we obtain

$$t^{\alpha m+2\beta-\alpha+1} (f_1^m)'' = \alpha f_1(\eta) + \beta\eta f_1'(\eta).$$

If we choose

$$\alpha (m-1) + 2\beta + 1 = 0,$$

we obtain

$$\beta = \frac{-1 - \alpha (m-1)}{2}.$$

We have

$$(f_1^m)'' = \alpha f_1 (\eta) + \beta \eta f_1' (\eta).$$

Therefore, we substitute the terme β into the equation to get

$$(f_1^m)'' = \alpha f_1 (\eta) - \frac{1}{2} [\alpha (m-1) + 1] \eta f_1' (\eta).$$

In the same way, from (1.3) and (1.4), we solve u_2 and u_3, and we get

$$(f_2^m)'' (\eta) = -\alpha f_2 (\eta) + \frac{1}{2} [\alpha (m-1) + 1] \eta f_2' (\eta)$$

and

$$(f_3^m)'' (\eta) = \alpha f_3 (\eta) - \frac{1}{2} \alpha (m-1) \eta f_3' (\eta).$$

At the boundaries, we impose the conditions

$$f_i (0) = U \geq 0, \qquad f_i (\infty) = 0, \qquad i = 1, 2, 3.$$

Accordingly, the solutions $u_i (x, t)$ satisfy the boundary conditions

$$u_1 (0, t) = (t + \tau)^{\alpha} U, \qquad u_2 (0, t) = (t + \tau)^{\alpha} U, \qquad u_3 (0, t) = e^{\alpha (t + \tau)} U$$

and

$$u_i (x, t) \to 0 \qquad \text{as} \qquad x \to \infty \qquad i = 1, 2, 3.$$

Thus, we observe from the previous cases that we can write all the equations in the present form

$$\begin{cases} (f^m)'' + p \eta f' = q f, & 0 < \eta < \infty, \\ f(0) = U, \quad f(\infty) = 0, \end{cases} \tag{2.4}$$

in which, p and q are arbitrary real constants.

2.1 Weak Solution with Compact Support

Definition 2.1 (Weak Solution [11]). *A function f is called a weak solution with compact support of the problem (2.4) if*

1. *f is continuous, bounded, and nonnegative on* $[0, \infty)$.

2. *f^m has a continuous derivative with respect to η on* $(0, \infty)$.

3. *f satisfies the identity*

$$\int_0^\infty \varphi' \left[(f^m)' + p\eta f \right] d\xi + (p+q) \int_0^\infty \varphi f d\xi = 0, \quad \forall \varphi \in C_0^1 (0, \infty) .$$

Let f be a weak solution of the problem (2.4) and positive with compact support. Then, we assume the existence of a positive constant $a > 0$, where $f > 0$ on the interval $(0, a)$ and $f = 0$ on the interval $[a, \infty)$ then $(f^m)' (a) = 0$ and we write:

$$f(a) = (f^m)'(a) = 0.$$

Lemma 2.1 ([11])**.** *The existence of a nontrivial weak solution of (2.4) with compact support implies one of the following properties:*

i. $p = 0$ *and* $q > 0$.

ii. $p > 0$, *for any* q.

Proof. Suppose f is a nontrivial solution of (2.4) with compact support; then there exists $a > 0$ such that $f > 0$ on $(a - \varepsilon, a)$ and $f = 0$ on $[a, \infty)$ for some $\varepsilon > 0$ and $f(a) = 0$. Then $\exists \eta_0 \in (a - \varepsilon, a)$ such that $f'(\eta_0) < 0$ because

$$\begin{aligned} f'(\eta_0) &= \frac{f(a) - f(a-\varepsilon)}{a - (a-\varepsilon)} \\ &= \frac{-f(a-\varepsilon)}{\varepsilon} < 0. \end{aligned}$$

Integration of (2.4) on an interval (η, a) , we get

$$\int_\eta^a (f^m)''(\xi) d\xi + p \int_\eta^a \xi f'(\xi) d\xi = q \int_\eta^a f(\xi) d\xi.$$

Therefore

$$\left[(f^m)'(\xi) \right]_\eta^a + p \left([\xi f(\xi)]_\eta^a - \int_\eta^a f(\xi) d\xi \right) = q \int_\eta^a f(\xi) d\xi.$$

Consequently, we get

$$-(f^m)'(\eta) = p\eta f(\eta) + (p+q) \int_\eta^a f(\xi) d\xi. \tag{2.5}$$

Then, at $\eta = \eta_0$, the left part of equation (2.5) is positive, and in order to say that the right part is positive, then we can not take p and $p+q$ both negative, and if $p = 0$, then $q > 0$.

Suppose that $p < 0$. Then, from equation (2.5) we have $p + q > 0$ and $q > 0$. And $f' < 0$ on $(a - \varepsilon, a)$ implies

$$\forall \eta \in (a - \varepsilon, a), \quad \forall \xi \in (\eta, a), \quad \text{we have} \quad f(\xi) < f(\eta)$$

and we conclude

$$\frac{f(\xi)}{f(\eta)} < 1,$$

also

$$\int_{\eta}^{a} \frac{f(\xi)}{f(\eta)} d\xi < a - \eta,$$

from equation (2.5) we get

$$-m f^{m-2} f'(\eta) - p\eta < (p + q)(a - \eta).$$

If we now let η tend to a, we obtain a contradiction. Then $p > 0$. $\square$

Explicit Solution for $p = 0$

From equation (2.5) we find

$$-(f^m)' = q \int_{\eta}^{a} f(\xi)\, d\xi, \quad \text{for} \quad q > 0.$$

We put

$$h(\eta) = -\int_{\eta}^{a} f(\xi)\, d\xi,$$

also

$$h(a) = 0, \quad h'(\eta) = f(\eta), \quad \text{and} \quad h''(\eta) = f'(\eta).$$

Therefore

$$-m f^{m-1} f'(\eta) = q \int_{\eta}^{a} f(\xi)\, d\xi.$$

Substituting h, we get

$$m (h')^{m-1}(\eta)\, h''(\eta) = q h(\eta).$$

If we multiply by h' and integrate, we obtain

$$\begin{aligned}
m \int_{\eta}^{a} (h')^{m}(\xi)\, h''(\xi)\, d\xi &= q \int_{\eta}^{a} h(\xi)\, h'(\xi)\, d\xi \\
\left[\frac{m}{m+1} (h')^{m+1}(\xi)\right]_{\eta}^{a} &= \left[\frac{q}{2} h^2(\xi)\right]_{\eta}^{a} \\
(h')^{m+1}(\eta) &= \frac{q(m+1)}{2m} h^2(\eta) \\
h'(\eta) &= \left[\frac{q(m+1)}{2m}\right]^{\frac{1}{m+1}} h^{\frac{2}{m+1}}(\eta).
\end{aligned}$$

We put

$$Z=\left[\frac{q\left(m+1\right)}{2m}\right]^{\frac{1}{m+1}},$$

we have

$$h^{\frac{-2}{m+1}}\left(\eta\right)h'\left(\eta\right)=Z.$$

With integration from η to a

$$\begin{aligned}-\frac{m+1}{m-1}h^{\frac{m-1}{m+1}}\left(\eta\right) &= Z\left(a-\eta\right)\\ -h\left(\eta\right) &= \left[\frac{Z\left(m-1\right)}{m+1}\left(a-\eta\right)\right]^{\frac{m+1}{m-1}},\end{aligned}$$

we differentiate h and substitute Z by its value to get

$$h'\left(\eta\right)=\left[\frac{q\left(m+1\right)}{2m}\right]^{\frac{1}{m-1}}\left[\left(\frac{m-1}{m+1}\right)^{2}\left(a-\eta\right)^{2}\right]^{\frac{1}{m-1}}.$$

Finally, the result is as follows

$$f\left(\eta\right)=\left[\frac{q\left(m-1\right)^{2}}{2m\left(m+1\right)}\left(a-\eta\right)^{2}\right]^{\frac{1}{m-1}}. \tag{2.6}$$

The solution to the problem (2.4), where f is a continuous and monotonic function with respect to a, the equation $f\left(a\right)=U$ is uniquely solvable for every $U\geq 0$. Let $a\left(U\right)$ be its solution. Then $f\left(\eta\right)$ is the unique solution of problem (2.4).

We have $f\left(a\right)=0$ and from the boundary conditions $f\left(0\right)=U$, Therefore

$$f\left(0\right)=U=\left[\frac{q\left(m-1\right)^{2}}{2m\left(m+1\right)}a^{2}\right]^{\frac{1}{m-1}}.$$

Thus, we find

$$a\left(U\right)=\sqrt{\frac{2m\left(m+1\right)U^{m-1}}{q\left(m-1\right)^{2}}}.$$

The problem (2.4) admits a nontrivial weak solution of the second type of self-similar solutions (see 1.3), with compact support in the interval $[0,a]$. Here we get $\beta=\alpha=-\frac{1}{m-1}$, and this holds when $p=0$ and $q=-\alpha>0$. Consequently, the solution takes the following form:

$$u\left(x,t\right)=\left(\tau-t\right)^{\alpha}\left[\frac{q\left(m-1\right)^{2}}{2m\left(m+1\right)}\left(a-x\left(\tau-t\right)^{\beta}\right)^{2}\right]^{\frac{1}{m-1}}.$$

Lemma 2.2 ([11]). *Let $b \in (0,a)$, and let f be a positive solution of problem (2.4) on $[b,a)$.*

i. *If $p+q \geq 0$, then $f'(\eta) < 0$ on $[b,a)$.*

ii. *If $p+q < 0$, and there exists an $\eta_0 \in [b,a)$ such that $f'(\eta_0) = 0$, then f has a maximum at η_0 and $\eta_0 \leq \frac{p+q}{q} a$.*

iii. *If f is a positive solution of (2.4) on $[0,a)$, then*

$$\begin{cases} p+q > 0 & \Rightarrow & f'(0) < 0, \\ p+q = 0 & \Rightarrow & f'(0) = 0, \\ p+q < 0 & \Rightarrow & f'(0) > 0. \end{cases}$$

Proof. Starting from (2.5), we find that

i. If $p+q \geq 0$, thus we conclude $-(f^m)'(\eta) > 0$ and as a result $f'(\eta) < 0$ on $[b,a)$.

ii. If $p+q < 0$, notice that $q < 0$ from which (2.4) we get

$$m(m-1) f^{m-2}(\eta) [f'(\eta)]^2 + m f^{m-1}(\eta) f''(\eta) + p\eta f'(\eta) = q f(\eta).$$

Suppose there exists $\eta_0 \in [b,a)$ such that $f'(\eta_0) = 0$ where we have

$$m f^{m-1}(\eta) f''(\eta) = q f(\eta),$$

therefore

$$f''(\eta_0) < 0,$$

it follows that f has a maximum at η_0 and $f'(\eta) < 0$ where it gives us (2.5)

$$0 = p\eta_0 f(\eta_0) + (p+q) \int_{\eta_0}^{a} f(\xi)\, d\xi,$$

we have

$$f(\xi) \leq f(\eta_0) \qquad \text{on} \qquad [\eta_0, a),$$

also

$$\begin{aligned} (p+q) \int_{\eta_0}^{a} f(\xi)\, d\xi &\geq (p+q) \int_{\eta_0}^{a} f(\eta_0)\, d\xi \\ &\geq (p+q) f(\eta_0) \int_{\eta_0}^{a} d\xi, \end{aligned}$$

we conclude that

$$\begin{aligned} p\eta_0 f(\eta_0) + (p+q) f(\eta_0)(a-\eta_0) &\leq 0 \\ p\eta_0 + (p+q)(a-\eta_0) &\leq 0 \\ (p+q)a &\leq q\eta_0 \end{aligned}$$

and we get

$$\eta_0 \leq \frac{p+q}{q} a.$$

iii. Finally, if $b=0$ then the interval becomes $[0,a)$ and from (2.5), we find

$$-(f^m)'(0) = (p+q)\int_0^a f(\xi)\,d\xi.$$

Thus, it becomes easy to determine the sign of f' at $b=0$.

The proof is complete □

2.2 Existence and Uniqueness of Solutions

Lemma 2.3 ([11]). *Let $p>0$ and q be arbitrary. For any $a>0$, there exists an $\varepsilon>0$ such that in $(a-\varepsilon,a)$, problem (2.4) has a unique positive solution.*

Proof. We write (2.5) and find

$$-(f^m)'(\eta) = p\eta f(\eta) + (p+q)\int_\eta^a f(\xi)\,d\xi,$$

where

$$\int_\eta^a f(\xi)\,d\xi = [\xi f]_\eta^a - \int_\eta^a \xi f'(\xi)\,d\xi,$$

hence

$$\begin{aligned} -(f^m)'(\eta) &= p\eta f(\eta) + (p+q)\left([\xi f]_\eta^a - \int_\eta^a \xi f'(\xi)\,d\xi\right) \\ &= p\eta f(\eta) - p\eta f(\eta) - q\eta f(\eta) - (p+q)\int_\eta^a \xi f'(\xi)\,d\xi, \end{aligned}$$

then

$$(f^m)'(\eta) = q\eta f(\eta) + (p+q)\int_\eta^a \xi f'(\xi)\,d\xi.$$

We put the inverse function

$$\eta = g(f),$$

when

$$\eta \to a, \quad f \to 0,$$

we have

$$\begin{aligned} mf^{m-1}(\eta)\frac{df}{dg} &= qg(f)f + (p+q)\int_f^0 g(\varphi)\frac{d\varphi}{dg}dg \\ \frac{df}{dg} &= \frac{qg(f)f + (p+q)\int_f^0 g(\varphi)d\varphi}{mf^{m-1}(\eta)}. \end{aligned}$$

Therefore

$$\frac{dg}{df} = \frac{mf^{m-1}(\eta)}{qg(f)f - (p+q)\int_0^f g(\varphi)d\varphi}.$$

By integrating from 0 to f, we get

$$\begin{aligned} \int_0^f \frac{dg}{df} &= \int_0^f \frac{m\varphi^{m-1}d\varphi}{qg(\varphi)\varphi - (p+q)\int_0^\varphi g(\psi)d\psi} \\ g(f) - a &= \int_0^f \frac{m\varphi^{m-1}d\varphi}{qg(\varphi)\varphi - (p+q)\int_0^\varphi g(\psi)d\psi}. \end{aligned}$$

Let now

$$T(f) = 1 - a^{-1}g(f),$$

then

$$g(f) = a - aT(f)$$

and we substitute

$$\begin{aligned} a - aT(f) - a &= \int_0^f \frac{m\varphi^{m-1}d\varphi}{q(a - aT(\varphi))\varphi - (p+q)\int_0^\varphi (a - aT(\psi))d\psi} \\ -aT(f) &= \int_0^f \frac{m\varphi^{m-1}d\varphi}{q(a - aT(\varphi))\varphi - (p+q)\int_0^\varphi (a - aT(\psi))d\psi}, \end{aligned}$$

we put

$$\begin{aligned} Z &= q(a - aT(\varphi))\varphi - (p+q)\int_0^\varphi (a - aT(\psi))d\psi \\ &= a\left[q(1 - T(\varphi))\varphi - (p+q)\int_0^\varphi (1 - T(\psi))d\psi\right] \\ &= a\left[q\varphi - q\varphi T(\varphi) - (p+q)\varphi + (p+q)\int_0^\varphi T(\psi)d\psi\right] \\ &= a\left[-q\varphi T(\varphi) - p\varphi + (p+q)\int_0^\varphi T(\psi)d\psi\right] \\ &= -a\left[p\varphi + q\varphi T(\varphi) - (p+q)\int_0^\varphi T(\psi)d\psi\right], \end{aligned}$$

then

$$\begin{aligned} -aT(f) &= \int_0^f \frac{m\varphi^{m-1}d\varphi}{-a\left[p\varphi + q\varphi T(\varphi) - (p+q)\int_0^\varphi T(\psi)\,d\psi\right]} \\ T(f) &= \frac{m}{a^2}\int_0^f \frac{\varphi^{m-1}d\varphi}{p\varphi + q\varphi T(\varphi) - (p+q)\int_0^\varphi T(\psi)\,d\psi}. \end{aligned}$$

Let $\gamma > 0$, and let X be the set of bounded functions $T(f)$ defined on $[0,\gamma]$ such that

$$0 \leq T(f) \leq \tau = \frac{p}{2\left(|p| + |p+q|\right)}.$$

We denote by $\|\cdot\|$ the supremum norm on X. Then X is a Banach space. We define on X the operator

$$A(T)(f) = \frac{m}{a^2}\int_0^f \frac{\varphi^{m-1}d\varphi}{p\varphi + q\varphi T(\varphi) - (p+q)\int_0^\varphi T(\psi)\,d\psi}.$$

Suppose $\tau \in X$. Then

$$p\varphi + q\varphi T(\varphi) \geq p\varphi - |q|\,\varphi\,\|T\|$$

and

$$-(p+q)\int_0^\varphi T(\psi)\,d\psi \geq -|p+q|\,\varphi\,\|T\|,$$

hence

$$\begin{aligned} p\varphi + q\varphi T(\varphi) - (p+q)\int_0^\varphi T(\psi)\,d\psi &\geq p\varphi - |q|\,\varphi\,\|T\| - |p+q|\,\varphi\,\|T\| \\ &\geq \varphi\left(p - \left(|q| + |p+q|\right)\|T\|\right) \\ &\geq \varphi\left(p - \left(|q| + |p+q|\right)\tau\right) \\ &\geq \varphi\left(p - \frac{p}{2}\right) \\ &\geq \frac{p}{2}\varphi, \end{aligned}$$

then

$$\begin{aligned} |A(T)(f)| &\leq \frac{m}{a^2}\int_0^f \frac{\varphi^{m-1}d\varphi}{\frac{p}{2}\varphi} \\ &\leq \frac{2m}{pa^2}\int_0^f \varphi^{m-2}d\varphi \\ &\leq \frac{2m}{pa^2(m-1)}f^{m-1} \\ &\leq \frac{2m}{pa^2(m-1)}\gamma^{m-1}. \end{aligned}$$

Thus, $A(T)$ is well defined on the whole of X. Clearly, if $\tau \in X$, $A(T) : [0,\gamma] \to R$ is nonnegative and continuous. Moreover, there exists a $\gamma_0 = \left[\frac{\tau pa^2(m-1)}{2m}\right]^{\frac{1}{m-1}} > 0$ such that if

$\gamma \leq \left[\frac{\tau p a^2(m-1)}{2m}\right]^{\frac{1}{m-1}}$ and $\tau \in X$, $\|A(T)\| \leq \tau$. Thus, if $\gamma \leq \gamma_0$, A is a map from X into X. Let now $T_1, T_2 \in X$ and let $\gamma \leq \gamma_0$. Then

$$A(T_1)(f) = \frac{m}{a^2}\int_0^f \frac{\varphi^{m-1}d\varphi}{p\varphi + q\varphi T_1(\varphi) - (p+q)\int_0^\varphi T_1(\psi)\,d\psi},$$

also

$$|A(T_1)(f)| \leq \frac{m}{a^2}\int_0^f \frac{\varphi^{m-1}d\varphi}{\frac{p}{2}\varphi},$$

in the same way with T_2

$$A(T_2)(f) = \frac{m}{a^2}\int_0^f \frac{\varphi^{m-1}d\varphi}{p\varphi + q\varphi T_2(\varphi) - (p+q)\int_0^\varphi T_2(\psi)\,d\psi}$$

and

$$|A(T_2)(f)| \leq \frac{m}{a^2}\int_0^f \frac{\varphi^{m-1}d\varphi}{\frac{p}{2}\varphi},$$

we put

$$k(T) = p\varphi + q\varphi T(\varphi) - (p+q)\int_0^\varphi T(\psi)\,d\psi.$$

Then

$$\begin{aligned}
|k(T_2) - k(T_1)| &= \bigg| p\varphi + q\varphi T_2(\varphi) - (p+q)\int_0^\varphi T_2(\psi)\,d\psi \\
&\quad - p\varphi - q\varphi T_1(\varphi) + (p+q)\int_0^\varphi T_1(\psi)\,d\psi \bigg| \\
&= \left| q\varphi\left(T_2(\varphi) - T_1(\varphi)\right) - (p+q)\int_0^\varphi \left(T_2(\varphi) - T_1(\varphi)\right)d\psi \right| \\
&\leq |q|\,\varphi\,\|T_2(\varphi) - T_1(\varphi)\| + |p+q|\,\varphi\,\|T_2(\varphi) - T_1(\varphi)\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
|A(T_1)(f) - A(T_2)(f)| &= \left| \frac{m}{a^2}\int_0^f \frac{\varphi^{m-1}d\varphi}{k(T_1)} - \frac{m}{a^2}\int_0^f \frac{\varphi^{m-1}d\varphi}{k(T_2)} \right| \\
&= \left| \frac{m}{a^2}\int_0^f \frac{\varphi^{m-1}\left(k(T_2) - k(T_1)\right)}{k(T_1)\,k(T_2)} \right| \\
&\leq \frac{m}{a^2}\int_0^f \frac{\varphi^{m-1}\left|k(T_2) - k(T_1)\right|}{\left(\frac{p}{2}\varphi\right)\left(\frac{p}{2}\varphi\right)}.
\end{aligned}$$

Thus

$$\begin{aligned}
|A(T_1)(f) - A(T_2)(f)| \leq\ & \tfrac{4m}{p^2a^2}\textstyle\int_0^f \varphi^{m-3}\times \\
& \left| q\varphi\left(T_2(\varphi) - T_1(\varphi)\right) - (p+q)\textstyle\int_0^\varphi \left(T_2(\varphi) - T_1(\varphi)\right)d\psi \right| d\varphi \\
\leq\ & \tfrac{4m}{p^2a^2}\textstyle\int_0^f \varphi^{m-3}\left(|q|\,\varphi\,\|T_2(\varphi) - T_1(\varphi)\|\right. \\
& \left. + |p+q|\,\varphi\,\|T_2(\varphi) - T_1(\varphi)\|\right) d\varphi \\
\leq\ & \left(\tfrac{4m(|q|+|p+q|)}{p^2a^2}\textstyle\int_0^f \varphi^{m-2}d\varphi\right)\|T_2(\varphi) - T_1(\varphi)\|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\left|A\left(T_1\right)(f)-A\left(T_2\right)(f)\right| &\leq \frac{4m\left(|q|+|p+q|\right)f^{m-1}}{p^2a^2\left(m-1\right)}\left\|T_2\left(\varphi\right)-T_1\left(\varphi\right)\right\| \\
&\leq \frac{4m\left(|q|+|p+q|\right)\gamma^{m-1}}{p^2a^2\left(m-1\right)}\left\|T_2\left(\varphi\right)-T_1\left(\varphi\right)\right\|.
\end{aligned}$$

Hence, if we take $\gamma<\left[\frac{pa^2(m-1)\tau}{2m}\right]^{\frac{1}{m-1}}$, we will find $\frac{4m(|q|+|p+q|)\gamma^{m-1}}{p^2a^2(m-1)}<1$. Therefore, we conclude that A is a contraction on X. Thus, by the Banach contraction mapping principle [12], A has a unique fixed point in X, and (2.4) has a unique solution. □

Explicit Solution for $p+q=0$

From equation (2.5) we find

$$\begin{aligned}
-\left(f^m\right)'(\eta) &= p\eta f(\eta) \\
-mf^{m-1}(\eta)\,f'(\eta) &= p\eta f(\eta) \\
-mf^{m-2}(\eta)\,f'(\eta) &= p\eta.
\end{aligned}$$

Integrate from η to a

$$\begin{aligned}
\int_\eta^a -mf^{m-2}(\xi)\,f'(\xi)\,d\xi &= \int_\eta^a p\xi d\xi \\
-\frac{m}{m-1}\left[f^{m-1}(\xi)\right]_\eta^a &= \frac{p}{2}\left[\xi^2\right]_\eta^a \\
f(\eta) &= \left[\frac{p\left(m-1\right)}{2m}\left(a^2-\eta^2\right)\right]^{\frac{1}{m-1}},
\end{aligned}$$

we put

$$f(0)=U=\left[\frac{p\left(m-1\right)}{2m}a^2\right]^{\frac{1}{m-1}},$$

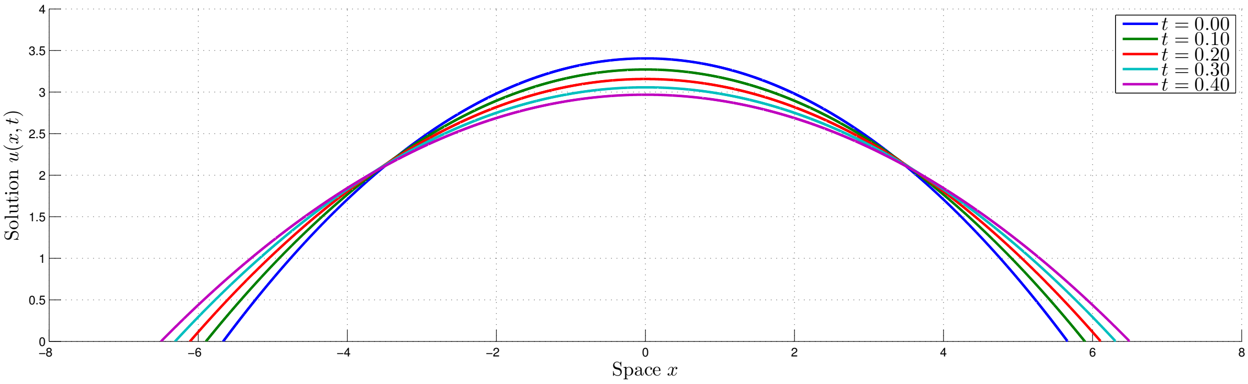
then

$$a(U)=\sqrt{\frac{2mU^{m-1}}{p\left(m-1\right)}}.$$

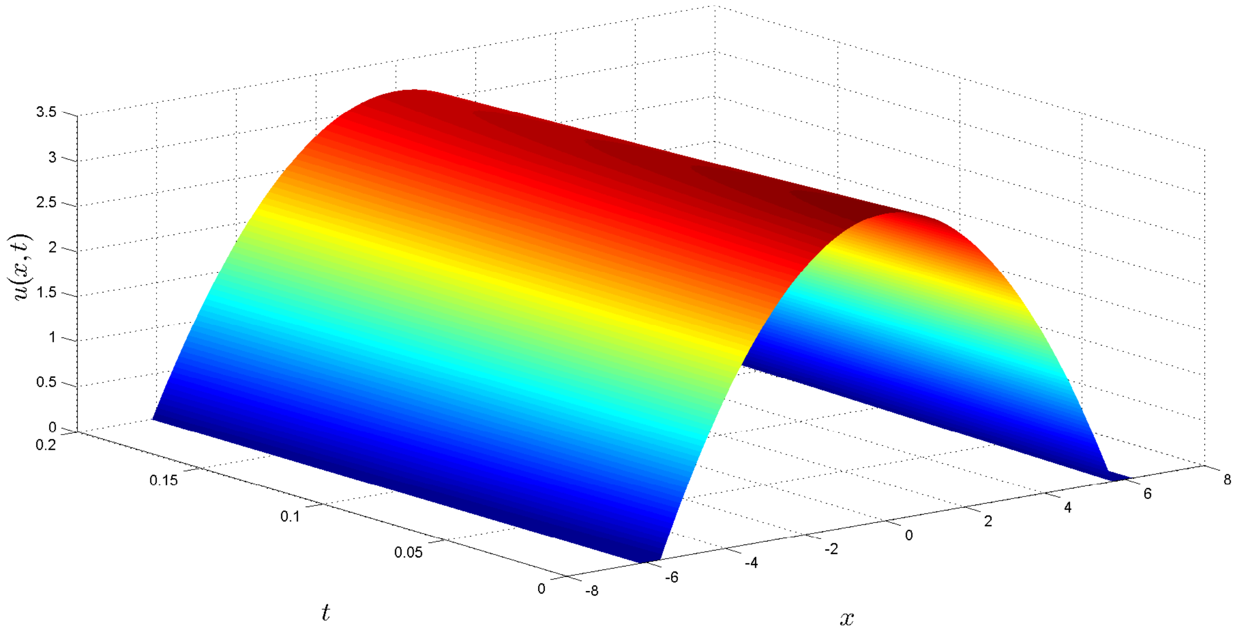
The problem (2.4) admits a nontrivial weak solution of the first type of self-similar solutions (see 1.2), with compact support in the interval $[0,a]$. Here we get $\beta=\alpha=-\dfrac{1}{m+1}$. this holds when $p+q=0$, where $p=-\beta=\dfrac{1}{m+1}$. Consequently, the solution takes the following form:

$$u(x,t)=(t+\tau)^\alpha\left[\frac{p\left(m-1\right)}{2m}\left(a-x\left(t+\tau\right)^\beta\right)^2\right]^{\frac{1}{m-1}}.$$

Based on these results, we performed a numerical simulation to solve the equation, which takes the following form:



(a) 2D Plot of the explicit solution $u(x,t)$ for different values of t.



(b) 3D Plot of the explicit solution $u(x,t)$

Figure 2.1: (a) and (b) represent the explicit solution $u(x,t)$ for the case $p+q=0$, when $m=2$.

2.3 Boundedness of the Solution

Lemma 2.4 ([11]). *Let $b \in [0,a)$ and let f be a positive solution of (2.4) on (b,a) then, if $p>0$*

$$\sup_{(b,a)} f(\eta) \leq \left[\frac{(m-1)\,a^2}{2m} \max\{p, 2p+q\}\right]^{\frac{1}{m-1}}.$$

Proof. We have

i. Assume that $p+q\geq 0$; then by lemma 2, $f'(\eta)<0$ on (b,a). Using this in (2.5) we obtain

$$-(f^m)'(\eta)=p\eta f(\eta)+(p+q)\int_\eta^a f(\xi)\,d\xi.$$

We have $f'(\eta)<0$ on (b,a), thus

$$\begin{aligned} f(\xi) &< f(\eta) \\ \frac{f(\xi)}{f(\eta)} &\leq 1 \\ \int_\eta^a \frac{f(\xi)}{f(\eta)}d\xi &\leq a-\eta, \end{aligned}$$

dividing (2.5) by $f(\eta)$, we obtain

$$-mf^{m-2}(\eta)\,f'(\eta)=p\eta+(p+q)\int_\eta^a \frac{f(\xi)}{f(\eta)}d\xi,$$

hence

$$\begin{aligned} -mf^{m-2}(\eta)\,f'(\eta) &\leq p\eta+(p+q)(a-\eta) \\ &\leq (p+q)\,a-q\eta. \end{aligned}$$

The integration from η to a gives us

$$\begin{aligned} -\textstyle\int_\eta^a mf^{m-2}(\xi)\,f'(\xi)\,d\xi=-\left[\frac{m}{m-1}f^{m-1}(\xi)\right]_\eta^a &\leq (p+q)\,a\textstyle\int_\eta^a d\xi-q\int_\eta^a \xi d\xi \\ &\leq (p+q)\,a\,(a-\eta)-\tfrac{q}{2}\left(a^2-\eta^2\right) \\ &\leq \left[(p+q)\,a-\tfrac{q}{2}(a-\eta)\right](a-\eta), \end{aligned}$$

then

$$\frac{m}{m-1}f^{m-1}(\eta)\leq\left[pa+qa-\frac{q}{2}a-\frac{q}{2}\eta\right](a-\eta), \quad b\leq\eta\leq a,$$

hence

$$\begin{aligned} \sup_{(b,a)} f^{m-1}(\eta) &\leq \frac{m-1}{m}\left[pa+\frac{q}{2}a\right]a \\ &\leq \frac{m-1}{m}\frac{a^2}{2}(2p+q). \end{aligned}$$

ii. Assume that $p+q<0$; then it follows from (2.5) that

$$-(f^m)'(\eta)\leq p\eta f(\eta),$$

dividing by $f(\eta)$

$$-mf^{m-2}(\eta)\,f'(\eta)\leq p\eta,$$

we integrate from η to a to obtain

$$\begin{array}{rcl} -\int_\eta^a mf^{m-2}(\xi) f'(\xi)\, d\xi = \frac{m}{m-1} f^{m-1}(\eta) & \leq & p\int_\eta^a \xi d\xi \\ & \leq & \frac{p}{2}[\xi]_\eta^a. \end{array}$$

Consequently,

$$\frac{m}{m-1} f^{m-1}(\eta) \leq \frac{p}{2}\left(a^2-\eta^2\right), \quad b \leq \eta \leq a.$$

Thus

$$\sup_{(b,a)} f^{m-1}(\eta) \leq \frac{m-1}{m}\frac{p}{2}a^2.$$

The proof is complete □

2.4 Main Results

Lemma 2.5 ([11]). *Let f be the positive solution of problem (2.4) in a left neighborhood of $\eta = a$. Assume that $p > 0$. then:*

i. *if $2p+q>0$, $f(\eta)>0$ on $[0,a)$.*

ii. *if $2p+q=0$, $f(\eta)>0$ on $(0,a)$ and $f(0)=0$.*

iii. *if $2p+q<0$, there exists an $\eta^* \in (0,a)$ such that $f(\eta)>0$ on (η^*,a) and $f(\eta^*)=0$.*

Proof. Integration of (2.5) from η to a gives

$$\int_\eta^a -(f^m)'(\xi)\, d\xi = p\int_\eta^a \xi f(\xi)\, d\xi + (p+q)\int_\eta^a \int_\xi^a f(t)\, dt$$

or

$$f^m(\eta) = p\int_\eta^a \xi f(\xi)\, d\xi + (p+q)\int_\eta^a \int_\xi^a f(t)\, dt d\xi.$$

Using integration by parts leads us to write

$$\begin{array}{rcl} f^m(\eta) & = & p\int_\eta^a \xi f(\xi)\, d\xi + (p+q)\int_\eta^a (\xi-\eta) f(\xi)\, d\xi \\ & = & p\eta \int_\eta^a f(\xi)\, d\xi + (2p+q)\int_\eta^a (\xi-\eta) f(\xi)\, d\xi. \end{array}$$

Thus, the lemma is now established. □

Explicit Solution for $2p+q=0$

We have

$$f^{m}(\eta)=p\eta\int_{\eta}^{a}f(\xi)\,d\xi,$$

we put

$$h(\eta)=-\int_{\eta}^{a}f(\xi)\,d\xi,$$

also

$$h(a)=0,\quad h'(\eta)=f(\eta),\quad \text{and}\quad h''(\eta)=f'(\eta).$$

Therefore

$$\begin{aligned}(h')^{m} &= -p\eta h(\eta)\\ h'(\eta) &= -p^{\frac{1}{m}}\eta^{\frac{1}{m}}h^{\frac{1}{m}}(\eta)\\ h'(\eta)\,h^{\frac{-1}{m}}(\eta) &= -p^{\frac{1}{m}}\eta^{\frac{1}{m}},\end{aligned}$$

by integrating from η to 0, we find

$$\begin{aligned}\int_{\eta}^{a}h'(\xi)\,h^{\frac{-1}{m}}(\xi)\,d\xi &= -p^{\frac{1}{m}}\int_{\eta}^{a}\xi^{\frac{1}{m}}d\xi\\ h^{\frac{m-1}{m}}(\eta) &= \frac{(m-1)\,p^{\frac{1}{m}}}{m+1}\left(a^{\frac{m+1}{m}}-\eta^{\frac{m+1}{m}}\right),\end{aligned}$$

then

$$h(\eta)=\left[\frac{(m-1)\,p^{\frac{1}{m}}}{m+1}\left(a^{\frac{m+1}{m}}-\eta^{\frac{m+1}{m}}\right)\right]^{\frac{m}{m-1}},$$

we have

$$h'(\eta)=f(\eta),$$

hence

$$\begin{aligned}h'(\eta) &= \frac{m}{m-1}\left[\frac{(m-1)\,p^{\frac{1}{m}}}{m+1}\left(a^{\frac{m+1}{m}}-\eta^{\frac{m+1}{m}}\right)\right]^{\frac{1}{m-1}}\left[\frac{(m-1)\,p^{\frac{1}{m}}}{m+1}\frac{(m+1)\,\eta^{\frac{1}{m}}}{m}\right]\\ &= \frac{m-1}{m+1}\eta^{\frac{1}{m}}p^{\frac{1}{m(m-1)}}p^{\frac{1}{m}}\left(a^{\frac{m+1}{m}}-\eta^{\frac{m+1}{m}}\right)^{\frac{1}{m-1}},\end{aligned}$$

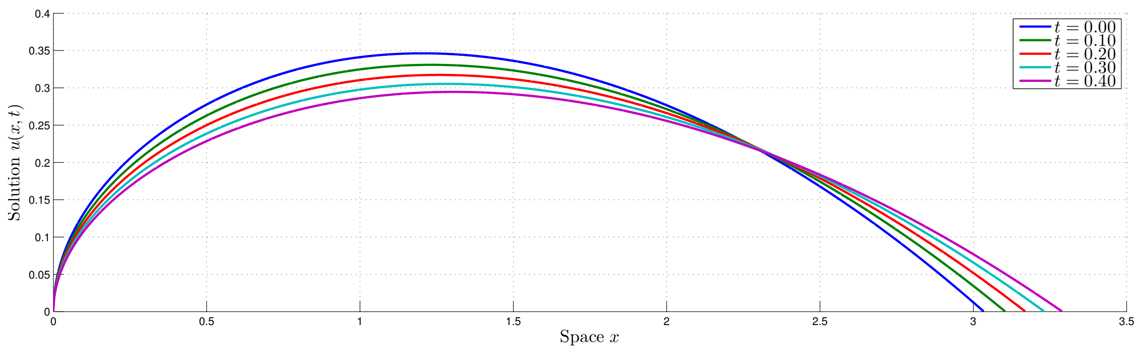
then

$$f(\eta)=\eta^{\frac{1}{m}}\left[\frac{p\,(m-1)}{m+1}\left(a^{\frac{m+1}{m}}-\eta^{\frac{m+1}{m}}\right)\right]^{\frac{1}{m-1}},\quad \text{for any } a>0.$$

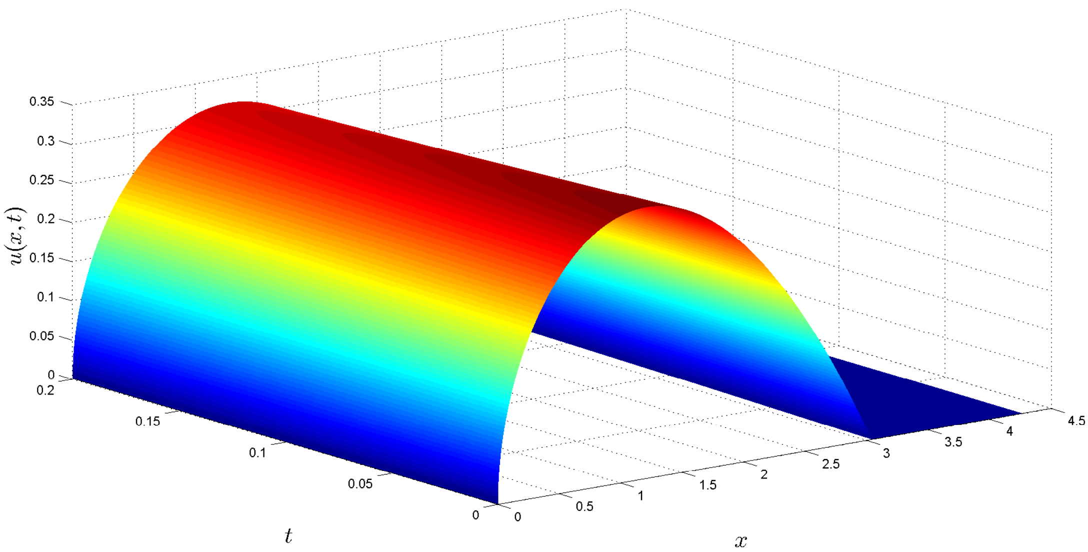
The problem (2.4) admits a nontrivial weak solution of the first type of self-similar solutions (see 1.2), with compact support in the interval $[0, a]$. Here we get $\alpha = -\dfrac{1}{m}$ and $\beta = -\dfrac{1}{2m}$. this holds when $2p + q = 0$, where $p = -\beta = \dfrac{1}{2m}$. Thus, the solution:

$$u(x,t) = x^{\frac{1}{m}} (t+\tau)^{\frac{\beta}{m}+\alpha} \left[\frac{p(m-1)}{m+1} \left(a^{\frac{m+1}{m}} - x^{\frac{m+1}{m}} (t+\tau)^{\frac{\beta(m+1)}{m}} \right) \right]^{\frac{1}{m-1}}.$$

Based on these results, we performed a numerical simulation to solve the equation, which takes the following form:



(a) 2D Plot of the explicit solution $u(x,t)$ for different values of t.



(b) 3D Plot of the explicit solution $u(x,t)$.

Figure 2.2: (a) and (b) represent the explicit solution $u(x,t)$ for the case $2p + q = 0$, when $m = 2$.

Theorem 2.1. *Both conditions* (i) *and* (ii) *are satisfied.*

i. *If* $U = 0$, *then for every* $a > 0$, *there exists a nontrivial weak solution with compact support for problem (2.4) which is positive in* $(0, a)$ *if and only if* $p > 0$ *and* $2p + q = 0$.

ii. *Assume that* $U > 0$. *Then, there exists a unique weak solution for* $a(U) > 0$ *such that* $f(\eta; a(U))$ *has compact support for problem (2.4). In which,* f *is positive on* $(0, a)$ *if and only if* $p > 0$ *and* $2p + q > 0$. *In both cases, the solution is bounded and unique.*

Proof. By Lemma 2.1 the necessary condition for the existence of such a solution is that $p > 0$ for any $m > 1$.
For $p = 0$ and $q > 0$, we already gave the solution (2.6) of (2.5) which is:

$$f(\eta; a) = \left[\frac{q(m-1)^2}{2m(m+1)}(a-\eta)^2\right]^{\frac{1}{m-1}}, \qquad 0 \leq \eta < a.$$

This solution is continuous and satisfies for every $U > 0$, that $f(0; a) = U$ and $f(a; a) = 0$, where

$$a(U) = \sqrt{\frac{2m(m+1)U^{m-1}}{q(m-1)^2}}.$$

In the following, for the case $p \neq 0$, we put $p > 0$ for any $m > 1$.
We already proved in Lemma 2.3 the local existence of a positive solution in the left neighborhood of $\eta = a$ of problem (2.4). This local solution is unique and can be continued back to $\eta = 0$ as positive solution with $f(0) > 0$ if and only if $2p + q > 0$, according to Lemma 2.5. Thus the boundary condition at $\eta = 0$ is satisfied if we can find an $a > 0$ such that

$$f(0; a) = U. \tag{2.7}$$

We distinguish two cases:

i. $U = 0$. Then by Lemma 2.5, the equation (2.7) can only be satisfied if $2p+q = 0$. Moreover, (2.7) is then satisfied for any $a > 0$.

ii. $U > 0$. It follows from Lemma 2.5 that now a necessary condition for (2.7) to have a solution is that $2p + q > 0$. To prove that it is also sufficient, we use an observation due to Barenblatt[1] [1]. Let $f(\eta; a)$ be a solution of problem (2.4) on $(0, a)$. Then for any

[1]Grigory Berneblatt, born in 1927, A prominent Russian scientist in fluid mechanics, mathematical physics, and applied engineering, known for self-similar solutions and dimensional analysis.

$\mu > 0$, the function $\mu^{\frac{-2}{m-1}} f(\mu\eta; \mu a)$ is a solution of problem (2.4) on $(0, \mu a)$. If $\mu = a^{-1}$, then

$$a^{\frac{2}{m-1}} f(0; 1) = U. \tag{2.8}$$

Because $f(0; 1) > 0$ in the cases $2p + q > 0$, we obtain a unique solution $a = a(U)$ of (2.8). Thus, the function $f(\eta; a(U))$ is a unique solution of (2.4), with the property

$$\begin{cases} f(\eta) > 0 & \text{on} \quad [0, a), \\ f(\eta) = 0 & \text{on} \quad [a, \infty). \end{cases}$$

Hence the theorem. $\square$

CHAPTER 3

NONLINEAR DIFFUSION EQUATION NOT IN DIVERGENCE FORM

In this chapter, we discuss a class of nonlinear diffusion equations that model diffusion processes in heterogeneous media. These equations are used to describe phenomena such as fluid flow and heat transfer in non-uniform materials. The general form of the equation is

$$\frac{\partial u}{\partial t} = f(u)\frac{\partial^2 u^m}{\partial x^2} + cu^m \frac{\partial^2 u}{\partial x^2}, \quad m > 1, \tag{3.1}$$

where $u > 0$, x is the spatial variable and t represents time. Also, f is a function that can be chosen. The second term represents an additional effect of nonlinear diffusion, which may account for phenomena such as diffusion in media with varying resistance; it will be written for $f \equiv 0$ and $c = 1$ as

$$\frac{\partial u}{\partial t} = u^m \frac{\partial^2 u}{\partial x^2}.$$

The global existence and blow-up of generalized self-similar solutions for this equation have been studied by Basti et al. in [2]. Equation (3.1) is a representation of a large number of diffusion equations. For $f \equiv 1$ and $c = 0$, (3.1) becomes the porous medium equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2},$$

which is studied in the second chapter. If we choose $f(u) = u$ and $c = -1$ the nonlinear diffusion equation (3.1) becomes

$$\frac{\partial u}{\partial t} = u\frac{\partial^2 u^m}{\partial x^2} - u^m \frac{\partial^2 u}{\partial x^2}, \quad m > 1. \tag{3.2}$$

This equation yields a solution with a self-similar structure, where the diffusion evolves according to a power law dependent on m. The parameter $m > 1$ governs the diffusion speed and the solution's nature, reflecting a nonlinear interaction between the two terms of the equation.

We equip equation (3.2) with the following boundary conditions:

$$u\left(0,t\right)=\left(t+\tau\right)^{\alpha}U,\quad \frac{\partial u}{\partial x}\left(a\left(t+\tau\right)^{\beta},t\right)=u\left(a\left(t+\tau\right)^{\beta},t\right)=0,\quad \forall\tau>0,\text{ and }t\geq 0.$$

From the types mentioned in the last part of the first chapter (part 1.2), we use (1.2) with a slight modification on the first type (we use $-\beta$ instead of β); Thus

$$u\left(x,t\right)=\left(t+\tau\right)^{\alpha}f(\eta),\quad \text{for}\quad \eta=\left(t+\tau\right)^{-\beta}x.$$

Using the same method as in chapter 2, we can find a solution u where

$$\frac{\partial u}{\partial t}=\left(t+\tau\right)^{\alpha-1}\left[\alpha f\left(\eta\right)-\beta\eta f'\left(\eta\right)\right]$$

and

$$\frac{\partial^2 u^m}{\partial x^2}=\left(t+\tau\right)^{m\alpha-2\beta}\left(f^m\left(\eta\right)\right)''.$$

Also

$$u\frac{\partial^2 u^m}{\partial x^2}-u^m\frac{\partial^2 u}{\partial x^2}=\left(t+\tau\right)^{\alpha(m+1)-2\beta}\left[\left(f^m\left(\eta\right)\right)''f\left(\eta\right)-f^m\left(\eta\right)f''\left(\eta\right)\right],$$

hence

$$\left(t+\tau\right)^{\alpha-1}\left[\alpha f\left(\eta\right)-\beta\eta f'\left(\eta\right)\right]=\left(t+\tau\right)^{\alpha(m+1)-2\beta}\left[\left(f^m\left(\eta\right)\right)''f\left(\eta\right)-f^m\left(\eta\right)f''\left(\eta\right)\right].$$

The condition related to similarity is

$$\alpha\left(m+1\right)-2\beta-\alpha+1=0,$$

then

$$\beta=\frac{\alpha m+1}{2}.$$

Consequently,

$$\alpha f\left(\eta\right)-\beta\eta f'\left(\eta\right)=\left(f^m\left(\eta\right)\right)''f\left(\eta\right)-f^m\left(\eta\right)f''\left(\eta\right).$$

We know that

$$\begin{aligned}\alpha f\left(\eta\right)-\beta\eta f'\left(\eta\right) &= m\left(m-1\right)f^{m-1}\left(\eta\right)\left(f'\left(\eta\right)\right)^2+mf^m\left(\eta\right)f''\left(\eta\right)-f^m\left(\eta\right)f''\left(\eta\right)\\ &= m\left(m-1\right)f^{m-1}\left(\eta\right)\left(f'\left(\eta\right)\right)^2+\left(m-1\right)f^m\left(\eta\right)f''\left(\eta\right)\\ &= \left(m-1\right)\left[mf^{m-1}\left(\eta\right)\left(f'\left(\eta\right)\right)^2+f^m\left(\eta\right)f''\left(\eta\right)\right].\end{aligned}$$

As

$$\left(f^m\left(\eta\right)f'\left(\eta\right)\right)'=mf^{m-1}\left(\eta\right)\left(f'\left(\eta\right)\right)^2+f^m\left(\eta\right)f''\left(\eta\right),$$

then

$$\left(m-1\right)\left(f^m\left(\eta\right)f'\left(\eta\right)\right)'=\alpha f\left(\eta\right)-\beta\eta f'\left(\eta\right).\tag{3.3}$$

The solution of (3.3) must satisfies the following conditions:

$$f\left(0\right)=U,\qquad f>0\text{ on }\left(0,a\right),\qquad f'\left(a\right)=f\left(a\right)=0.\tag{3.4}$$

3.1 Existence of Weak Solution with Compact Support

Lemma 3.1. *The function f is to be a nontrivial weak solution of problem (3.3)–(3.4) with compact support, which implies one of the following properties:*

i. $\beta > 0$ *for* $m > 1$.

ii. $\beta = 0$ *and* $\alpha > 0$.

Proof. Suppose f is a nontrivial solution of (3.3) with compact support; then there exists $a > 0$ such that $f > 0$ on $(a-\varepsilon, a)$ and $f = 0$ on $[a, \infty)$ for some $\varepsilon > 0$ and $f(a) = 0$. Then $\exists \eta_0 \in (a-\varepsilon, a)$ such that $f'(\eta_0) < 0$. Integration of (3.3) on an interval (η, a) gives

$$\begin{aligned} (m-1)\int_\eta^a \left(f^m(\xi)\, f'(\xi)\right)' d\xi &= \alpha \int_\eta^a f(\xi)\, d\xi - \beta \int_\eta^a \xi f'(\xi)\, d\xi \\ -(m-1)\, f^m(\eta)\, f'(\eta) &= \alpha \int_\eta^a f(\xi)\, d\xi - \beta \left([\xi f(\xi)]_\eta^a - \int_\eta^a f(\xi)\, d\xi \right) \\ &= \alpha \int_\eta^a f(\xi)\, d\xi + \beta \eta f(\eta) + \beta \int_\eta^a f(\xi)\, d\xi, \end{aligned}$$

then

$$-(m-1)\, f^m(\eta)\, f'(\eta) = \beta \eta f(\eta) + (\alpha+\beta) \int_\eta^a f(\xi)\, d\xi. \tag{3.5}$$

Hence, we begin verifying the previous cases where we have

i. Let us consider $\beta < 0$ for $m > 1$ where $\eta \in (a-\varepsilon, a)$ and we set $(\alpha+\beta) > 0$

$$\forall \eta \in (a-\varepsilon, a), \quad \forall \xi \in (\eta, a), \quad \text{we have} \quad f(\xi) < f(\eta)$$

and we conclude

$$\frac{f(\xi)}{f(\eta)} < 1,$$

also

$$\int_\eta^a \frac{f(\xi)}{f(\eta)} < a - \eta,$$

from equation (3.5) we find

$$-(m-1)\, f^{m-1}(\eta)\, f'(\eta) - \beta\eta < (\alpha+\beta)(a-\eta).$$

If $\eta \to a$ in the last equation, we notice that the left part is positive while the right part tends towards zero, which is a contradiction, implying that $\beta > 0$ for $m > 1$.

ii. we set $\beta = 0$ and $\alpha > 0$, we obtain

$$-(m-1) f^m(\eta) f'(\eta) = \alpha \int_\eta^a f(\xi)\, d\xi.$$

Hence, it is satisfied for $\alpha > 0$.

The proof is complete. $\square$

Lemma 3.2. *For any $m > 1$ and any $a > 0$, the problem (3.3)–(3.4) has a unique positive solution.*

Proof. We have

$$-(m-1) f^m(\eta) f'(\eta) = \beta\eta f(\eta) + (\alpha+\beta) \int_\eta^a f(\xi)\, d\xi,$$

where

$$\int_\eta^a f(\xi)\, d\xi = [\xi f]_\eta^a - \int_\eta^a \xi f'(\xi)\, d\xi,$$

thus

$$\begin{aligned} -(m-1) f^m(\eta) f'(\eta) &= \beta\eta f(\eta) + (\alpha+\beta)\left(-\eta f(\eta) - \int_\eta^a \xi f'(\xi)\, d\xi\right) \\ &= \beta\eta f(\eta) - (\alpha+\beta)\eta f(\eta) - (\alpha+\beta)\int_\eta^a \xi f'(\xi)\, d\xi, \end{aligned}$$

then

$$(m-1) f^m(\eta) f'(\eta) = \alpha\eta f(\eta) + (\alpha+\beta)\int_\eta^a \xi f'(\xi)\, d\xi.$$

We put the inverse function

$$G(f) = \eta,$$

when

$$\eta \to a, \qquad f \to 0.$$

By substituting it, we get

$$\begin{aligned} (m-1) f^m(\eta) \frac{df}{dG} &= \alpha G(f) f(\eta) + (\alpha+\beta)\int_f^0 G(\varphi) \frac{d\varphi}{dG} dG \\ \frac{df}{dG} &= \frac{\alpha G(f) f(\eta) + (\alpha+\beta)\int_f^0 G(\varphi)\, d\varphi}{(m-1) f^m(\eta)}. \end{aligned}$$

Therefore

$$\frac{dG}{df} = \frac{(m-1) f^m(\eta)}{\alpha G(f) f(\eta) - (\alpha+\beta)\int_0^f G(\varphi)\, d\varphi}.$$

By integrating from 0 to f , we get

$$\begin{aligned}\int_0^f \frac{dG}{df} &= \int_0^f \frac{(m-1)\,\varphi^m d\varphi}{\alpha G(\varphi)\,\varphi - (\alpha+\beta)\int_0^\varphi G(\psi)\,d\psi} \\ G(f) - a &= \int_0^f \frac{(m-1)\,\varphi^m d\varphi}{\alpha G(\varphi)\,\varphi - (\alpha+\beta)\int_0^\varphi G(\psi)\,d\psi}.\end{aligned}$$

We put

$$D(f) = 1 - a^{-1}G(f),$$

then

$$G(f) = a - aD(f),$$

By substituting it, we find

$$\begin{aligned}a - aD(f) - a &= \int_0^f \frac{(m-1)\,\varphi^m d\varphi}{\alpha\,(a - aD(\varphi))\,\varphi - (\alpha+\beta)\int_0^\varphi (a - aD(\psi))\,d\psi} \\ -aD(f) &= \int_0^f \frac{(m-1)\,\varphi^m d\varphi}{\alpha\,(a - aD(\varphi))\,\varphi - (\alpha+\beta)\int_0^\varphi (a - aD(\psi))\,d\psi} \\ &= \int_0^f \frac{(m-1)\,\varphi^m d\varphi}{a\left[\alpha\,(1 - D(\varphi))\,\varphi - (\alpha+\beta)\int_0^\varphi (1 - D(\psi))\,d\psi\right]}.\end{aligned}$$

Divide by a to obtain

$$\begin{aligned}D(f) &= \frac{1}{a^2}\int_0^f \frac{(m-1)\,\varphi^m d\varphi}{\left[\alpha\,(-1 + D(\varphi))\,\varphi + (\alpha+\beta)\int_0^\varphi (1 - D(\psi))\,d\psi\right]} \\ &= \frac{1}{a^2}\int_0^f \frac{(m-1)\,\varphi^m d\varphi}{-\alpha\varphi + \alpha D(\varphi)\,\varphi + (\alpha+\beta)\int_0^\varphi d\psi - (\alpha+\beta)\int_0^\varphi D(\psi)\,d\psi} \\ &= \frac{1}{a^2}\int_0^f \frac{(m-1)\,\varphi^m d\varphi}{-\alpha\varphi + \alpha D(\varphi)\,\varphi + \alpha\varphi + \beta\varphi - (\alpha+\beta)\int_0^\varphi D(\psi)\,d\psi} \\ &= \frac{1}{a^2}\int_0^f \frac{(m-1)\,\varphi^m d\varphi}{\alpha D(\varphi)\,\varphi + \beta\varphi - (\alpha+\beta)\int_0^\varphi D(\psi)\,d\psi}.\end{aligned}$$

Let $\delta > 0,$ and let X be the set of bounded functions $D(f)$ defined on $[0, \delta]$ such that

$$0 \le D(f) \le \tau = \frac{\beta}{2\,(|\alpha| + |\alpha+\beta|)}.$$

We denote by $\|\cdot\|$ the supremum norm on X. Then X is a Banach space. We define on X the operator

$$M(D)(f) = \frac{1}{a^2}\int_0^f \frac{(m-1)\,\varphi^m d\varphi}{\alpha D(\varphi)\,\varphi + \beta\varphi - (\alpha+\beta)\int_0^\varphi D(\psi)\,d\psi}.$$

Let $D \in X$ we have

$$\begin{aligned}
\beta D(\varphi)\varphi + \alpha\varphi - (\alpha+\beta)\int_0^{\varphi} D(\psi)\,d\psi &\geq \beta\varphi - |\alpha|\,\varphi\,\|D(\varphi)\| - |\alpha+\beta|\,\|D(\varphi)\|\int_0^{\varphi} d\psi \\
&\geq \beta\varphi - (|\alpha| + |\alpha+\beta|)\,\varphi\,\|D(\varphi)\| \\
&\geq \varphi\,[\beta - (|\alpha| + |\alpha+\beta|)\,\tau] \\
&\geq \varphi\left[\beta - (|\alpha| + |\alpha+\beta|)\frac{\beta}{2(|\alpha| + |\alpha+\beta|)}\right] \\
&\geq \frac{\beta}{2}\varphi.
\end{aligned}$$

Therefore

$$\begin{aligned}
M(D)(f) &\leq \frac{1}{a^2}\int_0^f \frac{(m-1)\,\varphi^m d\varphi}{\frac{\beta}{2}\varphi} \\
&\leq \frac{2(m-1)}{a^2\beta}\int_0^f \varphi^{m-1} d\varphi \\
&\leq \frac{2(m-1)\,f^m}{a^2\beta m} \\
&\leq \frac{2(m-1)\,\delta^m}{a^2\beta m}.
\end{aligned}$$

Thus, $M(D)$ is well defined on the whole of X. Moreover, there exists a $\delta_0 > 0$ such that if $\delta \leq \delta_0$, $\|M(D)\| \leq \delta$. Let now $D_1, D_2 \in X$ and let $\delta \leq \delta_0$.Then

$$M(D_1)(f) = \frac{1}{a^2}\int_0^f \frac{(m-1)\,\varphi^m d\varphi}{\alpha D_1(\varphi)\varphi + \beta\varphi - (\alpha+\beta)\int_0^{\varphi} D_1(\psi)\,d\psi}$$

and

$$M(D_2)(f) = \frac{1}{a^2}\int_0^f \frac{(m-1)\,\varphi^m d\varphi}{\alpha D_2(\varphi)\varphi + \beta\varphi - (\alpha+\beta)\int_0^{\varphi} D_2(\psi)\,d\psi}.$$

We put

$$S(D) = \alpha D(\varphi)\varphi + \beta\varphi - (\alpha+\beta)\int_0^{\varphi} D(\psi)\,d\psi.$$

Then

$$\begin{aligned}
|S(D_2) - S(D_1)| &= \Big|\alpha D_1(\varphi)\varphi + \beta\varphi - (\alpha+\beta)\int_0^{\varphi} D_1(\psi)\,d\psi \\
&\quad - \alpha D_2(\varphi)\varphi - \beta\varphi + (\alpha+\beta)\int_0^{\varphi} D_2(\psi)\,d\psi\Big| \\
&= |\alpha\varphi(D_2(\varphi) - D_1(\varphi)) + (\alpha+\beta)\varphi(D_2(\varphi) - D_1(\varphi))| \\
&\leq |\alpha|\,\varphi\,\|D_2(\varphi) - D_1(\varphi)\| + |\alpha+\beta|\,\varphi\,\|D_2(\varphi) - D_1(\varphi)\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
|M(D_1)(f) - M(D_2)(f)| &= \left| \frac{1}{a^2} \int_0^f \frac{(m-1)\varphi^m d\varphi}{S(D_1)} - \frac{1}{a^2} \int_0^f \frac{(m-1)\varphi^m d\varphi}{S(D_2)} \right| \\
&= \frac{1}{a^2} \left| \int_0^f \frac{(m-1)\varphi^m (S(D_2) - S(D_1)) d\varphi}{S(D_1) S(D_2)} \right| \\
&\leq \frac{(m-1)}{a^2} \left| \int_0^f \frac{\varphi^m (S(D_2) - S(D_1)) d\varphi}{\left(\frac{\beta}{2}\varphi\right)\left(\frac{\beta}{2}\varphi\right)} \right|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|M(D_1)(f) - M(D_2)(f)| &\leq \frac{4(m-1)}{a^2 m \beta^2} \int_0^f \varphi^{m-2} (|\alpha| \varphi \|D_2(\varphi) - D_1(\varphi)\| \\
&\quad + |\alpha + \beta| \varphi \|D_2(\varphi) - D_1(\varphi)\|) \\
&\leq \left(\frac{4(m-1)(|\alpha| + |\alpha+\beta|)}{a^2 \beta^2 m} \int_0^f \varphi^{m-1} \right) \|D_2(\varphi) - D_1(\varphi)\| \\
&\leq \left(\frac{4(m-1)(|\alpha| + |\alpha+\beta|) f^m}{a^2 \beta^2 m} \right) \|D_2(\varphi) - D_1(\varphi)\| \\
&\leq \left(\frac{4(m-1)(|\alpha| + |\alpha+\beta|) \delta^m}{a \beta^2 m} \right) \|D_2(\varphi) - D_1(\varphi)\|.
\end{aligned}$$

Hence, if we take $\delta < \left[\frac{a^2 \beta m \tau}{2(m-1)}\right]^{\frac{1}{m}}$ we get $\frac{4(m-1)(|\alpha|+|\alpha+\beta|)\delta^m}{a\beta^2 m} < 1$. Therefore, we conclude that M is a contraction on X. Thus, by Banach contraction mapping principle [12], M has a unique fixed point in X, and (3.3) has a unique solution. □

3.2 Explicit Solutions

Explicit Solution for $\alpha + \beta = 0$

From equation (3.5), we have

$$-(m-1) f^m(\eta) f'(\eta) = \beta \eta f(\eta),$$

then

$$-(m-1) f^{m-1}(\eta) f'(\eta) = \beta \eta.$$

By integrating the last equation from η to a, we get

$$f^m(\eta) = \frac{\beta m}{2(m-1)} \left(a^2 - \eta^2\right),$$

hence

$$f(\eta) = \left[\frac{\beta m}{2(m-1)} \left(a^2 - \eta^2\right)\right]^{\frac{1}{m}}.$$

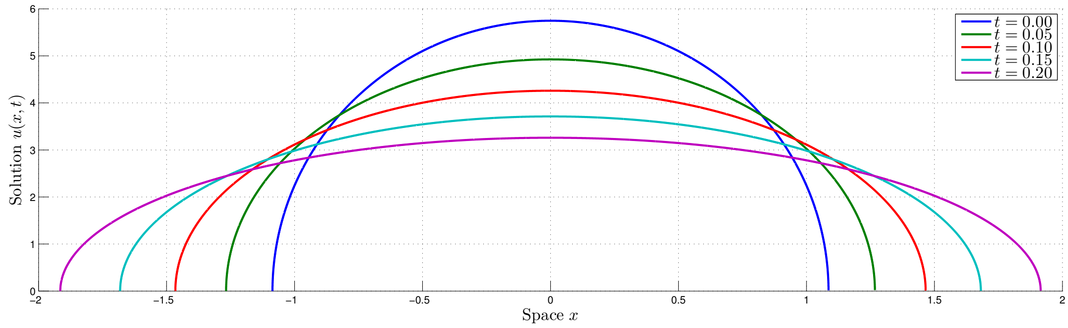
Using the boundary conditions, we obtain

$$a\left(U\right)=\sqrt{\frac{2\left(m-1\right)U^{m}}{\beta m}}.$$

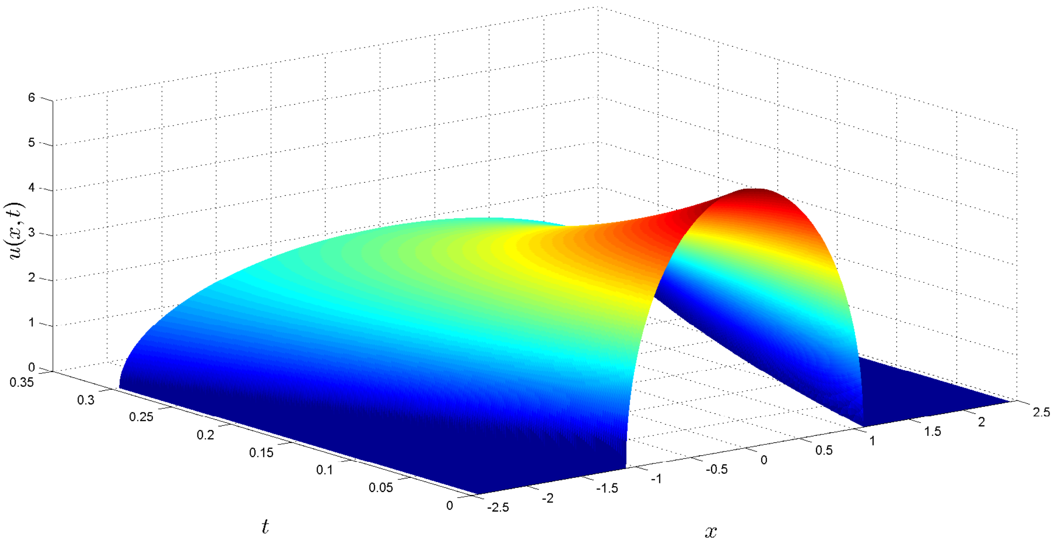
in which problem (3.3)–(3.4) admits a nontrivial weak solution with compact support $[0, a]$ when $\alpha = -\beta$. Therefore

$$u\left(x,t\right)=\left(t+\tau\right)^{-\beta}\left[\frac{\beta m}{2\left(m-1\right)}\left(a^{2}-\frac{x^{2}}{\left(t+\tau\right)^{2\beta}}\right)\right]^{\frac{1}{m}}.$$

Based on these results, we performed a numerical simulation to solve the equation, which takes the following form:



(a) 2D Plot of the explicit solution $u(x, t)$ for different values of t.



(b) 3D Plot of the explicit solution $u(x, t)$.

Figure 3.1: (a) and (b) represent the explicit solution $u(x, t)$ for the case $\alpha + \beta = 0$, when $m = 2$

Explicit Solution for $\beta = 0$

From equation (3.5), we have

$$-\left(m-1\right)f^{m}\left(\eta\right)f'\left(\eta\right)=\alpha\int_{\eta}^{a}f\left(\xi\right)d\xi.$$

If we put

$$h\left(\eta\right)=-\int_{\eta}^{a}f\left(\xi\right)d\xi,$$

then,

$$h'\left(\eta\right)=f\left(\eta\right),\quad \text{and}\quad h''\left(\eta\right)=f'\left(\eta\right),$$

hence

$$-\left(m-1\right)\left(h'\left(\eta\right)\right)^{m}h''\left(\eta\right)=-\alpha h\left(\eta\right).$$

If we multiply by h' and integrate, we obtain

$$-\left(m-1\right)\int_{\eta}^{a}\left(h'\left(\xi\right)\right)^{m+1}h''\left(\xi\right)d\xi=-\alpha\int_{\eta}^{a}h\left(\xi\right)h'\left(\xi\right)d\xi,$$

then

$$h'\left(\eta\right)=\left[\frac{\alpha\left(m+2\right)}{2\left(m-1\right)}\right]^{\frac{1}{m+2}}h^{\frac{2}{m+2}}\left(\eta\right).$$

Hence

$$-h\left(\eta\right)=\left[\frac{Fm}{m+2}\left(a-\eta\right)\right]^{\frac{m+2}{m}},$$

where

$$F=\left[\frac{\alpha\left(m+2\right)}{2\left(m-1\right)}\right]^{\frac{1}{m+2}}.$$

Substituting F and h, we find

$$f\left(\eta\right)=\left[\frac{\alpha m^{2}}{2\left(m-1\right)\left(m+2\right)}\left(a-\eta\right)^{2}\right]^{\frac{1}{m}}.$$

Using the boundary conditions, we obtain

$$a\left(U\right)=\sqrt{\frac{2\left(m-1\right)\left(m+2\right)U^{\frac{1}{m}}}{\alpha m^{2}}}.$$

in which problem (3.3)–(3.4) admits a nontrivial weak solution with compact support $[0,a]$ when $\beta = 0$. Therefore

$$u\left(x,t\right)=\left(t+\tau\right)^{\alpha}\left[\frac{\alpha m^{2}}{2\left(m-1\right)\left(m+2\right)}\left(a-x\right)^{2}\right]^{\frac{1}{m}}.$$

CHAPTER 4

INTRODUCTION TO MATHEMATICAL MODELING OF EPIDEMICS

The application of PDEs constitutes a robust mathematical framework for analyzing spatiotemporal disease transmission dynamics. Unlike conventional compartmental models (e.g., SIRS, SEIR) based on ODEs that assume homogeneous mixing, PDE formulations explicitly account for spatial heterogeneities in population distributions, geographical barriers, and environmental variations.

This investigation employs **reaction-diffusion** systems to model the spatiotemporal evolution of susceptible, infected, and recovered population densities. We conduct comparative analysis of two principal epidemiological frameworks: The SEIR model considering individuals in the exposed or latent stage, and the canonical SIRS model, utilizing PDE methodology to capture realistic disease spread patterns across heterogeneous environments with varying population densities and mobility patterns.

4.1 PDE-based SIRS Model

The SIRS model describes the spatial-temporal spread of an infectious disease in a population. It assumes recovered individuals can lose immunity and become susceptible again. Let $S(x,t)$, $I(x,t)$, and $R(x,t) \in \mathbb{R}^+$ denote the population densities of susceptible, infected, and recovered individuals, respectively, at spatial location $x \in \Omega \subseteq \mathbb{R}^n$ and time $t \geq 0$. It can be represented by the following chart:

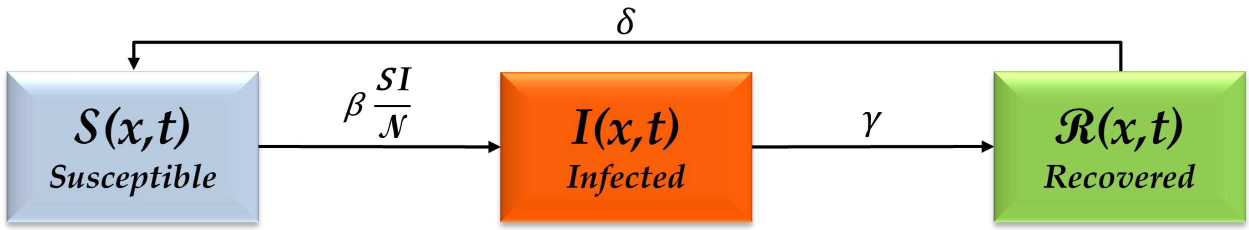


Figure 4.1: **SIRS Model**

The system of equations for the model is shown below:

$$\begin{cases} \frac{\partial S}{\partial t} = D_S \Delta S - \beta \frac{SI}{N} + \delta R, \\ \frac{\partial I}{\partial t} = D_I \Delta I + \beta \frac{SI}{N} - \gamma I, \\ \frac{\partial R}{\partial t} = D_R \Delta R + \gamma I - \delta R, \end{cases}$$

with initial conditions:

$$S(x,0) = S_0(x), \quad I(x,0) = I_0(x), \quad R(x,0) = R_0(x), \quad \text{for all} \quad x \in \Omega.$$

where

- Δ is the Laplace operator (e.g., $\Delta u = \nabla^2 u$),
- D_S, D_I, D_R are diffusion coefficients,
- β is the transmission rate,
- γ is the recovery rate,
- δ is the immunity loss rate.

When neglecting spatial effects (assuming spatial homogeneity) and eliminating diffusion terms ($\Delta S = \Delta I = \Delta R = 0$), the system transitions from PDEs to the classical ODEs framework used in traditional epidemiological models. The model then takes the form:

$$\begin{cases} \frac{dS}{dt} = -\beta \frac{SI}{N} + \delta R, \\ \frac{dI}{dt} = \beta \frac{SI}{N} - \gamma I, \\ \frac{dR}{dt} = \gamma I - \delta R. \end{cases}$$

In analyzing the classical SIRS model, we observe a fundamental conservation property where the total population ($N = S + I + R$) remains constant. This can be rigorously proved by summing the three governing equations:

$$\begin{aligned} \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} &= -\beta \frac{SI}{N} + \delta R + \beta \frac{SI}{N} - \gamma I + \gamma I - \delta R \\ &= 0 \end{aligned}$$

and thus

$$\frac{dN}{dt} = 0.$$

Hence, N is constant.

Given the population conservation ($N = S + I + R$ constant), we normalize the model using population fractions:

$$\mathcal{S}(t) = \frac{S(t)}{N}, \quad \mathcal{I}(t) = \frac{I(t)}{N}, \quad \mathcal{R}(t) = \frac{R(t)}{N}.$$

Substitution into the original equations yields the dimensionless system:

$$\begin{cases} \frac{d\mathcal{S}}{dt} = -\beta\mathcal{S}\mathcal{I} + \delta\mathcal{R}, \\ \frac{d\mathcal{I}}{dt} = \beta\mathcal{S}\mathcal{I} - \gamma\mathcal{I}, \\ \frac{d\mathcal{R}}{dt} = \gamma\mathcal{I} - \delta\mathcal{R}. \end{cases}$$

This normalized form preserves the original dynamics while simplifying mathematical analysis.

The Equilibrium Points

We now study the equilibrium points, starting with

Disease-Free Equilibrium

- If $\mathcal{I} = 0$, we have

$$\begin{cases} \frac{d\mathcal{S}}{dt} = -\beta\mathcal{S}\mathcal{I} + \delta\mathcal{R} = 0, \\ \frac{d\mathcal{I}}{dt} = \beta\mathcal{S}\mathcal{I} - \gamma\mathcal{I} = 0, \\ \frac{d\mathcal{R}}{dt} = \gamma\mathcal{I} - \delta\mathcal{R} = 0, \end{cases}$$

  hence $\mathfrak{Dfe} = (\mathcal{S}_0, \mathcal{I}_0, \mathcal{R}_0) = (\mathcal{S}_0, 0, 0) = (1, 0, 0)$.

Endemic Equilibrium

- If $\mathcal{I} > 0$, we have

$$\begin{aligned} \beta\mathcal{S}^*\mathcal{I}^* - \gamma\mathcal{I}^* &= 0 \\ \mathcal{I}^*(\beta\mathcal{S}^* - \gamma) &= 0, \end{aligned}$$

  hence

$$\mathcal{S}^* = \frac{\gamma}{\beta}.$$

And

$$\gamma\mathcal{I}^* - \delta\mathcal{R}^* = 0,$$

thus

$$\mathcal{R}^* = \frac{\gamma}{\delta}\mathcal{I}^*.$$

From the normalization condition $N = 1$, we derive:

$$\mathcal{S}^* + \mathcal{I}^* + \mathcal{R}^* = 1,$$

then

$$\begin{aligned}\mathcal{I}^* &= 1 - \mathcal{S}^* - \mathcal{R}^* \\ &= 1 - \frac{\gamma}{\beta} - \frac{\gamma}{\delta}\mathcal{I}^*.\end{aligned}$$

Therefore

$$\mathcal{I}^* = \frac{1 - \frac{\gamma}{\beta}}{1 + \frac{\gamma}{\delta}}.$$

Thus

$$\mathfrak{Eqp} = (\mathcal{S}^*, \mathcal{I}^*, \mathcal{R}^*) = \left(\frac{\gamma}{\beta}, \mathcal{I}^*, \frac{\gamma}{\delta}\mathcal{I}^*\right).$$

Basic Reproduction Number

To calculate the basic reproduction number $\mathfrak{Rb}$, we use the SIRS model with one infected compartment $\mathcal{I}$:

$$f - v = \beta\mathcal{S}(t)\mathcal{I}(t) - \gamma\mathcal{I}(t).$$

Accordingly,

$$f = \beta\mathcal{S}(t)\mathcal{I}(t), \quad v = \gamma\mathcal{I}(t).$$

Here, f is the rate of appearance of new infections, and v is the rate of other transitions. The new infection matrix $\mathcal{F}$ and transition matrix $\mathcal{V}$ are assessed at the disease-free equilibrium point, as follows:

$$\mathcal{F} = \frac{d}{d\mathcal{I}}\left[\beta\mathcal{S}(t)\mathcal{I}(t)\right]\Big|_{\mathfrak{Dfe}}, \quad \mathcal{V} = \frac{d}{d\mathcal{I}}\left[\gamma\mathcal{I}(t)\right]\Big|_{\mathfrak{Dfe}},$$

hence

$$\mathcal{F} = \beta\mathcal{S}_0, \quad \mathcal{V} = \gamma.$$

It follows that

$$\mathcal{V}^{-1} = \frac{1}{\gamma},$$

then

$$\mathcal{F}\mathcal{V}^{-1} = \frac{\beta\mathcal{S}_0}{\gamma},$$

thus

$$\mathfrak{Rb} = \frac{\beta}{\gamma}.$$

We can rewrite $\mathcal{I}^*$ of the endemic equilibrium point as follows

$$\mathcal{I}^* = \frac{1-\frac{\gamma}{\beta}}{1+\frac{\gamma}{\delta}} = \frac{\frac{\gamma}{\beta}\left(\frac{\beta}{\gamma}-1\right)}{\frac{\gamma}{\delta}\left(\frac{\delta}{\gamma}+1\right)},$$

therefore

$$\mathcal{I}^* = \frac{\delta\left(\mathfrak{Rb}-1\right)}{\delta\mathfrak{Rb}+\beta},$$

and

$$\mathfrak{Eqp} = (\mathcal{S}^*, \mathcal{I}^*, \mathcal{R}^*) = \left(\frac{\gamma}{\beta}, \frac{\delta\left(\mathfrak{Rb}-1\right)}{\delta\mathfrak{Rb}+\beta}, \frac{\gamma\left(\mathfrak{Rb}-1\right)}{\delta\mathfrak{Rb}+\beta}\right).$$

The endemic equilibrium point exists when $\mathfrak{Rb} > 1$.

Stability Analysis of the Equilibrium Points

The system's Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial(-\beta\mathcal{S}\mathcal{I}+\delta\mathcal{R})}{\partial\mathcal{S}} & \frac{\partial(-\beta\mathcal{S}\mathcal{I}+\delta\mathcal{R})}{\partial\mathcal{I}} & \frac{\partial(-\beta\mathcal{S}\mathcal{I}+\delta\mathcal{R})}{\partial\mathcal{R}} \\ \frac{\partial(\beta\mathcal{S}\mathcal{I}-\gamma\mathcal{I})}{\partial\mathcal{S}} & \frac{\partial(\beta\mathcal{S}\mathcal{I}-\gamma\mathcal{I})}{\partial\mathcal{I}} & \frac{\partial(\beta\mathcal{S}\mathcal{I}-\gamma\mathcal{I})}{\partial\mathcal{R}} \\ \frac{\partial(\gamma\mathcal{I}-\delta\mathcal{R})}{\partial\mathcal{S}} & \frac{\partial(\gamma\mathcal{I}-\delta\mathcal{R})}{\partial\mathcal{I}} & \frac{\partial(\gamma\mathcal{I}-\delta\mathcal{R})}{\partial\mathcal{R}} \end{pmatrix}.$$

Then

$$J = \begin{pmatrix} -\beta\mathcal{I} & -\beta\mathcal{S} & \delta \\ \beta\mathcal{I} & \beta\mathcal{S}-\gamma & 0 \\ 0 & \gamma & -\delta \end{pmatrix}$$

- At the disease-free equilibrium point $\mathfrak{Dfe}$, we have

$$J_{\mathfrak{Dfe}} = \begin{pmatrix} 0 & -\beta & \delta \\ 0 & \beta-\gamma & 0 \\ 0 & \gamma & -\delta \end{pmatrix}.$$

The stability of the $\mathfrak{Dfe}$ depends on the eigenvalues of the Jacobian matrix. Solving thecharacteristic equation $\det\left(J_{\mathfrak{Dfe}} - \lambda \times Id_{3\times3}\right) = 0$. The characteristic equation is by calculating:

$$\begin{vmatrix} -\lambda & -\beta & \delta \\ 0 & \beta-\gamma-\lambda & 0 \\ 0 & \gamma & -\delta-\lambda \end{vmatrix} = 0.$$

Thus

$$-\lambda\left[(\beta-\gamma-\lambda)(-\delta-\lambda)\right]=0,$$

hence

$$\lambda(\delta+\lambda)\left(\gamma\left(\frac{\beta}{\gamma}-1\right)-\lambda\right)=0.$$

Thus

$$\lambda(\delta+\lambda)(\gamma(\mathfrak{Rb}-1)-\lambda)=0.$$

If $\mathfrak{Rb}=\frac{\beta}{\gamma}<1$, then all the eigenvalues are negative, therefore the disease-free equilibrium is stable when $\mathfrak{Rb}<1$.

- At the endemic equilibrium point $\mathfrak{Eqp}$, we get

$$J_{\mathfrak{Eqp}}=\begin{pmatrix} -\frac{\delta\gamma(\mathfrak{Rb}-1)}{\delta+\gamma} & -\gamma & \delta \\ \frac{\delta\gamma(\mathfrak{Rb}-1)}{\delta+\gamma} & 0 & 0 \\ 0 & \gamma & -\delta \end{pmatrix}.$$

Let $\psi=\frac{\delta\gamma(\mathfrak{Rb}-1)}{\delta+\gamma}$, then

$$\begin{vmatrix} -\psi-\lambda & -\gamma & \delta \\ \psi & -\lambda & 0 \\ 0 & \gamma & -\delta-\lambda \end{vmatrix}=0,$$

the characteristic polynomial is given by

$$\begin{aligned} P(\lambda) &= (-\psi-\lambda)\left[-\lambda(-\delta-\lambda)\right]+\gamma\left[\psi(-\delta-\lambda)\right]+\delta\left[\psi\gamma\right] \\ &= (-\psi-\lambda)\left(\delta\lambda+\lambda^2\right)+\gamma\psi(-\delta-\lambda)+\delta\psi\gamma, \end{aligned}$$

by simplifying, we find

$$P(\lambda)=-\lambda Q(\lambda), \quad \text{with} \quad Q(\lambda)=\lambda^2+(\psi+\delta)\lambda+\psi(\delta+\gamma).$$

All the eigenvalues are negative because all the coefficients of $Q(\lambda)$ are positive. we note that

$$(\psi+\delta)>0, \quad \text{and} \quad \psi(\delta+\gamma)>0.$$

Since $\psi>0$ or $\psi>-\delta$, we choose $\psi>0$, where $\psi=\frac{\delta\gamma(\mathfrak{Rb}-1)}{\delta+\gamma}>0$ if and only if $\mathfrak{Rb}>1$. Therefore, the $\mathfrak{Eqp}$ is locally asymptotically stable when $\mathfrak{Rb}>1$.

4.2 Applications with Numerical Results

Within the framework of studying epidemic spread dynamics, we conduct a numerical simulation of the SIR (Susceptible-Infected-Recovered) model for a closed population with constant size $N = 1000$ individuals. The simulation is initialized with the following conditions: a single infected case $I(0) = 1$, zero recovered cases at baseline $R(0) = 0$, resulting in an initial susceptible population

$$S(0) = N - I(0) - R(0) = 999.$$

This simulation utilizes a set of epidemiological parameters governing model interactions, whose values and corresponding biological interpretations are detailed in the accompanying table

Table 4.1: **Initial and Model Parameters for the SIR Epidemiological Model**

Parameter	Interpretation	Assumed Value	Unit
$S(0)$	Initial number of susceptible individuals	999	individuals
$I(0)$	Initial number of infected individuals	1	individuals
$R(0)$	Initial number of recovered individuals	0	individuals
β	Transmission rate	0.3	$\frac{1}{\text{time}}$
γ	Recovery rate	0.1	$\frac{1}{\text{time}}$

Accordingly, the basic reproduction number $\mathfrak{Rb}$ is calculated based on the data provided in the table.

Based on these results, we performed a numerical simulation of the model, as follows:

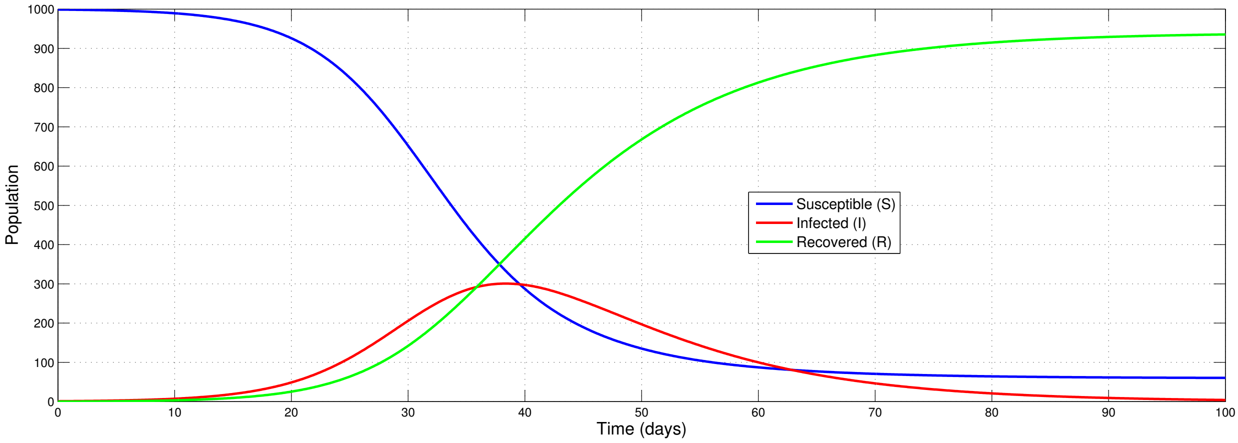


Figure 4.2: **SIR Model Dynamics**

From Figure 4.5, we observe that the number of susceptibles starts high and then gradually decreases as they move to the infected group. The number of infected individuals rises to a peak and then decreases as they transition to the recovered group. The number of recovered individuals starts from zero and increases over time. The graph shows a single peak of infection followed by a stabilization period and a continuous decline in the number of susceptibles.

By introducing the waning immunity parameter $\delta = 0.02$ into the classical SIR model, the system transitions to an SIRS model that captures the dynamics of diseases with temporary immunity. This parameter represents the rate at which recovered individuals R revert to the susceptible population S due to loss of acquired immunity.

To examine this model's behavior, we conducted numerical simulations under these specifications, with the results presented as follows:

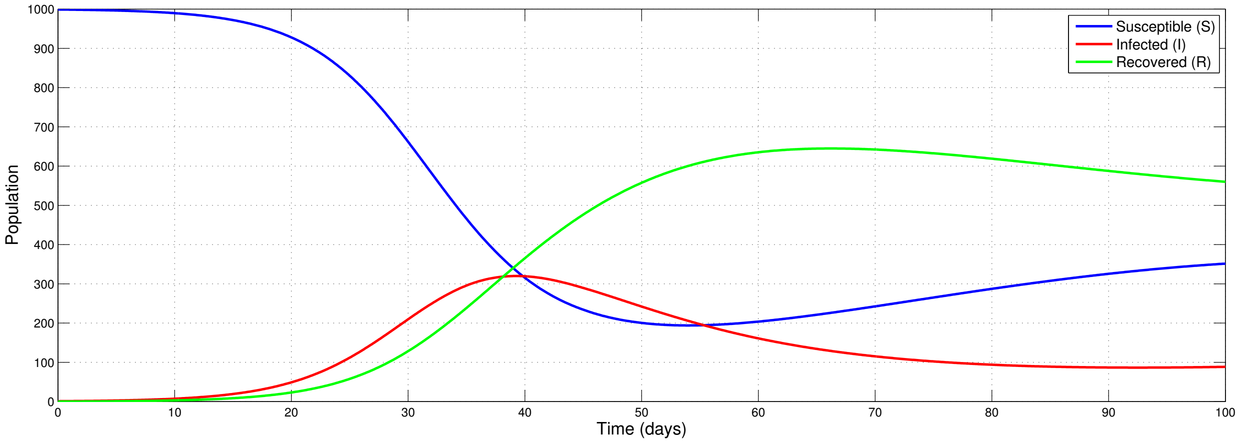


Figure 4.3: **SIRS Model Dynamics**

From Figure 4.3, we observe that the number of susceptibles starts high and then gradually decreases, with some fluctuations due to individuals recovering and returning to the susceptible group due to the effect of δ. The number of infected individuals rises to a peak and then decreases due to recovery, while the number of recovered individuals gradually increases. However, some individuals return to the susceptible group because of δ, leading to fluctuations in the population of the different groups.

4.3 PDE-based SEIR Model

The Partial Differential Equation-Based SEIR Model extends the classical SEIR framework by incorporating partial differential equations to account for spatial dynamics and heterogeneous disease transmission patterns, thereby enhancing the predictive accuracy of disease spread across populations and diverse environments. It can be represented by the following chart:

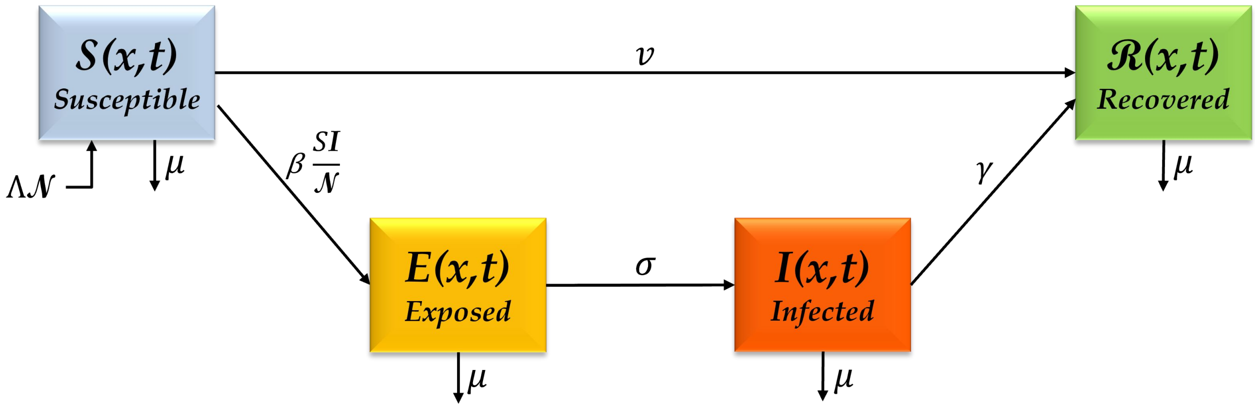


Figure 4.4: **SEIR Model**

The system of equations for the model is shown below:

$$\begin{cases} \frac{\partial S}{\partial t} = D_S \Delta S + \Lambda N - \left(\beta \frac{I}{N} - \upsilon - \mu\right) S, \\ \frac{\partial E}{\partial t} = D_E \Delta E + \beta \frac{SI}{N} - (\sigma + \mu) E, \\ \frac{\partial I}{\partial t} = D_I \Delta I + \sigma E - (\gamma +) I, \\ \frac{\partial R}{\partial t} = D_R \Delta R + \upsilon S + \gamma I - \mu R. \end{cases}$$

with initial conditions:

$$S(x,0) = S_0(x), \quad I(x,0) = I_0(x), \quad R(x,0) = R_0(x), \quad \text{for all} \quad x \in \Omega.$$

Here

- S is the susceptible individuals.
- E is the exposed individuals.
- I expresses the infected individuals.
- R presents the recovered individuals.
- Λ is the birth rate.
- β is the infection transmission rate.
- υ is represents the vaccination rate,
- σ is shows the transition rate from E to I.
- γ is presents the recovery rate.

- μ is expresses the death rate.
- N is the total population.

When neglecting spatial effects (assuming spatial homogeneity) and eliminating diffusion terms ($\Delta S = \Delta I = \Delta R = 0$), the system transitions from PDEs to the classical ODEs framework used in traditional epidemiological models. The model then takes the form:

$$\begin{cases} \frac{dS}{dt} = \Lambda N - \left(\beta \frac{I}{N} + \upsilon + \mu\right) S, \\ \frac{dE}{dt} = \beta \frac{SI}{N} - (\sigma + \mu) E, \\ \frac{dI}{dt} = \sigma E - (\gamma + \mu) I, \\ \frac{dR}{dt} = \upsilon S + \gamma I - \mu R. \end{cases}$$

Summing all equations, we obtain

$$\frac{dN}{dt} = (\Lambda - \mu) N(t).$$

Consequently,

$$N(t) \leq N_0 \exp\left((\Lambda - \mu)T\right).$$

Where T is the maximum time in our study to the model and N_0 is the initial total population at $t = 0$. Then, N is bounded.

In the subsequent sections of this study, we assume the existence of a positive constant

$$\mathcal{N} \leq N_0 \exp\left((\Lambda - \mu)T\right),$$

for which the total population N remains fixed throughout our study, which can be expressed as $N(t) = \mathcal{N}$, for any $t \in [0, T]$. This assumption is made to normalize the SEIR model. Therefore, we put:

$$\mathcal{S}(t) = \frac{S(t)}{\mathcal{N}}, \quad \mathcal{E}(t) = \frac{E(t)}{\mathcal{N}}, \quad \mathcal{I}(t) = \frac{I(t)}{\mathcal{N}}, \quad \mathcal{R}(t) = \frac{R(t)}{\mathcal{N}}.$$

Then we obtain

$$\begin{cases} \frac{d\mathcal{S}}{dt} = \Lambda - (\beta \mathcal{I} + \upsilon + \mu) \mathcal{S}, \\ \frac{d\mathcal{E}}{dt} = \beta \mathcal{S}\mathcal{I} - (\sigma + \mu) \mathcal{E}, \\ \frac{d\mathcal{I}}{dt} = \sigma \mathcal{E} - (\gamma + \mu) \mathcal{I}, \\ \frac{d\mathcal{R}}{dt} = \upsilon \mathcal{S} + \gamma \mathcal{I} - \mu \mathcal{R}. \end{cases}$$

Basic Reproduction Number

The SEIR model consists of two infection components, $\mathcal{E}$ and $\mathcal{I}$, therefore

$$f_i - v_i = \begin{pmatrix} \beta\mathcal{S}\mathcal{I} - (\sigma+\mu)\,\mathcal{E}, \\ \sigma\mathcal{E} - (\gamma+\mu)\,\mathcal{I}. \end{pmatrix}$$

Accordingly,

$$f_i = \begin{pmatrix} \beta\mathcal{S}\mathcal{I} \\ 0 \end{pmatrix}, \quad v_i = \begin{pmatrix} (\sigma+\mu)\,\mathcal{E} \\ -\sigma\mathcal{E} + (\gamma+\mu)\,\mathcal{I} \end{pmatrix}.$$

At the disease-free equilibrium, we compute the infection matrix $\mathcal{F}$ and transition matrix $\mathcal{V}$ as follows

$$\mathcal{F} = \begin{pmatrix} 0 & \beta\mathcal{S}_0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} \sigma+\mu & 0 \\ -\sigma & \gamma+\mu \end{pmatrix}.$$

It follows that

$$\mathcal{V}^{-1} = \begin{pmatrix} \frac{1}{\sigma+\mu} & 0 \\ \frac{\sigma}{(\sigma+\mu)(\gamma+\mu)} & \frac{1}{\gamma+\mu} \end{pmatrix}.$$

We have $\mathfrak{Rb} = \rho\left(\mathcal{F}\mathcal{V}^{-1}\right)$, then

$$\mathfrak{Rb} = \begin{pmatrix} 0 & \beta\mathcal{S}_0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma+\mu} & 0 \\ \frac{\sigma}{(\sigma+\mu)(\gamma+\mu)} & \frac{1}{\gamma+\mu} \end{pmatrix},$$

therefore

$$\mathfrak{Rb} = \begin{pmatrix} \frac{\beta\mathcal{S}_0\sigma}{(\sigma+\mu)(\gamma+\mu)} & \frac{\beta\mathcal{S}_0}{\gamma+\mu} \\ 0 & 0 \end{pmatrix},$$

thus

$$\mathfrak{Rb} = \frac{\beta\mathcal{S}_0\sigma}{(\sigma+\mu)\,(\gamma+\mu)}.$$

By substitution, we obtain

$$\mathfrak{Rb} = \frac{\beta\Lambda\sigma}{(\upsilon+\mu)\,(\sigma+\mu)\,(\gamma+\mu)}.$$

The Equilibrium Points

Disease-Free Equilibrium

- If $\mathcal{I} = \mathcal{E} = 0$, we have

$$\begin{cases} 0 = \Lambda - (\upsilon+\mu)\,\mathcal{S}, \\ 0 = \upsilon\mathcal{S} - \mu\mathcal{R}. \end{cases}$$

Using these equations, we deduce

$$\mathcal{S}_0 = \frac{\Lambda}{\upsilon + \mu}, \quad \mathcal{R}_0 = \frac{\Lambda \upsilon}{\mu \left(\upsilon + \mu\right)}.$$

Thus

$$\mathfrak{Dfe} = \left(\mathcal{S}_0, \mathcal{E}_0, \mathcal{I}_0, \mathcal{R}_0\right) = \left(\frac{\Lambda}{\upsilon + \mu}, 0, 0, \frac{\Lambda \upsilon}{\mu \left(\upsilon + \mu\right)}\right).$$

Endemic equilibruim

- If $\mathcal{I} > 0$ and $\mathcal{E} > 0$, we have

$$\begin{cases} 0 = \Lambda - \left(\beta \mathcal{I} + \upsilon + \mu\right) \mathcal{S}, \\ 0 = \beta \mathcal{S} \mathcal{I} - \left(\sigma + \mu\right) \mathcal{E}, \\ 0 = \sigma \mathcal{E} - \left(\gamma + \mu\right) \mathcal{I}, \\ 0 = \upsilon \mathcal{S} + \gamma \mathcal{I} - \mu \mathcal{R}. \end{cases}$$

From the equation

$$0 = \sigma \mathcal{E} - \left(\gamma + \mu\right) \mathcal{I} \quad \text{and} \quad 0 = \upsilon \mathcal{S} + \gamma \mathcal{I} - \mu \mathcal{R},$$

we deduce

$$\mathcal{I}^* = \frac{\sigma}{\gamma + \mu} \mathcal{E}^*, \quad \mathcal{R}^* = \frac{\upsilon}{\mu} \mathcal{S}^* + \frac{\gamma \sigma}{\mu \left(\gamma + \mu\right)} \mathcal{E}^*,$$

we have

$$\Lambda - \left(\beta \mathcal{I} + \upsilon + \mu\right) \mathcal{S} + \beta \mathcal{S} \mathcal{I} - \left(\sigma + \mu\right) \mathcal{E} = 0,$$

we obtain

$$\mathcal{S}^* = \frac{\Lambda}{\upsilon + \mu} - \frac{\sigma + \mu}{\upsilon + \mu} \mathcal{E}^*.$$

This finally gives us

$$0 = \beta \mathcal{S} \mathcal{I} - \left(\sigma + \mu\right) \mathcal{E}.$$

By substituting into the equation, we have

$$\mathcal{E}^* = \frac{\left(\upsilon + \mu\right) \left(\gamma + \mu\right) \left(\mathfrak{Rb} - 1\right)}{\beta \sigma},$$

then, we can conclude that

$$\mathcal{S}^* = \frac{\left(\sigma + \mu\right) \left(\gamma + \mu\right)}{\beta \sigma},$$

and

$$\mathfrak{Eqp} = \left(\mathcal{S}^*, \mathcal{E}^*, \mathcal{I}^*, \mathcal{R}^*\right) = \left(\mathcal{S}^*, \mathcal{E}^*, \frac{\sigma}{\gamma + \mu} \mathcal{E}^*, \frac{\upsilon}{\mu} \mathcal{S}^* + \frac{\gamma \sigma}{\mu \left(\gamma + \mu\right)} \mathcal{E}^*\right).$$

Stability Analysis of the Equilibrium Points

The system's Jacobian matrix is

$$J = \begin{pmatrix} -(\beta\mathcal{I}+\upsilon+\mu) & 0 & -\beta\mathcal{S}_0 & 0 \\ \beta\mathcal{I} & -(\sigma+\mu) & \beta\mathcal{S}_0 & 0 \\ 0 & \sigma & -(\gamma+\mu) & 0 \\ \upsilon & 0 & \gamma & -\mu \end{pmatrix}.$$

- At the disease-free equilibrium point $\mathfrak{Dfe}$, we have

$$J_{\mathfrak{Dfe}} = \begin{pmatrix} -(\upsilon+\mu) & 0 & \frac{-\beta\Lambda}{\upsilon+\mu} & 0 \\ 0 & -(\sigma+\mu) & \frac{\beta\Lambda}{\upsilon+\mu} & 0 \\ 0 & \sigma & -(\gamma+\mu) & 0 \\ \upsilon & 0 & \gamma & -\mu \end{pmatrix}.$$

The stability of the $\mathfrak{Dfe}$ depends on the eigenvalues of the Jacobian matrix. Solving thecharacteristic equation $\det(J_{\mathfrak{Dfe}} - \lambda \times Id_{3\times 3}) = 0$. The characteristic equation is by calculating:

$$\begin{vmatrix} -(\upsilon+\mu)-\lambda & 0 & \frac{-\beta\Lambda}{\upsilon+\mu} & 0 \\ 0 & -(\sigma+\mu)-\lambda & \frac{\beta\Lambda}{\upsilon+\mu} & 0 \\ 0 & \sigma & -(\gamma+\mu)-\lambda & 0 \\ \upsilon & 0 & \gamma & -\mu-\lambda \end{vmatrix} = 0.$$

Hence

$$(-\upsilon-\mu-\lambda)(-\mu-\lambda)\left[(-\sigma-\mu-\lambda)(-\gamma-\mu-\lambda) - \frac{\beta\Lambda\sigma}{\upsilon+\mu}\right] = 0.$$

Therefore

$$(-\upsilon-\mu-\lambda)(-\mu-\lambda)\left[\lambda^2 + (2\mu+\sigma+\gamma)\lambda + \left(\mu^2+\mu\gamma+\mu\sigma+\gamma\sigma\right) - \frac{\beta\Lambda\sigma}{\upsilon+\mu}\right] = 0,$$

from the term $\mathfrak{Rb}$, we obtain

$$(-\upsilon-\mu-\lambda)(-\mu-\lambda)\left[\lambda^2 + (2\mu+\sigma+\gamma)\lambda + (\sigma+\mu)(\gamma+\mu)(1-\mathfrak{Rb})\right] = 0,$$

we put

$$\lambda = -(\upsilon+\mu), \quad \lambda = -\mu, \quad \text{and} \quad Q(\lambda) = \lambda^2 + (2\mu+\sigma+\gamma)\lambda + (\sigma+\mu)(\gamma+\mu)(1-\mathfrak{Rb}).$$

If $\mathfrak{Rb} < 1$, all the coefficients of $Q(\lambda)$ are positive. Therefore, all the roots of $Q(\lambda)$ are negative reals or complexes of negative real parts, then the disease-free equilibrium is stable when $\mathfrak{Rb} < 1$.

- Following the same aforementioned approach, we conclude that the endemic equilibrium $\mathfrak{Eqp}$ is locally asymptotically stable when the basic reproduction number satisfies $\mathfrak{Rb} > 1$.

4.4 Applications with Numerical Results

In this study, we analyze the dynamics of epidemic spread through a numerical simulation of the SEIR model (Susceptible–Exposed–Infectious–Recovered) within a closed and constant-size population of $N = 1000$. The simulation begins with specific initial conditions: ten infectious cases $I(0) = 10$, ten individuals in the exposed compartment $E(0) = 10$, and no individuals in the recovered compartment $R(0) = 0$, which implies that the initial number of susceptible individuals is

$$S(0) = N - E(0) - I(0) - R(0) = 980.$$

The simulation is based on a set of epidemiological parameters, the precise values and biological interpretations of which are presented in Table

Table 4.2: **Initial and Model Parameters for the SEIR Epidemiological Model**

Parameter	Interpretation	Value	Reference	Unit
Λ	Recruitment rate	0.0224	[22]	$\frac{1}{\text{time}}$
$S(0)$	Initial susceptible individuals	999	Assumed	individuals
$I(0)$	Initial infected individuals	10	Assumed	individuals
$R(0)$	Initial recovered individuals	0	Assumed	individuals
$E(0)$	Initial Exposed individuals	10	Assumed	individuals
β	Transmission rate	0.5	Assumed	$\frac{1}{\text{time}}$
γ	Recovery rate	0.1	Assumed	$\frac{1}{\text{time}}$
υ	Vaccination rate	0.01	Assumed	$\frac{1}{\text{time}}$
σ	Transition rate (E to I)	0.2	Assumed	$\frac{1}{\text{time}}$
μ	Natural death rate	0.0224	[22]	$\frac{1}{\text{time}}$

Accordingly, the basic reproduction number $\mathfrak{Rb}$ is calculated based on the data provided in the table.

Based on these results, we performed a numerical simulation of the model, as follows:

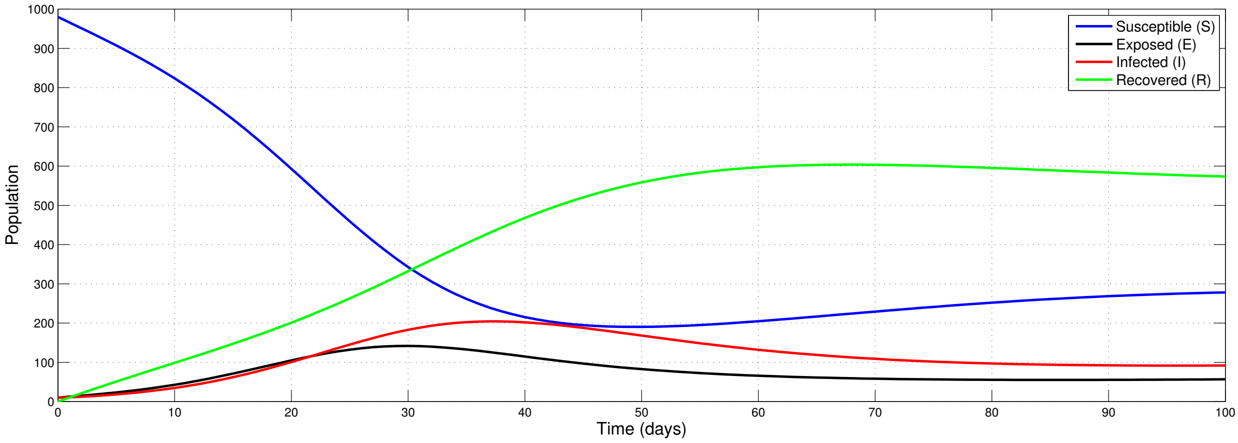


Figure 4.5: **SEIR Model Dynamics**

From Figure 4.5, we observe that the number of susceptible individuals is initially high and gradually decreases without reaching zero. The exposed group increases rapidly until peaking, then declines. Infected individuals start low, reach a peak later, and then decrease due to recovery or death. Meanwhile, the number of recovered individuals gradually increases through recovery or immunization and stabilizes at a high level.

By introducing the waning immunity parameter $\delta = 0.02$ into the classical SEIR model, the system transitions to an SEIRS model, which captures the dynamics of diseases with temporary immunity. This parameter represents the rate at which recovered individuals R revert to the susceptible population S due to the loss of acquired immunity. To analyze the model's behavior, we conducted numerical simulations under the specified parameters, and the results are presented as follows:

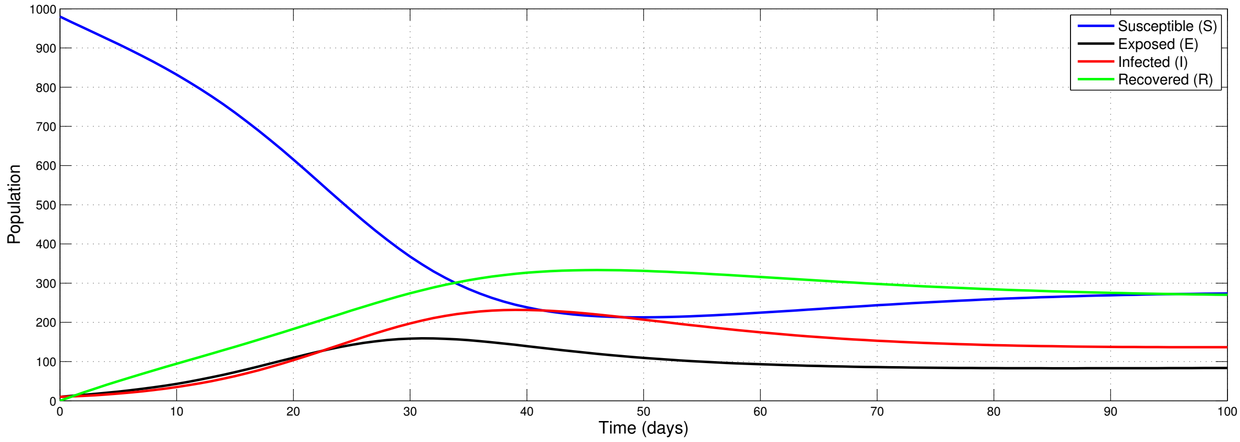


Figure 4.6: **SEIRS Model Dynamics**

From Figure 4.6, we observe that immunity loss leads to a lack of permanent stability across the population groups: the number of susceptibles decreases and then rises again, while exposed and infected individuals show fluctuations indicating repeated waves of infection. The recovered population increases initially, then gradually declines as individuals return to the susceptible class. This behavior reflects the continued spread of the disease due to the absence of lasting immunity.

4.5 Diffusion Equilibrium with Spatial Variation

In spatial epidemiological models, the movement of individuals within the geographic domain is taken into account, which necessitates the inclusion of diffusion terms in the equations, thereby transforming the system into a reaction-diffusion model. This fundamentally alters the analytical approach, particularly in the computation of the basic reproduction number $\mathfrak{R}\mathfrak{b}$.

To compute $\mathfrak{R}\mathfrak{b}$ in the SEIR model, we should mention that

$$\begin{cases} \frac{\partial E}{\partial t} = D_E \Delta E + \beta \frac{SI}{N} - (\sigma + \mu) E, \\ \frac{\partial I}{\partial t} = D_I \Delta I + \sigma E - (\gamma + \mu) I. \end{cases}$$

To solve equations involving the Laplacian Δ, we utilize spatial field expansions based on the eigenfunctions of the operator $-\Delta$, namely

$$-\Delta \phi_k = \lambda_k \phi_k, \quad \text{for} \quad k = 0, 1, \ldots, n \in \mathbb{N}_0.$$

where

- λ_k are the eigenvalues of the Laplacian operator $-\Delta$.
- ϕ_k are the corresponding eigenfunctions (or spatial modes) of the Laplacian.

As shown in the following graph, the Laplacian eigenvalues λ_k increase with the mode number k:

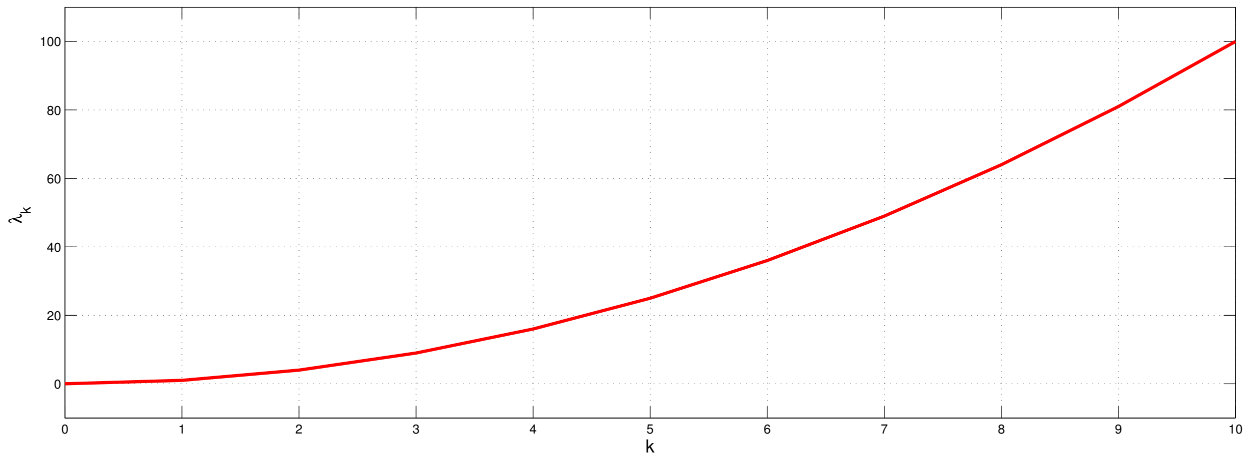


Figure 4.7: **Eigenvalues λ_k vs Mode Number**

The variables E and I are expanded using spectral decomposition at $S = S_0$, as follows

$$E(x,t) = \sum_{k=0}^{\infty} e_k(t)\,\phi_k(x), \quad \text{and} \quad I(x,t) = \sum_{k=0}^{\infty} i_k(t)\,\phi_k(x),$$

where

- $e_k(t)$ and $i_k(t)$ are the temporal coefficients.

Assuming that $D_E = D_I = D$, we substitute into the equation for E to obtain

$$\sum_{k=0}^{\infty} \frac{de_k(t)}{dt}\phi_k(x) = -D\sum_{k=0}^{\infty} e_k(t)\,\lambda_k\phi_k(x) + \beta\frac{S_0}{N}\sum_{k=0}^{\infty} i_k(t)\,\phi_k(x) - (\sigma+\mu)\sum_{k=0}^{\infty} e_k(t)\,\phi_k(x).$$

Since $\phi_k(x)$ is independent of time, then

$$\frac{de_k(t)}{dt} = \beta\frac{S_0}{N} i_k(t) - (D\lambda_k + \sigma + \mu)\, e_k(t).$$

Similarly, we obtain

$$\frac{di_k(t)}{dt} = \sigma e_k(t) - (D\lambda_k + \gamma + \mu)\, i_k(t).$$

We can write the system in matrix form as follows

$$\frac{d}{dt}\begin{pmatrix} e_k(t) \\ i_k(t) \end{pmatrix} = \begin{pmatrix} -(D\lambda_k + \sigma + \mu) & \beta\frac{S_0}{N} \\ \sigma & -(D\lambda_k + \gamma + \mu) \end{pmatrix}\begin{pmatrix} e_k(t) \\ i_k(t) \end{pmatrix}.$$

Calculation of $\mathfrak{Rb}^{PDE}$ for each mode

For our system, we have

$$\mathcal{F} = \begin{pmatrix} 0 & \beta\frac{S_0}{N} \\ 0 & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} D\lambda_k + \sigma + \mu & 0 \\ -\sigma & D\lambda_k + \gamma + \mu \end{pmatrix}.$$

Therefore

$$\mathfrak{Rb}^{PDE} = \rho\left(\mathcal{F}\mathcal{V}^{-1}\right) = \frac{1}{\left(D\lambda_k + \sigma + \mu\right)\left(D\lambda_k + \gamma + \mu\right)} \begin{pmatrix} 0 & \beta\frac{S_0}{N} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D\lambda_k + \gamma + \mu & 0 \\ \sigma & D\lambda_k + \sigma + \mu \end{pmatrix}.$$

Then

$$\mathfrak{Rb}^{PDE} = \begin{pmatrix} \frac{\beta\frac{S_0}{N}\sigma}{(D\lambda_k+\sigma+\mu)(D\lambda_k+\gamma+\mu)} & \frac{\beta\frac{S_0}{N}}{D\lambda_k+\gamma+\mu} \\ 0 & 0 \end{pmatrix},$$

hence

$$\mathfrak{Rb}^{PDE} = \frac{\beta\frac{S_0}{N}\sigma}{\left(D\lambda_k + \sigma + \mu\right)\left(D\lambda_k + \gamma + \mu\right)}.$$

Thus

$$\mathfrak{Rb}^{PDE} = \max_k \left(\frac{\beta\frac{S_0}{N}\sigma}{\left(D\lambda_k + \sigma + \mu\right)\left(D\lambda_k + \gamma + \mu\right)}\right).$$

Since $\lambda_0 = 0$ (the spatially constant eigenmode) and $\lambda_k > 0$ for $k \geq 1$, the maximum is achieved at $k = 0$. Hence

$$\mathfrak{Rb}^{PDE} = \frac{\beta\sigma S_0}{N\left(\sigma + \mu\right)\left(\gamma + \mu\right)}.$$

Conclusion

This thesis constituted a systematic study of a class of nonlinear partial differential equations used to model diffusion phenomena in both physical and biological contexts. The focus was placed on the fundamental mathematical properties of these equations, highlighting their complexity compared to linear models, both in terms of theoretical analysis and numerical treatment.

The results obtained showed that nonlinear models were capable of accurately capturing phenomena characterized by irregular or density-dependent diffusion, as observed in practical applications such as heat transfer in heterogeneous media or the spread of infectious diseases in spatially structured populations. The analyses enabled a deeper understanding of the solution dynamics and their long-term behavior, while also providing robust mathematical tools to investigate stability and convergence.

Based on these findings, this work laid a solid foundation for future research that may aim to extend the models to higher spatial dimensions, incorporate more complex boundary conditions, or introduce additional interactive mechanisms. Future directions could also involve bridging the gap between rigorous theoretical analysis and efficient numerical schemes to ensure both accuracy and computational feasibility in simulations.

Thus, this thesis contributed to clarifying the potential of nonlinear partial differential equations in describing diffusion-driven phenomena, while emphasizing the need for continued development and deeper investigation of these models within a rigorous mathematical framework.

Bibliography

[1] G. I. Barenblatt, *On Some Unsteady Motions of a Liquid or a Gas in a Porous Medium*, Akad. Nauk SSSR. Prikl. Mat. Meh., **16** (1952), 67–78.

[2] B. Basti and N. Benhamidouche, *Global Existence and Blow-up of Generalized Self-Similar Solutions to Nonlinear Degenerate Diffusion Equations Not in Divergence Form*, Appl. Math. E-Notes, **20** (2020), 367–387.

[3] B. Basti, B. Chennaf, M. A. Boubekeur and S. Bounouiga, *Fractional Mathematical Model for Exploring the Influence of Infectious Diseases on Population with Chronic Conditions,* Advanced Theory and Simulations, **7**(8) (2024), 2301285.

[4] S. Bounouiga, B. Basti and N. Benhamidouche, *Mathematical Exploration of Malaria Transmission Dynamics: Insights from Fractional Models and Numerical Simulation*, Advanced Theory and Simulations, **8**(2) (2025), 2400630.

[5] G. W. Bluman and J. D. Cole, *Similarity Methods for Differential Equations,* Springer, 1974.

[6] G. I. Barenblatt, *Scaling self-similarity, and intermediate asymptotics: Dimensional analysis and intermediate asymptotics,* Cambridge University Press, 1979.

[7] E. A. Bender, *An Introduction to Mathematical Modeling*, Courier Corporation, 2000.

[8] R. Cui, K. Y. Lam and Y. Lou, *Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments*, J. Differential Equations, **263** (2017), 2343–2373.

[9] R. Cui, Y. Lou, *A spatial SIS model in advective heterogeneous environments*, J. Differential Equations, **261** (2016), 3305–3343.

[10] L. Edelstein-Keshet, *Mathematical models in biology*, Society for Industrial and Applied Mathematics, 2005.

[11] B. H. Gilding and L. A. Peletier, *On a Class of Similarity Solutions of the Porous Media Equation*, Journal of Mathematical Analysis and Applications, **55** (1976), 351–364.

[12] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[13] J. Hulshof et J. L. Vàzquez, *Self-similar solutions of the second kind for the modified porous medium equation*, European J. Appl. Math., **5** (1994), 391–403.

[14] F. John, *Partial Differential Equations*, Springer, 1982.

[15] P. J. Olver, *Introduction to Partial Differential Equations*, Springer International Publishing, 2013.

[16] V. Pata, *Fixed Point Theorems and Applications*, Springer, 2019.

[17] R. Peng, X. Zhao, *A reaction-diffusion SIS epidemic model in a time-periodic environment*, Nonlinearity, **25** (2012), 1451–1471.

[18] D. R. Smart, *Fixed Point Theorems*, Cambridge, UK: Cambridge University Press, 1974.

[19] J. L. Vàzquez, *The Porous Medium Equation – Mathematical Theory,* Oxfored, 2007.

[20] J. L. Vàzquez, *Asymptotic behaviour for the porous medium equation posed in the whole space,* Birkhauser Verlag, Basel, 2002, 11–12.

[21] C. Wang et J. Yin, *Shrinking self-similar solution of a nonlinear diffusion equation with nondivergence form,* J. Math. Anl. Appl, **289** (2004), 387–404.

[22] Population growth in Algeria, algerie/croiss-ppl.php.

تَــمَ بِحَمْدِ اللهِ

"الْحَمْدُ لِلَّهِ الَّذِي لَهُ مَا فِي السَّمَاوَاتِ وَمَا فِي
الْأَرْضِ وَلَهُ الْحَمْدُ فِي الْآخِرَةِ وَهُوَ الْحَكِيمُ
الْخَبِيرُ" [سبأ: 01]

الملخص:

تهدف هذه المذكرة إلى دراسة وتحليل المعادلات التفاضلية الجزئية غير الخطية التي تُستخدم في نمذجة ظواهر الانتشار في الأنظمة الفيزيائية والبيولوجية. تم التركيز على معادلة الوسط المسامي والنماذج ذات الصيغة غير التباعدية، بالإضافة إلى أنظمة تفاعل-انتشار في سياق انتشار الأمراض. اعتمد البحث على تقنيات رياضية مثل مبرهنة باناخ والتشابه الذاتي لتحليل وجود الحلول وسلوكها الديناميكي. أظهرت النتائج قدرة هذه النماذج على تقديم وصف دقيق للسلوك المعقد للأنظمة الطبيعية في بيئات غير متجانسة، مما يعزز استخدامها في تطبيقات مثل مراقبة التلوث البيئي ونمذجة انتشار الأوبئة.

الكلمات المفتاحية: المعادلات التفاضلية الجزئية غير الخطية، مسائل الانتشار، الحلول الذاتية، الحلول الذاتية، الوجود والوحدانية، الأنظمة البيولوجية، نقاط التوازن، معدل التكاثر الاساسي.

Abstract:

This thesis aims to study and analyze nonlinear partial differential equations used in modeling diffusion phenomena in physical and biological systems. The focus is placed on the porous medium equation, non-divergence form equations, and reaction-diffusion systems in the context of disease spread. The research employs mathematical techniques such as Banach's fixed-point theorem and self-similarity to analyze the existence and behavior of solutions. The results demonstrate the ability of these models to provide accurate descriptions of the complex dynamics of natural systems in heterogeneous environments, enhancing their application in fields such as environmental pollution monitoring and epidemic spread modeling.

Key words : Nonlinear partial differential equations, diffusion problems, self-similar solutions, existence and uniqueness, biological systems, equilibrium points, basic reproduction number.