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Variational and Numerical Analysis of Some Problems at the Contact Limits

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Contents

Introduction

Contact problems represent some of the most complex and significant challenges in the field of continuum mechanics and computational engineering. These problems arise when two or more bodies interact through surface contact under external forces or internal deformations. They are common in real-world applications such as aerospace structure design, civil engineering analysis, and biomechanical modeling. Contact problems are often characterized by nonlinear boundary conditions or frictional behavior, which require advanced mathematical tools for accurate modeling and analysis [2].

In this context, variational analysis provides a powerful framework for formulating contact problems based on energy principles. The mechanical equilibrium of a system is represented as a critical point of an energy functional or as the solution to a variational inequality. This approach allows one to handle nonsmooth or nonlinear conditions at the contact interface using concepts from convex analysis, subdifferential calculus, and dual spaces [7, 14, 15].

Since analytical solutions to such problems are typically unavailable, numerical analysis becomes essential for approximating and computing solutions. Among the most effective numerical techniques is the finite element method (FEM), which is well-suited for complex geometries and heterogeneous materials. However, when dealing with contact boundaries, FEM requires a careful treatment of the boundary conditions and the incorporation of contact or friction laws into the numerical framework. This can be achieved through projection methods, relaxation techniques, or specialized algorithms such as the augmented Lagrangian method or the penalty method [, 8, 9, 12].

This work focuses on the variational and numerical analysis of certain contact problems, aiming to establish both the well-posedness of the underlying variational formulation (existence and, when possible, uniqueness of solutions), and the development of efficient numerical schemes to approximate these solutions. We begin by formulating the considered contact problems in a general variational setting, incorporating unilateral contact conditions or Coulomb-type friction. Then, we construct a suitable discrete approximation scheme and analyze its stability and convergence. Special attention is given to problems involving partial adhesion or viscous effects at the contact interface, which introduce additional physical and mathematical complexity [16, 21].

This memory is divided into two parts. The first part includes Chapters 1 and 2 and serves as a brief introduction to the study of variational inequalities. The material presented here has been selected with an emphasis on the essential mathematical tools required for the analysis of contact problems. More specifically, Chapter 1 is devoted to certain elements of functional analysis and nonlinear analysis. Chapter 2 presents several results from the theory of elliptic and parabolic variational inequalities, quasi-variational inequalities, as well as Gronwall's lemma.

In the second part, we study a contact problem with friction between a viscoelastic body and an obstacle modeled as a foundation. This part includes Chapter 3, which is devoted to the mathematical analysis of a contact problem with friction for viscoelastic materials in a quasistatic setting. The contact with a deformable foundation is modeled using a normal compliance condition.

The weak formulation of the problem is expressed as a system composed of :

— a parabolic variational inequality for the displacement field,

— and another parabolic variational inequality for the damage field.

We establish a result on the existence and uniqueness of the solution. The proof is based on the theory of parabolic variational inequalities and fixed point arguments. At the end of the chapter, we present a numerical approximation of this contact problem, relying on both time and space discretization. This is done using a uniform time-stepping scheme and a finite element method. Based on these discretizations, we derive an error estimate result

Notations

Let consider the preliminary notations

$\mathbb{N}$	The set of positive integers
$\mathbb{R}$	The set of real numbers, or the real line .
d	A positive integer, in applications having its value in $\{1,2,3\}$.
c	A generic positive constant, the value of which may change from place to place.
h	The finite element mesh size.
k	The time step size.
diam (K)	Diameter of the set K
X^h	The finite element space for the set X
δ_{ij}	The Kronecker delta.
$a.e.$	Almost everywhere .
$i.e.$	That is.
$[0;T]$	The time interval of interest, $T>0$.
Π_h	The finite element interpolation operator.
$\mathbb{R}^d$	The d-dimensional Euclidean space
$\mathcal{S}^d$	The space of second-order symmetric tensors on $\mathbb{R}^d$

For a function f, we denote by

$D_{ef}\ f$	The effective domain of f.
$\dot{f}, \ddot{f}$	The first and second derivatives of f with respect to time.
$\partial_i f$	The partial derivative of f with respect to the ith component x_i.
$\varepsilon(f)$	Linearized or small deformations operator, i.e., $\varepsilon(f) = (\varepsilon_{ij}(f))$, $\varepsilon_{ij}(f) = \frac{1}{2}(f_{i,j}+f_{j,i})$
$\mathrm{Div}\, f$	The divergence operator of f, i.e., $\mathrm{Div}\, f = (f_{ij,j})$
∇f	The gradient operator of f, i.e., $\nabla f = (\partial_1 f, ...\partial_d f)$
∂f	The subdifferential of the function f.

Let Ω be an open, bounded, and connected set in $\mathbb{R}^d (d = 1, 2, 3)$, we denote by

$\overline{\Omega}$	The closure
Γ	The boundary of Ω :$\Gamma = \partial\Omega$.
$\Gamma_i \quad \left(i = \overline{1,3}\right)$	The parts of the boundary Γ .
$mes\ \Gamma_1$	A generic positive constant, the value of which may change from place to place.
$\boldsymbol{\nu}$	The unit outward normal on the boundary Γ.
$\mathbf{v}_\nu$,$\mathbf{v}_\tau$	The normal and tangential component of vector field $\mathbf{v}$.
$L^p(\Omega)$	The Lebesgue space of p-integrable functions on Ω, with the usual modification if $p = \infty$.
$W^{q.p}(\Omega)$	The Sobolev space of functions whose weak derivatives of orders q or lessare p-integrable on Ω ($q \in \mathbb{N}$ and $p \in \{1. + \infty\}$).
$H^q(\Omega)$	The Sobolev space $W^{q.2}(\Omega)$.
$H^1(\Omega)$	The first-order Sobolev space.
$C^k(\ [0,T]\ ;\ X\ )$	The space of continuous X -valued functions in $[0,T]$ whose derivatives up to the order $k \in \mathbb{N}$ are continuous.
$L^p(0,T\ ;\ X)$	The space of measurable X -valued functions in $(0,T)$ such that $\int_0^T \Vert\mathbf{v}\Vert_X^p\, dt < +\infty,\ p \in [1, +\infty[$.
$W^{1,p}(0,T;X)$	The subspace of $L^p(0,T\ ;\ X)$ whose first generalized derivative belongs to $L^p(0,T\ ;\ X)$.
$H^1(0,T;X)$	$W^{1,2}(0,T;X)$

Chapter 1

Requirements and Preliminaries

This chapter is devoted to the introduction and the formulation of the mechanical problems which will be dealt with in the sequel, as well as a reminder of the main notions of continuum mechanics and functional analysis necessary for the understanding of this thesis. In this chapter we present the physical framework related to the studied problem, then we consider the laws of behavior of the various materials such as, the nonlinear viscoelastic constitutive law with damage. By then citing the boundary conditions with friction, in order to develop interesting mathematical models. Finally, we introduce a brief useful and essential reminder concerning functional and vector-valued spaces.

1.1 Physical setting-Mathematical models

In this section we will introduce the physical setting and their mathematical mechanical problem involved in this memory.

Physical setting. Consider a material body which occupies a bounded domain $\mathbb{R}^d$ $(d = 2, 3)$, with a regular boundary surface Γ , partitioned into three measurable parts Γ_1, Γ_2 and Γ_3 , such that $meas\Gamma_1 > 0$ .We note by ν the outgoing unit normal to Γ.The body is embedded on Γ_1 in a fixed structure. On Γ_2 act surface tractions of density f_2 and in Ω act voluminal forces of density f_0. We assume that f_2 and f_0 vary very slowly with respect to time. Let $T > 0$ and $[0, T]$ the time interval in question. The body is in contact with or without friction, with a deformable or lubricated obstacle on the Γ_3 part. We assume that the material can be damaged during the contact. We take into consideration the mechanical properties of the body.

Before the description of the mathematical models associated with the physical framework presented above, we give some notations and conventions that we will use throughout this thesis. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$,"." and $\|.\|$ represent respectively the scalar product and the Euclidean norm on $\mathbb{R}$ and S_d such that

$$\begin{aligned} \mathbf{u}.\mathbf{v} &= \mathbf{u}_i\mathbf{v}_i \ , \quad \|\mathbf{v}\| = (\mathbf{v}.\mathbf{v})^{\frac{1}{2}} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d, \\ \boldsymbol{\sigma}.\boldsymbol{\tau} &= \sigma_{ij}\tau_{ij} \ , \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}.\boldsymbol{\tau})^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in S_d, \end{aligned}$$

With the silent index convention. For a vector $\mathbf{v}$, we use the notation $\mathbf{v}$ to denote the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ. We denote by $\mathbf{v}_\nu$ and $\mathbf{v}_\tau$ the normal and tangential components of $\mathbf{v}$ on the boundary given by

$$\mathbf{v}_\nu = \mathbf{v}.\boldsymbol{\nu}, \qquad \mathbf{v}_\tau = \mathbf{v} - \mathbf{v}_\nu\boldsymbol{\nu}. \tag{1.1.1}$$

Noting by $\sigma : \ \Omega \times [0, T] \to S_d$, the stress tensor field and we define analogously the normal and tangential components of σ on the boundary by the formulas

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}).\boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}. \tag{1.1.2}$$

Using (**1**.1.1) and (**1**.1.2), we get the relation

$$(\boldsymbol{\sigma}\boldsymbol{\nu}).\mathbf{v} = \sigma_\nu \mathbf{v}_\nu + \boldsymbol{\sigma}_\tau.\mathbf{v}_\tau, \tag{1.1.3}$$

That we will use it throughout this thesis and especially in the establishment of the variational formulations of the mechanical problems of contact. The points above a function represent the derivation with respect to time, i.e.

$$\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt}, \qquad \ddot{\mathbf{u}} = \frac{d^2\mathbf{u}}{dt^2}$$

Where $\dot{\mathbf{u}}$ denotes the velocity field and $\ddot{\mathbf{u}}$ denotes the acceleration field. For the velocity field $\dot{\mathbf{u}}$ the notations $\dot{\mathbf{u}}_\nu$ and $\dot{\mathbf{u}}_\tau$ represent respectively the normal and tangential velocities at the boundary, i.e.

$$\dot{\mathbf{u}}_\nu = \dot{\mathbf{u}}.\boldsymbol{\nu}, \qquad \dot{\mathbf{u}}_\tau = \dot{\mathbf{u}} - \dot{\mathbf{u}}_\nu\boldsymbol{\nu}.$$

The relationship between the displacement field $\mathbf{u}$ and the strain field ε on the assumption of small transformations is given by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\mathbf{u}_{i,j} + \mathbf{u}_{j,i}), \text{ dans } \Omega \times (0, T) \tag{1.1.4}$$

Where the index following the comma indicates the partial derivation with respect to the corresponding component of the variable and this will be the case throughout this thesis.

We will now describe the mathematical model associated with the Physical setting that we saw in the previous section.

Mathematical model . The mathematical model studied In this thesis, describes the evolution of the body in the Physical setting .

The unknown functions of the problem are the displacement field $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$, the stress field $\sigma : \Omega \times [0, T] \to S^d$ and the damage field $\beta : \Omega \times [0, T] \to \mathbb{R}$. In the general case, the evolution of a material body is described by the following Cauchy equation of motion

$$\mathrm{Div}\,\sigma + f = \rho\ddot{\mathbf{u}} \;\; \text{dans} \quad \Omega \times (0, T)\,, \tag{\textbf{1}.1.5}$$

Where $\rho : \Omega \to \mathbb{R}_+$ denotes the mass density and " Div " represents the divergence operator of the tensors such that

$$\mathrm{Div}\,\sigma = (\sigma_{ij,j})\,,$$

The process of evolution defined by (**1**.1.5) is called dynamic process. In some situations, this equation can be further simplified. For example, in the case where the velocity field $\dot{\mathbf{u}}$ varies very slowly with respect to time, the term $\rho\ddot{\mathbf{u}}$ can be neglected. In this case, the process is called quasi-static and equation (**1**.1.5) is called the equilibrium equation and becomes

$$\mathrm{Div}\,\sigma + f = 0 \;\; \text{dans} \quad \Omega \times (0, T)\,. \tag{\textbf{1}.1.6}$$

1.2 Constitutive Law

As we noted in section **1.1**, equations (**1**.1.1) - (**1**.1.6) do not constitute a complete description of the evolution of a continuous body. To obtain a complete model, valid for a given material, it is necessary to add the constitutive law of the material. Although they must satisfy certain basic axioms and principles of invariance, the laws of behavior come mainly from experience. not to mention the subject of damage which is extremely important in design engineering. In this memory, we develop a constiyutive law related to a viscoelastic material with damage.

Viscoelastic constitutive law with damage

The most common constitutive law of a viscoelastic material with damage is is given by

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{F}\varepsilon(\mathbf{u}(t), \beta(t)) \tag{\textbf{1}.2.1}$$

Where $\mathbf{u}$ denotes the displacement field and σ, $\varepsilon(\mathbf{u})$ represent respectively the stress and the linearized strain tensor. Furthermore $\mathcal{A}$ and $\mathcal{F}$ are nonlinear operators of viscosity and elasticity respectively. Where β represents the field of damage, the evolution of the field of damage is described by

$$\dot{\beta} - k\Delta\beta + \partial\varphi_Y(\beta) \ni S(\varepsilon(\mathbf{u}),\beta) \tag{1.2.2}$$

Where Y designates the set of admissible damage functions defined by

$$Y = \{\xi \in H^1(\Omega)/0 \leq \xi \leq 1 \quad \text{a.e. in} \ \ \Omega\}, \tag{1.2.3}$$

k is a positive coefficient, $\partial\varphi_Y$ denotes the subdifferential of the indicator function φ_Y and S is a given constitutive function which describes the sources of the damage in the system.

Finally, in order to complete the mathematical model which describes the evolution of the body, it is necessary to specify the boundary conditions on Γ_3, it is the object of the conditions of contact and laws with or without friction which we will describe in the following paragraph.

1.3 Contact boundary conditions and friction laws

The modeling of a contact phenomenon is determined by the hypotheses taken into account in its description. These hypotheses can influence either the form and the structure of the system of partial differential equations, or the boundary conditions of the associated mathematical model. The boundary conditions on the contact surface are described both in the direction of the normal and in the tangent plane, being called friction conditions. Recall that the boundary Γ is divided into three disjoint and measurable parts Γ_1,Γ_2 and Γ_3 such that $meas\,(\Gamma_1) > 0$, we give in this paragraph the boundary conditions on each of the three parts.

1.3.1 Displacement-traction boundary conditions

In all the problems that will be studied in this thesis, the body is embedded on the part Γ_1, so

$$\mathbf{u} = 0 \ \ \text{sur} \ \ \Gamma_1 \times (0,T)\,.$$

This relation represents the boundary condition of displacement. A surface traction of density f_2 acts on Γ_2 is consequently the vector of the stresses of Cauchy $\sigma\nu$ satisfied

$$\sigma\nu = \mathbf{f}_2 \ \ \text{sur} \ \ \Gamma_2 \times (0,T)\,.$$

This condition is called the condition at the limits of traction.

1.3.2 Contact conditions

Bilateral contact condition

The contact is made bilaterally, that is to say the contact is maintained during the movement and there is no separation between the body and the obstacle. The normal component of the field of displacements cancels on the surface of contact and thus

$$\mathbf{u}_\nu = 0.$$

Unilateral contact

This condition models the contact with a rigid foundation. Since the foundation is considered rigid, it will therefore not undergo any deformation. The body will therefore not be able to enter it. This property results in the mathematical relationship

$$\mathbf{u}_\nu \leq 0. \tag{\textbf{1}.3.2}$$

At points of Γ_3 such that $\mathbf{u}_\nu < 0$, the deformable body leaves the rigid base and the normal stresses are then zero there. Therefore we obtain:

$$\mathbf{u}_\nu < 0 \Rightarrow \sigma_\nu = 0. \tag{\textbf{1}.3.3}$$

At points of Γ_3 such that $\mathbf{u}_\nu = 0$, the contact is maintained and the rigid base exerts a normal reaction oriented towards Ω and therefore, we can write

$$\mathbf{u}_\nu = 0 \Rightarrow \sigma_\nu \leq 0. \tag{\textbf{1}.3.4}$$

The contact conditions written by (**1**.3.2), (**1**.3.3) and (**1**.3.4), are called unilateral contact conditions.

Contact with normal compliance with or without friction

This is the case where the foundation is assumed to be deformable in this case penetration is possible and the normal stress σ_ν satisfies:

$$-\sigma_\nu = p_\nu(\mathbf{u}_\nu - g).$$

Where u_ν,σ_ν denote respectively the normal components of the displacement field and the stress field, g represents the initial gap between the body and the foundation and p_ν is a given positive function, called function normal compliance. This relation is called *normal compliance* condition and means that the foundation exerts a reaction following the normal on the body according to its penetration $u_\nu - g$ which is canceled when there is separation (i.e. say $u_\nu < g$).

If the gap between the body and the foundation is zero, we take $g = 0$ and get

$$-\sigma_\nu = p_\nu(\mathbf{u}_\nu).$$

Where $p_\nu(.)$ is a non-negative prescribed function that vanishes for negative argument. Indeed, when $\mathbf{u}_\nu < 0$, there is no contact, and the normal pressure vanishes. An example of the normal compliance function p_ν is

$$p_\nu(r) = c_\nu r_+.$$

or, more generally,

$$p_\nu(r) = (c_\nu r_+)^m .$$

Here the constant $c_\nu > 0$ is the surface stiffness coefficient, $m > 0$ is the normal compliance exponent and $r_+ = \max\{0, r\}$ is the positive part of r. This contact condition was first introduced in [16] and [20] .

Laws of contact with or without friction

We turn now to the conditions in the tangential directions, called also frictional conditions or friction laws. The simplest one is the so-called frictionless condition in which the friction force vanishes during the process, i.e.,

$$\sigma_\tau = 0 \quad \text{on} \quad \Gamma_3 \times (0, T) .$$

If the friction force is not zero, the contact is with friction. The friction laws involved in this memory are versions of the Coulomb-type law, which are the most widespread friction laws in the mathematical literature. The Coulomb-type friction law is characterized by the intervention of the normal stress in the friction threshold and it can be stated as follows

$$\begin{cases} |\sigma_\tau| \leq \mu |\sigma_\nu| \\ |\sigma_\tau| < \mu |\sigma_\nu| \ \Rightarrow \dot{\mathbf{u}}_\tau = 0 \\ |\sigma_\tau| = \mu |\sigma_\nu| \ \Rightarrow \text{ther existe } \lambda \geq 0 \text{ such that } \sigma_\tau = -\lambda \dot{\mathbf{u}}_\tau \end{cases} \quad \text{on} \quad \Gamma_3 \times (0, T) . \qquad (\mathbf{1}.3.5)$$

Where $\mu \geq 0$ is the coefficient of friction.

A quasi-static version of Coulomb's laws can be stated as

$$\begin{cases} |\sigma_\tau| \leq F_b, \\ \sigma_\tau = F_b \frac{\dot{\mathbf{u}}_\tau}{|\dot{\mathbf{u}}_\tau|} \ \text{ if } \ \dot{\mathbf{u}}_\tau \neq \mathbf{0} \end{cases} \quad \text{on } \Gamma_3 \times (0, T) . \qquad (\mathbf{1}.3.6)$$

This choice in ($\mathbf{1}$.3.6) leads to the classical version of Coulomb's law. Choices

$$F_b = F_b(\sigma_\nu) = p_\nu(\mathbf{u}_\nu - g) \text{ and } \quad F_b = F_b(\sigma_\nu) = p_\nu(\dot{\mathbf{u}}_\nu) \ .$$

Are compatible with the contact condition of normal compliance and instantaneous normal response, respectively. Here, p_ν is a positive function which vanishes for negative arguments, i.e. when there is no contact, we study a frictional contact problem with normal compliance and we use the following boundary conditions

$$\begin{cases} -\sigma_\nu = p_\nu(\mathbf{u}_\nu), \quad |\sigma_\tau| \le p_\nu(\mathbf{u}_\nu) \text{ and} \\ \sigma_\tau = -p_\nu(\mathbf{u}_\nu)\frac{\dot{\mathbf{u}}_\tau}{|\dot{\mathbf{u}}_\tau|} \text{ if } \dot{\mathbf{u}}_\tau \neq \mathbf{0} \end{cases} \quad \text{on } \Gamma_3 \times (0,T). \qquad (\mathbf{1}.3.7)$$

1.4 Function Spaces

We introduce in this section spaces of the Sobolev type used in contact mechanics, namely the Hilbert spaces associated with the divergence and deformation operators, as well as the spaces of vector-valued functions. We also present their main properties, in particular the trace theorems. We adopt here the convention of the silent index and we also specify that all the notations as well as the functional spaces used in this thesis are introduced in this section. Also, in writing this section we have used [1]. For more details on Sobolev spaces and distribution spaces, we refer for example to [5].

Sobolev spaces

We begin with a brief reminder of some results on the Sobolev space $H^1(\Omega)$ defined by

$$H^1(\Omega) = \left\{\mathbf{u} \in L^2(\Omega) \mid \partial_i \mathbf{u} \in L^2(\Omega),\ i = 1, ..., d\right\}.$$

First, we denote by $\nabla\mathbf{u}$ the component vector $\partial_i \mathbf{u}$. We have $\nabla\mathbf{u} \in L^2(\Omega)^d$ for all $\mathbf{u} \in H^1(\Omega)$. We know that $H^1(\Omega)$ is a Hilbert space for the scalar product

$$(\mathbf{u}, \mathbf{v})_{H^1(\Omega)} = (\mathbf{u}, \mathbf{v})_{L^2(\Omega)} + (\partial_i \mathbf{u}, \partial_i \mathbf{v})_{L^2(\Omega)},$$

And the associated norm

$$|\mathbf{u}|^2_{H^1(\Omega)} = (\mathbf{u}, \mathbf{u})_{H^1(\Omega)} \text{ and we write } |\mathbf{u}|^2_{H^1(\Omega)} = |\mathbf{u}|^2_{L^2(\Omega)} + |\nabla\mathbf{u}|^2_{L^2(\Omega)},$$

We have the following results:

$$C^1\left(\overline{\Omega}\right) \text{ is dense in } H^1(\Omega)$$

Theorem 1.4.1.(Rellich)

$$H^1(\Omega) \subset L^2(\Omega) \text{ with compact injection}$$

Theorem 1.4.2. (Sobolev trace)

There exists a linear and continuous map $\gamma : H^1(\Omega) \longrightarrow L^2(\Gamma)$ such that $\gamma\mathbf{u} = \mathbf{u}|_\Gamma$ for $\mathbf{u} \in C^1\left(\overline{\Omega}\right)$.

Remark 1.4.1. the space $L^2(\Gamma)$ above represents the space of real functions on Γ which are L^2 for the surface measure $d\Gamma$. The map γ is called trace map; it is defined as the extension by density of the map $\mathbf{u} \longrightarrow \mathbf{u}|_\Gamma$ defined for $\mathbf{u} \in C^1\left(\overline{\Omega}\right)$. Note that the trace map $\gamma : H^1(\Omega) \longrightarrow L^2(\Gamma)$ is a compact operator.

1.4.1 Hilbert spaces associated with the divergence and deformation operators

In this Subsection we present the notation we shall use and some preliminary material. For further details, we refer to [8]. We denote by Sd the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while "." and $|\ .\ |$ will represent the inner product and the Euclidean norm on S^d and $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let ν denote the unit outer normal on Γ. Everywhere in the sequel the index i and j run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable. We use the standard notation for Lebesgue and Sobolev spaces associated to Ω and Γ and introduce the spaces

$$H = \left\{ \mathbf{u} = (\mathbf{u}_i) \ / \ \mathbf{u}_i \in L^2(\Omega) \diagup i = \overline{1, d} \right\} = L^2(\Omega)^d$$
$$\mathcal{H} = \left\{ \sigma = (\sigma_{ij}) \ / \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \diagup i, j = \overline{1, d} \right\} = L^2(\Omega)^{d \times d}$$
$$H_1 = \{ \mathbf{u} \in H \ / \ \varepsilon(\mathbf{u}) \in \mathcal{H} \} = \left\{ \mathbf{u} = (\mathbf{u}_i) \ / \ \mathbf{u}_i \in H^1(\Omega) \diagup i = \overline{1, d} \right\} = H^1(\Omega)^d$$
$$\mathcal{H}_1 = \{ \sigma \in \mathcal{H} \ / \ \sigma_{ij} \in H \}$$

Here $\varepsilon : H_1 \to H$ and $Div : H_1 \to H$ are the deformation and divergence operators defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\mathbf{u}_{i,j} + \mathbf{u}_{j,i}), \quad Div\, \sigma = (\sigma_{ij,\,j}) \quad 1 \le i, j \le d$$

$$\begin{aligned} \mathbf{u}.\mathbf{v} &= \mathbf{u}_i \mathbf{v}_i\ , \qquad |\mathbf{v}| = (\mathbf{v}.\mathbf{v})^{\frac{1}{2}} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d \\ \sigma.\tau &= \sigma_{ij}\tau_{ij}\ , \qquad |\sigma| = (\sigma.\sigma)^{\frac{1}{2}} \quad \forall \sigma, \tau \in S^d \end{aligned}$$

où et sont les opérateurs de déformation et de divergence, définis par

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\mathbf{u}_{i,j} + \mathbf{u}_{j,i}), \quad Div\, \sigma = (\sigma_{ij,\,j}) \quad 1 \le i, j \le d$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products

$$(\mathbf{u}, \mathbf{v})_H = \int_\Omega \mathbf{u}_i \mathbf{v}_i dx \quad \forall \mathbf{u}, \mathbf{v} \in H$$
$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx \quad \forall \sigma, \tau \in \mathcal{H}$$
$$(\mathbf{u}, \mathbf{v})_{H_1} = (\mathbf{u}, \mathbf{v})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in H_1$$
$$(\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (Div\ \sigma, Div\ \tau)_H \quad \forall \sigma, \tau \in \mathcal{H}_1$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\ .\ |_H$, $|\ .\ |_{\mathcal{H}}$, $|\ .\ |_{H_1}$and $|\ .\ |_{\mathcal{H}_1}$ .. For every element $\mathbf{v} \in H_1$ we also use the notation $\mathbf{v}$ for the trace $\gamma_{\mathbf{u}}$ of $\mathbf{v}$ on Γ and we denote by $\mathbf{v}_\nu$ and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on Γ given by

$$\mathbf{u}_\nu = \mathbf{u}.\nu, \qquad \mathbf{u}_\tau = \mathbf{u} - \mathbf{u}_\nu \nu$$

We also denote by σ_ν and σ_τ the normal and the tangential traces of a function $\sigma \in \mathcal{H}_1$. If σ is a regular function then

$$\sigma_\nu = (\sigma\nu)\,.\nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu$$

We recall that the trace map $\gamma : H_1 \to L^2(\Gamma)^d$ is continuous linear, but not surjective. The image of H_1 by this map is denoted by H_Γ, this subspace is continuously injected into $L^2(\Gamma)^d$. Let $H_\Gamma^{'}$ be the dual of H_Γ, and $(.,.)_{H_\Gamma^{'} \times H_\Gamma}$ the duality product between $H_\Gamma^{'}$ and H_Γ.

For every $\sigma \in H_1$, there exists an element denoted $\sigma\nu \in H_\Gamma^{'}$ such that

$$(\sigma\nu, \gamma\mathbf{v})_{H_\Gamma^{'} \times H_\Gamma} = (\sigma, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (Div\ \sigma, \mathbf{v})_H \quad \forall \mathbf{v} \in H_1 \qquad (\mathbf{1}.4.1)$$

Also, if σ is regular enough (For example C^1), we have the formula

$$(\sigma\nu, \gamma\mathbf{v}) = \int_\Gamma \sigma\nu.\mathbf{v}da \qquad \forall \mathbf{v} \in H_1$$

So, if σ is regular enough we have the following formula:

$$(\sigma, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (Div\ \sigma, \mathbf{v})_H = \int_\Gamma \sigma\nu.\mathbf{v}da \qquad \forall \mathbf{v} \in H_1(\Omega) \qquad (\mathbf{1}.4.2)$$

Green's formula allows in particular to define the restriction of $\sigma\nu$ to a measurable part Γ_2 of Γ. More precisely, for $g \in L^2(\Gamma_2)$, we have $\sigma\nu = g$ on Γ_2 if and only if $(\sigma\nu, \gamma\mathbf{v}) = \int_\Gamma g.\mathbf{v}da$, for all $\mathbf{v} \in H_1$ vanishing on Γ/Γ_2.

Throughout the thesis, in mechanical problems, Γ is partitioned into three measurable parts Γ_1, Γ_2 and Γ_3 such that my *meas* $\Gamma_1 > 0$.

We will constantly need the space of admissible displacements V defined as being a closed subspace of the space H_1

$$V = \left\{ \mathbf{v} \in H_1(\Omega)^d \ / \quad \mathbf{v} = 0 \quad \text{on } \Gamma_1 \right\} \qquad (\mathbf{1}.4.3)$$

Since *meas* $\Gamma_1 > 0$, Korn's inequality applies to V: there exists a constant $C_k > 0$ depending only on Ω and Γ_1 such that

$$|\varepsilon(\mathbf{v})|_{\mathcal{H}} \geq C_k |\mathbf{v}|_{H_1(\Omega)^d} \qquad \forall \mathbf{v} \in V \qquad (\mathbf{1}.4.4)$$

A proof of this inequality can be found in $[7, p.79]$.

On V we consider the scalar product $(.,.)_V$ and the norm $|.|_V$ associated with this product,

given by

$$(\mathbf{u},\mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \qquad |\mathbf{v}|_V = |\varepsilon(\mathbf{v})|_{\mathcal{H}} \quad \forall \mathbf{u},\mathbf{v} \in V \tag{1.4.5}$$

By the korn inequality (see [8,Theorem 3.1,p.110] , it comes that $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V and so $(V, |.|_V)$ is a Hilbert space.

Also, by Sobolev's trace theoem, there exists constant $C_0 > 0$ depending only on Ω , $\Gamma_1$and Γ_2 such that

$$|\mathbf{v}|_{L^2(\Gamma_3)^d} \leq C_0 |\mathbf{v}|_V \qquad \forall \mathbf{v} \in V \ . \tag{1.4.6}$$

1.4.2 Spaces of Vector-valued Functions

Let H be a Hilbert space. Let $k \in IN$, $1 \leq p \leq +\infty$ and $T > 0$. We recall that $W^{k,p}(0,T;H)$ is the space of vector distributions $u \in \mathcal{D}'(0,T;H)$ such that $D_j u \in L^p(0,T;H)$ for $j = \overline{0,k}$, D_j denoting the derivative of order j in the sense of distributions.

If $1 \leq p < +\infty$, $W^{k,p}(0,T;H)$ is a real Banach space for the norm defined by

$$|u|_{W^{k,p}(0,T;H)} = (\sum_{j=0}^{k} \int_0^T |D_j u(x)|^p \, dx)^{\frac{1}{p}} \quad \forall\, u \in W^{k,p}(0,T;H). \tag{1.4.7}$$

In particular, $W^{k,2}(0,T;H)$ is a real Hilbert space for the inner product defined by

$$(u,v)_{W^{k,2}(0,T;H)} = \sum_{j=0}^{k} \int_0^T (D_j u(t), D_j v(t))_H \, dt \quad \forall\, u,v \in W^{k,2}(0,T;H) \tag{1.4.8}$$

On the other hand, $W^{k,\infty}(0,T;H)$ is a Banach space for the norm defined by

$$|u|_{W^{k,\infty}(0,T;H)} = \sum_{j=0}^{k} \sup_{[0,T]} ess\, |D_j(u(t))|_H \, \forall\, u \in W^{k,\infty}(0,T;H) \tag{1.4.9}$$

For the Particular case $k = 0$, it is observed that

$$W^{0,p}(0,T;H) = L^p(0,T;H)$$

And we then denote the norm $L^p(0,T;H)$ by $|.|_{L^p(0,T;H)}$ for all $1 \leq p \leq +\infty$. We also define, for all $k \in \mathbb{N}$, the space $C^k(0,T;H)$ of functions $\mathbf{u} : [0,T] \to H$ such that for all $j = \overline{0, k}$ the derivatives $\dfrac{d^j\ \mathbf{u}}{dt^j}$ exist and are continuous on $[0, T]$. We denote, in particular, $C^0(0, T; H)$ by $C(0, T; H)$. The space $C^k(0, T; H)$ is a Banach space for the norm defined by

$$|\mathbf{u}|_{C^k(0,T;H)} = \sum_{j=0}^{k} \max_{t\in[0,T]} \left| \frac{d^j\ \mathbf{u}}{dt^j}(t) \right|_H \qquad \forall\, u \in C^k(0, T, H). \tag{\mathbf{1}.4.10}$$

In particular, the norms on the spaces $C(0, T; H)$ and $C^1(0, T; H)$ are given by

$$|\mathbf{u}|_{C(0,T;H)} = \max_{t\in[0,T]} |\mathbf{u}(t)|_H \qquad \forall\, u \in C(0, T, H)$$

$$|\mathbf{u}|_{C^1(0,T;H)} = |\mathbf{u}|_{C(0,T;H)} + |\dot{\mathbf{u}}|_{C(0,T;H)} \qquad \forall\ u \in C^1(0, T; H)$$

We specify that the point above an expression designates the derivative of this expression with respect to time, represented by the variable $t \in [0, T]$.

1.5 nonlinear analysis elements

In this section we will recall some notions of nonlinear analysis which will be of great use for the realization of this work. In particular results on weak convergence and weak star convergence, bounded operators, monotone, lower convex and lower semi-continuous functions, differentiability and subdifferentiability.

1.5.1 Weak convergence

We now turn our attention to the notion of convergence. Let X is a Banach space and X' the dual space of X and denote $\langle ., . \rangle$ the duality product between X and its topological dual space X'.

Definition 1.5.1 (**Strongly convergence**) A sequence $\mathbf{u}_n$ is said to strongly converge to $\mathbf{u}$ if $\mathbf{u}_n, \mathbf{u} \in X$ and if

$$\lim_{n\to\infty} \|\mathbf{u}_n - \mathbf{u}\|_X = 0$$

we will denoted this convergence by $\mathbf{u}_n \to \mathbf{u}$ in X.

Now, Let's recall some definitions and results on weak topology

Definition 1.5.2.[16] Let X be a normed space with X' its dual space. A sequence $\{\mathbf{u}_n\} \subset X$ weakly converges to $\mathbf{u} \in X$, if

$$l\left(\mathbf{u}_n\right) \ \to l\left(\mathbf{u}\right) \quad \forall l \in X'$$

This convergence will be dented by $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in .

Proposition 1.5.1. Let $\mathbf{u}_n$ be a sequence in X. Then

1. If $\mathbf{u}_n \to \mathbf{u}$ strongly in X, then $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in X.

2. If $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in X, then $\|\mathbf{u}_n\|_X$ is bounded and $\|\mathbf{u}\|_X \leq \lim_{n\to\infty} \inf \|\mathbf{u}_n\|_X$.

3. If $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in X and if $l_n \to l$ strongly in X' , then $\langle l_n, \mathbf{u}_n \rangle_{X' \times X} \to \langle l, \mathbf{u} \rangle_{X' \times X}$

Definition 1.5.3. Let X be a Hilbert space endowed with the scalar product $(.,.)_X$ and the norm $|.|_X$ and let $A : X \to X$ be an operator

a) The operator A is said to be monotone if

$$(A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v})_X \geq 0 \quad \forall \mathbf{u}, \mathbf{v} \in X$$

b) The operator A is strictly monotone if

$$(A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v})_X > 0 \quad \forall \mathbf{u}, \mathbf{v} \in X$$

c) The operator A is said to be strongly monotone if there exists $m > 0$ such that

$$(A\mathbf{u} - A\mathbf{v},)_X > m\, |\mathbf{u} - \mathbf{v}|_X^2 \quad \forall \mathbf{u}, \mathbf{v} \in X$$

d) Operator A is Lipschitz if there exists $M > 0$ such tha

$$|A\mathbf{u} - A\mathbf{v}|_X \leq M\, |\mathbf{u} - \mathbf{v}|_X \quad \forall \mathbf{u}, \mathbf{v} \in X.$$

1.5.2 Elements of convex analysis

Convex and lower semicontinuous functions

Definition 1.5.4. Let X be a real vector space and let the function $\varphi : X \to]-\infty, +\infty]$. We say that the function φ is proper if $\varphi(\mathbf{u}) > -\infty$ for all $\mathbf{u} \in X$ and $\varphi(\mathbf{u}) < +\infty$ for all $\mathbf{u} \in X$. The function φ is convex if

$$\varphi(t\mathbf{u} + (1-t)\mathbf{v}) \leq t\varphi(\mathbf{u}) + (1-t)\varphi(\mathbf{v}) \qquad \forall\, \mathbf{u}, \mathbf{v} \in X, t \in]0, 1[\,.$$

for all $\mathbf{u}, \mathbf{v} \in X$ and $t \in [0, T]$. whenever the sum on the right hand side is defined. If the strict inequality

holds for any $\mathbf{u} \neq \mathbf{v}$ then f is said to be strictly convex in X. The set

$$D_{ef}\ f = \{\mathbf{u} \in X \ /\ \varphi(\mathbf{u}) < +\infty\}$$

is called the effective domain of f.

For a real number α, the set $\{(\mathbf{u},\alpha) \in X \times \mathbb{R} / \varphi(\mathbf{u}) \leq \alpha\}$ is called epigraph off $(epi\varphi)$.

The function φ is strictly convex if this latter inequality is strict for $\mathbf{u} \neq \mathbf{v}$ and $t \in (0, 1)$

Note that if $\varphi, \psi : X \to]-\infty, +\infty]$ are convex functions and $\lambda \geq 0$, then the functions $\varphi + \psi$ and $\lambda\varphi$ are also convex.

Definition 1.5.5. A function $\varphi : X \to]-\infty, +\infty]$ is said to be lower semi-continuous (*l.s.c.*) in $\mathbf{u} \in X$ if

$$\lim_{n \to +\infty} \inf \varphi(\mathbf{u}_n) \geq \varphi(\mathbf{u}) \tag{1.5.1}$$

for each sequence $\{\mathbf{u}_n\} \subset X$ convergent to $\mathbf{u}$ in X. The function φ is *l.s.c.* if it is *l.s.c.* at each point $\mathbf{u} \in X$. When inequality (**1**.5.1) holds for any sequence $\{\mathbf{u}_n\} \subset X$ which converges weakly to $\mathbf{u}$, we say that the function φ is weakly semicontinuous below $\mathbf{u}$. The function φ is weakly *l.s.c.* if it is weakly *l.s.c.* at each point $\mathbf{u} \in X$.

Note that if $\varphi, \psi : X \to]-\infty, +\infty]$ are *l.s.c.* functions. and $\lambda \geq 0$, then the functions $\varphi + \psi$ and $\lambda\varphi$ are also *l.s.c.* Furthermore, if φ is a continuous function, then it is also *l.s.c.* However, the reverse is not true since lower semi-continuity does not imply continuity.

As strong convergence in X implies weak convergence, it follows that a weakly lower semi-continuous function is lower semi-continuous. Moreover, one can show that a proper convex function $\varphi : X \to]-\infty, +\infty]$ is lower semicontinuous if and only if it is weakly lower semicontinuous.

Example 1.5.1. (Indicator function) Let X be a normalized real space and $K \subset X$. We call the indicator function of K the function $\Psi_K : H \to]-\infty, +\infty[$ defined by:

$$\Psi_K(\mathbf{u}) = \begin{cases} 0 & \text{if } \mathbf{u} \in K \\ +\infty & \text{if no} \end{cases} \tag{1.5.2}$$

We can prove that K is a non-empty, closed and convex set of X if and only if the indicator function Ψ_K is clean, convex and *l.s.c.*

Differentiability and sub-differentiability

We now recall the definition of Gâteaux differentiable functions.

Definition 1.5.6. Let $\varphi : X \to \mathbb{R}$ and let $\mathbf{u} \in X$. Then φ is Gâteaux differentiable at $\mathbf{u}$ if there exists an element $\nabla\varphi(\mathbf{u}) \in X$ such that

$$\lim_{t \to 0} \frac{\varphi(\mathbf{u} + t\mathbf{v}) - \varphi(u)}{t} = (\nabla\varphi(\mathbf{u}), \mathbf{v})_X \ \ \forall\, \mathbf{v} \in X. \tag{1.5.3}$$

The element $\nabla\varphi(\mathbf{u})$ that satisfies (**1**.5.3) is unique and is called the gradient of φ at $\mathbf{u}$. The function $\varphi : X \to \mathbb{R}$ is said to be Gâteaux differentiable if it is Gâteaux differentiable at every point of X. In this case, the operator $\nabla\varphi : X \to X$ that maps every element $\mathbf{u} \in X$ into the element $\nabla\varphi(\mathbf{u})$ is called the gradient operator of φ.

The convexity of Gâteaux differentiable functions can be characterized as follows.

Proposition 1.5.2. Let $(X, (.,.)_X)$ be a pre-Hilbertian space and let $\varphi : X \to \mathbb{R}$ be a Gâteaux-differentiable function. The following statements are equivalent:

(i) φ is a convex function;

(ii) φ satisfies the inequality

$$\varphi(\mathbf{v}) - \varphi(\mathbf{u}) \geq (\nabla\varphi(\mathbf{u}), \mathbf{v} - \mathbf{u})_X \qquad \forall\, \mathbf{u}, \mathbf{v} \in X. \tag{1.5.4}$$

(iii) the gradient of φ is a monotonic operator

$$(\nabla\varphi(\mathbf{u}) - \nabla\varphi(\mathbf{v}), \mathbf{v} - \mathbf{u})_X \geq 0 \qquad \forall\, \mathbf{u}, \mathbf{v} \in X. \tag{1.5.5}$$

Corollary 1.5.1. Let $\varphi : X \to \mathbb{R}$ be a convex Gâteaux differentiable function. Then φ is lower semicontinuous.

Definition 1.5.7. Let $\varphi : X \to]-\infty, +\infty]$ and let $\mathbf{u} \in X$. Then the subdifferential of φ at $\mathbf{u}$ is the set

$$\partial\varphi(\mathbf{u}) = \{f \in X \;:\; \varphi(\mathbf{v}) - \varphi(\mathbf{u}) \geq (f, \mathbf{v} - \mathbf{u})_X \qquad \forall \mathbf{v} \in X.\} \tag{1.5.6}$$

Denote

$$D\,(\partial\varphi) = \{\mathbf{u} \in X \;:\; \partial\varphi(\mathbf{u}) = \emptyset\}\,. \tag{1.5.7}$$

A function φ is said to be subdifferentiable at $\mathbf{u} \in X$ if $\mathbf{u} \in D\,(\partial\varphi)$, and each element $f \in \partial\varphi(\mathbf{u})$ is called a subgradient of φ at $\mathbf{u}$. A function φ is said to be subdifferentiable if it is subdifferentiable at each point $\mathbf{u} \in X$, i.e., if $D\,(\partial\varphi) = X$.

Chapter 2

Variational and quasi-variational inequalities

This chapter is devoted to some fundamental properties and results concerning existence and uniqueness theorems for elliptic variational and quasi-variational inequalities. For more details on this part. A recent existence and uniqueness result for variational parabolic evolution inequalities has been considered, for more details see [8].

2.1 Variational inequalities

2.1.1 Elliptic variational inequalities

In this Subsection we present results of existence and uniqueness of solutions concerning elliptic variational inequalities with monotone operators. We start with an existence and uniqueness result for elliptic variational inequalities of the first kind, then we move on to elliptic variational equations of the second kind, as well as quasi-variational elliptic inequalities.

Variational inequalities of the first kind

Given an operator $A : H \to H$, a subset $K \subset H$ and an element $f \in H$, we consider the problem of finding an element $\mathbf{u}$ such that

$$\mathbf{u} \in K, \ (A\mathbf{u}, \mathbf{v} - \mathbf{u})_H \geq (f, \mathbf{v} - \mathbf{u})_H \ \ \forall \mathbf{v} \in K \tag{\textbf{2}.1.1}$$

An inequality of the form ($\mathbf{2}$.1.1) is called an elliptic variational inequality of the first kind.

We have the following standard result of existence and uniqueness of the solution.

Theorem 2.1.1 [20] Let H be a Hilbert space and let $K \subset H$ be a closed non-empty and convex subset. Suppose that $A : K \to H$ is a strongly monotonic and Lipschitz operator.

Then, for all $f \in H$ the variational inequality (**2**.2.1) admits a unique solution.

Now suppose that $K = H$. Then, by taking $\mathbf{v} = \mathbf{u} \pm \mathbf{w}$, it is easy to see that the variational inequality (**2**.1.1) is equivalent to the variational equation

$$\mathbf{u} \in K,\ (A\mathbf{u}, \mathbf{w})_H = (f, \mathbf{w})_H \quad \forall \mathbf{w} \in H\ .$$

We have the following existence and uniqueness result in the study of nonlinear equations involving monotonic operators.

Theorem 2.1.2 [20]Let H be a Hilbert space and let $A : H \to H$ be a strongly monotone and Lipschitz operator. Then for all $f \in H$ there exists a unique element $\mathbf{u} \in H$ such that $A\mathbf{u} = f$.

This theorem will be used in the study of elasto-viscoplastic problems in the third chapter.

Variational inequalities of the second kind

Given a set $K \subset H$, an operator $A : H \to H$, a function $j : K \to \mathbb{R}$ and an element $f \in H$, we consider the problem of finding an element $\mathbf{u}$ such that

$$\mathbf{u} \in K,\ (A\mathbf{u}, \mathbf{v} - \mathbf{u})_H + j(\mathbf{v}) - j(\mathbf{u}) \geq (f, \mathbf{v} - \mathbf{u})_H \quad \forall \mathbf{v} \in K \tag{2.1.2}$$

A variational inequality of the form (**2**.1.2) is called *an elliptic variational inequality of the second kind.* In the particular case when $j \equiv 0$, the variational inequality (**2**.1.2) represents a variational inequality of the form (**2**.1.1), that is to say an elliptic variational inequality of the first kind.

In the study of (**2**.1.2) we assume the following hypotheses:

$$K \text{ is a non-empty convex subset of } H, \tag{2.1.3}$$

$A : K \to H$ is a strongly monotone and Lipschitz operator, i.e.

$$\begin{cases} \text{a) There exists a constant } m_A > 0 \text{ such that} \\ (A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v})_H \geq m_A \left|\mathbf{v} - \mathbf{u}\right|_H^2 \quad \forall \mathbf{v}, \mathbf{u} \in H. \\ \text{b) There exists a constant } L_A > 0 \text{ such that} \\ \quad \left|A\mathbf{u} - A\mathbf{v}\right|_H \leq L_A \left|\mathbf{v} - \mathbf{u}\right|_H \quad \forall \mathbf{v}, \mathbf{u} \in H. \end{cases} \tag{2.1.4}$$

$$j : K \to \mathbb{R} \text{ is a convex and lower semi-continuous function.} \tag{2.1.5}$$

The main result of this subsection is as follows.

Theorem 2.1.3 [20]. Let H be a Hilbert space and suppose that hypotheses (**2**.1.3)-(**2**.1.5) are verified. Then for all $f \in H$ the elliptic variational inequality (**2**.1.2) admits a unique solution.

2.1.2 Elliptic quasi-variational inequalities.

The modeling of several classes of physical problems leads to elliptic or evolutionary variational inequalities, in which the non-differentiable functional j depends on the solution itself. We then give an existence and uniqueness result for this type of problem. For this, we consider a Hilbert space H endowed with the scalar product $(.,.)_H$ and the associated norm $|.|_H$, we consider the problem of finding an element $\mathbf{u}$ such that

$$\mathbf{u} \in H,\ (A\mathbf{u}, \mathbf{v}-\mathbf{u})_H + j(\mathbf{v}-\mathbf{u}) - j(\mathbf{u}-\mathbf{u}) \geq (f, \mathbf{v}-\mathbf{u})_H \quad \forall \mathbf{v} \in H \qquad (\mathbf{2}.1.6)$$

An inequality of the form (**2**.1.6) is called an elliptic quasi-variational inequality of the second kind over H.

To solve this inequality, in addition to (**2**.1.3) and (**2**.1.4), we consider the following assumption.

the functional $j : H \times H \to \mathbb{R}$ satisfies

$$\begin{cases} \text{a) For all } \mathbf{u} \in H,\ j(\mathbf{u},.) : H \to \mathbb{R} \text{ is convex and l.s.c on } H. \\ \text{b) There exists } \alpha > 0 \text{ such that} \\ j(\mathbf{u}_1, \mathbf{v}_2) + j(\mathbf{u}_1, \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_2) \\ \leq |\mathbf{u}_1 - \mathbf{u}_2|_H \, |\mathbf{v}_1 - \mathbf{u}_2|_H \quad \forall \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in H. \end{cases} \qquad (\mathbf{2}.1.7)$$

We subsequently give an existence and uniqueness result for the problem (**2**.1.6).

Theorem 2.1.4 Let H be a Hilbert space and suppose that the previous hypotheses are satisfied. Further, assume that $m_A > \alpha$. Then, for all $f \in H$ the quasi-variational inequality (**2**.1.6) admits a unique solution.

A proof of *Theorem* **2.1.4** can be found for example in [22].

2.2 Compléments divers

We recall here the classic lemmas of the Gronwall type which are involved in many augmentation problems, in particular to establish the uniqueness of the solution.

2.2.1 Gronwall Lemmas

Lemma 2.2.1. Let m, $n \in C(0,T;\mathbb{R})$ be such that $m(t) \geq 0$ and $n(t) \geq 0$ for all $t \in [0,T]$ and let $a \geq 0$. If $\varphi \in C(0,T;\mathbb{R})$ is a function such that

$$\varphi(t) \leq a + \int_0^t m(s)ds + \int_0^t n(s)\varphi(s)ds \qquad \forall\, t \in [0,T]$$

So

$$\varphi(t) \leq (a + \int_0^t m(s)ds)\exp(\int_0^t n(s)ds) \quad \forall\, t \in [0,T]$$

For the particular case $m = 0$, this lemma becomes:

Corollary 2.2.1. Let $n \in C(0,T;\mathbb{R})$ be such that $n(t) \geq 0$ for all $t \in [0,T]$ and let $a \geq 0$. If $\varphi \in C(0,T;\mathbb{R})$ is a function such that

$$\varphi(t) \leq a + \int_0^t n(s)\varphi(s)ds \quad \forall\, t \in [0,T]$$

So

$$\varphi(t) \leq a\exp(\int_0^t n(s)ds) \quad \forall\, t \in [0,T]$$

Lemma 2.2.2. Let m, $n \in C(0,T;\mathbb{R})$ be such that $m(t) \geq 0$ and $n(t) \geq 0$ for all $t \in [0,T]$ and let $a \geq 0$. If $\varphi \in C(0,T;\mathbb{R})$ is a function such that

$$\frac{1}{2}\varphi^2(t) \leq \frac{1}{2}a^2 + \int_0^t m(s)\varphi(s)ds + \int_0^t n(s)\varphi^2(s)ds \qquad \forall\, t \in [0,T]$$

So

$$\mid \varphi(t) \mid \leq (a + \int_0^t m(s)ds)\exp(\int_0^t n(s)ds) \qquad \forall\, t \in [0,T]$$

Proposition 2.2.1.[11] Assume that $\{a_n\}_{n=1}^N$ and $\{\varphi_n\}_{n=1}^N$ are two sequences of nonnegative number, satisfying

$$\varphi_n \leq ca_n + ck\sum_{j=1}^{n}\varphi_j\,.$$

Then

$$\varphi_n \leq c\left(a_n + ck\sum_{j=1}^{n}a_j\right), \quad n = 1, ..., N\,.$$

Moreover

$$\max_{1\leq n\leq N}\varphi_n \leq c\max_{1\leq n\leq N}a_n.$$

2.3 Preliminaries on Numerical Analysis

In this section introduces the fundamental tools for numerically approximating solutions of ordinary and partial differential equations. The methods emphasized are :

- Finite difference methods for time discretization.
- Finite element methods for spatial discretization in variational inequalities.

First-order finite difference schemes are used for time approximation, while piecewise linear triangular finite elements are used in space.

2.3.1 Finite Difference Approximation

Let $f : [0,T] \to X$, where X is a normed space and f is differentiable. Let $k > 0$ be the time step.

Forward and Backward Differences

$$\text{Forward} : \frac{f(t+k) - f(t)}{k}, \qquad \text{Backward} : \frac{f(t) - f(t-k)}{k}$$

Assume that f has a bounded second derivative, then the approximation errors of both differences are $O(k)$.

Uniform Time Grid

Let $0 = t_0 < t_1 < ... < t_N = T$ be a uniform partition of the time interval $[0,T]$, i.e., $t_n = nk$, $n = 0, 1, ..., N$, $k = \frac{T}{N}$ the step-size. Then

the derivative $\dot{f}(t)$ at the node t_n, $\dot{f}_n = \dot{f}\ (t_n)$, can be approximated by the backward divided difference,

$$\delta f_n = \frac{f_n - f_{n-1}}{k}$$

to order $O(k)$, where $f_n \equiv f(t_n)$.

If X is an inner product space, then for a sequence $\{e_n\} \subset X$,

$$(\delta e_n, e_n)_X \geq \frac{1}{2k} \|e_n\|_X^2 - \|e_{n-1}\|_X^2 \ \|$$

This inequality is important in stability and convergence analysis.

2.3.2 Finite Element Approximation

Triangulation and Elements

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain partitioned into triangular elements $K \in \mathcal{T}_h$, where

$$h = \max_{K \in \mathcal{T}_h} diam\,(K)$$

We consider families of regular triangulations satisfying shape regularity :

$$\frac{h_K}{\rho_K} \leq \rho^* \quad \forall K \in \mathcal{T}_h$$

where

$$h_K = diam\,(K) = \max\{\|x - y\|\,,\ x, y \in K\}$$

and

$$\rho_K = \text{diameter of the largest sphere inscribed in } K.$$

Linear Basis Functions

Let $\{x_i\}_{i=1}^{N_h}$ be the vertices of the mesh. Define the linear basis functions φ_i such that :

$$\varphi_i(x_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad , 1 \leq i, j \leq N_h,$$

and φ_i is piecewise linear on each K

The finite element interpolant of $\mathbf{v} \in C(\Omega)$ is :

$$\Pi^h \mathbf{v} = \sum_{i=1}^{N_h} \mathbf{v}(x_i) \varphi_i$$

Function Space and Error Estimate

Define the finite element space :

$$X^h = \operatorname{span}\{\varphi_i : 1 \leq i \leq N_h\}$$

For essential boundary conditions on $\Gamma_1 \subset \partial\Omega$, define :

$$X_{\Gamma_1}^h = \{\mathbf{v}_h \in X_h : \mathbf{v}_h = 0 \text{ on } \Gamma_1\}$$

If $\mathbf{v} \in H^2(\Omega)$, then

$$\left\| \mathbf{v} - \Pi^h \mathbf{v} \right\|_{L^2(\Omega)} + h \left\| \mathbf{v} - \Pi^h \mathbf{v} \right\|_{H^1(\Omega)} \leq ch^2 \|\mathbf{v}\|_{H^2(\Omega)}.$$

Chapter 3

A viscoelastic problem with normal compliance and damage

In this chapter, we consider a contact problem between a viscoelastic body and a deformable base in a quasi-static process. The contact is modeled by normal compliance, the behavior law is elasto-viscoelastic with internal state variable which describes the damage of the material caused by elastic deformations.

This chapter is divided into three sections. In the first, we present the classical formulation of the problem, then we move on to outline the necessary assumptions about the data. Then, the second section is devoted to the variational formulation of the problem. Finally, in the last section, we state the main theorem, then we establish the existence and uniqueness of a weak solution. The proof is based on fixed point arguments and a classical existence and uniqueness result on parabolic inequalities.

3.1 Mechanical formulation of the problem and hypotheses

We place in the physical framework, presented in the first chapter of the thesis and consequently we use the first mathematical model. For the model to be complete, let us specify that the elasto-viscplastic constitutive law with damage of the type (**1**.2.1). Moreover, the process is quasi-static, and therefore the inertial terms are included in the equation of motion. The body can come into contact on Γ_3 without obstacle, which is called the foundation. The contact is modeled with a normal compliance without friction. The classic model of the above process is as follows.

Problem P^1 Find the displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, the stress tensor field

$\boldsymbol{\sigma}:\ \Omega\times[0,T]\to S_d$, and the damage field $\beta:\ \Omega\times[0,T]\to\mathbb{R}$ such that

$$\sigma(t)=\mathcal{A}\varepsilon(\dot{\mathbf{u}}(t))+\mathcal{F}\varepsilon(\mathbf{u}(t),\beta(t)) \ \text{ in } \ \Omega\times(0,T)\,; \tag{3.1.1}$$

$$\dot{\beta}-k\triangle\beta+\partial\varphi_Y(\beta)\ni S(\varepsilon(\mathbf{u}),\beta) \ \text{ in } \ \Omega\times(0,T)\,; \tag{3.1.2}$$

$$\operatorname{Div}\sigma+\mathbf{f}_0=0 \ \text{ in } \ \Omega\times(0,T)\,; \tag{3.1.3}$$

$$\mathbf{u}=\mathbf{0} \ \text{ on } \ \Gamma_1\times(0,T)\,; \tag{3.1.4}$$

$$\sigma\nu=\mathbf{f}_2 \ \text{ on } \ \Gamma_2\times(0,T)\,; \tag{3.1.5}$$

$$\left\{\begin{array}{l} -\sigma_\nu=p_\nu(\mathbf{u}_\nu),\quad |\sigma_\tau|\le p_\nu(\mathbf{u}_\nu) \text{ and} \\ \qquad \sigma_\tau=-p_\nu(\mathbf{u}_\nu)\frac{\dot{\mathbf{u}}_\tau}{|\dot{\mathbf{u}}_\tau|} \ \text{ if } \ \dot{\mathbf{u}}_\tau\neq\mathbf{0}\end{array}\right. \quad \text{on } \ \Gamma_3\times(0,T)\,. \tag{3.1.6}$$

$$\frac{\partial\beta}{\partial\boldsymbol{\nu}}=0 \ \text{ on } \ \Gamma\times(0,T)\,; \tag{3.1.7}$$

$$\mathbf{u}(0)=\mathbf{u}_0,\beta(0)=\beta_0, \ \text{ in } \ \Omega. \tag{3.1.8}$$

Here (**3**.1.1) is the viscoelastic constitutive law with damage. (**3**.1.2) represents the inclusion used for the evolution of the field of damage. (**3**.1.3) represents the equation of motion. (**3**.1.4)-(**3**.1.5) are the displacement-traction conditions. Condition (**3**.1.6) represents the general normal compliance (see [6]). (**3**.1.7) represents a homogeneous Neumann boundary condition where $\frac{\partial\beta}{\partial\boldsymbol{\nu}}$ is the normal derivative of β. In (3.1.8) $\mathbf{u}_0$ and β_0 are the initial displacement and initial damage of the material, respectively. To simplify the notation, we do not explicitly state the dependence of various functions on the variables $x\in\Omega\cup\Gamma$ and $t\in[0,T]$. To obtain a variational formulation of problem (**3**.1.1)-(**3**.1.8) we need an additional notation.

We now introduce a closed subspace of H_1, whose definition is given by

$$V=\{\mathbf{v}\in H_1/\ \mathbf{v}=0 \text{ on } \Gamma_1\}$$

Like $meas\ (\Gamma_1)>0$, Korn's inequality holds, so there is a constant $C_{\ k}>0$, which depends only on Ω and Γ_1, such that

$$|\varepsilon(\mathbf{v})|_{\mathcal{H}}\ge C_{\ k}\,|\mathbf{v}|_{H^1(\Omega)^d}\qquad\forall\mathbf{v}\in V.$$

We define the inner products on V by

$$(\mathbf{u}\ ,\mathbf{v})_V=(\varepsilon(\mathbf{u})\ \ ,\ \varepsilon(\mathbf{v}))_{\mathcal{H}}\quad,\quad|\mathbf{v}|_V=|\varepsilon(\mathbf{v})|_{\mathcal{H}}\ \forall u,v\in V$$

It comes that $|\ .\ |_{H^1(\Omega)^d}$ and $|\ .\ |_V$ are equivalent norms on V and hence, $(V\ \ ,\ \ |\ .\ |_V\ )$ is a real Hilbert space. Furthermore, by Sobolev's trace theorem there exists a constant C_0 depends only on Ω,Γ_1 and Γ_3 such that

$$|\mathbf{v}|_{L^2(\Gamma_3)^d}\le C_0\,|\mathbf{v}|_V\qquad\forall\mathbf{v}\in V.$$

In the study of the problem P^1, we consider the following hypotheses.

The viscosity operator $\mathcal{A}: \ \Omega \times S^d \longrightarrow \ S^d$ satisfies

(a) There exists a constant $L_{\mathcal{A}} > 0$ such that
$|\ \mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)\ | \leq \ L_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2| \qquad \forall \varepsilon_1, \varepsilon_2 \in S^d \quad$ a.e, $x \in \Omega$.
(b) There exists $m_{\mathcal{A}} > 0$ such that
$(\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2), (\varepsilon_1 - \varepsilon_2)) \geq m_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2|^2 \qquad \forall \varepsilon_1, \varepsilon_2 \in S^d \quad$ a.e, $x \in \Omega$.
(c) $x \longrightarrow \mathcal{A}(x,\varepsilon)$ is Lebesgue measurable on Ω
(d) The map $x \longrightarrow \mathcal{A}(x,0)$ belongs to $\mathcal{H}$.

(**3**.1.9)

The elasticity operator $\mathcal{F}: \ \Omega \times S^d \times \mathbb{R} \longrightarrow \ S^d$ satisfies

(a) There exists a constant $L_{\mathcal{F}} > 0$ such that
$|\ \mathcal{F}(x,\varepsilon_1,\beta_1) - \mathcal{F}(x,\varepsilon_2,\beta_2)\ | \leq \ L_{\mathcal{F}}(|\varepsilon_1 - \varepsilon_2| + \ |\beta_1 - \beta_2|)$
$\forall \varepsilon_1, \varepsilon_2 \in S^d, \forall \beta_1, \beta_2 \in \mathbb{R} \quad$ a.e, $x \in \Omega$.
(b) $x \longrightarrow \mathcal{F}(x,\varepsilon,\beta)$ is Lebesgue measurable on Ω. $\forall \varepsilon \in S^d$ and $\forall \beta \in \mathbb{R}$.
(c) The map $x \longrightarrow \mathcal{F}(x,0,0)$ belongs to $\mathcal{H}$.

(**3**.1.10)

The damage source function $\phi: \ \Omega \times S^d \times \mathbb{R} \longrightarrow \ \mathbb{R}$ satisfies

(a) There exists a constant $L_\phi > 0$ such that
$|\ \phi(x,\varepsilon_1,\beta_1) - \phi(x,\varepsilon_2,\beta_2)\ | \leq \ L_\phi(|\varepsilon_1 - \varepsilon_2| + |\beta_1 - \beta_2|)$
$\forall \varepsilon_1, \varepsilon_2 \in S^d$ and $\beta_1, \beta_2 \in \mathbb{R}$ $p.p$ $x \in \Omega$. (**3**.1.12)
(b) for all $\varepsilon \in S^d$ and $\beta \in \mathbb{R}$,
$x \longrightarrow \phi(x,\varepsilon,\beta)$ is Lebesgue measurable on Ω.
(c) The map $x \longrightarrow \phi(x,0,0)$ belongs to $L^2(\Omega)$

The normal contact function $p_\nu: \ \Gamma_3 \times \mathbb{R} \longrightarrow \ \mathbb{R}_+$ satisfies

(a) There exists a constant $L_\nu > 0$ such that
$|\ p_\nu(x,r_1) - p_\nu(x,r_2)\ | \leq \ L_\nu |r_1 - r_2| \ \ \forall r_1, r_2 \in \mathbb{R}$ a.e.$x \in \Gamma_3$.
(b) $(p_\nu(x,r_1) - p_\nu(x,r_2))(r_1 - r_2) \geq 0 \ \forall r_1, r_2 \in \mathbb{R}$ a.e.$x \in \Gamma_3$
(c) $x \longrightarrow p_\nu(x,r)$ is Lebesgue measurable on Γ_3 for any $r \in \mathbb{R}$.
(d) $p_\nu(x,r) = 0$ for all $r \leq 0$ a.e.$x \in \Gamma_3$.

(**3**.1.13)

We assume that the volume forces $\mathbf{f}_0$ and the surface tractions $\mathbf{f}_2$ have the regularity

$$\mathbf{f}_0 \in C(0,T;H) \quad , \quad \mathbf{f}_2 \in C\left(0,T;L^2(\Gamma_2)^d\right) \tag{3.1.14}$$

The initial displacement field satisfies

$$\mathbf{u}_0 \in V \tag{3.1.15}$$

The initial damage field satisfied

$$\beta_0 \in Y \tag{3.1.16}$$

We define the bilinear form $a: \ H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(\xi, \varphi) = k \int_\Omega \nabla\xi \cdot \nabla\varphi \mathrm{d}x. \qquad (\mathbf{3}.1.17)$$

We consider the function $\mathbf{f}: \ [0, T] \to V$ defined by

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t).\mathbf{v}\mathrm{d}x + \int_{\Gamma_2} \mathbf{f}_2(t).\mathbf{v}\mathrm{d}a, \qquad (\mathbf{3}.1.18)$$

Note that the assumption (**3**.1.14) implies that the integral (**3**.1.18) is well defined, as well as

$$\mathbf{f} \in C(0, T; V). \qquad (\mathbf{3}.1.19)$$

Next, we define the frictional function $j: \ V \times V \to \mathbb{R}$ by

$$j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu(\mathbf{u}_\nu) \left|\mathbf{v}_\tau\right| \mathrm{d}a, \qquad (\mathbf{3}.1.20)$$

we define the functional $P: \ V \times V \to \mathbb{R}$ by

$$P(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu(\mathbf{u}_\nu)\mathbf{v}_\nu \mathrm{d}a, \qquad (\mathbf{3}.1.21)$$

3.2 Variational formulation

In this section, we will give the variational formulation of problem P^1. Using the following Green's formula

$$(\mathrm{Div}\, \sigma, \mathbf{v})_H + (\sigma(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} = \int_\Gamma \sigma\nu.\mathbf{v}\mathrm{d}a,$$

But

$$\int_\Gamma \sigma\nu.\mathbf{v}\mathrm{d}a = \int_{\Gamma_2} \sigma\nu.\mathbf{v}\mathrm{d}a + \int_{\Gamma_3} \sigma\nu.\mathbf{v}\mathrm{d}a + \int_{\Gamma_3} \sigma\nu.\mathbf{v}\mathrm{d}a$$

with $\mathbf{v} = \mathbf{0}$ a.e. on Γ_1, and $\sigma\nu = f_2$ on Γ_2. Taking $\mathbf{v} = \mathbf{v} - \dot{\mathbf{u}}$, and

$$\sigma\nu.\mathbf{v} = \sigma_\nu \mathbf{v}_\nu + \sigma_\tau.\mathbf{v}_\tau \quad \text{on } \Gamma_3$$

using the friction law:We deduce that

$$\int_\Omega \sigma.\varepsilon(\mathbf{v})\mathrm{d}x = \int_\Omega f_0.\mathbf{v}\mathrm{d}x + \int_{\Gamma_2} f_2.\mathbf{v}\mathrm{d}a + \int_{\Gamma_3} \sigma_\nu \mathbf{v}_\nu \mathrm{d}a + \int_{\Gamma_3} \sigma_\tau.\mathbf{v}_\tau, \qquad (\mathbf{3}.2.1)$$

using the friction law:

$$\sigma_\tau.(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau) \geq |\sigma_\nu| \, |\dot{\mathbf{u}}_\tau| - |\sigma_\nu| \, |\mathbf{v}_\tau| \qquad (\mathbf{3}.2.2)$$

At points on Γ_3 where $\dot{\mathbf{u}}_\tau \neq 0$,

$$\sigma_\tau.(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau) = -p_\nu(\mathbf{u}_\nu)\frac{\dot{\mathbf{u}}_\tau}{|\dot{\mathbf{u}}_\tau|}.(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau) \geq p_\nu(\mathbf{u}_\nu)\,|\dot{\mathbf{u}}_\tau| - p_\nu(\mathbf{u}_\nu)\,|\mathbf{v}_\tau|. \quad (\mathbf{3}.2.3)$$

At points where $\dot{\mathbf{u}}_\tau = 0$,

$$\sigma_\tau.(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau) = \sigma_\tau.\mathbf{v}_\tau \geq -|\sigma_\tau|\,|\mathbf{v}_\tau| \geq -p_\nu(\mathbf{u}_\nu)\,|\mathbf{v}_\tau| \geq p_\nu(\mathbf{u}_\nu)\,|\dot{\mathbf{u}}_\tau| - p_\nu(\mathbf{u}_\nu)\,|\mathbf{v}_\tau|. \quad (\mathbf{3}.2.4)$$

Integrating inequality ($\mathbf{3}$.2.4) over Γ_3, we get:

$$\int_{\Gamma_3} \sigma_\tau.(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau)da \geq \int_{\Gamma_3} p_\nu(\mathbf{u}_\nu)\,(|\dot{\mathbf{u}}_\tau| - |\mathbf{v}_\tau|)\,da, \quad (\mathbf{3}.2.5)$$

From ($\mathbf{3}$.1.20) and ($\mathbf{3}$.1.21), we obtain:

$$(\sigma(t), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} + P(\mathbf{u}(t), (\mathbf{v} - \dot{\mathbf{u}}(t))) + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \geq (f(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad (\mathbf{3}.2.6)$$

On the other hand

$$\dot{\beta} - k\Delta\beta + \partial\varphi_Y(\beta) \ni S(\varepsilon(\mathbf{u}),\beta) \text{ in } \Omega \times (0,T);$$

So there exists $h \in \partial\varphi_Y(\beta)$ such that

$$\dot{\beta} - k \triangle \beta + h = S(\varepsilon(\mathbf{u}),\beta),$$

Knowing that the element h satisfies

$$(h, \xi - \beta(t))_{L^2(\Omega)} \leq \partial\varphi_Y(\xi) - \partial\varphi_Y(\beta) \text{ for all } \xi \in Y$$

So

$$(h, \xi - \beta(t))_{L^2(\Omega)} \leq 0 \text{ for all } \xi \in Y$$

Then for all $\xi \in Y$ we have

$$(\dot{\beta} - k\Delta\beta + h, \xi - \beta(t))_{L^2(\Omega)} = (S(\varepsilon(\mathbf{u}),\beta), \xi - \beta(t))_{L^2(\Omega)},$$
$$\Leftrightarrow (\dot{\beta} - k\Delta\beta, \xi - \beta(t))_{L^2(\Omega)} + (h, \xi - \beta(t))_{L^2(\Omega)} = (S(\varepsilon(\mathbf{u}),\beta), \xi - \beta(t))_{L^2(\Omega)},$$

It comes that

$$(\dot{\beta} - k\Delta\beta, \xi - \beta(t))_{L^2(\Omega)} \geq (S(\varepsilon(\mathbf{u}),\beta), \xi - \beta(t))_{L^2(\Omega)}, \text{ pour tout } \xi \in Y$$

But

$$(\dot{\beta} - k\Delta\beta, \xi - \beta(t))_{L^2(\Omega)} = (\dot{\beta}, \xi - \beta(t))_{L^2(\Omega)} - k(\Delta\beta, \xi - \beta(t))_{L^2(\Omega)}, \text{ pour tout } \xi \in Y$$

Where

$$(\Delta\beta(t), \xi - \beta(t))_{L^2(\Omega)} = \int_\Omega \Delta\beta(t)(\xi - \beta(t))dx,$$

We integrate by part $\int_\Omega \Delta\beta(t)(\xi-\beta(t))dx$ we obtain

$$(\Delta\beta(t),\xi-\beta(t))_{L^2(\Omega)}=\int_\Gamma \frac{\partial\beta}{\partial x}(\xi-\beta(t))dx-\int_\Omega \nabla\beta(t)\nabla(\xi-\beta(t))dx,$$

But by Neumann's condition (**3**.1.7) $\frac{\partial\beta}{\partial x}=\frac{\partial\beta}{\partial\nu}\frac{\partial\nu}{\partial x}=0$ and the notation (**3**.1.17), implies tha

$$(\dot\beta(t)-k\Delta\beta,\xi-\beta(t))_{L^2(\Omega)}=(\dot\beta(t),\xi-\beta(t))_{L^2(\Omega)}-a(\beta(t),\xi-\beta(t)),$$

Finally

$$(\dot\beta(t),\xi-\beta(t))_{L^2(\Omega)}-a(\beta(t),\xi-\beta(t))\geq(S(\varepsilon(\mathbf{u}),\beta),\xi-\beta(t))_{L^2(\Omega)},\ \text{for all } \xi\in Y \quad (\mathbf{3}.2.4)$$

From (**3**.1.1), (**3**.1.8), (**3**.2.6) and (**3**.2.7), we obtain the variational formulation of the mechanical problem P^1.

Problem PV^1 Find the displacement field $\mathbf{u}:\ [0,T]\to V$, the stress tensor field $\sigma:\ [0,T]\to\mathcal{H}$, and the damage field $\beta:\ [0,T]\to H^1(\Omega)$ such that

$$\begin{aligned}
&\sigma(t)=\mathcal{A}\varepsilon(\dot{\mathbf{u}}(t))+\mathcal{F}(\varepsilon(\mathbf{u}(t)),\beta(t)) \text{ in } \Omega\times(0,T)\\
&\qquad(\sigma(t),\varepsilon(\mathbf{v}-\dot{\mathbf{u}}(t)))_{\mathcal{H}}+P(\mathbf{u}(t),(\mathbf{v}-\dot{\mathbf{u}}(t)))+j(\mathbf{u}(t),\mathbf{v})-j(\mathbf{u}(t),\dot{\mathbf{u}}(t))\\
&\quad\geq\ (f(t),\mathbf{v}-\dot{\mathbf{u}}(t))_V \text{ for all } \mathbf{v}\in V,\text{a.e.} t\in(0,T) && (\mathbf{3}.2.8)\\
&\beta(t)\in Y,\ \ (\dot\beta(t),\xi-\beta(t))_{L^2(\Omega)}+a(\beta(t),\xi-\beta(t))\\
&\quad\geq\ (S(\varepsilon(\mathbf{u}(t)),\beta(t)),\xi-\beta(t))_{L^2(\Omega)} \text{ for all } \xi\in Y,\ \text{a.e. } t\in(0,T) && (\mathbf{3}.2.9)\\
&\mathbf{u}(0)=\mathbf{u}_0,\ \ \beta(0)=\beta_0. && (\mathbf{3}.2.10)
\end{aligned}$$

3.3 Existence and uniqueness of the solution

In the study of the PV^1 problem we have the following result

Theorem 3.3.1 We assume that the hypotheses Hypotheses (**3**.1.9)-(**3**.1.16), are satisfied. then there exists a unique solutionf $\{\mathbf{u},\sigma,\beta\}$ of the problem PV^1, having the following regularity:

$$\mathbf{u}\in C^1(0,T;V), \quad (\mathbf{3}.3.1)$$

$$\beta\in W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)), \quad (\mathbf{3}.3.2)$$

The triple $\{\mathbf{u},\sigma,\beta\}$ satisfying (**3**.1.1) and (**3**.2.8)-(**3**.2.10) is called weak solution of the normal compliance problem P Where satisfies

$$\sigma\in C([0,T];\mathcal{H}_1), \quad (\mathbf{3}.3.3)$$

Indeed, it follows from (**3**.2.8) that $\operatorname{Div}\sigma + \mathbf{f}_0 = 0$ for all $t \in [0,T]$ and, consequently, the regularity (**3**.3.1) of $\mathbf{u}$, combined with (**3**.1.9)-(**3**.1.15), and the regularity of $\mathbf{f}_2$ in (**3**.1.14), implies (**3**.3.3).

The proof of *Theorem* **3.3.1** is done in several steps. It is based on parabolic variational inequatities with monotonic operators and fixed point arguments, but with a different choice of operators, see references [6] , [17] , [18] and [19] .

3.4 Numerical Analysis on viscoelasticity

In this section, we introduce a discrete numerical scheme for Problem PV. Throughout what follows, we assume that conditions (**3**.1.9)-(**3**.1.16) are satisfied. It then follows from *Theorem 3.3.1* that Problem PV admits a unique solution. More precisely, we are interested in solving Problem PV over a fixed time interval $[0,T]$, with $T > 0$ arbitrary but fixed.

3.4.1 Notations and Discrete Spaces

Let N be a positive integer. We define the time step size as

$$k = \frac{T}{N}$$

and consider a uniform temporal discretization :

$$t_n = nk, \quad 0 \leq n \leq N,$$

where N is a sufficiently large integer. For a continuous function $\varphi(t)$ taking values in a functional space, we write

$$\varphi_j = \varphi(t_j), \quad 0 \leq j \leq N.$$

For the spatial discretization, we consider a polygonal domain Ω. To discretize the integrals in Problem PV, we use the rectangle (midpoint) rule :

$$\int_{t_j}^{t_{j+1}} \varphi(s)ds = k\varphi_j.$$

Let H^h and B^h denote the finite element spaces of piecewise constant functions. The spaces H and $L^2(\Omega)3$ are approximated by H^h and B^h, respectively.

The spaces V and Y are approximated by the following finite element spaces :

$$V^h = \left\{ \mathbf{v}^h \in \left[C(\overline{\Omega})\right]^d \mid \mathbf{v}^h|_K \in \left[P_1(K)\right]^d, \forall K \in T^h, \mathbf{v}^h = 0 \text{ on } \overline{\Gamma}_1 \right\},$$

$$Y^h = \left\{ \xi^h \in Z_1^h : 0 \le \xi^h \le 1 \quad \text{in } \Omega \right\},$$

where T^h is a triangulation of the domain Ω, $P_1(K)$ is the space of polynomials of degree less than or equal to one on the element K, and h denotes the spatial discretization parameter defined by

$$h = \max_{K \in T^h} diam(K),$$

with

$$diam(K) = \max\{\| x - y \| : \; x, y \in K\}.$$

For every $\varphi \in H$, we denote by $P_{H^h}\varphi$ its finite element projection onto H^h, defined by :

$$(P_{H^h}\varphi, \psi_h)_H = (\varphi, \psi_h)_H, \forall \psi_h \in H^h.$$

It is convenient to introduce the velocity field $\mathbf{v}(t) = \dot{\mathbf{u}}(t)$, so that :

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s) ds, \qquad t \in [0, T].$$

It follows from *Theorem 3.3.1* that $\mathbf{v} \in C(0,T;V)$, and for all $t \in [0,T]$, we have :

$$\begin{aligned} &\sigma(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{F}(\varepsilon(\mathbf{u}(t)), \beta(t)) \;\; \text{in} \;\; \Omega \times (0,T) \\ &(\sigma(t), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} + P(\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) + j(\mathbf{u}(t), \mathbf{v}) \\ &-j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \ge (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \text{for all } \mathbf{v} \in V, \text{a.e.} t \in (0,T) \end{aligned} \tag{3.4.1}$$

We now consider the following approximate variational problem :

3.4.2 Discrete Problem

Problem PV^{hk}. Find a discrete velocity field $\mathbf{v}^{hk} = \{\mathbf{v}_n^{hk}\}_{n=0}^N \subset V^h$, a discrete stress field $\sigma^{hk} = \{\sigma_n^{hk}\}_{n=0}^N \subset \mathcal{H}^h$ and a discrete damage field $\beta^{hk} = \{\beta_n^{hk}\}_{n=0}^N \subset Y^h$ such that

$$\begin{aligned} &(\sigma_n^{hk}, \varepsilon(\mathbf{w}^h - \mathbf{v}_n^{hk}))_{\mathcal{H}} + P(\mathbf{u}_{n-1}^{hk}, \mathbf{w}^h - \mathbf{v}_n^{hk}) + j(\mathbf{u}_{n-1}^{hk}, \mathbf{w}^h) \\ &-j(\mathbf{u}_{n-1}^{hk}, \mathbf{v}_n^{hk}) \ge (\mathbf{f}_n, \mathbf{w}^h - \mathbf{v}_n^{hk})_V \;\; \text{for all } \mathbf{w}^h \in V^h, \end{aligned} \tag{3.4.2}$$

$$(\delta\beta_n^{hk}, \xi^h - \beta_n^{hk})_{L^2(\Omega)} + a(\beta_n^{hk}, \xi^h - \beta_n^{hk}) \ge (\phi(\varepsilon(\mathbf{u}_{n-1}^{hk}), \beta_{n-1}^{hk}), \xi^h - \beta_n^{hk})_{L^2(\Omega)} \;\; \forall \xi^h \in Y^h \tag{3.4.3}$$

$$\text{for } n = 1, ..., N, \text{ and } \quad \mathbf{u}_0^{hk} = \mathbf{u}_0^h, \;\; \beta_0^{hk} = \beta_0^h. \tag{3.4.4}$$

$$\sigma_0^{hk}(t) = P_{\mathcal{H}^h}\mathcal{A}\varepsilon(\mathbf{v}_0^h) + P_{\mathcal{H}^h}\mathcal{F}\varepsilon(\mathbf{v}_0^h, \beta_0^h)$$

Here $\mathbf{u}_0^h \in V^h$ and $\beta_0^h \in Y^h$ are approoriate approximations of $\mathbf{u}_0$ and β_0, and $\{\mathbf{u}_n^{hk}\}_{n=0}^N$and $\{\mathbf{v}_n^{hk}\}_{n=0}^N$ are related by

$$\mathbf{v}_n^{hk} = \delta \mathbf{u}_n^{hk} \quad \text{and} \quad \mathbf{u}_n^{hk} = \mathbf{u}_0^h + k \sum_{j=1}^{n} \mathbf{v}_j^{hk}.$$

We discover that Problem PV^{hk} admits a single solution using classic results of nonlinear variational equations (see [10]), which we summarize below.

Theorem 3.4.1 Assume that (**3**.1.9)-(**3**.1.16) hold. Then, Problem PV^{hk} has a unique solution $(\mathbf{u}^{hk}, \sigma^{hk}, \beta^{hk}) \subset V^h \times \mathcal{H}^h \times Y^h$. We now state the following existence and uniqueness result.

3.4.3 Numerical Approximation

The purpose of this section is to to estimate the numerical errors $\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V$, $\|\sigma_n - \sigma_n^{hk}\|_{\mathcal{H}}$and $\|\beta_n - \beta_n^{hk}\|_Y$. To this end, we make the following assumptions on the regularity of the solution $\{\mathbf{u}, \sigma, \beta\}$ for integer $\lambda \geq 1$:

$$\mathbf{u}_0 \in H^{1+\lambda}(\Omega)^d, \quad \sigma\nu \in C([0,T]; L^2(\Gamma)^d), \tag{3.4.5}$$

$$\mathbf{v} \in W^{1,1}(0,T;V) \cap C([0,T]; H^{1+\lambda}(\Omega)^d), \mathbf{v}_\tau \in C([0,T]; H^{1+\lambda}(\Gamma_3)^d) \tag{3.4.6}$$

$$\beta \in C([0,T]; H^{1+\lambda}(\Omega)) \cap H^2(0,T; L^2(\Omega)), \dot{\beta} \in L^2(0,T; H^{\lambda}(\Omega)). \tag{3.4.7}$$

It isnot difficult to see that $\dot{\beta}, \phi\left(\varepsilon(\mathbf{u}),\beta\right), \Delta\beta \in C([0,T]; L^2(\Omega))$.

a) From the assumption:

$$\beta \in H^2(0,T; L^2(\Omega)),$$

we use the Sobolev embedding:

$$\beta \in H^2(0,T; L^2(\Omega)) \subset C^1([0,T]; L^2(\Omega))$$

.

This implies:

β is once differentiable in time, its time derivative $\dot{\beta}$ is continuous with values in $L^2(\Omega)$.

Therefore,

$$\dot{\beta} \in C([0,T]; L^2(\Omega))$$

b) We are given that:

$$\begin{aligned} \mathbf{u} &\in C([0,T]; H^{1+\lambda}(\Omega)^d) \Rightarrow \varepsilon(\mathbf{u}) \in C([0,T]; L^2(\Omega)) \\ \beta &\in C([0,T]; H^{1+\lambda}(\Omega)) \Rightarrow \beta \in C([0,T]; L^2(\Omega)). \end{aligned}$$

If ϕ is a smooth nonlinear function (e.g., Lipschitz continuous in its arguments), then the composition $\phi\left(\varepsilon(\mathbf{u}),\beta\right)$ will also be continuous in time and lie in $L^2(\Omega)$ at each t.

So:

$$\phi\left(\varepsilon(\mathbf{u}),\beta\right) \in C([0,T]\,;L^2(\Omega))$$

c) Since

$$\beta \in C([0,T]\,;H^{1+\lambda}(\Omega)) \text{ with } \lambda \geq 1 \Rightarrow \beta \in C([0,T]\,;H^2(\Omega)).$$

, and the Laplacian is continuous from $H^2 \longrightarrow L^2$, then:

$$\Delta\beta \in C([0,T]\,;L^2(\Omega)).$$

Let's start by getting an error estimation on the displacement field. In order to have, we therefore replace (3.1.1) in (3.2.8).

$$\begin{array}{l}(\mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{F}(\varepsilon\left(\mathbf{u}(t)\right),\beta(t)),\varepsilon(\mathbf{v}-\dot{\mathbf{u}}(t)))_{\mathcal{H}} + P(\mathbf{u}(t),\mathbf{v}-\dot{\mathbf{u}}(t)) + j(\mathbf{u}(t),\mathbf{v}) \\ -j(\mathbf{u}(t),\dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t),\mathbf{v}-\dot{\mathbf{u}}(t))_V \quad \text{for all } \mathbf{v}\in V, \text{a.e.} t\in(0,T)\end{array} \qquad (\mathbf{3}.4.8)$$

We rewrite ($\mathbf{3}$.4.8) at time $t = t_n$ for $\mathbf{v} = \mathbf{v}_n^{hk}$ to find

$$\begin{array}{l}(\sigma_n, \varepsilon(\mathbf{v}_n^{hk} - \mathbf{v}_n))_{\mathcal{H}} + P(\mathbf{u}_n, \mathbf{v}_n^{hk} - \mathbf{v}_n) + j(\mathbf{u}_n, \mathbf{v}_n^{hk}) \\ -j(\mathbf{u}_n,\mathbf{v}_n) \geq (\mathbf{f}_n, \mathbf{v}_n^{hk} - \mathbf{v}_n)_V.\end{array} \qquad (\mathbf{3}.4.9)$$

We add this inequality to ($\mathbf{3}$.4.8) with $\mathbf{w}^h = \mathbf{w}_n^h \in V^h$. After a rearrangement, we obtain

$$(\sigma_n - \sigma_n^{hk}, \varepsilon(\mathbf{w}^h - \mathbf{v}_n^{hk}))_{\mathcal{H}} \leq R_n\left(\mathbf{w}^h, \mathbf{v}_n\right) + J\left(\mathbf{u}_n, \mathbf{u}_{n-1}^{hk}; \mathbf{v}_n^{hk}, \mathbf{w}^h\right) + I_n \qquad (\mathbf{3}.4.10)$$

then

$$\begin{array}{rl}\Leftrightarrow & \left(\mathcal{A}\varepsilon(\mathbf{v}_n) - \mathcal{A}\varepsilon(\mathbf{v}_n^{hk}), \varepsilon(\mathbf{v}_n) - \varepsilon(\mathbf{v}_n^{hk})\right)_{\mathcal{H}} \\ \leq & \left(\mathcal{A}\varepsilon(\mathbf{v}_n) - \mathcal{A}\varepsilon(\mathbf{v}_n^{hk}), \varepsilon(\mathbf{v}_n - \mathbf{w}^h)\right)_{\mathcal{H}} \\ & -\left(\mathcal{F}\left(\varepsilon(\mathbf{u}_n),\beta_n\right) - \mathcal{F}\left(\varepsilon(\mathbf{u}_{n-1}^{hk}),\beta_{n-1}^{hk}\right), \varepsilon\left(\left(\mathbf{v}_n - \mathbf{v}_n^{hk}\right) - \varepsilon(\mathbf{v}_n - \mathbf{w}^h)\right)_{\mathcal{H}} \\ & +R_n\left(\mathbf{w}^h,\mathbf{v}_n\right) + J\left(\mathbf{u}_n, \mathbf{u}_{n-1}^{hk}; \mathbf{v}_n^{hk}, \mathbf{w}^h\right) + I_n\end{array} \qquad (\mathbf{3}.4.11)$$

where

$$R_n\left(\mathbf{w}^h,\mathbf{v}_n\right) = (\sigma_n, \varepsilon\left(\mathbf{w}^h - \mathbf{v}_n\right))_{\mathcal{H}} + j(\mathbf{u}_n, \mathbf{w}^h) - j(\mathbf{u}_n, \mathbf{v}_n)$$

Using relations ($\mathbf{3}$.1.3), ($\mathbf{3}$.1.5)-($\mathbf{3}$.1.6) and the boundary condition $\mathbf{w}^h - \mathbf{v}_n = 0$ on Γ_1, we have

$$R_n\left(\mathbf{w}^h,\mathbf{v}_n\right) = \int_{\Gamma_3}\left((\sigma_n)_\tau . \varepsilon\left(\left(\mathbf{w}^h\right)_\tau - (\mathbf{v}_n)_\tau\right) + p_\nu\left((\mathbf{u}_n)_\nu\right)\left(\left\|\left(\mathbf{w}^h\right)_\tau\right\| - \|(\mathbf{v}_n)_\tau\|\right)\right) da$$

Thus,we obtain the estimate

$$\left|R_n\left(\mathbf{w}^h,\mathbf{v}_n\right)\right| \le c\left\|\left(\mathbf{w}^h\right)_\tau-(\mathbf{v}_n)_\tau\right\|_{L^2(\Gamma_3)^d}, \qquad (\mathbf{3}.4.12)$$

where the constant c depends on the solution.

The term $J\left(\mathbf{u}_n,\mathbf{u}^{hk}_{n-1};\mathbf{v}^{hk}_n,\mathbf{w}^h\right)$ is defined as

$$J\left(\mathbf{u}_n,\mathbf{u}^{hk}_{n-1};\mathbf{v}^{hk}_n,\mathbf{w}^h\right)=j(\mathbf{u}_n,\mathbf{v}^{hk}_n)-j(\mathbf{u}_n,\mathbf{w}^h_n)+j(\mathbf{u}^{hk}_{n-1},\mathbf{w}^h_n)-j(\mathbf{u}^{hk}_{n-1},\mathbf{v}^{hk}_n).$$

Using ($\mathbf{3}$.1.20) and ($\mathbf{3}$.1.13) we get

$$J\left(\mathbf{u}_n,\mathbf{u}^{hk}_{n-1};\mathbf{v}^{hk}_n,\mathbf{w}^h\right)\le c\left\|\mathbf{u}_n-\mathbf{u}^{hk}_{n-1}\right\|_V\left(\left\|\mathbf{v}_n-\mathbf{v}^{hk}_n\right\|_V+\left\|\mathbf{v}_n-\mathbf{w}^h\right\|_V\right). \qquad (\mathbf{3}.4.13)$$

And

$$\begin{aligned} I_n &= P(\mathbf{u}_n,\mathbf{v}^{hk}_n-\mathbf{w}^h)-P(\mathbf{u}^{hk}_n,\mathbf{v}^{hk}_n-\mathbf{w}^h)\\ I_n &= P(\mathbf{u}_n,\mathbf{v}^{hk}_n-\mathbf{u}_n)-P(\mathbf{u}^{hk}_n,\mathbf{v}^{hk}_n-\mathbf{u}_n)+P(\mathbf{u}_n,\mathbf{u}_n-\mathbf{w}^h)-P(\mathbf{u}^{hk}_n,\mathbf{u}_n-\mathbf{w}^h). \end{aligned}$$

Note that

$$P(\mathbf{u}_n,\mathbf{v}^{hk}_n-\mathbf{u}_n)-P(\mathbf{u}^{hk}_n,\mathbf{v}^{hk}_n-\mathbf{u}_n)=-\int_{\Gamma_3}\left[p_\nu\left((\mathbf{u}_n)_\nu\right)-p_\nu\left(\left(\mathbf{u}^{hk}_n\right)_\nu\right)\right]\left((\mathbf{u}_n)_\nu-\left(\mathbf{u}^{hk}_n\right)_\nu\right)da\le 0$$

Hence

$$\begin{aligned} I_n &\le P(\mathbf{u}_n,\mathbf{u}_n-\mathbf{w}^h)-P(\mathbf{u}^{hk}_n,\mathbf{u}_n-\mathbf{w}^h)=\int_{\Gamma_3}\left[p_\nu\left((\mathbf{u}_n)_\nu\right)-p_\nu\left(\left(\mathbf{u}^{hk}_n\right)_\nu\right)\right]\left((\mathbf{u}_n)_\nu-\left(\mathbf{w}^h\right)_\nu\right)da\\ &\le L_\nu\left\|(\mathbf{u}_n)_\nu-\left(\mathbf{u}^{hk}_n\right)_\nu\right\|_{L^2(\Gamma_3)}\left\|(\mathbf{u}_n)_\nu-\left(\mathbf{w}^h\right)_\nu\right\|_{L^2(\Gamma_3)}, \end{aligned}$$

and then

$$I_n\le c_0^2L_\nu\left\|\mathbf{u}_n-\mathbf{u}^{hk}_n\right\|_V\left\|\mathbf{u}_n-\mathbf{w}^h\right\|_V. \qquad (\mathbf{3}.4.14)$$

By using assumption ($\mathbf{3}$.1.9) and ($\mathbf{3}$.1.10), we find from ($\mathbf{3}$.4.11) and ($\mathbf{3}$.4.13)- ($\mathbf{3}$.4.14) that

$$\begin{aligned} m_{\mathcal{A}}\left\|\mathbf{v}_n-\mathbf{v}^{hk}_n\right\|_V^2 \le\ & c(\left\|\mathbf{v}_n-\mathbf{v}^{hk}_n\right\|_V\left\|\mathbf{v}_n-\mathbf{w}^h\right\|_V\\ &+\left(\left\|\mathbf{u}_n-\mathbf{u}^{hk}_{n-1}\right\|_V+\left\|\beta_n-\beta^{hk}_{n-1}\right\|_{L^2(\Gamma_3)}\right)\left(\left\|\mathbf{v}_n-\mathbf{v}^{hk}_n\right\|_V+\left\|\mathbf{v}_n-\mathbf{w}^h\right\|_V\right)\\ &+\left\|\mathbf{u}_n-\mathbf{u}^{hk}_n\right\|_V\left\|\mathbf{u}_n-\mathbf{w}^h\right\|_V)+\left|R_n\left(\mathbf{w}^h,\mathbf{v}_n\right)\right|. \qquad (\mathbf{3}.4.15) \end{aligned}$$

From this,and using the inequality $ab\le\left(a^2+b^2\right)/2$ we obtain that

$$\left\|\mathbf{v}_n-\mathbf{v}^{hk}_n\right\|_V^2\le c\left(\left\|\mathbf{v}_n-\mathbf{w}^h\right\|_V^2+\left\|\mathbf{u}_n-\mathbf{u}^{hk}_{n-1}\right\|_V^2+\left\|\beta_n-\beta^{hk}_{n-1}\right\|_{L^2(\Gamma_3)}^2\right)+\left|R_n\left(\mathbf{w}^h,\mathbf{v}_n\right)\right|. \qquad (\mathbf{3}.4.16)$$

We estimate the term $\left\|\mathbf{u}_n - \mathbf{u}_{n-1}^{hk}\right\|_V$ .We know that

$W^{1,1}(0,T;V) \subset C([0,T];V)$ and $\|\mathbf{v}\|_{C([0,T];V)} \leq c\,\|\mathbf{v}\|_{W^{1,1}(0,T;V)}$ $\forall \mathbf{v} \in W^{1,1}(0,T;V)$.

Since

$$\begin{aligned}
\mathbf{u}_n - \mathbf{u}_{n-1}^{hk} &= \mathbf{u}_0 + \int_0^{t_n} \mathbf{v}(s)\,ds - \mathbf{u}_0^h - k\sum_{j=1}^{n-1} \mathbf{v}_j^{hk} \\
&= k\sum_{j=1}^{n-1}\left(\mathbf{v}_j - \mathbf{v}_j^{hk}\right) + \mathbf{u}_0 - \mathbf{u}_0^h + \sum_{j=1}^{n-1}\left(\int_{t_{j-1}}^{t_j} \mathbf{v}(s)\,ds - k\mathbf{v}_j\right) + \int_{t_{n-1}}^{t_n} \mathbf{v}(s)\,ds,
\end{aligned}$$

we have

$$\begin{aligned}
\left\|\sum_{j=1}^{n-1}\left(k\mathbf{v}_j - \int_{t_{j-1}}^{t_j} \mathbf{v}(s)\,ds\right)\right\|_V &= \left\|\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} (\mathbf{v}_j - \mathbf{v}(s))\,ds\right\|_V \\
&= \left\|\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} (\mathbf{v}_j - \mathbf{v}(s))\,ds\right\|_V \\
&\leq \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\int_s^{t_j} \|\dot{\mathbf{v}}(\tau)\|_V\,d\tau ds \leq ck\,\|\dot{\mathbf{v}}\|_{L^1(0,T;V)}
\end{aligned}$$

and

$$\left\|\int_{t_{n-1}}^{t_n} \mathbf{v}(s)\,ds\right\|_V^2 \leq \left(\int_{t_{n-1}}^{t_n} \mathbf{v}(s)\,ds\right)^2 \leq k^2\,\|\dot{\mathbf{v}}\|_{C([0,T];V)},$$

we have

$$\left\|\mathbf{u}_n - \mathbf{u}_{n-1}^{hk}\right\|_V \leq k\sum_{j=1}^{n-1}\left\|\mathbf{v}_j - \mathbf{v}_j^{hk}\right\|_V + \left\|\mathbf{u}_0 - \mathbf{u}_0^h\right\|_V + ck\,\|\mathbf{v}\|_{W^1(0,T;V)}.$$

Now

$$\left(k\sum_{j=1}^{n-1}\left\|\mathbf{v}_j - \mathbf{v}_j^{hk}\right\|_V\right)^2 \leq ck\sum_{j=1}^{n-1}\left\|\mathbf{v}_j - \mathbf{v}_j^{hk}\right\|_V^2,$$

thus

$$\left\|\mathbf{u}_n - \mathbf{u}_{n-1}^{hk}\right\|_V^2 \leq c\left[k\sum_{j=1}^{n-1}\left\|\mathbf{v}_j - \mathbf{v}_j^{hk}\right\|_V^2 + \left\|\mathbf{u}_0 - \mathbf{u}_0^h\right\|_V^2 + k^2\,\|\mathbf{v}\|_{W^1(0,T;V)}^2\right]. \quad (\mathbf{3}.4.17)$$

Similarly, we have

$$\left\|\mathbf{u}_n - \mathbf{u}_{n-1}^{hk}\right\|_V^2 \leq c\left[k\sum_{j=1}^{n-1}\left\|\mathbf{v}_j - \mathbf{v}_j^{hk}\right\|_V^2 + \left\|\mathbf{u}_0 - \mathbf{u}_0^h\right\|_V^2\right]. \quad (\mathbf{3}.4.18)$$

If folloows from (**3**.4.7) that $\dot{\beta} \in C([0,T]\,;L^1(\Omega))$, therefore

$$\left\|\beta_n - \beta_{n-1}^{hk}\right\|_{L^2(\Gamma_3)} \leq \left\|\beta_{n-1} - \beta_{n-1}^{hk}\right\|_{L^2(\Gamma_3)} + k\left\|\dot{\beta}\right\|_{C([0,T],L^1(\Omega))}. \qquad (\mathbf{3}.4.19)$$

We combine (**3**.4.16)-(**3**.4.19)and obtain

$$\begin{aligned}\left\|\mathbf{v}_n - \mathbf{v}_n^{hk}\right\|_V^2 \;\leq\; & c(\left\|\mathbf{v}_n - \mathbf{w}^h\right\|_V^2 + \left\|\mathbf{u}_0 - \mathbf{u}_0^h\right\|_V^2 + k^2\left\|\mathbf{v}\right\|^2_{W^{1,1}(0,T:V)} \\ & + k\left\|\dot{\beta}\right\|^2_{C([0,T],L^1(\Omega))} + \left|R_n\left(\mathbf{w}^h, \mathbf{v}_n\right)\right| \\ & + k\sum_{j=1}^{n-1}\left\|\mathbf{v}_j - \mathbf{v}_j^{hk}\right\|_V^2 + \left\|\beta_{n-1} - \beta_{n-1}^{hk}\right\|_{L^2(\Gamma_3)}) \qquad (\mathbf{3}.4.20)\end{aligned}$$

We turn to estimate $\left\|\beta_n - \beta_n^{hk}\right\|_{L^2(\Gamma_3)}$. We choosev $\xi = \beta_n^{hk}$ in (**3**.2.9) at $t = t_n$:

$$(\dot{\beta}_n, \beta_n^{hk} - \beta_n)_{L^2(\Omega)} + a(\beta_n, \beta_n^{hk} - \beta_n) \geq (\phi(\varepsilon(\mathbf{u}_n),\beta_n), \beta_n^{hk} - \beta_n)_{L^2(\Omega)}.$$

Adding this inequality to , with $\xi^h = \xi_n^h \in Y^h$, we finid

$$\begin{aligned}& \left\langle\delta(\beta_n - \beta_n^{hk}), \beta_n - \beta_n^{hk}\right\rangle_{L^2(\Omega)} + a(\beta_n - \beta_n^{hk}, \beta_n - \beta_n^{hk}) \\ \leq\; & \left\langle\delta\beta_n - \dot{\beta}_n, \beta_n - \beta_n^{hk}\right\rangle_{L^2(\Omega)} + \left\langle\delta(\beta_n - \beta_n^{hk}), \beta_n - \xi_n^h\right\rangle_{L^2(\Omega)} \\ & + a(\beta_n - \beta_n^{kk}, \beta_n - \xi_n^h) - \left\langle\delta\beta_n, \beta_n - \xi_n^h\right\rangle_{L^2(\Omega)} - a(\beta_n, \beta_n - \xi_n^h) \\ & + \left\langle\phi(\varepsilon(\mathbf{u}_n),\beta_n), \beta_n - \xi_n^h)\right\rangle_{L^2(\Omega)}. \qquad (\mathbf{3}.4.21)\end{aligned}$$

we estimate each term on the right-hand side. For he term

$$\left\langle\delta(\beta_n - \beta_n^{hk}), \beta_n - \beta_n^{hk}\right\rangle_{L^2(\Omega)} = \frac{1}{k}(\left\|\beta_n - \beta_n^{hk}\right\|^2_{L^2(\Omega)} - \left\langle\beta_n - \beta_n^{hk}), \beta_{n-1} - \beta_{n-1}^{hk}\right\rangle_{L^2(\Omega)}),$$

we have

$$\left\langle\delta(\beta_n - \beta_n^{hk}), \beta_n - \beta_n^{hk}\right\rangle_{L^2(\Omega)} \geq \frac{1}{2k}(\left\|\beta_n - \beta_n^{hk}\right\|^2_{L^2(\Omega)} - \left\|\beta_{n-1} - \beta_{n-1}^{hk}\right\|^2_{L^2(\Omega)}). \qquad (\mathbf{3}.4.22)$$

We use (**3**.4.22), replace n by j in (**3**.4.21) and the sum over $j = 1, ..., n$ to obtain

$$\begin{aligned}
& \left\|\beta_n - \beta_n^{hk}\right\|_{L^2(\Omega)}^2 - \left\|\beta_0 - \beta_0^{hk}\right\|_{L^2(\Omega)}^2 + \sum_{j=1}^{n} a(\beta_j - \beta_j^{kk}, \beta_j - \beta_j^{kk}) \\
\leq\ & k \sum_{j=1}^{n} \left\langle \delta\beta_j - \dot{\beta}_j, \beta_j - \beta_j^{hk} \right\rangle_{L^2(\Omega)} + \left\langle \beta_n - \beta_n^{hk}, \beta_n - \xi_n^h \right\rangle_{L^2(\Omega)} \\
& - \sum_{j=1}^{n-1} \left\langle \beta_j - \beta_j^{kk}, \left(\beta_{j+1} - \xi_{j+1}^h\right) - \left(\beta_j - \xi_j^h\right) \right\rangle_{L^2(\Omega)} - \left\langle \beta_0 - \beta_0^{hk}, \beta_1 - \xi_1^h \right\rangle_{L^2(\Omega)} \\
& + k \sum_{j=1}^{n} a(\beta_j - \beta_j^{kk}, \beta_j - \xi_j^h) \\
& + \sum_{j=1}^{n} \left[\left\langle \phi(\varepsilon(\mathbf{u}_j), \beta_j), \beta_j - \xi_j^h) \right\rangle_{L^2(\Omega)} - \left\langle \delta\beta_j, \beta_j - \xi_j^h \right\rangle_{L^2(\Omega)} - a\left(\beta_j, \beta_j - \xi_j^h\right) \right] \\
& + \sum_{j=1}^{n} \left\langle \phi(\varepsilon(\mathbf{u}_j), \beta_j) - \phi(\varepsilon(\mathbf{u}_{j-1}^{hk}), \beta_{j-1}^{hk}), \xi_j^h - \beta_j^{kk}) \right\rangle_{L^2(\Omega)}.
\end{aligned}$$

Then

$$\begin{aligned}
& \left\|\beta_n - \beta_n^{hk}\right\|_{L^2(\Omega)}^2 + \sum_{j=1}^{n} \left\|\nabla\left(\beta_j - \beta_j^{kk}\right)\right\|_{L^2(\Omega)}^2 \\
\leq\ & c \Bigg(\left\|\beta_0 - \beta_0^{hk}\right\|_{L^2(\Omega)}^2 + k \sum_{j=1}^{n} \left\|\delta\beta_j - \dot{\beta}_j\right\|_{L^2(\Omega)} \left\|\beta_j - \beta_j^{hk}\right\|_{L^2(\Omega)} \\
& + \left\|\beta_n - \beta_n^{hk}\right\|_{L^2(\Omega)} \left\|\beta_n - \xi_n^h\right\|_{L^2(\Omega)} + \left\|\beta_0 - \beta_0^{hk}\right\|_{L^2(\Omega)} \left\|\beta_1 - \xi_1^h\right\|_{L^2(\Omega)} \\
& + k \sum_{j=1}^{n-1} \left\|\beta_j - \beta_j^{hk}\right\|_{L^2(\Omega)} \left\|\left(\beta_{j+1} - \xi_{j+1}^h\right) - \left(\beta_j - \xi_j^h\right)\right\|_{L^2(\Omega)} \\
& + k \sum_{j=1}^{n} \left\|\nabla\left(\beta_j - \beta_j^{kk}\right)\right\|_{L^2(\Omega)} \left\|\nabla\left(\beta_j - \xi_j^h\right)\right\|_{L^2(\Omega)} \\
& + \sum_{j=1}^{n} \left\|\phi(\varepsilon(\mathbf{u}_j), \beta_j) - \delta\beta_j + \kappa\Delta\beta_j\right\|_{L^2(\Omega)L^2(\Omega)} \left\|\beta_j - \xi_j^h\right\|_{L^2(\Omega)} \\
& + k \sum_{j=1}^{n} \left(\left\|\mathbf{u}_j - \mathbf{u}_{j-1}^{hk}\right\|_{L^2(\Omega)} + \left\|\beta_j - \beta_j^{hk}\right\|_{L^2(\Omega)} \right) \left(\left\|\beta_j - \beta_j^{hk}\right\|_{L^2(\Omega)} + \left\|\beta_j - \xi_j^h\right\|_{L^2(\Omega)} \right) \Bigg).
\end{aligned}$$

Using (**3**.4.17) and (**3**.4.19) we find

$$
\begin{aligned}
&\left\|\beta_n-\beta_n^{hk}\right\|_{L^2(\Omega)}^2+k\sum_{j=1}^{n}\left\|\nabla\left(\beta_j-\beta_j^{kk}\right)\right\|_{L^2(\Omega)}^2\\
\leq\ & c\Bigg(\left\|\beta_0-\beta_0^{hk}\right\|_{L^2(\Omega)}^2+\left\|\beta_1-\xi_1^h\right\|_{L^2(\Omega)}^2+\left\|\mathbf{u}_0-\mathbf{u}_0^h\right\|_V^2\\
&+k^2\left(\|\mathbf{v}\|_{W^{1,1}(0,T:V)}^2+\left\|\dot\beta\right\|_{C([0,T],L^1(\Omega))}^2\right)+k\sum_{j=1}^{n}\left\|\delta\beta_j-\dot\beta_j\right\|_{L^2(\Omega)}^2\\
&+\left\|\beta_n-\xi_n^h\right\|_{L^2(\Omega)}^2+k\sum_{j=1}^{n}\left\|\phi(\varepsilon(\mathbf{u}_j),\beta_j)-\delta\beta_j+\kappa\Delta\beta_j\right\|_{L^2(\Omega)L^2(\Omega)}\left\|\beta_j-\xi_j^h\right\|_{L^2(\Omega)}\\
&+k\sum_{j=1}^{n}\left\|\nabla\left(\beta_j-\xi_j^h\right)\right\|_{L^2(\Omega)}^2+k^{-1}\sum_{j=1}^{n}\left\|\left(\beta_{j+1}-\xi_{j+1}^h\right)-\left(\beta_j-\xi_j^h\right)\right\|_{L^2(\Omega)}^2\\
&+k\sum_{j=1}^{n}\left\|\beta_j-\beta_j^{kk}\right\|_{L^2(\Omega)}^2+k\sum_{j=1}^{n-1}\left\|\mathbf{v}_j-\mathbf{v}_j^{kk}\right\|_V^2\Bigg).
\end{aligned}
\tag{3.4.23}
$$

Now, (**3**.4.20) and (**3**.4.23) imply

$$
\begin{aligned}
&\left\|\mathbf{v}_n-\mathbf{v}_n^{hk}\right\|_V^2+\left\|\beta_n-\beta_n^{hk}\right\|_{L^2(\Omega)}^2+k\sum_{j=1}^{n}\left\|\nabla\left(\beta_j-\beta_j^{hk}\right)\right\|_{L^2(\Omega)}^2\\
\leq\ & c\Bigg(\left\|\beta_0-\beta_0^{hk}\right\|_{L^2(\Omega)}^2+\left\|\beta_1-\xi_1^h\right\|_{L^2(\Omega)}^2+\left\|\mathbf{u}_0-\mathbf{u}_0^h\right\|_V^2\\
&+k^2\left(\|\mathbf{v}\|_{W^{1,1}(0,T:V)}^2+\left\|\dot\beta\right\|_{C([0,T],L^1(\Omega))}^2\right)+k\sum_{j=1}^{n}\left\|\delta\beta_j-\dot\beta_j\right\|_{L^2(\Omega)}^2\\
&+\left\|\mathbf{v}_n-\mathbf{w}_n^h\right\|_V^2+\left\|\beta_n-\xi_n^h\right\|_{L^2(\Omega)}^2\\
&+k\sum_{j=1}^{n}\left\|\phi(\varepsilon(\mathbf{u}_j),\beta_j)-\delta\beta_j+\kappa\Delta\beta_j\right\|_{L^2(\Omega)}\left\|\beta_j-\xi_j^h\right\|_{L^2(\Omega)}\\
&+k\sum_{j=1}^{n}\left\|\nabla\left(\beta_j-\xi_j^h\right)\right\|_{L^2(\Omega)}^2+k^{-1}\sum_{j=1}^{n-1}\left\|\left(\beta_{j+1}-\xi_{j+1}^h\right)-\left(\beta_j-\xi_j^h\right)\right\|_{L^2(\Omega)}^2\\
&+\left|R_n\left(\mathbf{w}_n^h,\mathbf{v}_n\right)\right|+k\sum_{j=1}^{n}\left\|\beta_j-\beta_j^{hk}\right\|_{L^2(\Omega)}^2+k\sum_{j=1}^{n-1}\left\|\mathbf{v}_j-\mathbf{v}_j^{hk}\right\|_V^2\Bigg).
\end{aligned}
\tag{3.4.24}
$$

Applying *Proposition* **2.2.1.** to (**3**.4.24) yields

$$\max_n \Big(\underbrace{\left\| \mathbf{v}_n - \mathbf{v}_n^{hk} \right\|_V^2 + \left\| \beta_n - \beta_n^{hk} \right\|_{L^2(\Omega)}^2}_{\varphi_n} \Big) + k \sum_{j=1}^{n} \left\| \nabla \left(\beta_j - \beta_j^{hk} \right) \right\|_{L^2(\Omega)}^2$$

$$\leq c \Bigg(\left\| \beta_0 - \beta_0^{hk} \right\|_{L^2(\Omega)}^2 + \left\| \beta_1 - \xi_1^h \right\|_{L^2(\Omega)}^2 + \left\| \mathbf{u}_0 - \mathbf{u}_0^h \right\|_V^2 + + k^2 \left(\|\mathbf{v}\|_{W^{1,1}(0,T:V)}^2 + \left\| \dot{\beta} \right\|_{C([0,T],L^1(\Omega))}^2 \right)$$

$$+ k \sum_{j=1}^{n} \left\| \delta\beta_j - \dot{\beta}_j \right\|_{L^2(\Omega)}^2 + \max_n \left[\left\| \mathbf{v}_n - \mathbf{w}_n^h \right\|_V^2 + \left| R_n \left(\mathbf{w}_n^h, \mathbf{v}_n \right) \right| \right]$$

$$+ \max_n \Bigg[\left\| \left(\beta_n - \xi_n^h \right) \right\|_{L^2(\Omega)}^2 + k \sum_{j=1}^{n} \left\| \nabla \left(\beta_j - \xi_j^h \right) \right\|_{L^2(\Omega)}^2 + k^{-1} \sum_{j=1}^{n-1} \left\| \left(\beta_{j+1} - \xi_{j+1}^h \right) - \left(\beta_j - \xi_j^h \right) \right\|_{L^2(\Omega)}^2$$

$$+ k \sum_{j=1}^{n} \left\| \phi(\varepsilon(\mathbf{u}_j),\beta_j) - \delta\beta_j + \kappa\Delta\beta_j \right\|_{L^2(\Omega)} \left\| \beta_j - \xi_j^h \right\|_{L^2(\Omega)} \Bigg] \Bigg) \qquad (\mathbf{3}.4.2$$

for each $\mathbf{w}_n^h \in V^h, \xi_j^h \in Y^h, n = 0, 1, ..., N$.

The finite element method is employed to construct the discrete spaces V^h and Y^h. To do this, a finite element space is first introduced to approximate the Sobolev space $H^1(\Omega)$, assuming for simplicity that Ω is a polygon. A regular finite element mesh $\mathcal{T}^h$ is defined over Ω, such that any edge lying on the boundary of an element fully belongs to one of the designated boundary parts Γ_1, Γ_2, or Γ_3. The symbol h refers to the maximal diameter of the mesh elements.

A finite element space $X^h \subset H^1(\Omega)$ is defined, consisting of piecewise polynomials of degree less than or equal to λ, based on the mesh $\mathcal{T}^h$. The discrete spaces are then defined as :

$$V^h = (X^h)^d \cap V, \qquad Y^h = X^h \cap Y.$$

It is noted that the polynomial degrees in V^h and Y^h may differ, and the construction can easily be extended to such cases.

Next, finite element interpolation operators Π^h are introduced :

— If $\mathbf{v}(t) \in C(\overline{\Omega})$, then $\Pi^h \mathbf{v}(t)$ denotes the *standard finite element interpolant* (see [3]).

— If $\mathbf{v}(t) \in C(\Omega)$, then $\Pi^h \mathbf{v}(t)$ denotes the *Clément interpolant* (see [4]).

The same notation Π^h is used for interpolating $C(t)$ on Γ_3 and for interpolating $\eta(t)$ onto Y^h. It can be verified that if $\eta(t) \in Y$, then $\Pi^h \eta(t) \in Y^h$, and

$$\frac{d}{dt} \Pi^h \eta(t) = \Pi^h \dot{\eta}(t).$$

Under the regularity assumptions (**3**.4.5)–(**3**.4.11), interpolation error estimates are available.

$$\left\|\mathbf{u}_0 - \Pi^h \mathbf{u}_0\right\|_V \leq ch^\lambda \left|\mathbf{u}_0\right|_{H^{\lambda+1}(\Omega)^d}, \quad (\mathbf{3}.4.26)$$

$$\left\|\beta_0 - \Pi^h \beta_0\right\|_{L^2(\Omega)} \leq ch^\lambda \left|\beta_0\right|_{H^\lambda(\Omega)} \quad (\mathbf{3}.4.27)$$

and for all $t \in [0, T]$,

$$\left\|\mathbf{v}(t) - \Pi^h \mathbf{v}(t)\right\|_V \leq ch^\lambda \left|\mathbf{v}(t)\right|_{H^{\lambda+1}(\Omega)^d}, \quad (\mathbf{3}.4.28)$$

$$\left\|\mathbf{v}_\tau(t) - \Pi^h \mathbf{v}_\tau(t)\right\|_{L^2(\Gamma_3)^d} \leq ch^{\lambda+1} \left|\mathbf{v}_\tau(t)\right|_{H^{\lambda+1}(\Omega)^d}, \quad (\mathbf{3}.4.29)$$

$$\left\|\dot{\beta}(t) - \Pi^h \dot{\beta}(t)\right\|_{L^2(\Omega)} \leq ch^\lambda \left|\dot{\beta}(t)\right|_{H^\lambda(\Omega)}, \quad (\mathbf{3}.4.30)$$

$$\left\|\beta(t) - \Pi^h \beta(t)\right\|_{L^2(\Omega)} \leq ch^{\lambda+1} \left|\beta(t)\right|_{H^{\lambda+1}(\Omega)}, \quad (\mathbf{3}.4.31)$$

$$\left\|\beta(t) - \Pi^h \beta(t)\right\|_{H^1(\Omega)} \leq ch^\lambda \left|\beta(t)\right|_{H^{\lambda+1}(\Omega)}. \quad (\mathbf{3}.4.32)$$

Now, we choose the initial values to be

$$\mathbf{u}_0^h = \Pi^h \mathbf{u}_0, \quad \beta_0^h = \Pi^h \beta_0 \quad (\mathbf{3}.4.33)$$

Next, we choose $\mathbf{w}_n^h = \Pi^h \mathbf{v}_n$, $\xi_n^h = \Pi^h \beta_n$, $n = 1, ..., N$ in (**3**.4.6). We use the regularity of solutions (**3**.4.5) - (**3**.4.7) to estimate terms in inequality (**3**.4.25). We write

$$\delta\beta_j - \dot{\beta}_j = \frac{1}{k}\int_{t_{j-1}}^{t_j} \left(\dot{\beta}(t) - \dot{\beta}(t_j)\right) dt = \frac{1}{k}\int_{t_{j-1}}^{t_j}\int_{t_j}^{t} \ddot{\beta}(s)\, ds dt.$$

Then , it is easy to see that

$$+k\sum_{j=1}^{n} \left\|\delta\beta_j - \dot{\beta}_j\right\|_{L^2(\Omega)}^2 \leq k^2 \left\|\ddot{\beta}\right\|_{L^2(0,T;L^2(\Omega))}^2.$$

With $\xi_j^h = \Pi^h \beta(t_j)$, we have

$$\left(\beta_{j+1} - \xi_{j+1}^h\right) - \left(\beta_j - \xi_j^h\right) = \left(\beta_{j+1} - \beta_j\right) - \Pi^h\left(\beta_{j+1} - \beta_j\right) = k\left(\delta\beta_{j+1} - \Pi^h \delta\beta_{j+1}\right).$$

Now, $\delta\beta_{j+1} = \frac{1}{k}\int_{t_j}^{t_{j+1}} \dot{\beta}(t)\, dt$, and so

$$\left\|\delta\beta_{j+1}\right\|_{H^\lambda(\Omega)}^2 \leq \frac{1}{k}\int_{t_j}^{t_{j+1}} \left\|\dot{\beta}(t)\right\|_{H^\lambda(\Omega)}^2 dt.$$

From the error estimate (**3**.4.30) we infer that

$$\left\|\left(\beta_{j+1} - \xi_{j+1}^h\right) - \left(\beta_j - \xi_j^h\right)\right\|_{L^2(\Omega)}^2 \leq ckh^{2\lambda}\int_{t_j}^{t_{j+1}} \left\|\dot{\beta}(t)\right\|_{H^\lambda(\Omega)}^2 dt.$$

Therefore

$$k^{-1}\sum_{j=1}^{n-1}\left\|\left(\beta_{j+1}-\xi_{j+1}^{h}\right)-\left(\beta_{j}-\xi_{j}^{h}\right)\right\|_{L^2(\Omega)}^2\leq ch^{2\lambda}\left\|\dot{\beta}\left(t\right)\right\|_{L^2(0,T;H^{\lambda}(\Omega))}^2.$$

The remaining terms on the right-hand side of equation (**3**.4.25) can be directly estimated using the interpolation error bounds given in (**3**.4.26)-(**3**.4.32).

Finally, by combining all the estimates above, we derive the following error estimate from equation (**3**.4.25):

$$\max_n\left(\left\|\mathbf{v}_n-\mathbf{v}_n^{hk}\right\|_V^2+\left\|\beta_n-\beta_n^{hk}\right\|_{L^2(\Omega)}^2\right)+k\sum_{j=1}^{n}\left\|\nabla\left(\beta_j-\beta_j^{hk}\right)\right\|_{L^2(\Omega)}^2\leq c\left(k^2+h^{\min(2\lambda,\lambda+1)}\right),$$

wich, using (**3**.4.18), can be replaced by

$$\begin{aligned}&\max_n\left(\left\|\mathbf{u}_n-\mathbf{u}_n^{hk}\right\|_V^2+\left\|\delta\mathbf{u}_n-\delta\mathbf{u}_n^{hk}\right\|_V^2+\left\|\beta_n-\beta_n^{hk}\right\|_{L^2(\Omega)}^2\right)+k\sum_{j=1}^{n}\left\|\nabla\left(\beta_j-\beta_j^{hk}\right)\right\|_{L^2(\Omega)}^2\\ \leq\ & c\left(k^2+h^{\min(2\lambda,\lambda+1)}\right).\end{aligned}$$

Then, we have the following result for the discrete scheme PV^{hk}.

Theorem 3.4.2. The approximate problem PV^{hk} has aunique solution. Undr assumptions (**3**.4.5)-(**3**.4.7) the following error estmate holds:

$$\begin{aligned}&\max_n\left(\left\|\mathbf{u}_n-\mathbf{u}_n^{hk}\right\|_V+\left\|\delta\mathbf{u}_n-\delta\mathbf{u}_n^{hk}\right\|_V+\left\|\beta_n-\beta_n^{hk}\right\|_{L^2(\Omega)}\right)+k\left(\sum_{j=1}^{n}\left\|\nabla\left(\beta_j-\beta_j^{hk}\right)\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}\\ \leq\ & c\left(k+h^{\{\min(2\lambda,(\lambda+1))\}/2}\right).\end{aligned}$$

The constant c depends on norms $\|\mathbf{u}_0\|_{H^{\lambda+1}(\Omega)^d}$, $\left\|\phi(\varepsilon(\mathbf{u}),\beta)-\dot{\beta}+\kappa\Delta\beta\right\|_{C([0,T];L^2(\Omega))}$, $\|\mathbf{v}\|_{W^{1,1}(0,T;V)}$, $\|\mathbf{v}\|_{C([0,T];H^{\lambda+1}(\Omega)^d)}$, $\|\mathbf{v}\|_{C([0,T];H^{\lambda+1}(\Gamma_3))}$, $\|\beta\|_{C([0,T];H^{\lambda+1}(\Omega))}$, $\|\beta\|_{H^2(0,T;L^2(\Omega))}$, and $\left\|\dot{\beta}\right\|_{L^2(0,T;H^{\lambda}(\Omega))}$.

In particlar, when $\lambda=1$ and (**3**.4.5)-(**3**.4.7) holdwe have the optimal- order error estimate

$$\begin{aligned}&\max_n\left(\left\|\mathbf{u}_n-\mathbf{u}_n^{hk}\right\|_V+\left\|\delta\mathbf{u}_n-\delta\mathbf{u}_n^{hk}\right\|_V+\left\|\beta_n-\beta_n^{hk}\right\|_{L^2(\Omega)}\right)+k\left(\sum_{j=1}^{n}\left\|\nabla\left(\beta_j-\beta_j^{hk}\right)\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}\\ \leq\ & c\left(k+h\right).\end{aligned}$$

Conclusion

In this work, we have conducted a comprehensive variational and numerical analysis of mechanical problems governed by contact conditions at their physical or geometrical limits. The study emphasized the formulation of these problems as variational inequalities or hemivariational inequalities, capturing the nonlinearity and nonsmoothness inherent in contact and friction phenomena. We established existence and uniqueness results under appropriate assumptions, using tools from functional analysis, convex analysis, and monotone operator theory.

On the numerical side, we developed finite element approximations suited for the nonsmooth nature of the problem, and provided a rigorous analysis of convergence and error estimates. Special attention was given to the discretization of contact and friction conditions, including penalization, regularization, and augmented Lagrangian techniques. The proposed methods were validated by numerical experiments, which confirmed the theoretical predictions and demonstrated the effectiveness of the approach in capturing complex contact behaviors such as sticking, sliding, and separation.

This work not only contributes to the mathematical understanding of contact problems but also provides reliable numerical tools for their simulation in engineering applications. Future directions include the extension to dynamic and thermomechanical contact problems, the incorporation of damage and wear, and the development of efficient algorithms for large-scale computations.

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Abstract

the variational and numerical study of some contact problems with or without friction, between a deformable body and a foundation. Here we consider nonlinear laws of behavior for viscoplastic materials. For these problems we obtain variational formulations followed by existence and uniqueness results of weak solutions. The techniques employed are based on the theory of monotone operators followed by a version of the Cauchy-Lipschitz theorem and Banachís fixed point arguments. Finally, we propose a numerical approximation of the purely mechanical problem using discrete schemes. For these schemes, we obtain error estimation results.

Key words : Viscoelasticity , damage, , normal compliance, fixed point, Approximation, Finite Difference, Finite elements.

Résumé

Cette étude porte sur l'analyse variationnelle et numérique de certains problèmes de contact, avec ou sans frottement, entre un corps déformable et une fondation. Nous considérons ici des lois de comportement non linéaires pour des matériaux visco- élastiques. Pour ces problèmes, nous établissons des formulations variationnelles, suivies de résultats d'existence et d'unicité de solutions faibles. Les techniques employées s'appuient sur la théorie des opérateurs monotones, ainsi que sur une version du théorème de Cauchy-Lipschitz et des arguments du point fixe de Banach. Enfin, nous proposons une approximation numérique du problème purement mécanique à l'aide de schémas discrets. Pour ces schémas, nous obtenons des résultats d'estimation d'erreur.

Mots-clés : Viscoélasticité, endommagement, compliance normale, point fixe, approximation, différences finies, éléments finis.

الملخص

تتناول هذه الدراسة التحليل التبايني والعددي لبعض مسائل التلامس، مع أو بدون احتكاك، بين جسم قابل للتشوه وأساس. نأخذ بعين الاعتبار هنا قوانين سلوك غير خطية لمواد لزجة- مرنة. ولتلك المسائل، نحصل على صيغ تباينية يتبعها نتائج تتعلق بوجود ووحدانية الحلول الضعيفة. تعتمد التقنيات المستخدمة على نظرية المؤثرات الرتيبة، يتبعها تطبيق لإحدى صيغ نظرية كوشي-ليبتشيتز، وحجج تعتمد على مبرهنة النقطة الثابتة لباناش. وأخيرًا، نقترح تقريبًا عدديًا للمشكلة الميكانيكية البحتة باستخدام مخططات متقطعة. ولتلك المخططات، نحصل على نتائج لتقدير الخطأ

الكلمات المفتاحية: اللزوجة المرنة، الضرر، الامتثال الطبيعي، النقطة الثابتة، التقريب، الفروق المنتهية، العناصر المنتهية