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NORELHOUDA BAKRI

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Closures and openings of ternary relations

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Closures and openings of ternary relations

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Thesis submitted in fulfillment of the requirements for the degree of Doctor of Mathematics

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Table of contents

Introduction

Relationships between the elements of a same set or between the elements of different sets can be represented at a very basic level through the use of the mathematical construct called relation. Relations come in many flavours, such as binary or ternary, crisp or fuzzy, et cetera.

Surprisingly, in contrast to binary relations, ternary and, more generally, n-ary relations, have received far less attention. However, in recent years, the interest in ternary relations is on the rise, for instance in the theory of dependence spaces [50] and (fuzzy) triadic formal concept analysis [16, 40, 46]. From a theoretical point of view, ternary relations have been studied in algebra (e.g., in median algebras [5]), group theory [21, 22], order theory (e.g., in the study of cyclic orders [19, 48]) and logic (e.g., in the Routley-Meyer semantics of relevant logic [12, 33, 51]). Recently, betweenness relations, a specific type of ternary relation of geometric origin, also came to play a pivotal role in models for decision making [55] and aggregation [54]. Other instances of ternary relations can be found in biology (e.g., modelling of phylogenies [62]), qualitative spatial reasoning [41] and string matching [44]. Also, there are many different uses of ternary relations in the field of information modelling (e.g., in the Resource Description Framework (RDF) [59], entity-relationship or class diagrams [1] and the Ternary Relations Model [58]).

Ternary relations can display various interesting properties, such as reflexivity, symmetry, cyclicity and transitivity (see, e.g., [21, 50, 67, 73, 75]), some of which do not exist in the binary case (such as cyclicity) or come in a multitude of variations in the ternary case (such as transitivity). For instance, Pitcher and Smiley [56] introduced several types of four-point transitivity and five-point transitivity of betweenness relations. Additional types of four-point transitivity of ternary relations were proposed by Novák and Novotný [48]. Recently, Zedam et al. [71] characterized various properties of ternary relations using the notion of traces of a ternary relation.

Transitivity is by far the most interesting property of (crisp and fuzzy) binary relations, both in the case of finite universes (e.g. applications in fuzzy preference modelling and multi-criteria decision making [37]) and in the case of infinite universes (e.g. indistinguishability relations or equality relations on the real line [27, 42, 45, 66]). Note that transitivity of a fuzzy relation is usually understood as T-transitivity, with T a triangular norm [61] playing the role of generalized conjunction. Thanks to the notion of composition of binary relations, the transitivity of a binary relation can conveniently be expressed as a relational inclusion.

The composition of two relations is one of the main concepts in relational calculus, dating back to the nineteenth century (see, e.g., [52]). In the twentieth century, Bandler and Kohout [7] introduced two additional relational compositions, arising from the substitution of the underlying existential quantifier by the universal quantifier. Some decades earlier, already in his seminal work on fuzzy sets [70], Zadeh realized that also crisp relations (allowing to model relationship and non-relationship only) lacked expressivity, giving rise to the study of fuzzy relations. Goguen further generalized fuzzy relations to the lattice-theoretical framework [39]. This notion of a fuzzy relation spread very quickly and it is thus not surprising that Bandler and Kohout presented fuzzy versions of their relational compositions in tandem with the crisp versions. Some further modifications were suggested by De Baets and Kerre [25]. More recently, Štěpnička and Holčapek [65] revisited the fuzzy relational compositions by employing more general fuzzy quantifiers than the existential and universal ones. Noteworthy is also the recent work on fuzzy relations in the context of fuzzy class theory [13, 14].

In addition to the role of relational compositions in the study and characterization of various properties of binary crisp or fuzzy relations [32, 36, 69], they also appear in many branches of mathematics, for instance in the study of fuzzy relational equations [24, 31], formal concept analysis [15, 38] and relation algebras (e.g., in temporal and spatial reasoning [34]). Compositions of crisp and fuzzy relations also appear in applications, for instance, in medical diagnosis [8], fuzzy inference systems [63, 64] and relational databases [57].

Motivated by the usefulness of relational compositions of binary relations and the importance of ternary relations, in this thesis we introduce several types of composition of ternary relations considering four points (resp. considering five points).

In practice, however, the transitivity of a binary relation is quite often violated, even by decision makers who accept transitivity as a kind of consistency or rationality condition. The question then arises how such violations can be repaired. More generally, in case a binary relation R does not possess a desired property P, the question arises whether it is possible to find (if it exists) the smallest binary relation including R and possessing property P, which is called its P-closure; and, dually, whether it is possible to find (if it exists) the greatest binary relation that is included in R and possesses property P, which is called its P-opening. Such closures and openings play an important role in many different mathematical areas, such as geometry [68] and logic [47]; many computational areas, such as database theory [30] and data analysis [35, 38]; and various applications such as medical diagnosis [6, 9, 10] and handwriting classification [43]. General results applicable to both crisp and fuzzy binary relations have been presented by Bandler and Kohout [11]. More specific results dealing with the transitivity property can be found in [28], for instance.

The main aim of the present thesis is to derive results similar to those of Bandler and Kohout for the setting of ternary relations. For a given potential property P of ternary relations, we will lay bare necessary and sufficient conditions for the existence of a P-closure and/or P-opening. Although we will discuss various properties, the main focus will be on the property of transitivity. As mentioned earlier, for ternary relations there does not exist a single notion of transitivity, so the results might depend on the notion considered, as will become clear later on. Also, given the link between transitivity and relational composition, we will take both points of view into account. In particular, we will study several new types of transitivity of a ternary relation based on the types of composition of ternary relations.

This dissertation is structured as follows.

- In Chapter 1, we provide generalities on binary relations and ternary relations that we need throughout this thesis.
- In Chapter 2, we focus on the compositions of ternary relations. First, we introduce several types of composition of ternary relations considering four points, including three associative ones. Also, we introduce six types of composition of ternary relations considering five points, including four associative ones. These compositions are based on two types of composition of a ternary relation with a binary relation recently introduced in [71]. We investigate the most important properties of these compositions, paying particular attention to show the link with the usual composition of binary relations through the use of the operations of projection and cylindrical extension. Also, we study the interaction of these compositions with the basic set operations, permutations and traces of ternary relations. The link between all these compositions is discussed. Moreover, we introduce the subcompositions and supercompositions of ternary relations based on the associative four-point compositions. Furthermore, we show some relational inequalities based on these compositions.

- In Chapter 3, we focus on the transitivity of ternary relations. First, we introduce several types of transitivity of a ternary relation based on the compositions of ternary relations, and discuss the interaction of these transitivity properties with the binary projections, cylindrical extensions and traces. Also, the link between these transitivity properties is discussed.
- In Chapter 4, we focus on the closures and openings of ternary relations. On the one hand, we study closures in general and apply the results to some basic properties of ternary relations. On the other hand, we characterize the closures for the transitivity properties, and discuss in particular the interaction of these transitive closures with the binary projections and cylindrical extensions. Next, we consider the setting of finite universes. Moreover, we develop parallel results on openings.
- Finally, general conclusions and future research are drawn.

Most parts of results presented in this thesis have already been published or submitted for publication in peer-reviewed international journals. Results included in Chapter 2 have been described in [2] and those included in Chapters 3 and 4 have been described in [74].

1 Generalities on binary and ternary relations

In this chapter, we recall the necessary basic properties and concepts of binary relations as well as of ternary relations that will be needed throughout this thesis.

1.1. Binary relations

In this section, we recall the basic definitions and properties of binary relations. Also, we recall the closures and openings for theses properties of binary relations.

1.1.1. Definitions and properties

A binary relation R on a set X is a subset of X^2, i.e., it is a set of couples $(x, y) \in X^2$. Three special binary relations on X are the null relation $O_{X^2} = \emptyset$, the binary identity relation $I_{X^2} = \{(x, x) \mid x \in X\}$ and the universal binary relation X^2.

Two elements x and y of a set X equipped with a binary relation R are called comparable elements if it holds that $(x, y) \in R$ or $(y, x) \in R$, otherwise, they are called incomparable elements. A binary relation R_1 on a set X is said to be included in a binary relation R_2 on the same set X, denoted by $R_1 \subseteq R_2$, if, for any $x, y \in X$, $(x, y) \in R_1$ implies that $(x, y) \in R_2$. The intersection of two binary relations R_1 and R_2 on X is the binary relation $R_1 \cap R_2$ on X defined as $R_1 \cap R_2 = \{(x, y) \in X^2 \mid (x, y) \in R_1 \wedge (x, y) \in R_2\}$. If $R_1 \cap R_2 = \emptyset$, then R_1 and R_2 are called disjoint relations. Also, the union of two binary relations R_1 and R_2 on X is the binary relation $R_1 \cup R_2$ on X defined as $R_1 \cup R_2 = \{(x, y) \in X^2 \mid (x, y) \in R_1 \vee (x, y) \in R_2\}$. The composition of two relations R_1 and R_2 on X is the relation $R_1 \circ R_2$ on X defined as:

$$R_1 \circ R_2 = \{(x, z) \in X^2 \mid (\exists y \in X)((x, y) \in R_1 \wedge (y, z) \in R_2)\} .$$

Let $\mathbb{N}^*$ be the set of non-zero natural numbers. The n-th power R^n of a binary relation R on X is recursively defined as

$$R^1 = R \quad \text{and} \quad R^n = R^{n-1} \circ R\,,$$

for any $n \in \mathbb{N}^*$ and $n > 1$.

For a given binary relation R on a set X, we denote the *transpose* (*converse*) of R by R^t, i.e., for any $x, y \in X$, $(x, y) \in R^t$ means that $(y, x) \in R$. Also, we denote the *complement* of R by R^c, i.e., for any $x, y \in X$, $(x, y) \in R^c$ means that $(x, y) \notin R$. We denote the *dual* of R by R^d, i.e., for any $x, y \in X$, $(x, y) \in R^d$ means that $(y, x) \notin R$.

A binary relation R on a set X is called:

(i) *reflexive*, if, for any $x \in X$, it holds that $(x, x) \in R$;

(ii) *irreflexive*, if, for any $x \in X$, it holds that $(x, x) \in R^c$;

(iii) *symmetric*, if, for any $x, y \in X$, it holds that $(x, y) \in R$ implies $(y, x) \in R$;

(iv) *asymmetric*, if, for any $x, y \in X$, it holds that $(x, y) \in R$ implies $(y, x) \in R^c$;

(v) *antisymmetric*, if, for any $x, y \in X$, it holds that $(x, y) \in R \wedge (y, x) \in R$ implies $x = y$;

(vi) *transitive*, if, for any $x, y, z \in X$, it holds that $(x, y) \in R \wedge (y, z) \in R$ implies $(x, z) \in R$;

(vii) *complete*, if, for any $x, y \in X$, it holds that $(x, y) \in R \vee (y, x) \in R$.

A preorder is binary relation over a set X which is reflexive and transitive. A partial order (order for short) is a binary relation on a set X that is reflexive, antisymmetric and transitive, a set with an order relation is called an ordered set (also called a poset). A total order is a partial order that satisfies the complete condition. A tolerance is a binary relation on a set X that is reflexive and symmetric. A binary relation on a set X is said to be an equivalence relation if is it reflexive, symmetric and transitive.

1.1.2. Closures of a binary relation

Where P is any property which a binary relation R on a set X may have or fail to have, the P-closure of R is defined to be the smallest relation S containing R

and possessing P. Bandler and Kohout [11] have discussed the concept of closure of a given binary relation with respect to a property P as follows.

Definition 1.1. *If P is a property which a binary relation R on a set X may have or fail to have, then a ternary relation S is the P-closure of R, written $S = P^{\mathrm{cl}}(R)$, if and only if S satisfies all of*

(i) S has property P;

(ii) $R \subseteq S$;

(iii) If $R \subseteq T$ and T has property P, then $S \subseteq T$.

It is clear that a P-closure, if it exists, must be unique.

Corollary 1.1. *A binary relation R on a set X possesses property P if and only if $R = P^{\mathrm{cl}}(R)$.*

For many properties P, a P-closure exists for some binary relations but not for others. Thus, all R that already possess P have a P-closure trivially (by the above corollary), but this guarantees nothing for other R. The interesting closures are those for properties where every R has a P-closure, because in these cases, and of course only in these, there is a P-closure operator on the entire set of binary relations on X. The following theorem states the conditions for this to occur.

Theorem 1.1. *A P-closure exists for all binary relations R on a set X if and only if*

(i) The universal relation X^2 possesses P;

(ii) The intersection of every (non-empty) family of binary relations, each of which possesses P, also possesses P.

Theorem 1.2. *If P and P' are properties for which closures exist (satisfying the conditions of Theorem 1.1), and if P^{cl} and $\mathrm{P}'^{\mathrm{cl}}$ commute with each other, then $(P \wedge P')$ also satisfies the conditions of Theorem 1.1, and has a closure given by*

$$(P \wedge P')^{\mathrm{cl}} = P^{\mathrm{cl}}(P'^{\mathrm{cl}}) = P'^{\mathrm{cl}}(P^{\mathrm{cl}})\,.$$

In [11], Bandler and Kohout gave a list of closures for the properties of a binary relation as follows.

Theorem 1.3. *Let R be a binary relation on a set X. It holds that:*

(i) The reflexive closure of R is $\mathrm{Ref}^{\mathrm{cl}}(R) = R \cup I_{X^2}$;

(ii) The symmetric closure of R is $\mathrm{Sym}^{\mathrm{cl}}(R) = R \cup R^t$;

(iii) The transitive closure of R is $\mathrm{Tr}^{\mathrm{cl}}(R) = \bigcup_{n\geq 1} R^n$*;*

(iv) The tolerance closure of R is

$$\begin{aligned}\mathrm{Tol}^{\mathrm{cl}}(R) &= \mathrm{Ref}^{\mathrm{cl}}(\mathrm{Sym}^{\mathrm{cl}}(R)) \\ &= \mathrm{Sym}^{\mathrm{cl}}(\mathrm{Ref}^{\mathrm{cl}}(R));\end{aligned}$$

(v) The preorder closure of R is

$$\begin{aligned}\mathrm{Pre}^{\mathrm{cl}}(R) &= \mathrm{Ref}^{\mathrm{cl}}(\mathrm{Tr}^{\mathrm{cl}}(R)) \\ &= \mathrm{Tr}^{\mathrm{cl}}(\mathrm{Ref}^{\mathrm{cl}}(R));\end{aligned}$$

(vi) The equivalence closure of R is

$$\begin{aligned}\mathrm{Equ}^{\mathrm{cl}}(R) &= \mathrm{Tr}^{\mathrm{cl}}(\mathrm{Tol}^{\mathrm{cl}}(R)) \\ &= \mathrm{Tr}^{\mathrm{cl}}(\mathrm{Sym}^{\mathrm{cl}}(\mathrm{Ref}^{\mathrm{cl}}(R))) \\ &= \mathrm{Tr}^{\mathrm{cl}}(\mathrm{Ref}^{\mathrm{cl}}(\mathrm{Sym}^{\mathrm{cl}}(R))) \\ &= \mathrm{Ref}^{\mathrm{cl}}(\mathrm{Tr}^{\mathrm{cl}}(\mathrm{Sym}^{\mathrm{cl}}(R)))\,.\end{aligned}$$

The property of being complete fails to meet the second criterion of Theorem 1.1, while asymmetry and antisymmetry fail to meet even the first.

1.1.3. Openings of a binary relation

Where P is any property which a binary relation R on X may have or fail to have, the P-opening of R is defined to be the greatest relation Q that possesses P and is contained in R. Bandler and Kohout [11] have discussed the concept of opening of a given binary relation with respect to a property P as follows.

Definition 1.2. *Where P is any property which a binary relation T on X may have or fail to have, then a binary relation Q is the P-opening of R, written $Q = P^{\mathrm{op}}(R)$, if and only if Q satisfies all of*

(i) Q has property P;

(ii) $Q \subseteq R$;

(iii) If $M \subseteq R$ and M possesses P, then $M \subseteq Q$.

It is clear that a P-opening, if it exists, must be unique.

Corollary 1.2. *A binary relation R possesses property P if and only if $R = P^{\mathrm{op}}(R)$.*

As with closures, openings are of interest chiefly for those properties P for which every R has a P-opening, and where accordingly there is a P-opening operator on the whole set of binary relations on X. Necessary and sufficient conditions for this are as follows.

Theorem 1.4. *A P-opening exists for all binary relations R on X if and only if*

(i) The null relation $O_{X^2} = \emptyset$ possesses P;

(ii) The union of every (non-empty) family of binary relations, each of which possesses P, also possesses P.

Theorem 1.5. *If P and P' are properties for which openings exist (satisfying the conditions of Theorem 1.4), and if P^{op} and $\mathrm{P}'^{\mathrm{op}}$ commute with each other, then $(P \wedge P')$ also satisfies the conditions of Theorem 1.4, and has an opening given by*

$$(P \wedge P')^{\mathrm{op}} = P^{\mathrm{op}}(P'^{\mathrm{op}}) = P'^{\mathrm{op}}(P^{\mathrm{op}})\,.$$

In [11], Bandler and Kohout discuss the openings for the properties of a binary relation as follows.

Theorem 1.6. *Let R be a binary relation on a set X. It holds that the symmetric opening of R is $\mathrm{Sym}^{\mathrm{op}}(R) = R \cap R^t$.*

It is clear that, of the fundamental properties of binary relations, only the symmetry meets both criteria of Theorem 1.4. Reflexivity and being complete fail in the first criterion, asymmetric, antisymmetry and transitivity fail in the second.

For more details on binary relations, we refer to [11, 17, 18, 23, 29, 60, 72].

1.2. Ternary relations

In this section, we recall the basic definitions and properties of ternary relations.

1.2.1. Definitions and examples

A ternary relation T on a set X is a subset of X^3, i.e., it is a set of triplets $(x, y, z) \in X^3$. Three special ternary relations on X are the null relation $O_{X^3} = \emptyset$, the ternary identity relation $I_{X^3} = \{(x,x,x) \mid x \in X\}$ and the universal ternary relation X^3.

Example 1.1. *Let T be the ternary relation on $\mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers given as follows: $(a,b,c) \in T$ if $a \geq b \geq c$. Then, it is clear that $(15,8,3) \in T$, whereas $(3,10,2) \notin T$.*

Example 1.2. *Given any set X whose elements are arranged on a circle, one can define a ternary relation T on X, i.e., a subset of X^3, by stipulating that $(x,y,z) \in T$ holds if and only if the elements x, y and z are pairwise different and when going from x to z in a clockwise direction one passes through y. For example, if $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ represents the hours on a clock face, then $(8,12,4) \in T$ holds and $(12,8,4) \in T$ does not hold.*

A ternary relation T_1 on a set X is said to be included in a ternary relation T_2 on the same set X, denoted by $T_1 \subseteq T_2$, if, for any $x,y,z \in X$, $(x,y,z) \in T_1$ implies that $(x,y,z) \in T_2$. The intersection of two ternary relations T_1 and T_2 on X is the ternary relation $T_1 \cap T_2$ on X defined as $T_1 \cap T_2 = \{(x,y,z) \in X^3 \mid (x,y,z) \in T_1 \wedge (x,y,z) \in T_2\}$. If $T_1 \cap T_2 = \emptyset$, then T_1 and T_2 are called disjoint ternary relations. Also, the union of two ternary relations T_1 and T_2 on X is the ternary relation $T_1 \cup T_2$ on X defined as $T_1 \cup T_2 = \{(x,y,z) \in X^3 \mid (x,y,z) \in T_1 \vee (x,y,z) \in T_2\}$.

Example 1.3.

(i) Let T_1 and T_2 be two ternary relations on $X = \{x_1,x_2,x_3,x_4,x_5,x_6\}$ given by

$$T_1 = \{(x_1,x_2,x_4),(x_1,x_3,x_2),(x_5,x_1,x_6)\}\,,$$
$$T_2 = \{(x_1,x_2,x_4),(x_1,x_3,x_2),(x_5,x_1,x_6),(x_6,x_6,x_3)\}\,.$$

It is clear that $T_1 \subseteq T_2$.

(ii) Let T_1 and T_2 be two ternary relations on $X = \{x_1,x_2,x_3,x_4,x_5\}$ given by

$$T_1 = \{(x_1,x_2,x_4),(x_1,x_3,x_2),(x_5,x_1,x_4)\}\,,$$
$$T_2 = \{(x_1,x_1,x_2),(x_1,x_2,x_4)\}\,.$$

One easily verifies that

$$T_1 \cap T_2 = \{(x_1,x_2,x_4)\}\,;$$
$$T_1 \cup T_2 = \{(x_1,x_1,x_2),(x_1,x_2,x_4),(x_1,x_3,x_2),(x_5,x_1,x_4)\}\,.$$

For a given ternary relation T on a set X, we denote the *transpose* of T by T^t, i.e., for any $x, y, z \in X$, $(x, y, z) \in T^t$ means that $(z, y, x) \in T$. Also, we denote the *complement* of T by T^c, i.e., for any $x, y, z \in X$, $(x, y, z) \in T^c$ means that $(x, y, z) \notin T$. We denote the *dual* of T by T^d, i.e., for any $x, y, z \in X$, $(x, y, z) \in T^d$ means that $(z, y, x) \notin T$.

1.2.2. Ternary relations obtained by permutation

A permutation σ of a 3-element set $U = \{u, v, w\}$ is a bijection from U to itself. We use the shorthand notation $\sigma(u, v, w)$ instead of $(\sigma(u), \sigma(v), \sigma(w))$. The six permutations of U are given by:

$$\sigma_0(u, v, w) = (u, v, w), \quad \sigma_1(u, v, w) = (u, w, v), \quad \sigma_2(u, v, w) = (v, u, w)\,,$$

$$\sigma_3(u, v, w) = (v, w, u), \quad \sigma_4(u, v, w) = (w, u, v), \quad \sigma_5(u, v, w) = (w, v, u)\,.$$

For a ternary relation T on X and any of the above six permutations σ, the ternary relation T^σ on X is defined as [71]:

$$T^\sigma = \{\sigma(x, y, z) \in X^3 \mid (x, y, z) \in T\}\,.$$

Note that

$$T^\sigma = \{(x, y, z) \in X^3 \mid \sigma^{-1}(x, y, z) \in T\}\,,$$

with $\sigma_i^{-1} = \sigma_i$ for any $i \in \{0, 1, 2, 5\}$ and $\sigma_3^{-1} = \sigma_4$. It is clear that $T^{\sigma_0} = T$ and $T^{\sigma_5} = T^t$.

Remark 1.1. *For any family of ternary relations $(T_j)_{j\in J}$ on a set X and a permutation σ_i, $i \in \{0, \ldots, 5\}$, the following equalities hold:*

$$\Big(\bigcup_{j\in J} T_j\Big)^{\sigma_i} = \bigcup_{j\in J} T_j^{\sigma_i} \quad \text{and} \quad \Big(\bigcap_{j\in J} T_j\Big)^{\sigma_i} = \bigcap_{j\in J} T_j^{\sigma_i}\,,$$

for any $i \in \{0, \ldots, 5\}$.

Definition 1.3. [71] *Let T be a ternary relation on a set X.*

(i) The right-converse of T is the ternary relation $T^{\dashv}$ on X defined as $T^{\dashv} = T^{\sigma_1}$;

(ii) The left-converse of T is the ternary relation $T^{\vdash}$ on X defined as $T^{\vdash} = T^{\sigma_2}$;

(iii) The right-rotation of T is the ternary relation T^{+} on X defined as $T^{+} = T^{\sigma_3}$;

(iv) The left-rotation of T is the ternary relation T^{-} on X defined as $T^{-} = T^{\sigma_4}$.

Note that the table in the following proposition is the corrected version from [71].

Proposition 1.1. *Let T be a ternary relation on a set X. The composition table for the considered permutations reads as follows: for given σ_i and σ_j, it lists $(T^{\sigma_i})^{\sigma_j}$.*

σ_i \ σ_j	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5
σ_0	T	$T^{\dashv}$	$T^{\vdash}$	T^+	T^-	T^t
σ_1	$T^{\dashv}$	T	T^-	T^t	$T^{\vdash}$	T^+
σ_2	$T^{\vdash}$	T^+	T	$T^{\dashv}$	T^t	T^-
σ_3	T^+	$T^{\vdash}$	T^t	T^-	T	$T^{\dashv}$
σ_4	T^-	T^t	$T^{\dashv}$	T	T^+	$T^{\vdash}$
σ_5	T^t	T^-	T^+	$T^{\vdash}$	$T^{\dashv}$	T

A ternary relation T on a set X is called:

(i) *reflexive*, if, for any $x \in X$, it holds that $(x, x, x) \in T$;

(ii) *strongly reflexive*, if, for any $x, y, z \in X$ with $\text{card}\{x, y, z\} \leq 2$, it holds that $(x, y, z) \in T$;

(iii) *irreflexive*, if, for any $x \in X$, it holds that $(x, x, x) \notin T$;

(iv) *strongly irreflexive*, if, for any $x, y, z \in X$ with $\text{card}\{x, y, z\} \leq 2$, it holds that $(x, y, z) \notin T$;

(v) *symmetric*, if, for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ implies $(z, y, x) \in T$, i.e., $T = T^t$;

(vi) *strongly symmetric*, if $T = T^{\sigma_i}$, for any $i \in \{1, \ldots, 5\}$;

(vii) *asymmetric*, if, for any $x, y, z \in X$ such that $\text{card}\{x, y, z\} \geq 2$, it holds that $(x, y, z) \in T$ implies $(z, y, x) \notin T$;

(viii) *strongly asymmetric*, if, for any $x, y, z \in X$ such that $\text{card}\{x, y, z\} \geq 2$, it holds that $(x, y, z) \in T$ implies $\sigma_i(x, y, z) \notin T$, for any $i \in \{1, \ldots, 5\}$;

(ix) *cyclic*, if, for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ implies $(y, z, x) \in T$;

(x) *complete*, if, for any $x, y, z \in X$ such that $\text{card}\{x, y, z\} \geq 2$, it holds that $(x, y, z) \in T \vee (z, y, x) \in T$;

(xi) *strongly complete*, if, for any $x, y, z \in X$ such that $\text{card}\{x, y, z\} \geq 2$, it holds that $\sigma_i(x, y, z) \in T$, for some $i \in \{0, ..., 5\}$.

It is clear that a ternary relation T on X is called:

(a) *left reflexive*, if, for any $x, y \in X$, it holds that $(x, y, y) \in T$;

(b) *middle reflexive*, if, for any $x, y \in X$, it holds that $(x, y, x) \in T$;

(c) *right reflexive*, if, for any $x, y \in X$, it holds that $(x, x, y) \in T$.

A cyclic order is a ternary relation that is asymmetric, transitive and cyclic. A ternary relation T on a set X is a betweenness relation if it satisfies the following conditions:

(i) for any $x, y, z \in X$, $(x, y, z) \in T$ if and only if $(z, y, x) \in T$;

(ii) for any $x, y, z \in X$, $(x, y, z) \in T$ and $(x, z, y) \in T$ if and only if $y = z$;

(iii) for any $x, y, z, u \in X$, $(x, y, u) \in T$ and $(x, u, z) \in T$ implies $(x, y, z) \in T$.

Also, a ternary relation T is said to be a strict order-betweenness relation if, for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ if and only if $x < y < z$ or $z < y < x$. An order-betweenness relation is a ternary relation T such that for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ if and only if $x \leq y \leq z$ or $z \leq y \leq x$.

For more details on ternary relations, we refer to [3, 4, 20, 21, 22, 49, 50, 71].

1.2.3. Traces of ternary relations

In this section, we recall the definitions and properties of traces of a ternary relation introduced in [71]. As in the binary case, these traces facilitate the study and characterization of properties of a ternary relation. Interestingly, the traces themselves turn out to be the greatest solutions of relational inequalities associated with newly introduced compositions of ternary relations.

Definition 1.4. [71] *Let T be a ternary relation on a set X.*

(i) The left trace of T is the binary relation T^ℓ on X defined as

$$T^\ell = \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((x, a, b) \in T \Rightarrow (y, a, b) \in T)\}\,;$$

(ii) The middle trace of T is the binary relation T^m on X defined as

$$T^m = \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((a, x, b) \in T \Rightarrow (a, y, b) \in T)\}\,;$$

(iii) The right trace of T is the binary relation T^r on X defined as

$$T^r = \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((a, b, x) \in T \Rightarrow (a, b, y) \in T)\}\,.$$

The following result discusses the traces of some particular ternary relations.

Proposition 1.2. [71] *Let T be a ternary relation on a set X. The following statements hold:*

(i) If $T = X^3$ or $T = \emptyset$, then $T^\ell = T^m = T^r = X^2$;

(ii) If $T = I_{X^3}$, then $T^\ell = T^m = T^r = I_{X^2} = \{(x, x) \mid x \in X\}$;

(iii) If $T^\ell = T^m = T^r = X^2$, then $T = X^3$ or $T = \emptyset$.

The following proposition discusses the traces of ternary relations obtained by permutation.

Proposition 1.3. [71] *Let T be a ternary relation on a set X. The left, middle and right traces of the corresponding ternary relations obtained by permutation are listed in the following table:*

	$(.)^\ell$	$(.)^m$	$(.)^r$
$T^\dashv$	T^ℓ	T^r	T^m
$T^\vdash$	T^m	T^ℓ	T^r
T^+	T^r	T^ℓ	T^m
T^-	T^m	T^r	T^ℓ
T^t	T^r	T^m	T^ℓ

2 Compositions of ternary relations

The main aim of this chapter is to study the compositions of (crisp) ternary relations. First, we introduce several types of composition of ternary relations considering four points (four-point compositions, for short), including three associative ones. Also, we introduce six types of composition of ternary relations considering five points (five-point compositions, for short), including four associative ones. These compositions are based on two types of composition of a ternary relation with a binary relation recently introduced in [71].

We study the properties of these compositions, in particular the link with the usual composition of binary relations through the use of the operations of projection and cylindrical extension. Note that these compositions will be play a key role in the characterization of different types of transitivity of a ternary relation, as we shall see in the next chapter of this thesis.

Also, we introduce the subcompositions and supercompositions of ternary relations based on the associative four-point compositions. Furthermore, we show some relational inequalities based on these compositions.

2.1. Compositions of relations

In contrast to binary relations, there is much less agreement in literature on the definition of composition of ternary relations. Indeed, various alternative definitions, each with its own motivation, have been proposed, for instance in [41]. In this section, we recall the definitions of the compositions of (binary and ternary) relations.

2.1.1. Composition of binary relations

In the theory of binary relations, a major role is played by the composition of relations, as it is the most important operation that allows to combine relations. In this section, an explanation of the usual notion of composition of two binary relations is given. One easily observes that there does not exist a single type of composition of two binary relations. In the following, all the possible types of composition of two binary relations R and P are given as follows:

$$R \circ_1 P = \{(x,z) \in X^2 \mid (\exists t \in X)((x,t) \in R \wedge (t,z) \in P)\};$$

$$R \circ_2 P = \{(x,z) \in X^2 \mid (\exists t \in X)((x,t) \in R \wedge (z,t) \in P)\};$$
$$R \circ_3 P = \{(x,z) \in X^2 \mid (\exists t \in X)((t,x) \in R \wedge (t,z) \in P)\};$$
$$R \circ_4 P = \{(x,z) \in X^2 \mid (\exists t \in X)((t,x) \in R \wedge (z,t) \in P)\};$$
$$R \circ_5 P = \{(x,z) \in X^2 \mid (\exists t \in X)((t,z) \in R \wedge (x,t) \in P)\};$$
$$R \circ_6 P = \{(x,z) \in X^2 \mid (\exists t \in X)((z,t) \in R \wedge (x,t) \in P)\};$$
$$R \circ_7 P = \{(x,z) \in X^2 \mid (\exists t \in X)((t,z) \in R \wedge (t,x) \in P)\};$$
$$R \circ_8 P = \{(x,z) \in X^2 \mid (\exists t \in X)((z,t) \in R \wedge (t,x) \in P)\}.$$

Only the two types of composition ($\circ_i$, $i \in \{1,2\}$) of binary relations are considered, as the other types can be defined by using the same formulas based on the commutativity of the conjunction and the notion of transpose. i.e.,

$$R \circ_i P = P \circ_{i-4} R, \text{ for any } i \in \{5,\ldots,8\},$$
$$R \circ_i P = R^t \circ_{5-i} P^t, \text{ for any } i \in \{3,4\}.$$

It is clear that associativity is the most important property of the compositions of relations, then, among the two compositions ($\circ_i$, $i \in \{1,2\}$) of binary relations, only the first composition $\circ_1$ is associative. In literature, the composition $\circ_1$ is the usual type of composition of two binary relations R and P and it is denoted by $\circ$, i.e.,

$$R \circ P = \{(x,z) \in X^2 \mid (\exists t \in X)((x,t) \in R \wedge (t,z) \in P)\}.$$

In [7], Bandler and Kohout have noticed that the above associative composition $R \circ S$ of the binary relations R and P can be defined or can be expressed as:

$$R \circ P = \{(x,z) \in X^2 \mid {}_xR \cap P_z \neq \emptyset\},$$

${}_xR = \{y \in x \mid (x,y) \in R\}$ is called the afterset of the element $x \in X$ and $P_z = \{y \in X \mid (y,z) \in P\}$ is called the foreset of the element $z \in X$.

Inspired by this style of the notation, these authors have introduced three new compositions called: the subcomposition, the supercomposition and the square composition, which are defined, respectively, as

$$\begin{aligned} R \lhd P &= \{(x,z) \in X^2 \mid {}_xR \subseteq P_z\}\,, \\ R \rhd P &= \{(x,z) \in X^2 \mid P_z \subseteq {}_xR\}\,, \\ R \diamond P &= \{(x,z) \in X^2 \mid {}_xR = P_z\}\,. \end{aligned}$$

In [26], De Baets and Kerre have shown that the above definitions are not acceptable for empty foresets and aftersets, because $R \lhd P$, $R \rhd P$ and $R \diamond P$ can contain a lot of unwanted couples. Therefore they have redefined them in the following way

$$\begin{aligned} R \lhd P &= \{(x,z) \in X^2 \mid \emptyset \neq {}_xR \subseteq P_z\}\,, \\ R \rhd P &= \{(x,z) \in X^2 \mid \emptyset \neq P_z \subseteq {}_xR\}\,, \\ R \diamond P &= \{(x,z) \in X^2 \mid \emptyset \neq {}_xR = P_z\}\,. \end{aligned}$$

2.1.2. Compositions of ternary and binary relations

In this subsection, we recall the definition of two types of composition of a ternary relation with a binary relation and vice versa. These compositions will turn out to be intimately linked to the definition of composition of ternary relations.

Definition 2.1 ([71]). *Let T be a ternary relation and R be a binary relation on a set X.*

(i) The $\ltimes$-composition of T and R is the ternary relation $T \ltimes R$ on X defined as

$$T \ltimes R = \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (t,z) \in R)\}\,;$$

(ii) The $\rtimes$-composition of R and T is the ternary relation $R \rtimes T$ on X defined as

$$R \rtimes T = \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t) \in R \wedge (t,y,z) \in T)\}\,.$$

The following proposition lists some properties of the compositions of a ternary relation with a binary relation.

Proposition 2.1 ([71]). *Let T, T_1 and T_2 be three ternary relations on a set X and R, R_1 and R_2 be three binary relations on X. The following statements hold:*

(i) *If* $R_1 \subseteq R_2$ *and* $T_1 \subseteq T_2$*, then* $T_1 \ltimes R_1 \subseteq T_2 \ltimes R_2$ *and* $R_1 \rtimes T_1 \subseteq R_2 \rtimes T_2$*;*

(ii) $(T \ltimes R)^t = R^t \rtimes T^t$*;*

(iii) $(R \rtimes T)^t = T^t \ltimes R^t$*;*

(iv) $(T_1 \cup T_2) \ltimes R = (T_1 \ltimes R) \cup (T_2 \ltimes R)$ *and* $(T_1 \cap T_2) \ltimes R = (T_1 \ltimes R) \cap (T_2 \ltimes R)$*;*

(v) $(R_1 \cup R_2) \rtimes T = (R_1 \rtimes T) \cup (R_2 \rtimes T)$ *and* $(R_1 \cap R_2) \rtimes T = (R_1 \rtimes T) \cap (R_2 \rtimes T)$*;*

(vi) $(T \ltimes R_1) \ltimes R_2 = T \ltimes (R_1 \circ R_2)$*;*

(vii) $R_1 \rtimes (R_2 \rtimes T) = (R_1 \circ R_2) \rtimes T$*.*

2.2. Binary projections and cylindrical extensions

In this section, we discuss some properties of the binary projections of a ternary relation and the cylindrical extensions of a binary relation that will be needed throughout this work.

2.2.1. Binary projections of a ternary relation

Definition 2.2. [71] *Let* T *be a ternary relation on a set* X*.*

(i) *The left projection of* T *is the binary relation* $P_\ell(T)$ *on* X *defined as*

$$P_\ell(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((z, x, y) \in T)\}\,;$$

(ii) *The middle projection of* T *is the binary relation* $P_m(T)$ *on* X *defined as*

$$P_m(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((x, z, y) \in T)\}\,;$$

(iii) *The right projection of* T *is the binary relation* $P_r(T)$ *on* X *defined as*

$$P_r(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((x, y, z) \in T)\}\,.$$

For a ternary relation T*, we write* $P(T) = P_\ell(T) \cup P_m(T) \cup P_r(T)$*.*

The following proposition is immediate.

Proposition 2.2. [71] *Let* T *be a ternary relation on a set* X*. The left, middle and right projections of its transpose* T^t *are listed in the following table:*

	$P_\ell(.)$	$P_m(.)$	$P_r(.)$
T^t	$(P_r(T))^t$	$(P_m(T))^t$	$(P_\ell(T))^t$

The following proposition shows the interaction of the projections of a ternary relation with inclusion and basic set-theoretical operations.

Proposition 2.3. *Let T_1 and T_2 be two ternary relations on a set X. For any $\lambda \in \{\ell, m, r\}$, the following statements hold:*

(i) If $T_1 \subseteq T_2$, then $P_\lambda(T_1) \subseteq P_\lambda(T_2)$;

(ii) $P_\lambda(T_1 \cap T_2) \subseteq P_\lambda(T_1) \cap P_\lambda(T_2)$;

(iii) $P_\lambda(T_1 \cup T_2) = P_\lambda(T_1) \cup P_\lambda(T_2)$.

Proof. We only give the proof for the case $\lambda = \ell$, as the other cases can be proved analogously.

(i) Suppose that $T_1 \subseteq T_2$ and let $(x, y) \in P_\ell(T_1)$. Then there exists $z \in X$ such that $(z, x, y) \in T_1$. This implies that $(z, x, y) \in T_2$. Hence, $(x, y) \in P_\ell(T_2)$. Thus, $P_\ell(T_1) \subseteq P_\ell(T_2)$.

(ii) Let $(x, y) \in P_\ell(T_1 \cap T_2)$. Then there exists $z \in X$ such that $(z, x, y) \in T_1 \cap T_2$. This implies that $(z, x, y) \in T_1$ and $(z, x, y) \in T_2$. Hence, $(x, y) \in P_\ell(T_1) \cap P_\ell(T_2)$. Thus, $P_\ell(T_1 \cap T_2) \subseteq P_\ell(T_1) \cap P_\ell(T_2)$.

(iii) We easily verify that

$$\begin{aligned} P_\ell(T_1 \cup T_2) &= \{(x, y) \in X^2 \mid (\exists z \in X)((z, x, y) \in T_1 \cup T_2)\} \\ &= \{(x, y) \in X^2 \mid (\exists z \in X)((z, x, y) \in T_1 \vee (z, x, y) \in T_2)\} \\ &= P_\ell(T_1) \cup P_\ell(T_2)\,. \end{aligned}$$

□

The following proposition shows the interaction of $P(T)$ with inclusion and basic set-theoretical operations.

Proposition 2.4. *Let T_1 and T_2 be two ternary relations on a set X. The following statements hold:*

(i) If $T_1 \subseteq T_2$, then $P(T_1) \subseteq P(T_2)$;

(ii) $P(T_1 \cap T_2) \subseteq P(T_1) \cap P(T_2)$;

(iii) $P(T_1 \cup T_2) = P(T_1) \cup P(T_2)$.

Proof. (i) Suppose that $T_1 \subseteq T_2$, Proposition 2.3 then guarantees that $P_\ell(T_1) \cup P_m(T_1) \cup P_r(T_1) \subseteq P_\ell(T_2) \cup P_m(T_2) \cup P_r(T_2)$. Thus, $P(T_1) \subseteq P(T_2)$.

(ii) From Proposition 2.3, it follows that

$$\begin{aligned} P(T_1 \cap T_2) &= P_\ell(T_1 \cap T_2) \cup P_m(T_1 \cap T_2) \cup P_r(T_1 \cap T_2) \\ &\subseteq (P_\ell(T_1) \cap P_\ell(T_2)) \cup (P_m(T_1) \cap P_m(T_2)) \cup (P_r(T_1) \cap P_r(T_2)) \,. \end{aligned}$$

The distributivity of $\cap$ and $\cup$ guarantees that

$$\begin{aligned} P(T_1 \cap T_2) &\subseteq (P_\ell(T_1) \cup P_m(T_1) \cup P_r(T_1)) \cap (P_\ell(T_2) \cup P_m(T_2) \cup P_r(T_2)) \\ &= P(T_1) \cap P(T_2) \,. \end{aligned}$$

(iii) Also, from Proposition 2.3, we easily verify that

$$\begin{aligned} P(T_1 \cup T_2) &= P_\ell(T_1 \cup T_2) \cup P_m(T_1 \cup T_2) \cup P_r(T_1 \cup T_2) \\ &= (P_\ell(T_1) \cup P_\ell(T_2)) \cup (P_m(T_1) \cup P_m(T_2)) \cup (P_r(T_1) \cup P_r(T_2)) \\ &= (P_\ell(T_1) \cup P_m(T_1) \cup P_r(T_1)) \cup (P_\ell(T_2) \cup P_m(T_2) \cup P_r(T_2)) \\ &= P(T_1) \cup P(T_2) \,. \end{aligned}$$

□

2.2.2. Cylindrical extensions of a binary relation

Definition 2.3. [71] *Let R be a binary relation on a set X.*

(i) The left cylindrical extension of R is the ternary relation $C_\ell(R)$ on X defined as

$$C_\ell(R) = \{(x, y, z) \in X^3 \mid (y, z) \in R\} \,;$$

(ii) The middle cylindrical extension of R is the ternary relation $C_m(R)$ on X defined as

$$C_m(R) = \{(x, y, z) \in X^3 \mid (x, z) \in R\} \,;$$

(iii) The right cylindrical extension of R is the ternary relation $C_r(R)$ on X defined as

$$C_r(R) = \{(x, y, z) \in X^3 \mid (x, y) \in R\} \,.$$

The following proposition is immediate.

Proposition 2.5. [71] *Let R be a binary relation on a set X. The left, middle and right cylindrical extensions of its transpose R^t, its complement R^c and its dual R^d are listed in the following table:*

	$C_\ell(.)$	$C_m(.)$	$C_r(.)$
R^t	$(C_r(R))^t$	$(C_m(R))^t$	$(C_\ell(R))^t$
R^c	$(C_\ell(R))^c$	$(C_m(R))^c$	$(C_r(R))^c$
R^d	$(C_r(R))^d$	$(C_m(R))^d$	$(C_\ell(R))^d$

The following proposition shows the interaction of the cylindrical extensions of a binary relation with inclusion and set-theoretical operations.

Proposition 2.6. *Let R_1 and R_2 be two binary relations on a set X. For any $\lambda \in \{\ell, m, r\}$, the following statements hold:*

(i) If $R_1 \subseteq R_2$, then $C_\lambda(R_1) \subseteq C_\lambda(R_2)$;

(ii) $C_\lambda(R_1 \cap R_2) = C_\lambda(R_1) \cap C_\lambda(R_2)$;

(iii) $C_\lambda(R_1 \cup R_2) = C_\lambda(R_1) \cup C_\lambda(R_2)$.

Proof. We only give the proof for the case $\lambda = \ell$, as the other cases can be proved analogously.

(i) Suppose that $R_1 \subseteq R_2$. Let $(x, y, z) \in C_\ell(R_1)$, then it holds that $(y, z) \in R_1$. Since $R_1 \subseteq R_2$, it follows that $(y, z) \in R_2$. Hence, $(x, y, z) \in C_\ell(R_2)$.

(ii) We easily verify that

$$\begin{aligned} C_\ell(R_1 \cap R_2) &= \{(x, y, z) \in X^3 \mid (y, z) \in R_1 \cap R_2\} \\ &= \{(x, y, z) \in X^3 \mid (y, z) \in R_1 \wedge (y, z) \in R_2\} \\ &= \{(x, y, z) \in X^3 \mid (x, y, z) \in C_\ell(R_1) \wedge (x, y, z) \in C_\ell(R_2)\} \\ &= C_\ell(R_1) \cap C_\ell(R_2)\,. \end{aligned}$$

(iii) The proof is analogous to that of (ii).

□

The following proposition shows that any binary relation coincides with binary projections of its cylindrical extensions and any ternary relation is included in the cylindrical extensions of its projections.

Proposition 2.7. *Let T be a ternary relation and R be a binary relation on a set X. For any $\lambda \in \{\ell, m, r\}$, the following statements hold:*

(i) $R = P_\lambda(C_\lambda(R))$;

(ii) $T \subseteq C_\lambda(P_\lambda(T))$.

Proof. We only give the proof for the case $\lambda = \ell$, as the other cases can be proved analogously.

(i) We easily verify that

$$P_\ell(C_\ell(R)) = \{(x, y) \in X^2 \mid (\exists z \in X)((z, x, y) \in C_\ell(R))\} = R\,.$$

(ii) Let $(x, y, z) \in T$, then it holds that $(y, z) \in P_\ell(T)$. Hence, $(x, y, z) \in C_\ell(P_\ell(T))$. Thus, $T \subseteq C_\ell(P_\ell(T))$.

□

Remark 2.1. *The following example shows that in Proposition 2.7 (ii) the equality does not hold in general. Indeed, let T be the ternary relation on $X = \{x_1, x_2, x_3, x_4\}$ given by:*

$$T = \{(x_1, x_1, x_2), (x_1, x_2, x_3)\}\,.$$

It holds that

$$P_\ell(T) = \{(x_1, x_2), (x_2, x_3)\}\,,$$

and thus

$$C_\ell(P_\ell(T)) = \{(x_1, x_1, x_2), (x_2, x_1, x_2), (x_3, x_1, x_2), (x_4, x_1, x_2), (x_1, x_2, x_3), (x_2, x_2, x_3), (x_3, x_2, x_3), (x_4, x_2, x_3)\}\,.$$

It is clear that $C_\ell(P_\ell(T)) \nsubseteq T$.

2.3. Four-point compositions of ternary relations

In this section, we introduce several types of four-point composition of ternary relations. Also, we investigate their properties and interactions with basic set operations, permutations, binary projections, cylindrical extensions and traces.

2.3.1. Four-point compositions

In this subsection, we introduce the compositions of ternary relations considering four points. Using the same reasoning of constructing the above compositions of binary relations, one easily finds all the four-point compositions of two ternary relations T and S. This reasoning yields 216 four-point compositions of ternary relations.

$$
\begin{aligned}
T \circ_1 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (x,t,z) \in S)\};\\
T \circ_2 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (x,z,t) \in S)\};\\
T \circ_3 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (t,x,z) \in S)\};\\
T \circ_4 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (t,z,x) \in S)\};\\
T \circ_5 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (z,x,t) \in S)\};\\
T \circ_6 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (z,t,x) \in S)\};\\
T \circ_7 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (t,y,z) \in S)\};\\
T \circ_8 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (t,z,y) \in S)\};\\
T \circ_9 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (y,t,z) \in S)\};\\
T \circ_{10} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (y,z,t) \in S)\};\\
T \circ_{11} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{12} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (z,y,t) \in S)\};\\
T \circ_{13} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (t,y,z) \in S)\};\\
T \circ_{14} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (t,z,y) \in S)\};\\
T \circ_{15} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (y,t,z) \in S)\};\\
T \circ_{16} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (y,z,t) \in S)\};\\
T \circ_{17} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{18} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (z,y,t) \in S)\};\\
T \circ_{19} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (x,y,t) \in S)\};\\
T \circ_{20} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,z,t) \in T \wedge (x,y,t) \in S)\};\\
T \circ_{21} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (x,y,t) \in S)\};\\
T \circ_{22} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (x,y,t) \in S)\};\\
T \circ_{23} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (x,y,t) \in S)\};\\
T \circ_{24} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,x) \in T \wedge (x,y,t) \in S)\};\\
T \circ_{25} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,z) \in T \wedge (x,y,t) \in S)\};\\
T \circ_{26} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (x,y,t) \in S)\};
\end{aligned}
$$

$$
\begin{aligned}
T \circ_{27} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,z) \in T \wedge (x,y,t) \in S)\};\\
T \circ_{28} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (x,y,t) \in S)\};\\
T \circ_{29} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,y) \in T \wedge (x,y,t) \in S)\};\\
T \circ_{30} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,y,t) \in T \wedge (x,y,t) \in S)\};\\
T \circ_{31} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,z) \in T \wedge (x,t,z) \in S)\};\\
T \circ_{32} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (x,t,z) \in S)\};\\
T \circ_{33} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,z) \in T \wedge (x,t,z) \in S)\};\\
T \circ_{34} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (x,t,z) \in S)\};\\
T \circ_{35} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,y) \in T \wedge (x,t,z) \in S)\};\\
T \circ_{36} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,y,t) \in T \wedge (x,t,z) \in S)\};\\
T \circ_{37} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,y) \in T \wedge (x,z,t) \in S)\};\\
T \circ_{38} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,y) \in T \wedge (x,t,z) \in S)\};\\
T \circ_{39} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,y) \in T \wedge (t,z,x) \in S)\};\\
T \circ_{40} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,y) \in T \wedge (t,x,z) \in S)\};\\
T \circ_{41} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,y) \in T \wedge (z,t,x) \in S)\};\\
T \circ_{42} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,y) \in T \wedge (z,x,t) \in S)\};\\
T \circ_{43} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,y) \in T \wedge (t,z,y) \in S)\};\\
T \circ_{44} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,y) \in T \wedge (t,y,z) \in S)\};\\
T \circ_{45} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,y) \in T \wedge (y,z,t) \in S)\};\\
T \circ_{46} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,y) \in T \wedge (y,t,z) \in S)\};\\
T \circ_{47} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,y) \in T \wedge (z,y,t) \in S)\};\\
T \circ_{48} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,y) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{49} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,z,t) \in T \wedge (t,z,y) \in S)\};\\
T \circ_{50} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,z,t) \in T \wedge (t,y,z) \in S)\};\\
T \circ_{51} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,z,t) \in T \wedge (y,z,t) \in S)\};\\
T \circ_{52} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,z,t) \in T \wedge (y,t,z) \in S)\};\\
T \circ_{53} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,z,t) \in T \wedge (z,y,t) \in S)\};\\
T \circ_{54} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,z,t) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{55} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,z,t) \in T \wedge (x,t,y) \in S)\};\\
T \circ_{56} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (x,t,y) \in S)\};\\
T \circ_{57} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (x,t,y) \in S)\};\\
T \circ_{58} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (x,t,y) \in S)\};\\
T \circ_{59} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,x) \in T \wedge (x,t,y) \in S)\};
\end{aligned}
$$

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\begin{aligned}
T \circ_{60} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (x,t,y) \in S)\};\\
T \circ_{61} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (x,t,y) \in S)\};\\
T \circ_{62} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,z) \in T \wedge (x,t,y) \in S)\};\\
T \circ_{63} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (x,t,y) \in S)\};\\
T \circ_{64} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,z) \in T \wedge (x,t,y) \in S)\};\\
T \circ_{65} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,y,t) \in T \wedge (x,t,y) \in S)\};\\
T \circ_{66} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,y) \in T \wedge (x,t,y) \in S)\};\\
T \circ_{67} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (x,z,t) \in S)\};\\
T \circ_{68} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,z) \in T \wedge (x,z,t) \in S)\};\\
T \circ_{69} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (x,z,t) \in S)\};\\
T \circ_{70} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,z) \in T \wedge (x,z,t) \in S)\};\\
T \circ_{71} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,y,t) \in T \wedge (x,z,t) \in S)\};\\
T \circ_{72} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,y) \in T \wedge (x,z,t) \in S)\};\\
T \circ_{73} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (t,x,z) \in S)\};\\
T \circ_{74} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (z,x,t) \in S)\};\\
T \circ_{75} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (x,t,z) \in S)\};\\
T \circ_{76} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (z,t,x) \in S)\};\\
T \circ_{77} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (x,z,t) \in S)\};\\
T \circ_{78} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (t,z,x) \in S)\};\\
T \circ_{79} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (y,t,z) \in S)\};\\
T \circ_{80} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{81} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (t,y,z) \in S)\};\\
T \circ_{82} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (z,y,t) \in S)\};\\
T \circ_{83} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (t,z,y) \in S)\};\\
T \circ_{84} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (y,z,t) \in S)\};\\
T \circ_{85} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (y,t,z) \in S)\};\\
T \circ_{86} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{87} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (t,y,z) \in S)\};\\
T \circ_{88} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (z,y,t) \in S)\};\\
T \circ_{89} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (t,z,y) \in S)\};\\
T \circ_{90} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (y,z,t) \in S)\};\\
T \circ_{91} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (y,x,t) \in S)\};\\
T \circ_{92} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (y,x,t) \in S)\};
\end{aligned}
$$

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\begin{aligned}
T \circ_{93} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (y,x,t) \in S)\};\\
T \circ_{94} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,x) \in T \wedge (y,x,t) \in S)\};\\
T \circ_{95} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,z,t) \in T \wedge (y,x,t) \in S)\};\\
T \circ_{96} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (y,x,t) \in S)\};\\
T \circ_{97} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,z) \in T \wedge (y,x,t) \in S)\};\\
T \circ_{98} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,y) \in T \wedge (y,x,t) \in S)\};\\
T \circ_{99} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,z) \in T \wedge (y,x,t) \in S)\};\\
T \circ_{100} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,y,t) \in T \wedge (y,x,t) \in S)\};\\
T \circ_{101} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (y,x,t) \in S)\};\\
T \circ_{102} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (y,x,t) \in S)\};\\
T \circ_{103} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,z) \in T \wedge (t,x,z) \in S)\};\\
T \circ_{104} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,y) \in T \wedge (t,x,z) \in S)\};\\
T \circ_{105} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,z) \in T \wedge (t,x,z) \in S)\};\\
T \circ_{106} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,y,t) \in T \wedge (t,x,z) \in S)\};\\
T \circ_{107} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (t,x,z) \in S)\};\\
T \circ_{108} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (t,x,z) \in S)\};\\
T \circ_{109} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (z,x,t) \in S)\};\\
T \circ_{110} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (t,x,z) \in S)\};\\
T \circ_{111} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (z,t,x) \in S)\};\\
T \circ_{112} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (x,t,z) \in S)\};\\
T \circ_{113} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (t,z,x) \in S)\};\\
T \circ_{114} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (x,z,t) \in S)\};\\
T \circ_{115} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{116} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (y,t,z) \in S)\};\\
T \circ_{117} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (z,y,t) \in S)\};\\
T \circ_{118} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (t,y,z) \in S)\};\\
T \circ_{119} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (y,z,t) \in S)\};\\
T \circ_{120} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (t,z,y) \in S)\};\\
T \circ_{121} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{122} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (y,t,z) \in S)\};\\
T \circ_{123} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (z,y,t) \in S)\};\\
T \circ_{124} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (t,y,z) \in S)\};\\
T \circ_{125} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (y,z,t) \in S)\};
\end{aligned}
$$

$$
\begin{aligned}
T \circ_{126} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (t,z,y) \in S)\};\\
T \circ_{127} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (t,x,y) \in S)\};\\
T \circ_{128} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (t,x,y) \in S)\};\\
T \circ_{129} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,x) \in T \wedge (t,x,y) \in S)\};\\
T \circ_{130} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (t,x,y) \in S)\};\\
T \circ_{131} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (t,x,y) \in S)\};\\
T \circ_{132} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,z,t) \in T \wedge (t,x,y) \in S)\};\\
T \circ_{133} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,y) \in T \wedge (t,x,y) \in S)\};\\
T \circ_{134} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,z) \in T \wedge (t,x,y) \in S)\};\\
T \circ_{135} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,y,t) \in T \wedge (t,x,y) \in S)\};\\
T \circ_{136} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,z) \in T \wedge (t,x,y) \in S)\};\\
T \circ_{137} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (t,x,y) \in S)\};\\
T \circ_{138} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (t,x,y) \in S)\};\\
T \circ_{139} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,y) \in T \wedge (z,x,t) \in S)\};\\
T \circ_{140} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,z) \in T \wedge (z,x,t) \in S)\};\\
T \circ_{141} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,y,t) \in T \wedge (z,x,t) \in S)\};\\
T \circ_{142} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,z) \in T \wedge (z,x,t) \in S)\};\\
T \circ_{143} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (z,x,t) \in S)\};\\
T \circ_{144} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (z,x,t) \in S)\};\\
T \circ_{145} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,x) \in T \wedge (t,z,x) \in S)\};\\
T \circ_{146} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,x) \in T \wedge (z,t,x) \in S)\};\\
T \circ_{147} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,x) \in T \wedge (x,z,t) \in S)\};\\
T \circ_{148} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,x) \in T \wedge (z,x,t) \in S)\};\\
T \circ_{149} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,x) \in T \wedge (x,t,z) \in S)\};\\
T \circ_{150} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,x) \in T \wedge (t,x,z) \in S)\};\\
T \circ_{151} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,x) \in T \wedge (y,z,t) \in S)\};\\
T \circ_{152} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,x) \in T \wedge (z,y,t) \in S)\};\\
T \circ_{153} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,x) \in T \wedge (t,z,y) \in S)\};\\
T \circ_{154} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,x) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{155} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,x) \in T \wedge (t,y,z) \in S)\};\\
T \circ_{156} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,x) \in T \wedge (y,t,z) \in S)\};\\
T \circ_{157} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (y,z,t) \in S)\};\\
T \circ_{158} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (z,y,t) \in S)\};
\end{aligned}
$$

$$
\begin{aligned}
T \circ_{159} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (t,z,y) \in S)\};\\
T \circ_{160} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{161} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (t,y,z) \in S)\};\\
T \circ_{162} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (y,t,z) \in S)\};\\
T \circ_{163} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (y,t,x) \in S)\};\\
T \circ_{164} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,x) \in T \wedge (y,t,x) \in S)\};\\
T \circ_{165} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,z,t) \in T \wedge (y,t,x) \in S)\};\\
T \circ_{166} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (y,t,x) \in S)\};\\
T \circ_{167} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (y,t,x) \in S)\};\\
T \circ_{168} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (y,t,x) \in S)\};\\
T \circ_{169} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (y,t,x) \in S)\};\\
T \circ_{170} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,y,t) \in T \wedge (y,t,x) \in S)\};\\
T \circ_{171} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (y,t,x) \in S)\};\\
T \circ_{172} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,y) \in T \wedge (y,t,x) \in S)\};\\
T \circ_{173} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,z) \in T \wedge (y,t,x) \in S)\};\\
T \circ_{174} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,z) \in T \wedge (y,t,x) \in S)\};\\
T \circ_{175} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (t,z,x) \in S)\};\\
T \circ_{176} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,y,t) \in T \wedge (t,z,x) \in S)\};\\
T \circ_{177} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (t,z,x) \in S)\};\\
T \circ_{178} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,y) \in T \wedge (t,z,x) \in S)\};\\
T \circ_{179} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,z) \in T \wedge (t,z,x) \in S)\};\\
T \circ_{180} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,z) \in T \wedge (t,z,x) \in S)\};\\
T \circ_{181} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (z,t,x) \in S)\};\\
T \circ_{182} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (t,z,x) \in S)\};\\
T \circ_{183} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (z,x,t) \in S)\};\\
T \circ_{184} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (x,z,t) \in S)\};\\
T \circ_{185} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (t,x,z) \in S)\};\\
T \circ_{186} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (x,t,z) \in S)\};\\
T \circ_{187} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (z,y,t) \in S)\};\\
T \circ_{188} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (y,z,t) \in S)\};\\
T \circ_{189} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{190} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (t,z,y) \in S)\};\\
T \circ_{191} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (y,t,z) \in S)\};
\end{aligned}
$$

$$
\begin{aligned}
T \circ_{192} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (t,y,z) \in S)\};\\
T \circ_{193} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,x) \in T \wedge (z,y,t) \in S)\};\\
T \circ_{194} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,x) \in T \wedge (y,z,t) \in S)\};\\
T \circ_{195} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,x) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{196} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,x) \in T \wedge (t,z,y) \in S)\};\\
T \circ_{197} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,x) \in T \wedge (y,t,z) \in S)\};\\
T \circ_{198} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,x) \in T \wedge (t,y,z) \in S)\};\\
T \circ_{199} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,x) \in T \wedge (t,y,x) \in S)\};\\
T \circ_{200} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (t,y,x) \in S)\};\\
T \circ_{201} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (t,y,x) \in S)\};\\
T \circ_{202} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,z,t) \in T \wedge (t,y,x) \in S)\};\\
T \circ_{203} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (t,y,x) \in S)\};\\
T \circ_{204} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (t,y,x) \in S)\};\\
T \circ_{205} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,y,t) \in T \wedge (t,y,x) \in S)\};\\
T \circ_{206} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (t,y,x) \in S)\};\\
T \circ_{207} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,y) \in T \wedge (t,y,x) \in S)\};\\
T \circ_{208} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (t,y,x) \in S)\};\\
T \circ_{209} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,z) \in T \wedge (t,y,x) \in S)\};\\
T \circ_{210} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,z) \in T \wedge (t,y,x) \in S)\};\\
T \circ_{211} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,y,t) \in T \wedge (z,t,x) \in S)\};\\
T \circ_{212} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (z,t,x) \in S)\};\\
T \circ_{213} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,t,y) \in T \wedge (z,t,x) \in S)\};\\
T \circ_{214} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (z,t,x) \in S)\};\\
T \circ_{215} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t,z) \in T \wedge (z,t,x) \in S)\};\\
T \circ_{216} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,y,z) \in T \wedge (z,t,x) \in S)\}.
\end{aligned}
$$

Note that the $\circ_1$-composition is defined in [41]. In the following, we discuss the associativity of only the following eighteen four-point compositions of ternary relations, as all the other compositions can be defined by using the same formulas based on the commutativity of the conjunction and permutations.

$$
\begin{aligned}
T \circ_1 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (x,t,z) \in S)\};\\
T \circ_2 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (x,z,t) \in S)\};\\
T \circ_3 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (t,x,z) \in S)\};
\end{aligned}
$$

$$\begin{aligned}
T \circ_4 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (t,z,x) \in S)\};\\
T \circ_5 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (z,x,t) \in S)\};\\
T \circ_6 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (z,t,x) \in S)\};\\
T \circ_7 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (t,y,z) \in S)\};\\
T \circ_8 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (t,z,y) \in S)\};\\
T \circ_9 S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (y,t,z) \in S)\};\\
T \circ_{10} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (y,z,t) \in S)\};\\
T \circ_{11} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{12} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (z,y,t) \in S)\};\\
T \circ_{13} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (t,y,z) \in S)\};\\
T \circ_{14} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (t,z,y) \in S)\};\\
T \circ_{15} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (y,t,z) \in S)\};\\
T \circ_{16} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (y,z,t) \in S)\};\\
T \circ_{17} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (z,t,y) \in S)\};\\
T \circ_{18} S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (z,y,t) \in S)\}.
\end{aligned}$$

Also, note that these 18 compositions we choosed are different from each others, which means that, for any $\circ_p, \circ_q \in \{1\ldots, 18\}$, there does not exist $j \in \{0, \ldots, 5\}$ such that $T \circ_p S = T^{\sigma_j} \circ_q S^{\sigma_j}$.

Proposition 2.8. *Let T and S be two ternary relations on a set X. It is clear that by using the commutativity of the conjunction and permutations, one easily defines all the rest of 216 four-point compositions of ternary relations. i.e.,*

$$T \circ_i S = S \circ_{i-18} T, \text{ for any } i \in \{19, \ldots, 36\},$$
$$T \circ_i S = T^{\sigma_j} \circ_{i \bmod 36} S^{\sigma_j}, \text{ for any } i \in \{37, \ldots, 216\}, \text{ and } j = \lfloor \frac{i-1}{36} \rfloor.$$

Proof. In the following, we show how to find $\circ_i$-composition, for a given $i \in \{1, \ldots, 216\}$. Indeed, let $i = 60$, one easily verifies that

$$\begin{aligned}
T \circ_{60} S &= T^{\lfloor \frac{60-1}{36} \rfloor} \circ_{60 \bmod 36} S^{\lfloor \frac{60-1}{36} \rfloor}\\
&= T^{\sigma_1} \circ_{24} S^{\sigma_1}.
\end{aligned}$$

From the above formula of $\circ_i$-composition, it is clear that $T^{\sigma_1} \circ_{24} S^{\sigma_1} = S^{\sigma_1} \circ_6 T^{\sigma_1}$.

Hence

$$\begin{aligned} T \circ_{60} S &= S^{\sigma_1} \circ_6 T^{\sigma_1} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in S^{\sigma_1} \wedge (z,t,x) \in T^{\sigma_1})\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,y) \in S \wedge (z,x,t) \in T)\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (x,t,y) \in S)\}. \end{aligned}$$

Therefore, $\circ_{60}$-composition can be written in a unique way. A similar example can be given for the other compositions.

Also, we show how to find $i \in \{1, \ldots, 216\}$, for a given $\circ_i$-composition. Indeed, let T and S be two ternary relations on a set X, and let $\circ_i$-composition defined as follows:

$$T \circ_i S = \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (t,z,x) \in S)\}.$$

One easily verifies that

$$\begin{aligned} T \circ_i S &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T^{\sigma_2} \wedge (z,t,x) \in S^{\sigma_2})\} \\ &= T^{\sigma_2} \circ_6 S^{\sigma_2}. \end{aligned}$$

Hence, one easily finds that $2 = \lfloor \frac{i-1}{36} \rfloor$ which yields that $73 \leq i \leq 108$ and $i \bmod 36 = 6$ which yields that $i = 78$. Therefore

$$T \circ_{78} S = \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (t,z,x) \in S)\}.$$

A similar example can be given for the other compositions. □

The following example shows that the four-point compositions introduced are different.

Example 2.1. *Let T_1 and T_2 be two ternary relations on $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ given by*

$$\begin{aligned} T_1 &= \{(x_1,x_2,x_3),(x_1,x_4,x_5)\}, \\ T_2 &= \{(x_1,x_3,x_4),(x_1,x_5,x_3),(x_1,x_6,x_3),(x_5,x_1,x_6)\}. \end{aligned}$$

One easily verifies that

$$\begin{aligned} T_1 \circ_1 T_2 &= \{(x_1,x_2,x_4),(x_1,x_4,x_3)\}, \\ T_1 \circ_2 T_2 &= \{(x_1,x_2,x_5),(x_1,x_2,x_6)\}, \end{aligned}$$

$$T_1 \circ_3 T_2 = \{(x_1, x_4, x_6)\}\,.$$

A similar example can be given for the other compositions .

2.3.2. Properties of the four-point compositions of ternary relations

In the following, we express the associativity of the four-point compositions of ternary relations.

Proposition 2.9. *Among the above 18 four-point compositions of ternary relations, only the compositions $\circ_1, \circ_7$ and $\circ_{13}$ are associative, i.e., for any ternary relations T_1, T_2 and T_3 on a set X, it holds that*

$$(T_1 \circ_i T_2) \circ_i T_3 = T_1 \circ_i (T_2 \circ_i T_3),$$

with $i \in \{1, 7, 13\}$.

Proof. We only give the proof for the case $i = 1$, as the other cases can be proved analogously.

$$\begin{aligned}
(T_1 \circ_1 T_2) \circ_1 T_3 &= \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in T_1 \circ_1 T_2 \wedge (x, t, z) \in T_3)\} \\
&= \{(x, y, z) \in X^3 \mid (\exists t, s \in X)((x, y, s) \in T_1 \wedge (x, s, t) \in T_2 \wedge \\
&\qquad (x, t, z) \in T_3)\} \\
&= \{(x, y, z) \in X^3 \mid (\exists s \in X)((x, y, s) \in T_1 \wedge (x, s, z) \in T_2 \circ_1 T_3)\} \\
&= \{(x, y, z) \in X^3 \mid (x, y, z) \in T_1 \circ_1 (T_2 \circ_1 T_3)\} \\
&= T_1 \circ_1 (T_2 \circ_1 T_3).
\end{aligned}$$

□

Remark 2.2. *One easily gives counterexamples that the rest of the remaining 15 compositions are not associative. For instance, the following example shows that $\circ_2$ is not associative. Indeed, let T_1, T_2 and T_3 be the ternary relations on $X = \{x_1, x_2, x_3, x_4\}$ given by:*

$$\begin{aligned}
T_1 &= \{(x_1, x_2, x_3)\}, \\
T_2 &= \{(x_1, x_1, x_3), (x_1, x_4, x_2)\}, \\
T_3 &= \{(x_1, x_4, x_1), (x_2, x_3, x_2)\}.
\end{aligned}$$

One easily verifies that

$$(T_1 \circ_2 T_2) \circ_2 T_3 = \{(x_1, x_2, x_4)\},$$
$$T_1 \circ_2 (T_2 \circ_2 T_3) = \emptyset.$$

It is clear that

$$(T_1 \circ_2 T_2) \circ_2 T_3 \neq T_1 \circ_2 (T_2 \circ_2 T_3).$$

Remark 2.3. *It is clear that by using the commutativity of the conjunction, one proves that the following compositions are also associative.*

$$\begin{aligned} T \circ_{19} S &= S \circ_1 T; \\ T \circ_{25} S &= S \circ_7 T; \\ T \circ_{31} S &= S \circ_{13} T. \end{aligned}$$

Note that throughout this work, the associative four-point compositions $\circ_1, \circ_7$ and $\circ_{13}$ will be denoted by $\diamond_1, \diamond_2$ and $\diamond_3$ respectively. Thus, the associative compositions of two ternary relations T and S are defined as

$$\begin{aligned} T \diamond_1 S &= \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in T \wedge (x, t, z) \in S)\}; \\ T \diamond_2 S &= \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in T \wedge (t, y, z) \in S)\}; \\ T \diamond_3 S &= \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, t, z) \in T \wedge (t, y, z) \in S)\}. \end{aligned}$$

Note that the (n, i)-th powers $T^{n,\diamond_i}$ of a ternary relation T on X are recursively defined as $T^{1,\diamond_i} = T$ and $T^{n,\diamond_i} = T^{n-1,\diamond_i} \diamond_i T$, for any $n \in \mathbb{N}^*$, $n > 1$ and $i \in \{1, 2, 3\}$.

Remark 2.4. *In the binary case, it is clear that any reflexive binary relation is included in its powers. In the following, for $i \in \{1, 2, 3\}$, we show that any reflexive ternary relation is not necessarily included in its $\diamond_i$-powers. Indeed, let T be the reflexive ternary relation on $X = \{x_1, x_2, x_3, x_4\}$ given by:*

$$T = \{(x_1, x_1, x_1), (x_1, x_2, x_3), (x_2, x_2, x_2), (x_1, x_3, x_4), (x_3, x_3, x_3), (x_4, x_4, x_4)\}.$$

One easily computes the $\diamond_1$-powers of T:

$$\begin{aligned} T^{2,\diamond_1} &= \{(x_1, x_1, x_1), (x_1, x_2, x_4), (x_2, x_2, x_2), (x_3, x_3, x_3), (x_4, x_4, x_4)\}, \\ T^{3,\diamond_1} &= \{(x_1, x_1, x_1), (x_2, x_2, x_2), (x_3, x_3, x_3), (x_4, x_4, x_4)\}, \\ T^{4,\diamond_1} &= \{(x_1, x_1, x_1), (x_2, x_2, x_2), (x_3, x_3, x_3), (x_4, x_4, x_4)\}. \end{aligned}$$

It is clear that $T \not\subseteq T^{n,\diamond_1}$, for $n \in \mathbb{N}^*$. With a similar example, one easily proves the other cases.

The following proposition shows that any left (resp. middle and right) reflexive ternary relation is included in its $\diamond_1$-powers (resp. $\diamond_2$-powers and $\diamond_3$-powers). The proof is straightforward.

Proposition 2.10. *Let T be a ternary relation on a set X. The following implications hold:*

(i) The left reflexivity of T implies that $T \subseteq T^{n,\diamond_1}$, for any $n \in \mathbb{N}^*$;

(ii) The middle reflexivity of T implies that $T \subseteq T^{n,\diamond_2}$, for any $n \in \mathbb{N}^*$;

(iii) The right reflexivity of T implies that $T \subseteq T^{n,\diamond_3}$, for any $n \in \mathbb{N}^*$.

The following proposition identifies the neutral element of the $\diamond_i$-compositions of two ternary relations, with $i \in \{1, 2, 3\}$. The proof is straightforward.

Proposition 2.11. *Let T be a ternary relation on a set X.*

(i) The neutral element of the $\diamond_1$-composition is $E_\ell = \{(x, y, y) \in X^3 \mid x, y \in X\}$, i.e.,

$$T \diamond_1 E_\ell = E_\ell \diamond_1 T = T.$$

(ii) The neutral element of the $\diamond_2$-composition is $E_m = \{(x, y, x) \in X^3 \mid x, y \in X\}$, i.e.,

$$T \diamond_2 E_m = E_m \diamond_2 T = T.$$

(iii) The neutral element of the $\diamond_3$-composition is $E_r = \{(x, x, y) \in X^3 \mid x, y \in X\}$, i.e.,

$$T \diamond_3 E_r = E_r \diamond_3 T = T.$$

Remark 2.5. *It is clear that a given ternary relation T on a set X is strongly reflexive if and only if $\bigcup_{\lambda} E_\lambda \subseteq T$, for $\lambda \in \{\ell, m, r\}$.*

Inspired by the reasoning in the case of the composition of binary relations, the above associative compositions of ternary relations can be rewritten as follows. First, we need to recall the 1-ary projections of a given ternary relation.

Definition 2.4. *Let T be a ternary relation on a set X. The afterset ${}_{xy}T$, the middleset ${}_xT_y$ and the foreset T_{xy} of the elements $x, y \in X$ are defined as:*

$$\begin{aligned} {}_{xy}T &= \{t \in x \mid (x, y, t) \in T\}; \\ {}_xT_y &= \{t \in x \mid (x, t, y) \in T\}; \end{aligned}$$

$$T_{xy} = \{t \in x \mid (t, x, y) \in T\}.$$

Having in mind the above sets, the four-point compositions of ternary relations can be written as follows.

Proposition 2.12. *Let T and S be two ternary relations on a set X. The four-point compositions of T and S can be written as:*

$$\begin{aligned} T \diamond_1 S &= \{(x, y, z) \in X^3 \mid {}_{xy}T \cap {}_{x}S_{z} \neq \emptyset\}; \\ T \diamond_2 S &= \{(x, y, z) \in X^3 \mid {}_{xy}T \cap S_{yz} \neq \emptyset\}; \\ T \diamond_3 S &= \{(x, y, z) \in X^3 \mid {}_{x}T_{z} \cap S_{yz} \neq \emptyset\}. \end{aligned}$$

Remark 2.6. *(i) Note that the above sets can be used to express all the above 18 compositions.*

(ii) The above reasoning can be also used to define all the $(n+1)$-point compositions of two n-ary relations and also the reasoning to get the associative ones.

2.3.3. Interaction of the four-point compositions with basic set operations and permutations

In this subsection, we study the interaction of the four-point compositions of ternary relations with basic set operations and permutations. First, the following proposition shows the interaction of the $\diamond_i$-compositions with inclusion and set-theoretical operations, for any $i \in \{1, 2, 3\}$.

Proposition 2.13. *Let T_1, T_2, S_1, S_2 and S be ternary relations on a set X. For any $i \in \{1, 2, 3\}$, the following statements hold:*

(i) If $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$, then $T_1 \diamond_i S_1 \subseteq T_2 \diamond_i S_2$;

(ii) $(T_1 \cap T_2) \diamond_i S = (T_1 \diamond_i S) \cap (T_2 \diamond_i S)$ and $S \diamond_i (T_1 \cap T_2) = (S \diamond_i T_1) \cap (S \diamond_i T_2)$;

(iii) $(T_1 \cup T_2) \diamond_i S = (T_1 \diamond_i S) \cup (T_2 \diamond_i S)$ and $S \diamond_i (T_1 \cup T_2) = (S \diamond_i T_1) \cup (S \diamond_i T_2)$.

Proof. We only give the proof for the $\diamond_1$-composition and property (i), as the other results can be proved similarly. Suppose that $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$. Let $(x, y, z) \in T_1 \diamond_1 S_1$, then there exists $t \in X$ such that $(x, y, t) \in T_1$ and $(x, t, z) \in S_1$. Since $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$, it follows that $(x, y, z) \in T_2 \diamond_1 S_2$. Thus, $T_1 \diamond_1 S_1 \subseteq T_2 \diamond_1 S_2$. □

The following proposition shows the interaction of the $\diamond_i$-compositions of ternary relations with permutations, for any $i \in \{1, 2, 3\}$.

Proposition 2.14. *Let T and S be two ternary relations on a set X. The following equalities hold:*

(i) *(a)* $(T \diamond_1 S)^{\dashv} = S^{\dashv} \diamond_1 T^{\dashv}$;

(b) $(T \diamond_2 S)^{\dashv} = T^{\dashv} \diamond_3 S^{\dashv}$;

(c) $(T \diamond_3 S)^{\dashv} = T^{\dashv} \diamond_2 S^{\dashv}$.

(ii) *(a)* $(T \diamond_1 S)^{\vdash} = T^{\vdash} \diamond_2 S^{\vdash}$;

(b) $(T \diamond_2 S)^{\vdash} = T^{\vdash} \diamond_1 S^{\vdash}$;

(c) $(T \diamond_3 S)^{\vdash} = S^{\vdash} \diamond_3 T^{\vdash}$.

(iii) *(a)* $(T \diamond_1 S)^{+} = T^{+} \diamond_3 S^{+}$;

(b) $(T \diamond_2 S)^{+} = S^{+} \diamond_1 T^{+}$;

(c) $(T \diamond_3 S)^{+} = S^{+} \diamond_2 T^{+}$.

(iv) *(a)* $(T \diamond_1 S)^{-} = S^{-} \diamond_2 T^{-}$;

(b) $(T \diamond_2 S)^{-} = S^{-} \diamond_3 T^{-}$;

(c) $(T \diamond_3 S)^{-} = T^{-} \diamond_1 S^{-}$.

(v) *(a)* $(T \diamond_1 S)^{t} = S^{t} \diamond_3 T^{t}$;

(b) $(T \diamond_2 S)^{t} = S^{t} \diamond_2 T^{t}$;

(c) $(T \diamond_3 S)^{t} = S^{t} \diamond_1 T^{t}$.

Proof. (i) We only prove that $(T \diamond_1 S)^{\dashv} = S^{\dashv} \diamond_1 T^{\dashv}$, as the other cases are analogous.

$$\begin{aligned}
(T \diamond_1 S)^{\dashv} &= \Big\{(x, y, z) \in X^3 \mid (x, z, y) \in T \diamond_1 S\Big\} \\
&= \Big\{(x, y, z) \in X^3 \mid {}_{xz}T \cap {}_{x}S_{y} \neq \emptyset\Big\} \\
&= \Big\{(x, y, z) \in X^3 \mid (\exists t \in X)((x, z, t) \in T \wedge (x, t, y) \in S)\Big\} \\
&= \Big\{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in S^{\dashv} \wedge (x, t, z) \in T^{\dashv})\Big\} \\
&= \Big\{(x, y, z) \in X^3 \mid (x, y, z) \in S^{\dashv} \diamond_1 T^{\dashv}\Big\} \\
&= S^{\dashv} \diamond_1 T^{\dashv}.
\end{aligned}$$

(ii) We only prove that $(T \diamond_1 S)^{\vdash} = T^{\vdash} \diamond_2 S^{\vdash}$, as the other cases are analogous.

$$\begin{aligned}
(T \diamond_1 S)^{\vdash} &= \Big\{(x,y,z) \in X^3 \mid (y,x,z) \in T \diamond_1 S\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid {}_{yx}T \cap {}_{y}S_{z} \neq \emptyset\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (y,t,z) \in S)\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T^{\vdash} \wedge (t,y,z) \in S^{\vdash})\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid (x,y,z) \in T^{\vdash} \diamond_2 S^{\vdash}\Big\} \\
&= T^{\vdash} \diamond_2 S^{\vdash}.
\end{aligned}$$

(iii) We only prove that $(T \diamond_1 S)^{+} = T^{+} \diamond_3 S^{+}$, as the other cases are analogous.

$$\begin{aligned}
(T \diamond_1 S)^{+} &= \Big\{(x,y,z) \in X^3 \mid (z,x,y) \in T \diamond_1 S\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid {}_{zx}T \cap {}_{z}S_{y} \neq \emptyset\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (z,t,y) \in S)\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T^{+} \wedge (t,y,z) \in S^{+})\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid (x,y,z) \in T^{+} \diamond_3 S^{+}\Big\} \\
&= T^{+} \diamond_3 S^{+}.
\end{aligned}$$

(iv) We only prove that $(T \diamond_1 S)^{-} = S^{-} \diamond_2 T^{-}$, as the other cases are analogous.

$$\begin{aligned}
(T \diamond_1 S)^{-} &= \Big\{(x,y,z) \in X^3 \mid (y,z,x) \in T \diamond_1 S\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid {}_{yz}T \cap {}_{y}S_{x} \neq \emptyset\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (y,t,x) \in S)\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in S^{-} \wedge (t,y,z) \in T^{-})\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid (x,y,z) \in S^{-} \diamond_2 T^{-}\Big\} \\
&= S^{-} \diamond_2 T^{-}.
\end{aligned}$$

(v) We only prove that $(T \diamond_1 S)^{t} = S^{t} \diamond_3 T^{t}$, as the other cases are analogous.

$$\begin{aligned}
(T \diamond_1 S)^{t} &= \Big\{(x,y,z) \in X^3 \mid (z,y,x) \in T \diamond_1 S\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid {}_{zy}T \cap {}_{z}S_{x} \neq \emptyset\Big\} \\
&= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((z,y,t) \in T \wedge (z,t,x) \in S)\Big\}
\end{aligned}$$

$$= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in S^t \wedge (t,y,z) \in T^t)\Big\}$$
$$= \Big\{(x,y,z) \in X^3 \mid (x,y,z) \in S^t \diamond_3 T^t\Big\}$$
$$= S^t \diamond_3 T^t.$$

□

Proposition 2.14 leads to the following result.

Corollary 2.1. *Let T be a ternary relation on a set X. The following equalities hold:*

(i) *(a)* $(T^{n,\diamond_1})^{\dashv} = (T^{\dashv})^{n,\diamond_1}$;

(b) $(T^{n,\diamond_2})^{\dashv} = (T^{\dashv})^{n,\diamond_3}$;

(c) $(T^{n,\diamond_3})^{\dashv} = (T^{\dashv})^{n,\diamond_2}$.

(ii) *(a)* $(T^{n,\diamond_1})^{\vdash} = (T^{\vdash})^{n,\diamond_2}$;

(b) $(T^{n,\diamond_2})^{\vdash} = (T^{\vdash})^{n,\diamond_1}$;

(c) $(T^{n,\diamond_3})^{\vdash} = (T^{\vdash})^{n,\diamond_3}$.

(iii) *(a)* $(T^{n,\diamond_1})^{+} = (T^{+})^{n,\diamond_3}$;

(b) $(T^{n,\diamond_2})^{+} = (T^{+})^{n,\diamond_1}$;

(c) $(T^{n,\diamond_3})^{+} = (T^{+})^{n,\diamond_2}$.

(iv) *(a)* $(T^{n,\diamond_1})^{-} = (T^{-})^{n,\diamond_2}$;

(b) $(T^{n,\diamond_2})^{-} = (T^{-})^{n,\diamond_3}$;

(c) $(T^{n,\diamond_3})^{-} = (T^{-})^{n,\diamond_1}$.

(v) *(a)* $(T^{n,\diamond_1})^{t} = (T^{t})^{n,\diamond_3}$;

(b) $(T^{n,\diamond_2})^{t} = (T^{t})^{n,\diamond_2}$;

(c) $(T^{n,\diamond_3})^{t} = (T^{t})^{n,\diamond_1}$.

2.3.4. Interaction of the four-point compositions with binary projections and cylindrical extensions

In this subsection, we study the interaction of the four-point compositions of ternary relations with binary projections and cylindrical extensions. The following proposition investigates the projections of the four-point compositions of ternary relations in terms of the compositions of their binary projections.

Proposition 2.15. *Let T and S be two ternary relations on a set X. The following table shows the inclusion of the left, middle and right projections of the compositions $(T \diamond_i S)$ of ternary relations in the compositions of their binary projections, for any $i \in \{1, 2, 3\}$.*

Comp. \ Proj.	$P_\ell(.)$	$P_m(.)$	$P_r(.)$
$T \diamond_1 S$	$P_\ell(T) \circ P_\ell(S)$	$P_m(T) \circ P_\ell(S)$	$-$
$T \diamond_2 S$	$P_\ell(T) \circ P_m(S)$	$P_m(T) \circ P_m(S), P_r(T) \circ P_\ell(S)$	$P_m(T) \circ P_r(S)$
$T \diamond_3 S$	$-$	$P_r(T) \circ P_m(S)$	$P_r(T) \circ P_r(S)$

Proof. We only prove that $P_\ell(T \diamond_1 S) \subseteq P_\ell(T) \circ P_\ell(S)$, as the other cases are analogous. Let $(x, y) \in P_\ell(T \diamond_1 S)$. Then there exists $z, t \in X$ such that $(z, x, t) \in T$ and $(z, t, y) \in S$. This implies that $(x, t) \in P_\ell(T)$ and $(t, y) \in P_\ell(S)$. Hence, $(x, y) \in P_\ell(T) \circ P_\ell(S)$. Thus, $P_\ell(T \diamond_1 S) \subseteq P_\ell(T) \circ P_\ell(S)$. □

Remark 2.7. *The following example shows that the projections of the $\diamond_1$-composition of two ternary relations are not equal to the composition of their binary projections. Indeed, let T and S be the ternary relations on $X = \{x_1, x_2, x_3, x_4\}$ given by:*

$$T = \{(x_1, x_1, x_2), (x_1, x_2, x_3)\},$$
$$S = \{(x_1, x_2, x_4), (x_2, x_4, x_1), (x_3, x_2, x_2)\}.$$

It holds that

$$T \diamond_1 S = \{(x_1, x_1, x_4)\},$$

and thus

$$P_\ell(T \diamond_1 S) = \{(x_1, x_4)\},$$
$$P_m(T \diamond_1 S) = \{(x_1, x_4)\}.$$

Further, it holds that

	$P_\ell(.)$	$P_m(.)$	$P_r(.)$
T	$\{(x_1, x_2), (x_2, x_3)\}$	$\{(x_1, x_2), (x_1, x_3)\}$	$\{(x_1, x_1), (x_1, x_2)\}$
S	$\{(x_2, x_2), (x_2, x_4), (x_4, x_1)\}$	$\{(x_1, x_4), (x_2, x_1), (x_3, x_2)\}$	$\{(x_1, x_2), (x_2, x_4), (x_3, x_2)\}$

and

$\circ$	$P_\ell(S)$	$P_m(S)$	$P_r(S)$
$P_\ell(T)$	$\{(x_1,x_2),(x_1,x_4)\}$	$\{(x_1,x_1),(x_2,x_2)\}$	$\{(x_1,x_4),(x_2,x_2)\}$
$P_m(T)$	$\{(x_1,x_2),(x_1,x_4)\}$	$\{(x_1,x_1),(x_1,x_2)\}$	$\{(x_1,x_2),(x_1,x_4)\}$
$P_r(T)$	$\{(x_1,x_2),(x_1,x_4)\}$	$\{(x_1,x_1),(x_1,x_4)\}$	$\{(x_1,x_2),(x_1,x_4)\}$

It is clear that

$$P_\ell(T \diamond_1 S) \neq P_\ell(T) \circ P_\ell(S),$$

$$P_m(T \diamond_1 S) \neq P_m(T) \circ P_\ell(S).$$

In a similar way, one proves the other cases.

The following proposition expresses the cylindrical extensions of the composition of binary relations in terms of the four-point compositions of their cylindrical extensions.

Proposition 2.16. *Let R_1 and R_2 be two binary relations on a set X. The left, middle and right cylindrical extensions of the composition $R_1 \circ R_2$ are listed in the following table:*

Cyl. ext. / *Comp.*	$C_\ell(.)$	$C_m(.)$	$C_r(.)$
$R_1 \circ R_2$	$C_\ell(R_1) \diamond_1 C_\ell(R_2)$ $C_\ell(R_1) \diamond_2 C_m(R_2)$	$C_m(R_1) \diamond_1 C_\ell(R_2)$ $C_m(R_1) \diamond_2 C_m(R_2)$ $C_r(R_1) \diamond_3 C_m(R_2)$	$C_m(R_1) \diamond_2 C_r(R_2)$ $C_r(R_1) \diamond_3 C_r(R_2)$

Proof. We only prove that $C_\ell(R_1 \circ R_2) = C_\ell(R_1) \diamond_1 C_\ell(R_2)$, as the other cases are analogous.

$$\begin{aligned} C_\ell(R_1 \circ R_2) &= \{(x,y,z) \in X^3 \mid (y,z) \in R_1 \circ R_2\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t) \in R_1 \wedge (t,z) \in R_2)\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in C_\ell(R_1) \wedge (x,t,z) \in C_\ell(R_2))\} \\ &= \{(x,y,z) \in X^3 \mid (x,y,z) \in C_\ell(R_1) \diamond_1 C_\ell(R_2)\} \\ &= C_\ell(R_1) \diamond_1 C_\ell(R_2). \end{aligned}$$

□

Remark 2.8. *The following example shows that the left cylindrical extension of the composition of two binary relations is not equal to the $\diamond_3$-composition of any of their cylindrical extensions. Indeed, let R_1 and R_2 be the binary relations on $X = \{x_1, x_2, x_3\}$ given by:*

$$R_1 = \{(x_1,x_2)\},$$
$$R_2 = \{(x_2,x_3),(x_3,x_3)\}.$$

It holds that

$$R_1 \circ R_2 = \{(x_1, x_3)\},$$

and thus

$$C_\ell(R_1 \circ R_2) = \{(x_1, x_1, x_3), (x_2, x_1, x_3), (x_3, x_1, x_3)\}.$$

Further, it holds that

	$C_\ell(.)$	$C_m(.)$	$C_r(.)$
R_1	$\{(x_1,x_1,x_2),(x_2,x_1,x_2),$ $(x_3,x_1,x_2)\}$	$\{(x_1,x_1,x_2),(x_1,x_2,x_2),$ $(x_1,x_3,x_2)\}$	$\{(x_1,x_2,x_1),(x_1,x_2,x_2),$ $(x_1,x_2,x_3)\}$
R_2	$\{(x_1,x_2,x_3),(x_1,x_3,x_3),$ $(x_2,x_2,x_3),(x_2,x_3,x_3),$ $(x_3,x_2,x_3),(x_3,x_3,x_3)\}$	$\{(x_2,x_1,x_3),(x_2,x_2,x_3),$ $(x_2,x_3,x_3),(x_3,x_1,x_3),$ $(x_3,x_2,x_3),(x_3,x_3,x_3)\}$	$\{(x_2,x_3,x_1),(x_2,x_3,x_2),$ $(x_2,x_3,x_3),(x_3,x_3,x_1),$ $(x_3,x_3,x_2),(x_3,x_3,x_3)\}$

and

$\diamond_3$	$C_\ell(R_2)$	$C_m(R_2)$	$C_r(R_2)$
$C_\ell(R_1)$	$\emptyset$	$\emptyset$	$\emptyset$
$C_m(R_1)$	$\emptyset$	$\emptyset$	$\{(x_1,x_3,x_2)\}$
$C_r(R_1)$	$\{(x_1,x_2,x_3),(x_1,x_3,x_3)\}$	$\{(x_1,x_1,x_3),(x_1,x_2,x_3),$ $(x_1,x_3,x_3)\}$	$\{(x_1,x_3,x_1),(x_1,x_3,x_2),$ $(x_1,x_3,x_3)\}$

It is clear that for any $\lambda_1, \lambda_2 \in \{\ell, m, r\}$ it holds that

$$C_\ell(R_1 \circ R_2) \neq C_{\lambda_1}(R_1) \diamond_3 C_{\lambda_2}(R_2).$$

In a similar way, one proves the other case.

The following proposition expresses the compositions of a ternary relation with a binary relation introduced in [71] in terms of the associative four-point compositions of ternary relations.

Proposition 2.17. *Let T be a ternary relation and R be a binary relation on a set X. The following equalities hold:*

(i) $T \ltimes R = T \diamond_1 C_\ell(R)$;

(ii) $T \ltimes R = T \diamond_2 C_m(R)$;

(v) $R \rtimes T = C_m(R) \diamond_2 T$;

(vi) $R \rtimes T = C_r(R) \diamond_3 T$.

Proof. We only give the proof for the first equality, as the other equalities can be proved analogously.

$$T \ltimes R = \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in T \wedge (t, z) \in R)\}$$

$$= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (x,t,z) \in C_\ell(R))\}$$
$$= T \diamond_1 C_\ell(R)\,.$$

□

2.3.5. Interaction of the four-point compositions with traces

In the following, we study the interaction of the four-point compositions of ternary relations with left, middle and right traces.

Proposition 2.18. *Let T and S be two ternary relations on a set X. The following inclusions hold:*

(i) $T^\ell \subseteq (T \diamond_i S)^\ell$, *for any* $i \in \{2,3\}$*;*

(ii) $T^m \subseteq (T \diamond_1 S)^m$*;*

(iii) $S^m \subseteq (T \diamond_3 S)^m$*;*

(iv) $S^r \subseteq (T \diamond_i S)^r$, *for any* $i \in \{1,2\}$*.*

Proof. We only give the proof for first inclusion and $i = 2$, as the other inclusions can be proved analogously. Let $(x,y) \in T^\ell$ and $(x,a,b) \in T \diamond_2 S$. Then there exists $t \in X$ such that $(x,a,t) \in T$ and $(t,a,b) \in S$. Since $(x,y) \in T^\ell$, it follows that $(y,a,t) \in T$. This implies that $(y,a,b) \in T \diamond_2 S$. Hence, $(x,y) \in (T \diamond_2 S)^\ell$. Thus, $T^\ell \subseteq (T \diamond_2 S)^\ell$. □

Next, we will show that the traces of a ternary relation are the greatest binary relations that satisfy some relational inclusions. First, we need to recall the following two theorems.

Theorem 2.1. [71] *Let T be a ternary relation on a set X. It holds that*

(i) T^ℓ *is the greatest binary relation* R *that satisfies* $R^t \rtimes T \subseteq T$*;*

(ii) T^r *is the greatest binary relation* R *that satisfies* $T \ltimes R \subseteq T$*.*

Theorem 2.2. [71] *Let T be a ternary relation on a set X. It holds that*

$$(T^\ell)^t \rtimes T = T \ltimes T^r = T.$$

The following result shows that the traces of a ternary relation are the greatest binary relations that satisfy the following relational inclusions corresponding to the relational compositions introduced above.

Proposition 2.19. *Let T be a ternary relation on a set X. It holds that*

(i) T^ℓ *is the greatest binary relation R that satisfies the following inclusions:*

(a) $C_m(R^t) \diamond_2 T \subseteq T$;

(b) $C_r(R^t) \diamond_3 T \subseteq T$.

(ii) T^m *is the greatest binary relation R that satisfies the following inclusions:*

(a) $C_\ell(R^t) \diamond_1 T \subseteq T$;

(b) $T \diamond_3 C_r(R) \subseteq T$.

(iii) T^r *is the greatest binary relation R that satisfies the following inclusions:*

(a) $T \diamond_1 C_\ell(R) \subseteq T$;

(b) $T \diamond_2 C_m(R) \subseteq T$.

Proof. We only give the proof for the inclusion (i) (a), as the other inclusions can be proved analogously. Note that Proposition 2.17 implies that $R^t \rtimes T = C_m(R^t) \diamond_2 T$. Theorem 2.1 guarantees that T^ℓ is the greatest binary relation R that satisfies $R^t \rtimes T \subseteq R$. Hence, T^ℓ is the greatest binary relation R that satisfies $C_m(R^t) \diamond_2 T \subseteq T$. □

Combining Proposition 2.17 and Theorem 2.2 leads to the following result.

Corollary 2.2. *Let T be a ternary relation on a set X. The following equalities hold:*

(i) $T \diamond_1 C_\ell(T^r) = T$;

(ii) $T \diamond_2 C_m(T^r) = T$;

(iii) $T \diamond_3 C_r(T^m) = T$;

(iv) $C_\ell((T^m)^t) \diamond_1 T = T$;

(v) $C_m((T^\ell)^t) \diamond_2 T = T$;

(vi) $C_r((T^\ell)^t) \diamond_3 T = T$.

2.4. Five-point compositions of ternary relations

In this section, based on the compositions of a ternary relation with a binary relation introduced by Zedam et al. [71], we introduce several types of composition of ternary relations considering five points through the use of the binary projections. Also, we investigate their properties and interactions with basic set operations, permutations, binary projections, cylindrical extensions and traces.

2.4.1. Five-point compositions

Based on the compositions of a ternary relation with a binary relation, we introduce six types of five-point composition of ternary relations through the use of the binary projections.

Definition 2.5. *Let T and S be two ternary relations on a set X. The $\circ_i$-compositions of T and S, with $i \in \{1, \ldots, 6\}$, are defined as*

(i) $T \circ_1 S := T \ltimes P_\ell(S) = \{(x, y, z) \in X^3 \mid (\exists t, s \in X)((x, y, t) \in T \wedge (s, t, z) \in S)\}$;

(ii) $T \circ_2 S := T \ltimes P_m(S) = \{(x, y, z) \in X^3 \mid (\exists t, s \in X)((x, y, t) \in T \wedge (t, s, z) \in S)\}$;

(iii) $T \circ_3 S := T \ltimes P_r(S) = \{(x, y, z) \in X^3 \mid (\exists t, s \in X)((x, y, t) \in T \wedge (t, z, s) \in S)\}$;

(iv) $T \circ_4 S := P_\ell(T) \rtimes S = \{(x, y, z) \in X^3 \mid (\exists t, s \in X)((s, x, t) \in T \wedge (t, y, z) \in S)\}$;

(v) $T \circ_5 S := P_m(T) \rtimes S = \{(x, y, z) \in X^3 \mid (\exists t, s \in X)((x, s, t) \in T \wedge (t, y, z) \in S)\}$;

(vi) $T \circ_6 S := P_r(T) \rtimes S = \{(x, y, z) \in X^3 \mid (\exists t, s \in X)((x, s, t) \in T \wedge (s, y, z) \in S)\}$.

Next, we introduce a composition that is composed of all of the above six five-point compositions.

Definition 2.6. *Let T and S be two ternary relations on a set X. The $\circ$-composition of T and S is defined as*

$$T \circ S = (T \ltimes P(S)) \cup (P(T) \rtimes S) .$$

Clearly, it holds that $T \circ S = \bigcup_{i=1}^{6} T \circ_i S$.

The following example shows that the compositions introduced are different.

Example 2.2. *Let* T *and* S *be the ternary relations on* $X = \{x_1, x_2, x_3, x_4\}$ *given by:*

$$T = \{(x_1, x_1, x_2), (x_1, x_2, x_3)\}\,,$$
$$S = \{(x_1, x_1, x_1), (x_2, x_4, x_1), (x_3, x_2, x_2)\}\,.$$

One easily verifies that

$$\begin{aligned}
T \circ_1 S &= \{(x_1, x_1, x_2)\}\,,\\
T \circ_2 S &= \{(x_1, x_1, x_1), (x_1, x_2, x_2)\}\,,\\
T \circ_3 S &= \{(x_1, x_1, x_4), (x_1, x_2, x_2)\}\,,\\
T \circ_4 S &= \{(x_1, x_4, x_1), (x_2, x_2, x_2)\}\,,\\
T \circ_5 S &= \{(x_1, x_4, x_1), (x_1, x_2, x_2)\}\,,\\
T \circ_6 S &= \{(x_1, x_1, x_1), (x_1, x_4, x_1)\}\,,
\end{aligned}$$

and

$$T \circ S = \{(x_1, x_4, x_1), (x_2, x_2, x_2), (x_1, x_2, x_2), (x_1, x_1, x_1), (x_1, x_1, x_2), (x_1, x_1, x_4)\}\,.$$

2.4.2. Properties of the five-point compositions of ternary relations

In this subsection, we investigate some properties of the five-point compositions of ternary relations. First of all, we show that four of these compositions are associative.

Proposition 2.20. *The compositions* $\circ_i$, *with* $i \in \{1, 2, 5, 6\}$, *are associative, i.e., for any ternary relations* T_1, T_2 *and* T_3 *on a set* X, *it holds that*

$$(T_1 \circ_i T_2) \circ_i T_3 = T_1 \circ_i (T_2 \circ_i T_3)\,.$$

Proof. We only give the proof for the case $i = 1$, as the other cases can be proved analogously.

$$\begin{aligned}
(T_1 \circ_1 T_2) \circ_1 T_3 &= \{(x, y, z) \in X^3 \mid (\exists t, s \in X)((x, y, t) \in T_1 \circ_1 T_2 \wedge (s, t, z) \in T_3)\}\\
&= \{(x, y, z) \in X^3 \mid (\exists t, s, m, n \in X)((x, y, m) \in T_1 \wedge (n, m, t) \in\\
&\qquad T_2 \wedge (s, t, z) \in T_3)\}\\
&= \{(x, y, z) \in X^3 \mid (\exists m, n \in X)((x, y, m) \in T_1 \wedge (n, m, z) \in
\end{aligned}$$

$$\begin{aligned}
&T_2 \circ_1 T_3)\} \\
&= \{(x,y,z) \in X^3 \mid (x,y,z) \in T_1 \circ_1 (T_2 \circ_1 T_3)\} \\
&= T_1 \circ_1 (T_2 \circ_1 T_3)\,.
\end{aligned}$$

□

In the following example, we show that the compositions $\circ_3$ and $\circ_4$ are not associative.

Example 2.3. *Let T_1, T_2 and T_3 be the ternary relations on $X = \{x_1, x_2, x_3, x_4\}$ given by:*

$$\begin{aligned}
T_1 &= \{(x_1,x_1,x_2),(x_1,x_2,x_3)\}\,, \\
T_2 &= \{(x_1,x_1,x_1),(x_2,x_4,x_1),(x_3,x_2,x_2)\}\,, \\
T_3 &= \{(x_1,x_1,x_1),(x_4,x_4,x_2),(x_2,x_3,x_1)\}\,.
\end{aligned}$$

One easily verifies that

$$\begin{aligned}
(T_1 \circ_3 T_2) \circ_3 T_3 &= \{(x_1,x_1,x_4),(x_1,x_2,x_3)\}\,, \\
T_1 \circ_3 (T_2 \circ_3 T_3) &= \{(x_1,x_1,x_4),(x_1,x_2,x_2)\}\,, \\
(T_1 \circ_4 T_2) \circ_4 T_3 &= \{(x_4,x_1,x_1),(x_2,x_3,x_1)\}\,, \\
T_1 \circ_4 (T_2 \circ_4 T_3) &= \{(x_1,x_3,x_1)\}\,.
\end{aligned}$$

It is clear that

$$(T_1 \circ_3 T_2) \circ_3 T_3 \neq T_1 \circ_3 (T_2 \circ_3 T_3) \quad \text{and} \quad (T_1 \circ_4 T_2) \circ_4 T_3 \neq T_1 \circ_4 (T_2 \circ_4 T_3)\,.$$

Obviously, since $\circ_3$ and $\circ_4$ are not associative, the composition $\circ$ is not associative as well.

The following proposition shows that the ternary identity relation I_{X^3} is a right neutral element of three of the five-point compositions of ternary relations, while it is a left neutral element of the other compositions. The proof of this proposition is straightforward.

Proposition 2.21. *Let T be a ternary relation on a set X. It holds that*

(i) $T \circ_i I_{X^3} = T$, *for any* $i \in \{1,2,3\}$;

(ii) $I_{X^3} \circ_i T = T$, *for any* $i \in \{4,5,6\}$.

Next, we give a counterexample showing that I_{X^3} is not a left (resp. right) neutral element of the $\circ_i$-composition, for any $i \in \{1,2,3\}$ (resp. for any $i \in$

$\{4, 5, 6\}$).

Example 2.4. *Let T be the ternary relation on $X = \{x_1, x_2, x_3, x_4\}$ given by:*

$$T = \{(x_1, x_1, x_2), (x_1, x_2, x_3)\}\,.$$

One easily verifies that

$$\begin{aligned} I_{X^3} \circ_1 T &= \{(x_1, x_1, x_2), (x_2, x_2, x_3)\}\,, \\ I_{X^3} \circ_2 T &= \{(x_1, x_1, x_2), (x_1, x_1, x_3)\}\,, \\ I_{X^3} \circ_3 T &= \{(x_1, x_1, x_1), (x_1, x_1, x_2)\}\,, \\ T \circ_4 I_{X^3} &= \{(x_1, x_2, x_2), (x_2, x_3, x_3)\}\,, \\ T \circ_5 I_{X^3} &= \{(x_1, x_2, x_2), (x_1, x_3, x_3)\}\,, \\ T \circ_6 I_{X^3} &= \{(x_1, x_1, x_1), (x_1, x_2, x_2)\}\,. \end{aligned}$$

It is clear that

$$I_{X^3} \circ_1 T \neq T, I_{X^3} \circ_2 T \neq T \quad \text{and} \quad I_{X^3} \circ_3 T \neq T\,,$$

$$T \circ_4 I_{X^3} \neq T, T \circ_5 I_{X^3} \neq T \quad \text{and} \quad T \circ_6 I_{X^3} \neq T\,,$$

and

$$I_{X^3} \circ T \neq T \circ I_{X^3} \neq T\,.$$

In the following proposition, we show that the empty relation $\emptyset$ is an absorbing element of the five-point compositions of ternary relations. The proof is straightforward.

Proposition 2.22. *Let T be a ternary relation on a set X. It holds that*

$$T \circ_i \emptyset = \emptyset \circ_i T = \emptyset\,,$$

for any $i \in \{1, \ldots, 6\}$, and hence also $T \circ \emptyset = \emptyset \circ T = \emptyset$.

2.4.3. Interaction of the five-point compositions with basic set operations and permutations

In this subsection, we study the interaction of the five-point compositions of ternary relations with basic set operations and permutations. First, the following proposition shows the interaction of the $\circ_i$-compositions with inclusion and set-theoretical operations, for any $i \in \{1, \ldots, 6\}$.

Proposition 2.23. *Let T_1, T_2, S_1, S_2 and S be ternary relations on a set X. For any $i \in \{1, \ldots, 6\}$, the following statements hold:*

(i) If $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$, then $T_1 \circ_i S_1 \subseteq T_2 \circ_i S_2$;

(ii) $(T_1 \cap T_2) \circ_i S = (T_1 \circ_i S) \cap (T_2 \circ_i S)$ and $S \circ_i (T_1 \cap T_2) = (S \circ_i T_1) \cap (S \circ_i T_2)$;

(iii) $(T_1 \cup T_2) \circ_i S = (T_1 \circ_i S) \cup (T_2 \circ_i S)$ and $S \circ_i (T_1 \cup T_2) = (S \circ_i T_1) \cup (S \circ_i T_2)$.

Proof. We only give the proof for the $\circ_1$-composition and property (i) as the other results are analogous. Suppose that $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$. Let $(x, y, z) \in T_1 \circ_1 S_1$, then it holds that $(x, y, z) \in T_1 \ltimes P_\ell(S_1)$. Then there exists $t \in X$ such that $(x, y, t) \in T_1$ and $(t, z) \in P_\ell(S_1)$. Since $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$, it follows that $(x, y, z) \in T_2 \ltimes P_\ell(S_2)$. Hence, $(x, y, z) \in T_2 \circ_1 S_2$. Thus, $T_1 \circ_1 S_1 \subseteq T_2 \circ_1 S_2$. □

The following proposition shows the interaction of the $\circ$-composition with inclusion and set-theoretical operations.

Proposition 2.24. *Let T_1, T_2, S_1, S_2 and S be ternary relations on a set X. The following statements hold:*

(i) If $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$, then $T_1 \circ S_1 \subseteq T_2 \circ S_2$;

(ii) $(T_1 \cap T_2) \circ S \subseteq (T_1 \circ S) \cap (T_2 \circ S)$;

(iii) $(T_1 \cup T_2) \circ S \subseteq (T_1 \circ S) \cup (T_2 \circ S)$.

Proof.

(i) Suppose that $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$. Let $(x, y, z) \in T_1 \circ S_1$, then it holds that $(x, y, z) \in \bigcup_{i=1}^{6} T_1 \circ_i S_1$. Then there exists $j \in \{1, \ldots, 6\}$ such that $(x, y, z) \in T_1 \circ_j S_1$. Since $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$, it follows from Proposition 2.23 that $T_1 \circ_j S_1 \subseteq T_2 \circ_j S_2$. Hence, $(x, y, z) \in T_2 \circ_j S_2$. Thus, $(x, y, z) \in T_2 \circ S_2$.

(ii) Let $(x, y, z) \in (T_1 \cap T_2) \circ S$. It holds that $(x, y, z) \in \bigcup_{i=1}^{6} (T_1 \cap T_2) \circ_i S$. Then there exists $j \in \{1, \ldots, 6\}$ such that $(x, y, z) \in (T_1 \cap T_2) \circ_j S$. From Proposition 2.23, it follows that $(x, y, z) \in (T_1 \circ_j S) \cap (T_2 \circ_j S)$, and, hence, $(x, y, z) \in \bigcup_{i=1}^{6} T_1 \circ_i S$ and $(x, y, z) \in \bigcup_{i=1}^{6} T_2 \circ_i S$. Hence, $(x, y, z) \in (T_1 \circ S) \cap (T_2 \circ S)$. Thus, $(T_1 \cap T_2) \circ S \subseteq (T_1 \circ S) \cap (T_2 \circ S)$.

(iii) The proof is analogous to that of (ii).

□

The following example shows that the equality in properties (ii) and (iii) of Proposition 2.24 does not hold in general.

Example 2.5. *Let T_1, T_2 and T_3 be the ternary relations on $X = \{x_1, x_2, x_3, x_4\}$ given by:*

$$\begin{aligned} T_1 &= \{(x_1, x_1, x_2), (x_1, x_2, x_3)\}\,, \\ T_2 &= \{(x_1, x_1, x_1), (x_2, x_4, x_1), (x_3, x_2, x_2)\}\,, \\ T_3 &= \{(x_1, x_1, x_2), (x_4, x_4, x_2), (x_2, x_3, x_1)\}\,. \end{aligned}$$

One easily verifies that

$$\begin{aligned} (T_1 \cap T_3) \circ T_2 &= \{(x_1, x_1, x_1), (x_1, x_1, x_2), (x_1, x_4, x_1), (x_1, x_1, x_4)\}\,, \\ (T_1 \circ T_2) \cap (T_3 \circ T_2) &= \{(x_1, x_1, x_1), (x_1, x_1, x_2), (x_1, x_4, x_1), (x_1, x_1, x_4), (x_2, x_2, x_2)\}\,. \end{aligned}$$

It is clear that $(T_1 \cap T_3) \circ T_2 \neq (T_1 \circ T_2) \cap (T_3 \circ T_2)$.

In the following, we investigate the interaction of the five-point compositions with permutations.

Proposition 2.25. *Let T and S be two ternary relations on a set X. The following equalities hold:*

(i) $(T \circ_i S)^{\dashv} = T \circ_i S^{\dashv}$, *for any* $i \in \{4, 5, 6\}$*;*

(ii) $(T \circ_i S)^{\vdash} = T^{\vdash} \circ_i S$, *for any* $i \in \{1, 2, 3\}$*;*

(iii) $(T \circ_i S)^{+} = (S^t \circ_{7-i} T^t)^{\vdash}$, *for any* $i \in \{1, \ldots, 6\}$*;*

(iv) $(T \circ_i S)^{+} = S^{+} \circ_{i-4} T^{+}$, *for any* $i \in \{5, 6\}$*;*

(v) $(T \circ_i S)^{-} = (S^t \circ_{7-i} T^t)^{\dashv}$, *for any* $i \in \{1, \ldots, 6\}$*;*

(vi) $(T \circ_i S)^{-} = S^{-} \circ_{i+4} T^{-}$, *for any* $i \in \{1, 2\}$*;*

(vii) $(T \circ_i S)^{t} = S^t \circ_{7-i} T^t$, *for any* $i \in \{1, \ldots, 6\}$.

Proof. (i) We only prove that $(T \circ_4 S)^{\dashv} = T \circ_4 S^{\dashv}$, as the other cases can be proved analogously.

$$\begin{aligned} (T \circ_4 S)^{\dashv} &= \Big\{(x, y, z) \in X^3 \mid (x, z, y) \in T \circ_4 S\Big\} \\ &= \Big\{(x, y, z) \in X^3 \mid (x, z, y) \in P_\ell(T) \rtimes S\Big\} \\ &= \Big\{(x, y, z) \in X^3 \mid (\exists t \in X)((x, t) \in P_\ell(T) \wedge (t, z, y) \in S)\Big\} \\ &= \Big\{(x, y, z) \in X^3 \mid (\exists t \in X)((x, t) \in P_\ell(T) \wedge (t, y, z) \in S^{\dashv})\Big\} \end{aligned}$$

$$= \Big\{(x,y,z) \in X^3 \mid (x,y,z) \in P_\ell(T) \rtimes S^{\dashv}\Big\}$$
$$= T \circ_4 S^{\dashv}.$$

(ii) We only prove that $(T \circ_1 S)^{\vdash} = T^{\vdash} \circ_1 S$, as the other cases can be proved analogously.

$$\begin{aligned}
(T \circ_1 S)^{\vdash} &= \Big\{(x,y,z) \in X^3 \mid (y,x,z) \in T \circ_1 S\Big\}\\
&= \Big\{(x,y,z) \in X^3 \mid (y,x,z) \in T \ltimes P_\ell(S)\Big\}\\
&= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (t,z) \in P_\ell(S))\Big\}\\
&= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T^{\vdash} \wedge (t,z) \in P_\ell(S))\Big\}\\
&= \Big\{(x,y,z) \in X^3 \mid (x,y,z) \in T^{\vdash} \ltimes P_\ell(S)\Big\}\\
&= T^{\vdash} \circ_1 S.
\end{aligned}$$

(iii) We only prove that $(T \circ_1 S)^{+} = (S^t \circ_6 T^t)^{\vdash}$, as the other cases can be proved analogously.

$$\begin{aligned}
(T \circ_1 S)^{+} &= \Big\{(x,y,z) \in X^3 \mid (z,x,y) \in T \circ_1 S\Big\}\\
&= \Big\{(x,y,z) \in X^3 \mid (z,x,y) \in T \ltimes P_\ell(S)\Big\}\\
&= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (t,y) \in P_\ell(S))\Big\}\\
&= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t) \in P_r(S^t) \wedge (t,x,z) \in T^t)\Big\}\\
&= \Big\{(x,y,z) \in X^3 \mid (y,x,z) \in P_r(S^t) \rtimes T^t\Big\}\\
&= \Big\{(x,y,z) \in X^3 \mid (x,y,z) \in (S^t \circ_6 T^t)^{\vdash}\Big\}\\
&= (S^t \circ_6 T^t)^{\vdash}.
\end{aligned}$$

(iv) We only prove that $(T \circ_5 S)^{+} = S^{+} \circ_1 T^{+}$, as the other cases can be proved analogously.

$$\begin{aligned}
(T \circ_5 S)^{+} &= \{(x,y,z) \in X^3 \mid (x,y,z) \in (T \circ_5 S)^{+}\}\\
&= \{(x,y,z) \in X^3 \mid (z,x,y) \in T \circ_5 S\}\\
&= \{(x,y,z) \in X^3 \mid (\exists t,s \in X)((z,s,t) \in T \wedge (t,x,y) \in S)\}\\
&= \{(x,y,z) \in X^3 \mid (\exists t,s \in X)((x,y,t) \in S^{+} \wedge (s,t,z) \in T^{+})\}\\
&= S^{+} \circ_1 T^{+}.
\end{aligned}$$

(v) We only prove that $(T \circ_1 S)^- = (S^t \circ_6 T^t)^{\dashv}$, as the other cases can be proved analogously.

$$\begin{aligned}(T \circ_1 S)^- &= \Big\{(x,y,z) \in X^3 \mid (y,z,x) \in T \circ_1 S\Big\}\\ &= \Big\{(x,y,z) \in X^3 \mid (y,z,x) \in T \ltimes P_\ell(S)\Big\}\\ &= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (t,x) \in P_\ell(S))\Big\}\\ &= \Big\{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t) \in P_r(S^t) \wedge (t,z,y) \in T^t)\Big\}\\ &= \Big\{(x,y,z) \in X^3 \mid (x,z,y) \in P_r(S^t) \rtimes T^t\Big\}\\ &= \Big\{(x,y,z) \in X^3 \mid (x,y,z) \in (S^t \circ_6 T^t)^{\dashv}\Big\}\\ &= (S^t \circ_6 T^t)^{\dashv}.\end{aligned}$$

(vi) We only prove that $(T \circ_1 S)^- = S^- \circ_5 T^-$, as the other cases can be proved analogously.

$$\begin{aligned}(T \circ_1 S)^- &= \{(x,y,z) \in X^3 \mid (x,y,z) \in (T \circ_1 S)^-\}\\ &= \{(x,y,z) \in X^3 \mid (y,z,x) \in T \circ_1 S\}\\ &= \{(x,y,z) \in X^3 \mid (\exists t,s \in X)((y,z,t) \in T \wedge (s,t,x) \in S)\}\\ &= \{(x,y,z) \in X^3 \mid (\exists t,s \in X)((x,s,t) \in S^- \wedge (t,y,z) \in T^-)\}\\ &= S^- \circ_5 T^- .\end{aligned}$$

(vii) We only prove that $(T \circ_1 S)^t = S^t \circ_6 T^t$, as the other cases can be proved analogously.

$$\begin{aligned}(T \circ_1 S)^t &= (T \ltimes P_\ell(S))^t\\ &= (P_\ell(S))^t \rtimes T^t\\ &= P_r(S^t) \rtimes T^t\\ &= S^t \circ_6 T^t.\end{aligned}$$

□

Corollary 2.3. *Let T be a ternary relation on a set X. It holds that:*

(i) $(T^+)^{n,\circ_i} = (T^{n,i+4})^+$, *for any* $i \in \{1,2\}$*;*

(ii) $(T^-)^{n,\circ_i} = (T^{n,i-4})^-$, *for any* $i \in \{5,6\}$*;*

(iii) $(T^t)^{n,\circ_i} = (T^{n,7-i})^t$, *for any* $i \in \{1,2,5,6\}$.

In the following proposition, we investigate the interaction of the $\circ$-composition with permutations.

Proposition 2.26. *Let T and S be two ternary relations on a set X. The following equalities hold:*

(i) $(T \circ S)^+ = (S^t \circ T^t)^{\vdash}$;

(ii) $(T \circ S)^- = (S^t \circ T^t)^{\dashv}$;

(iii) $(T \circ S)^t = S^t \circ T^t$.

Proof. We only prove (i), as the other cases are analogous. From Remark 1.1 and Proposition 2.25, it follows that

$$(T\circ S)^+ = (\bigcup_{i=1}^{6} T\circ_i S)^+ = \bigcup_{i=1}^{6}(T\circ_i S)^+ = \bigcup_{i=1}^{6}(S^t\circ_{7-i}T^t)^{\vdash} = (\bigcup_{i=1}^{6} S^t\circ_{7-i}T^t)^{\vdash} = (S^t\circ T^t)^{\vdash}.$$

□

2.4.4. Interaction of the five-point compositions with binary projections and cylindrical extensions

In this subsection, we study the interaction of the five-point compositions of ternary relations with binary projections and cylindrical extensions.

First, the following proposition investigates the projections of the five-point compositions of ternary relations in terms of the compositions of their binary projections.

Proposition 2.27. *Let T and S be two ternary relations on a set X. The left, middle and right projections of the compositions $(T \circ_i S)$, for any $i \in \{1, ..., 6\}$, are listed in the following table:*

Comp. \ *Proj.*	$P_\ell(.)$	$P_m(.)$	$P_r(.)$
$T \circ_1 S$	$P_\ell(T) \circ P_\ell(S)$	$P_m(T) \circ P_\ell(S)$	–
$T \circ_2 S$	$P_\ell(T) \circ P_m(S)$	$P_m(T) \circ P_m(S)$	–
$T \circ_3 S$	$P_\ell(T) \circ P_r(S)$	$P_m(T) \circ P_r(S)$	–
$T \circ_4 S$	–	$P_\ell(T) \circ P_m(S)$	$P_\ell(T) \circ P_r(S)$
$T \circ_5 S$	–	$P_m(T) \circ P_m(S)$	$P_m(T) \circ P_r(S)$
$T \circ_6 S$	–	$P_r(T) \circ P_m(S)$	$P_r(T) \circ P_r(S)$

Proof. We only prove that $P_\ell(T \circ_1 S) = P_\ell(T) \circ P_\ell(S)$, as the other cases are analogous.

$$\begin{aligned}
P_\ell(T \circ_1 S) &= \{(x,y) \in X^2 \mid (\exists z \in X)((z,x,y) \in T \circ_1 S)\} \\
&= \{(x,y) \in X^2 \mid (\exists z \in X)((z,x,y) \in T \ltimes P_\ell(S))\} \\
&= \{(x,y) \in X^2 \mid (\exists z,t \in X)((z,x,t) \in T \wedge (t,y) \in P_\ell(S))\} \\
&= \{(x,y) \in X^2 \mid (\exists t \in X)((x,t) \in P_\ell(T) \wedge (t,y) \in P_\ell(S))\} \\
&= \{(x,y) \in X^2 \mid (x,y) \in P_\ell(T) \circ P_\ell(S)\} \\
&= P_\ell(T) \circ P_\ell(S)\,.
\end{aligned}$$

□

Remark 2.9. *The following example shows that the left projection of the* $\circ_4$*-composition of two ternary relations is not equal to the composition of any of their binary projections. Indeed, let* T *and* S *be the ternary relations on* $X = \{x_1, x_2, x_3, x_4\}$ *given by:*

$$\begin{aligned}
T &= \{(x_1,x_1,x_2),(x_1,x_2,x_3)\}\,, \\
S &= \{(x_1,x_1,x_1),(x_2,x_4,x_1),(x_3,x_2,x_2)\}\,.
\end{aligned}$$

It holds that

$$T \circ_4 S = \{(x_1,x_4,x_1),(x_2,x_2,x_2)\}\,,$$

and thus

$$P_\ell(T \circ_4 S) = \{(x_4,x_1),(x_2,x_2)\}\,.$$

Further, it holds that

	$P_\ell(.)$	$P_m(.)$	$P_r(.)$
T	$\{(x_1,x_2),(x_2,x_3)\}$	$\{(x_1,x_2),(x_1,x_3)\}$	$\{(x_1,x_1),(x_1,x_2)\}$
S	$\{(x_1,x_1),(x_2,x_2),(x_4,x_1)\}$	$\{(x_1,x_1),(x_2,x_1),(x_3,x_2)\}$	$\{(x_1,x_1),(x_2,x_4),(x_3,x_2)\}$

and

$\circ$	$P_\ell(S)$	$P_m(S)$	$P_r(S)$
$P_\ell(T)$	$\{(x_1,x_2)\}$	$\{(x_1,x_1),(x_2,x_2)\}$	$\{(x_1,x_4),(x_2,x_2)\}$
$P_m(T)$	$\{(x_1,x_2)\}$	$\{(x_1,x_1),(x_1,x_2)\}$	$\{(x_1,x_2),(x_1,x_4)\}$
$P_r(T)$	$\{(x_1,x_1),(x_1,x_2)\}$	$\{(x_1,x_1)\}$	$\{(x_1,x_1),(x_1,x_4)\}$

It is clear that for any $\lambda_1, \lambda_2 \in \{\ell, m, r\}$ *it holds that*

$$P_\ell(T \circ_4 S) \neq P_{\lambda_1}(T) \circ P_{\lambda_2}(S)\,.$$

In a similar way, one can easily prove the other cases.

In the following, we investigate the interaction of the five-point compositions of ternary relations with left, middle and right cylindrical extensions of a binary relation. First, we need the following proposition which expresses the compositions of a ternary relation with a binary relation in terms of the six five-point compositions of ternary relations.

Proposition 2.28. *Let T be a ternary relation and R be a binary relation on a set X. The following equalities hold:*

(i) $T \ltimes R = T \circ_1 C_\ell(R)$;

(ii) $T \ltimes R = T \circ_2 C_m(R)$;

(iii) $T \ltimes R = T \circ_3 C_r(R)$;

(iv) $R \rtimes T = C_\ell(R) \circ_4 T$;

(v) $R \rtimes T = C_m(R) \circ_5 T$;

(vi) $R \rtimes T = C_r(R) \circ_6 T$.

***Proof*.** We only give the proof for the first equality, as the other equalities can be proved analogously.

$$\begin{aligned} T \ltimes R &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (t,z) \in R)\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t,s \in X)((x,y,t) \in T \wedge (s,t,z) \in C_\ell(R))\} \\ &= T \circ_1 C_\ell(R)\,. \end{aligned}$$

□

The following proposition investigates the cylindrical extensions of the composition of binary relations in terms of the five-point compositions of their cylindrical extensions.

Proposition 2.29. *Let R_1 and R_2 be two binary relations on a set X. The left, middle and right cylindrical extensions of the composition $R_1 \circ R_2$ are listed in the following table:*

Cyl. ext. / *Comp.*	$C_\ell(.)$	$C_m(.)$	$C_r(.)$
$R_1 \circ R_2$	$C_\ell(R_1) \circ_1 C_\ell(R_2)$ $C_\ell(R_1) \circ_2 C_m(R_2)$ $C_\ell(R_1) \circ_3 C_r(R_2)$	$C_m(R_1) \circ_1 C_\ell(R_2)$ $C_m(R_1) \circ_2 C_m(R_2)$ $C_m(R_1) \circ_3 C_r(R_2)$ $C_\ell(R_1) \circ_4 C_m(R_2)$ $C_m(R_1) \circ_5 C_m(R_2)$ $C_r(R_1) \circ_6 C_m(R_2)$	$C_\ell(R_1) \circ_4 C_r(R_2)$ $C_m(R_1) \circ_5 C_r(R_2)$ $C_r(R_1) \circ_6 C_r(R_2)$

Proof. We only prove that $C_\ell(R_1 \circ R_2) = C_\ell(R_1) \circ_1 C_\ell(R_2)$, as the other cases can be proved analogously.

$$\begin{aligned}
C_\ell(R_1 \circ R_2) &= \{(x,y,z) \in X^3 \mid (y,z) \in R_1 \circ R_2\} \\
&= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t) \in R_1 \wedge (t,z) \in R_2)\} \\
&= \{(x,y,z) \in X^3 \mid (\exists t,s \in X)((x,y,t) \in C_\ell(R_1) \wedge (s,t,z) \in C_\ell(R_2))\} \\
&= \{(x,y,z) \in X^3 \mid (x,y,z) \in C_\ell(R_1) \circ_1 C_\ell(R_2)\} \\
&= C_\ell(R_1) \circ_1 C_\ell(R_2)\,.
\end{aligned}$$

□

Remark 2.10. *The following example shows that the left cylindrical extension of the composition of two binary relations is not equal to the $\circ_4$-composition of any of their cylindrical extensions. Indeed, let R_1 and R_2 be the binary relations on $X = \{x_1, x_2, x_3\}$ given by:*

$$\begin{aligned}
R_1 &= \{(x_1,x_2)\}\,, \\
R_2 &= \{(x_2,x_2),(x_3,x_1)\}\,.
\end{aligned}$$

It holds that

$$R_1 \circ R_2 = \{(x_1,x_2)\}\,,$$

and thus

$$C_\ell(R_1 \circ R_2) = \{(x_1,x_1,x_2),(x_2,x_1,x_2),(x_3,x_1,x_2)\}\,.$$

Further, it holds that

	$C_\ell(.)$	$C_m(.)$	$C_r(.)$
R_1	$\{(x_1,x_1,x_2),(x_2,x_1,x_2),$ $(x_3,x_1,x_2)\}$	$\{(x_1,x_1,x_2),(x_1,x_2,x_2),$ $(x_1,x_3,x_2)\}$	$\{(x_1,x_2,x_1),(x_1,x_2,x_2),$ $(x_1,x_2,x_3)\}$
R_2	$\{(x_1,x_2,x_2),(x_1,x_3,x_1),$ $(x_2,x_2,x_2),(x_2,x_3,x_1),$ $(x_3,x_2,x_2),(x_3,x_3,x_1)\}$	$\{(x_2,x_1,x_2),(x_2,x_2,x_2),$ $(x_2,x_3,x_2),(x_3,x_1,x_1),$ $(x_3,x_2,x_1),(x_3,x_3,x_1)\}$	$\{(x_2,x_2,x_1),(x_2,x_2,x_2),$ $(x_2,x_2,x_3),(x_3,x_1,x_1),$ $(x_3,x_1,x_2),(x_3,x_1,x_3)\}$

and

$\circ_4$	$C_\ell(R_2)$	$C_m(R_2)$	$C_r(R_2)$
$C_\ell(R_1)$	$\{(x_1,x_2,x_2),(x_1,x_3,x_1)\}$	$\{(x_1,x_1,x_2),(x_1,x_2,x_2),(x_1,x_3,x_2)\}$	$\{(x_1,x_2,x_1),(x_1,x_2,x_2),(x_1,x_2,x_3)\}$
$C_m(R_1)$	$\{(x_1,x_2,x_2),(x_1,x_3,x_1),(x_2,x_2,x_2),(x_2,x_3,x_1),(x_3,x_2,x_2),(x_3,x_3,x_1)\}$	$\{(x_1,x_1,x_2),(x_1,x_2,x_2),(x_1,x_3,x_2),(x_2,x_1,x_2),(x_2,x_2,x_2),(x_2,x_3,x_2),(x_3,x_1,x_2),(x_3,x_2,x_2),(x_3,x_3,x_2)\}$	$\{(x_1,x_2,x_1),(x_1,x_2,x_2),(x_1,x_2,x_3),(x_2,x_2,x_1),(x_2,x_2,x_2),(x_2,x_2,x_3),(x_3,x_2,x_1),(x_3,x_2,x_2),(x_3,x_2,x_3)\}$
$C_r(R_1)$	$\{(x_2,x_2,x_2),(x_2,x_3,x_1)\}$	$\{(x_2,x_1,x_1),(x_2,x_1,x_2),(x_2,x_2,x_1),(x_2,x_2,x_2),(x_2,x_3,x_1),(x_2,x_3,x_2)\}$	$\{(x_2,x_1,x_1),(x_2,x_1,x_2),(x_2,x_1,x_3),(x_2,x_2,x_1),(x_2,x_2,x_2),(x_2,x_2,x_3)\}$

It is clear that for any $\lambda_1, \lambda_2 \in \{\ell, m, r\}$ it holds that

$$C_\ell(R_1 \circ R_2) \neq C_{\lambda_1}(R_1) \circ_4 C_{\lambda_2}(R_2)\,.$$

In a similar way, one can easily prove the other cases.

2.4.5. Interaction of the five-point compositions with traces

The following proposition shows the interaction of the five-point compositions of ternary relations with left, middle and right traces.

Proposition 2.30. *Let T and S be two ternary relations on a set X. The following inclusions hold:*

(i) $T^\ell \subseteq (T \circ_i S)^\ell$, *for any* $i \in \{1,2,3,5,6\}$*;*

(ii) $T^m \subseteq (T \circ_i S)^m$, *for any* $i \in \{1,2,3\}$*;*

(iii) $S^m \subseteq (T \circ_i S)^m$, *for any* $i \in \{4,5,6\}$*;*

(iv) $T^r \subseteq (T \circ_i S)^r$, *for any* $i \in \{1,2,4,5,6\}$*.*

Proof. We only give the proof for the first inclusion and $i = 1$, as the other inclusions can be proved analogously. Let $(x,y) \in T^\ell$ and $(x,a,b) \in T \circ_1 S$. Then there exists $t,s \in X$ such that $(x,a,t) \in T$ and $(s,t,b) \in S$. Since $(x,y) \in T^\ell$, it follows that $(y,a,t) \in T$. This implies that $(y,a,b) \in T \circ_1 S$. Hence, $(x,y) \in (T \circ_1 S)^\ell$. Thus, $T^\ell \subseteq (T \circ_1 S)^\ell$. □

The following result shows that the traces of a ternary relation are the greatest binary relations that satisfy the following relational inclusions corresponding to the relational five-point compositions introduced above.

Proposition 2.31. *Let T be a ternary relation on a set X. It holds that*

(i) T^ℓ is the greatest binary relation R that satisfies the following inclusions:

(a) $C_\ell(R^t) \circ_4 T \subseteq T$;

(b) $C_m(R^t) \circ_5 T \subseteq T$;

(c) $C_r(R^t) \circ_6 T \subseteq T$.

(ii) T^r is the greatest binary relation R that satisfies the following inclusions:

(a) $T \circ_1 C_\ell(R) \subseteq T$;

(b) $T \circ_2 C_m(R) \subseteq T$;

(c) $T \circ_3 C_r(R) \subseteq T$.

Proof. We only give the proof for the inclusion (i) (a), as the other inclusions can be proved analogously. Note that Proposition 2.28 implies that $C_\ell(R^t) \circ_4 T = R^t \rtimes T$. Theorem 2.1 guarantees that T^ℓ is the greatest binary relation R that satisfies $R^t \rtimes T \subseteq R$. Hence, T^ℓ is the greatest binary relation R that satisfies $C_\ell(R^t) \circ_4 T \subseteq T$. □

Combining Proposition 2.28 and Theorem 2.2 leads to the following result.

Corollary 2.4. *Let T be a ternary relation on a set X. The following equalities hold:*

(i) $T \circ_1 C_\ell(T^r) = T$;

(ii) $T \circ_2 C_m(T^r) = T$;

(iii) $T \circ_3 C_r(T^r) = T$;

(iv) $C_\ell((T^\ell)^t) \circ_4 T = T$;

(v) $C_m((T^\ell)^t) \circ_5 T = T$;

(vi) $C_r((T^\ell)^t) \circ_6 T = T$.

Along the same lines of Proposition 2.31, we obtain the following equivalences.

Proposition 2.32. *Let T and S be two ternary relations on a set X. The following equivalences hold:*

(i) $T \circ_1 S \subseteq T$ if and only if $S \subseteq C_\ell(T^r)$;

(ii) $T \circ_2 S \subseteq T$ if and only if $S \subseteq C_m(T^r)$;

(iii) $T \circ_3 S \subseteq T$ *if and only if* $S \subseteq C_r(T^r)$;

(iv) $S \circ_4 T \subseteq T$ *if and only if* $S \subseteq C_\ell((T^\ell)^t)$;

(v) $S \circ_5 T \subseteq T$ *if and only if* $S \subseteq C_m((T^\ell)^t)$;

(vi) $S \circ_6 T \subseteq T$ *if and only if* $S \subseteq C_r((T^\ell)^t)$.

Proof. We only prove the first equivalence, as the other equivalences can be proved analogously. Suppose that $T \circ_1 S \subseteq T$, then we need to prove that $S \subseteq C_\ell(T^r)$. From Proposition 2.28, it follows that $T \circ_1 S = T \circ_1 C_\ell(P_\ell(S))$. Since $T \circ_1 S \subseteq T$, it follows that $T \circ_1 C_\ell(P_\ell(S)) \subseteq T$. We know from Proposition 2.31 (ii) that T^r is the greatest binary relation R that satisfies $T \circ_1 C_\ell(R) \subseteq T$, and, hence, $P_\ell(S) \subseteq T^r$, which implies that $C_\ell(P_\ell(S)) \subseteq C_\ell(T^r)$. Proposition 2.7 guarantees that $S \subseteq C_\ell(P_\ell(S))$. Thus, $S \subseteq C_\ell(T^r)$. Conversely, assume that $S \subseteq C_\ell(T^r)$. From Proposition 2.23, it follows that $T \circ_1 S \subseteq T \circ_1 C_\ell(T^r)$. Proposition 2.31 guarantees that $T \circ_1 C_\ell(T^r) \subseteq T$. Since $T \circ_1 S \subseteq T \circ_1 C_\ell(T^r)$ and $T \circ_1 C_\ell(T^r) \subseteq T$, it follows that $T \circ_1 S \subseteq T$. □

2.5. Subcompositions and supercompositions of ternary relations

Inspired by the notions of afterset and foreset, Bandler and Kohout in [7] have introduced several new compositions of relations called subcomposition and supercomposition. In this section, we introduce the subcompositions and supercompositions of ternary relations based on the associative four-point compositions. Furthermore, we show some relational inequalities based on these compositions.

Definition 2.7. *Let T and S be two ternary relations on a set X. The subcompositions and supercompositions of T and S are defined as follows:*

(i) (a) $T \lhd_1 S = \{(x,y,z) \in X^3 \mid {}_{xy}T \subseteq {}_xS_z\}$;

(b) $T \rhd_1 S = \{(x,y,z) \in X^3 \mid {}_{xy}T \supseteq {}_xS_z\}$;

(ii) (a) $T \lhd_2 S = \{(x,y,z) \in X^3 \mid {}_{xy}T \subseteq S_{yz}\}$;

(b) $T \rhd_2 S = \{(x,y,z) \in X^3 \mid {}_{xy}T \supseteq S_{yz}\}$;

(iii) (a) $T \lhd_3 S = \{(x,y,z) \in X^3 \mid {}_xT_z \subseteq S_{yz}\}$;

(b) $T \rhd_3 S = \{(x,y,z) \in X^3 \mid {}_xT_z \supseteq S_{yz}\}$.

The compositions $\lhd_i$ are called subcompositions and the compositions $\rhd_i$ are called supercompositions, for $i \in \{1,2,3\}$. It is clear that

(i) (a) $T \lhd_1 S = \{(x,y,z) \in X^3 \mid (\forall t \in X)((x,y,t) \in T \Rightarrow (x,t,z) \in S)\}$;

(b) $T \rhd_1 S = \{(x,y,z) \in X^3 \mid (\forall t \in X)((x,y,t) \in T \Leftarrow (x,t,z) \in S)\}$;

(ii) (a) $T \lhd_2 S = \{(x,y,z) \in X^3 \mid (\forall t \in X)((x,y,t) \in T \Rightarrow (t,y,z) \in S)\}$;

(b) $T \rhd_2 S = \{(x,y,z) \in X^3 \mid (\forall t \in X)((x,y,t) \in T \Leftarrow (t,y,z) \in S)\}$;

(iii) (a) $T \lhd_3 S = \{(x,y,z) \in X^3 \mid (\forall t \in X)((x,t,z) \in T \Rightarrow (t,y,z) \in S)\}$;

(b) $T \rhd_3 S = \{(x,y,z) \in X^3 \mid (\forall t \in X)((x,t,z) \in T \Leftarrow (t,y,z) \in S)\}$.

In the following, we show some relational inequalities based on the above $(\lhd_i, \rhd_i)$-compositions of ternary relations, for $i \in \{1,2,3\}$.

Theorem 2.3. *Let R, S and T be three ternary relations on a set X. It holds that*

(i) $R \diamond_1 S \subseteq T \Longleftrightarrow R \subseteq T \rhd_1 S^{\dashv} \Longleftrightarrow S \subseteq R^{\dashv} \lhd_1 T$;

(ii) $R \diamond_2 S \subseteq T \Longleftrightarrow R \subseteq T \rhd_2 S^{t} \Longleftrightarrow S \subseteq R^{t} \lhd_2 T$;

(iii) $R \diamond_3 S \subseteq T \Longleftrightarrow R \subseteq T \rhd_3 S^{\vdash} \Longleftrightarrow S \subseteq R^{\vdash} \lhd_3 T$.

Proof. We only give the proof for the first equivalence, as the other equivalences can be proved analogously. Suppose that $R \diamond_1 S \subseteq T$. Then, for any $x, y, z \in X$, it holds that

$$(x,y,z) \in R \diamond_1 S \Longrightarrow (x,y,z) \in T.$$

Then there exists $t \in X$ such that

$$(x,y,t) \in R \wedge (x,t,z) \in S \Longrightarrow (x,y,z) \in T.$$

This implies that

$$(x,y,t) \in R \Longrightarrow ((x,t,z) \in S \Longrightarrow (x,y,z) \in T).$$

It is clear that

$$(x,y,t) \in R \Longrightarrow ((x,z,t) \in S^{\dashv} \Longrightarrow (x,y,z) \in T).$$

Hence, $(x,y,t) \in R$ implies that $(x,y,t) \in T \rhd_1 S^{\dashv}$. Thus, $R \subseteq T \rhd_1 S^{\dashv}$. Conversely, suppose that $R \subseteq T \rhd_1 S^{\dashv}$. Let $(x,y,z) \in R \diamond_1 S$, then there exists $t \in X$ such that

$(x, y, t) \in R$ and $(x, t, z) \in S$. Since $R \subseteq T \triangleright_1 S^{\dashv}$, it follows that $(x, y, t) \in T \triangleright_1 S^{\dashv}$ and $(x, t, z) \in S$. Then, for any $s \in X$, it holds that

$$((x, t, s) \in S \Longrightarrow (x, y, s) \in T) \wedge (x, t, z) \in S.$$

It is clear that $(x, y, z) \in T$. Thus, $R \diamond_1 S \subseteq T$. Therefore, $R \diamond_1 S \subseteq T \Longleftrightarrow R \subseteq T \triangleright_1 S^{\dashv}$. $\square$

3 Transitivity of ternary relations

Transitivity is by far the most interesting property of (crisp and fuzzy) binary relations. While the transitivity property of a binary relation is quite standard, it comes in a multitude of variations in the ternary case. Also, the fact that the transitivity property can conveniently be expressed as a relational inclusion using the notion of composition, leads us to introduce several notions of four-point (resp. five-point) transitivity properties.

In this chapter, we discuss the four-point (resp. five-point) transitivity of a ternary relation from the point of view of four-point (resp. five-point) compositions of ternary relations and investigate their properties and interactions with binary projections, cylindrical extensions and traces.

3.1. Transitivity properties of ternary relations in literature

While the definitions of reflexivity and symmetry are quite standard, various notions of transitivity of ternary relations have been proposed in literature. For instance, Pitcher and Smiley [56] introduced several four-point transitivity properties of betweenness relations. Explicitly, a ternary relation T on X is called

(i) t_1-transitive, if, for any $x, y, z, t \in X$, it holds that $(t, x, y) \in T$ and $(t, y, z) \in T$ imply $(x, y, z) \in T$;

(ii) t_2-transitive, if, for any $x, y, z, t \in X$, it holds that $(x, y, t) \in T$ and $(x, t, z) \in T$ imply $(x, y, z) \in T$;

(iii) t_3-transitive, if, for any $x, y, z, t \in X$ such that $y \neq t$, it holds that $(x, y, t) \in T$ and $(y, t, z) \in T$ imply $(x, y, z) \in T$.

Note that t_2-transitivity is also used by Pérez-Fernández and De Baets [54] as the end-point transitivity condition of a betweenness relation. Novák and Novotný [48] introduced several other four-point transitivity properties: a ternary relation T on X is called

(iv) t_4-transitive, if, for any $x, y, z, t \in X$, it holds that $(x, y, t) \in T$ and $(y, z, t) \in T$ imply $(x, y, z) \in T$;

(v) t_5-transitive, if, for any $x, y, z, t \in X$, it holds that $(x, t, z) \in T$ and $(t, y, z) \in T$ imply $(x, y, z) \in T$;

(vi) t_6-transitive, if, for any $x, y, z, t \in X$, it holds that $(x, y, t) \in T$ and $(t, y, z) \in T$ imply $(x, y, z) \in T$.

It is clear that these six types of transitivity correspond to six types of composition of ternary relations, e.g.

$$T \circ_{t_1} S = \{(x, y, z) \in X^3 \mid (\exists t \in X)((t, x, y) \in T \wedge (t, y, z) \in S)\} .$$

Note that only the $\circ_{t_i}$-compositions with $i \in \{2, 5, 6\}$ are associative.

The transitivity of a binary relation R can be concisely written as $R \circ R \subseteq R$, with $\circ$ the usual composition of binary relations. Obviously, for the above types of transitivity of ternary relations it holds by definition of the compositions that a ternary relation T is $\circ_{t_i}$-transitive if and only if $T \circ_{t_i} T \subseteq T$, for any $i \in \{1, \ldots, 6\}$. Conversely, with each of the several types of composition of ternary relations introduced in Chapter 2, we can associate two types of transitivity, four-point transitivity and five-point transitivity.

3.2. Four-point transitivity properties

In this section, we discuss the four-point transitivity properties of ternary relations from the point of view of associative four-point compositions of ternary relations and investigate their properties and interactions with binary projections, cylindrical extensions and traces.

3.2.1. Definitions and properties

Definition 3.1. *Let T be a ternary relation on a set X. For any $i \in \{1, 2, 3\}$, T is called $\diamond_i$-transitive if $T \diamond_i T \subseteq T$.*

The following proposition shows, for any $i \in \{1, 2, 3\}$, that the powers of a $\diamond_i$-transitive ternary relation are included in this ternary relation. The proof is straightforward.

Proposition 3.1. *Let T be a ternary relation on a set X. For any $i \in \{1, 2, 3\}$, T is $\diamond_i$-transitive if and only if $T^{n,\diamond_i} \subseteq T$, for any $n \in \mathbb{N}^*$.*

Combining Propositions 2.10 and 3.1 leads to the following result.

Corollary 3.1. *Let T be a ternary relation on a set X. The following equivalences hold:*

(i) The left reflexivity of T implies that T is $\diamond_1$-transitive if and only if $T^{n,\diamond_1} = T$, for any $n \in \mathbb{N}^*$;

(ii) The middle reflexivity of T implies that T is $\diamond_2$-transitive if and only if $T^{n,\diamond_2} = T$, for any $n \in \mathbb{N}^*$;

(iii) The right reflexivity of T implies that T is $\diamond_3$-transitive if and only if $T^{n,\diamond_3} = T$, for any $n \in \mathbb{N}^*$.

The following proposition shows, for any $i \in \{1, 2, 3\}$, the equivalence between the $\diamond_i$-transitivity of a ternary relation and that of its powers. The proof is straightforward.

Proposition 3.2. *Let T be a ternary relation on a set X. For any $i \in \{1, 2, 3\}$, it holds that T is $\diamond_i$-transitive if and only if $T^{n,\diamond_i}$ is $\diamond_i$-transitive, for any $n \in \mathbb{N}^*$.*

The following proposition shows that, for any $i \in \{1, 2, 3\}$, $\diamond_i$-transitivity is preserved under intersection.

Proposition 3.3. *Let $(T_j)_{j\in J}$ be a family of ternary relations on a set X. For any $i \in \{1, 2, 3\}$, the $\diamond_i$-transitivity of $(T_j)_{j\in J}$ implies the $\diamond_i$-transitivity of $\bigcap\limits_{j\in J} T_j$.*

Proof. To prove that $\bigcap\limits_{j\in J} T_j$ is $\diamond_i$-transitive, it suffices to prove that $\bigcap\limits_{j\in J} T_j \diamond_i \bigcap\limits_{j\in J} T_j \subseteq \bigcap\limits_{j\in J} T_j$. Suppose that $(T_j)_{j\in J}$ is $\diamond_i$-transitive, i.e., $T_j \diamond_i T_j \subseteq T_j$, for any $i \in \{1, 2, 3\}$. This implies that $\bigcap\limits_{j\in J}(T_j \diamond_i T_j) \subseteq \bigcap\limits_{j\in J} T_j$. It is clear that $\bigcap\limits_{j\in J} T_j \diamond_i \bigcap\limits_{j\in J} T_j \subseteq \bigcap\limits_{j\in J}(T_j \diamond_i T_j)$, for any $i \in \{1, 2, 3\}$, and hence $\bigcap\limits_{j\in J} T_j \diamond_i \bigcap\limits_{j\in J} T_j \subseteq \bigcap\limits_{j\in J} T_j$. Hence, $\bigcap\limits_{j\in J} T_j$ is $\diamond_i$-transitive, for any $i \in \{1, 2, 3\}$. □

3.2.2. Interaction of the four-point transitivity properties with permutations

In this subsection, we study the interaction of the four-point transitivity properties with permutations.

Proposition 3.4. *Let T be a ternary relation on a set X. The following statements hold:*

(i) (a) T is $\diamond_1$-transitive if and only if $T^{\dashv}$ is $\diamond_1$-transitive;

(b) T is $\diamond_2$-transitive if and only if $T^{\dashv}$ is $\diamond_3$-transitive.

(ii) *(a)* *T is $\diamond_1$-transitive if and only if $T^{\vdash}$ is $\diamond_2$-transitive;*

(b) *T is $\diamond_3$-transitive if and only if $T^{\vdash}$ is $\diamond_3$-transitive.*

(iii) *(a)* *The $\diamond_1$-transitivity of T implies the $\diamond_3$-transitivity of T^+;*

(b) *The $\diamond_2$-transitivity of T implies the $\diamond_1$-transitivity of T^+;*

(c) *The $\diamond_3$-transitivity of T implies the $\diamond_2$-transitivity of T^+.*

(iv) *(a)* *The $\diamond_1$-transitivity of T implies the $\diamond_2$-transitivity of T^-;*

(b) *The $\diamond_2$-transitivity of T implies the $\diamond_3$-transitivity of T^-;*

(c) *The $\diamond_3$-transitivity of T implies the $\diamond_1$-transitivity of T^-.*

(v) *(a)* *T is $\diamond_1$-transitive if and only if T^t is $\diamond_3$-transitive;*

(b) *T is $\diamond_2$-transitive if and only if T^t is $\diamond_2$-transitive.*

Proof. We only give the proof for the first statement, as the other statements are analogous. Suppose that T is $\diamond_1$-transitive. Let $x, y, z, t \in X$ such that $(x, y, t) \in T^{\dashv}$ and $(x, t, z) \in T^{\dashv}$. This implies that $(x, z, t) \in T$ and $(x, t, y) \in T$. Since T is $\diamond_1$-transitive, it follows that $(x, z, y) \in T$. Hence, $(x, y, z) \in T^{\dashv}$. Thus, $T^{\dashv}$ is $\diamond_1$-transitive. Conversely, suppose that $T^{\dashv}$ is $\diamond_1$-transitive. Let $x, y, z, t \in X$ such that $(x, y, t) \in T$ and $(x, t, z) \in T$. This implies that $(x, z, t) \in T^{\dashv}$ and $(x, t, y) \in T^{\dashv}$. Since $T^{\dashv}$ is $\diamond_1$-transitive, it follows that $(x, z, y) \in T^{\dashv}$. Hence, $(x, y, z) \in T$. Thus, T is $\diamond_1$-transitive. □

3.2.3. Interaction of the four-point transitivity properties with binary projections and cylindrical extensions

In this subsection, we study the interaction of the four-point transitivity properties with binary projections and cylindrical extensions. The following proposition shows, for any $i \in \{1, 2, 3\}$, that for a given $\diamond_i$-transitive ternary relation, the transitivity of its binary projections does not hold in general. The proof is straightforward and follows from Proposition 2.15.

Proposition 3.5. *Let T be a ternary relation on a set X. For any $i \in \{1, 2, 3\}$, if T is $\diamond_i$-transitive then, for any $\lambda \in \{\ell, m, r\}$, the transitivity of $P_\lambda(T)$ does not hold in general.*

Remark 3.1. *The following example shows that for a given ternary relation T, if T is $\diamond_1$-transitive then, for any $\lambda \in \{\ell, m, r\}$, the transitivity of $P_\lambda(T)$ does not* *hold in general. Indeed, let T be the following $\diamond_1$-transitive ternary relation on $X = \{x_1, x_2, x_3, x_4\}$:*

$$T = \{(x_1, x_1, x_2), (x_1, x_1, x_3), (x_1, x_2, x_3), (x_4, x_3, x_1)\}.$$

It holds that

$$\begin{aligned} P_\ell(T) &= \{(x_1, x_2), (x_1, x_3), (x_2, x_3), (x_3, x_1)\}, \\ P_m(T) &= \{(x_1, x_2), (x_1, x_3), (x_4, x_1)\}, \\ P_r(T) &= \{(x_1, x_1), (x_1, x_2), (x_4, x_3)\}. \end{aligned}$$

It is clear that the projections $P_\ell(T), P_m(T)$ and $P_r(T)$ are not transitive. A similar example can be given for the other transitivity properties.

In the following proposition, we show that the transitivity of a given binary relation is equivalent with $\diamond_i$-transitivity of an appropriate cylindrical extension.

Proposition 3.6. *Let R be a binary relation on a set X. The following equivalences hold:*

(i) R is transitive if and only if $C_\ell(R)$ is $\diamond_1$-transitive;

(ii) R is transitive if and only if $C_m(R)$ is $\diamond_2$-transitive;

(iii) R is transitive if and only if $C_r(R)$ is $\diamond_3$-transitive.

***Proof*.** We only give the proof for the first equivalence, as the other equivalences are analogous. Suppose that R is transitive, i.e., $R \circ R \subseteq R$. From Proposition 2.16, it is clear that $C_\ell(R \circ R) = C_\ell(R) \diamond_1 C_\ell(R) \subseteq C_\ell(R)$. Thus, $C_\ell(R)$ is $\diamond_1$-transitive. Conversely, suppose that $C_\ell(R)$ is $\diamond_1$-transitive, then it holds that $C_\ell(R) \diamond_1 C_\ell(R) \subseteq C_\ell(R)$. This implies that $P_\ell(C_\ell(R) \diamond_1 C_\ell(R)) \subseteq P_\ell(C_\ell(R))$. Proposition 2.15 guarantees that $P_\ell(C_\ell(R)) \circ P_\ell(C_\ell(R)) \subseteq P_\ell(C_\ell(R))$. Since R coincides with binary projections of its cylindrical extensions, it follows that $R \circ R \subseteq R$. Thus, R is transitive. □

3.3. Five-point transitivity properties

In this section, we discuss the five-point transitivity properties of ternary relations from the point of view of five-point compositions of ternary relations and investigate their properties and interactions with binary projections, cylindrical extensions and traces.

3.3.1. Definitions and properties

Definition 3.2. *Let T be a ternary relation on a set X. For any $i \in \{1, \dots, 6\}$, T is called $\circ_i$-transitive if $T \circ_i T \subseteq T$.*

As in the binary case, the following propositions show, for the associative compositions, that the powers of a $\circ_i$-transitive ternary relation are included in this ternary relation and that any reflexive ternary relation is included in its powers. The proof are straightforward.

Proposition 3.7. *Let T be a ternary relation on a set X. For any $i \in \{1, 2, 5, 6\}$, it holds that T is $\circ_i$-transitive if and only if $T^{n,\circ_i} \subseteq T$, for any $n \in \mathbb{N}^*$.*

Proposition 3.8. *Let T be a ternary relation on a set X. For any $i \in \{1, 2, 5, 6\}$, if T is reflexive, then it holds that $T \subseteq T^{n,\circ_i}$, for any $n \in \mathbb{N}^*$.*

Combining Propositions 3.7 and 3.8 leads to the following corollary.

Corollary 3.2. *Let T be a reflexive ternary relation on a set X. For any $i \in \{1, 2, 5, 6\}$, it holds that T is $\circ_i$-transitive if and only if $T^{n,\circ_i} = T$, for any $n \in \mathbb{N}^*$.*

Remark 3.2. *One easily verifies that for the non-associative compositions ($i \in \{3, 4\}$), the above results also hold when considering either the left or the right powers.*

The following proposition shows, for the associative compositions, the equivalence between the $\circ_i$-transitivity of a ternary relation and that of its powers. The proof is straightforward.

Proposition 3.9. *Let T be a ternary relation on a set X. For any $i \in \{1, 2, 5, 6\}$, it holds that T is $\circ_i$-transitive if and only if $T^{n,\circ_i}$ is $\circ_i$-transitive, for any $n \in \mathbb{N}^*$.*

Remark 3.3. *Similarly, one easily verifies the equivalence between the $\circ_i$-transitivity of a ternary relation T and that of its left or right (n, i)-th powers for the non-associative compositions. For any $i \in \{3, 4\}$, the following statements are equivalent:*

(i) *T is $\circ_i$-transitive;*

(ii) *${}^{\ell}T^{n,\circ_i}$ is $\circ_i$-transitive, for any $n \in \mathbb{N}^*$;*

(iii) *${}^{r}T^{n,\circ_i}$ is $\circ_i$-transitive, for any $n \in \mathbb{N}^*$.*

The following proposition shows that for any $i \in \{1, \dots, 6\}$, $\circ_i$-transitivity is preserved under intersection.

Proposition 3.10. *Let $(T_j)_{j\in J}$ be a family of ternary relations on a set X. For any $i \in \{1,\ldots,6\}$, the $\circ_i$-transitivity of $(T_j)_{j\in J}$ implies the $\circ_i$-transitivity of $\bigcap\limits_{j\in J} T_j$.*

Proof. To prove that $\bigcap\limits_{j\in J} T_j$ is $\circ_i$-transitive, it suffices to prove that $\bigcap\limits_{j\in J} T_j \circ_i \bigcap\limits_{j\in J} T_j \subseteq \bigcap\limits_{j\in J} T_j$. Suppose that $(T_j)_{j\in J}$ is $\circ_i$-transitive, i.e., $T_j \circ_i T_j \subseteq T_j$, for any $i \in \{1,\ldots,6\}$. This implies that $\bigcap\limits_{j\in J}(T_j \circ_i T_j) \subseteq \bigcap\limits_{j\in J} T_j$. It is clear that $\bigcap\limits_{j\in J} T_j \circ_i \bigcap\limits_{j\in J} T_j \subseteq \bigcap\limits_{j\in J}(T_j \circ_i T_j)$, for any $i \in \{1,\ldots,6\}$, and hence $\bigcap\limits_{j\in J} T_j \circ_i \bigcap\limits_{j\in J} T_j \subseteq \bigcap\limits_{j\in J} T_j$. Hence, $\bigcap\limits_{j\in J} T_j$ is $\circ_i$-transitive, for any $i \in \{1,\ldots,6\}$. □

3.3.2. Interaction of the five-point transitivity properties with permutations

In this subsection, we investigate the $\circ_i$-transitivity of the permutations of a ternary relation T, for $i \in \{1,\ldots,6\}$.

Proposition 3.11. *Let T be a ternary relation on a set X. It holds that*

(i) T is $\circ_i$-transitive if and only if T^+ is $\circ_{i-4}$-transitive, for $i \in \{5,6\}$;

(ii) T is $\circ_i$-transitive if and only if T^- is $\circ_{i+4}$-transitive, for $i \in \{1,2\}$;

(iii) T is $\circ_i$-transitive if and only if T^t is $\circ_{7-i}$-transitive, for $i \in \{1,\ldots,6\}$.

Proof. (i) We only give the proof for the case $i = 5$, as the other cases can be proved similarly. It is clear that T is $\circ_5$-transitive if and only if $T \circ_5 T \subseteq T$. Hence, $(T \circ_5 T)^+ \subseteq T^+$. From Proposition 2.25, it follows that $T^+ \circ_1 T^+ \subseteq T^+$. Hence, T^+ is $\circ_1$- transitive.

(ii) We only give the proof for the case $i = 1$, as the other cases can be proved similarly. It is clear that T is $\circ_1$-transitive if and only if $T \circ_1 T \subseteq T$. Hence, $(T \circ_1 T)^- \subseteq T^-$. From Proposition 2.25, it follows that $T^- \circ_5 T^- \subseteq T^-$. Hence, T^- is $\circ_5$- transitive.

(iii) We only give the proof for the case $i = 1$, as the other cases can be proved similarly. It is clear that T is $\circ_1$-transitive if and only if $T \circ_1 T \subseteq T$. Hence, $(T \circ_1 T)^t \subseteq T^t$. From Proposition 2.25, it follows that $T^t \circ_6 T^t \subseteq T^t$. Hence, T^t is $\circ_6$- transitive.

□

3.3.3. Interaction of the five-point transitivity properties with binary projections and cylindrical extensions

In this subsection, we study the interaction of the five-point transitivity properties with binary projections and cylindrical extensions.

The following proposition expresses the interaction of the $\circ_i$-transitivity of a ternary relation with transitivity of its binary projections.

Proposition 3.12. *Let T be a ternary relation on a set X. The following implications hold:*

(i) The $\circ_1$-transitivity of T implies the transitivity of $P_\ell(T)$;

(ii) The $\circ_2$-transitivity of T implies the transitivity of $P_m(T)$;

(iii) The $\circ_5$-transitivity of T implies the transitivity of $P_m(T)$;

(iv) The $\circ_6$-transitivity of T implies the transitivity of $P_r(T)$.

Proof. We only give the proof for the first implication, as the other implications can be proved similarly. Suppose that T is $\circ_1$-transitive. From Proposition 2.27 , it is clear that $P_\ell(T \circ_1 T) = P_\ell(T) \circ P_\ell(T)$. The $\circ_1$-transitivity of T guarantees that $P_\ell(T \circ_1 T) \subseteq P_\ell(T)$. Hence, $P_\ell(T) \circ P_\ell(T) \subseteq P_\ell(T)$. Thus, $P_\ell(T)$ is transitive. $\square$

Remark 3.4. *The following example shows that for a given ternary relation T, if T is $\circ_3$-transitive then, for any $\lambda \in \{\ell, m, r\}$, the transitivity of $P_\lambda(T)$ does not hold in general. Indeed, let T be the following $\circ_3$-transitive ternary relation on $X = \{x_1, x_2, x_3, x_4, x_5\}$:*

$$T = \{(x_1, x_2, x_3), (x_1, x_2, x_4), (x_1, x_3, x_2), (x_3, x_4, x_5)\}.$$

It holds that

$$\begin{aligned} P_\ell(T) &= \{(x_2, x_3), (x_2, x_4), (x_3, x_2), (x_4, x_5)\}, \\ P_m(T) &= \{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_3, x_5)\}, \\ P_r(T) &= \{(x_1, x_2), (x_1, x_3), (x_3, x_4)\}. \end{aligned}$$

It is clear that the projections $P_\ell(T), P_m(T)$ and $P_r(T)$ are not transitive. A similar example can be given for the $\circ_4$-transitive relation .

The following proposition shows, for the associative compositions, that the transitivity of a given binary relation is equivalent with the $\circ_i$-transitivity of an appropriate cylindrical extension.

Proposition 3.13. *Let R be a binary relation on a set X. The following equivalences hold:*

(i) R is transitive if and only if $C_\ell(R)$ is $\circ_1$-transitive;

(ii) R is transitive if and only if $C_m(R)$ is $\circ_2$-transitive;

(iii) R is transitive if and only if $C_m(R)$ is $\circ_5$-transitive;

(iv) R is transitive if and only if $C_r(R)$ is $\circ_6$-transitive.

Proof. We only give the proof for the first equivalence, as the other equivalences can be proved similarly. Suppose that R is transitive, i.e., $R \circ R \subseteq R$. From Proposition 2.29, it is clear that $C_\ell(R \circ R) = C_\ell(R) \circ_1 C_\ell(R) \subseteq C_\ell(R)$. Thus, $C_\ell(R)$ is $\circ_1$-transitive. Conversely, suppose that $C_\ell(R)$ is $\circ_1$-transitive, then it holds that $C_\ell(R) \circ_1 C_\ell(R) \subseteq C_\ell(R)$. This implies that $P_\ell(C_\ell(R) \circ_1 C_\ell(R)) \subseteq P_\ell(C_\ell(R))$. Proposition 2.27 guarantees that $P_\ell(C_\ell(R)) \circ P_\ell(C_\ell(R)) \subseteq P_\ell(C_\ell(R))$. Since R coincides with the binary projections of its cylindrical extensions, it follows that $R \circ R \subseteq R$. Thus, R is transitive. □

Remark 3.5. *From Proposition 2.29, it is clear that there exists no $\lambda \in \{\ell, m, r\}$ such that*

$$C_\lambda(R) \circ_3 C_\lambda(R) = C_\lambda(R) \quad \textit{or} \quad C_\lambda(R) \circ_4 C_\lambda(R) = C_\lambda(R)\,.$$

Hence, if R is transitive, then, for any $i \in \{3, 4\}$ and $\lambda \in \{\ell, m, r\}$, the $\circ_i$-transitivity of $C_\lambda(R)$ does not hold in general.

3.3.4. Interaction of the five-point transitivity properties with traces

The following proposition investigates the interaction of the five-point transitivity properties with left, middle and right traces.

Proposition 3.14. *Let T be a ternary relation on a set X. The following equivalences hold:*

(i) T is $\circ_1$-transitive if and only if $T \subseteq C_\ell(T^r)$;

(ii) T is $\circ_2$-transitive if and only if $T \subseteq C_m(T^r)$;

(iii) T *is* $\circ_3$*-transitive if and only if* $T \subseteq C_r(T^r)$*;*

(iv) T *is* $\circ_4$*-transitive if and only if* $T \subseteq C_\ell((T^\ell)^t)$*;*

(v) T *is* $\circ_5$*-transitive if and only if* $T \subseteq C_m((T^\ell)^t)$*;*

(vi) T *is* $\circ_6$*-transitive if and only if* $T \subseteq C_r((T^\ell)^t)$*.*

Proof. We only give the proof for the first equivalence, as the other equivalences can be proved analogously. Suppose that T is $\circ_1$-transitive and let $(x, y, z) \in T$ and $(a, b, y) \in T$. Since T is $\circ_1$-transitive, it follows that $(a, b, z) \in T$. Hence, $(y, z) \in T^r$. Thus, $(x, y, z) \in C_\ell(T^r)$. Therefore, $T \subseteq C_\ell(T^r)$. Conversely, suppose that $T \subseteq C_\ell(T^r)$. From Proposition 2.23, it follows that $T \circ_1 T \subseteq T \circ_1 C_\ell(T^r)$. Proposition 2.31 guarantees that $T \circ_1 C_\ell(T^r) \subseteq T$. Hence, $T \circ_1 T \subseteq T$. Thus, T is $\circ_1$-transitive. □

Combining Proposition 3.14 and the fact that any binary relation coincides with the binary projection of its cylindrical extensions leads to the following corollary.

Corollary 3.3. *Let* T *be a ternary relation on a set* X*. The following equivalences hold:*

(i) T *is* $\circ_1$*-transitive if and only if* $P_\ell(T) \subseteq T^r$*;*

(ii) T *is* $\circ_2$*-transitive if and only if* $P_m(T) \subseteq T^r$*;*

(iii) T *is* $\circ_3$*-transitive if and only if* $P_r(T) \subseteq T^r$*;*

(iv) T *is* $\circ_4$*-transitive if and only if* $P_\ell(T) \subseteq (T^\ell)^t$*;*

(v) T *is* $\circ_5$*-transitive if and only if* $P_m(T) \subseteq (T^\ell)^t$*;*

(vi) T *is* $\circ_6$*-transitive if and only if* $P_r(T) \subseteq (T^\ell)^t$*.*

3.4. Link between four-point transitivity and five-point transitivity properties

The following proposition discusses the link between the associative four-point transitivity properties and the associative five-point transitivity properties of a given ternary relation.

Proposition 3.15. *Let* T *be a ternary relation on a set* X*. It holds that*

(i) The $\circ_1$*-transitivity of* T *implies the* $\diamond_1$*-transitivity of* T*;*

(ii) The $\circ_2$-transitivity of T implies the $\diamond_2$-transitivity of T;

(iii) The $\circ_5$-transitivity of T implies the $\diamond_2$-transitivity of T;

(iv) The $\circ_6$-transitivity of T implies the $\diamond_3$-transitivity of T.

Proof. (i) Suppose that T is $\circ_1$-transitive. Let $x, y, z, t \in X$ such that $(x, y, t) \in T$ and $(x, t, z) \in T$. Since T is $\circ_1$-transitive, it follows that $(x, y, z) \in T$. Hence, T is $\diamond_1$-transitive.

(ii)-(iv) Analogous to that of (i).

□

4 Closures and openings of ternary relations

In this chapter, we study the problem of closing or opening a ternary relation with respect to various relational properties, with a focus on the many transitivity properties that have been proposed for ternary relations over the past years. In particular, we consider the transitivity properties corresponding to the four-point compositions and five-point compositions of ternary relations making a careful distinction between the associative ones and the non-associative ones.

4.1. Closures of a ternary relation

For a property P which a ternary relation T on a set X may have or fail to have, the P-closure of T is defined to be the smallest relation S containing T and possessing P. In this section, we investigate the closures for various properties of a ternary relation.

4.1.1. General results

In this subsection, all the results are obtained in the same way as Bandler and Kohout [11] did for binary relations.

Definition 4.1. *If P is a property which a ternary relation T on a set X may have or fail to have, then a ternary relation S is the P-closure of T, written $S = P^{\mathrm{cl}}(T)$, if and only if S satisfies all of*

(i) S possesses property P;

(ii) $T \subseteq S$;

(iii) If $T \subseteq T'$ and T' possesses property P, then $S \subseteq T'$.

It is clear that a P-closure, if it exists, must be unique.

Corollary 4.1. *A ternary relation T on a set X possesses property P if and only if $T = P^{\mathrm{cl}}(T)$.*

For many properties P, a P-closure exists for some ternary relations but not for others. The interesting properties P are those for which every ternary relation has a P-closure. The following theorem states the conditions for this to occur.

Theorem 4.1. *A P-closure exists for all ternary relations T on a set X if and only if*

(i) The universal relation X^3 possesses P;

(ii) The intersection of every (non-empty) family of ternary relations, each of which possesses P, also possesses P.

This characterization can be rephrased in the following corollary.

Corollary 4.2. *Let T be a ternary relation on a set X, then*

$$P^{\mathrm{cl}}(T) = \bigcap\{S \mid T \subseteq S \wedge S \textit{ has property } P\}\,.$$

An example of a property P for which a P-closure does not exist in general is the completeness property. Indeed, this property is not preserved under intersection. Even more, there does not exist any non-complete ternary relation T that can be closed for completeness.

Theorem 4.2. *If P and P' are properties for which closures exist (satisfying the conditions of Theorem 4.1), and if P^{cl} and $\mathrm{P}'^{\mathrm{cl}}$ commute with each other, then $(P \wedge P')$ also satisfies the conditions of Theorem 4.1, and has a closure given by*

$$(P \wedge P')^{\mathrm{cl}} = P^{\mathrm{cl}}(P'^{\mathrm{cl}}) = P'^{\mathrm{cl}}(P^{\mathrm{cl}})\,.$$

4.1.2. Closures for some special properties of ternary relations

Bandler and Kohout [11] studied the closures for some special properties of binary relations, such as the symmetric closure, transitive closure, preorder closure and equivalence closure. In the following proposition, we follow the same direction and express the closures of a ternary relation for the reflexivity, symmetry, strong symmetry and cyclicity properties. Note that these properties are all preserved under intersection and that the universal relation X^3 possesses all of them. Hence, the closures for these properties always exist.

First, we need to give the following result that expresses the relationships between the properties of a ternary relation (symmetry and cyclicity) and those of its permutations (transpose, right rotation and left rotation). The proof is straightforward.

Proposition 4.1. *For a given ternary relation T, it holds that*

(i) T is symmetric if and only if $T = T^t$;

(ii) T is cyclic if and only if $T = T^+ = T^-$.

Proposition 4.2. *Let T be a ternary relation on a set X. It holds that:*

(i) The reflexive closure is $\mathrm{Ref}^{\mathrm{cl}}(T) = T \cup I_{X^3}$;

(ii) The symmetric closure is $\mathrm{Sym}^{\mathrm{cl}}(T) = T \cup T^t$;

(iii) The strongly symmetric closure is $\mathrm{SSym}^{\mathrm{cl}}(T) = \bigcup_{i=0}^{5} T^{\sigma_i}$;

(iv) The cyclic closure is $\mathrm{Cyc}^{\mathrm{cl}}(T) = T \cup T^+ \cup T^-$.

Proof. Items (i) and (ii) are immediate.

(iii) To prove that $\mathrm{SSym}^{\mathrm{cl}}(T) = \bigcup_{i=0}^{5} T^{\sigma_i}$, it suffices to prove that $\bigcup_{i=0}^{5} T^{\sigma_i}$ is the smallest strongly symmetric ternary relation containing T. First, it is clear that $T \subseteq \bigcup_{i=0}^{5} T^{\sigma_i}$. From Remark 1.1, it follows that $\bigcup_{i=0}^{5} T^{\sigma_i}$ is strongly symmetric. Next, if $T \subseteq S$ and S is strongly symmetric, then for any $i \in \{0, \ldots, 5\}$, it holds that $T^{\sigma_i} \subseteq S^{\sigma_i}$. Hence, $\bigcup_{i=0}^{5} T^{\sigma_i} \subseteq \bigcup_{i=0}^{5} S^{\sigma_i}$. Since S is strongly symmetric, it follows from Corollary 4.1 that $\mathrm{SSym}^{\mathrm{cl}}(S) = S$. Thus, $\bigcup_{i=0}^{5} T^{\sigma_i} \subseteq S$. Therefore, $\mathrm{SSym}^{\mathrm{cl}}(T) = \bigcup_{i=0}^{5} T^{\sigma_i}$.

(iv) We show that $\mathrm{Cyc}^{\mathrm{cl}}(T) = T \cup T^+ \cup T^-$. First, we need to prove that $T \cup T^+ \cup T^-$ is cyclic. In view of Proposition 4.1, it suffices to prove that $T \cup T^+ \cup T^- = (T \cup T^+ \cup T^-)^+ = (T \cup T^+ \cup T^-)^-$. From Proposition 1.1 and Remark 1.1, it indeed follows that

$$\begin{aligned}(T \cup T^+ \cup T^-)^+ &= T^+ \cup (T^+)^+ \cup (T^-)^+ \\ &= T^+ \cup T^- \cup T \\ &= T \cup T^+ \cup T^-\end{aligned}$$

and

$$(T \cup T^+ \cup T^-)^- = T^- \cup (T^+)^- \cup (T^-)^-$$

$$= T^- \cup T \cup T^+$$
$$= T \cup T^+ \cup T^- \,.$$

Furthermore, it is clear that $T \cup T^+ \cup T^-$ is the smallest cyclic ternary relation containing T. Thus, $\mathrm{Cyc}^{\mathrm{cl}}(T) = T \cup T^+ \cup T^-$.

□

4.2. Transitive closures

In this section, we focus on one of the most important types of closure, the transitive closures of a ternary relation. From a practical point of view, the important question often arises whether it is possible to find (if it exists), for a given type of transitivity, the smallest transitive ternary relation including a given ternary relation.

4.2.1. Four-point transitive closures

In this subsection, we characterize the four-point transitive closures of a ternary relation. Note that, for any $i \in \{1,2,3\}$, the $\diamond_i$-transitive closure always exist, as the universal relation X^3 is $\diamond_i$-transitive and intersection preserves $\diamond_i$-transitivity. Since $\diamond_i$-composition is associative, for any $i \in \{1,2,3\}$, the $\diamond_i$-transitive closure of a ternary relation can be written as the union of powers. First we give the following proposition, the proof of which is straightforward.

Proposition 4.3. *Let T and S be two ternary relations on a set X. If $T \subseteq S$, then it holds that $T^{n,\diamond_i} \subseteq S^{n,\diamond_i}$, for any $i \in \{1,2,3\}$.*

In the following result, we characterize the four-point transitive closures of a ternary relation.

Theorem 4.3. *Let T be a ternary relation on a set X. It holds that the $\diamond_i$-transitive closure of T is $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T) = \bigcup\limits_{n\geq 1} T^{n,\diamond_i}$, for any $i \in \{1,2,3\}$.*

Proof. Let $i \in \{1,2,3\}$. First, we prove that $\bigcup\limits_{n\geq 1} T^{n,\diamond_i} \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T)$. Since $T \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T)$, it follows from Proposition 4.3 that $T^{n,\diamond_i} \subseteq (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T))^{n,\diamond_i}$, for any $i \in \{1,2,3\}$. Since $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T)$ is $\diamond_i$-transitive, it follows from Proposition 3.1 that $(\mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T))^{n,\diamond_i} \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T)$, for any $i \in \{1,2,3\}$, and hence, $T^{n,\diamond_i} \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T)$.

Conversely, to prove that $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T) \subseteq \bigcup_{n\geq 1} T^{n,\diamond_i}$, it suffices to prove that $M = \bigcup_{n\geq 1} T^{n,\diamond_i}$ is $\diamond_i$-transitive. We only give the prove for $i=1$, as the other cases can be proved analogously. Let $(x,y,z) \in M \diamond_1 M$, then there exist $t \in X$ such that $(x,y,t) \in M$ and $(x,t,z) \in M$. Hence, there exist $m_1, m_2 \in \mathbb{N}^*$ such that $(x,y,t) \in T^{m_1,\diamond_1}$ and $(x,t,z) \in T^{m_2,\diamond_1}$. It then follows that $(x,y,z) \in T^{m_1,\diamond_1} \diamond_1 T^{m_2,\diamond_1}$, which implies that $(x,y,z) \in T^{m_1+m_2,\diamond_1}$, and, hence, $(x,y,z) \in M$. Thus, M is $\diamond_1$-transitive. Since $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T)$ is the smallest $\diamond_1$-transitive ternary relation including T, it follows that $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T) \subseteq \bigcup_{n\geq 1} T^{n,\diamond_1}$. Therefore, $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T) = \bigcup_{n\geq 1} T^{n,\diamond_i}$, for any $i \in \{1,2,3\}$. □

In the following, we investigate the interaction of the four-point transitive closures with the permutations.

Proposition 4.4. *Let T be a ternary relation on a set X. The following equalities hold:*

- *(i)* *(a)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T^{\dashv}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T))^{\dashv}$;
  - *(b)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(T^{\dashv}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(T))^{\dashv}$;
  - *(c)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(T^{\dashv}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(T))^{\dashv}$.
- *(ii)* *(a)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T^{\vdash}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(T))^{\vdash}$;
  - *(b)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(T^{\vdash}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T))^{\vdash}$;
  - *(c)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(T^{\vdash}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(T))^{\vdash}$.
- *(iii)* *(a)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T^{+}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(T))^{+}$;
  - *(b)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(T^{+}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(T))^{+}$;
  - *(c)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(T^{+}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T))^{+}$.
- *(iv)* *(a)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T^{-}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(T))^{-}$;
  - *(b)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(T^{-}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T))^{-}$;
  - *(c)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(T^{-}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(T))^{-}$.
- *(v)* *(a)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T^{t}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(T))^{t}$;
  - *(b)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(T^{t}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(T))^{t}$;
  - *(c)* $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(T^{t}) = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T))^{t}$.

Proof. We only give the proof for the first equality, as the other equalities can be proved analogously. From Remark 1.1, Corollary 2.1 and Theorem 4.3, it holds that:

$$\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T^{\dashv}) = \bigcup_{n\geq 1}(T^{\dashv})^{n,\diamond_1} = \bigcup_{n\geq 1}(T^{n,\diamond_1})^{\dashv} = (\bigcup_{n\geq 1} T^{n,\diamond_1})^{\dashv} = (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T))^{\dashv}.$$

□

The following proposition shows that the left, middle and right projections of the four-point transitive closures of a given ternary relation are included in the transitive closures of the binary projections of this ternary relation.

Proposition 4.5. *Let T be a ternary relation on a set X. The following inclusions hold:*

(i) $P_\ell(\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T)) \subseteq \mathrm{Tr}^{\mathrm{cl}}(P_\ell(T))$;

(ii) $P_m(\mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(T)) \subseteq \mathrm{Tr}^{\mathrm{cl}}(P_m(T))$;

(iii) $P_r(\mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(T)) \subseteq \mathrm{Tr}^{\mathrm{cl}}(P_r(T))$.

Proof. We only give the proof for the first inclusion, as the other inclusions can be proved analogously. From Proposition 2.15 and Theorem 4.3, it follows that:

$$P_\ell(\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T)) = P_\ell(\bigcup_{n\geq 1} T^{n,\diamond_1}) = \bigcup_{n\geq 1} P_\ell(T^{n,\diamond_1}) \subseteq \bigcup_{n\geq 1}(P_\ell(T))^n \subseteq \mathrm{Tr}^{\mathrm{cl}}(P_\ell(T)).$$

□

Remark 4.1. *The binary projections of the $\diamond_i$-transitive closure of a ternary relation T on X are in general included but not equal to the transitive closure of the corresponding projection of T, for any $i \in \{1,2,3\}$. This will be illustrated later in Example 4.3.*

The following proposition expresses the left, middle and right cylindrical extensions of the transitive closure of a given binary relation in terms of the four-point transitive closures of the cylindrical extensions of this relation.

Proposition 4.6. *Let R be a binary relation on a set X. The following equalities hold:*

(i) $C_\ell(\mathrm{Tr}^{\mathrm{cl}}(R)) = \mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(C_\ell(R))$;

(ii) $C_m(\mathrm{Tr}^{\mathrm{cl}}(R)) = \mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(C_m(R))$;

(iii) $C_r(\mathrm{Tr}^{\mathrm{cl}}(R)) = \mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(C_r(R))$.

Proof. We only prove that $C_\ell(\mathrm{Tr}^{\mathrm{cl}}(R)) = \mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(C_\ell(R))$, as the other equalities can be proved analogously. From Proposition 2.16, it follows that:

$$C_\ell(\mathrm{Tr}^{\mathrm{cl}}(R)) = C_\ell(\bigcup_{n\geq 1} R^n) = \bigcup_{n\geq 1} C_\ell(R^n) = \bigcup_{n\geq 1} (C_\ell(R))^{n,\diamond_1} = \mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(C_\ell(R)).$$

□

4.2.2. Five-point transitive closures

In this subsection, we characterize the five-point transitive closures of a ternary relation. Note that, for any $i \in \{1, \dots, 6\}$, the $\circ_i$-transitive closure always exists, as the universal relation X^3 is $\circ_i$-transitive and intersection preserves $\circ_i$-transitivity. As in the binary case, the transitive closure of a ternary relation can be written as the union of powers. However, this only holds for the associative five-point compositions. First, we give the following proposition, the proof of which is straightforward.

Proposition 4.7. *Let T and S be two ternary relations on a set X. If $T \subseteq S$, then it holds that $T^{n,\circ_i} \subseteq S^{n,\circ_i}$, for any $i \in \{1, 2, 5, 6\}$.*

In the following theorem, we characterize the five-point transitive closures of a ternary relation.

Theorem 4.4. *Let T be a ternary relation on a set X. It holds that the $\circ_i$-transitive closure of T is $\mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T) = \bigcup_{n\geq 1} T^{n,\circ_i}$, for any $i \in \{1, 2, 5, 6\}$.*

Proof. Let $i \in \{1, 2, 5, 6\}$. First, we prove that $\bigcup_{n\geq 1} T^{n,\circ_i} \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T)$. Since $T \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T)$, it follows from Proposition 4.7 that $T^{n,\circ_i} \subseteq (\mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T))^{n,\circ_i}$. Since $\mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T)$ is $\circ_i$-transitive, it follows from Proposition 3.7 that $(\mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T))^{n,\circ_i} \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T)$, and hence, $T^{n,\circ_i} \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T)$.

Conversely, to prove that $\mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T) \subseteq \bigcup_{n\geq 1} T^{n,\circ_i}$, it suffices to prove that $S = \bigcup_{n\geq 1} T^{n,\circ_i}$ is $\circ_i$-transitive. We only give the proof for $i = 1$, as the other cases can be proved analogously. Let $(x, y, z) \in S \circ_1 S$, then there exist $t, s \in X$ such that $(x, y, t) \in S$ and $(s, t, z) \in S$. Hence, there exist $n_0, m_0 \in \mathbb{N}^*$ such that $(x, y, t) \in T^{n_0,\circ_1}$ and $(s, t, z) \in T^{m_0,\circ_1}$. It then follows that $(x, y, z) \in T^{n_0,\circ_1} \circ_1 T^{m_0,\circ_1}$, which implies that $(x, y, z) \in T^{n_0+m_0,\circ_1}$, and, hence, $(x, y, z) \in S$. Thus, S is $\circ_1$-transitive. Since $\mathrm{Tr}^{\mathrm{cl}}_{\circ_1}(T)$ is the smallest $\circ_1$-transitive ternary relation including T, it follows that $\mathrm{Tr}^{\mathrm{cl}}_{\circ_1}(T) \subseteq \bigcup_{n\geq 1} T^{n,\circ_1}$.

Therefore, $\mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T) = \bigcup_{n\geq 1} T^{n,\circ_i}$ for any $i \in \{1,2,5,6\}$. □

Remark 4.2. *It is clear that we cannot compute the $\circ_3$-transitive (resp. $\circ_4$-transitive) closure of a ternary relation T as the union of powers of T, since the notion of $\circ_3$-powers (resp. $\circ_4$-powers) of T cannot be defined unambiguously.*

The following proposition shows the interaction of the five-point transitive closures with permutations.

Proposition 4.8. *Let T be a ternary relation on a set X. It holds that:*

(i) $\mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T^+) = (\mathrm{Tr}^{\mathrm{cl}}_{\circ_{i+4}}(T))^+$, *for any* $i \in \{1,2\}$;

(ii) $\mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T^-) = (\mathrm{Tr}^{\mathrm{cl}}_{\circ_{i-4}}(T))^-$, *for any* $i \in \{5,6\}$;

(iii) $\mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T^t) = (\mathrm{Tr}^{\mathrm{cl}}_{\circ_{7-\mathrm{i}}}(T))^t$, *for any* $i \in \{1,2,5,6\}$.

Proof. We only give the proof for the first equality, as the other equalities can be proved analogously. For any $i \in \{1,2\}$, it follows from Remark 1.1, Corollary 2.3 and Theorem 4.4 that

$$\mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T^+) = \bigcup_{n\geq 1}(T^+)^{n,\circ_i} = \bigcup_{n\geq 1}(T^{n,\circ_{i+4}})^+ = (\bigcup_{n\geq 1} T^{n,\circ_{i+4}})^+ = (\mathrm{Tr}^{\mathrm{cl}}_{\circ_{i+4}}(T))^+ .$$

□

In the following, we investigate the left, middle and right projections of the five-point transitive closures of a given ternary relation in terms of the transitive closures of the binary projections of this ternary relation.

Proposition 4.9. *Let T be a ternary relation on a set X. The following equalities hold:*

(i) $P_\ell(\mathrm{Tr}^{\mathrm{cl}}_{\circ_1}(T)) = \mathrm{Tr}^{\mathrm{cl}}(P_\ell(T))$;

(ii) $P_m(\mathrm{Tr}^{\mathrm{cl}}_{\circ_2}(T)) = P_m(\mathrm{Tr}^{\mathrm{cl}}_{\circ_5}(T)) = \mathrm{Tr}^{\mathrm{cl}}(P_m(T))$;

(iii) $P_r(\mathrm{Tr}^{\mathrm{cl}}_{\circ_6}(T)) = \mathrm{Tr}^{\mathrm{cl}}(P_r(T))$.

Proof. We only give the proof for the first equality, as the other equalities can be proved analogously. From Proposition 2.27 and Theorem 4.4, it follows that:

$$P_\ell(\mathrm{Tr}^{\mathrm{cl}}_{\circ_1}(T)) = P_\ell(\bigcup_{n\geq 1} T^{n,\circ_1}) = \bigcup_{n\geq 1} P_\ell(T^{n,\circ_1}) = \bigcup_{n\geq 1}(P_\ell(T))^n = \mathrm{Tr}^{\mathrm{cl}}(P_\ell(T)) .$$

□

Remark 4.3. *For the non-associative composition $\circ_3$ (resp. $\circ_4$), the binary projections of the $\circ_3$-transitive (resp. $\circ_4$-transitive) closure of a ternary relation T are in general not equal to the transitive closure of any projection of T. This will be illustrated later in Example 4.6.*

In the following, we express the left, middle and right cylindrical extensions of the transitive closure of a given binary relation in terms of the five-point transitive closures of the cylindrical extensions of this relation.

Proposition 4.10. *Let R be a binary relation on a set X. The following equalities hold:*

(i) $C_\ell(\mathrm{Tr}^{\mathrm{cl}}(R)) = \mathrm{Tr}^{\mathrm{cl}}_{\circ_1}(C_\ell(R))$;

(ii) $C_m(\mathrm{Tr}^{\mathrm{cl}}(R)) = \mathrm{Tr}^{\mathrm{cl}}_{\circ_2}(C_m(R)) = \mathrm{Tr}^{\mathrm{cl}}_{\circ_5}(C_m(R))$;

(iii) $C_r(\mathrm{Tr}^{\mathrm{cl}}(R)) = \mathrm{Tr}^{\mathrm{cl}}_{\circ_6}(C_r(R))$.

Proof. We only prove that $C_\ell(\mathrm{Tr}^{\mathrm{cl}}(R)) = \mathrm{Tr}^{\mathrm{cl}}_{\circ_1}(C_\ell(R))$, as the other equalities can be proved analogously. From Proposition 2.29, it follows that:

$$C_\ell(\mathrm{Tr}^{\mathrm{cl}}(R)) = C_\ell(\bigcup_{n\geq 1} R^n) = \bigcup_{n\geq 1} C_\ell(R^n) = \bigcup_{n\geq 1} (C_\ell(R))^{n,\circ_1} = \mathrm{Tr}^{\mathrm{cl}}_{\circ_1}(C_\ell(R))\,.$$

□

Remark 4.4. *For the non-associative composition $\circ_3$ (resp. $\circ_4$), the cylindrical extensions of the transitive closure of a binary relation R are in general not equal to the $\circ_3$-transitive (resp. $\circ_4$-transitive) closure of any cylindrical extension of R. This will be illustrated later in Example 4.7.*

4.3. Transitive closures of ternary relations on a finite set

In this section, we characterize the transitive closures of a ternary relation on a finite set, and provide some examples.

4.3.1. Four-point transitive closures on a finite set

In this subsection, we characterize the four-point transitive closures on a finite set, and provide some examples.

Proposition 4.11. *Let T be a ternary relation on a finite set X with cardinality n. For any $i \in \{1,2,3\}$, the $\diamond_i$-transitive closure of T is given by:*

$$\mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T) = \bigcup_{k=1}^{n} T^{k,\diamond_i}.$$

Proof. We only give the proof for the case $i = 1$, as the other cases can be proved analogously. From Theorem 4.3, we know that $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T) = \bigcup_{k\geq 1} T^{k,\diamond_1}$. We will now show that the powers with $k > n$ do not bring along any new triplets.

Let $(x,y,z) \in X^3$. If $(x,y,z) \in T^{2,\diamond_1}$, then there exists $t_1 \in X$ such that $(x,y,t_1) \in T$ and $(x,t_1,z) \in T$. Similarly, if $(x,y,z) \in T^{3,\diamond_1}$, then there exist $t_1,t_2 \in X$ such that $(x,y,t_1) \in T$, $(x,t_1,t_2) \in T$ and $(x,t_2,z) \in T$. More generally, denote $t_0 := y$. If $(x,y,z) \in T^{k,\diamond_1}$, then there exist $(t_i)_{i=0}^{k-1} \in X$ such that $(x,t_i,t_{i+1}) \in T$, for any $i = 0,\ldots,k-1$, and $(x,t_{k-1},z) \in T$. In other words, if $(x,y,z) \in T^{k,\diamond_1}$, then there exists a path of length $k-1$ in X starting in y and ending in t_{k-1}, such that $(x,y,t_1) \in T$ and $(x,t_{k-1},z) \in T$.

Hence, since the cardinality of X equals n, it holds in case $k > n$ that such a path must contain at least one cycle. By deleting subcycles, this path can be reduced to either

(a) a cycle-free path of length l ($l \leq n-1$), implying that $(x,y,z) \in T^{l+1,\diamond_1}$;

(b) a single cycle of length n, i.e., $t_n = t_0$, implying that $(x,t_n,z) = (x,t_0,z) = (x,y,z) \in T$.

This analysis show that the cases $k > n$ do not need to be considered. □

The following example illustrates the $\diamond_i$-transitive closures of a ternary relation on a finite set, for any $i \in \{1,2,3\}$.

Example 4.1. *Let T be the ternary relation on $X = \{x_1,x_2,x_3,x_4\}$ given by:*

$$T = \{(x_1,x_2,x_4),(x_1,x_4,x_3),(x_2,x_1,x_4),(x_4,x_2,x_2)\}.$$

One easily computes the $\diamond_1$-powers of T:

$$\begin{aligned} T^{2,\diamond_1} &= \{(x_1,x_2,x_3),(x_4,x_2,x_2)\},\\ T^{3,\diamond_1} &= \{(x_4,x_2,x_2)\},\\ T^{4,\diamond_1} &= \{(x_4,x_2,x_2)\}. \end{aligned}$$

Thus

$$\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T) = \bigcup_{k=1}^{4} T^{n,\diamond_1} = \{(x_1,x_2,x_3),(x_1,x_2,x_4),(x_1,x_4,x_3),(x_2,x_1,x_4),(x_4,x_2,x_2)\}.$$

In a similar way, one computes the transitive closures of T corresponding to the other associative compositions as a union of powers:

$$\mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(T) = \{(x_1,x_2,x_2),(x_1,x_2,x_4),(x_1,x_4,x_3),(x_2,x_1,x_4),(x_4,x_2,x_2)\},$$
$$\mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(T) = \{(x_1,x_1,x_4),(x_1,x_2,x_4),(x_1,x_4,x_3),(x_2,x_1,x_4),(x_2,x_2,x_4),(x_4,x_2,x_2)\}.$$

The following proposition shows how to compute the $\diamond_1$-transitive (resp. $\diamond_2$-transitive and $\diamond_3$-transitive) closure of a left-(resp. middle- and right-) reflexive ternary relation on a finite set. The proof is straightforward and follows from Propositions 2.10 and 4.11.

Proposition 4.12. *Let T be a ternary relation on a finite set X with cardinality n. The following implications hold:*

(i) The left reflexivity of T implies $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T) = T^{k,\diamond_1}$, where $k \leq n$ and $T^{k,\diamond_1}$ is the first power satisfying $T^{k,\diamond_1} = T^{k+1,\diamond_1}$;

(ii) The middle reflexivity of T implies $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_2}(T) = T^{k,\diamond_2}$, where $k \leq n$ and $T^{k,\diamond_2}$ is the first power satisfying $T^{k,\diamond_2} = T^{k+1,\diamond_2}$;

(iii) The right reflexivity of T implies $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_3}(T) = T^{k,\diamond_3}$, where $k \leq n$ and $T^{k,\diamond_3}$ is the first power satisfying $T^{k,\diamond_3} = T^{k+1,\diamond_3}$.

The following example illustrates the $\diamond_1$-transitive closures of a left reflexive ternary relation on a finite set.

Example 4.2. *Let T be the left reflexive ternary relation on $X = \{x_1,x_2,x_3\}$ given by:*

$$T = \{(x_1,x_1,x_1),(x_1,x_2,x_2),(x_1,x_2,x_3),(x_1,x_3,x_1),(x_1,x_3,x_3),(x_2,x_1,x_1),$$
$$(x_2,x_2,x_2),(x_2,x_3,x_3),(x_3,x_1,x_1),(x_3,x_2,x_2),(x_3,x_3,x_3)\}.$$

One easily computes the $\diamond_1$-powers of T:

$$T^{2,\diamond_1} = \{(x_1,x_1,x_1),(x_1,x_2,x_1),(x_1,x_2,x_2),(x_1,x_2,x_3),(x_1,x_3,x_1),(x_1,x_3,x_3),$$
$$(x_2,x_1,x_1),(x_2,x_2,x_2),(x_2,x_3,x_3),(x_3,x_1,x_1),(x_3,x_2,x_2),(x_3,x_3,x_3)\},$$

$$T^{3,\diamond_1} = \{(x_1,x_1,x_1),(x_1,x_2,x_1),(x_1,x_2,x_2),(x_1,x_2,x_3),(x_1,x_3,x_1),(x_1,x_3,x_3),$$

$$(x_2,x_1,x_1),(x_2,x_2,x_2),(x_2,x_3,x_3),(x_3,x_1,x_1),(x_3,x_2,x_2),(x_3,x_3,x_3)\},$$

Since $T^{2,\diamond_1} = T^{3,\diamond_1}$, *it follows that*

$$\begin{aligned}\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T) = T^{2,\diamond_1} = \{&(x_1,x_1,x_1),(x_1,x_2,x_1),(x_1,x_2,x_2),(x_1,x_2,x_3),(x_1,x_3,x_1),\\ &(x_1,x_3,x_3),(x_2,x_1,x_1),(x_2,x_2,x_2),(x_2,x_3,x_3),(x_3,x_1,x_1),\\ &(x_3,x_2,x_2),(x_3,x_3,x_3)\}.\end{aligned}$$

In a similar way, one computes the $\diamond_2$*-transitive (resp.* $\diamond_3$*-transitive) closures of a given middle (resp. right) reflexive ternary relation* T.

The following example shows that, for any $i \in \{1,2,3\}$, the binary projections of the $\diamond_i$-transitive closure of a ternary relation T on X are not equal to the transitive closure of the corresponding projection of T in general.

Example 4.3. *Let* T *be the ternary relation on* $X = \{x_1,x_2,x_3,x_4\}$ *given by:*

$$T = \{(x_1,x_2,x_3),(x_1,x_3,x_4),(x_3,x_4,x_2)\}.$$

One easily computes the $\diamond_1$*-transitive closure of* T:

$$\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T) = \{(x_1,x_2,x_3),(x_1,x_2,x_4),(x_1,x_3,x_4),(x_3,x_4,x_2)\}.$$

Hence

$$P_\ell(\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T)) = \{(x_2,x_3),(x_2,x_4),(x_3,x_4),(x_4,x_2)\}.$$

Further, one easily verifies that

$$\begin{aligned}\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(P_\ell(T)) = \{&(x_2,x_2),(x_2,x_3),(x_2,x_4),(x_3,x_2),(x_3,x_3),(x_3,x_4),(x_4,x_2),\\ &(x_4,x_3),(x_4,x_4)\}.\end{aligned}$$

It is clear that

$$P_\ell(\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T)) \neq \mathrm{Tr}^{\mathrm{cl}}(P_\ell(T)).$$

A similar example can be given for the other transitive closures.

4.3.2. Five-point transitive closures on a finite set

In this subsection, we characterize the five-point transitive closures of a ternary relation on a finite set, and provide some examples.

Proposition 4.13. *Let* T *be a ternary relation on a finite set* X *with cardinality* n.

For any $i \in \{1,2,5,6\}$, the $\circ_i$-transitive closure of T is given by:

$$\mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T) = \bigcup_{k=1}^{n^2} T^{k,\circ_i}\,.$$

***Proof*.** We only give the proof for the case $i = 1$, as the other cases can be proved analogously. From Theorem 4.4, we know that $\mathrm{Tr}^{\mathrm{cl}}_{\circ_1}(T) = \bigcup_{k\geq 1} T^{k,\circ_1}$. We will now show that the powers with $k > n^2$ do not bring along any new triplets.

Let $(x,y,z) \in X^3$. If $(x,y,z) \in T^{2,\circ_1}$, then there exists $(t_1,s_1) \in X^2$ such that $(x,y,t_1) \in T$ and $(s_1,t_1,z) \in T$. Similarly, if $(x,y,z) \in T^{3,\circ_1}$, then there exist $(t_1,s_1),(t_2,s_2) \in X^2$ such that $(x,y,t_1) \in T$, $(s_1,t_1,t_2) \in T$ and $(s_2,t_2,z) \in T$. More generally, denote $(t_0,s_0) := (y,x)$. If $(x,y,z) \in T^{k,\circ_1}$, then there exist $((t_i,s_i))_{i=0}^{k-1} \in X^2$ such that $(s_i,t_i,t_{i+1}) \in T$, for any $i = 0,\ldots,k-1$, and $(s_{k-1},t_{k-1},z) \in T$. In other words, if $(x,y,z) \in T^{k,\circ_1}$, then there exists a path of length $k-1$ in X^2 (i.e., containing k vertices and $k-1$ edges) starting in (y,x) and ending in (t_{k-1},s_{k-1}), such that $(x,y,t_1) \in T$ and $(s_{k-1},t_{k-1},z) \in T$.

Hence, since the cardinality of X^2 equals n^2, it holds in case $k > n^2$ that such a path must contain at least one cycle. By deleting subcycles, this path can be reduced to either

(a) a cycle-free path of length l ($l \leq n^2-1$), implying that $(x,y,z) \in T^{l+1,\circ_1}$;

(b) a single cycle of length n^2, i.e., $(t_{n^2},s_{n^2}) = (t_0,s_0)$, implying that $(s_{n^2},t_{n^2},z) = (s_0,t_0,z) = (x,y,z) \in T$.

This analysis show that the cases $k > n^2$ do not need to be considered. □

The following example illustrates the $\circ_i$-transitive closures of a ternary relation on a finite set, for any $i \in \{1,\ldots,6\}$.

Example 4.4. *Let T be the ternary relation on $X = \{x_1,x_2,x_3,x_4\}$ given by:*

$$T = \{(x_1,x_1,x_2),(x_2,x_4,x_1),(x_3,x_2,x_3)\}\,.$$

One easily computes the $\circ_1$-powers of T:

$$\begin{aligned}
T^{2,\circ_1} &= \{(x_1,x_1,x_3),(x_2,x_4,x_2)\}\,,\\
T^{3,\circ_1} &= \{(x_2,x_4,x_3)\}\,,\\
T^{4,\circ_1} &= \emptyset\,.
\end{aligned}$$

Thus

$$\mathrm{Tr}^{\mathrm{cl}}_{\circ_1}(T) = \bigcup_{k=1}^{3} T^{n,\circ_1} = \{(x_1,x_1,x_2),(x_1,x_1,x_3),(x_2,x_4,x_1),(x_2,x_4,x_2),(x_2,x_4,x_3),(x_3,x_2,x_3)\}\,.$$

In a similar way, one computes the transitive closures of T corresponding to the other associative five-point compositions as a union of powers:

$$\mathrm{Tr}^{\mathrm{cl}}_{\circ_2}(T) = \{(x_1,x_1,x_1),(x_1,x_1,x_2),(x_2,x_4,x_1),(x_2,x_4,x_2),(x_3,x_2,x_3)\}\,,$$
$$\mathrm{Tr}^{\mathrm{cl}}_{\circ_5}(T) = \{(x_1,x_1,x_2),(x_1,x_4,x_1),(x_2,x_1,x_2),(x_2,x_4,x_1),(x_3,x_2,x_3)\}\,,$$
$$\mathrm{Tr}^{\mathrm{cl}}_{\circ_6}(T) = \{(x_1,x_1,x_2),(x_2,x_4,x_1),(x_3,x_2,x_3),(x_3,x_4,x_1)\}\,.$$

For the non-associative compositions ($i \in \{3,4\}$), one computes in an iterative manner the smallest $\circ_i$-transitive ternary relation containing T without computing a union of powers:

$$\mathrm{Tr}^{\mathrm{cl}}_{\circ_3}(T) = \{(x_1,x_1,x_2),(x_1,x_1,x_4),(x_2,x_4,x_1),(x_3,x_2,x_2),(x_3,x_2,x_3),(x_3,x_2,x_4)\}\,,$$
$$\mathrm{Tr}^{\mathrm{cl}}_{\circ_4}(T) = \{(x_1,x_1,x_2),(x_1,x_2,x_3),(x_1,x_4,x_1),(x_2,x_2,x_3),(x_2,x_4,x_1),(x_3,x_2,x_3),(x_4,x_1,x_2),(x_4,x_2,x_3),(x_4,x_4,x_1)\}\,.$$

The following proposition shows how to compute the $\circ_i$-transitive closure of a reflexive ternary relation on a finite set, for any $i \in \{1,2,5,6\}$. The proof is straightforward and follows from Propositions 3.8 and 4.13.

Proposition 4.14. *Let T be a reflexive ternary relation on a finite set X with cardinality n. For any $i \in \{1,2,5,6\}$, the $\circ_i$-transitive closure of T is given by:*

$$\mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T) = T^{k,\circ_i}\,,$$

where $k \le n^2$ and $T^{k,\circ_i}$ is the first power satisfying $T^{k,\circ_i} = T^{k+1,\circ_i}$.

The following example illustrates the $\circ_i$-transitive closures of a reflexive ternary relation on a finite set, for any $i \in \{1,\dots,6\}$.

Example 4.5. *Let T be the reflexive ternary relation on $X = \{x_1,x_2,x_3,x_4\}$ given by:*

$$T = \{(x_1,x_1,x_1),(x_2,x_1,x_3),(x_2,x_2,x_2),(x_3,x_3,x_3),(x_4,x_3,x_2),(x_4,x_4,x_4)\}\,.$$

One easily computes the $\circ_1$-powers of T:

$$\begin{aligned}T^{2,\circ_1} &= \{(x_1,x_1,x_1),(x_1,x_1,x_3),(x_2,x_1,x_2),(x_2,x_1,x_3),(x_2,x_2,x_2),(x_3,x_3,x_2),\\ &\quad (x_3,x_3,x_3),(x_4,x_3,x_2),(x_4,x_4,x_4)\}\,,\\ T^{3,\circ_1} &= \{(x_1,x_1,x_1),(x_1,x_1,x_2),(x_1,x_1,x_3),(x_2,x_1,x_2),(x_2,x_1,x_3),(x_2,x_2,x_2),\\ &\quad (x_3,x_3,x_2),(x_3,x_3,x_3),(x_4,x_3,x_2),(x_4,x_4,x_4)\}\,,\\ T^{4,\circ_1} &= \{(x_1,x_1,x_1),(x_1,x_1,x_2),(x_1,x_1,x_3),(x_2,x_1,x_2),(x_2,x_1,x_3),(x_2,x_2,x_2),\\ &\quad (x_3,x_3,x_2),(x_3,x_3,x_3),(x_4,x_3,x_2),(x_4,x_4,x_4)\}\,.\end{aligned}$$

Since $T^{3,\circ_1} = T^{4,\circ_1}$, it follows that

$$\begin{aligned}\mathrm{Tr}^{\mathrm{cl}}_{\circ_1}(T) = T^{3,\circ_1} &= \{(x_1,x_1,x_1),(x_1,x_1,x_2),(x_1,x_1,x_3),(x_2,x_1,x_2),(x_2,x_1,x_3),\\ &\quad (x_2,x_2,x_2),(x_3,x_3,x_2),(x_3,x_3,x_3),(x_4,x_3,x_2),(x_4,x_4,x_4)\}\,.\end{aligned}$$

In a similar way, one computes the transitive closures of T corresponding to the other associative five-point compositions

$$\begin{aligned}\mathrm{Tr}^{\mathrm{cl}}_{\circ_2}(T) = T^{3,\circ_2} &= \{(x_1,x_1,x_1),(x_2,x_1,x_3),(x_2,x_2,x_2),(x_2,x_2,x_3),(x_3,x_3,x_3),\\ &\quad (x_4,x_3,x_2),(x_4,x_3,x_3),(x_4,x_4,x_2),(x_4,x_4,x_4)\}\,,\\ \mathrm{Tr}^{\mathrm{cl}}_{\circ_5}(T) = T^{3,\circ_5} &= \{(x_1,x_1,x_1),(x_2,x_1,x_3),(x_2,x_2,x_2),(x_2,x_3,x_3),(x_3,x_3,x_3),\\ &\quad (x_4,x_1,x_3),(x_4,x_2,x_2),(x_4,x_3,x_2),(x_4,x_3,x_3),(x_4,x_4,x_4)\}\,,\\ \mathrm{Tr}^{\mathrm{cl}}_{\circ_6}(T) = T^{2,\circ_6} &= \{(x_1,x_1,x_1),(x_2,x_1,x_1),(x_2,x_1,x_3),(x_2,x_2,x_2),(x_3,x_3,x_3),\\ &\quad (x_4,x_3,x_2),(x_4,x_3,x_3),(x_4,x_4,x_4)\}\,.\end{aligned}$$

For the non-associative compositions ($i \in \{3,4\}$), one computes the smallest $\circ_i$-transitive ternary relation containing T without computing a union of powers:

$$\begin{aligned}\mathrm{Tr}^{\mathrm{cl}}_{\circ_3}(T) &= \{(x_1,x_1,x_1),(x_2,x_1,x_3),(x_2,x_2,x_1),(x_2,x_2,x_2),(x_3,x_3,x_3),(x_4,x_3,x_1),\\ &\quad (x_4,x_3,x_2),(x_4,x_4,x_3),(x_4,x_4,x_4)\}\,,\\ \mathrm{Tr}^{\mathrm{cl}}_{\circ_4}(T) &= \{(x_1,x_1,x_1),(x_1,x_1,x_3),(x_1,x_2,x_2),(x_1,x_3,x_3),(x_2,x_1,x_3),(x_2,x_2,x_2),\\ &\quad (x_3,x_1,x_3),(x_3,x_2,x_2),(x_3,x_3,x_3),(x_4,x_3,x_2),(x_4,x_4,x_4)\}\,.\end{aligned}$$

Proposition 4.9 shows that, for any $i \in \{1,2,5,6\}$, the binary projections of the $\circ_i$-transitive closure of a ternary relation T on X coincide with the transitive closure of the corresponding projection of T. The following example shows that this does not hold for $i \in \{3,4\}$.

Example 4.6. *Let T be the ternary relation on $X = \{x_1, x_2, x_3, x_4\}$ given by:*

$$T = \{(x_1, x_2, x_3), (x_3, x_1, x_2)\}\,.$$

One easily computes the $\circ_3$-transitive and $\circ_4$-transitive closures of T:

$$\mathrm{Tr}^{\mathrm{cl}}_{\circ_3}(T) = \{(x_1, x_2, x_1), (x_1, x_2, x_2), (x_1, x_2, x_3), (x_3, x_1, x_2)\}\,,$$
$$\mathrm{Tr}^{\mathrm{cl}}_{\circ_4}(T) = \{(x_1, x_1, x_2), (x_1, x_2, x_3), (x_2, x_1, x_2), (x_3, x_1, x_2)\}\,.$$

Thus

	$P_\ell(.)$	$P_m(.)$	$P_r(.)$
$\mathrm{Tr}^{\mathrm{cl}}_{\circ_3}(T)$	$\{(x_1, x_2), (x_2, x_1), (x_2, x_2), (x_2, x_3)\}$	$\{(x_1, x_1), (x_1, x_2), (x_1, x_3), (x_3, x_2)\}$	$\{(x_1, x_2), (x_3, x_1)\}$
$\mathrm{Tr}^{\mathrm{cl}}_{\circ_4}(T)$	$\{(x_1, x_2), (x_2, x_3)\}$	$\{(x_1, x_2), (x_1, x_3), (x_2, x_2), (x_3, x_2)\}$	$\{(x_1, x_1), (x_1, x_2), (x_2, x_1), (x_3, x_1)\}$

Further, one easily verifies that

	$P_\ell(.)$	$P_m(.)$	$P_r(.)$
T	$\{(x_1, x_2), (x_2, x_3)\}$	$\{(x_1, x_3), (x_3, x_2)\}$	$\{(x_1, x_2), (x_3, x_1)\}$

Hence

	$P_\ell(T)$	$P_m(T)$	$P_r(T)$
$\mathrm{Tr}^{\mathrm{cl}}(.)$	$\{(x_1, x_2), (x_1, x_3), (x_2, x_3)\}$	$\{(x_1, x_2), (x_1, x_3), (x_3, x_2)\}$	$\{(x_1, x_2), (x_3, x_1), (x_3, x_2)\}$

It is clear that

$$P_\lambda(\mathrm{Tr}^{\mathrm{cl}}_{\circ_3}(T)) \neq \mathrm{Tr}^{\mathrm{cl}}(P_\lambda(T)),\ \textit{for any}\ \lambda \in \{\ell, m, r\}$$

and

$$P_\lambda(\mathrm{Tr}^{\mathrm{cl}}_{\circ_4}(T)) \neq \mathrm{Tr}^{\mathrm{cl}}(P_\lambda(T)),\ \textit{for any}\ \lambda \in \{\ell, m, r\}\,.$$

Proposition 4.10 shows that any cylindrical extension of the transitive closure of a binary relation R on X coincides with the $\circ_i$-transitive closure, for an appropriate $i \in \{1, 2, 5, 6\}$, of the same cylindrical extension of R. The following example shows that no such result holds for $i \in \{3, 4\}$.

Example 4.7. *Let R be the binary relation on $X = \{x_1, x_2, x_3\}$ given by:*

$$R = \{(x_1, x_3), (x_3, x_2)\}\,.$$

One easily verifies that $\mathrm{Tr}^{\mathrm{cl}}(R) = \{(x_1, x_2), (x_1, x_3), (x_3, x_2)\}$*. Thus*

	$C_\ell(.)$	$C_m(.)$	$C_r(.)$
$\mathrm{Tr}^{\mathrm{cl}}(R)$	$\{(x_1,x_1,x_2),(x_1,x_1,x_3),$ $(x_1,x_3,x_2),(x_2,x_1,x_2),$ $(x_2,x_1,x_3),(x_2,x_3,x_2),$ $(x_3,x_1,x_2),(x_3,x_1,x_3),$ $(x_3,x_3,x_2)\}$	$\{(x_1,x_1,x_2),(x_1,x_1,x_3),$ $(x_1,x_2,x_2),(x_1,x_2,x_3),$ $(x_1,x_3,x_2),(x_1,x_3,x_3),$ $(x_3,x_1,x_2),(x_3,x_2,x_2),$ $(x_3,x_3,x_2)\}$	$\{(x_1,x_2,x_1),(x_1,x_2,x_2),$ $(x_1,x_2,x_3),(x_1,x_3,x_1),$ $(x_1,x_3,x_2),(x_1,x_3,x_3),$ $(x_3,x_2,x_1),(x_3,x_2,x_2),$ $(x_3,x_2,x_3)\}$

Further, one easily verifies that

	$C_\ell(.)$	$C_m(.)$	$C_r(.)$
R	$\{(x_1,x_1,x_3),(x_1,x_3,x_2),$ $(x_2,x_1,x_3),(x_2,x_3,x_2),$ $(x_3,x_1,x_3),(x_3,x_3,x_2)\}$	$\{(x_1,x_1,x_3),(x_1,x_2,x_3),$ $(x_1,x_3,x_3),(x_3,x_1,x_2),$ $(x_3,x_2,x_2),(x_3,x_3,x_2)\}$	$\{(x_1,x_3,x_1),(x_1,x_3,x_2),$ $(x_1,x_3,x_3),(x_3,x_2,x_1),$ $(x_3,x_2,x_2),(x_3,x_2,x_3)\}$

Hence

	$C_\ell(R)$	$C_m(R)$	$C_r(R)$
$\mathrm{Tr}^{\mathrm{cl}}_{\circ_4}(.)$	$\{(x_1,x_1,x_3),(x_1,x_3,x_2),$ $(x_2,x_1,x_3),(x_2,x_3,x_2),$ $(x_3,x_1,x_3),(x_3,x_3,x_2)\}$	$\{(x_1,x_1,x_2),(x_1,x_1,x_3),$ $(x_1,x_2,x_2),(x_1,x_2,x_3),$ $(x_1,x_3,x_2),(x_1,x_3,x_3),$ $(x_2,x_1,x_2),(x_2,x_2,x_2),$ $(x_2,x_3,x_2),(x_3,x_1,x_2),$ $(x_3,x_2,x_2),(x_3,x_3,x_2)\}$	$\{(x_1,x_3,x_1),(x_1,x_3,x_2),$ $(x_1,x_3,x_3),(x_2,x_2,x_1),$ $(x_2,x_2,x_2),(x_2,x_3,x_1),$ $(x_2,x_3,x_2),(x_2,x_3,x_3),$ $(x_3,x_2,x_1),(x_3,x_2,x_2),$ $(x_3,x_2,x_3),(x_3,x_3,x_1),$ $(x_3,x_3,x_2),(x_3,x_3,x_3)\}$

It is clear that

$$C_\lambda(\mathrm{Tr}^{\mathrm{cl}}(R)) \neq \mathrm{Tr}^{\mathrm{cl}}_{\circ_4}(C_\lambda(R)),\ \textit{for any}\ \lambda \in \{\ell, m, r\}\,.$$

A similar example can be given for the $\circ_3$*-transitive closure.*

4.4. Openings of a ternary relation

While the P-closure of a given ternary relation is the smallest relation containing this ternary relation and possessing a property P, the P-opening of a ternary relation is the greatest relation that possesses P and is contained in this ternary relation.

4.4.1. General results

In this subsection, the results are obtained in the same way as in [11] and Subsection 4.1.

Definition 4.2. *If P is a property which a ternary relation T on X may have or fail to have, then a ternary relation Q is the P-opening of T, written $Q = P^{\mathrm{op}}(T)$, if and only if Q satisfies all of*

(i) Q possesses property P;

(ii) $Q \subseteq T$;

(iii) If $T' \subseteq T$ and T' possesses P, then $T' \subseteq Q$.

The uniqueness of the P-opening, if it exists, is clear.

Corollary 4.3. *A ternary relation T on a set X possesses property P if and only if $T = P^{\mathrm{op}}(T)$.*

As with closures, openings are of interest for those properties P for which every ternary relation T has a P-opening. The following theorem states the conditions for this to occur.

Theorem 4.5. *A P-opening exists for all ternary relations T on X if and only if*

(i) The null relation $O_{X^3} = \emptyset$ possesses P;

(ii) The union of every (non-empty) family of ternary relations, each of which possesses P, also possesses P.

This characterization can be rephrased in the following corollary.

Corollary 4.4. *Let T be a ternary relation on a set X, then*

$$P^{\mathrm{op}}(T) = \bigcup \{S \mid S \subseteq T \wedge S \text{ has property } P\} .$$

An example of a property P for which a P-opening does not exist in general is again the completeness property. Indeed, the null relation does not have this property. Obviously, there does not exist any non-complete ternary relation T that can be opened for completeness.

Theorem 4.6. *If P and P' are properties for which openings exist (satisfying the conditions of Theorem 4.5), and if P^{op} and $\mathrm{P}'^{\mathrm{op}}$ commute with each other, then $(P \wedge P')$ also satisfies the conditions of Theorem 4.5, and has an opening given by*

$$(P \wedge P')^{\mathrm{op}} = P^{\mathrm{op}}(P'^{\mathrm{op}}) = P'^{\mathrm{op}}(P^{\mathrm{op}}) .$$

4.4.2. Openings for some special properties of ternary relations

From the properties of ternary relations listed in Chapter 1, only symmetry, strong symmetry and cyclicity meet both criteria of Theorem 4.5, while reflexivity fails for the first criterion and $\circ_i$-transitivity fails for the second, for any $i \in \{1, \ldots, 6\}$. The following proposition expresses the symmetric opening, strongly symmetric opening and cyclic opening of a ternary relation.

Proposition 4.15. *Let T be a ternary relation on a set X. It holds that:*

(i) The symmetric opening is $\mathrm{Sym}^{\mathrm{op}}(T) = T \cap T^t$*;*

(ii) The strongly symmetric opening is $\mathrm{SSym}^{\mathrm{op}}(T) = \bigcap_{i=0}^{5} T^{\sigma_i}$*;*

(iii) The cyclic opening is $\mathrm{Cyc}^{\mathrm{op}}(T) = T \cap T^+ \cap T^-$*.*

Proof. (i) To prove that $\mathrm{Sym}^{\mathrm{op}}(T) = T \cap T^t$, it suffices to prove that $T \cap T^t$ is the greatest symmetric ternary relation contained in T. First, it is clear that $T \cap T^t \subseteq T$. From Proposition 4.1, it follows that $T \cap T^t$ is symmetric. Next, if $M \subseteq T$ and M is symmetric, then it holds that $M^t \subseteq T^t$. Hence, $M \cap M^t \subseteq T \cap T^t$. Since M is symmetric, it follows from Corollary 4.3 that $\mathrm{Sym}^{\mathrm{op}}(M) = M$. Thus, $M \subseteq T \cap T^t$. Therefore, $\mathrm{Sym}^{\mathrm{op}}(T) = T \cap T^t$.

(ii) To prove that $\mathrm{SSym}^{\mathrm{op}}(T) = \bigcap_{i=0}^{5} T^{\sigma_i}$, it suffices to prove that $\bigcap_{i=0}^{5} T^{\sigma_i}$ is the greatest strongly symmetric ternary relation contained in T. First, it is clear that $\bigcap_{i=0}^{5} T^{\sigma_i} \subseteq T$. From Remark 1.1, it follows that $\bigcap_{i=0}^{5} T^{\sigma_i}$ is strongly symmetric. Next, if $M \subseteq T$ and M is strongly symmetric, then for any $i \in \{0, \ldots, 5\}$, it holds that $M^{\sigma_i} \subseteq T^{\sigma_i}$. Hence, $\bigcap_{i=0}^{5} M^{\sigma_i} \subseteq \bigcap_{i=0}^{5} T^{\sigma_i}$. Since M is strongly symmetric, it follows from Corollary 4.3 that $\mathrm{SSym}^{\mathrm{op}}(M) = M$. Thus, $M \subseteq \bigcap_{i=0}^{5} T^{\sigma_i}$. Therefore, $\mathrm{SSym}^{\mathrm{op}}(T) = \bigcap_{i=0}^{5} T^{\sigma_i}$.

(iii) We show that $\mathrm{Cyc}^{\mathrm{op}}(T) = T \cap T^+ \cap T^-$. First, we need to prove that $T \cap T^+ \cap T^-$ is cyclic. In view of Proposition 4.1, it suffices to prove that $T \cap T^+ \cap T^- = (T \cap T^+ \cap T^-)^+ = (T \cap T^+ \cap T^-)^-$. From Proposition 1.1 and Remark 1.1, it indeed follows that

$$(T \cap T^+ \cap T^-)^+ = T^+ \cap (T^+)^+ \cap (T^-)^+$$

$$
\begin{aligned}
&= T^{+} \cap T^{-} \cap T \\
&= T \cap T^{+} \cap T^{-}
\end{aligned}
$$

and

$$
\begin{aligned}
(T \cap T^{+} \cap T^{-})^{-} &= T^{-} \cap (T^{+})^{-} \cap (T^{-})^{-} \\
&= T^{-} \cap T \cap T^{+} \\
&= T \cap T^{+} \cap T^{-} \,.
\end{aligned}
$$

Furthermore, it is clear that $T \cap T^{+} \cap T^{-}$ is the greatest cyclic ternary relation contained in T. Thus, $\mathrm{Cyc}^{\mathrm{op}}(T) = T \cap T^{+} \cap T^{-}$.

□

General conclusions and future research

In this work, we have extended the composition of binary relations to the setting of ternary relations. First, we have introduced several types of four points composition of ternary relations, including three associative ones. Also, we have introduced six types of five points composition of ternary relations, including four associative ones; these compositions are based on the composition of a ternary relation with a binary relation and vice versa, and we have investigated their properties in detail. Moreover, we have studied the interaction of these compositions with the binary projections of ternary relations, cylindrical extensions of binary relations and traces of ternary relations.

These compositions of ternary relations facilitated our study of the different notions of transitivity of a ternary relation, among others. More precisely, we have discussed the four-point (resp. five-point) transitivity of a ternary relation from the point of view of four-point (resp. five-point) compositions of ternary relations, and we have investigated their properties and interactions with binary projections, cylindrical extensions and traces.

Throughout this thesis, we have studied closures and openings of ternary relations for some basic properties. More specifically, we have paid particular attention to the transitive closures of a ternary relation and we have provided several examples in the finite case.

Given the importance of fuzzy relations, as amply illustrated in the introduction for binary relations, future efforts will be directed to the study of fuzzy ternary relations as well. We anticipate that it will be interesting to extend the closures and openings of ternary relations to the fuzzy setting.

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إغلاقات وفتاحات العلاقات الثلاثية

ملـخص: من المثير للاهتمام، على عكس العلاقات الثنائية، أن العلاقات الثلاثية، وبشكل أعم، العلاقات n-ary، حظيت باهتمام أقل بكثير. ومع ذلك، في السنوات الأخيرة، كان الاهتمام بالعلاقات الثلاثية في ازدياد.

في هذه الأطروحة، قمنا بتعميم مفهوم تركيب العلاقة الثنائية الى حالة العلاقة الثلاثية. لقد درسنا كل التركيبات المحتملة للعلاقات الثلاثية المكونة من أربع وخمس نقاط، وقد أولينا اهتمامًا خاصًا للتركيبات التجميعية. بالإضافة إلى ذلك، قمنا بدراسة الإغلاقات (Closures) والفتاحات (Openings) لأهم خصائص العلاقات الثلاثية.

الكلمات المفتاحية : العلاقات الثنائية؛ العلاقات الثلاثية؛ تركيب العلاقات؛ علاقة التعدي؛ إغلاقات؛ فتاحات.

Fermetures et ouvertures de relations ternaires

Résumé: Étonnamment, contrairement aux relations binaires, les relations ternaires et, plus généralement, les relations n-aires, ont reçu beaucoup moins d'attention. Cependant, ces dernières années, l'intérêt pour les relations ternaires est en hausse.

Dans cette thèse, nous avons généralisé la notion de composition de relation binaire au cadre ternaire. Nous avons étudié tous les types possibles de compositions de relations ternaires à quatre et cinq points, et nous avons porté une attention particulière aux compositions associatives. De plus, nous avons étudié les fermetures et ouvertures de leurs propriétés intéressantes.

Mots-clés : Relation binaire; Relation ternaire; Compositions relationnelles; Transitivité; Fermeture; Ouverture.

Closures and openings of ternary relations

Abstract: Surprisingly, in contrast to binary relations, ternary and, more generally, n-ary relations, have received far less attention. However, in recent years, the interest in ternary relations is on the rise.

In this thesis, we have generalized the notion of composition of binary relation to the ternary setting. We have studied all possible types of four- and five-point compositions of ternary relations, and we have paid particular attention to the associative ones. Further, we have studied the closures and openings of their interesting properties.

Keywords: Binary relation; Ternary relation; Relational compositions; Transitivity; Closure; opening.