



N° d'ordre:.....

THESIS

*Presented for obtaining the diploma
of doctoral degree 3rd cycle LMD*

Speciality

Mathematics

Option

Algebra And Discrete Mathematics

By

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Theme

Representation of the solutions of Diophantine equations containing arithmetic functions

Thesis defended publicly on: 18/04/2024, before a jury composed of:

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Thanks

I cannot begin and finish my work without thanking the greatest and the most powerful "Allah."

At the end of this modest work, I would like to extend my sincere thanks to the people who have helped me and who have contributed to the preparation of this brief.

I would like to express my heartfelt thanks to my supervisor Professor BOUDAOU ABDELMADJID and Professor BELLAOUAR DJAMEL for their patience, their competence, and their precious help during the realization of this work and their remarks and their suggestions which allowed me to finalize this thesis.

My sincere thanks also go to the Jury members who will be kind enough to read and judge this modest work. My special thanks to Professor DOUADI MIHOUBI who kindly who has agreed to chair this jury.

I wish to express my sincere thanks to Professors NASSER GHADBANE, LMNAWAR NOUI and AHMED BENDJEDDOU who kindly accepted to be jury members of my thesis.

Dedications

I dedicate this thesis to:

My Mother

My Father

My Husband.

My daughter :

Hiba.

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Notations

\mathbb{N} : Set of natural numbers.

\mathbb{Z} : Set of integer numbers.

\mathbb{Q} : Set of rational numbers.

(a, b) : The greatest common divisor of a and b .

$\varphi(n)$: Euler's function of n .

$\tau(n)$: The number of divisors of n .

$\sigma(n)$: The sum of the divisors of n .

$\omega(n)$: The number of distinct prime factors of n .

$\Omega(n)$: The total number of prime factors of n .

$\psi(n)$: Dedekind psi function of n .

$a \mid b$: a divides b .

$a \nmid b$: Means that b is not divisible by a .

$q^k \parallel n$: It means that q^k fully divides n , that is, q^k divides n but $q^{k+1} \nmid n$.

F_n : Fermat number.

$S := \{n \in \mathbb{N} : \tau(n) = \tau(\varphi(n))\}$.

$\mathbb{S} := \{n \in \mathbb{N} : \tau(n^2) = \tau(\varphi(n))\}$.

$\mathbb{E}_k := \{n \in \mathbb{Z}^+ : \tau(kn) = \varphi(\varphi(n))\}, k \in \mathbb{Z}$.

$\mathbb{L}_k := \{n \in \mathbb{Z}^+ : \tau(kn) < \varphi(\varphi(n))\}, k \in \mathbb{Z}$.

$\mathbb{G}_k := \{n \in \mathbb{Z}^+ : \tau(kn) > \varphi(\varphi(n))\}, k \in \mathbb{Z}$.

$E_k := \{n \in \mathbb{N} : \sigma(n) = \tau(k \cdot \varphi(n^2))\}, k \in \mathbb{Z}$.

$L_k := \{n \in \mathbb{N} : \sigma(n) < \tau(k \cdot \varphi(n^2))\}, k \in \mathbb{Z}$.

$G_k := \{n \in \mathbb{N} : \sigma(n) > \tau(k \cdot \varphi(n^2))\}, k \in \mathbb{Z}$.

Introduction

An equation that is to be solved in integers is called a Diophantine equation. For example

$$\tau(n) = \sigma(n) + n \quad \text{or} \quad xyz + x^2 + z^3 = 8y^7z^2. \quad (1)$$

Unaware of their name, we've all been manipulating Diophantine equations since we were children. Over time, they have become increasingly complex and abstract. Some of us have had to solve a problem of this type: find the radius of a circle such that the difference between the perimeter of a circle and its area is equal to the number π . After simplification, this problem is equivalent to solving the following equation:

$$x^2 - 2x + 1 = 0. \quad (2)$$

Diophantine equations are named after the Greek mathematician Diophantus of Alexandria, also known as Diophantus, who dedicated a series of works.

Diophantus of Alexandria (born probably between 200 and 214 A.D.; died aged 84 years, probably between 284 and 298) was an Alexandrian mathematician, author of a series of books entitled *Arithmetica*, many of which are now lost. His texts deal with solving algebraic equations.

Historically, the first example of Diophantine equations is probably the Pythagorean equation:

$$x^2 + y^2 = z^2 \quad (3)$$

which consists in finding the triangle whose sides are integers. After Diophantus, eminent mathematicians responded to the call, either to formulate conjectures as those Pierre Fermat, Catalan, etc., either to solve problems (conjectures) like those Euler, Gauss, de Lagrange, Dedekind, etc.. to build what we have today as number theory.

Solving equations involving arithmetic functions is an interesting problem. In 1966

Nicol [21] studied Diophantine equations of the form

$$c_1 f_1(n) + c_2 f_2(n) \cdots + c_k f_k(n) = c, \quad (4)$$

where $c, c_1, c_2, \dots, c_k \in \mathbb{Z}$ and f_1, f_2, \dots, f_k are arithmetic functions. He has observed that many such equations of this type are exactly satisfied when n is prime.

As a continuation of this work, several authors have been treated particular cases of (4). For more details, see [9, pages 78-80], [26, pages 108-118], [13, e.g., § B-13, §B-18].

In [4] D. Bellaouar, A. Boudaoud and R. Jakimczuk have been worked on Diophantine equation containing the arithmetic functions $\tau(n)$ and $\varphi(n)$, where these functions denote the number of positive integers dividing the positive integer n and the Euler's phi function representing the numbers less than and prime to n , respectively. Their first goal is found all natural numbers $n \in \mathbb{N}$ having the property that n and $\varphi(n)$ have the same number of positive divisors, i.e., $\tau(n) = \tau(\varphi(n))$. Moreover, the related inequalities are studied. And there are many other papers in the literature dealing with Diophantine equations involving the arithmetic functions. For example, in [15] it is shown that $n = 2$ is the only known solution of the equation

$$\varphi(n) = \sigma(n) - n, \quad (5)$$

where $\sigma(n)$ denotes the sum of the positive divisors of n . And in [17], it is shown that the equation

$$\varphi(n!) = a \cdot m! \quad (6)$$

has only finitely many solutions (m, n) , where a is any positive rational number.

Other similar problems have been discussed in publications such as Guy [13]; e.g., § B-38 $\varphi(m) = \sigma(n)$, and § B-42, $\sigma(\varphi(n)) = \varphi(\sigma(n))$, $\varphi(\sigma(n)) = n$, $\varphi(\sigma(n)) = \varphi(n)$.

Some such problems have been solved and are popular topics of discussion on many internet forums; e.g., $\tau(\varphi(n)) = \varphi(\tau(n))$ has infinitely many solutions [9, page 78].

In the same sense, the equation

$$\tau(n) = \varphi(n) \quad (7)$$

has only six solutions: 1, 3, 8, 10, 24 and 30, where $\varphi(n) > \tau(n)$ for $n \geq 31$ due to Sándor ([26], pages 110-111). See also [6, 12, 14].

However, many Diophantine equations containing arithmetic functions remain unsolved to this day. We propose to solve a few of them. In this thesis, we have seek to

prove that the Diophantine equation

$$\tau(n^2) = \tau(\varphi(n)) \quad (8)$$

has infinitely many integer solutions as well as we identify large families of solutions. The first few terms of them are:

$$1, 5, 57, 74, 202, 292, 394, 514, 652, 1354, 2114, 2125, \dots,$$

For this purpose, we define

$$\mathbb{S} := \{n \in \mathbb{N} : \tau(n^2) = \tau(\varphi(n))\}.$$

We have characterized the elements of \mathbb{S} that have at most three distinct prime factors. The problem is interesting because it can force us to solve some Diophantine equations involving prime numbers. Note also that the proofs are all on the elementary side and depend on long case by case analysis type arguments.

If $n = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k} \in \mathbb{S}$, where $q_1 < q_2 < \cdots < q_k$ are prime numbers and a_1, a_2, \dots, a_k are positive integers, then a_k must be odd. In fact, since $(q_1 \cdots q_{k-1}(q_1 - 1) \cdots (q_k - 1), q_k) = 1$ we conclude that

$$\tau(\varphi(n)) = \tau(q_1^{a_1-1} q_2^{a_2-1} \cdots q_k^{a_k-1} (q_1 - 1)(q_2 - 1) \cdots (q_k - 1)) a_k,$$

and so a_k must be odd since $\tau(n^2) = \prod_{i=1}^k (2a_i + 1)$.

Now, let n as above and put

$$\prod_{i=1}^k (q_i - 1) = 2^{x_1+x_2+\cdots+x_k} \cdot q_1^{\alpha_1^{(2)}+\alpha_1^{(3)}+\cdots+\alpha_1^{(k)}} \cdots q_{k-2}^{\alpha_{k-2}^{(k-1)}+\alpha_{k-2}^{(k)}} \cdot q_{k-1}^{\alpha_{k-1}^{(k)}} \cdot m,$$

where $x_1, x_2, \dots, x_k, m \geq 1$ and $\alpha_1^{(2)}, \dots, \alpha_1^{(k)}, \alpha_2^{(3)}, \dots, \alpha_2^{(k)}, \dots, \alpha_{k-2}^{(k-1)}, \alpha_{k-2}^{(k)}, \alpha_{k-1}^{(k)}$ are non-negative integers with $(2q_1 q_2 \cdots q_{k-1}, m) = 1$. Thus in order to prove that n satisfies (8), it suffices to confirm that the exponents of the prime factors of n and the above variables satisfy the following Diophantine equation:

$$\prod_{i=1}^k (2a_i + 1) = \left(1 + \sum_{i=1}^k x_i\right) \left(a_1 + \sum_{i=2}^k \alpha_1^{(i)}\right) \left(a_2 + \sum_{i=3}^k \alpha_2^{(i)}\right) \cdots \left(a_{k-1} + \alpha_{k-1}^{(k)}\right) a_k \tau(m). \quad (9)$$

In particular, if $k = 3$ and n is odd then we need to solve the Diophantine equation

$$(2a + 1)(2b + 1)(2c + 1) = (x_1 + x_2 + x_3 + 1)(a_1 + \alpha_1 + \alpha_2)(b + \alpha_3) c \cdot \tau(m),$$

where $m, x_1, x_2, x_3 \geq 1$ and $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $(2q_1q_2, m) = 1$. This work was published in the journal NNTMD [2].

As a second work, we first define for any positive integer k the following sets:

$$\mathbb{E}_k = \{n \in \mathbb{N} : \tau(kn) = \varphi(\varphi(n))\},$$

$$\mathbb{L}_k = \{n \in \mathbb{N} : \tau(kn) < \varphi(\varphi(n))\},$$

$$\mathbb{G}_k = \{n \in \mathbb{N} : \tau(kn) > \varphi(\varphi(n))\}.$$

So, the main focus of this work is to examine the set \mathbb{E}_k of solutions n of the equation $\tau(kn) = \varphi(\varphi(n))$ and related inequalities. In fact, we have characterized all the elements of $\mathbb{E}_1, \mathbb{L}_1$ and \mathbb{G}_1 , respectively. Then we have proved that the sets \mathbb{G}_k and \mathbb{E}_k are finite, while \mathbb{L}_k is infinite. Moreover, we have proved that if p is a prime number with $p \geq 23$ and $p \neq 31$, then $\mathbb{E}_p = \{11, 13, 33, 34, 35, 39, 62, 63, 76, 88, 98, 102, 104, 105, 110, 130, 154, 186, 228, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840\}$. This work has been accepted for publication in Jordan Journal of Mathematics and Statistics.

The following text is also Another work has been submitted for publication.

Define for any positive integer k the following sets:

$$E_k := \{n \in \mathbb{N} : \sigma(n) = \tau(k \cdot \varphi(n^2))\},$$

$$L_k := \{n \in \mathbb{N} : \sigma(n) < \tau(k \cdot \varphi(n^2))\},$$

$$G_k := \{n \in \mathbb{N} : \sigma(n) > \tau(k \cdot \varphi(n^2))\}.$$

Here we have found all the solutions n of the equation $\sigma(n) = \tau(k \cdot \varphi(n^2))$. In effect, we have characterized the elements of $E_1, L_1, G_1, E_2, L_2, G_2, E_3, L_3$, and G_3 , and then we have shown that E_k and L_k are finite, while G_k is infinite ($k \geq 1$). Furthermore, we have proved that if p is prime with $p \geq 13$ and $p \neq 31$, then $E_p = \{6, 14, 31, 33, 77\}$ and if $p \geq 23$, then $L_p = \{1, 2, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 21, 35\}$.

This thesis is divided into 5 chapters.

Chapter 01 presents some preliminaries on arithmetic functions, introduces the Diophantine equations and the main elementary methods to solve them. These include the factorization method, the use of inequalities, the parametric method and mathematical induction.

Chapter 02 presents some new Diophantine equation involving arithmetic functions since 2019.

Chapter 03 presents a class of solutions that have at most three distinct prime factors of the equation $\tau(n^2) = \tau(\varphi(n))$, Moreover, it show that Dickson's conjecture implies that

$\tau(n^2) = \tau(\varphi(n))$ infinitely often.

Chapter 04 and 05 present the same work as chapter 03, but here the sets of the solutions of the Diophantine equations $\tau(kn) = \varphi(\varphi(n))$ and $\sigma(n) = \tau(kn\varphi(n))$ are finite.

PRELIMINARIES

In this chapter, we first define some most basic arithmetic functions, we also introduce the notion of Diophantine equations and their different types, finally we analyse the various classical methods for solving them.

1.1 Arithmetic functions

In this section, we look at some of the most basic arithmetic functions using in the next chapters.

Definition 1.1. An arithmetic function or a number-theoretic function is a real or complex valued function defined on positive integers (or all integers).

Example 1.1. These are some examples of arithmetic functions.

- The identity function

$$I_n = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

- The unit function

$$u(n) = 1, \quad n \geq 1.$$

- The function $N^s(n)$ such that

$$N^s(n) = n^s, \quad n \geq 1, s \in \mathbb{N}.$$

In the following we give two important class of arithmetic functions which are: multiplicative functions and additive functions.

1.1.1 Multiplicative functions

Definition 1.2. An arithmetic function f which is not identically zero is said to be multiplicative if

$$f(mn) = f(m)f(n) \tag{1.1}$$

whenever $(m, n) = 1$. Moreover, if (1.1) holds for all m, n , then f is called completely multiplicative.

Theorem 1.1. *If f is multiplicative then $f(1) = 1$.*

Proof. Since f is not identically zero, there exists $n \in \mathbb{N}$ such that $f(n) \neq 0$. We have $f(n) = f(n)f(1)$ as f is multiplicative. Hence $f(1) = 1$. \square

Theorem 1.2. [22, page 250] *If f is a multiplicative function, then the summatory function of f , namely, $F(n) = \sum_{d|n} f(d)$, is also multiplicative.*

In the following, we give some basic examples of multiplicative functions and discuss some of their properties.

Divisor function τ .

Definition 1.3. The divisor function $\tau(n)$ is defined as the number of positive divisors of n , i.e.,

$$\tau(n) = \sum_{d|n} 1.$$

As a simple consequence we have the following

Corollary 1.1. The function τ is multiplicative.

Proof. According to theorem 1.2 and because the function $f(n) = 1, n \geq 1$ is multiplicative, we conclude that the function τ is multiplicative. \square

Theorem 1.3. *If $n = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$, then*

$$\tau(n) = \prod_{i=1}^k (a_i + 1).$$

Divisor sum function σ .

Definition 1.4. The divisor power sum function $\sigma_s(n)$ (with $s \in \mathbb{N}$) is defined as the sum of s power of all positive divisors of n , i.e.,

$$\sigma_s(n) = \sum_{d|n} d^s.$$

we have the following corollary.

Corollary 1.2. The function $\sigma_s, s \geq 1$ is multiplicative.

Proof. According to theorem 1.2 and because the function $f(n) = n^s$, $n, s \geq 1$ is multiplicative, we conclude that the function τ is multiplicative. \square

Theorem 1.4. If $n = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$, then

$$\sigma_s(n) = \prod_{i=1}^k \frac{p_i^{(a_i+1)s} - 1}{p_i^s - 1}.$$

We denote σ_1 by σ . We say $n \in \mathbb{N}$ is a perfect number if $\sigma(n) = 2n$, which means the number is equal to the sum of its proper divisors. For examples, 6 and 28.

Euler's totient function φ .

Definition 1.5. The Euler totient function φ is defined as

$$\varphi(n) = \sum_{1 \leq k \leq n; (k,n)=1} 1.$$

Note that $\varphi(p) = p - 1$ if p is prime.

Theorem 1.5. If $n \geq 1$, then we have

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Proposition 1.1. We have

$$\varphi(n) = \prod_{p^a || n} (p^a - p^{a-1})$$

Corollary 1.3. The function φ is multiplicative .

Example 1.2. For $n = 66200$, we obtain

$$\varphi(66200) = \varphi(2^3 \cdot 5^2 \cdot 331) = 2^2(2 - 1)5(5 - 1)(331 - 1) = 26400.$$

Dedekind psi function ψ .

Definition 1.6. the Dedekind ψ function is the multiplicative function on the positive integers defined by

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right).$$

Proposition 1.2. We have

$$\psi(n) = \prod_{p^a || n} (p^a + p^{a-1}).$$

Corollary 1.4. The function ψ is multiplicative .

1.1.2 Additive functions

An arithmetic function f which is not identically zero is said to be additive if

$$f(mn) = f(m) + f(n) \quad (1.2)$$

whenever $(m, n) = 1$. Moreover, if (1.2) holds for all m, n , then f is called completely additive.

Omega function ω

Definition 1.7. The omega function $\omega(n)$ is defined as the number of distinct prime factors of n , i.e., if $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the prime-power decomposition, then

$$\omega(n) = k.$$

Corollary 1.5. The function ω is additive.

Omega function Ω

Definition 1.8. The Omega function $\Omega(n)$ is defined as the total number of prime factors of n , i.e., if $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the prime-power decomposition, then

$$\Omega(n) = a_1 + a_2 + \cdots + a_k.$$

Corollary 1.6. The function Ω is completely additive.

1.1.3 Some important inequalities of arithmetic functions

In our main results, we will use the following inequalities (for details, we refer the reader to [9, Problem 522] and [26, pages 110, 116, 117, 183]). At first, for all positive integers m and n we have

$$\tau(n) \leq 2\sqrt{n} \quad (1.3)$$

and

$$\tau(mn) \leq \tau(m)\tau(n). \quad (1.4)$$

Moreover, if $(m, n) > 1$, then

$$\tau(mn) < \tau(m)\tau(n). \quad (1.5)$$

If m divides n , then

$$\tau(n) \geq \tau(m). \quad (1.6)$$

Next, for $n \neq 2$ and $n \neq 6$, we have

$$\varphi(n) \geq \sqrt{n}. \quad (1.7)$$

For all positive integers m and n , we also have

$$\varphi(mn) = \varphi(m)\varphi(n)\frac{d}{\varphi(d)}, \quad (1.8)$$

where $d = (m, n)$. From which it follows that

$$\varphi(m)\varphi(n) \leq \varphi(mn) \quad (1.9)$$

In particular, if $d > 1$ then (1.8) and (1.9) give us the inequality

$$\varphi(m)\varphi(n) < \varphi(mn). \quad (1.10)$$

1.2 Diophantine Equations

Definition 1.9. A Diophantine equation is an equation of the form

$$f(x_1, x_2, \dots, x_n) = 0 \quad (1.11)$$

where f is an n -variable function with $n \geq 1$. If f is a polynomial with integral coefficients, then (1.11) is an algebraic Diophantine equation.

1.2.1 Diophantine Equations of Degree 1

Definition 1.10. Diophantine equations of degree 1 are Diophantine equations of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = c, \quad (1.12)$$

where a_1, a_2, \dots, a_n are integers.

We have the following theorem on the existence of solutions for Diophantine equations of degree 1

Theorem 1.6. *The equation (1.12) has a solution in integers x_1, x_2, \dots, x_n if and only if the greatest common divisor of a_1, a_2, \dots, a_n divides c i.e. $(a_1, a_2, \dots, a_n) \mid c$.*

In the case of two unknowns, we have the following theorem:

Theorem 1.7. *If the linear equation*

$$ax + by = c \quad (1.13)$$

admits α and β as solutions, then all integer solutions x and y of equation (1.13) are in the form:

$$x = \alpha + \frac{b}{(a,b)}t, \quad y = \beta - \frac{a}{(a,b)}t, \quad t \in \mathbb{Z} \quad (1.14)$$

Proof. It is easy to see that x and y given by expressions (1.14) satisfy equation (1.13) when we consider that $a\alpha + b\beta = c$. Assume that x, y are arbitrary integral roots of (1.13). In this case, we have,

$$ax + by = c = a\alpha + b\beta.$$

So

$$a(x - \alpha) = b(\beta - y). \quad (1.15)$$

If $d = (a, b)$, it follows that $\frac{b}{d}$ is a divisor of $x - \alpha$. By definition, there exists an integer t such that $x - \alpha = \frac{b}{d}t$.

To obtain the expression for y , simply replace the expression for $x - \alpha$ obtained previously in equation (1.15). We obtain $y - \beta = -\frac{a}{d}t$. So equation (1.13) has an infinite number of integral solutions. \square

Euclid's algorithm [22] can also be used to find the particulate solution to (1.13).

Example 1.3. If $a = 21, b = 13, c = 1$, Euclid's algorithm is

$$21 - 1 \cdot 13 = 8,$$

$$13 - 1 \cdot 8 = 5,$$

$$8 - 1 \cdot 5 = 3$$

$$5 - 1 \cdot 3 = 2$$

$$3 - 1 \cdot 2 = 1.$$

By elimination, we obtain

$$8 \cdot 5 - 13 \cdot 3 = 1.$$

$$21 \cdot 5 - 13 \cdot 8 = 1.$$

and we have $\alpha = 5$ and $\beta = -8$ as a particular solutions to the equation $21x + 13y = 1$. Since $(a, b) = (21, 13) = 1$, then all other solutions are $x = 5 + 13t, y = -8 + 21t, t \in \mathbb{Z}$.

Example 1.4. Let $a = 7, b = 12$ and $c = 5$. Euclid's algorithm finds $\alpha = -1$ and $\beta = 1$ as a particular solution. Since $(a, b) = (7, 12) = 1$, then all other solutions are $x = -1 + 12t, y =$

$1 + 7t, t \in \mathbb{Z}$.

Let's now turn to non-linear Diophantine equations, those in which some of the degree greater than or equal to two.

1.2.2 Diophantine Equations of Degree 2

In this subsection, we'll give some results on the Pythagorean equation and its generalization to four unknowns.

Theorem 1.8. [3] *The integral solutions of the equation*

$$x^2 + y^2 = z^2$$

are given by

$$x = k(m^2 - n^2), \quad y = 2kmn, \quad z = k(m^2 + n^2) \quad \text{with } k, m, n \in \mathbb{Z}. \quad (1.16)$$

More generally, we have the following result:

Theorem 1.9. [3] *All integer solutions x, y, z, t with y and z even of the equation*

$$x^2 + y^2 + z^2 = t^2$$

are given by

$$x = \frac{l^2 + m^2 - n^2}{n}, \quad y = 2l, \quad z = 2m, \quad t = \frac{l^2 + m^2 + n^2}{n},$$

with l and m arbitrary positive integers and n any divisor of $l^2 + m^2$.

We continue this section by examining the Diophantine equation

$$x^2 + axy + y^2 = z^2, \quad (1.17)$$

where a is a given integer. The Pythagorean equation is a special case of this equation ($a = 0$).

Theorem 1.10. [3] *All integral solutions to (1.17) are given by*

$$x = k(an^2 - 2mn), \quad y = k(m^2 - n^2), \quad z = \pm k(amn - m^2 - n^2),$$

where $m, n \in \mathbb{Z}$ are relatively prime and $k \in \mathbb{Q}$ such that $(a^2 - 4)k \in \mathbb{Z}$.

The case $a = 1$ Equation (1.17) becomes

$$x^2 + xy + y^2 = z^2, \quad (1.18)$$

The solutions of (1.18) are given by [3]

$$x = k(2mn + n^2), \quad y = k(m^2 - n^2) \quad z = k(m^2 + mn + n^2),$$

where $m, n \in \mathbb{Z}_+^*$, $m > n$ are relatively prime and $k \in \mathbb{Q}_+^*$ such that $3k \in \mathbb{Z}$

The case $a = -1$ Equation (1.17) becomes

$$x^2 - xy + y^2 = z^2, \quad (1.19)$$

The solutions of (1.19) are given by [3] as follows:

$$x = k(2mn - n^2), \quad y = k(m^2 - n^2) \quad , z = k(m^2 - mn + n^2),$$

where $m, n \in \mathbb{Z}_+^*$, $m > n$ are relatively prime and $k \in \mathbb{Q}_+^*$ such that $3k \in \mathbb{Z}$

1.2.3 Diophantine equations of degree greater than or equal to 3

In this subsection, we consider equations of degree greater than or equal to 3. Let's start with some equations of degree 3 and 4.

In [1] S. Alaca and K. S. Williams have used results from algebraic number theory to determine all integer solutions x and y of $y^2 = x^3 + k$ for certain classes of integers k .

The following theorem gives the conditions for solving the equation $y^2 = x^3 + k$.

Theorem 1.11. [1] *Let k be an integer such that $k < -1$, k is square-free, $k \equiv 2, 3 \pmod{4}$, $h(\mathbb{Q}\sqrt{k}) \not\equiv 0 \pmod{3}$.*

(a) *If there is an integer a such that*

$$k = 1 - 3a^2,$$

then the only integer solutions of $y^2 = x^3 + k$ are

$$x = 4a^2 - 1 \quad \text{and} \quad y = \pm(3a - 8a^3).$$

(b) *If there is an integer a such that*

$$k = -1 - 3a^2,$$

then the only integer solutions of $y^2 = x^3 + k$ are

$$x = 4a^2 + 1 \quad \text{and} \quad y = \pm(3a + 8a^3).$$

(c) If $k \neq \pm 1 - 3a^2$, for any integer a , then $y^2 = x^3 + k$ has no integer solutions in integers x and y .

Example 1.5. The integer $k = -11 = 1 - 3 \cdot 2^2$ satisfies the conditions of theorem 1.11(a) because $h(\mathbb{Q}\sqrt{-11}) = 1$ see [1, table 07 page 325]. Then the only integer solutions to the equation $y^2 = x^3 - k$ are $(15, \pm 58)$.

In [19] Nagell have proved the following three theorems

Theorem 1.12. *The Diophantine equation*

$$x^4 - y^4 = z^2 \tag{1.20}$$

has no integer solutions x, y and z .

Theorem 1.13. *The Diophantine equation:*

$$x^4 + y^4 = 2z^2$$

has no integer solutions x, y and z .

Theorem 1.14. *The Diophantine equation:*

$$x^4 + y^4 = pz^2,$$

where p is a prime $\equiv 3 \pmod{8}$, has no solutions in natural numbers x, y and z .

Theorem 1.15. *The Diophantine equation*

$$2x^4 - y^4 = z^2 \tag{1.21}$$

has infinitely many solutions according to Nagell (Page 232, [19]).

For more examples of this type, see [1, 3, 18, 19]

1.2.4 Exponential Diophantine equations

are equations of the type

$$f(x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n}) = 0, \tag{1.22}$$

where f is a polynomial in n variables with integer coefficients, whose solutions are the integers $(x_1, \dots, x_n, m_1, \dots, m_n)$.

Example 1.6. The Diophantine equation:

$$x^2 + 1 = y^n$$

has no solution for $n > 1$ and $y > 1$ (V. A. Lebesgues, 1850).

Example 1.7. The Diophantine equation:

$$x^2 - 1 = y^n$$

has no solution for $n > 1$ and $y > 3$ (Chao Ko, 1964).

1.3 Classical methods for solving Diophantine equations

We discuss different standard methods to solving Diophantine equations in this section. Each situation is supported by examples.

1.3.1 The Factoring Method

Let's solve an equation of the form $f(x_1, x_2, \dots, x_n) = 0$. We rewrite this equation in its equivalent form

$$f_1(x_1, x_2, \dots, x_n) f_2(x_1, x_2, \dots, x_n) \cdots f_k(x_1, x_2, \dots, x_n) = a, \quad (1.23)$$

with $f_1, f_2, \dots, f_k \in \mathbb{Z}[X_1, X_2, \dots, X_n]$ and $a \in \mathbb{Z}$. By decomposing a into a product of prime factors we obtain the decomposition of a into k prime factors of integers a_1, a_2, \dots, a_k . Each of these factorizations gives a system of equations

$$\begin{cases} f_1(x_1, x_2, \dots, x_n) = a_1 \\ f_2(x_1, x_2, \dots, x_n) = a_2 \\ \dots \\ f_k(x_1, x_2, \dots, x_n) = a_k. \end{cases}$$

Solving all these systems gives the set of solutions to equation (1.23). We illustrate this method with a few examples.

Example 1.8. Find all of the equation's integral solutions.

$$(x^2 + 9)(y^2 + 4) - 2(6 - xy)(2x - 3y) = 3(3 + 8xy).$$

Proof. Equation need to be written in the form

$$(xy - 6)^2 + (2x - 3y)^2 + 2(2x - 3y)(xy - 6) = 9,$$

This corresponds to

$$[xy - 6 + (2x - 3y)]^2 = 9,$$

or

$$(x - 3)(y + 2) = \pm 3.$$

If $(x - 3)(y + 2) = 3$, We get the equation systems

$$\begin{cases} x - 3 = 3 \\ y + 2 = 1 \end{cases} \quad \begin{cases} x - 3 = -3 \\ y + 2 = -1 \end{cases}$$

$$\begin{cases} x - 3 = 1 \\ y + 2 = 3 \end{cases} \quad \begin{cases} x - 3 = -1 \\ y + 2 = -3 \end{cases}$$

yielding the solutions $(6, -1)$, $(0, -1)$, $(4, 1)$ and $(2, 5)$.

If $(x - 3)(y + 2) = -3$, We get the equation systems

$$\begin{cases} x - 3 = 3 \\ y + 2 = -1 \end{cases} \quad \begin{cases} x - 3 = -3 \\ y + 2 = 1 \end{cases}$$

$$\begin{cases} x - 3 = 1 \\ y + 2 = -3 \end{cases} \quad \begin{cases} x - 3 = -1 \\ y + 2 = 3 \end{cases}$$

yielding the solutions $(6, -3)$, $(0, -3)$, $(4, -5)$ and $(2, 1)$.

All eight pairs that we have found satisfy the given equation. \square

Example 1.9. [3, Exercise 05 page 12] Determining the solutions of the following Diophantine equation

$$p - y^4 = 4,$$

where p is prime

Proof. Write the equation in the form

$$x = (y^2 + 2)^2 - (2y)^2,$$

This corresponds to

$$x = [(y - 1)^2 + 1][(y + 1)^2 + 1].$$

If $y \neq \pm 1$, x is the result of the product of two positive numbers > 1 ; hence it is not a prime. The solutions are $(5, 1)$ and $(5, -1)$. \square

1.3.2 Solving using inequalities

This method consists in restricting the intervals in which the variables lie using appropriate inequalities. Generally, this process leads to only finitely many possibilities for all variables or for some of them.

Example 1.10. [3, Exercise 01 page 17] Solve in positive integers the equation

$$3(xy + yz + zx) = 4xyz.$$

Proof. The equation is equivalent to

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{3}.$$

Assuming that $x \leq y \leq z$. Thus, it follows $\frac{3}{x} \geq \frac{4}{3}$ i.e., $x \leq \frac{9}{4}$. Consequently, $x \in (1, 2)$. The solutions $(1, 4, 12)$, $(1, 6, 6)$, and $(2, 2, 3)$, as well as all of their permutations, are obtained by analysing both of the cases. \square

1.3.3 The parametric method

In many situations the integral solutions to a Diophantine equation

$$f(x_1, x_2, \dots, x_n) = 0$$

can be represented in parametric form as follows :

$$x_1 = g_1(k_1, \dots, k_l), x_2 = g_2(k_1, \dots, k_l), \dots, x_n = g_n(k_1, \dots, k_l),$$

where g_1, g_2, \dots, g_n are integer functions of l -variable and $k_1, k_2, \dots, k_l \in \mathbb{Z}$. The set of solutions to some Diophantine equations might have multiple parametric representations.

For most Diophantine equations it is not possible to find all solutions explicitly. In many such cases the parametric method provides a proof of the existence of infinitely many solutions.

Example 1.11. [3, Example 03 page 22] Determine all the positive integer triplets (x, y, z) such that :

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$

Proof. The equation is equivalent to :

$$z = \frac{xy}{x + y}.$$

Let $d = (x, y)$. Then $x = dm, y = dn$, where $(m, n) = 1$. It follows that the $(mn, m + n) = 1$. Hence

$$z = \frac{dmn}{m + n},$$

which implies $(m + n) | d$, i.e., $d = k(m + n)$, $k \in \mathbb{Z}_+$.

The possible solutions to the equation are represented by

$$x = km(m + n), \quad y = kn(m + n), \quad z = kmn,$$

where $k, m, n \in \mathbb{Z}_+$. □

1.3.4 Method of mathematical induction

A strong and elegant approach for proving a proposition dependent on positive integers is mathematical induction.

Let $(P(n))_{n \geq 0}$ be a sequence of propositions. The method of mathematical induction assists us in proving that $P(n)$ is true for all $n \geq n_0$, where n_0 is a given positive integer.

• **Mathematical induction (weak form) :** [3] Assume that :

- $P(n_0)$ is true;
- For all $k \geq n_0$, $P(k)$ is true implies $P(k + 1)$ is true.

Then $P(n)$ is true for all $n \geq n_0$.

• **Mathematical induction (with s step) :** [3] Let s be a fixed positive integer. Assume that :

- $P(n_0), P(n_0 + 1), \dots, P(n_0 + s - 1)$ are true;
- For all $k \geq n_0$, $P(k)$ is true implies $P(k + s)$ is true.

Then $P(n)$ is true for all $n \geq n_0$.

• **Mathematical induction (strong form) :** [3] Assume that :

- $P(n_0)$ is true;
- For all $k \geq n_0$, $P(m)$ is true for all m with $n_0 \leq m \leq k$ implies $P(k + 1)$ is true.

Then $P(n)$ is true for all $n \geq n_0$.

This method is widely used in various fields of mathematics, including number theory.

The following example is intended to show how mathematical induction works in the study of Diophantine equations.

Example 1.12. [3, Exercise 02 page 45] Show that the following problem can be solved in integers for all positive integer n :

$$x^2 + xy + y^2 = 7^n.$$

Proof. If $n = 1$, we have the solution $x_1 = 2, y_1 = 1$. Assume that there exist positive integers x_n, y_n satisfying

$$x_n^2 + x_n y_n + y_n^2 = 7^n.$$

and define

$$x_{n+1} = 2x_n - y_n, \quad y_{n+1} = x_n + 3y_n.$$

Therefore

$$x_{n+1}^2 + x_{n+1} y_{n+1} + y_{n+1}^2 = 7(x_n^2 + x_n y_n + y_n^2) = 7 \cdot 7^n = 7^{n+1}.$$

□

1.3.5 The modular arithmetic method

In many situations, simple modular arithmetic considerations are employed in proving that certain Diophantine equations are not solvable or in reducing the range of their possible solutions.

Example 1.13. [3] Demonstrate that the equation

$$(x + 1)^2 + (x + 2)^2 + \cdots + (x + 2001)^2 = y^2$$

has no solution.

Proof. Let $x = z - 1001$. The previous equation becomes

$$(z - 1000)^2 + \cdots + (z - 1)^2 + z^2 + (z + 1)^2 + \cdots + (z + 1000)^2 = y^2,$$

or

$$2001z^2 + 2(1^2 + 2^2 + \cdots + 1000^2) = y^2.$$

It therefore follows that

$$2001z^2 + 2 \frac{1000 \cdot 1001 \cdot 2001}{6} = y^2,$$

or equivalent,

$$2001z^2 + 1000 \cdot 1001 \cdot 667 = y^2.$$

The term on the left-hand side is congruent to $2 \pmod{3}$, so it's not a perfect square. \square

NEW DIOPHANTINE EQUATIONS INVOLVING ARITHMETIC FUNCTIONS

In this chapter, we present some Diophantine equations involving arithmetic functions. In some new papers (since 2019). One of the most important mathematicians to have worked on this type of equation is Sándor, in addition to other works, we will discuss the most important aspects of his recent works.

2.1 Notes on the equation $\tau(n) = \tau(\varphi(n))$ and related inequalities

In this section, we show some notes on the equation $\tau(n) = \tau(\varphi(n))$. Several results on the related inequalities are also shown; All of these results are in the paper of D. Bellaouar, A. Boudaoud and R. Jakimczuk [4](2023).

Define

$$S := \{n \in \mathbb{N} : \tau(n) = \tau(\varphi(n))\},$$

$$S^+ := \{n \in \mathbb{N} : \tau(n) > \tau(\varphi(n))\},$$

and

$$S^- := \{n \in \mathbb{N} : \tau(n) < \tau(\varphi(n))\},$$

where $\tau(n)$ is the number of positive integers function, and $\varphi(n)$ is Euler's function.

Let p be a prime number. Then it is easy to see that $p \in S$ if and only if $p = 3$ and $p \in S^+$ if and only if $p = 2$. Hence S and S^+ are nonempty and S^- has infinite number of elements. Recall that an integer of the form $F_n = 2^{2^n} + 1$ is called a *Fermat number*. The numbers $F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257$ and $F_4 = 65537$ are the only known Fermat primes. If F_t is a Fermat prime, then $\tau(F_t) = 2$, whereas $\tau(\varphi(F_t)) = 2^t + 1$. Thus all Fermat primes excluding $F_0 = 3$ belong to S^- . A prime p is said to be a *Sophie Germain prime* [11] if $2p + 1$ is also a prime, in which case, this last prime is called a *safe prime*. It has been conjectured that there are infinitely many Sophie Germain primes, but this remains unproved [16]. So, if $2p + 1$ is a safe prime with $p \geq 3$, then $\tau(\varphi(2p + 1)) - \tau(2p + 1) = 2$. Hence all safe

primes belong to S^- . Moreover, if $n = 2^a$ for some $a \geq 1$, then $\tau(2^a) - \tau(\varphi(2^a)) = 1$. Consequently, S^+ is also an infinite set. A natural question: Does S also contain infinite number of elements?

Define for any $k \geq 1$ the subsets $S_k \subseteq S$, $S_k^+ \subseteq S^+$ and $S_k^- \subseteq S^-$ as follows:

$$S_k = S \cap \mathbb{W}_k, S_k^+ = S^+ \cap \mathbb{W}_k \text{ and } S_k^- = S^- \cap \mathbb{W}_k,$$

where $\mathbb{W}_k = \{n \in \mathbb{N} : \omega(n) = k\}$. For more results on the set \mathbb{W}_k , we refer to [5].

2.1.1 On the solutions of $\tau(n) = \tau(\varphi(n))$

We see that the first elements of S are:

$$1, 3, 14, 15, 22, 28, 44, 46, 50, 56, 68, 70, 78, 88, \dots$$

Also, the first few odd elements of S are:

$$3, 15, 255, 65535, 77805, 161595, 331695, 575025, 664335, \dots$$

Proposition 2.1. *Let $k \geq 2$ and let $q_1 < q_2 < \dots < q_k$ be primes. Consider the number $n = q_1^{a_1} q_2^{a_2} \dots q_k^{a_k}$, where a_1, a_2, \dots, a_k are positive integers with $a_k \geq 2$. If $n \in S_k$, then*

$$a_k \mid \prod_{i=1}^{k-1} (a_i + 1). \quad (2.1)$$

Remark 2.1. Note that (2.1) is not a sufficient condition for a number n to be in S_k . In fact, $n_1 = 2 \cdot 3^2$ and $n_2 = 3 \cdot 5^2$ are the smallest natural numbers that satisfy (2.1) for $k = 2$, where $\tau(n_1) > \tau(\varphi(n_1))$ and $\tau(n_2) < \tau(\varphi(n_2))$.

Corollary 2.1. Let $k, r, n \geq 2$. Then $n^r \notin S_k$. That is, the set S_k does not contain perfect powers. In particular, no square satisfies $\tau(n) = \tau(\varphi(n))$.

Remark 2.2. Let $n = q_1^{a_1} \dots q_k^{a_k}$, where q_1, \dots, q_k are distinct primes and a_1, \dots, a_k are positive integers ($k \geq 2$). For each even $r \geq 2$ we can also prove that $n^r \notin S_k$. In fact, if $n^r \in S_k$, then $\prod_{i=1}^k (ra_i + 1) = \tau(q_1^{ra_1-1} \dots q_{k-1}^{ra_{k-1}-1} (q_1 - 1) \dots (q_k - 1)) ra_k$, and so $ra_k \mid \prod_{i=1}^{k-1} (ra_i + 1)$. This is also a contradiction since ra_k is even and $ra_i + 1$ is odd for $i = 1, \dots, k - 1$.

Next, we see all the elements of S_1 and S_2 .

Theorem 2.1. *The only elements of S_1 are: 1 and 3 and the only elements of S_2 are: $15, 2^{2^i-2} F_i$, where F_i is a Fermat prime with $i \geq 2, 2^a q$, where $a \geq 1$ and $q \geq 7$ is a safe prime, and $2^{(2^i-1)j-1} F_i^j$, where F_i is a Fermat prime with $i \geq 1$ and $j \geq 2$.*

let n be an odd square-free integer such $\omega(n) \leq 5$. In the following corollary, we see that if $n \in S$, then all the prime factors of n are Fermat primes.

Corollary 2.2. In fact, we have

- (i) If $n = q$, where $q \geq 3$ is prime, then $\tau(n) / \tau(\varphi(n)) \neq 1$ except for $q = F_0$.
- (ii) If $n = q_1 q_2$, where q_1, q_2 are prime numbers with $q_1 < q_2$, then $\tau(n) / \tau(\varphi(n)) \neq 1$ except for $(q_1, q_2) = (F_0, F_1)$.
- (iii) If $n = q_1 q_2 q_3$, where q_1, q_2, q_3 are prime numbers with $q_1 < q_2 < q_3$, then $\tau(n) / \tau(\varphi(n)) \neq 1$ except for $(q_1, q_2, q_3) = (F_0, F_1, F_2)$.
- (iv) If $n = q_1 q_2 q_3 q_4$, where q_1, \dots, q_4 are prime numbers with $q_1 < \dots < q_4$, then $\tau(n) / \tau(\varphi(n)) \neq 1$ except for $(q_1, q_2, q_3, q_4) = (F_0, F_1, F_2, F_3)$.
- (v) If $n = q_1 q_2 q_3 q_4 q_5$, where q_1, \dots, q_5 are primes with $q_1 < \dots < q_5$, then $\tau(n) / \tau(\varphi(n)) \neq 1$ except for $(q_1, q_2, q_3, q_4, q_5) = (F_0, F_1, F_2, F_3, F_4)$.

We remark that the result in the above corollary is not true whenever $\omega(n) \geq 6$. For example, the number $n = 2250885 = 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 97 \in S_6$, which is odd and square-free; but not all the prime factors of n are Fermat primes.

Now, we show the solutions n of the equation $\tau(n) = \tau(\varphi(n))$, where n has three or more distinct prime factors.

First, assume that $n = p^a q^b r^c \in S_3$, where p, q, r are distinct primes with $2 \leq p < q < r$ and $a, b, c \geq 1$.

Proposition 2.2. *The only square-free solutions of the form pqr are:*

- $n = F_0 F_1 F_2$.
- $n = 2 \cdot 3 \cdot (4p + 1)$, where p is a prime such that $4p + 1$ is also prime.
- $n = 2 \cdot 5 \cdot (2p + 1)$, where p is a prime such that $2p + 1$ is also prime.

Proposition 2.3. *The only possible solutions of the form $2^a \cdot q \cdot r$, where $a \geq 2$ and $3 \leq q < r$ are:*

- $n = 2^{3(x+y)-4} (2^x p + 1) (2^y p + 1)$, where $2^x p + 1$ and $2^y p + 1$ are odd primes with $x < y$.
- $n = 2^{3(x+y)-4} (2^x + 1) (2^y p^2 + 1)$, where $2^x + 1$ and $2^y p^2 + 1$ are odd primes with $2^x < 2^y p^2$.
- $n = 2^{3(x+y)-4} (2^x p^2 + 1) (2^y + 1)$, where $2^y + 1$ and $2^x p^2 + 1$ are odd primes with $2^x p^2 < 2^y$.
- $n = 2^{x+y-2} (2^x + 1) (2^y p + 1)$, where $2^x + 1$ and $2^y p + 1$ are odd primes with $2^x < 2^y p$.

- $n = 2^{x+y-2} (2^x p + 1) (2^y + 1)$, where $2^y + 1$ and $2^x p + 1$ are odd primes with $2^x p < 2^y$.
- $n = 2^{(x+y-4)/3} (2^x + 1) (2^y + 1)$, where $2^x + 1$ and $2^y + 1$ are Fermat primes with $x < y$.

Proposition 2.4. *The only possible solutions of the form $2 \cdot q^b \cdot r$, where $b \geq 2$ and $3 \leq q < r$ are:*

- $n = 2 \cdot 3^2 \cdot 17$.
- $n = 2 \cdot 3^b \cdot 13$.
- $n = 2 \cdot 5^b \cdot 11$.
- $n = 2 \cdot 3^2 \cdot (2p + 1)$, where $p \geq 5$ is a prime such $2p + 1$ is also prime.
- $n = 2 (2^x + 1)^b (2^y (2^x + 1)^t + 1)$, where $b(4 - x - y - 1) = (x + y + 1)t - 4$.
- $n = 2 \cdot 3^{3t-4} \cdot (2 \cdot 3^t + 1)$, where $2 \cdot 3^t + 1$ is prime.

Proposition 2.5. *The only possible solutions of the form $2 \cdot q \cdot r^c$, where $c \geq 2$ and $3 \leq q < r$ are:*

- $n = 2 \cdot 3 \cdot 17^2$.
- $n = 2 \cdot 3 \cdot (2p + 1)^2$, where p is a prime such that $2p + 1$ is also prime.

Proposition 2.6. *The only possible solutions of the form $n = 2^a q^b r$, where $a, b \geq 2$ and $3 \leq q < r$ are:*

- $n = 2^{i(x+y-1)-1} (2^x p + 1)^i (2^y + 1)$, where $i \geq 2$, $2^x p + 1$ and $2^y + 1$ are odd primes with $2^x p < 2^y$.
- $n = 2^{i(x+y-1)-1} (2^x + 1)^i (2^y p + 1)$, where $i \geq 2$, $2^x p + 1$ and $2^y + 1$ are odd primes with $2^x < 2^y p$.
- $n = 2^a (2^x + 1)^b (2^y + 1)$, where $2^x + 1$ and $2^y + 1$ are Fermat primes with $x < y$.
- $n = 2^a (2^x + 1)^b (2^y q^t + 1)$, where $2^x + 1$ and $2^y q^t + 1$ are odd primes with $2^x < 2^y q^t$.

Proposition 2.7. *The only possible solutions of the form $n = 2q^b r^c$, where $b, c \geq 2$ and $3 \leq q < r$ are:*

- $n = 2F_0^3 F_1^2$.
- $n = 2F_0^2 F_1^3$.
- $n = 2 \cdot 3^b \cdot 7^2$ ($b \geq 2$).

Proposition 2.8. *The only possible solutions of the form $n = 2^a \cdot q \cdot r^c$, where $a, c \geq 2$ and $3 \leq q < r$ are:*

- $n = 2^a \cdot F_i \cdot F_j^c$
- $n = 2^a (2^x + 1) (2^y p + 1)^c$, where $2^x + 1$ and $2^y p + 1$ are odd primes with $2^x < 2^y p$.
- $n = 2^a (2^x p + 1) (2^y + 1)^c$, where $2^x p + 1$ and $2^y + 1$ are odd primes with $2^x p < 2^y$.

Proposition 2.9. *The only possible solutions of the form $n = 2^a \cdot q^b \cdot r^c$, where $a, b, c \geq 2$ and $3 \leq q < r$ are:*

- $n = 2^{8(x+y)-9} (2^x + 1)^2 (2^y p + 1)^2$, where $2^x + 1$ and $2^y p + 1$ are odd primes with $2^x < 2^y p$.
- $n = 2^{8(x+y)-9} (2^x p + 1)^2 (2^y + 1)^2$, where $2^x p + 1$ and $2^y + 1$ are odd primes with $2^x p < 2^y$.
- $n = 2^a (2^x + 1)^b (2^y + 1)^c$, where $2^x + 1$ and $2^y + 1$ are odd primes with $x < y$.
- $n = 2^a (2^x + 1)^b (2^y (2^x + 1)^t + 1)^c$, where $2^x + 1$ and $2^y (2^x + 1)^t + 1$ are odd primes.

Theorem 2.2. $n = F_0 F_1 F_2$ is the only odd solution of the equation $\tau(n) = \tau(\varphi(n))$, where n has three distinct prime factors.

Next, we see an example to show that there are infinitely many positive integers n such that $n \in S_k$, where $2 \leq k \leq 7$.

Example 2.1. Each of the following cases provides an infinite family of solutions.

1. For every $a \geq 2$, $n = 2^{a-1} \cdot 5^a \in S_2$.
2. For every $b \geq 1$, $n = 2 \cdot 3^b \cdot 7^2 \in S_3$.
3. For every $b \geq 1$, $n = 2 \cdot 3^b \cdot 7 \cdot 11 \in S_4$.
4. For every $c \geq 1$, $n = 2 \cdot 3 \cdot 5^c \cdot 7 \cdot 41 \in S_5$ and for every $b \geq 1$, $n = 2 \cdot 3^b \cdot 7 \cdot 17 \cdot (2p + 1) \in S_5$, where $p \geq 5$ is a prime such that $2p + 1$ is also prime.
5. For every $b \geq 1$, $n = 2 \cdot 3^b \cdot 7 \cdot 17 \cdot 257 \cdot (2p + 1) \in S_6$, where $p \geq 5$ is a prime such that $2p + 1$ is also prime.
6. For every $b \geq 1$, $n = 2 \cdot 3^b \cdot 7 \cdot 17 \cdot 257 \cdot 65537 \cdot (2p + 1) \in S_7$, where $p \geq 5$ is a prime such that $2p + 1$ is also prime.

Corollary 2.3. Let $k \geq 2$. Assume that there exist infinitely many Sophie Germain primes (or equivalently infinitely many safe primes), then there are infinitely many even positive integers n such that $n \in S_{k+2}$.

Corollary 2.4. Assume that $2 \leq k \leq 7$. The Sophie Germain prime conjecture implies that there exist infinitely many square-free numbers n such that $n \in S_k$.

proof. This follows immediately from the fact that S contains the following numbers:

1. $n = 2 \cdot (2p + 1)$, where $p \geq 3$ is a prime such that $2p + 1$ is also prime.
2. $n = 2 \cdot 5 \cdot (2p + 1)$, where $p \geq 3$ is a prime such that $2p + 1$ is also prime.
3. $n = 2 \cdot 5 \cdot 17 \cdot (2p + 1)$, where $p \geq 3$ is a prime such that $2p + 1$ is also prime.
4. $n = 2 \cdot 3 \cdot 17 \cdot (2p + 1)(2q + 1)$, where p, q are odd distinct primes such that $2p + 1$ and $2q + 1$ are also primes.
5. $n = 2 \cdot 5 \cdot 17 \cdot 257 \cdot 65537 \cdot (2p + 1)$, where $p \geq 3$ is a prime such that $2p + 1$ is also prime.
6. $n = 2 \cdot 3 \cdot 17 \cdot 257 \cdot 65537 \cdot (2p + 1)(2q + 1)$, where p, q are odd distinct primes such that $2p + 1$ and $2q + 1$ are also primes.

□

Remark 2.3. Assume that there are infinitely many safe primes and there are infinitely many primes of the form $4p + 1$, where p is also prime. Then there are infinitely many square-free numbers n such that $n \in S_9$. Indeed, if p, q, r, s and ω are distinct odd primes such that $2p + 1, 2q + 1, 4r + 1, 4s + 1$ and $4\omega + 1$ are also primes, then clearly

$$n = 2 \cdot 3 \cdot 5 \cdot 17 \cdot (2p + 1)(2q + 1)(4r + 1)(4s + 1)(4\omega + 1) \in S_9.$$

Finally, we see some examples to show that S_k contains large odd numbers for some $k \geq 4$.

Example 2.2. Let l be a positive integer and let $2^l + 1$ be a Fermat prime. Assume that $2^{r_i}(2^t + 1)^{s_i} + 1$ are different primes for some positive integers r_i, s_i ($i = 1, \dots, l$), where

$$\sum_{i=1}^l r_i = 2^l - t - 2. \quad (2.2)$$

If we put $n = (2^t + 1)^{(2^l - 1)(\sum_{i=1}^l s_i) - 2^l} \prod_{i=1}^l (2^{r_i}(2^t + 1)^{s_i} + 1)$, then $n \in S_{l+1}$. In fact, we see that

$$\tau(n) = \left((2^l - 1) \left(\sum_{i=1}^l s_i \right) - 2^l + 1 \right) 2^l = (2^l - 1) \left(-1 + \sum_{i=1}^l s_i \right) 2^l,$$

and $\varphi(n) = (2^t + 1)^{(2^l - 1)(\sum_{i=1}^l s_i) - 2^l - 1 + \sum_{i=1}^l s_i} 2^{t + \sum_{i=1}^l r_i}$. It follows from (2.2) that

$$\tau(\varphi(n)) = \left((2^l - 1) \left(\sum_{i=1}^l s_i \right) - 2^l + \sum_{i=1}^l s_i \right) (t + 1 + 2^l - t - 2) = \tau(n).$$

As an application, if $t = 1, l = 3, r_1 = r_2 = 2$ and $r_3 = 1$, then from above

$$n = 3^{7(s_1+s_2+s_3)-8} (2^2 \cdot 3^{s_1} + 1) (2^2 \cdot 3^{s_2} + 1) (2 \cdot 3^{s_3} + 1),$$

where $2^2 \cdot 3^{s_1} + 1, 2^2 \cdot 3^{s_2} + 1$ and $2 \cdot 3^{s_3} + 1$ are distinct primes. Thus, if $s_1 = s_3 = 1$ and $s_2 = 2$, then $n = 3^{20} \cdot 7 \cdot 13 \cdot 37$, which is odd and has four distinct prime factors.

Example 2.3. Let p be an odd prime. We put $p - 1 = 2^t \prod_{i=1}^h q_i^{e_i}$, where q_1, q_2, \dots, q_h are odd primes and t, e_1, e_2, \dots, e_h are positive integers ($h \geq 1$). Let l be a positive integer and assume that r_1, \dots, r_l are positive, not necessarily different integers such that $\prod_{i=1}^l (e_i + 1)$ divides $2^l - 1$ and

$$\sum_{i=1}^l r_i = \frac{2^l - 1}{\prod_{i=1}^l (e_i + 1)} - t - 1. \quad (2.3)$$

Assume further that s_1, \dots, s_l are arbitrary positive, not necessarily different integers for which $2^{r_1} p^{s_1} + 1, \dots, 2^{r_l} p^{s_l} + 1$ are different primes. If we put

$$n = p^{(2^l - 1)(\sum_{i=1}^l s_i) - 2^l} \prod_{i=1}^l (2^{r_i} p^{s_i} + 1), \quad (2.4)$$

then $n \in S_{l+1}$. As an application, we let $p = 37$, i.e., $p - 1 = 36 = 2^2 \cdot 3^2$ and choose $l = 6$. Consequently, equation (2.3) becomes $\sum_{i=1}^6 r_i = \frac{2^6 - 1}{3} - 2 - 1 = 18$. For example, if we choose $r_1 = r_2 = r_3 = 2$ and $r_4 = r_5 = r_6 = 4$, then by (2.4)

$$n = 37^{63(\sum_{i=1}^6 s_i) - 64} \prod_{i=1}^6 (2^{r_i} \cdot 37^{s_i} + 1),$$

where $2^2 \cdot 37^{s_1} + 1, 2^2 \cdot 37^{s_2} + 1, 2^2 \cdot 37^{s_3} + 1, (2^4 \cdot 37^{s_4} + 1), 2^4 \cdot 37^{s_5} + 1$ and $2^4 \cdot 37^{s_6} + 1$ are distinct primes. For $s_1 = s_4 = 1, s_2 = 2, s_3 = 3, s_5 = 4$ and $s_6 = 5$ we obtain the primes 149, 593, 5477, 202613, 29986577 and 1109503313, in which case $n = 37^{944} \cdot 149 \cdot 593 \cdot 5477 \cdot 202613 \cdot 29986577 \cdot 1109503313$, which is odd and has seven distinct prime factors.

2.1.2 On the inequality $\tau(n) > \tau(\varphi(n))$

The first terms of S^+ are:

$$2, 4, 6, 8, 10, 12, 16, 18, 20, 24, 30, 32, 36, 40, 42, 48, \dots$$

We can easily verify the following particular cases:

- (a) Let F_i be a Fermat prime with $i \geq 1$. Then $2^a F_i \in S_2^+$ if and only if $a \geq 2^i - 1$.

- (b) $n = 3^{6+2a} \cdot 5 \cdot 7 \cdot 19 \in S_4^+$, for $a \geq 1$.
- (c) $n = 3^a \cdot 5 \cdot 7 \cdot 19 \cdot 31 \in S_5^+$, for $a \geq 21$.
- (d) $n = 3^a \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 97 \in S_6^+$, for $a \geq 12$.
- (e) $n = 5^a \cdot 11 \cdot 31 \cdot 41 \cdot 61 \cdot 101 \cdot 251 \in S_7^+$, for $a \geq 12$.
- (f) $n = 5^a \cdot 11 \cdot 41 \cdot 61 \cdot 101 \cdot 251 \in S_8^+$, for $a \geq 21$.

Remark 2.4. The numbers 69615, 208845, 233415, 305235, 440895, 454545, 484785, 626535, 700245, 915705, 995085 are the only odd positive integers $\leq 10^6$ that belong to S^+ . It seems that the prime factorization of an odd integer $n \in S^+$ is a product of primorials. That is, $915705 = 3^4 \cdot 5 \cdot 7 \cdot 17 \cdot 19$, $995085 = 3^7 \cdot 5 \cdot 7 \cdot 13, \dots$, etc.

The following result is a generalization of the above cases: (a), (b), (c) and (e).

Proposition 2.10. Let q_1, q_2, \dots, q_k be distinct primes such that

$$\tau \left(\frac{(q_1 - 1)(q_2 - 1) \cdots (q_k - 1)}{q_1^t} \right) \leq 2^{k-1}, \quad (2.5)$$

where $k \geq 3$ and t is the largest nonnegative integer for which q_1^t divides $(q_1 - 1)(q_2 - 1) \cdots (q_k - 1)$. Then there exists a real number R such that $n = q_1^a q_2 \cdots q_k \in S_k^+$ for every $a \geq R$.

Remark 2.5. In view of (2.5), one can prove that $k = 1$ if and only $t = 0$, and so $n = 2^a \in S_1^+$ for every $a \geq 1$. Moreover, if $k = 2$, then by (2.5), $\tau((q_1 - 1)(q_2 - 1)/q_1^t) \leq 2$. In the case when $\tau((q_1 - 1)(q_2 - 1)/q_1^t) = 1$, we must have $q_1 = 2$ and q_2 is a Fermat prime. This applies to the particular case (a) as above. However, if $\tau((q_1 - 1)(q_2 - 1)/q_1^t) = 2$, then $q_1 = 2$ and q_2 is a prime of the form $2^x p + 1$, where p is also prime. In this case, $n = 2^a q_2 \notin S_2^+$ for every $a \geq 1$.

Proposition 2.11. For every $k \geq 1$, there exists a positive integer n such that $n \in S_k^+$.

In the following result we see that if $n \in S^+$ is odd, then n must have at least four prime factors, i.e., $\omega(n) \geq 4$. Note that $n = 484785 = 3^6 \cdot 5 \cdot 7 \cdot 19$ is the smallest odd element of S^+ .

Proposition 2.12. Let n be an odd positive integer. If $n \in S^+$, then $\omega(n) \geq 4$.

Now, let $k \geq 1$ and let $n \in S_k^+$. Does $\tau(\varphi(n))$ properly divide $\tau(n)$? Let us define the subset ${}^d S_k^+ \subseteq S_k^+$ as follows:

$${}^d S_k^+ := \{n \in S_k^+ : \tau(\varphi(n)) \mid \tau(n), \text{ where } n \neq 2\}.$$

In the following result we determine the elements of ${}^d S_1^+$ and ${}^d S_2^+$. In fact, we have

Theorem 2.3. ${}^d S_1^+ = \{\emptyset\}$ and ${}^d S_2^+ = \{2^a \cdot 3, a \geq 1\}$.

2.1.3 On the inequality $\tau(n) < \tau(\varphi(n))$

Note that the first terms of S^- are:

$$5, 7, 9, 11, 13, 17, 19, 21, 23, 25, 26, 27, 29, 31, 33, 34, 35, 37, 38, 39, 41, 43, 45, 47, 49, \dots$$

We can easily verify the following statements:

1. The set $\{123553 + a, \text{ for } a = 1, \dots, 52\}$ consists of a sequence of consecutive integers whose members are all in S^- .
2. Let p be an odd prime. For every $a \geq 2$, $p^a \in S_1^-$.
3. If $n = 2^a p^b$ with $a \geq 1$, $b \geq 2$ and p is a safe prime with $p \neq 5$, then $n \in S_2^-$.
4. Let p be a safe prime, and let q be an odd prime with $p \neq q$. For every $a \geq 1$, $pq^a \in S_2^-$.
5. Let q_1, \dots, q_k be distinct safe primes for some $k \geq 2$. For every $a \geq 1$, $2^a q_1 q_2 \dots q_k \in S_k^-$.

Lemma 2.1. *Let $k \geq 1$. There exist infinitely many k -tuples (q_1, \dots, q_k) of distinct odd primes such that*

$$\omega(q_1 \dots q_k (q_1 - 1) \dots (q_k - 1)) \geq 2k + 1. \quad (2.6)$$

Theorem 2.4. *Let $k \geq 1$. There are infinitely many odd positive integers n such that $n \in S_k^-$.*

Theorem 2.5. *Let $k \geq 1$ and let (q_1, \dots, q_k) be any k -tuple of distinct odd primes. There are infinitely many k -tuples (a_1, \dots, a_k) of positive integers such that for $n = q_1^{a_1} \dots q_k^{a_k}$ we have $n \in S_k^-$.*

Lemma 2.2. *Let $k \geq 1$. There exist infinitely many k -tuples (q_1, \dots, q_k) of distinct primes with $q_1 = 2$ such that*

$$\omega(q_1 \dots q_k (q_2 - 1) \dots (q_k - 1)) \geq 2k - 1. \quad (2.7)$$

Proposition 2.13. *Let $k \geq 3$. There are infinitely many even positive integers n such that $n \in S_k^-$.*

The following result is similar to that of Theorem 2.5.

Theorem 2.6. *Let $k \geq 3$ and let (q_1, q_2, \dots, q_k) be any k -tuple of distinct primes with $q_1 = 2$. There are infinitely many k -tuples (a_1, \dots, a_k) of positive integers such that for $n = q_1^{a_1} \dots q_k^{a_k}$ we have $n \in S_k^-$.*

In the next theorem, we see a sequence $(n_k)_{k \geq 1} \subset \mathbb{W}_k$ such that

$$\lim_{k \rightarrow \infty} \frac{\tau(n_k)}{\tau(\varphi(n_k))} = 0.$$

Consequently, there exists a positive integer k_0 such that if $k \geq k_0$, then $n_k \in S_k^-$.

Theorem 2.7. *Let $2 \leq q_1 < q_2 < q_3 < \dots$ be any sequence of primes. Let $(a_i)_{i \geq 1}$ be an infinite sequence of positive integers such that $\sum_{i=1}^{\infty} 1/a_i$ is convergent. Let $(n_k)_{k \geq 1}$ be the sequence of positive integers defined by $n_k = \prod_{i=1}^k q_i^{a_i}$ ($k \geq 1$). Then*

$$\lim_{k \rightarrow \infty} \frac{\tau(n_k)}{\tau(\varphi(n_k))} = 0. \tag{2.8}$$

Theorem 2.8. *There are infinitely many positive integers n such that $(n, n^2) \in S_k^- \times S_k^-$ for some $k \geq 3$.*

2.2 On certain equations and inequalities involving the arithmetical functions $\varphi(n)$ and $\tau(n)$

In this section, we show some important results on Diophantine equations and inequalities involving the arithmetic functions $\varphi(n)$ and $\tau(n)$, namely sándor’s papers [31], [32] and [35].

2.2.1 On the equation $\varphi(n) + \tau(n) = n$

and related inequalities

Theorem 2.9. *$n = 8$ and $n = 9$ are the only solutions to the equation*

$$\varphi(n) + \tau(n) = n. \tag{2.9}$$

In [29], we have this inequality

$$\varphi(n) + \tau(n) \leq n, \tag{2.10}$$

for any $n \neq 4$ different from prime.

Remark 2.6. From Theorem 2.9 and relation (2.10) it follows that

$$\varphi(n) + \tau(n) < n, \tag{2.11}$$

for any $n \neq 4, 8, 9$ and n being composite.

Remark 2.7. Thus, for any $n \geq 1262$ composite, we get the following refinement of (2.10):

$$\varphi(n) \leq n - \sqrt{n} < n - \tau(n). \tag{2.12}$$

Proposition 2.14. *The equation*

$$\varphi(n) + \omega(n) = n \quad (2.13)$$

has the only solutions as $n = \text{prime}$.

Other similar equations and inequalities are studied in [30] and [25].

2.2.2 The equations $\varphi(n) + \tau(n) = \frac{n}{2}$ and $\varphi(n) + \tau^2(n) = 2n$.

The aim of the paper is to study certain new equations, the first one being

$$\varphi(n) + \tau(n) = \frac{n}{2} \quad (2.14)$$

and the related inequalities.

Theorem 2.10. *$n = 72$ is the only solution to the equation (2.14).*

Theorem 2.11. *The inequality*

$$\varphi(n) + \tau(n) > \frac{n}{2} \quad (2.15)$$

is hold if we have one of these cases:

- $n = 18, 30, 36, 50$.
- $n = 2^k$ ($k \geq 1$).
- $n = 2^m \cdot p$ ($m = 1, 2, 3, 4$), p is odd prime.

If n is even and does not belong to the president cases, then we have

$$\varphi(n) + \tau(n) < \frac{n}{2}. \quad (2.16)$$

Theorem 2.12. *$n = 3, 10, 40, 84$ are the only solutions to the equation*

$$\varphi(n) + \tau^2(n) = 2n. \quad (2.17)$$

2.2.3 The equations $\varphi(n) + 2\tau(n) = f(n)$ and $2\varphi(n) + \tau(n) = f(n)$.

Theorem 2.13. *$n = 3$, and $n = 4$ are the only solutions to the equation*

$$\varphi(n) + 2\tau(n) = 2n. \quad (2.18)$$

The equation

$$2\varphi(n) + \tau(n) = 2n. \quad (2.19)$$

has the only solutions as $n = \text{primes}$.

Theorem 2.14. $n = 18$ and $n = 8 \cdot p$ ($p \geq 3$ is prime) are the solutions to the equation

$$2\varphi(n) + \tau(n) = n. \quad (2.20)$$

Theorem 2.15. $n = 14, 18, 20, 24$ are the solutions to the equation

$$\varphi(n) + 2\tau(n) = n. \quad (2.21)$$

Theorem 2.16. The equation

$$(\varphi(n))^{\varphi(n)} + (\tau(n))^{\tau(n)} = n^n \quad (2.22)$$

has no solutions.

$n = 2$ is the only solution to the equation

$$(\varphi(n))^{\varphi(n)} \cdot (\tau(n))^{\tau(n)} = n^n \quad (2.23)$$

Theorem 2.17. $n = 68, 128, 384, 864$ are the solutions to the equation

$$\varphi(n) + (\tau(n))^2 = n. \quad (2.24)$$

Theorem 2.18. The equation

$$\varphi(n) + \tau(n) = \frac{n}{4} \quad (2.25)$$

has all solutions of the form $n = 4m$, where m is an even number with $\omega(m) \geq 4$.

Theorem 2.19. Let the equation

$$\varphi(4n + \tau(4n)) = n. \quad (2.26)$$

we have $n = 2^2 \cdot 3^2 \cdot 5 \cdot 11$, $n = 2^2 \cdot 3 \cdot 5^2 \cdot 13$ and $n = 2^3 \cdot 3^2 \cdot 5 \cdot 13$ are all solutions to (2.26) of the type $n = 2^a \cdot 3^b \cdot 5^c \cdot p$, where $p \geq 7$ is a prime.

A solution with $\omega(n) = 5$ is $n = 2^4 \cdot 3 \cdot 5 \cdot 17 \cdot 251$.

2.3 On certain rational perfect numbers

In this section, we show some important results on Diophantine equations and inequalities involving the arithmetic functions $\sigma(n)$, $\psi(n)$ and $\varphi(n)$, namely sándor's papers [33] and [34].

2.3.1 The equation $\sigma(n) = \frac{k+1}{k}n + a$.

We see equations of type

$$\sigma(n) = \frac{k+1}{k}n + a, \quad (2.27)$$

where $a \in \{0, 1, 2, 3\}$, k and n are positive integers, while $\sigma(n)$ denotes the sum of divisors of n .

The equation (2.27), where $k = 1$ and $a = 1$, has a long history in mathematics, as these are the famous perfect numbers n for which $\sigma(n) = 2n$ i.e., $k = 1$ and $a = 0$. (see, [23,24]).

A similar equation has been considered in [27,28].

Theorem 2.20. *If the equation (2.27) with $a = 0$ is solvable, then we must have $n = k = p$, where p is a prime number.*

Theorem 2.21. *If the equation (2.27) with $a = 1$ is solvable, then k must be a prime ($k = p$) and $n = p^2$.*

Remark 2.8. Particularly, when $k = 2$, we get that only solution of (2.27) with $a = 1$ is $n = 4$.

Theorem 2.22. *Equation (2.27) with $a = 2$ is solvable only if $n = k = 4$.*

Theorem 2.23. *All solutions of Equation (2.27) with $a = 3$ are $n = 9$ and $n = 2q$, where $q \geq 3$ is an arbitrary prime.*

Theorem 2.24. *The equation*

$$\sigma(n) = \frac{k+1}{k}n + r, \quad (2.28)$$

where r is a given positive integer can have at most a finite number of solutions n , and/or an infinite number solutions of type $n = q \cdot N$, where q is an arbitrary prime, and N can take at most a finite number of values.

2.3.2 The equations $\psi(n) = \frac{k+1}{k}n + a$ and The equation $\varphi(n) = \frac{k+1}{k}n - a$.

In this subsection, we see equations of type $\psi(n) = \frac{k+1}{k}n + a$, $a \in \{0, 1, 2, 3\}$, and $\varphi(n) = \frac{k+1}{k}n - a$, $a \in \{0, 1, 2, 3\}$, where $\psi(n)$ and $\varphi(n)$ denote the Dedekind, respectively Euler's, arithmetical functions.

Theorem 2.25. *The equation*

$$\psi(n) = \frac{k+1}{k}n \quad (2.29)$$

is solvable only if $k = p = \text{prime}$ and $n = p^{s+1}$ ($s \geq 0$ integer).

Theorem 2.26. *The equation*

$$\psi(n) = \frac{k+1}{k}n + 1 \quad (2.30)$$

is solvable for $k > 1$ only if $n = k = 4$.

Theorem 2.27. *The equation*

$$\psi(n) = \frac{k+1}{k}n + 2 \quad (2.31)$$

is solvable for $k > 1$ only if $k = 4$ and $n = 8$ or $n = k = 9$.

Theorem 2.28. *The equation*

$$\psi(n) = \frac{k+1}{k}n + 3 \quad (2.32)$$

is solvable for $k > 1$ only if $n = k = 8$ and $n = 18, k = 2$.

Theorem 2.29. *The equation*

$$\varphi(n) = \frac{k+1}{k}n \quad (2.33)$$

is solvable for $k > 1$ only if $k = p = \text{prime}$ and $n = p^{s+1}$ ($s \geq 0$ arbitrary integer).

Theorem 2.30. *The equation*

$$\varphi(n) = \frac{k+1}{k}n - 1 \quad (2.34)$$

is solvable only if $k = 4$ and $n = 2p$, where $p \geq 3$ is a prime.

Theorem 2.31. *The equation*

$$\varphi(n) = \frac{k+1}{k}n - 2 \quad (2.35)$$

is solvable for $k > 1$ only if $k = 4$ and $n = 8$ or $n = k = 9$ or $n = 4 \cdot p$ ($p \geq 3$ prime), $k = 2$ or $n = 3 \cdot p$ ($p \geq 5$ prime), $k = 3$.

Theorem 2.32. *$n = k = 6$ is the only solution to equation*

$$\varphi(n) = \frac{k+1}{k}n - 3. \quad (2.36)$$

A CLASS OF SOLUTIONS OF THE EQUATION $\tau(n^2) = \tau(\varphi(n))$

In this chapter we present some notes on the equation $\tau(n^2) = \tau(\varphi(n))$. In fact, we characterize a class of solutions that have at most three distinct prime factors. Moreover, we show that the assumption of the truth of Dickson's conjecture implies that $\tau(n^2) = \tau(\varphi(n))$ infinitely often.

First we define the set of solution S

$$\mathbb{S} := \{n \in \mathbb{N} : \tau(n^2) = \tau(\varphi(n))\}.$$

If n is a prime power, namely $n = p^a$ with $p \geq 2$ and $a \geq 1$, then $n \in \mathbb{S}$ implies $(2a + 1) = \tau(p - 1)a$. But the last equation is only true for $a = 1$ and $p = 5$. Hence, $n = 5$.

3.1 Solutions having two distinct prime factors

Assume that $n = q_1^a q_2^b \in \mathbb{S}$, where q_1, q_2 are distinct primes with $2 \leq q_1 < q_2$ and $a, b \geq 1$. Since $((q_1 - 1)(q_2 - 1), q_2) = 1$, we obtain

$$(2a + 1)(2b + 1) = \tau((q_1 - 1)(q_2 - 1)q_1^{a-1})b. \quad (3.1)$$

3.1.1 n is square-free

We have the following result:

Proposition 3.1. *The only square-free solutions of the form $q_1 q_2$ are:*

- i) $n = 3 \cdot 19$.
- ii) $n = 2F_3$, where $F_3 = 257$.
- iii) $n = 2(4p^2 + 1)$, where p and $4p^2 + 1$ are simultaneously primes.

We need the following lemma.

Lemma 3.1. *Let p be a prime number with $p \geq 5$. Then the number $2p^{2a} + 1$ is composite for every $a \geq 1$. In particular, if $p \equiv 1 \pmod{3}$, then the number $2p^a + 1$ is composite for every $a \geq 2$.*

proof. First, it is clear that if $p = 3k + 1$ with $k \geq 2$, then $2p^{2a} + 1 = 2(3k + 1)^{2a} + 1 \equiv 0 \pmod{3}$. That is, $2p^{2a} + 1$ is a multiple of 3. But, if $p = 3k + 2$ for some $k \geq 1$, then $p = 3k' - 1$ with $k' = k + 1$ and so

$$2p^{2a} + 1 = 2(3k' - 1)^{2a} + 1 = 2 \left[\sum_{i=0}^{2a} (-1)^{2a-i} \binom{2a}{i} (3k')^i \right] + 1 \equiv 0 \pmod{3},$$

which is also a multiple of 3. By the same way, if $a \geq 2$ and $p \equiv 1 \pmod{3}$, then $2p^a + 1$ is a multiple of 3. This completes the proof. \square

Remark 3.1. Let p be a prime number with $p \geq 5$ and let $a \geq 1$. Similar to what we have shown in Lemma 3.1, if r is odd then the number $2^r p^{2a} + 1$ is composite. However, if r is odd and $p \equiv 1 \pmod{3}$, then the number $2^r p^a + 1$ is also composite.

Proof of Proposition 3.1. Suppose that $n = pq$, where p and q are odd primes with $p < q$. By (3.1), we have

$$9 = \tau((p-1)(q-1)). \quad (3.2)$$

We put $p-1 = 2^s m_1$ and $q-1 = 2^{s'} m_2$, where m_1, m_2 are odd and $s, s' \geq 1$. From (3.2), we obtain $9 = (s + s' + 1) \tau(m_1 m_2)$. We distinguish the following cases:

Case 1. $s + s' = 2$ and $\tau(m_1 m_2) = 3$. That is, $s = s' = 1$ and $m_1 m_2 = r^2$ for some prime $r \geq 3$. On the other hand, since $p < q$ we conclude that $m_1 = 1$ and $m_2 = r^2$. Hence, $p = 3$ and $q = 2r^2 + 1$. But, by Lemma 3.1, the number $2r^2 + 1$ is a multiple of 3 for $r \geq 5$, in which case $n = 3 \cdot 19$ is the only solution of this form.

Case 2. $s + s' = 8$ and $\tau(m_1 m_2) = 1$. That is, $m_1 = m_2 = 1$. Therefore, p and q are Fermat primes and hence s, s' are powers of 2. This case is not valid since $p < q$.

Now, assume that $n = 2q$ with $q \geq 3$ is prime. By (3.1), $9 = \tau(q-1)$ from which it follows that $q-1$ is either 2^8 or $2^2 p^2$ for some prime $p \geq 3$.

Hence, $n = 2 \cdot 257 = 2F_3$ or $n = 2(4p^2 + 1)$ with p and $4p^2 + 1$ are simultaneously primes. For example $p = 3, 5 \dots$

This completes the proof. \square

3.1.2 n is not square-free with n odd

Assume that n is odd. We have the following results:

Proposition 3.2. *Let $n = q_1^a q_2$, where $3 \leq q_1 < q_2$ and $a \geq 2$. If $n \in \mathbb{S}$, then n is one of the numbers:*

- $n = F_1^3 F_2$.
- $n = 3^{5t-3} (2^3 \cdot 3^t + 1)$, where $t \geq 2$ and $2^3 \cdot 3^t + 1$ is prime.
- $n = 5^{5t-3} (2^2 \cdot 5^t + 1)$, where $t \geq 2$ and $2^2 \cdot 5^t + 1$ is prime.
- $n = 3^{t-1} (2 \cdot 3^t + 1)$, where $t \geq 4$ and $2 \cdot 3^t + 1$ is prime.

proof. By (3.1), we have

$$3(2a + 1) = \tau((q_1 - 1)(q_2 - 1)q_1^{a-1}). \quad (3.3)$$

There are two cases:

Case 1. $(q_2 - 1, q_1) = 1$. We put $q_1 - 1 = 2^s m_1$ and $q_2 - 1 = 2^{s'} m_2$, where $(2, m_1 m_2) = 1$ and $s, s' \geq 1$. From (3.3), we have $3 = a(\tau(m_1 m_2)(s + s' + 1) - 6)$. Since $a \geq 2$, it follows that $a = 3$ and $s + s' = 6$.

Hence, $m_1 = m_2 = 1$ and so we must have $s = 2$ and $s' = 4$.

That is, $q_1 = 5$ and $q_2 = 17$, in which case $n = 5^3 \cdot 17$.

Case 2. $(q_2 - 1, q_1) = q_1$. As above, we put $q_1 - 1 = 2^s m_1$ and $q_2 - 1 = 2^{s'} q_1^t m_2$, where $(2, m_1 m_2) = 1$ and $s, s', t \geq 1$. By (3.3) we have

$$3(2a + 1) = \tau(m_1 m_2)(s + s' + 1)(a + t). \quad (3.4)$$

It is clear that $\tau(m_1 m_2)$ cannot be ≥ 3 , otherwise

$$3(2a + 1) \geq 3(a + t)(s + s' + 1) \geq 9(a + 1) > 6a + 3,$$

a contradiction. Moreover, if $\tau(m_1 m_2) = 2$, then the right-hand side of (3.4) is even, while its left-hand side is odd, also a contradiction. Therefore, $\tau(m_1 m_2) = 1$ and so by (3.4) once again, $3(2a + 1) = (2^i + s' + 1)(a + t)$ for some $i \geq 0$. Note also that $2^i + s' + 1$ cannot be ≥ 6 , otherwise $3(2a + 1) \geq 6(a + t) > 6a + 3$, a contradiction.

Consequently, we have either $2^i + s' = 4$ or $2^i + s' = 2$

(i) $2^i + s' = 4$. There are two possibilities:

- $i = 0$ and $s' = 3$. It follows that $a = 5t - 3$, $q_1 = 3$ and $q_2 = 2^3 \cdot 3^t + 1$, thus $n = 3^{5t-3} (2^3 \cdot 3^t + 1)$, where $2^3 \cdot 3^t + 1$ is prime.
For example, for $t = 2$, we get $n = 3^7 \cdot 73$.

- $i = 1$ and $s' = 2$. That is, $a = 5t - 3$, $q_1 = 5$ and $q_2 = 2^2 \cdot 5^t + 1$, thus $n = 5^{5t-3} (2^2 \cdot 5^t + 1)$, where $2^2 \cdot 5^t + 1$ is prime.

For example, for $t = 2$, we have $n = 5^7 \cdot 101$.

- (ii) $2^i + s' = 2$. That is, $i = 0$, $s' = 1$ and so $a = t - 1$. Hence, $q_1 = 3$ and $q_2 = 2 \cdot 3^t + 1$. Consequently, $n = 3^{t-1} (2 \cdot 3^t + 1)$, where $(2 \cdot 3^t + 1)$ is prime.

For example, for $t = 4$, we have $n = 3^3 \cdot 163$.

This completes the proof. \square

Proposition 3.3. *The number $n = F_1 F_2^3$ is the only solution of the form $q_1 q_2^b$, where $3 \leq q_1 < q_2$ and $b \geq 2$.*

proof. Assume that $n = q_1 q_2^b \in \mathbb{S}$, where $3 \leq q_1 < q_2$ and $b \geq 2$. Applying (3.1), we obtain

$$3(2b + 1) = \tau((q_1 - 1)(q_2 - 1))b,$$

from which it follows that

$$b(\tau((q_1 - 1)(q_2 - 1)) - 6) = 3,$$

and so

$$\tau((q_1 - 1)(q_2 - 1)) = 7 \text{ and } b = 3.$$

Or, equivalently,

$$(q_1 - 1)(q_2 - 1) = 2^6.$$

Thus, $q_1 = 5$ and $q_2 = 17$, in which case $n = 5 \cdot 17^3$. \square

Theorem 3.1. *Let $n = q_1^a q_2^b$, where $3 \leq q_1 < q_2$ and $a, b \geq 2$. If $n \in \mathbb{S}$, then $n = 3^a (2 \cdot 3^t + 1)^b$, where $2 \cdot 3^t + 1$ is prime and $ab + 2a + 2b + 1 = 3bt$.*

proof. By (3.1), we have

$$(2a + 1)(2b + 1) = \tau((q_1 - 1)(q_2 - 1)q_1^{a-1})b. \quad (3.5)$$

There are two cases:

Case 1. $(q_2 - 1, q_1) = 1$. We put $q_1 - 1 = 2^x m_1$ and $q_2 - 1 = 2^y m_2$, where $x, y \geq 1$ and $(2, m_1 m_2) = 1$. It then follows from (3.5) that

$$(2a + 1)(2b + 1) = (x + y + 1)\tau(m_1 m_2)ab. \quad (3.6)$$

It is clear that $\tau(m_1 m_2)$ cannot be ≥ 3 . Otherwise, $(x + y + 1)a \geq 3a > 2a + 1$ and $\tau(m_1 m_2)b \geq 3b > 2b + 1$, a contradiction. Moreover, if $\tau(m_1 m_2) = 2$, then the right-hand side of (3.6) is even, while its left-hand side is odd, also a contradiction. Therefore,

$m_1 = m_2 = 1$ and so by (3.6) once again, $(2a + 1)(2b + 1) = (2^j + 2^i + 1)ab$ for some $j > i \geq 0$. Note also that $2^j + 2^i + 1$ cannot be ≥ 6 , and hence $i = 0$ and $j = 1$. That is,

$$(2a + 1)(2b + 1) = 4ab,$$

which is impossible.

Case 2. $(q_2 - 1, q_1) = q_1$. We put $q_1 - 1 = 2^x m_1$ and $q_2 - 1 = 2^y q_1^t m_2$, where $x, y, t \geq 1$ and $(2, m_1 m_2) = 1$. By (3.5),

$$(2a + 1)(2b + 1) = (x + y + 1) \tau(m_1 m_2) (a + t) b,$$

from which it follows that $m_1 = m_2 = 1$ and $b \geq 3$. Moreover, we see that $x + y$ is even and $x + y \leq 4$. Therefore, if $x + y = 4$, then

$$a(b - 2) = b(2 - 5t) + 1. \quad (3.7)$$

This is impossible since the left-hand side of (3.7) is positive, while its right-hand side is negative. If $x + y = 2$, then $x = y = 1$. It follows that $n = 3^a (2 \cdot 3^t + 1)^b$, where $2 \cdot 3^t + 1$ is prime and $ab + 2a + 2b + 1 = 3bt$.

For example, for $a = 10$, $b = 7$ and $t = 5$, we obtain $n = 3^{10} (2 \cdot 3^5 + 1)^7 = 3^{10} \cdot 487^7$. \square

3.1.3 n is not square-free with n even

Now, assume that n is **even**. We have the following notes:

Lemma 3.2. $2^x - 3$ is divisible by 5 if and only if $x \equiv 3 \pmod{4}$.

proof. Clearly, $2^{4k} \equiv 1 \pmod{5}$ for every $k \geq 0$. Hence, $2^x \equiv 3 \pmod{5}$ if and only if $x \equiv 3 \pmod{4}$. \square

Proposition 3.4. Let $n = 2^a q_2$, where $q_2 \geq 3$ and $a \geq 2$. If $n \in \mathbb{S}$, then n is one of the numbers:

- $n = 2^{5t-3} (2^t \cdot p^4 + 1)$, where p is an odd prime with $2^t \cdot p^4 + 1$ is also prime.
- $n = 2^{t-1} (2^t \cdot p^2 + 1)$, where p is an odd prime with $2^t \cdot p^2 + 1$ is also prime.
- $n = 2^{(2^i-3)/5} F_i$, where $i \equiv 3 \pmod{4}$ and F_i is a Fermat prime.

proof. By (3.1), we have

$$3(2a + 1) = \tau((q - 1)2^{a-1}).$$

Put $q - 1 = 2^t m$, where $(2, m) = 1$ and $t \geq 1$. It follows that

$$3(2a + 1) = \tau(m)(a + t),$$

and hence

$$(6 - \tau(m))a = \tau(m)t - 3. \quad (3.8)$$

It is clear from (3.8) that $\tau(m)$ is odd and cannot be ≥ 6 . Now, we consider separately the following possibilities:

- $\tau(m) = 5$. It follows that $m = p^4$, where $p \geq 3$ is prime and $a = 5t + 3$. Thus, $n = 2^{5t-3}(2^t \cdot p^4 + 1)$, where $2^t \cdot p^4 + 1$ is also prime.
For example, for $t = 1$ and $p = 3$, we get $n = 2^2(2 \cdot 3^4 + 1) = 2^2 \cdot 163$.
- $\tau(m) = 3$. It follows that $m = p^2$, where $p \geq 3$ is prime and $a = t - 1$. Thus, $n = 2^{t-1}(2^t \cdot p^2 + 1)$, where $2^t \cdot p^2 + 1$ is also prime.
For example, for $t = p = 3$, we have $n = 2^2(2^3 \cdot 3^2 + 1) = 2^2 \cdot 73$.
- $\tau(m) = 1$. Then $t = 2^i$ for some $i \geq 0$, and by (3.8) we have $a = (2^i - 3)/5$. By Lemma 3.2, $i \equiv 3 \pmod{4}$.
Hence, $n = 2^{(2^i-3)/5} F_i$, where F_i is a Fermat prime.

□

Proposition 3.5. *Let $n = 2q_2^b$, where q_2 is odd prime and $b \geq 2$. Then $n \notin \mathbb{S}$.*

proof. By (3.1), we have

$$3(2b + 1) = \tau(q_2 - 1)b.$$

We put $q_2 - 1 = 2^s m$, where $(2, m) = 1$ and $s \geq 1$. It follows that

$$3(2b + 1) = \tau(m)(s + 1)b,$$

and so

$$3 = (\tau(m)(s + 1) - 6)b. \quad (3.9)$$

By (3.9), we must have $b = 3$, $\tau(m) = 7$ and $s = 0$ or $\tau(m) = 1$ and so $s = 2^i = 6$ for some i . Thus there is no solution in both cases. □

Proposition 3.6. *Let $n = 2^a q_2^b$, where $q_2 \geq 3$ and $a, b \geq 2$. If $n \in \mathbb{S}$, then n is one of the numbers:*

- $n = 2^a(2^s \cdot p^2 + 1)^b$, where p and $2^s \cdot p^2 + 1$ are simultaneously primes with $ab + 2a + 2b + 1 = 3bs$,

- $n = 2^a \left(2^{(3ab+2a+2b+1)/b} + 1\right)^b$, where b divides $2a + 1$ and $2^{(3ab+2a+2b+1)/b} + 1$ is prime.

proof. By (3.1), we have

$$(2a + 1)(2b + 1) = \tau(2^{a-1}(q_2 - 1))b.$$

If we put $q_2 - 1 = 2^s m$, where $(2, m) = 1$ and $s \geq 1$, it follows that

$$(2b + 1)(2b + 1) = \tau(m)(s + a)b. \quad (3.10)$$

By (3.10), $\tau(m)$ cannot be ≥ 6 ; otherwise, $\tau(m)(s + a)b \geq (3b)(2(a + s)) > (2a + 1)(2b + 1)$, a contradiction. Moreover, $\tau(m)$ cannot be even. So the rest cases are:

- $\tau(m) = 5$. By (3.10), we have $b(2 - 5s) = a(b - 2) - 1$, which has no sense.
- $\tau(m) = 3$. Then $m = p^2$ for some prime $p \geq 3$, and by (3.10) we have $ab + 2a + 2b + 1 = 3bs$. Thus, $n = 2^a (2^s \cdot p^2 + 1)^b$ where p and $2^s \cdot p^2 + 1$ are simultaneously primes with $ab + 2a + 2b + 1 = 3bs$.
For example, for $a = 13$, $b = 9$ and $s = 6$, we get $n = 2^{13} \cdot 577^9$.
- $\tau(m) = 1$. It follows that $q = 2^{2^i} + 1$ for some $i \geq 0$, and so $n = 2^a F_i^b$, where $3ab + 2a + 2b + 1 = 2^i b$ by (3.10). Or equivalently, $n = 2^a \left(2^{(3ab+2a+2b+1)/b} + 1\right)^b$, where b divides $2a + 1$ and $2^{(3ab+2a+2b+1)/b} + 1$ is prime.

The proof is finished. □

3.2 Solutions having three distinct prime factors

Let $n = q_1^a q_2^b q_3^c \in \mathbb{S}$, where q_1, q_2, q_3 are distinct primes with $2 \leq q_1 < q_2 < q_3$ and $a, b, c \geq 1$.
Be definition, we see that

$$(2a + 1)(2b + 1)(2c + 1) = \tau((q_1 - 1)(q_2 - 1)(q_3 - 1)q_1^{a-1}q_2^{b-1})c. \quad (3.11)$$

We consider separately the cases n is square-free and n is not square-free (odd and even).

3.2.1 n is square-free

First, assume that n is **square-free odd**.

Proposition 3.7. *The only possible solutions of the form $q_1 q_2 q_3$ are:*

- $F_1 F_3 F_4, F_0 F_2 \cdot 73, 11 \cdot F_2 \cdot 41$.

- $F_1 F_2 (4p^2 + 1)$, where p is an odd prime with $4p^2 + 1$ is prime.
- $F_1 (2p + 1) (2p^5 + 1)$, where p is an odd prime with $2p + 1$ and $2p^5 + 1$ are prime numbers.
- $F_1 (2^2p + 1) (2^4p + 1)$, where p is an odd prime with $2^2p + 1$ and $2^4p + 1$ are prime numbers.
- $F_2 (2p + 1) (2^3p + 1)$, where p is an odd prime with $2p + 1$ and $2^3p + 1$ are prime numbers.

For the proof we need the following lemma:

Lemma 3.3. *If p is prime greater than 3, then the numbers $2^{2\alpha+1}p + 1$ and $2^{2\beta}p + 1$ with $\alpha \geq 0$ and $\beta \geq 1$ cannot be simultaneously primes.*

proof. First we note that $2^{2\alpha+1} \equiv 2 \pmod{3}$ and $2^{2\beta} \equiv 1 \pmod{3}$. Thus if p is of the form $3k + 1$, then $2^{2\alpha+1}p + 1 = 2^{2\alpha+1}(3k + 1) + 1 \equiv 0 \pmod{3}$ and if p is of the form $3k + 2$, then $2^{2\beta}p + 1 = 2^{2\beta}(3k + 2) + 1 \equiv 0 \pmod{3}$. \square

Proof of Proposition 3.7. We put $q_1 - 1 = 2^x m_1$, $q_2 - 1 = 2^y m_2$, and $q_3 - 1 = 2^z m_3$, where m_i is odd and $x, y, z \geq 1$. Let $m = m_1 m_2 m_3$. Applying (3.11), we have

$$27 = \tau(m)(x + y + z + 1)$$

We distinguish two cases:

Case 1. $\tau(m) = 1$. Then $2^i + 2^j + 2^k = 26$ for some $0 \leq i < j < k$, which is only true for $i = 1, j = 3$ and $k = 4$. Hence, $n = F_1 F_3 F_4$.

Case 2. $\tau(m) = 3$ and $x + y + z = 8$. It follows that $m = p^2$, where $p \geq 3$ is prime. Also we consider the following subcases:

Case 2.1. $m_1 = 1, m_2 = 1$ and $m_3 = p^2$. Since $x < y$, by Lemma 3.1, there are only two possibilities:

- $n = 3 \cdot 17 \cdot 73$.
- $n = 5 \cdot 17 \cdot (4p^2 + 1)$, where p and $4p^2 + 1$ are both primes. For example, $p = 3$.

Case 2.2. $m_1 = 1$ and $m_2 = m_3 = p$. It follows that $x = 2^i$, where $i \geq 0$ and $y < z$. By Lemma 3.3, we have three possibilities:

- $n = 5(2p + 1)(2^5p + 1)$, where $p, 2p + 1$ and $2^5p + 1$ are simultaneously primes. For example, for $p = 11$, we obtain $n = 5 \cdot 23 \cdot 353$.
- $n = 5(4p + 1)(16p + 1)$, where $p, 4p + 1$ and $16p + 1$ are simultaneously primes. For example, for $p = 7$, we have $n = 5 \cdot 29 \cdot 113$.

- $n = 17(2p + 1)(2^3p + 1)$, where p , $2p + 1$ and $2^3p + 1$ are simultaneously primes.
For example, for $p = 11$, we have $n = 17 \cdot 23 \cdot 89$.

Case 2.3. $m_1 = m_2 = p$ and $m_3 = 1$. This case is not valid. We have the same for $m_1 = m_3 = 1$ and $m_2 = p^2$ or $m_2 = m_3 = 1$ and $m_1 = p^2$.

Case 2.4. $m_1 = m_3 = p$ and $m_2 = 1$. It follows that $y = 2^i$, where $i \geq 0$ and $x < z$. Thus we must have $p = 5$.

Hence, $n = 11 \cdot 17 \cdot 41$. □

Second, assume that n is **square-free even**.

Proposition 3.8. *The only possible solutions of the form $2q_2q_3$ are:*

- $2 \cdot 3 \cdot 1153, 2 \cdot 19 \cdot 1459$.
- $2 \cdot 5(2^6p^2 + 1)$, where p is an odd prime with $2^6p^2 + 1$ is prime.
- $2 \cdot 17(2^4p^2 + 1)$, where p is an odd prime with $2^4p^2 + 1$ is prime.
- $2(2p + 1)(2^7p + 1)$, where p is an odd prime with $2p + 1$ and $2^7p + 1$ are prime numbers.
- $2(2^2p + 1)(2^6p + 1)$, where p is an odd prime with $2^2p + 1$ and $2^6p + 1$ are prime numbers.
- $2(2^3p + 1)(2^5p + 1)$, where p is an odd prime with $2^3p + 1$ and $2^5p + 1$ are prime numbers.
- $2(2p + 1)(2p^7 + 1)$, where p is an odd prime with $2p + 1$ and $2p^7 + 1$ are prime numbers.
- $2(2p^3 + 1)(2p^5 + 1)$, where p is an odd prime with $2p^3 + 1$ and $2p^5 + 1$ are prime numbers.
- $2(2p_1 + 1)(2p_1p_2^2 + 1)$, where $p_1, p_2, 2p_1 + 1$ and $2p_1p_2^2 + 1$ are simultaneously primes.

For the proof, we need the following lemma:

Lemma 3.4. *If p_1 and p_2 are prime numbers greater than 3, then $2p_1^2p_2^2 + 1$ is composite.*

proof. This follows immediately from the proof of Lemma 3.1. □

Proof of Proposition 3.8. We put $q_1 - 1 = 2^x m_1$ and $q_2 - 1 = 2^y m_2$, where $(2, m_1 m_2) = 1$ and $x, y \geq 1$. By (3.11),

$$27 = \tau(m_1 m_2)(x + y + 1).$$

There are three cases to consider.

Case 1. $\tau(m_1 m_2) = 1$. That is, $2^i + 2^j = 26$ for some $i, j \geq 0$. This is impossible.

Case 2. $\tau(m_1 m_2) = 3$ and $x + y = 8$. It follows that $m_1 m_2 = p^2$, where $p \geq 3$ is prime. We have three subcases:

Case 2.1. $m_1 = 1$ and $m_2 = p^2$. Then the solutions are given by:

- $n = 2 \cdot 3(2^7p^2 + 1)$ with p and $2^7p^2 + 1$ are both prime, and by Lemma 3.1, $p = 3$ is the only prime with this property, in which case $n = 2 \cdot 3 \cdot 1153$.
- $n = 2 \cdot (5(2^6p^2 + 1))$ with p and $2^6p^2 + 1$ are both prime.
For example, for $p = 3$, we have $n = 2 \cdot 5 \cdot 577$.
- $n = 2 \cdot 17(16p^2 + 1)$ with p and $16p^2 + 1$ are both prime.
For example, for $p = 5$, we have $n = 2 \cdot 17 \cdot 401$.

Case 2.2. $m_1 = m_2 = p$. Then $x < y$, and the solutions are:

- $2 \cdot (2p + 1)(2^7p + 1)$, where p , $2p + 1$ and $2^7p + 1$ are primes.
For example, for $p = 5$, we get $n = 2 \cdot 11 \cdot 641$.
- $n = 2(4p + 1)(2^6p + 1)$, where p , $4p + 1$ and $2^6p + 1$ are prime numbers.
For example, for $p = 7$, we get $n = 2 \cdot 29 \cdot 449$.
- $n = 2(8p + 1)(2^5p + 1)$, where p , $8p + 1$ and $2^5p + 1$ are prime numbers.
For example, for $p = 11$, we have $n = 2 \cdot 89 \cdot 353$.

Case 2.3. $m_1 = p^2$ and $m_2 = 1$. This case is not valid.

Case 3. $\tau(m_1m_2) = 9$ and $x + y = 2$. We have two subcases:

Case 3.1. $m_1m_2 = p^8$, where $p \geq 3$ is prime. Then the solutions are:

- $n = 2(2p + 1)(2p^7 + 1)$, where p , $2p + 1$ and $2p^7 + 1$ are prime numbers.
- $n = 2(2p^2 + 1)(2p^6 + 1)$, where p , $2p^2 + 1$ and $2p^6 + 1$ are primes, which is only true for $p = 3$, and so $n = 2 \cdot 19 \cdot 1459$.
- $n = 2(2p^3 + 1)(2p^5 + 1)$, where p , $2p^3 + 1$ and $2p^5 + 1$ are primes.
For example, for $p = 29$, we have $n = 2 \cdot 48779 \cdot 41022299$.

Case 3.2. $m_1m_2 = p_1^2p_2^2$, where p_1, p_2 are odd primes with $p_1 < p_2$. By Lemma 3.4, the number $2 \cdot 3 \cdot (2p_1^2p_2^2 + 1)$ is composite and by Lemma 3.1, we obtain

- $n = 2(2p_1 + 1)(2p_1p_2^2 + 1)$ with $p_1, p_2, 2p_1 + 1$ and $2p_1p_2^2 + 1$ are prime numbers.
For example, for $p_1 = 3$ and $p_2 = 5$, $n = 2 \cdot 7 \cdot 151$.
- $n = 2(2p_2 + 1)(2p_1^2p_2 + 1)$ with $p_1, p_2, 2p_2 + 1$ and $2p_1^2p_2 + 1$ are prime numbers.
For example, for $p_1 = 3$ and $p_2 = 11$ we obtain $n = 2 \cdot 23 \cdot 199$.

This completes the proof. □

3.2.2 n is not square-free with n odd

Assume that n is **not square-free** with n is **odd**. Here, we characterize all odd solutions having only one power prime.

Proposition 3.9. *The only possible solutions of the form $q_1^a q_2 q_3$, where $a \geq 2$ and $3 \leq q_1 < q_2 < q_3$ are:*

- $5^3(2p+1)(8p+1)$, where p , $2p+1$ and $8p+1$ are prime numbers.
- $q_1^{17(t+t')-9}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$, where q_1 is a Fermat prime and y, z are positive integers with $(q_1, y+z) \in \{(3, 15), (5, 14), (17, 12), (257, 8)\}$ and $2^y q_1^t + 1, 2^z q_1^{t'} + 1$ are prime numbers.
- $q_1^{5(t+t')-3}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$, where q_1 is a Fermat prime and y, z, t, t' are positive integers with $(q_1, y+z) \in \{(3, 13), (5, 12), (17, 10), (257, 6)\}$ and $2^y q_1^t + 1, 2^z q_1^{t'} + 1$ are prime numbers.
- $q_1^{\frac{13(t+t')-9}{5}}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$, where q_1 is a Fermat prime and y, z, t, t' are positive integers, with $(q_1, y+z) \in \{(3, 11), (5, 10), (17, 8), (257, 4)\}$ and $2^y q_1^t + 1, 2^z q_1^{t'} + 1$ are prime numbers.
- $q_1^{\frac{11(t+t')-9}{7}}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$, where q_1 is a Fermat prime and y, z, t, t' are positive integers with $(q_1, y+z) \in \{(3, 9), (5, 8), (17, 6)\}$ and $2^y q_1^t + 1, 2^z q_1^{t'} + 1$ are prime numbers.
- $q_1^{t+t'-1}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$, where q_1 is a Fermat prime and y, z, t, t' are positive integers with $(q_1, y+z) \in \{(3, 7), (5, 6), (17, 4)\}$ and $2^y q_1^t + 1, 2^z q_1^{t'} + 1$ are prime numbers.
- $q_1^{\frac{7(t+t')-9}{11}}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$, where q_1 is a Fermat prime and y, z, t, t' are positive integers with $(q_1, y+z) \in \{(3, 5), (5, 4)\}$ and $2^y q_1^t + 1, 2^z q_1^{t'} + 1$ are primes.
- $q_1^{\frac{5(t+t')-9}{13}}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$, where q_1 is a Fermat prime and y, z, t, t' are positive integers with $(q_1, y+z) \in \{(3, 3), (5, 2)\}$ and $2^y q_1^t + 1, 2^z q_1^{t'} + 1$ are primes.
- $n = (2^x q + 1)^{5(t+t')-3} (2^y (2^x q + 1)^t q + 1) (2^z (2^x q + 1)^{t'} + 1)$, where $q, 2^x q + 1, 2^y (2^x q + 1)^t q + 1, 2^z (2^x q + 1)^{t'} + 1$ are prime numbers and x, y, z, t, t' are positive integers with $x + y + z = 4$.
- $n = (2^x q + 1)^{5(t+t')-3} (2^y (2^x q + 1)^t + 1) (2^z (2^x q + 1)^{t'} q + 1)$, where $q, 2^x q + 1, 2^y (2^x q + 1)^t + 1, 2^z (2^x q + 1)^{t'} q + 1$ are prime numbers and x, y, z, t, t' are positive integers with $x + y + z = 4$.
- $n = (2^x + 1)^{5(t+t')-3} (2^y (2^x + 1)^t q + 1) (2^z (2^x + 1)^{t'} q + 1)$, where $q, 2^x + 1, 2^y (2^x + 1)^t q + 1, 2^z (2^x + 1)^{t'} q + 1$ are prime numbers and x, y, z, t, t' are positive integers with $x + y + z = 4$.

- $n = (2^x + 1)^{5(t+t')-3} (2^y(2^x + 1)^t q^2 + 1) (2^z(2^x + 1)^{t'} + 1)$, where $q, 2^x + 1, 2^y(2^x q^2 + 1)^t q + 1, 2^z(2^x q + 1)^{t'} + 1$ are prime numbers and x, y, z, t, t' are positive integers with $x + y + z = 4$.
- $n = (2^x + 1)^{5(t+t')-3} (2^y(2^x + 1)^t + 1) (2^z(2^x + 1)^{t'} q^2 + 1)$, where $q, 2^x + 1, 2^y(2^x q + 1)^t + 1, 2^z(2^x q + 1)^{t'} q^2 + 1$ are primes and x, y, z, t, t' are positive integers with $x + y + z = 4$.

proof. Let $n = q_1^a q_2 q_3 \in \mathbb{S}$. Therefore, by (3.11) we get

$$9(2a + 1) = \tau((q_1 - 1)(q_2 - 1)(q_3 - 1)q_1^{a-1}). \quad (3.12)$$

There are two cases:

Case 1. Suppose that $((q_2 - 1)(q_3 - 1), q_1) = 1$. It follows from (3.12) that

$$9 = (\tau((q_1 - 1)(q_2 - 1)(q_3 - 1)) - 18)a. \quad (3.13)$$

We distinguish the following subcases:

Case 1.1. $a = 9$. From (3.13), $\tau((q_1 - 1)(q_2 - 1)(q_3 - 1)) = 19$. We must have

$$(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^{18},$$

and hence q_1, q_2 and q_3 are Fermat primes. This is impossible.

Case 1.2. $a = 3$. From (3.13), $\tau((q_1 - 1)(q_2 - 1)(q_3 - 1)) = 21$. If

$$(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^{20},$$

then q_1, q_2 and q_3 are Fermat primes, which is impossible. Thus, we must have

$$(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^6 q^2,$$

where q is odd prime, with $(q_1, q) = 1$. By Lemma 3.1, $8q^2 + 1$ and $2q^2 + 1$ are composite, and by Lemma 3.3, $2q + 1$ and $2^4 q + 1$ are not simultaneously primes and the same for $2^2 q + 1$ and $2^3 q + 1$, so we conclude that n is of the form: $n = 5^3(2q + 1)(8q + 1)$, where $q, 2q + 1$ and $8q + 1$ are prime numbers with $q > 5$. For example, if $q = 11$ then $n = 5^3 \cdot 23 \cdot 89$.

Case 2. Suppose that $((q_2 - 1)(q_3 - 1), q_1) = q_1$. We put $q_1 - 1 = 2^x m_1, q_2 - 1 = 2^y q_1^t m_2$ and $q_3 - 1 = 2^z q_1^{t'} m_3$, where $x, y, z \geq 1, \max(t, t') \geq 1$ and $(2p_1, m_1 m_2 m_3) = 1$. Let us take $m = m_1 m_2 m_3$. It follows from (3.12) that

$$(18 - \tau(m)(x + y + z + 1))a = \tau(m)(x + y + z + 1)(t + t') - 9. \quad (3.14)$$

As above, $\tau(m)(x+y+z+1)$ cannot be ≥ 18 . Thus, we distinguish the following sub-cases:

Case 2.1. $\tau(m)(x+y+z+1) = 17$. Thus, $\tau(m) = 1$ and so $x+y+z = 16$. Or equivalently, $m_1 = m_2 = m_3 = 1$, q_1 is a Fermat prime and by (3.14), $a = 17(t+t') - 9$. Therefore, we get

$$n = q_1^{17(t+t')-9} (2^y \cdot q_1^t + 1) (2^z \cdot q_1^{t'} + 1),$$

where $2^y \cdot q_1^t + 1$ and $2^z \cdot q_1^{t'} + 1$ are primes with $2^y \cdot q_1^t < 2^z \cdot q_1^{t'}$, and we have

$$(q_1, y+z) \in \{(3, 15), (5, 14), (17, 12), (257, 8)\}.$$

For example, for $q_1 = 3, y = 2, z = 13, t = 2$ and $t' = 5$ we obtain $n = 3^{110} \cdot 37 \cdot 1990657$.

Case 2.2. $\tau(m)(x+y+z+1) = 15$. We will consider separately the two cases $\tau(m) = 1$ and $\tau(m) = 3$.

- $\tau(m) = 1$. Then $x+y+z = 14$, q_1 is a Fermat prime and by (3.14), $a = 5(t+t') - 3$. Thus, we have

$$n = q_1^{5(t+t')-3} (2^y \cdot q_1^t + 1) (2^z \cdot q_1^{t'} + 1),$$

where $2^y \cdot q_1^t + 1$ and $2^z \cdot q_1^{t'} + 1$ are primes with $2^y \cdot q_1^t < 2^z \cdot q_1^{t'}$, and we have $(q_1, y+z) \in \{(3, 13), (5, 12), (17, 10), (257, 6)\}$.

For example, for $x = 1, y = 2, z = 11, t = 1$ and $t' = 2$ we obtain $n = 3^{12} \cdot 13 \cdot 18433$.

- $\tau(m) = 3$. Then $x+y+z = 4$ and $m = q^2$, where q is an odd prime with $(q_1, q) = 1$. Thus, by (3.14), $a = 5(t+t') - 3$. In this case, n is one of the numbers:
 - $n = (2^x q + 1)^{5(t+t')-3} (2^y q_1^t q + 1) (2^z q_1^{t'} + 1)$, where $2^x q + 1, 2^y q_1^t q + 1$ and $2^z q_1^{t'} + 1$ are prime numbers.
For example, for $x = y = 1, z = 2, q = 3, t = 1$ and $t' = 2$, we get $n = 7^{12} \cdot 43 \cdot 197$.
 - $n = (2^x q + 1)^{5(t+t')-3} (2^y q_1^t + 1) (2^z q_1^{t'} q + 1)$, where $2^x q + 1, 2^y q_1^t + 1$ and $2^z q_1^{t'} q + 1$ are prime numbers.
For example, for $x = z = 1, y = 2, q = 3, t = 1$ and $t' = 1$, we get $n = 7^7 \cdot 29 \cdot 43$.
 - $n = (2^x + 1)^{5(t+t')-3} (2^y q_1^t q + 1) (2^z q_1^{t'} q + 1)$, where $2^x + 1, 2^y q_1^t q + 1$ and $2^z q_1^{t'} q + 1$ are prime numbers.
For example, for $x = z = 1, y = 2, q = 13, t = 0$ and $t' = 1$, we get $n = 3^2 \cdot 53 \cdot 79$.

- $n = (2^x + 1)^{5(t+t')-3} (2^y q_1^t q^2 + 1) (2^z q_1^{t'} + 1)$, where $2^x + 1$, $2^y q_1^t q^2 + 1$ and $2^z q_1^{t'} + 1$ are prime numbers.

For example, for $x = y = 1$, $z = 2$, $q = 5$, $t = 1$ and $t' = 6$, we get $n = 3^{32} \cdot 151 \cdot 2917$.

- $n = (2^x + 1)^{5(t+t')-3} (2^y q_1^t + 1) (2^z q_1^{t'} q^2 + 1)$, where $2^x + 1$, $2^y q_1^t + 1$ and $2^z q_1^{t'} q^2 + 1$ are prime numbers.

For example, for $y = z = t = 1$, $x = 2$, $q = 3$ and $t' = 0$, we get $n = 5^2 \cdot 11 \cdot 19$.

Case 2.3. $\tau(m)(x + y + z + 1) = 13$. Then $\tau(m) = 1$ and $x + y + z = 12$. Thus, $m_1 = m_2 = m_3 = 1$ and q_1 is a Fermat prime. By (3.14), $5a = 13(t + t') - 9$. That is,

$$n = (q_1)^{\frac{13(t+t')-9}{5}} (2^y q_1^t + 1) (2^z q_1^{t'} + 1),$$

where $2^x + 1$, $2^y q_1^t + 1$ and $2^z q_1^{t'} + 1$ are primes with $2^y q_1^t < 2^z q_1^{t'}$ and $5 \mid 13(t + t') - 9$, and we have $(q_1, y + z) \in \{(3, 11), (5, 10), (17, 8), (257, 4)\}$.

For example, for $x = t = 1$, $y = 5$, $z = 6$ and $t' = 2$ we get $n = 3^6 \cdot 97 \cdot 577$.

Case 2.4. $\tau(m)(x + y + z + 1) = 11$. We also have $m = 1$ and $x + y + z = 10$. So, q_1 is a Fermat prime and by (3.14), $7a = 11(t + t') - 9$. Thus,

$$n = q_1^{\frac{11(t+t')-9}{7}} (2^y q_1^t + 1) (2^z q_1^{t'} + 1),$$

where $2^y q_1^t + 1$ and $2^z q_1^{t'} + 1$ are prime numbers with $2^y q_1^t < 2^z q_1^{t'}$ and $7 \mid 11(t + t') - 9$, and we have $(q_1, x + y) \in \{(3, 9), (5, 8), (17, 6)\}$.

For example, for $x = t = 1$, $y = 2$, $z = 7$ and $t' = 3$ we have $n = 3^5 \cdot 13 \cdot 3457$.

Case 2.5. $\tau(m)(x + y + z + 1) = 9$. Here, we must have $\tau(m) = 1$ and $x + y + z = 8$, from which it follows that $m = 1$ and q_1 is a Fermat prime. Thus, by (3.14), $a = t + t' - 1$. Hence,

$$n = q_1^{t+t'-1} (2^y q_1^t + 1) (2^z q_1^{t'} + 1),$$

where $2^y q_1^t + 1$ and $2^z q_1^{t'} + 1$ are prime numbers with $2^y q_1^t < 2^z q_1^{t'}$, and we have

$$(q_1, x + y) \in \{(3, 7), (5, 6), (17, 4)\}.$$

For example, for $x = t = 1$, $y = 2$, $z = 5$ and $t' = 4$ we have $n = 3^4 \cdot 13 \cdot 2593$.

Case 2.6. $\tau(m)(x + y + z + 1) = 7$. As above $m = 1$ and $x + y + z = 6$. By (3.14), $11a = 7(t + t') - 9$. That is,

$$n = q_1^{\frac{7(t+t')-9}{11}} (2^y q_1^t + 1) (2^z q_1^{t'} + 1),$$

where $2^y q_1^t + 1$ and $2^z q_1^{t'} + 1$ are prime numbers with $2^y q_1^t < 2^z q_1^{t'}$ and $11 \mid 7(t+t') - 9$, and we have $(q_1, x+y) \in \{(3, 5), (5, 4)\}$.

For example, for $x = y = 1, z = t' = 4$ and $t = 2$ we get $n = 3^3 \cdot 19 \cdot 1297$.

Case 2.7. $\tau(m)(x+y+z+1) = 5$. We must have $m = 1$ and $x+y+z = 4$. By (3.14), we deduce that $13a = 5(t+t') - 9$, and so

$$n = q_1^{\frac{5(t+t')-9}{13}} (2^y q_1^t + 1) (2^z q_1^{t'} + 1),$$

where $2^y q_1^t + 1$ and $2^z q_1^{t'} + 1$ are prime numbers with $2^y q_1^t < 2^z q_1^{t'}$ and $13 \mid 5(t+t') - 9$, and we have $(q_1, x+y) \in \{(3, 3), (5, 2)\}$.

For example, for $x = t = 1, y = 1, z = 2$ and $t' = 6$ we get $n = 3^2 \cdot 7 \cdot 2917$.

□

Proposition 3.10. *The only possible solutions of the form $q_1 q_2^b q_3$, where $b \geq 2$ and $3 \leq q_1 < q_2 < q_3$ are:*

- $5(2q+1)^3(8q+1)$, where $q, 2q+1, 8q+1$ are prime numbers with $(q, 2q+1) = 1$.
- $q_1 q_2^{17t-9} (2^z q_2^t + 1)$, where q_1, q_2 are Fermat primes and z, t are positive integers with $(q_1, q_2, z) \in \{(3, 5, 13), (3, 17, 11), (3, 257, 7), (5, 17, 10), (5, 257, 6), (17, 257, 4)\}$ and $2^z q_2^t + 1$ is prime.
- $n = q_1 q_2^{5t-3} (2^z \cdot q_2^t + 1)$, where q_1, q_2 are Fermat primes and z, t are positive integers with $(q_1, q_2, z) \in \{(3, 5, 11), (3, 17, 9), (3, 257, 5), (5, 17, 8), (5, 257, 4), (17, 257, 2)\}$ and $2^z q_2^t + 1$ is prime.
- $n = q_1 q_2^{\frac{13t-9}{5}} (2^z \cdot q_2^t + 1)$, where q_1, q_2 are Fermat primes and z, t are positive integers with $(q_1, q_2, z) \in \{(3, 5, 9), (3, 17, 7), (3, 257, 3), (5, 17, 6), (5, 257, 2)\}$ and $2^z q_2^t + 1$ is prime.
- $n = q_1 q_2^{\frac{11t-9}{7}} (2^z \cdot q_2^t + 1)$, where q_1, q_2 are Fermat primes and z, t are positive integers with $(q_1, q_2, z) \in \{(3, 5, 7), (3, 17, 5), (3, 257, 1), (5, 17, 4)\}$ and $2^z q_2^t + 1$ is prime.
- $n = q_1 q_2^{t-1} (2^z \cdot q_2^t + 1)$, where q_1, q_2 are Fermat primes and z, t are positive integers with $(q_1, q_2, z) \in \{(3, 5, 5), (3, 17, 3), (5, 17, 2)\}$ and $2^z q_2^t + 1$ is prime.
- $n = q_1 q_2^{\frac{7t-9}{11}} (2^z \cdot q_2^t + 1)$, where q_1, q_2 are Fermat primes and z, t are positive integers with $(q_1, q_2, z) \in \{(3, 5, 3), (3, 17, 1)\}$ and $2^z q_2^t + 1$ is prime.
- $n = (2^x + 1) (2^y q + 1)^{5t-3} (2^z (2^y q + 1)^t q + 1)$, where $q, 2^x + 1, 2^y q + 1, 2^z (2^y q + 1)^t q + 1$ are prime numbers.

- $n = (2^x + 1)(2^y q^2 + 1)^{5t-3} (2^z (2^y q^2 + 1)^t + 1)$, where q , $2^x + 1$, $2^y q^2 + 1$ and $2^z (2^y q^2 + 1)^t + 1$ are prime numbers.
- $n = 3 \cdot 5^{5t-3} (2 \cdot 5^t q^2 + 1)$, where q and $2 \cdot 5^t q^2$ are prime numbers.
- $n = 3 \cdot 5^{\frac{5t-9}{13}} (2 \cdot 5^t + 1)$, where $2 \cdot 5^t + 1$ is prime.

proof. Let $n = q_1 q_2^b q_3$, where $b \geq 2$ and $3 \leq q_1 < q_2 < q_3$. Assume further that $n \in \mathbb{S}$. It follows that

$$9(2b + 1) = \tau((q_1 - 1)(q_2 - 1)(q_3 - 1)q_2^{b-1}). \quad (3.15)$$

There are two cases:

Case 1. Assume that $((q_3 - 1), q_2) = 1$. By (3.15), we have

$$9 = (\tau((q_1 - 1)(q_2 - 1)(q_3 - 1)) - 18)b. \quad (3.16)$$

We distinguish the following subcases:

Case 1.1. $b = 9$. From (3.16), $\tau((q_1 - 1)(q_2 - 1)(q_3 - 1)) = 19$. We must have

$$(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^{18},$$

and hence q_1, q_2 and q_3 are Fermat numbers. This is impossible.

Case 1.2. $b = 3$. By (3.16), $\tau((q_1 - 1)(q_2 - 1)(q_3 - 1)) = 21$. If $(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^{20}$, then q_1, q_2 and q_3 are Fermat numbers, which is impossible. Thus, we must have

$$(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^6 q^2,$$

where q is odd prime with $(q_2, q) = 1$. By Lemma 3.1, $8q^2 + 1$ and $2q^2 + 1$ are composite and by Lemma 3.3, $2q + 1$ and $2^4 q + 1$ are not simultaneously primes. A similar argument holds for $2^2 q + 1$ and $2^3 q + 1$, so we conclude that n is of the form:

$$n = 5(2q + 1)^3(8q + 1),$$

where $2q + 1$ and $8q + 1$ are simultaneously primes. For example, if $q = 5$ then $n = 5 \cdot 11^3 \cdot 41$.

Case 2. Assume that $((q_3 - 1), q_2) = q_2$. We put $q_1 - 1 = 2^x m_1$, $q_2 - 1 = 2^y m_2$ and $q_3 - 1 = 2^z q_2^t m_3$, where $x, y, z \geq 1, t \geq 1$ and $(2q_2, m_1 m_2 m_3) = 1$. Put $m = m_1 m_2 m_3$. It follows from (3.15) that

$$(18 - \tau(m)(x + y + z + 1))b = (\tau(m)(x + y + z + 1)t - 9). \quad (3.17)$$

Note that $\tau(m)(x + y + z + 1)$ cannot ≥ 18 . Thus, we distinguish the following subcases:

Case 2.1. $\tau(m)(x+y+z+1) = 17$. Then $\tau(m) = 1$, and so $x+y+z = 16$. Or equivalently, $m_1 = m_2 = m_3 = 1$, q_1 and q_2 are Fermat numbers and by (3.17), $b = 17t - 9$. Therefore, $n = q_1 q_2^{17t-9} (2^z q_2^t + 1)$, where $2^z q_2^t + 1$ is prime with

$$(q_1, q_2, z) \in \{(3, 5, 13), (3, 17, 11), (3, 257, 7), (5, 17, 10), (5, 257, 6), (17, 257, 4)\}.$$

For example, for $x = 1, y = 2, z = 13, t = 1$ we have $n = 3 \cdot 5^8 \cdot 40961$.

Case 2.2. $\tau(m)(x+y+z+1) = 15$. We will consider separately the two cases $\tau(m) = 1$ and $\tau(m) = 3$.

- When $\tau(m) = 1$. Then $x+y+z = 14$, q_1 and q_2 are Fermat numbers and by (3.17), $b = 5t - 3$. Thus, $n = q_1 q_2^{5t-3} (2^z \cdot q_2^t + 1)$, where $2^z \cdot q_2^t + 1$ is prime with

$$(q_1, q_2, z) \in \{(3, 5, 11), (3, 17, 9), (3, 257, 5), (5, 17, 8), (5, 257, 4), (17, 257, 2)\}.$$

For example, for $x = 1, y = 2, z = 11$, we have $t = 15$ which is the first value with this property. That is, $n = 3 \cdot 5^{72} \cdot 62500000000001$.

- When $\tau(m) = 3$. Then $x+y+z = 4$ and $m = q^2$, where q is an odd prime with $(q_2, q) = 1$. Thus, by (3.17), $b = 5t - 3$. Hence,
 - $n = (2^x + 1)(2^y q + 1)^{5t-3} (2^z q_2^t q + 1)$.
For example, for $x = z = 1, y = 2$ and $t = 1$ we have $n = 3 \cdot 13^2 \cdot 79$.
 - $n = (2^x + 1)(2^y q^2 + 1)^{5t-3} (2^z q_2^t + 1)$.
For example, for $x = y = 1, z = 2$ and $t = 3$ we have $n = 3 \cdot 19^{12} \cdot 27437$.
 - $n = (2^x + 1)(2^y + 1)^{5t-3} (2^z q_2^t q^2 + 1)$, we must have $x = z = 1, y = 2$, hence $n = (3)(5)^{5t-3} (2 \cdot 5^t q^2 + 1)$.
For example for $q = 7$ and $t = 1$ we get $n = 3 \cdot 5^2 \cdot 491$.

Case 2.3. $\tau(m)(x+y+z+1) = 13$. Then $\tau(m) = 1$ and $x+y+z = 12$. Thus, $m_1 = m_2 = m_3 = 1$ and q_1, q_2 are Fermat primes. By (3.17), $5b = 13t - 9$. That is,

$$n = q_1 q_2^{(13t-9)/5} (2^z \cdot q_2^t + 1),$$

where $5 \mid 13t - 9$ and $2^z \cdot q_2^t + 1$ is prime and we have

$$(q_1, q_2, z) \in \{(3, 5, 9), (3, 17, 7), (3, 257, 5), (5, 17, 6), (5, 257, 2)\}.$$

Case 2.4. $\tau(m)(x+y+z+1) = 11$. We also have $m = 1$ and $x+y+z = 10$. So, q_1 and q_2 are Fermat numbers and by (3.17), $7b = 11t - 9$. Thus, $n = q_1 q_2^{(11t-9)/7} (2^z \cdot q_2^t + 1)$, where

$7 \mid 11t - 9$ and $2^z \cdot q_2^t + 1$ is prime and

$$(q_1, q_2, z) \in \{(3, 5, 7), (3, 17, 5), (3, 257, 1), (5, 17, 4)\}.$$

For example, if $x = 1, y = 2, z = 7$ and $t = 333$ then $n = 3 \cdot 5^{522} \cdot (2^7 \cdot 5^{333} + 1)$.

Case 2.5. $\tau(m)(x + y + z + 1) = 9$. Here, we must have $m = 1$ and $x + y + z = 8$, from which it follows that q_1 and q_2 are Fermat primes. Thus, by (3.17), $b = t - 1$ and so

$$n = q_1 q_2^{t-1} (2^z \cdot q_2^t + 1),$$

where $2^z \cdot q_2^t + 1$ is prime and $(q_1, q_2, z) \in \{(3, 5, 5), (3, 17, 3), (5, 17, 2)\}$.

For example, for $x = 1, y = 2, z = 5$. For $t = 3$, we get $n = 3 \cdot 5^2 \cdot 4001$.

Case 2.6. $\tau(m)(x + y + z + 1) = 7$. Obviously $m = 1$ and $x + y + z = 6$. By (3.14), $11b = 7t - 9$. That is, $n = q_1 q_2^{(7t-9)/11} (2^z \cdot q_2^t + 1)$, where $11 \mid 7t - 9$ and $2^z \cdot q_2^t + 1$ is prime and we have

$$(q_1, q_2, z) \in \{(3, 5, 3), (3, 17, 1)\}.$$

Case 2.7. $\tau(m)(x + y + z + 1) = 5$. We must have $m = 1$ and $x + y + z = 4$. By (3.17), we deduce that $13b = 5t - 9$, and so $n = 3 \cdot 5^{(5t-9)/13} (2 \cdot 5^t + 1)$, where $13 \mid 5t - 9$ and $2 \cdot 5^t + 1$ is prime. After computation, $t = 3699$ is the first value with this property. That is, $n = 3 \cdot 5^{1422} \cdot (2 \cdot 5^{3699} + 1)$. \square

Proposition 3.11. *The only solutions of the form $q_1 q_2 q_3^c$, where $c \geq 2$ and $3 \leq q_1 < q_2 < q_3$ are: $3 \cdot 5 \cdot 73^3, 3 \cdot 17 \cdot 19^3$ and $5(2q + 1)(8q + 1)^3$, where $q, 2q + 1$ and $8q + 1$ are prime numbers.*

proof. Let $n = q_1 q_2 q_3^c$, where $c \geq 2$ and $3 \leq q_1 < q_2 < q_3$. Since $n \in \mathbb{S}$, then

$$9 = (\tau((q_1 - 1)(q_2 - 1)(q_3 - 1)) - 18) c. \quad (3.18)$$

We distinguish the following two cases:

Case 1. $c = 9$ and $\tau((q_1 - 1)(q_2 - 1)(q_3 - 1)) = 19$. It follows that

$$(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^{18}$$

and so q_1, q_2, q_3 are Fermat numbers, which is impossible.

Case 2. $c = 3$ and $\tau((q_1 - 1)(q_2 - 1)(q_3 - 1)) = 21$. As above, if $(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^{20}$, then q_1, q_2, q_3 are also Fermat number, which is impossible. But, if $(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^6 q^2$, with q is odd prime, then by applying Lemma 3.3, $2q + 1$ and $16q + 1$ are not simultaneously primes (the same for $4q + 1$ and $8q + 1$). Then n is one of the numbers:

- $n = 3 \cdot 5 \cdot (8q^2 + 1)^3$, where $8q^2 + 1$ is prime. By Lemma 3.1, $q = 3$ is the only prime with this property, hence $n = 3 \cdot 5 \cdot 73^3$.
- $n = 3 \cdot 17 \cdot (2q^2 + 1)^3$, where $2q^2 + 1$ is prime. From Lemma 3.1, $q = 3$ is the only prime with this property, hence $n = 3 \cdot 17 \cdot 19^3$.
- $n = 5 \cdot (2q + 1)(8q + 1)^3$, where $2q + 1$ and $8q + 1$ are prime numbers. The first primes with these properties are $q = 5, 11, 29, 131, 179, 239, 431, 491, \dots$

□

3.2.3 n is not square-free with n even

Now, assume that n is **not square-free** with n **is even**. Here, we characterize all even solutions having only one power prime.

Proposition 3.12. *The only possible solutions of the form $2^a pq$, where $a \geq 2$ and $3 \leq p < q$ are:*

- $2^{17(x+y)-9}(2^x m_1 + 1)(2^y m_2 + 1)$, where $2^x m_1 + 1, 2^y m_2 + 1$ are prime numbers with $2^x m_1 < 2^y m_2$ and $m_1 m_2 = r^{16}$ such that r is an odd prime.
- $2^{5(x+y)-3}(2^x m_1 + 1)(2^y m_2 + 1)$, where $2^x m_1 + 1, 2^y m_2 + 1$ are prime numbers with $2^x m_1 < 2^y m_2$ and $m_1 m_2 = r^{14}$ or $m_1 m_2 = r_1^4 r_2^2$ such that r, r_1, r_2 are odd primes.
- $2^{\frac{13(x+y)-9}{5}}(2^x m_1 + 1)(2^y m_2 + 1)$, where $2^x m_1 + 1, 2^y m_2 + 1$ are prime numbers with $2^x m_1 < 2^y m_2$ and $m_1 m_2 = r^{12}$ such that r is an odd prime.
- $2^{\frac{11(x+y)-9}{7}}(2^x m_1 + 1)(2^y m_2 + 1)$, where $2^x m_1 + 1, 2^y m_2 + 1$ are prime numbers with $2^x m_1 < 2^y m_2$ and $m_1 m_2 = r^{10}$ such that r is an odd prime.
- $2^{(x+y)-1}(2^x m_1 + 1)(2^y m_2 + 1)$, where $2^x m_1 + 1, 2^y m_2 + 1$ are prime numbers with $2^x m_1 < 2^y m_2$ and $m_1 m_2 = r^8$ or $m_1 m_2 = r_1^2 r_2^2$ such that r, r_1, r_2 are odd primes.
- $2^{\frac{7(x+y)-9}{11}}(2^x m_1 + 1)(2^y m_2 + 1)$, where $2^x m_1 + 1, 2^y m_2 + 1$ are prime numbers with $2^x m_1 < 2^y m_2$ and where $m_1 m_2 = r^6$ such that r is an odd prime.
- $2^{\frac{5(x+y)-9}{13}}(2^x m_1 + 1)(2^y m_2 + 1)$, where $2^x m_1 + 1, 2^y m_2 + 1$ are prime numbers with $2^x m_1 < 2^y m_2$ and $m_1 m_2 = r^4$, such that r is an odd prime.
- $2^{\frac{(x+y)-3}{5}}(2^x m_1 + 1)(2^y m_2 + 1)$, where $2^x m_1 + 1, 2^y m_2 + 1$ are prime numbers with $2^x m_1 < 2^y m_2$ and $m_1 m_2 = r^2$ such that r is an odd prime.
- $2^{\frac{(x+y)-1}{9}}(2^x + 1)(2^y + 1)$, where $2^x + 1$ and $2^y + 1$ are prime numbers with $x < y$.

proof. Let $n = 2^a pq$, where $a \geq 2$ and $3 \leq p < q$. Since $n \in \mathbb{S}$, then

$$9(2a + 1) = \tau((p - 1)(q - 1)2^{a-1}). \quad (3.19)$$

We put $p - 1 = 2^x m_1$, $q - 1 = 2^y m_2$, where $x, y \geq 1$ and $(2, m_1 m_2) = 1$, it follow from (3.19) that

$$(18 - \tau(m_1 m_2))a = \tau(m_1 m_2)(x + y) - 9. \quad (3.20)$$

We observe that $\tau(m_1 m_2)$ is odd and cannot be ≥ 18 , so we distinguish the following cases:

Case 1. $\tau(m_1 m_2) = 17$. It follows that $m_1 m_2 = r^{16}$, where $r \geq 3$ is prime and so, by (3.20), $a = 17(x + y) - 9$. Then $n = 2^{17(x+y)-9}(2^x m_1 + 1)(2^y m_2 + 1)$, where $2^x m_1 + 1$ and $2^y m_2 + 1$ are prime numbers with $2^x m_1 < 2^y m_2$.

For example, for $m_1 = r$, $m_2 = r^{15}$, $y = 2$, $x = 2$ and $r = 3$, we get $n = 2^{59} \cdot 13 \cdot 57395629$.

Case 2. $\tau(m_1 m_2) = 15$. It follows that $m_1 m_2 = r^{14}$ or $m_1 m_2 = r_1^4 r_2^2$, where r_1 and r_2 are distinct odd primes and by (3.20), $a = 5(x + y) - 3$. Therefore,

$$n = 2^{5(x+y)-3}(2^x m_1 + 1)(2^y m_2 + 1),$$

where $2^x m_1 + 1$ and $2^y m_2 + 1$ are prime numbers with $2^x m_1 < 2^y m_2$.

For example, for $m_1 = 5^2$, $m_2 = 3^4$, $x = 2$, $y = 1$, $r_1 = 5$ and $r_2 = 3$ we have $n = 2^{12} \cdot 101 \cdot 163$.

Case 3. $\tau(m_1 m_2) = 13$. It follows that $m_1 m_2 = r^{12}$, where $r \geq 3$ is prime and by (3.20), $5a = 13(x + y) - 9$. Thus we obtain $n = 2^{(13(x+y)-9)/5}(2^x m_1 + 1)(2^y m_2 + 1)$, where $5|13(x + y) - 9$, $2^x m_1 + 1$ and $2^y m_2 + 1$ are prime numbers with $2^x m_1 < 2^y m_2$.

For example, for $x = 1$, $y = 2$ and $m_1 = m_2 = 3^6$ we obtain $n = 2^6 \cdot 1459 \cdot 2917$.

Case 4. $\tau(m_1 m_2) = 11$. Therefore, $m_1 m_2 = r^{10}$, where $r \geq 3$ is prime. From (3.20), $7a = 11(x + y) - 9$. Hence, $n = 2^{(11(x+y)-9)/7}(2^x m_1 + 1)(2^y m_2 + 1)$, where $7|11(x + y) - 9$, $2^x m_1 + 1$ and $2^y m_2 + 1$ are prime numbers with $2^x m_1 < 2^y m_2$.

For example, for $x = 2$, $y = 16$, $m_1 = 3^6$ and $m_2 = 3^4$ we obtain $n = 2^{27} \cdot 2917 \cdot 5308417$.

Case 5. $\tau(m_1 m_2) = 9$. It follows that $m_1 m_2 = r^8$ or $m_1 m_2 = r_1^2 r_2^2$, where r_1 and r_2 are distinct odd primes. By (3.20), $a = (x + y) - 1$. Hence, $n = 2^{(x+y)-1}(2^x m_1 + 1)(2^y m_2 + 1)$, where $2^x m_1 + 1$ and $2^y m_2 + 1$ are prime numbers with $2^x m_1 < 2^y m_2$.

For example, for $x = 1$, $y = 4$ and $m_1 = m_2 = 3^4$ we have $n = 2^4 \cdot 163 \cdot 1297$. Also, for $x = 1$, $y = 2$, $m_1 = 3^2$ and $m_2 = 5^2$ we get $n = 2^2 \cdot 19 \cdot 101$.

Case 6. $\tau(m_1m_2) = 7$. That is, $m_1m_2 = r^6$, where $r \geq 3$ is prime, and by (3.20), $11a = 7(x+y) - 9$. Hence, $n = 2^{(7(x+y)-9)/11}(2^xm_1 + 1)(2^ym_2 + 1)$, where $11|7(x+y) - 9$, $(2^xm_1 + 1)$ and $(2^ym_2 + 1)$ are prime numbers with $2^xm_1 < 2^ym_2$.
For example, for $x = 2, y = 4, m_1 = 3^2$ and $m_2 = 3^4$ we get $n = 2^3 \cdot 37 \cdot 1297$.

Case 7. $\tau(m_1m_2) = 5$. Then $m_1m_2 = r^4$, where $r \geq 3$ is prime and by (3.20), $13a = 5(x+y) - 9$. Hence, $n = 2^{(5(x+y)-9)/13}(2^xm_1 + 1)(2^ym_2 + 1)$, where $13|5(x+y) - 9$, $(2^xm_1 + 1)$ and $(2^ym_2 + 1)$ are prime numbers with $2^xm_1 < 2^ym_2$.
For example, for $x = 1, y = 6$ and $m_1 = m_2 = 3^2$ we have $n = 2^2 \cdot 19 \cdot 577$.

Case 8. $\tau(m_1m_2) = 3$. Then $m_1m_2 = r^2$, where $r \geq 3$ is prime. By (3.20), $5a = (x+y) - 3$. Hence, $n = 2^{((x+y)-3)/5}(2^xm_1 + 1)(2^ym_2 + 1)$, where $5|(x+y) - 3$, $2^xm_1 + 1$ and $2^ym_2 + 1$ are prime numbers with $2^xm_1 < 2^ym_2$.
For example, for $x = 1, y = 12$ and $m_1 = 1, m_2 = 11^2$ we have $n = 2^2 \cdot 3 \cdot 495617$.

Case 9. $\tau(m_1m_2) = 1$. That $m_1 = m_2 = 1$ and by (3.20), $9a = (x+y) - 1$. Hence,

$$n = 2^{(x+y-1)/9}(2^x + 1)(2^y + 1),$$

where $9|(x+y) - 1$, $(2^x + 1)$ and $(2^y + 1)$ are Fermat primes with $2^x + 1 < 2^y + 1$. \square

Proposition 3.13. *The only possible solutions of the form $2p^bq$, where $b \geq 2$ and $3 \leq p < q$ are:*

- $2F_1^9F_4, 2F_2^3F_4, 2 \cdot 19^3 \cdot 163$.
- $2 \cdot 5^3(2^4r^2 + 1)$, where r and $2^4r^2 + 1$ are prime numbers.
- $2 \cdot 17^3(2^2r^2 + 1)$, where r and $2^2r^2 + 1$ are prime numbers.
- $2(2r + 1)^3(2^5r + 1)$, where $r, 2r + 1$ and $2^5r + 1$ are prime numbers.
- $2(2r + 1)^3(2r^5 + 1)$, where $r, 2r + 1$ and $2r^5 + 1$ are prime numbers.
- $2(2^x + 1)^{17t-9}(2^y(2^x + 1)^t + 1)$, where $2^x + 1, 2^y(2^x + 1)^t + 1$ are prime numbers and x, y are positive integers with $x + y = 16$.
- $2(2^x + 1)^{5t-3}(2^y(2^x + 1)^t + 1)$, where $2^x + 1, 2^y(2^x + 1)^t + 1$ are prime numbers and x, y are positive integers with $x + y = 14$.
- $2 \cdot 3^{5t-3}(2^3 \cdot 3^t \cdot r^2 + 1)$, where r and $2^3 \cdot 3^t \cdot r^2 + 1$ are prime numbers with $(3, r) = 1$.
- $2 \cdot 5^{5t-3}(2^2 \cdot 5^t \cdot r^2 + 1)$, where r and $2^3 \cdot 5^t \cdot r^2 + 1$ are prime numbers with $(5, r) = 1$.
- $2(2r + 1)^{5t-3}(2^3r(2r + 1)^t + 1)$, where $r, 2r + 1$ and $2^3r(2r + 1)^t + 1$ are prime numbers.

- $2(4r + 1)^{5t-3}(2^2r(4r + 1)^t + 1)$, where r , $4r + 1$ and $2^2r(4r + 1)^t + 1$ are prime numbers.
- $2(8r + 1)^{5t-3}(2r(8r + 1)^t + 1)$, where r , $8r + 1$ and $2r(8r + 1)^t + 1$ are prime numbers.
- $2(4r^2 + 1)^{5t-3}(4(4r^2 + 1)^t + 1)$, where r , $4r^2 + 1$ and $4(4r^2 + 1)^t + 1$ are prime numbers.
- $2 \cdot 3^{5t-3}(2 \cdot 3^t \cdot r^4 + 1)$, where r and $2 \cdot 3^t \cdot r^4 + 1$ are prime numbers with $(3, r) = 1$.
- $2(2r + 1)^{5t-3}(2r^3(2r + 1)^t + 1)$, where r , $2r + 1$ and $2r^3(2r + 1)^t + 1$ are prime numbers.
- $2 \cdot 19^{5t-3}(2 \cdot 19^t r^2 + 1)$, where r and $2 \cdot 19^t r^2 + 1$ are prime numbers with $(19, r) = 1$.
- $2(2r^3 + 1)^{5t-3}(2r(2r^3 + 1)^t + 1)$, where r , $2r^3 + 1$ and $2r(2r^3 + 1)^t + 1$ are prime numbers.
- $2(2r^4 + 1)^{5t-3}(2(2r^4 + 1)^t + 1)$, where r , $2r^4 + 1$ and $2(2r^4 + 1)^t + 1$ are prime numbers.
- $2(2^x + 1)^{\frac{13t-9}{5}}(2^y(2^x + 1)^t + 1)$, where $2^x + 1$, $2^y(2^x + 1)^t + 1$ are prime numbers and x, y, t are positive integers with $x + y = 12$.
- $2(2^x + 1)^{\frac{11t-9}{7}}(2^y(2^x + 1)^t + 1)$, where $2^x + 1$, $2^y(2^x + 1)^t + 1$ are prime numbers and x, y, t are positive integers with $x + y = 10$.
- $2(2^x + 1)^{t-1}(2^y(2^x + 1)^t + 1)$, where $2^x + 1$, $2^y(2^x + 1)^t + 1$ are prime numbers and x, y, t are positive integers with $x + y = 8$.
- $2 \cdot 3^{t-1}(2 \cdot 3^t r^2 + 1)$ where r and $2 \cdot 3^t r^2 + 1$ are prime numbers with $(3, r) = 1$.
- $2(2r + 1)^{t-1}(2r(2r + 1)^t + 1)$, where r , $2r + 1$ and $2r(2r + 1)^t + 1$ are prime numbers.
- $2(2^x + 1)^{\frac{7t-9}{11}}(2^y(2^x + 1)^t + 1)$ where $2^x + 1$, $2^y(2^x + 1)^t + 1$ are prime numbers and x, y, t are positive integers with $x + y = 6$.
- $2(2^x + 1)^{\frac{5t-9}{13}}(2^y(2^x + 1)^t + 1)$, where $2^x + 1$, $2^y(2^x + 1)^t + 1$ are prime numbers and x, y, t are positive integers with $x + y = 4$.

proof. Let $n = 2p^bq$, where $b \geq 2$ and $3 \leq p < q$. Since $n \in \mathbb{S}$, we have

$$9(2b + 1) = \tau((p - 1)(q - 1)p^{b-1}) \quad (3.21)$$

Case 1. Assume that $(q - 1, p) = 1$. It follows from (3.21) that $9 = (\tau((p - 1)(q - 1)) - 18)b$.

We distinguish the following subcases:

Case 1.1. $b = 9$ and $\tau((p - 1)(q - 1)) = 19$. Thus we must have $(p - 1)(q - 1) = 2^{18}$ and p, q are Fermat primes. Hence, $n = 2F_1^9F_4$.

Case 1.2. $b = 3$ and $\tau((p - 1)(q - 1)) = 21$. Here we have the following possibilities:

- $(p - 1)(q - 1) = 2^{20}$ and p, q are Fermat number. As above, $n = 2F_2^3F_4$.
- $(p - 1)(q - 1) = 2^6r^2$, where $r \geq 3$ is prime. By Lemma 3.1, $2^2 \cdot r + 1$ and $2^4 \cdot r^2 + 1$ cannot be simultaneously primes. Thus, n is one of the numbers:
 - $n = 2 \cdot 5^3(2^4r^2 + 1)$, where $2^4 \cdot r^2 + 1$ is prime.
For example, for $r = 29$ we have $n = 2 \cdot 5^3 \cdot 13457$.
 - $n = 2 \cdot 17^3(2^2r^2 + 1)$, where $2^2 \cdot r^2 + 1$ is prime.
For example, for $r = 3$ we have $n = 2 \cdot 17^3 \cdot 37$.
 - $n = 2(2r + 1)^3(2^5r + 1)$, where $2 \cdot r + 1$ and $2^5 \cdot r + 1$ are both prime.
For example, for $r = 3$ we have $n = 2 \cdot 7^3 \cdot 97$.
- $(p - 1)(q - 1) = 2^2r^6$, where $r \geq 3$ is prime with $(r, q) = 1$. Thus, n is one of the numbers:
 - $n = 2(2 \cdot r + 1)^3(2 \cdot r^5 + 1)$, where $2 \cdot r + 1$ and $2r^5 + 1$ are both prime.
For example, for $r = 3$ we get $n = 2 \cdot 7^3 \cdot 487$.
 - $n = 2(2 \cdot r^2 + 1)^3(2 \cdot r^4 + 1)$, where $2 \cdot r^2 + 1$ and $2 \cdot r^4 + 1$ are both prime.
By Lemma 3.1, $r = 3$ is the only solution for this case, hence we get $n = 2 \cdot 19^3 \cdot 163$.

Case 2. Assume that $(q - 1, p) = p$. We put $p - 1 = 2^x m_1$ and $q - 1 = 2^y p^t m_2$, where $x, y \geq 1$, $t \geq 1$ and $(2p, m_1 m_2) = 1$. Let $m = m_1 m_2$, it follows from (3.21) that

$$(18 - \tau(m)(x + y + 1))b = (\tau(m)(x + y + 1)t - 9). \quad (3.22)$$

We observe that $\tau(m)(x + y + 1)$ is odd and cannot be ≥ 18 , so we have the following possibilities:

Case 2.1. $\tau(m)(x + y + 1) = 17$. That is, $m = 1$ and $x + y = 16$. So, p is a Fermat prime. By (3.22), $b = 17t - 9$ and therefore $n = 2(2^x + 1)^{17t-9}(2^y(2^x + 1)^t + 1)$, where $(2^x + 1)$ and $(2^y(2^x + 1)^t + 1)$ are prime numbers.

For example, for $x = 1$, $t = 4$ and $y = 15$ we have $n = 2 \cdot 3^{59} \cdot 2654209$.

Case 2.2. $\tau(m)(x + y + 1) = 15$. There are three possibilities:

- $\tau(m) = 1$ and $x + y = 14$. So, $m_1 = m_2 = 1$ and p is a Fermat prime. By (3.22), $b = 5t - 3$, in which case $n = 2(2^x + 1)^{5t-3}(2^y(2^x + 1)^t + 1)$, where $2^x + 1$ and $2^y(2^x + 1)^t + 1$ are prime numbers.
For example, for $x = 1$, $y = 13$, $t = 5$ we have $n = 2 \cdot 3^{22} \cdot 1990657$.
- $\tau(m) = 3$ and $x + y = 4$. Therefore, $m = r^2$, where $r \geq 3$ is prime. From (3.22), $b = 5t - 3$. By Lemma 3.1, $2 \cdot 73^t + 1$ is composite. Also, by Remark 3.1, the number $2^3 \cdot 19^t + 1$ is composite. Thus, n is one of the numbers:

- $n = 2 \cdot 3^{5t-3}(2^3 \cdot 3^t \cdot r^2 + 1)$, where $2^3 \cdot 3^t \cdot r^2 + 1$ is prime.
For example, for $r = 7$ and $t = 2$ we obtain $n = 2 \cdot 3^7 \cdot 3529$.
- $n = 2 \cdot 5^{5t-3}(2^2 \cdot 5^t \cdot r^2 + 1)$, where $2^2 \cdot 5^t \cdot r^2 + 1$ is prime.
For example, for $r = 7$ and $t = 4$ we obtain $n = 2 \cdot 5^{17} \cdot 122501$.
- $n = 2(2r + 1)^{5t-3}(2^3 \cdot r \cdot (2r + 1)^t + 1)$, where $2r + 1$ and $2^3 \cdot r \cdot (2r + 1)^t + 1$ are prime numbers.
For example, for $r = t = 3$ we have $n = 2 \cdot 7^{12} \cdot 8233$.
- $n = 2(4r + 1)^{5t-3}(2^2 \cdot r \cdot (4r + 1)^t + 1)$, where $4r + 1$ and $2^2 \cdot r \cdot (4r + 1)^t + 1$ are prime numbers.
For example, for $r = 3$ and $t = 1$ we have $n = 2 \cdot 13^2 \cdot 157$.
- $n = 2(8r + 1)^{5t-3}(2 \cdot r \cdot (8r + 1)^t + 1)$, where $8r + 1$ and $2 \cdot r \cdot (8r + 1)^t + 1$ are prime numbers.
For example, for $r = 5$ and $t = 2$ we get $n = 2 \cdot 41^7 \cdot 16811$.
- $n = 2(4r^2 + 1)^{5t-3}(4 \cdot (4r^2 + 1)^t + 1)$, where $4r^2 + 1$ and $4 \cdot r \cdot (4r^2 + 1)^t + 1$ are prime numbers.
For example, for $r = 7$ and $t = 6$, we get $n = 2 \cdot 197^{27} \cdot 233806913236517$.
- $\tau(m) = 5$ and $x = y = 1$. Thus $m = r^4$, where $r \geq 3$ is prime. It follows from (3.22) that $b = 5t - 3$. Therefore, by Lemma 3.1, n is one of the numbers:
 - $n = 2 \cdot 3^{5t-3}(2 \cdot 3^t \cdot r^4 + 1)$, where $2 \cdot 3^t \cdot r^4 + 1$ is prime.
For example, if $r = 5$ and $t = 2$ then $n = 2 \cdot 3^7 \cdot 11251$.
 - $n = 2(2r + 1)^{5t-3}(2 \cdot r^3 \cdot (2r + 1)^t + 1)$, where $2r + 1$ and $2 \cdot r^3 \cdot (2r + 1)^t + 1$ are prime numbers.
For example, for $r = 3$ and $t = 1$ we have $n = 2 \cdot 7^2 \cdot 379$.
 - $n = 2 \cdot 19^{5t-3}(2 \cdot 19^t \cdot r^2 + 1)$, where $2 \cdot 19^t \cdot r^2 + 1$ is prime.
For example, if $r = 3$ and $t = 29$, then

$$n = 2 \cdot 19^{142} \cdot 218336795902605993201009018384568383223.$$

- $n = 2(2r^3 + 1)^{5t-3}(2 \cdot r \cdot (2r^3 + 1)^t + 1)$, where $2r^3 + 1$ and $2 \cdot r \cdot (2r^3 + 1)^t + 1$ are prime numbers.

For example, if $r = 5$ and $t = 12$, then

$$n = 2 \cdot 251^{57} \cdot 625294570645574159995353780011.$$

- $n = 2(2r^4 + 1)^{5t-3}(2 \cdot (2r^4 + 1)^t + 1)$, where $2r^4 + 1$ and $2 \cdot (2r^4 + 1)^t + 1$ are prime numbers.

Case 2.3. $\tau(m)(x + y + 1) = 13$. It follows that $m = 1$ and $x + y = 12$, which gives that p is a

Fermat prime. By (3.22), $5b = 13t - 9$ and so $n = 2(2^x + 1)^{(13t-9)/5}(2^y(2^x + 1)^t + 1)$, where $2^x + 1$ and $2^y(2^x + 1)^t + 1$ are primes, with $5|13t - 9$.

Case 2.4. $\tau(m)(x + y + 1) = 11$. Obviously, $m = 1$ and $x + y = 12$. So, p is a Fermat prime. By (3.22), $7b = 11t - 9$ and therefore $n = 2(2^x + 1)^{(11t-9)/7}(2^y(2^x + 1)^t + 1)$, where $2^x + 1$ and $2^y(2^x + 1)^t + 1$ are both prime, with $7|11t - 9$.

Case 2.5. $\tau(m)(x + y + 1) = 9$. There are two possibilities to consider:

- $\tau(m) = 1$ and so $x + y = 8$. Thus, $m_1 = m_2 = 1$ and q_2 is a Fermat prime By (3.22), $b = t - 1$, from which it follows that $n = 2(2^x + 1)^{t-1}(2^y(2^x + 1)^t + 1)$, where $2^x + 1$ and $2^y(2^x + 1)^t + 1$ are prime.

For example, for $x = 2$, $y = 6$ and $t = 14$ we have $n = 2 \cdot 5^{13} \cdot 390625000001$.

- $\tau(m) = 3$ and $x + y = 2$. So, $m_1 m_2 = r^2$, where r is odd prime and $(p, r) = 1$, $x = y = 1$ and $b = t - 1$. Therefore, by Lemma 3.1, it follows that n is one of the numbers:

○ $n = 2 \cdot 3^{t-1}(2 \cdot 3^t \cdot r^2 + 1)$, where $2 \cdot 3^t \cdot r^2 + 1$ is prime.

For example, for $r = 5$ and $t = 4$ we get $n = 2 \cdot 3^3 \cdot 4051$.

○ $n = 2(2r + 1)^{t-1}(2 \cdot r \cdot (2r + 1)^t + 1)$, where $2r + 1$ and $2 \cdot r \cdot (2r + 1)^t + 1$ are prime numbers.

For example for $r = 3$ and $t = 4$, we have $n = 2 \cdot 7^3 \cdot 14407$.

○ $n = 2 \cdot 19^{t-1}(2 \cdot 19^t + 1)$. By Lemma 3.1, the number $2 \cdot 19^t + 1$ is divisible by 3.

Case 2.6. $\tau(m)(x + y + 1) = 7$, it follows that $\tau(m) = 1$, and $(x + y + 1) = 7$. So, $m_1 = m_2 = 1$, p is Fermat numbers, $x + y = 6$. From (3.22), $11b = 7t - 9$ and hence

$$n = 2(2^x + 1)^{(7t-9)/11}(2^y(2^x + 1)^t + 1),$$

where $2^x + 1$ and $2^y(2^x + 1)^t + 1$ are prime numbers, with $11|7t - 9$.

For example, for $x = 4$, $y = 2$ and $t = 6$ we have $n = 2 \cdot 17^3 \cdot 96550277$.

Case 2.7. $\tau(m)(x + y + 1) = 5$. It follows that $m_1 = m_2 = 1$ and $x + y = 4$. So, p is a Fermat prime and by (3.22), $13b = 5t - 9$. Therefore, $n = 2(2^x + 1)^{(5t-9)/13}(2^y(2^x + 1)^t + 1)$, where $(2^x + 1)$ and $(2^y(2^x + 1)^t + 1)$ are both prime with $13|5t - 9$.

For example, for $x = 1$, $y = 3$ and $t = 7$ we have $n = 2 \cdot 3^2 \cdot 17497$.

□

Proposition 3.14. *The only possible solutions of the form $2pq^c$, with $c \geq 2$ and $3 \leq p < q$ are:*

- $2F_1F_4^9, 2F_2F_4^3, 2 \cdot 3 \cdot 1459^3, 2 \cdot 19 \cdot 163^3$.
- $2 \cdot 5 \cdot (2^4r^2 + 1)^3$, where r and $2^4r^2 + 1$ are odd primes.
- $2 \cdot 17 \cdot (2^2r^2 + 1)^3$, where r and $2^2r^2 + 1$ are odd primes.
- $2(4r + 1)(2^4r + 1)^3$, where $r, 4r + 1$ and $2^4r + 1$ are odd primes.
- $2(2r + 1)(2r^5 + 1)^3$, where $r, 2r + 1$ and $2r^5 + 1$ are odd primes.

proof. Let $n = 2pq^c$ where $c \geq 2$ and $3 \leq p < q$. Since $n \in \mathbb{S}$, we have

$$9 = (\tau((p-1)(q-1)) - 18)c.$$

We distinguish the following cases:

Case 1. $\tau((p-1)(q-1)) = 19$ and $c = 9$. It follows that $(p-1)(q-1) = 2^{18}$ and so p, q are Fermat primes. Then $n = 2F_1F_4^9$ is the only solution.

Case 2. $\tau((p-1)(q-1)) = 21$ and $c = 3$. Here $(p-1)(q-1)$ is either $2^{20}, 2^6 \cdot r^2$ or $2^2 \cdot r^6$ where $r \geq 3$ is prime. We study these subcases separately.

Case 2.1. $(p-1)(q-1) = 2^{20}$. Then p, q are Fermat primes, in which case $n = 2F_2F_4^3$.

Case 2.2. $(p-1)(q-1) = 2^6 \cdot r^2$. Then n is one of the numbers:

- $n = 2 \cdot 5(2^4 \cdot r^2 + 1)^3$, where $2^4 \cdot r^2 + 1$ is also prime.
For example, for $r = 5$ we have $n = 2 \cdot 5 \cdot 401^3$.
- $n = 2 \cdot 17(2^2 \cdot r^2 + 1)^3$, where $2^2 \cdot r^2 + 1$ is prime.
For example, for $r = 3$ we get $n = 2 \cdot 17 \cdot 37^3$.
- $n = 2(2^2 \cdot r + 1)(2^4 \cdot r + 1)^3$, where $2^2 \cdot r + 1$ and $2^4 \cdot r + 1$ are prime numbers.
For example, for $r = 7$ we get $n = 2 \cdot 29 \cdot 113^3$.

Case 2.3. $(p-1)(q-1) = 2^2 \cdot r^6$. Then n is one of the numbers:

- $n = 2 \cdot 3(2 \cdot r^6 + 1)^3$, where $2 \cdot r^6 + 1$ is prime. By Lemma 3.1, $r = 3$ is the only possible value, i.e., $n = 2 \cdot 3 \cdot 1459^3$.
- $n = 2(2 \cdot r^2 + 1)(2 \cdot r^4 + 1)^3$, where $2 \cdot r^2 + 1$ and $2 \cdot r^4 + 1$ are prime numbers. By Lemma 3.1, we get $r = 3$ and so $n = 2 \cdot 19 \cdot 163^3$.
- $n = 2(2 \cdot r + 1)(2 \cdot r^5 + 1)^3$, where $r, 2 \cdot r + 1$ and $2 \cdot r^5 + 1$ are prime numbers.
The first primes r with these properties are $r = 3, 23, 29, 53, 251, 443, 953, \dots$

□

3.3 Are there infinitely many n such that $\tau(n^2) = \tau(\varphi(n))$?

The crucial question that remains: Is the set \mathbb{S} infinite? The answer to this question seems difficult because we have, in the previous section, a system of polynomials in which, for a given prime p , each polynomial must take in p a value which is also a prime number.

Recall that Dickson's Conjecture was formulated by Leonard Dickson in [10]: *Let $s \geq 1$ and let $f_i(x) = a_i \cdot x + b_i$ with a_i, b_i integers, $b_i \geq 1$ for $i = 1, 2, \dots, s$. If there does not exist any integer $n > 1$ dividing all the products $f_1(k) f_2(k) \cdots f_s(k)$, for every integer k , then there exist infinitely many natural numbers m such that all numbers $f_1(m), f_2(m), \dots, f_s(m)$ are prime numbers.*

As in [7], let us take the system of integer valued polynomials whose leading coefficients are positive:

$$\begin{cases} f_1(x) = x, \\ f_2(x) = 4x + 1, \\ f_3(x) = 16x + 1. \end{cases}$$

Assume further that there exists an integer $n > 1$ which is a common divisor for the integers

$$f_1(k) f_2(k) f_3(k), \text{ for } k \in \mathbb{Z}.$$

That is, n is a common divisor for the integers $k(4k+1)(16k+1)$, $k \in \mathbb{Z}$. Then n is a common divisor for the integers $(n+1)(4n+5)(16n+17)$. Since $n \nmid (n+1)$, n divide $4n+5$ or n divide $16n+17$. This means that $n = 5$ or $n = 17$. But either $n = 5$ or $n = 17$ does not divide $f_1(2) f_2(2) f_3(2) = 2 \cdot 3^3 \cdot 11$. So there is no integer $n > 1$ which is a common divisor for the integers $f_1(k) f_2(k) f_3(k)$, $k \in \mathbb{Z}$. Consequently, we have the following result:

Theorem 3.2. *Assuming Dickson's conjecture, there exist infinitely many primes p such that $4p+1$ and $16p+1$ are prime numbers.*

Corollary 3.1. There exist infinitely many positive integers n such that $n \in \mathbb{S}$.

proof. Recall that the integer $n = 5(4p+1)(16p+1)$ with p , $4p+1$ and $16p+1$ primes are in \mathbb{S} . Since, by the above theorem there exist infinitely many primes p such that $4p+1$ and $16p+1$ are prime numbers. Thus, we have infinitely many integers $n = 5(4p+1)(16p+1) \in \mathbb{S}$. \square

We also use Dickson's conjecture to create families of prime numbers

$$\begin{cases} f_1(x) = a_1x + 1 \\ f_2(x) = a_2x + 1 \\ \vdots \\ f_s(x) = a_sx + 1 \end{cases}$$

where a_1, \dots, a_s are positive integers. We can easily check that the above polynomials verify Dickson's hypothesis. Indeed, suppose that there exists an integer $n > 1$ such that n is a common divisor of all the integers $f_1(k) f_2(k) \cdots f_s(k)$, $k \in \mathbb{Z}$. Then $n \mid f_1(0) f_2(0) \cdots f_s(0)$, i.e., $n \mid 1$ which implies that $n = 1$. Then there exist infinitely many positive integers n such that $f_1(n), f_2(n), \dots, f_s(n)$ are simultaneously primes.

3.4 Miscellaneous examples

In the following, we present some examples of solutions that cannot be deduced from the previous theorems and propositions.

Example 3.1. The set \mathbb{S} contains the following numbers:

- $n = F_1^a \cdot F_2^b \cdot F_3$, $n = F_1^a \cdot F_2 \cdot F_3^b$ and $n = F_1 \cdot F_2^a \cdot F_3^b$, where F_n is the n -th Fermat prime and $(a, b) = (3, 7)$ or $(7, 3)$.

- $n = F_1^a \cdot F_2^b \cdot F_3 \cdot F_4$, $n = F_1^a \cdot F_2 \cdot F_3^b \cdot F_4$, $n = F_1^a \cdot F_2 \cdot F_3 \cdot F_4^b$, $n = F_1 \cdot F_2^a \cdot F_3^b \cdot F_4$,
 $n = F_1 \cdot F_2^a \cdot F_3 \cdot F_4^b$ and $n = F_1 \cdot F_2 \cdot F_3^a \cdot F_4^b$, where $(a, b) = (3, 7)$.

Example 3.2. We have:

1. Let p, q, r be distinct primes such that $2p + 1$, $4q + 1$ and $2pqr^2 + 1$ are prime numbers. Then $n = 2 \cdot 17 \cdot (2p + 1)(4q + 1)(2pqr^2 + 1) \in \mathbb{S}$.
2. Let q_1, q_1, \dots, q_k be distinct primes such that $4q_1 + 1, \dots, 4q_k + 1$ and $4q_1 \cdots q_k + 1$ are prime numbers for some $k \geq 1$. If $n = 2(4q_1 + 1) \cdots (4q_k + 1)(4q_1 \cdots q_k + 1) \in \mathbb{S}$, then $k = 3$. For example, for $(q_1, q_2, q_3) = (7, 13, 57)$ we get

$$n = 2(4q_1 + 1)(4q_2 + 1)(4q_3 + 1)(4q_1q_2q_3 + 1) = 2 \cdot 29 \cdot 53 \cdot 229 \cdot 20749 \in \mathbb{S}.$$

3. If $n \geq 7$ is the product of safe primes¹, then $n \notin \mathbb{S}$.

Let q_1, q_1, \dots, q_k be Sophie Germain primes for some $k \geq 1$ such that $2q_1 \cdots q_k + 1$ is also prime.

Example 3.3. We have:

- If $n = 2 \cdot (2q_1 + 1) \cdots (2q_k + 1)(2q_1 \cdots q_k + 1) \in \mathbb{S}$, then $k = 7$. For example, if

$$(q_1, q_2, q_3, q_4, q_5, q_6, q_7) = (3, 5, 11, 23, 29, 41, 131),$$

$$\text{then } n = 2 \cdot (2q_1 + 1) \cdots (2q_7 + 1)(2q_1 \cdots q_7 + 1) = 253470367109666245154 \in \mathbb{S}.$$

- If $n = 2 \cdot F_0 \cdot (2q_1 + 1) \cdots (2q_k + 1)(2q_1 \cdots q_k + 1) \in \mathbb{S}$, then $k = 24$. For example, let q_i ($1 \leq i \leq 24$) be the following Sophie Germain primes: 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 191, 233, 239, 251, 281, 293, 359, 719, 1439, 1481, 3413. After computation, the number:

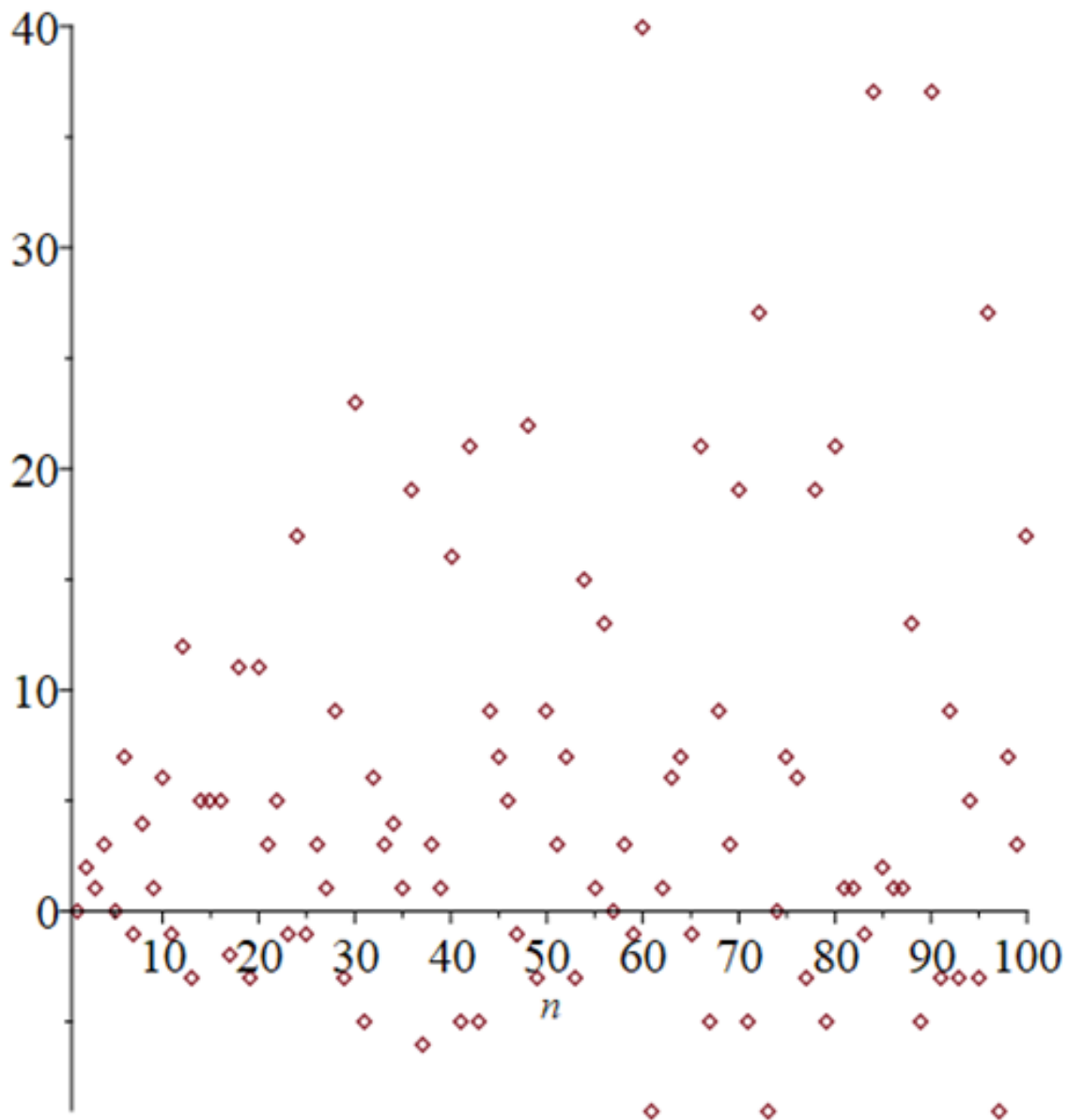
$$\begin{aligned} n &= 2 \cdot F_0 \cdot (2q_1 + 1) \cdots (2q_k + 1)(2q_1 \cdots q_k + 1) \\ &= 2 \cdot 3 \cdot 7 \cdot 11 \cdot 23 \cdot 47 \cdot 59 \cdot 83 \cdot 107 \cdot 167 \cdot 179 \cdot 227 \cdot 263 \\ &\quad \cdot 347 \cdot 359 \cdot 383 \cdot 467 \cdot 479 \cdot 503 \cdot 563 \cdot 587 \cdot 719 \cdot 1439 \cdot 2879 \cdot 2963 \\ &\quad \cdot 6827 \cdot 668385166547574839150402388419262454473804930401971. \end{aligned}$$

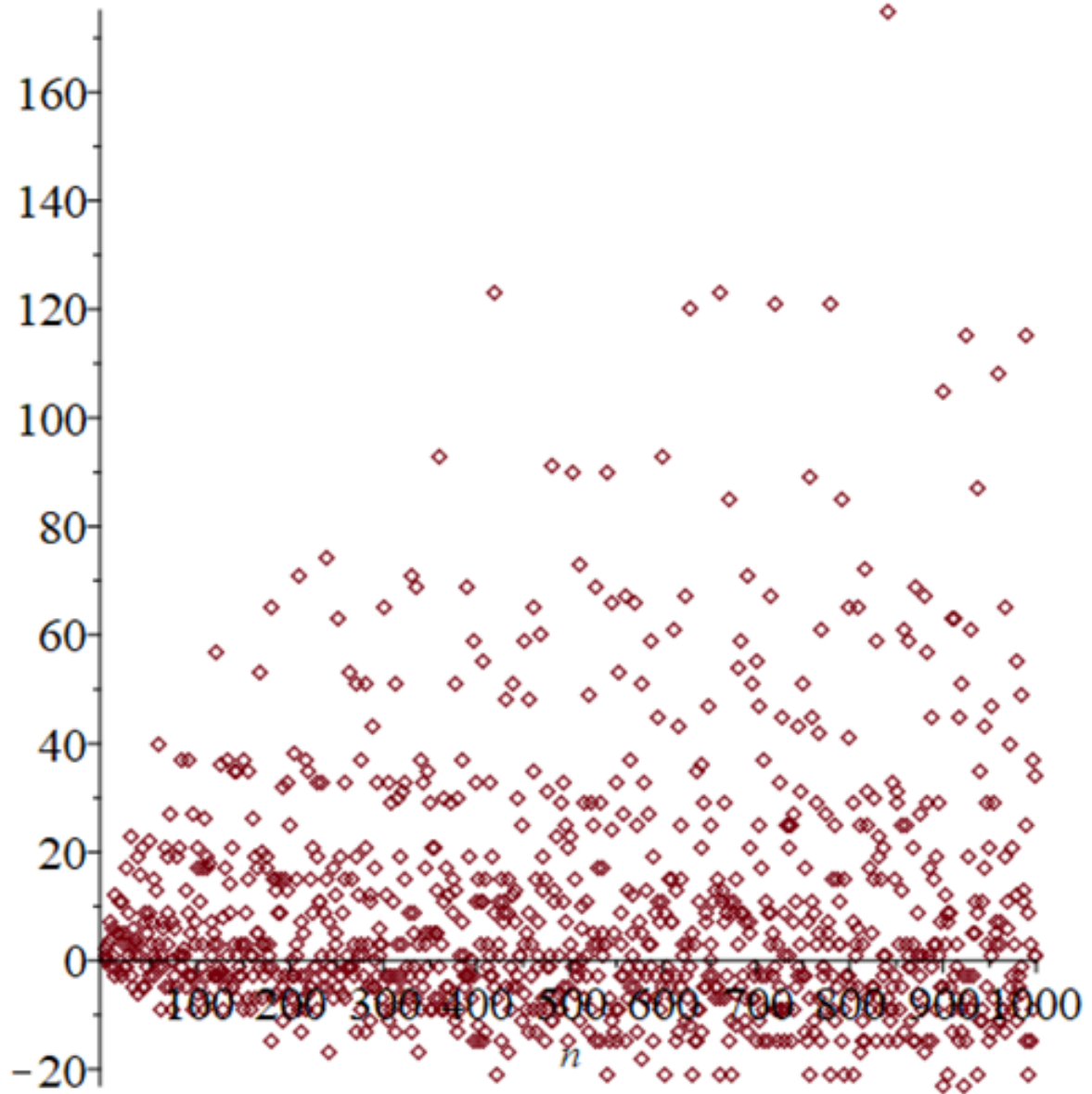
is an element of \mathbb{S} .

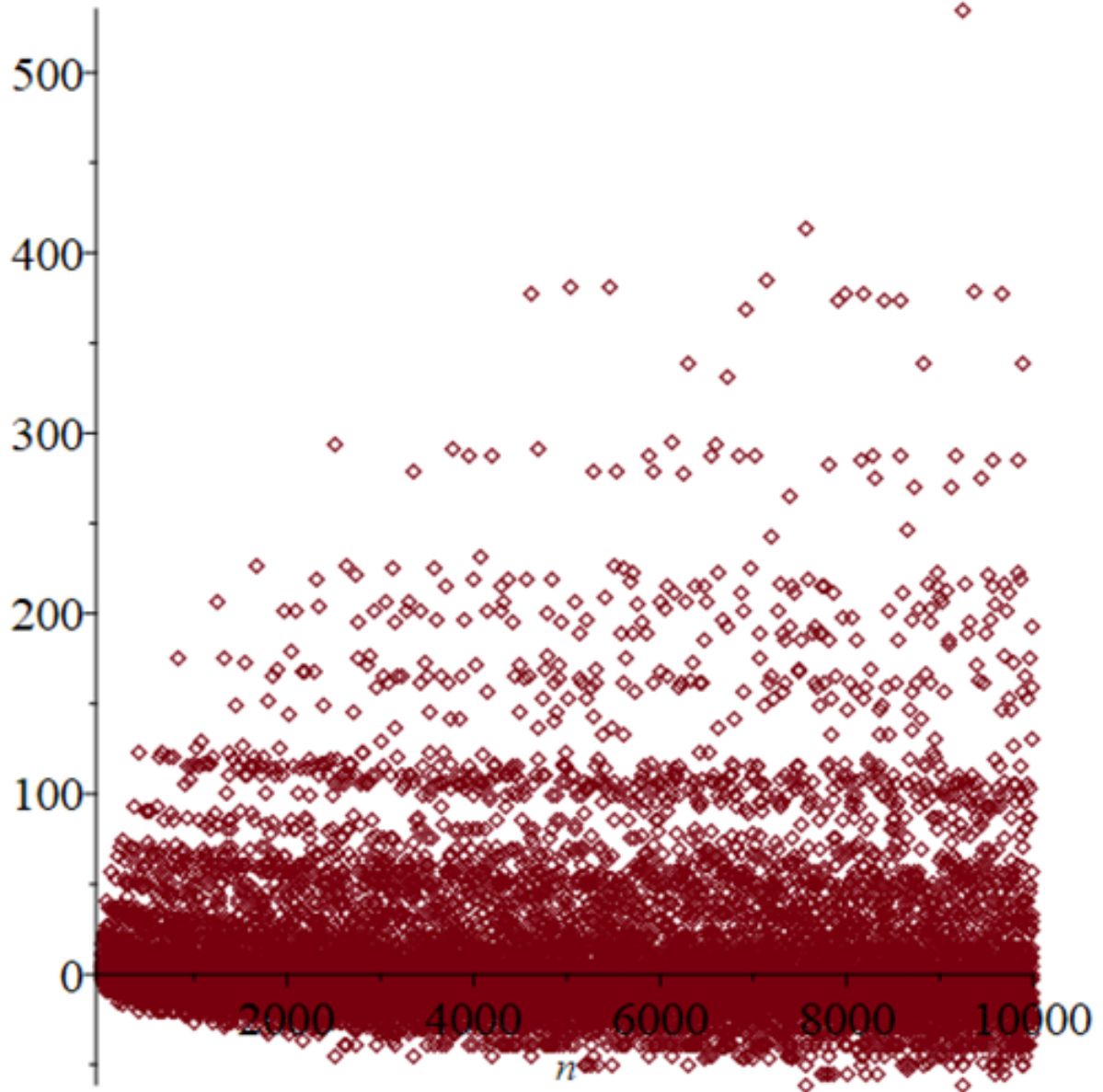
¹A prime p is said to be a *Sophie Germain prime* if $2p + 1$ is also a prime, in which case, this last prime is called a *safe prime*. It has been conjectured that there are infinitely many Sophie Germain primes, but this remains unproved, see [16].

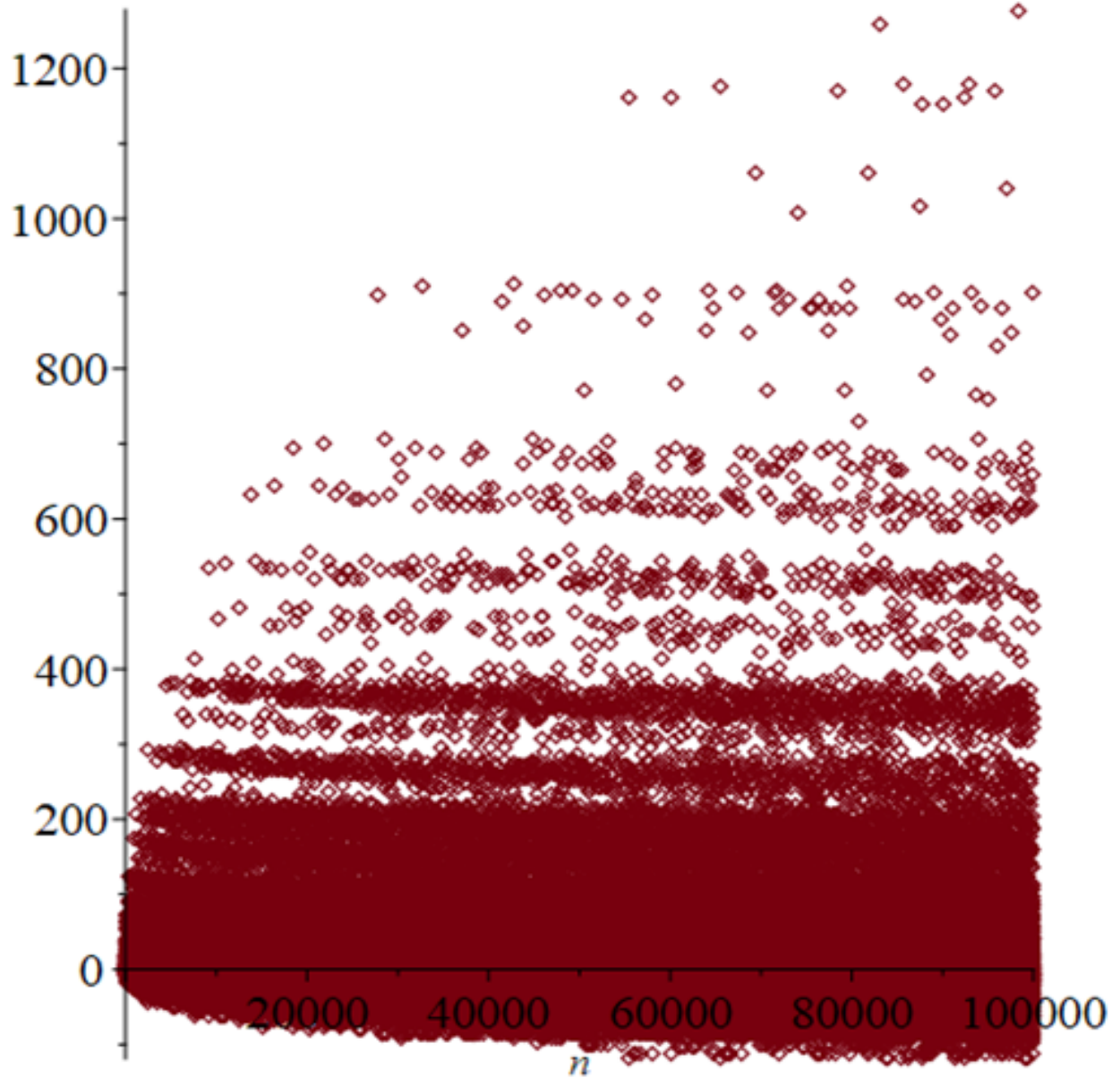
- If $n = 2 \cdot F_0 \cdot F_2 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$, then $k = 74$. Let us take q_1, q_2, \dots, q_{73} be the first odd Sophie Germain primes. That is, $(q_1, q_2, \dots, q_{73}) = (3, 5, \dots, 2945)$. Then the result holds for $q_{74} = 3863$.
- If $n = 2 \cdot F_0 \cdot F_1 \cdot F_2 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$, then $k = 234$.
- If $n = 2 \cdot F_0 \cdot F_1 \cdot F_2 \cdot F_3 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$, then $k = 712$.
- If $n = 2 \cdot F_0 \cdot F_1 \cdot F_2 \cdot F_3 \cdot F_4 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$, then $k = 2153$.

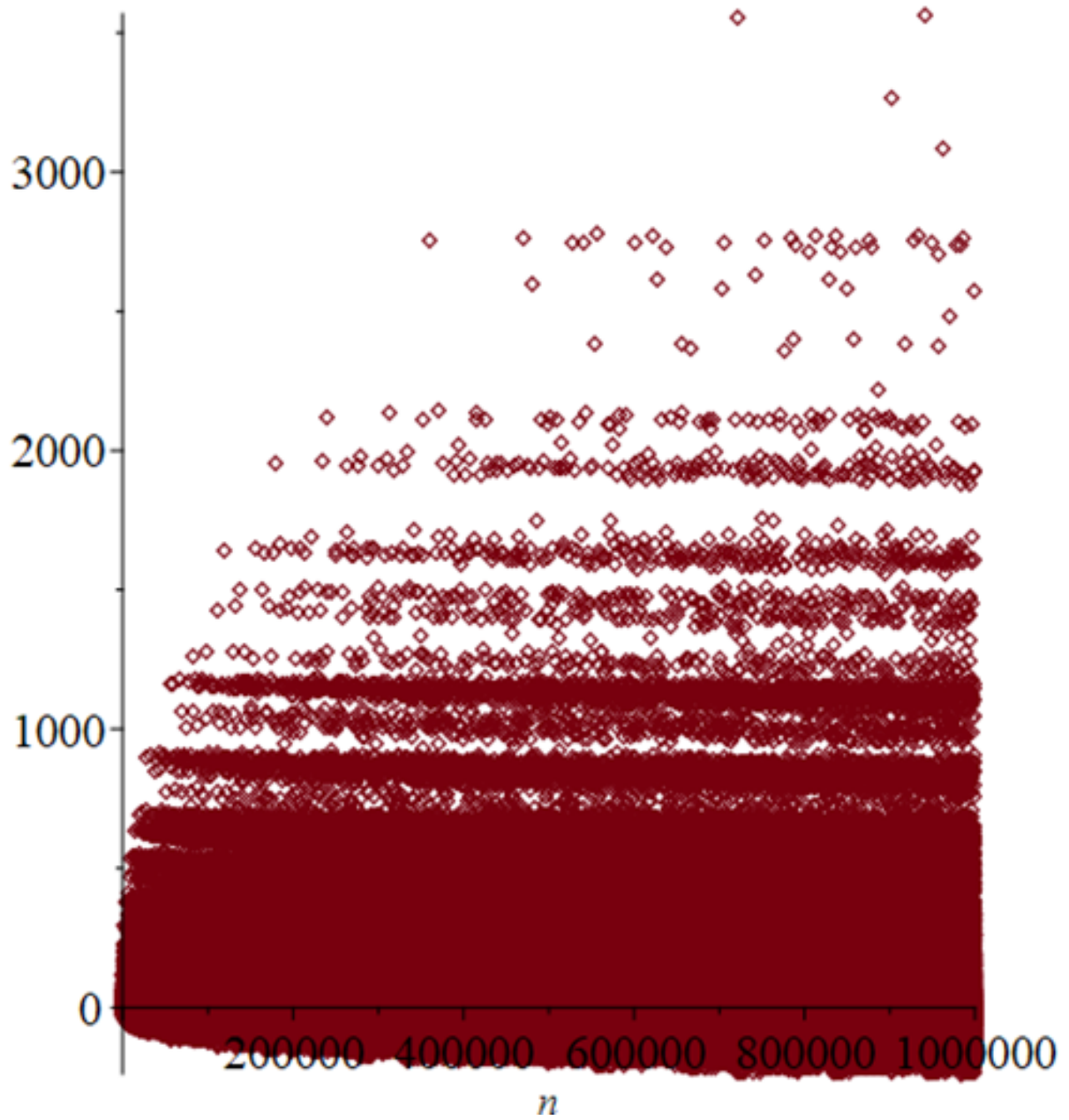
Representing the sequence $n \mapsto \tau(n^2) - \tau(\varphi(n))$ for n from 1 to 1000000











SOLUTIONS OF THE DIOPHANTINE EQUATION $\tau(kn) = \varphi(\varphi(n))$

In this chapter, we determine all solutions of the equation $\tau(n) = \varphi(\varphi(n))$ and we prove that the equation $\tau(kn) = \varphi(\varphi(n))$ has a finite number of solutions for any $k \geq 1$. Further, we characterize all solutions of the last equation when k is prime.

We first define for any positive integer k the following sets:

$$\mathbb{E}_k = \{n \in \mathbb{Z}^+ : \tau(kn) = \varphi(\varphi(n))\}, \quad (4.1)$$

$$\mathbb{L}_k = \{n \in \mathbb{Z}^+ : \tau(kn) < \varphi(\varphi(n))\}, \quad (4.2)$$

$$\mathbb{G}_k = \{n \in \mathbb{Z}^+ : \tau(kn) > \varphi(\varphi(n))\}. \quad (4.3)$$

4.1 The equation $\tau(n) = \varphi(\varphi(n))$

This section is devoted to investigate the elements of (4.1), (4.2) and (4.3) for $k = 1$.

Theorem 4.1. *The numbers 1, 5, 7, 15, 22, 26, 40, 56, 66, 70, 78, 108, 120, 126, 168, 210 are the only solutions of the equation $\tau(n) = \varphi(\varphi(n))$. Moreover, the numbers 2, 3, 4, 6, 8, 9, 10, 12, 14, 16, 18, 20, 24, 28, 30, 36, 42, 48, 54, 60, 72, 84, 90, 180 are the only solutions of the inequality $\tau(n) > \varphi(\varphi(n))$.*

For the proof, we start by proving the following results in which, if the solution $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$ with $q_1 < q_2 < \cdots < q_s$ and α_i are positive integers, then $s < 5$, $\alpha_i \leq 3$ for $1 \leq i \leq s$ and $q_s \leq 17$. This means that the number of solutions is finite. Let us start with square-free solutions.

4.1.1 Square-free solutions

Proposition 4.1. *The only prime numbers that satisfy the equation $\tau(n) = \varphi(\varphi(n))$ are 5 and 7.*

proof. Let p be a prime number. If $p \in \mathbb{E}_1$, then $\tau(p) = 2 = \varphi(p - 1)$. If $p - 1 \neq 2, 6$, then by (1.7), $p \leq 5$. In this case, $p = 5$ is the only solution. If $p - 1 = 2$ or 6 , it follows that $p = 3$ or 7 , where $7 \in \mathbb{E}_1$ while $3 \notin \mathbb{E}_1$. Finally, we conclude that if $p \in \mathbb{E}_1$, then p is either 5 or 7. \square

We deduce the following corollary.

Corollary 4.1. Let p be a prime number. We have:

- If p is either 5 or 7, then $p \in \mathbb{E}_1$.
- If p is either 2 or 3, then $p \in \mathbb{G}_1$.
- If $p \geq 11$, then $p \in \mathbb{L}_1$.

proof. By the same way of the proof of Proposition 4.1, we conclude that if $n = p$ is prime with $p \geq 11$, then the inequality $\tau(n) \geq \varphi(\varphi(n))$ cannot be true, and so $p \in \mathbb{L}_1$. \square

Proposition 4.2. *The only square-free solutions of the form q_1q_2 , where q_1 and q_2 are distinct primes, are*

- $2 \cdot 11, 2 \cdot 13, 3 \cdot 5$ and $3 \cdot 7$.

proof. Suppose that $q_1q_2 \in \mathbb{E}_1$, where q_1, q_2 are distinct primes with $2 \leq q_1 < q_2$. We obtain from (4.1) that $\varphi((q_1 - 1)(q_2 - 1)) = 4$. If $(q_1 - 1)(q_2 - 1) \neq 2, 6$, then by (1.7) we have $(q_1 - 1)(q_2 - 1) \leq 16$. Here, q_1 cannot be ≥ 5 . There are two possibilities:

1. If $q_1 = 2$, then q_2 must be in $\{5, 7, 11, 13, 17\}$.
2. If $q_1 = 3$, then q_2 must be in $\{5, 7\}$.

Thus, since $2 \cdot 5, 2 \cdot 7$ and $2 \cdot 17$ are in \mathbb{G}_1 , we get

$$(q_1, q_2) \in \{(2, 11), (2, 13), (3, 5), (3, 7)\}.$$

\square

By the same way we can prove the following proposition:

Proposition 4.3. *If a solution n is square-free and has 3 or 4 distinct primes, then n is one the numbers*

- $2 \cdot 3 \cdot 11, 2 \cdot 5 \cdot 7, 2 \cdot 3 \cdot 13$ and $2 \cdot 3 \cdot 5 \cdot 7$.

Proposition 4.4. *If a solution n is square-free and has more than 4 distinct primes, then $n \in \mathbb{L}_1$.*

For the proof we need the following lemma:

Lemma 4.1. *Let $k \geq 6$ and let q_1, q_2, \dots, q_k be distinct primes. Then*

$$(q_1 - 1)(q_2 - 1) \cdots (q_k - 1) > 2^{2k}.$$

proof. The proof holds by induction on k , since $(q_1 - 1)(q_2 - 1) \cdots (q_6 - 1) \geq 2^{2 \cdot 6}$ for every 6-tuple (q_1, q_2, \dots, q_6) of distinct primes. \square

Proof of proposition 4.4. Let $n = q_1 q_2 \cdots q_k$, where q_1, q_2, \dots, q_k are distinct primes. We assume further that $n \in \mathbb{E}_1 \cup \mathbb{G}_1$. In the case when $k = 5$. We see that

$$32 = \tau(q_1 q_2 \cdots q_5) \geq \varphi((q_1 - 1)(q_2 - 1) \cdots (q_5 - 1)).$$

By (1.7), we obtain

$$\varphi((q_1 - 1)(q_2 - 1) \cdots (q_5 - 1)) \geq \sqrt{(q_1 - 1)(q_2 - 1) \cdots (q_5 - 1)},$$

and hence $(q_1 - 1)(q_2 - 1) \cdots (q_5 - 1) \leq 1024$. Since $q_1 < q_2 < \cdots < q_5$, we distinguish the following cases:

- $q_1 = 2$ and (q_2, q_3, q_4, q_5) must belong to the set

$$\{(3, 5, 7, 11), (3, 5, 7, 13), (3, 5, 7, 17), (3, 5, 7, 19), (3, 5, 11, 13)\}.$$

But, the corresponding numbers $n = q_1 q_2 \cdots q_5$ belong to \mathbb{L}_1 , which is impossible since $n \in \mathbb{E}_1 \cup \mathbb{G}_1$.

- $q_1 \geq 3$. We get $(q_1 - 1)(q_2 - 1) \cdots (q_5 - 1) \geq 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 > 1024$, which is impossible as well. Thus, q_1 cannot be greater than 3.

In the case when $k \geq 6$, we also see that

$$2^k = \tau(q_1 q_2 \cdots q_k) \geq \varphi((q_1 - 1)(q_2 - 1) \cdots (q_k - 1)),$$

and by (1.7) we have

$$\varphi((q_1 - 1)(q_2 - 1) \cdots (q_k - 1)) \geq \sqrt{(q_1 - 1)(q_2 - 1) \cdots (q_k - 1)}.$$

It follows that

$$2^{2k} \geq (q_1 - 1)(q_2 - 1) \cdots (q_k - 1).$$

This contradicts Lemma 4.1. Hence, $\tau(n) < \varphi(\varphi(n))$. \square

4.1.2 Nonsquare-free odd solutions

Proposition 4.5. *Let $s \geq 2$ and let n_1, n_2, \dots, n_s be relatively prime positive integers with $n_i \geq 3$ for $i = 1, 2, \dots, s$. If $n_1, n_2, \dots, n_s \in \mathbb{E}_1 \cup \mathbb{L}_1$, then $n_1 n_2 \cdots n_s \in \mathbb{L}_1$.*

proof. Let n_1, n_2, \dots, n_s be as above and assume that $n_1, n_2, \dots, n_s \in \mathbb{E}_1 \cup \mathbb{L}_1$. Since $(\varphi(n_i), \varphi(n_j)) > 1$ for $1 \leq i, j \leq s$, it follows from (1.10) and the multiplicativity of τ and φ that

$$\begin{aligned} \tau(n_1 n_2 \cdots n_s) &= \tau(n_1) \tau(n_2) \cdots \tau(n_s) \leq \varphi(\varphi(n_1)) \varphi(\varphi(n_2)) \cdots \varphi(\varphi(n_s)) \\ &< \varphi(\varphi(n_1) \varphi(n_2) \cdots \varphi(n_s)) = \varphi(\varphi(n_1 n_2 \cdots n_s)). \end{aligned}$$

Hence, $n_1 n_2 \cdots n_s \in \mathbb{L}_1$. □

Proposition 4.6. *Let $\alpha \geq 2$ and let $n = q^\alpha$ be a prime power with q is odd. Then $n \in \mathbb{L}_1$ except for $n = 3^2 \in \mathbb{G}_1$.*

For the proof we need the following lemma.

Lemma 4.2. *Let n be a positive integer with $n \geq 3$ and let q be an odd prime number. Then*

$$(n + 1)^2 < q^{n-1}(q - 1).$$

proof. This follows immediately using mathematical induction. □

Proof of proposition 4.6. Assume by way of contradiction that $\tau(n) \geq \varphi(\varphi(n))$. Therefore, $\alpha + 1 \geq \varphi(q^{\alpha-1}(q - 1))$. Also, by (1.7), we get

$$\varphi(q^{\alpha-1}(q - 1)) \geq \sqrt{q^{\alpha-1}(q - 1)},$$

and so $(\alpha + 1)^2 \geq q^{\alpha-1}(q - 1)$, which is impossible by Lemma 4.2 except for $(q, \alpha) = (3, 2)$. □

Proposition 4.7. *Let $s \geq 2$ and let $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$ be an odd number with $\max(\alpha_1, \alpha_2, \dots, \alpha_s) \geq 2$. Then $n \in \mathbb{L}_1$.*

proof. We distinguish three cases:

Case 1. $s = 2$. There are two subcases:

Subcase 1.1. $q_1 = 3$. This means that $n = 3^{\alpha_1} q_2^{\alpha_2}$, and we have three possibilities:

- $\alpha_1 = 1$, so α_2 must be > 1 because $q_2 \geq 5$. Thus, it is clear that

$$2^2(\alpha_2 + 1)^2 < 2q_2^{\alpha_2-1}(q_2 - 1).$$

If we assume that $n \in \mathbb{E}_1 \cup \mathbb{G}_1$, this means that $2(\alpha_2 + 1) \geq \varphi(2q_2^{\alpha_2-1}(q_2 - 1))$, and by (1.7) we obtain

$$\varphi(2q_2^{\alpha_2-1}(q_2 - 1)) \geq \sqrt{2q_2^{\alpha_2-1}(q_2 - 1)}.$$

Thus, $2^2(\alpha_2 + 1)^2 \geq 2q_2^{\alpha_2-1}(q_2 - 1)$. This is a contradiction. Hence, $n \in \mathbb{L}_1$.

- $\alpha_1 = 2$. Since $q_2 \geq 5$, we conclude that $3^2(\alpha_2 + 1)^2 < 2 \cdot 3 \cdot q_2^{\alpha_2-1}(q_2 - 1)$. Hence, $n \in \mathbb{L}_1$.
- $\alpha_1 > 2$. By Corollary 4.1 and Proposition 4.6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$, so by Proposition 4.5, $n \in \mathbb{L}_1$.

Subcase 1.2. $q_1 > 3$. By Corollary 4.1 and Proposition 4.6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$, so it is clear by Proposition 4.5 that $n \in \mathbb{L}_1$.

Case 2. $s = 3$. Here, we also have two subcases:

Subcase 2.1. $q_1 = 3$. We also distinguish three possibilities:

- $\alpha_1 = 1$. That is, $\alpha_2 > 1$ or $\alpha_3 > 1$. Then we can easily check that

$$2^2(\alpha_2 + 1)^2(\alpha_3 + 1)^2 < 2(q_2 - 1)q_2^{\alpha_2-1}(q_3 - 1)q_3^{\alpha_3-1},$$

and hence $n \in \mathbb{L}_1$.

- $\alpha_1 = 2$. By the same way we find $n \in \mathbb{L}_1$.
- $\alpha_1 > 2$. By Corollary 4.1 and Proposition 4.6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$. So by Proposition 4.5, $n \in \mathbb{L}_1$.

Subcase 2.2. $q_1 > 3$. By Corollary 4.1 and Proposition 4.6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$. So by Proposition 4.5, $n \in \mathbb{L}_1$.

Case 3. $s \geq 4$. Here, we distinguish two subcases:

Subcase 3.1. $q_1 = 3$. By induction on s we can easily prove the following inequality

$$(\alpha_1 + 1)^2(\alpha_2 + 1)^2 \cdots (\alpha_s + 1)^2 < 2 \cdot 3^{\alpha_1-1}(q_2 - 1)q_2^{\alpha_2-1} \cdots (q_s - 1)q_s^{\alpha_s-1},$$

which gives $n \in \mathbb{L}_1$.

Subcase 3.2. $q_1 > 3$. By Corollary 4.1 and Proposition 4.6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$. Similarly, by Proposition 4.5, $n \in \mathbb{L}_1$. \square

4.1.3 Nonsquare-free even solutions

In this subsection, we can prove the following proposition as those appearing with the case when n is nonsquare-free odd.

Proposition 4.8. *Let $n = 2^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$ be an even number such that $\max(\alpha_1, \alpha_2, \dots, \alpha_s) \geq 2$. There are four possibilities:*

- $s = 1$. Here n is a prime power, where $n = 2^{\alpha_1}$ with $\alpha_1 \geq 2$. We have:
 - If n is either 2^2 or 2^3 or 2^4 , then $n \in \mathbb{G}_1$.
 - If $\alpha_1 \geq 5$, then $n \in \mathbb{L}_1$.
- $s = 2$. We have:
 - If $n = 2^2 \cdot 3, 2 \cdot 3^2, 2^2 \cdot 5, 2^3 \cdot 3, 2^2 \cdot 7, 2^2 \cdot 3^2, 2^4 \cdot 3, 2 \cdot 3^3, 2^3 \cdot 3^2$, then $n \in \mathbb{G}_1$.
 - If $n = 2^3 \cdot 5, 2^3 \cdot 7, 2^2 \cdot 3^3$, then $n \in \mathbb{E}_1$.
 - If n is different from the previous mentioned numbers, then $n \in \mathbb{L}_1$.
- $s = 3$. We have:
 - If $n = 2^2 \cdot 3 \cdot 5, 2^2 \cdot 3 \cdot 7, 2 \cdot 3^2 \cdot 5, 2^2 \cdot 3^2 \cdot 5$, then $n \in \mathbb{G}_1$.
 - If $n = 2^3 \cdot 3 \cdot 5, 2 \cdot 3^2 \cdot 7, 2^3 \cdot 3 \cdot 7$, then $n \in \mathbb{E}_1$.
 - If n is different from the previous mentioned numbers, then $n \in \mathbb{L}_1$.
- $s \geq 4$. Here, we have $n \in \mathbb{L}_1$.

Now, we are ready to prove Theorem 4.1.

Theorem 4.1. Clearly, $1 \in E_1$. Assume that $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$ with $n \neq 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 26, 28, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 84, 90, 108, 120, 126, 168, 180$ and 210 , where $q_1 < q_2 < \cdots < q_s$ are prime numbers and $\alpha_1, \alpha_2, \dots, \alpha_s$ are positive integers. There are five cases to consider:

Case 1. $s = 1$. There are two possibilities:

- $\alpha_1 = 1$. Since $n \neq 2, 3, 5, 7$, it follows from Corollary 4.1 that $n \in \mathbb{L}_1$.
- $\alpha_1 \geq 2$. Since $n \neq 2^2, 2^3, 3^2, 2^4$, it follows from Propositions 4.6 and 4.8 that $n \in \mathbb{L}_1$.

Case 2. $s = 2$. There are two possibilities:

- $\alpha_1 = \alpha_2 = 1$. Since $n \neq 10, 14, 15, 21, 22, 26, 34$, by Proposition 4.2 we conclude that $n \in \mathbb{L}_1$.
- α_1 or $\alpha_2 \geq 2$. Since $n \neq 12, 18, 20, 24, 28, 36, 40, 48, 54, 56, 72, 108$, it follows from Proposition 4.7 and Proposition 4.8 that $n \in \mathbb{L}_1$.

Case 3. $s = 3$.

- n is square-free. Since $n \neq 30, 42, 66, 70, 78$, by Propositions 4.3 and 4.4 we have $n \in \mathbb{L}_1$.
- n is not square-free. Since $n \neq 60, 84, 90, 120, 126, 168, 180$, it follows from Propositions 4.7 and 4.8 that $n \in \mathbb{L}_1$.

Case 4. $s = 4$.

- n is square-free. Since $n \neq 210$, it follows from Propositions 4.3 and 4.4 that $n \in \mathbb{L}_1$.
- n is not square-free. It follows from Propositions 4.7 and 4.8 that $n \in \mathbb{L}_1$.

Case 5. $s \geq 5$. By Propositions 4.4, 4.7 and 4.8 we have $n \in \mathbb{L}_1$.

The proof is finished. □

4.2 On the equation $\tau(k \cdot n) = \varphi(\varphi(n))$ with $k \geq 2$

Proposition 4.9. *Let $s, k \geq 2$ and let n_1, n_2, \dots, n_s be relatively prime positive integers with $n_i \geq 3$ for $i = 1, 2, \dots, s$. If $n_1, n_2, \dots, n_s \in \mathbb{E}_k \cup \mathbb{L}_k$, then $n_1 n_2 \cdots n_s \in \mathbb{L}_k$.*

proof. This is similar to the proof of Proposition 4.5. □

Theorem 4.2. *Let $k \geq 2$. The sets \mathbb{G}_k and \mathbb{E}_k are finite, while \mathbb{L}_k is infinite.*

proof. At first, we prove that for any prime p there exists an exponent α_0 such that $p^x \in \mathbb{L}_k$ for every $x \geq \alpha_0$. Let $n = p^\alpha$ be a prime power such that $p^\alpha \in \mathbb{E}_k \cup \mathbb{G}_k$. Then

$$\tau(k \cdot p^\alpha) \geq \varphi(p^{\alpha-1}(p-1)).$$

Put $k = p^a m$, where $a \geq 0, m \geq 1$ and $(p, m) = 1$. Then

$$\tau(m)(a + \alpha + 1) \geq p^{\alpha-2}(p-1)\varphi(p-1)$$

and by (1.3) and (1.7) we have

$$2\sqrt{m}(a + \alpha + 1) \geq \tau(m)(a + \alpha + 1) \geq p^{\alpha-2}(p-1)\varphi(p-1) \geq p^{\alpha-2}(p-1)\sqrt{p-1}. \quad (4.4)$$

Therefore,

$$4m(a + \alpha + 1)^2 \geq p^{2\alpha-4}(p-1)^3.$$

Since a and m are fixed and α is the exponent of p , the last inequality has only finite number of solutions. Thus, the set $\mathbb{E}_k \cup \mathbb{G}_k$ contains only finitely many prime powers, namely $l_1^{\gamma_1}, l_2^{\gamma_2}, \dots, l_s^{\gamma_s}$. Consequently, it suffices to choose $\alpha_0 = \max(\gamma_1 + 1, \gamma_2 + 1, \dots, \gamma_s + 1)$.

Now, let $n = q_1 q_2 \cdots q_m$, where $q_1 < q_2 < \cdots < q_m$ are prime numbers ($m \geq 2$). Assume that $n \in \mathbb{E}_k \cup \mathbb{G}_k$. Therefore,

$$\tau(k \cdot q_1 q_2 \cdots q_m) \geq \varphi(\varphi(q_1 q_2 \cdots q_m)).$$

By (1.3), (1.4) and (1.7) we obtain

$$(2\sqrt{k})(2^{m+1}) \geq \tau(k)\tau(q_1 \cdots q_m) \geq \tau(k \cdot q_1 \cdots q_m) \geq \varphi(\varphi(q_1 \cdots q_m)) \geq \sqrt{(q_1 - 1) \cdots (q_m - 1)},$$

from which it follows that

$$2^{2m+2}4k \geq (q_1 - 1)(q_2 - 1) \cdots (q_m - 1). \quad (4.5)$$

Since k is fixed and $(q_1 - 1)(q_2 - 1) \cdots (q_m - 1)$ can be sufficiently large, we deduce that there exists a positive integer m_0 for which the inequality (4.5) is not true for every $m \geq m_0$. Thus the inequality $\tau(k \cdot n) \geq \varphi(\varphi(n))$ holds for finitely many square-free integers.

Finally, let $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_m^{\alpha_m}$, where $q_1 < q_2 < \cdots < q_m$ are prime numbers and $\alpha_1, \alpha_2, \dots, \alpha_m$ are positive integers. Note that the number m cannot be sufficiently large as we wish such that $\tau(k \cdot s) \geq \varphi(\varphi(s))$ for $s = q_1 q_2 \cdots q_m$. Moreover, from above there exist positive integers $\alpha_1^{(0)}, \alpha_2^{(0)}, \dots, \alpha_m^{(0)}$ such that the numbers $q_1^{\alpha_1'}, q_2^{\alpha_2'}, \dots, q_m^{\alpha_m'}$ satisfy the inequality

$$\tau(k \cdot q_i^{\alpha_i'}) < \varphi(\varphi(q_i^{\alpha_i'}))$$

for every $\alpha_i' \geq \alpha_i^{(0)}$ ($1 \leq i \leq m$). Applying Proposition 4.9, the numbers $n' = q_1^{\alpha_1'} q_2^{\alpha_2'} \cdots q_m^{\alpha_m'}$ with $\alpha_i' \geq \alpha_i^{(0)}$ ($1 \leq i \leq m$) satisfy the inequality $\tau(k \cdot n') < \varphi(\varphi(n'))$. Thus, the inequality $\tau(k \cdot n) \geq \varphi(\varphi(n))$ has only a finite number of solutions, while the inequality $\tau(n) < \varphi(\varphi(n))$ has infinitely many solutions. \square

Proposition 4.10. We have $\bigcap_{k \geq 1} \mathbb{G}_k = \mathbb{G}_1$.

proof. It suffices to prove that $\mathbb{G}_1 \subset \bigcap_{k \geq 1} \mathbb{G}_k$. As we have already seen

$$\mathbb{G}_1 = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 16, 18, 20, 24, 28, 30, 36, 42, 48, 54, 60, 72, 84, 90, 180\}.$$

Then for every $k \geq 1$ and x in \mathbb{G}_1 we see that $\tau(kx) > \varphi(\varphi(x))$. This proves $\mathbb{G}_1 \subset \mathbb{G}_k$. This completes the proof. \square

Proposition 4.11. *Let $r, s \geq 2$. If r divides s , then $\mathbb{G}_r \subset \mathbb{G}_s$.*

proof. Let $n \in \mathbb{G}_r$. Since r divides s , we see by (1.6) that

$$\tau(s \cdot n) \geq \tau(r \cdot n) > \varphi(\varphi(n)),$$

and so $n \in \mathbb{G}_s$, as required. \square

Theorem 4.3. *Let p be a prime number with $p \geq 23$ and $p \neq 31$. Then $\mathbb{E}_p = \{11, 13, 33, 34, 35, 39, 62, 63, 76, 88, 98, 102, 104, 105, 110, 130, 154, 186, 228, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.*

proof. Let $p \geq 23$ be prime with $p \neq 31$. Clearly, if n is one of the above numbers, then we can easily check that $\tau(p \cdot n) = 2\tau(n) = \varphi(\varphi(n))$. Then the numbers mentioned in this theorem are part of the solution set.

Now, let $n \geq 1$ such that $n \in \mathbb{E}_p \cup \mathbb{G}_p$. It follows that

$$2\tau(n) \geq \tau(p \cdot n) \geq \varphi(\varphi(n)).$$

As in the proof of Theorem 4.1, we can show that the only solutions of $2\tau(n) \geq \varphi(\varphi(n))$ are $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 48, 50, 52, 54, 56, 60, 62, 63, 66, 70, 72, 76, 78, 80, 84, 88, 90, 96, 98, 100, 102, 104, 105, 108, 110, 112, 114, 120, 126, 130, 132, 140, 144, 150, 154, 156, 162, 168, 180, 186, 198, 210, 216, 228, 234, 240, 252, 264, 270, 280, 288, 294, 300, 312, 330, 336, 360, 390, 396, 420, 450, 462, 504, 540, 630$ and 840 .

In addition, it is easy to check that the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 30, 32, 36, 38, 40, 42, 44, 45, 48, 50, 52, 54, 56, 60, 66, 70, 72, 78, 80, 84, 90, 96, 100, 108, 112, 114, 120, 126, 132, 140, 144, 150, 156, 162, 168, 180, 198, 210, 216, 240, 252, 270, 288, 300, 336, 360, 396, 420$ and 450 satisfy the inequality $\tau(p \cdot n) > \varphi(\varphi(n))$.

Thus, the numbers quoted in the text of the present theorem are the only solutions of the equation $\tau(p \cdot n) = \varphi(\varphi(n))$, where $p \geq 23$ is prime with $p \neq 31$. This completes the proof. \square

Remark 4.1. By the same argument as above and by a brute force search with Maple in the range $1 \leq n \leq 10^{10}$, one can show that

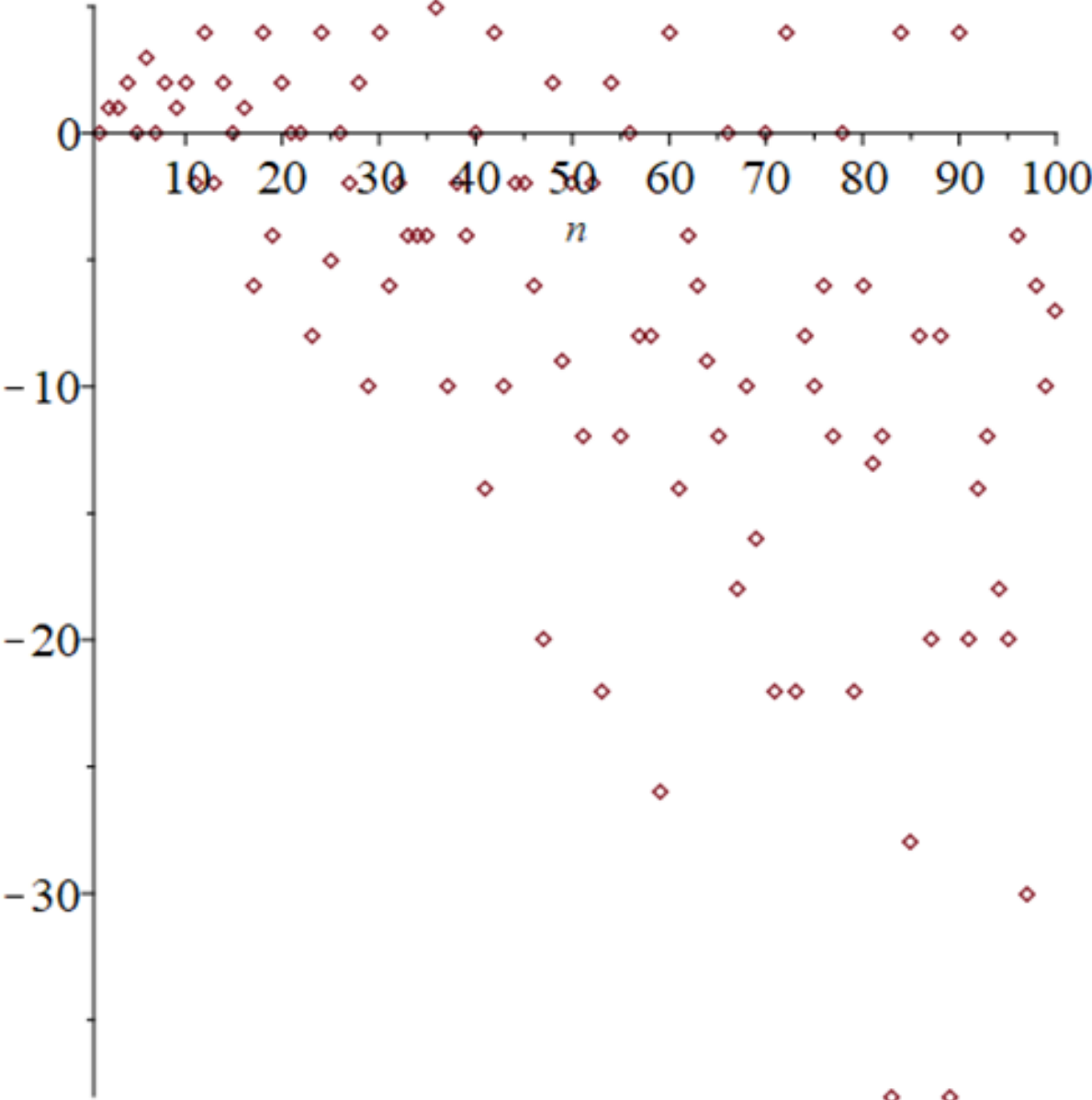
- $\mathbb{E}_2 = \{11, 13, 33, 35, 38, 39, 44, 52, 63, 105, 114, 132, 140, 156, 252, 270, 420\}$.
- $\mathbb{E}_3 = \{11, 13, 34, 35, 45, 62, 76, 88, 98, 104, 110, 114, 130, 154, 198, 252, 280, 360\}$.
- $\mathbb{E}_5 = \{11, 13, 33, 34, 39, 50, 62, 63, 76, 88, 98, 102, 104, 150, 154, 186, 228, 234, 264, 270, 294, 312, 462, 504\}$.

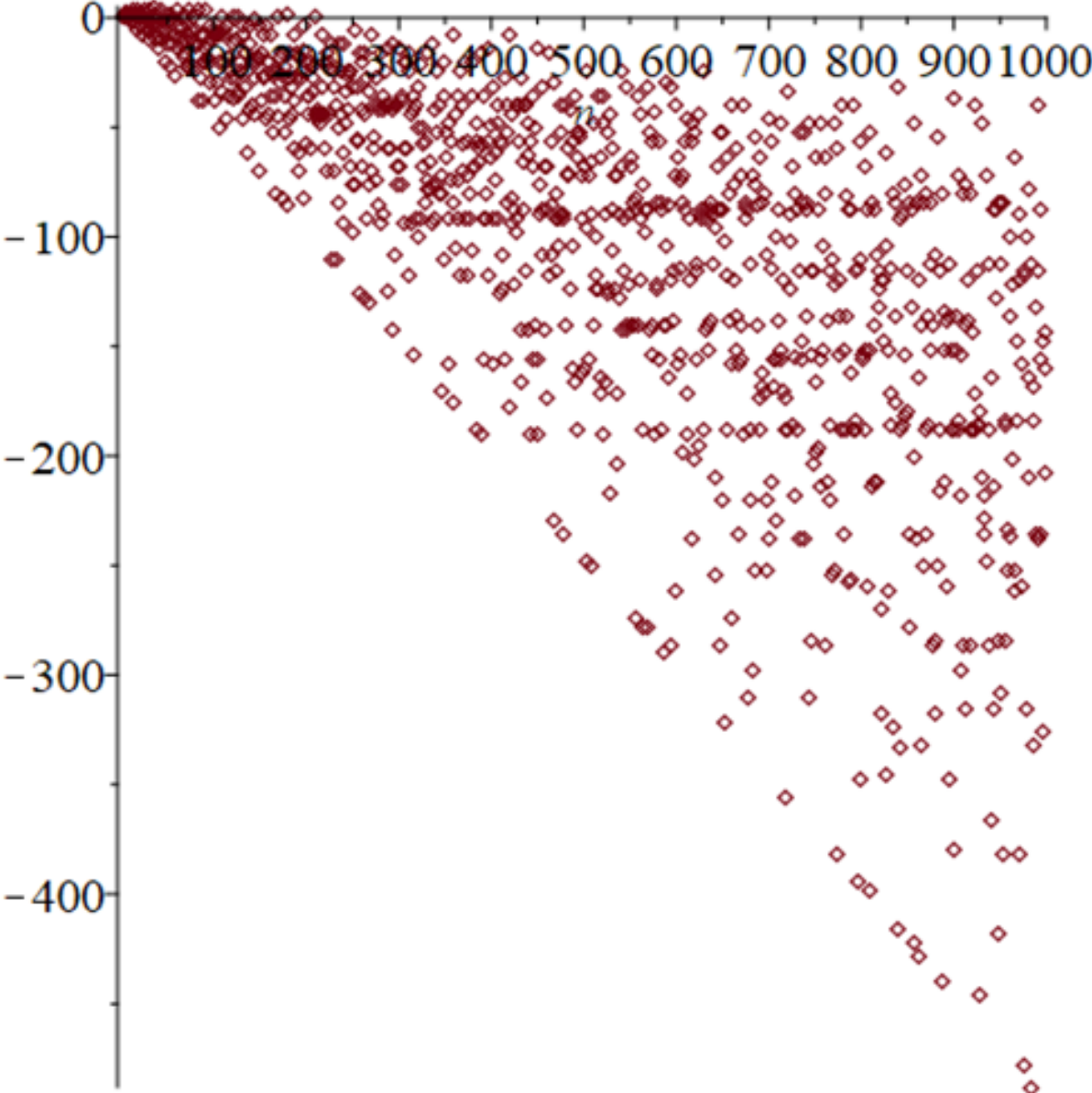
- $\mathbb{E}_7 = \{11, 13, 33, 34, 39, 62, 76, 88, 102, 104, 110, 130, 186, 228, 234, 264, 312, 330, 390, 540\}$.
- $\mathbb{E}_{11} = \{13, 34, 35, 39, 62, 63, 76, 98, 102, 104, 105, 130, 186, 228, 234, 280, 294, 312, 390, 504, 540, 630, 840\}$.
- $\mathbb{E}_{13} = \{11, 33, 34, 35, 62, 63, 76, 88, 98, 102, 105, 110, 154, 186, 228, 264, 280, 294, 330, 462, 504, 540, 630, 840\}$.
- $\mathbb{E}_{17} = \{11, 13, 33, 35, 39, 62, 63, 76, 88, 98, 104, 105, 110, 130, 154, 186, 228, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.
- $\mathbb{E}_{19} = \{11, 13, 33, 34, 35, 38, 39, 62, 63, 88, 98, 102, 104, 105, 110, 114, 130, 154, 186, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.
- $\mathbb{E}_{31} = \{11, 13, 33, 34, 35, 39, 63, 76, 88, 98, 102, 104, 105, 110, 130, 154, 228, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.

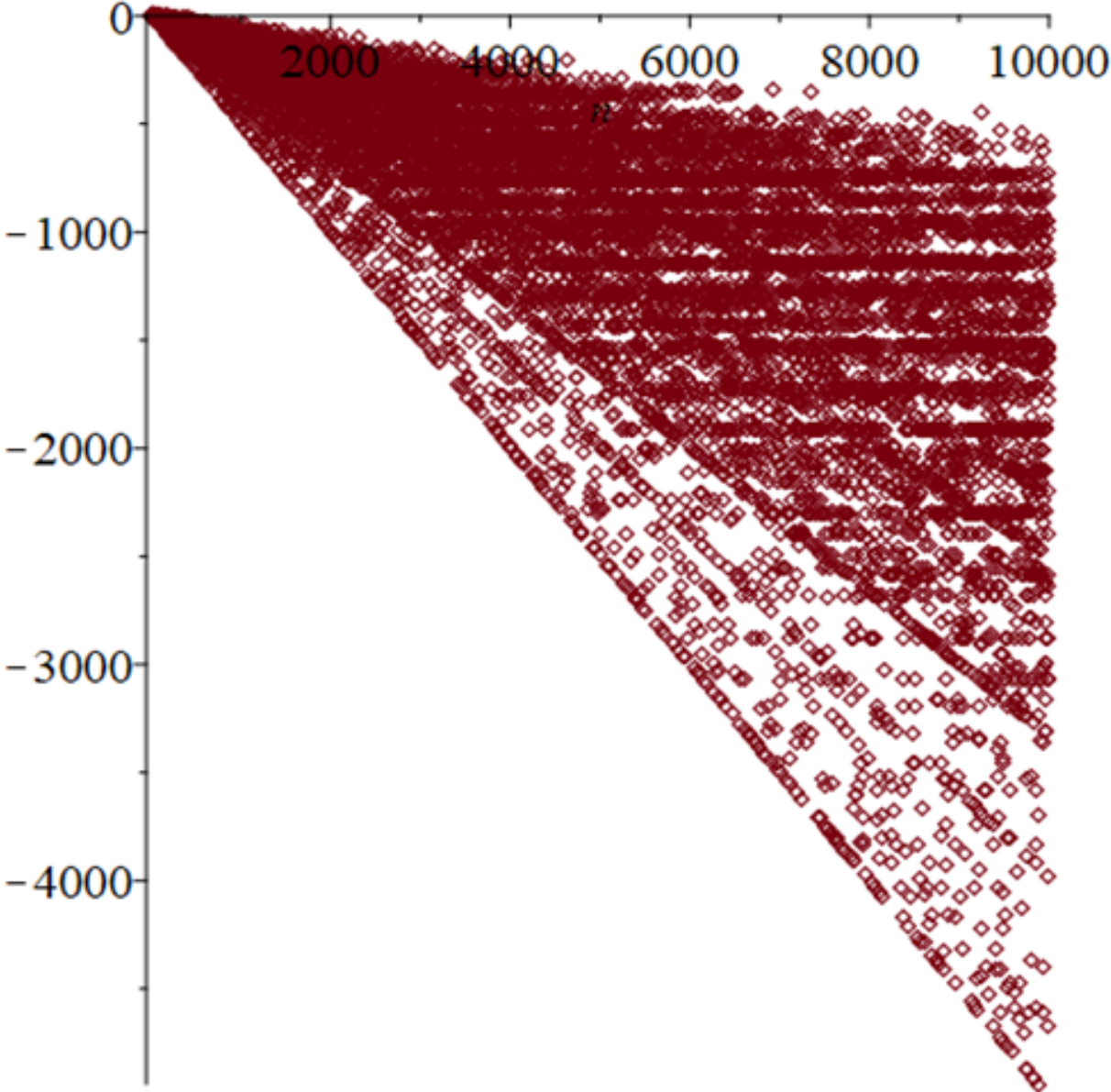
Moreover, from the proof of Theorem 4.3, for any prime $p \geq 23$ with $p \neq 31$ we deduce that $\mathbb{G}_p = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 30, 32, 36, 38, 40, 42, 44, 45, 48, 50, 52, 54, 56, 60, 66, 70, 72, 78, 80, 84, 90, 96, 100, 108, 112, 114, 120, 126, 132, 140, 144, 150, 156, 162, 168, 180, 198, 210, 216, 240, 252, 270, 288, 300, 336, 360, 396, 420, 450\}$.

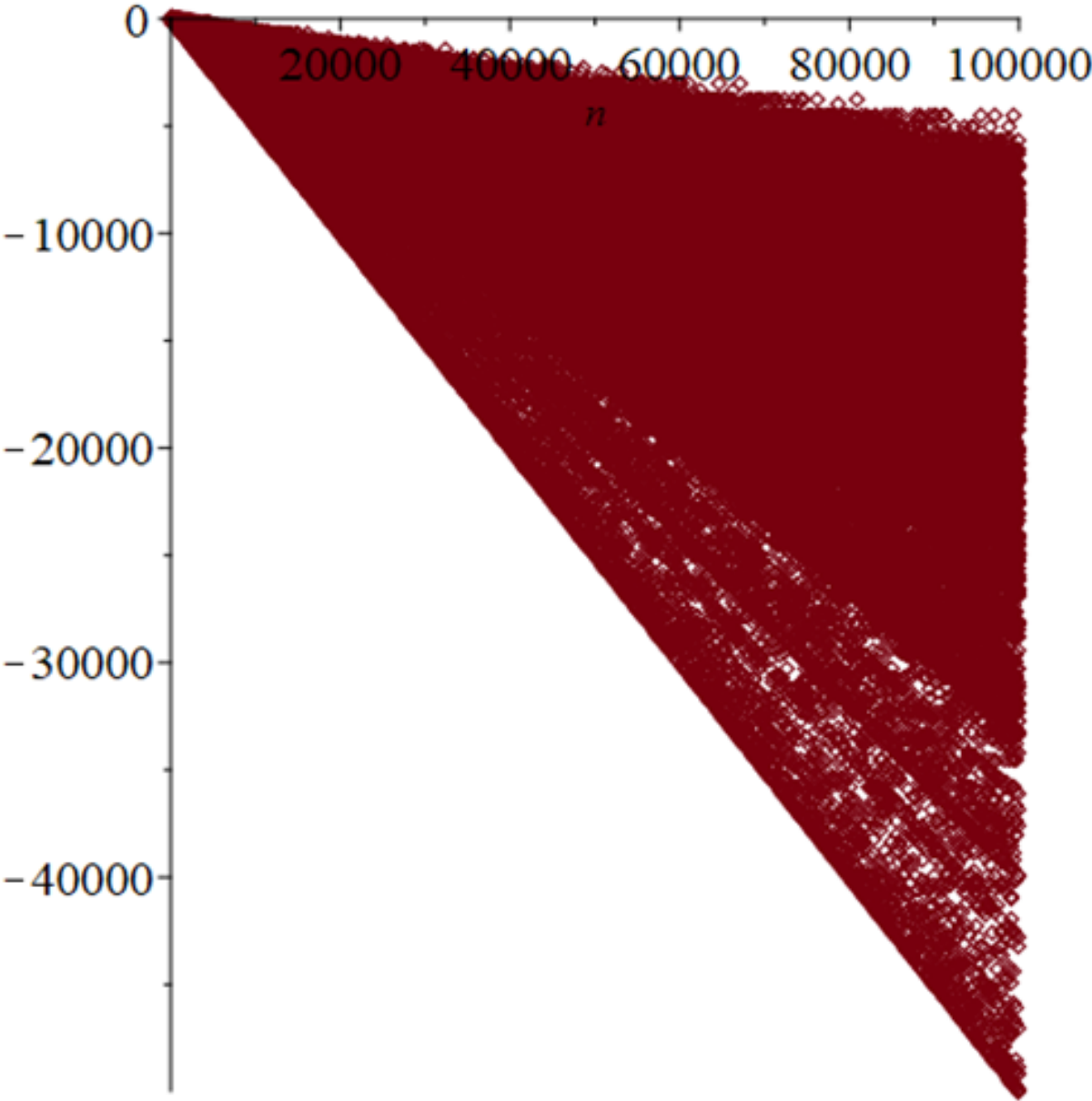
Thus, if I is a finite subset of positive integers, say $I = \{1, 2, \dots, N\}$ with $N \geq 100$, then $\mathbb{L}_p \cap I = I - \mathbb{E}_p - \mathbb{G}_p = \{17, 19, 23, 25, 29, 31, 37, 41, 43, 46, 47, 49, 51, 53, 55, 57, 58, 59, 61, 64, 65, 67, 68, 69, 71, 73, 74, 75, 77, 79, 81, 82, 83, 85, 86, 87, 89, 91, 92, 93, 94, 95, 97, 99, \dots\}$.

Representing the sequence $n \mapsto \tau(n) - \varphi(\varphi(n))$ for n from 1 to 100000

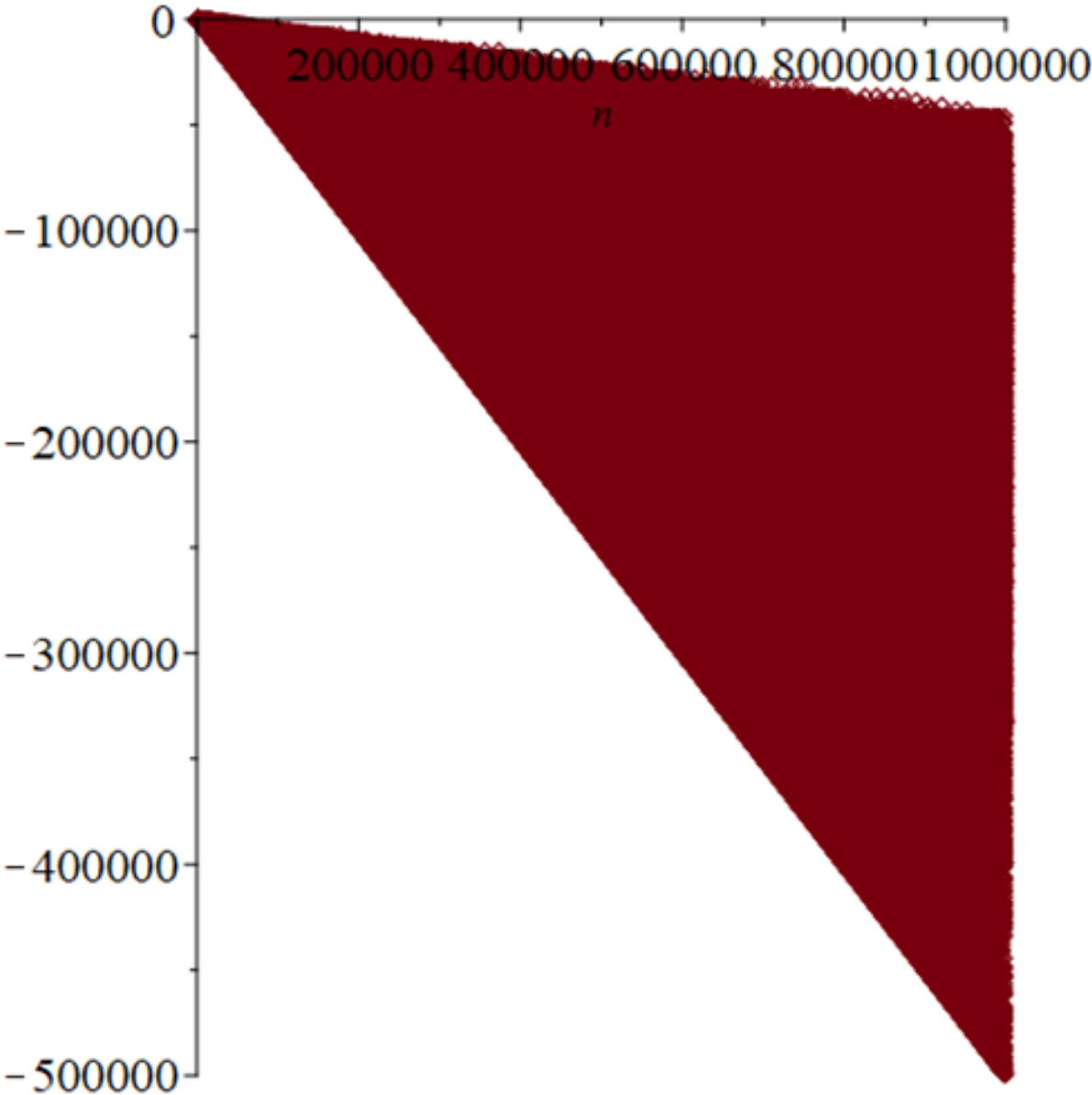








Representing the sequence $n \mapsto \tau(kn) - \varphi(\varphi(n))$ for k from 1 to 100 and n from 1 to 1000000



ON THE DIOPHANTINE EQUATION

$$\sigma(n) = \tau(kn \cdot \varphi(n))$$

In this chapter, we show that if $f(n) = kn$ with k is a fixed positive integer then the diophantine equation $\sigma(n) = \tau(f(n) \cdot \varphi(n))$ has only finitely many solutions. We also find all solutions of this equation and related inequalities when $k = 1, 2, 3$ and k is prime. Note that this choice of f makes $\sigma(n) > \tau(f(n) \cdot \varphi(n))$ for infinitely many n .

We first define for any positive integer k the following sets:

$$E_k := \{n \in \mathbb{N} : \sigma(n) = \tau(k \cdot \varphi(n^2))\}, \quad (5.1)$$

$$L_k := \{n \in \mathbb{N} : \sigma(n) < \tau(k \cdot \varphi(n^2))\}, \quad (5.2)$$

$$G_k := \{n \in \mathbb{N} : \sigma(n) > \tau(k \cdot \varphi(n^2))\}. \quad (5.3)$$

5.1 The equation $\sigma(n) = \tau(\varphi(n^2))$

This section is devoted to investigate the elements of (5.1), (5.2) and (5.3) for $k = 1$.

Theorem 5.1. *The numbers 1, 3, 5 and 7 are the only solutions of the equation $\sigma(n) = \tau(\varphi(n^2))$. Moreover, for every $n \neq 1, 3, 5, 7$ we have $\sigma(n) > \tau(\varphi(n^2))$.*

We start by proving the following results:

Proposition 5.1. *The only prime numbers n such that $\sigma(n) = \tau(\varphi(n^2))$ are 3, 5 and 7.*

proof. Assume that $n = p$ is prime with $\sigma(p) = \tau(\varphi(p^2))$. Since $(p, p-1) = 1$, we conclude that

$$\frac{1+p}{2} = \tau(p-1), \quad (5.4)$$

where p must be odd. On the other hand, by combining (1.3) and (5.4) we obtain

$$\frac{1+p}{2} \leq 2\sqrt{p-1},$$

from which it follows that $p^2 - 14p + 17 \leq 0$. Thus, $p \in \{2, \dots, 12\}$. Since p is prime and 11 does not satisfy (5.4), we deduce that $p \in \{3, 5, 7\}$. This completes the proof. \square

Corollary 5.1. If p is prime with $p \neq 3, 5, 7$, then $p \in G_1$.

proof. From the proof of Proposition 5.1, we deduce that if $n = p$ is prime with $p \geq 13$ then $\sigma(n) \leq \tau\tau(\varphi(n^2))$ cannot be true, and so $n = p \in G_1$. If $p = 2$ or $p = 11$, then clearly $p \in G_1$. \square

Proposition 5.2. Let $\alpha \geq 2$ and let $n = p^\alpha$ be a prime power. Then $n \in G_1$.

proof. Assume by the way of contradiction that $\sigma(n) \leq \tau(\varphi(n^2))$. Thus, we must have

$$\frac{1 + p + \dots + p^\alpha}{2\alpha} \leq \tau(p-1) \leq 2\sqrt{p-1}, \quad (5.5)$$

implying that $(1 + p + \dots + p^\alpha)^2 \leq 16\alpha^2(p-1)$, which is impossible except for the pair $(p, \alpha) = (2, 2)$. But the number $n = 2^2$ does not satisfy the left hand-side of (5.5), i.e., $2^2 \notin E_1 \cup L_1$. The proof is finished. \square

Proposition 5.3. Let $s \geq 2$ and let $n = q_1 q_2 \dots q_s$ be a square-free integer. Then $n \in G_1$.

For the proof we need the following lemma.

Lemma 5.1. Let $s \geq 2$ and let q_1, \dots, q_s be distinct primes. Then

$$(1 + q_1)^2 \dots (1 + q_s)^2 > 2^{2s+2} (q_1 - 1) \dots (q_s - 1).$$

proof. The proof holds by induction on s since $(1 + p)^2 (1 + q)^2 > 2^6 (p - 1)(q - 1)$ for every prime p . \square

Proof of Proposition 5.3. Assume, by way of contradiction, that $\sigma(n) \leq \tau(\varphi(n^2))$. It follows from (1.3) and (1.4) that

$$\begin{aligned} (1 + q_1) \dots (1 + q_s) &\leq \tau(q_1 \dots q_s ((q_1 - 1) \dots (q_s - 1))) \\ &\leq 2^s \tau((q_1 - 1) \dots (q_s - 1)) \\ &\leq 2^{s+1} \sqrt{(q_1 - 1) \dots (q_s - 1)}. \end{aligned}$$

Therefore, $(1 + q_1)^2 \dots (1 + q_s)^2 \leq 2^{2s+2} (q_1 - 1) \dots (q_s - 1)$, but this contradicts Lemma 5.1, and hence $n \notin E_1 \cup L_1$. \square

Proposition 5.4. Let $s \geq 2$ and let n_1, \dots, n_s be pairwise relatively prime positive integers with $n_i \geq 2$ for $i = 1, 2, \dots, s$. If $n_1, \dots, n_s \in E_1 \cup G_1$, then $n_1 \dots n_s \in G_1$.

proof. Let n_i be as above with $n_1, \dots, n_s \in E_1 \cup G_1$. Since

$$(\varphi(n_i), \varphi(n_j)) > 1$$

for $1 \leq i, j \leq s$, it follows from (1.5) and the multiplicativity of σ and φ that

$$\begin{aligned} \sigma(n_1 \cdots n_s) &= \sigma(n_1) \cdots \sigma(n_s) \geq \tau(n_1 \varphi(n_1)) \cdots \tau(n_s \varphi(n_s)) \\ &> \tau(n_1 \varphi(n_1)) \cdots \tau(n_s \varphi(n_s)) = \tau(n_1 \cdots n_s \varphi(n_1 \cdots n_s)). \end{aligned}$$

Hence, $n_1 \cdots n_s \in G_1$. □

Proof of Theorem 5.1. Clearly, $1 \in E_1$. Assume that $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$ with $n \neq 1, 3, 5, 7$, where $q_1 < q_2 < \cdots < q_s$ are prime numbers and $\alpha_1, \alpha_2, \dots, \alpha_s$ are positive integers. There are three cases to consider:

1. $s = \alpha_1 = 1$. Since $n \neq 3, 5, 7$, it follows from Corollary 5.1 that $n \in G_1$.
2. $s = 1$ and $\alpha_1 \geq 2$. By Proposition 5.2, $n \in G_1$.
3. $s \geq 2$. By applying Propositions 5.2, 5.3 and 5.4, we have $n = q_1^{\alpha_1} \cdots q_s^{\alpha_s} \in G_1$ since $q_1^{\alpha_1}, \dots, q_s^{\alpha_s} \in E_1 \cup G_1$.

The proof is finished. □

5.2 On the equation $\sigma(n) = \tau(kn \cdot \varphi(n))$, $k \geq 2$

Proposition 5.5. *Let $s, k \geq 2$ and let n_1, \dots, n_s be relatively prime positive integers with $n_i \geq 2$ for $i = 1, 2, \dots, s$. If $n_1, \dots, n_s \in E_k \cup G_k$, then $n_1 \cdots n_s \in G_k$.*

proof. The proof is the same as the proof of Proposition 5.4 since $(kn_i, kn_j) > 1$ for $1 \leq i, j \leq s$. □

Lemma 5.2. *Let q_1, \dots, q_s be distinct primes. If one of the following conditions:*

- (i) $s = 2$ and $q_2 \geq 29$
- (ii) $s = 3$ and $q_3 \geq 11$
- (iii) $s \geq 4$,

is satisfied then the inequality

$$(1 + q_1)^2 \cdots (1 + q_s)^2 > 2^{2s+4} (q_1 - 1) \cdots (q_s - 1)$$

holds.

proof. We can easily prove the claim by induction on s . □

Proposition 5.6. *We have $E_2 = \{2, 11\}$ and $L_2 = \{1, 3, 5, 7, 13\}$.*

proof. Clearly, $1 \in L_2$. Assume that $n = p$ is prime with $p \in E_2$. Then $1 + p = \tau(2p(p - 1))$. One can easily check that $p = 2$ is a solution. But, if p is odd, and we suppose that $p \in L_2 \cup E_2$, then we have

$$\frac{1+p}{4} \leq \tau(p-1), \quad (5.6)$$

from which it follows that $p^2 - 62p + 65 \leq 0$. This inequality is not valid whenever $p \geq 61$. However, for $p \in \{3, \dots, 59\}$ we have $n = p = 11$ is the only solution of $\sigma(n) = \tau(2\varphi(n^2))$. Thus, if $p \in E_2$ then p is either 2 or 11. Consequently, for every prime $p \geq 61$, we have $p \in G_2$. By simple computation, for every prime $p \leq 59$ with $p \neq 2, 3, 5, 7, 11, 13$, we have $p \in G_2$. Thus, the only primes of L_2 are 3, 5, 7 and 13.

Next, let $n = p^\alpha$ with $\alpha \geq 2$ and assume that $\sigma(n) \leq \tau(2\varphi(n^2))$. First, assume that p is odd. It follows that

$$\frac{1+p+\dots+p^\alpha}{2\alpha} \leq \tau(2(p-1)) \leq 2\sqrt{2(p-1)}, \quad (5.7)$$

and so $(1+p+\dots+p^\alpha)^2 \leq 32\alpha^2(p-1)$, which is only true for $p = 3$ and $\alpha = 2$. However, 3^2 is not in $E_2 \cup L_2$. Second, assume that $p = 2$. Since $\sigma(n) \leq \tau(2n\varphi(n))$, we conclude that $1+2+\dots+2^\alpha \leq 2\alpha+1$. This is impossible, and so $2^\alpha \in G_2$ for every $\alpha \geq 2$.

Let $s \geq 2$ and let $n = q_1 \cdots q_s$ be a square-free integer. Then $n \in G_2$. Assume, for sake of contradiction, that $n \notin G_2$. Then

$$\begin{aligned} (1+q_1) \cdots (1+q_s) &\leq \tau(2q_1 \cdots q_s ((q_1-1) \cdots (q_s-1))) \\ &\leq 2^{s+1} \tau((q_1-1) \cdots (q_s-1)) \\ &\leq 2^{s+2} \sqrt{(q_1-1) \cdots (q_s-1)}, \end{aligned}$$

and hence $(1+q_1)^2 \cdots (1+q_s)^2 \leq 2^{2s+4} (q_1-1) \cdots (q_s-1)$. This contradicts Lemma 5.2 whenever $s = 2$ and $q_2 \geq 29$ or $s = 3$ and $q_3 \geq 11$ or $s \geq 4$. In these cases, $n \in G_2$. On the other hand, by a simple computation, if $n = q_1 q_2$ with $q_2 \leq 23$ or $n = q_1 q_2 q_3$ with $q_3 \leq 7$, then $n \in G_2$. We deduce that if $n = q_1 q_2 \cdots q_s$ is a square-free with $s \geq 2$, then $n \in G_2$.

Now, let $a \in \{2, 3, 5, 7, 11, 13\}$ and $n \in G_2$. We show that $an \in G_2$. There are two cases:

1. $(a, n) = 1$. Since $1 + a \geq \tau((a-1)a)$, we conclude that

$$\begin{aligned} \sigma(an) &= (1+a)\sigma(n) \\ &> (1+a)\tau(2\varphi(n^2)) \\ &\geq \tau((a-1)a)\tau(2\varphi(n^2)) \\ &\geq \tau(2(a-1)a\varphi(n^2)) \\ &= \tau(2\varphi(a^2n^2)), \end{aligned}$$

and so $an \in G_2$.

2. $(a, n) > 1$. We put $an = ma^s$, where $s \geq 2$ and $(a, m) = 1$. As we have seen, we have $a^s \in G_2$. Note that if $m \in E_2 \cup G_2$, then $ma^s \in G_2$. In the case when $m \in L_2$, i.e., $m \in \{1, 3, 5, 7, 13\}$, we have two possibilities:

(i) If $m = 1$, then clearly $ma^s = a^s \in G_2$.

(ii) If $m \in \{3, 5, 7, 13\}$, then we can see by a similar argument that $\sigma(ma^s) > \tau(2\varphi((ma^s)^2))$. That is, $ma^s \in G_2$.

Finally, we deduce that if $n = q_1^{\alpha_1} \cdots q_s^{\alpha_s}$ with $n \neq 1, 2, 3, 5, 7, 11, 13$, then $n \in G_2$. This completes the proof of Proposition 5.6. \square

Remark 5.1. Similarly, we can prove that $E_3 := \{15, 35\}$ and

$$L_3 := \{1, 2, 3, 4, 5, 7, 11, 13, 17\}.$$

Theorem 5.2. Let $k \geq 2$. The sets L_k and E_k are finite, while G_k is infinite.

proof. At first, we prove that for any prime p there exists an exponent α_0 such that $p^x \in G_k$ for every $x \geq \alpha_0$. Let $n = p^\alpha$ be a prime power such that $p^\alpha \in E_k \cup L_k$. Then

$$1 + p + \cdots + p^\alpha \leq \tau(kp^{2\alpha-1}(p-1)).$$

Put $k = p^a m$, where $a \geq 0$, $m \geq 1$ and $(p, m) = 1$. Then

$$\frac{1 + p + \cdots + p^\alpha}{a + 2\alpha} \leq \tau(m(p-1)) \leq 2\sqrt{m(p-1)}, \quad (5.8)$$

and therefore,

$$(1 + p + \cdots + p^\alpha)^2 \leq 4(a + 2\alpha)^2 m(p-1).$$

Since a and m are fixed and α is the exponent of p , the last inequality has only finite number of solutions. Thus, $E_k \cup L_k$ contains only finitely many prime powers, namely $l_1^{\gamma_1}, l_2^{\gamma_2}, \dots, l_s^{\gamma_s}$. Consequently, it suffices to choose $\alpha_0 = \max(\gamma_1 + 1, \gamma_2 + 1, \dots, \gamma_s + 1)$.

Let $n = q_1 q_2 \cdots q_m$, where $q_1 < q_2 < \cdots < q_m$ are prime numbers ($m \geq 2$). Assume that $n \in E_k \cup L_k$. Therefore by (1.3) and (1.4) we obtained

$$\prod_{i=1}^m \frac{(1+q_i)^2}{q_i-1} \leq 2^{2m+2} \tau^2(k). \quad (5.9)$$

Since k is fixed and $(1+q_m)^2 / (q_m-1)$ can be sufficiently large, we deduce that there exists a positive integer m_0 for which the inequality (5.9) is not true for every $m \geq m_0$. Thus the inequality $\sigma(n) \leq \tau(kn \cdot \varphi(n))$ holds for finitely many square-free integers.

Finally, let $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_m^{\alpha_m}$, where $q_1 < q_2 < \cdots < q_m$ are prime numbers and $\alpha_1, \alpha_2, \dots, \alpha_m$ are positive integers. Note that the number m cannot be sufficiently large as we wish such that $\sigma(s) \leq \tau(k \cdot \varphi(s^2))$ for $s = q_1 q_2 \cdots q_m$. Moreover, from above there exist positive integers $\alpha_1^{(0)}, \alpha_2^{(0)}, \dots, \alpha_m^{(0)}$ such that the numbers

$$n' = q_1^{\alpha'_1}, \dots, q_m^{\alpha'_m}$$

satisfy the inequality $\sigma(n') > \tau(kn' \cdot \varphi(n'))$ for every $\alpha'_i \geq \alpha_i^{(0)}$ ($1 \leq i \leq m$). Applying Proposition 5.5, the numbers $q_1^{\alpha'_1} \cdots q_m^{\alpha'_m}$ with $\alpha'_i \geq \alpha_i^{(0)}$ ($1 \leq i \leq m$) satisfy the inequality $\sigma(n') > \tau(kn' \cdot \varphi(n'))$. Thus, the inequality $\sigma(n) \leq \tau(kn \cdot \varphi(n))$ has only a finite number of solutions, while the inequality $\sigma(n) > \tau(kn \cdot \varphi(n))$ has infinitely many solutions. \square

Proposition 5.7. We have $\bigcap_{k \geq 2} L_k = L_2$.

proof. It suffices to prove that $L_2 \subset \bigcap_{k \geq 2} L_k$. As we have already seen in Proposition 5.6, $L_2 = \{1, 3, 5, 7, 13\}$. Moreover, for every $k \geq 2$ we see that

$$\begin{aligned} \sigma(1) &= 1 < \tau(k) = \tau(k \cdot \varphi(1^2)) \\ \sigma(3) &= 4 < \tau(6k) = \tau(k \cdot \varphi(3^2)) \\ \sigma(5) &= 6 < \tau(20k) = \tau(k \cdot \varphi(5^2)) \\ \sigma(7) &= 8 < \tau(42k) = \tau(k \cdot \varphi(7^2)) \\ \sigma(13) &= 14 < \tau(156k) = \tau(k \cdot \varphi(13^2)). \end{aligned}$$

Thus, $1, 3, 5, 7, 13 \in L_k$ for every $k \geq 2$. This completes the proof. \square

Proposition 5.8. Let $r, s \geq 2$. If r divides s , then $L_r \subset L_s$.

proof. Let $n \in L_r$. Since r divides s , we see by (1.6) that

$$\sigma(n) < \tau(r \cdot \varphi(n^2)) \leq \tau(s \cdot \varphi(n^2)),$$

and so $n \in L_s$. \square

Corollary 5.2. There are infinitely many k such that $L_k \subset L_{k+1}$.

proof. Let k be a positive integer such that $\tau(k) \leq \tau(k+1)$. If $n \in L_k$, then

$$\sigma(n) < \tau(k \cdot \varphi(n^2)) \leq \tau((k+1) \cdot \varphi(n^2)).$$

Hence, $n \in L_{k+1}$. □

Remark 5.2. Let p be prime. Since $\tau(p) = 2 \leq \tau(p+1)$, we conclude that $L_p \subset L_{p+1}$. Also, we present the following notes:

1. One can easily check that if r divides s , then $E_r \cap E_s = \{\emptyset\}$. In particular, if α, β are two positive integers with $\alpha \neq \beta$, then $E_{p^\alpha} \cap E_{p^\beta} = \{\emptyset\}$. More generally, we can prove that if k, k' are two distinct positive integers having the same prime factors, then $E_k \cap E_{k'} = \{\emptyset\}$.
2. Using the same argument as the one in the proof of Theorem 5.1 and Proposition 5.6, the sets E_{27} and E_{51} are empty, i.e., $E_{27} = E_{51} = \{\emptyset\}$.

5.3 When does $k \in E_k$?

Theorem 5.3. We have $k \in E_k$ iff $k = 1, 2, 6, 11, 14$ and $k \in L_k$ iff $k = 3, 5, 7, 13, 15, 21, 33, 35, 65$ and 77 .

proof. Clearly, $1 \in E_1$ and by Proposition 5.6, we have $2 \in E_2$. Let $k \geq 3$ such that $k \in E_k$. We obtain

$$\sigma(k) = \tau(\varphi(k^3)). \quad (5.10)$$

Moreover, consider the related inequalities:

$$\sigma(k) < \tau(\varphi(k^3)), \quad (5.11)$$

and

$$\sigma(k) > \tau(\varphi(k^3)). \quad (5.12)$$

There are four parts:

1. Let p be a prime number satisfying (5.10). Then $p^2 - 34p + 37 \leq 0$, which means that $p \leq 31$ and so $p = 2$ or $p = 11$. Next, let $k = pq$ be a square-free integer satisfying (5.10), where $p < q$. Then $(1+p)^2(1+q)^2 \leq 324(p-1)(q-1)$. This means that $q \leq 31$. A quick research of all possibilities shows that the only solutions of (5.10) of the form $k = pq$ are 6 and 14. In addition, the numbers $k = pq$ satisfies (5.12) for every $(p, q) \neq (2, 3), (2, 7), (3, 5), (3, 7), (3, 11), (3, 7), (5, 13), (7, 11)$.

2. Let $s \geq 3$ and let $k = q_1 \cdots q_s$ be a square-free integer such that $\sigma(k) = \tau(\varphi(k^3))$. Then

$$(1 + q_1)^2 \cdots (1 + q_s)^2 \leq 3^{2s} \cdot 4 (q_1 - 1) \cdots (q_s - 1). \quad (5.13)$$

On the other hand, using the proof by induction we can prove that for $s \geq 6$, we have

$$(1 + q_1)^2 \cdots (1 + q_s)^2 > 3^{2s} \cdot 4 (q_1 - 1) \cdots (q_s - 1).$$

In the case when $3 \leq s \leq 5$, the above inequality holds for $q_s \geq 41$. This means that all the prime numbers $2 \leq q_1 < q_2 < \cdots < q_s \leq 41$ satisfy (5.13); but by simple computations they fail to satisfy the equation (5.10) or the inequality (5.11) for $k = q_1 \cdots q_s$. Hence, $k = q_1 \cdots q_s$ satisfies (5.12) for $s \geq 3$.

3. Let $k = p^\alpha$ with $\alpha \geq 2$. If $\sigma(k) \leq \tau(\varphi(k^3))$, then

$$(1 + p + \cdots + p^\alpha)^2 \leq 36\alpha^2 (p - 1).$$

This is only true for $p^\alpha \in \{2^2, 2^3, 3^2\}$. But these prime powers satisfy (5.12) and so does p^α for $\alpha \geq 2$.

4. Let $s \geq 2$ and let n_1, n_2, \dots, n_s be relatively prime positive integers with $n_i \geq 2$ for $i = 1, 2, \dots, s$. We can check that if n_1, n_2, \dots, n_s satisfy (5.10) or (5.12), then so does $n_1 n_2 \cdots n_s$.

Consequently, if $k \in E_k$ then $k = 1, 2, 6, 11, 14$ and if $k \in L_k$ then $k = 3, 5, 7, 13, 15, 21, 33, 35, 65$ and 77 . \square

In the following result we characterize the sets E_p and L_p , where p is prime.

Theorem 5.4. *Let p be a prime number with $p \geq 13$ and $p \neq 31$. Then $E_p = \{6, 14, 31, 33, 77\}$.*

proof. Let $p \geq 13$ be prime with $p \neq 31$. Clearly, we see that

$$\begin{aligned} \sigma(6) &= 12 = \tau(12p) = \tau(p \cdot \varphi(6^2)) \\ \sigma(14) &= 24 = \tau(84p) = \tau(p \cdot \varphi(14^2)) \\ \sigma(31) &= 32 = \tau(930p) = \tau(p \cdot \varphi(31^2)) \\ \sigma(33) &= 48 = \tau(660p) = \tau(p \cdot \varphi(33^2)) \\ \sigma(77) &= 96 = \tau(4620p) = \tau(p \cdot \varphi(77^2)), \end{aligned}$$

and so $6, 14, 31, 33, 77 \in E_p$.

Now, let $n \geq 1$ such that $n \in E_p \cup L_p$. Therefore,

$$\sigma(n) \leq \tau(p \cdot \varphi(n^2)) \leq \tau(p) \tau(\varphi(n^2)) = 2\tau(\varphi(n^2)).$$

As in the proof of Theorem 5.3, the only solutions of $\sigma(n) \leq 2\tau(\varphi(n^2))$ are $n = 1, 2, 3, 4, 5, 6, 7, 9, 11, 13, 14, 15, 17, 19, 21, 31, 33, 35$ and 77 . In the case when $p \geq 23$, it is easy to check that the numbers $1, 2, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 21$ and 35 satisfy the inequality $\sigma(n) < \tau(p \cdot \varphi(n^2))$. Thus, $6, 14, 31, 33, 77$ are the only solutions of $\sigma(n) = \tau(p \cdot \varphi(n^2))$, where $p \geq 13$ is prime with $p \neq 31$. This completes the proof. \square

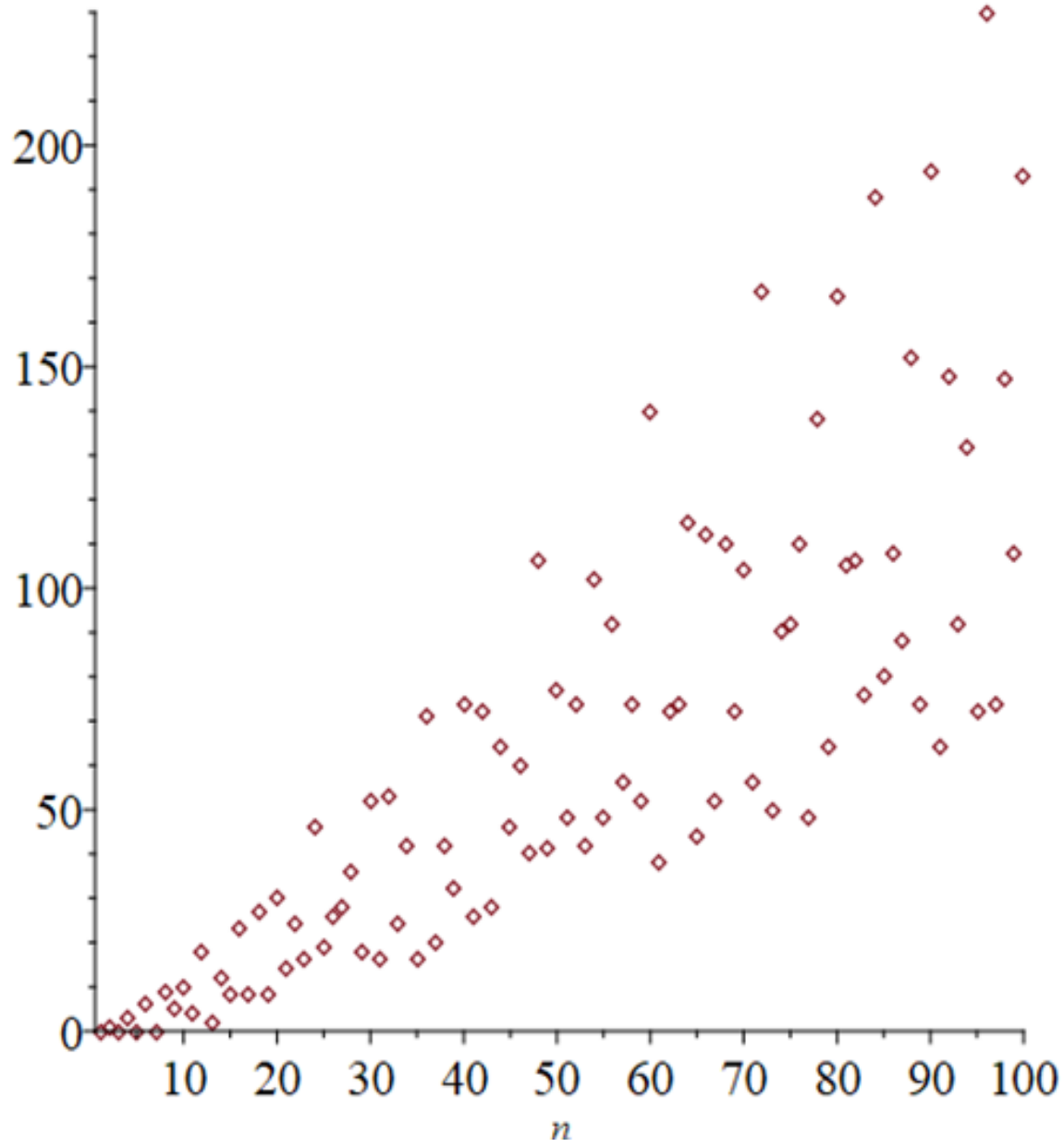
Remark 5.3. By the same reason, one can prove that

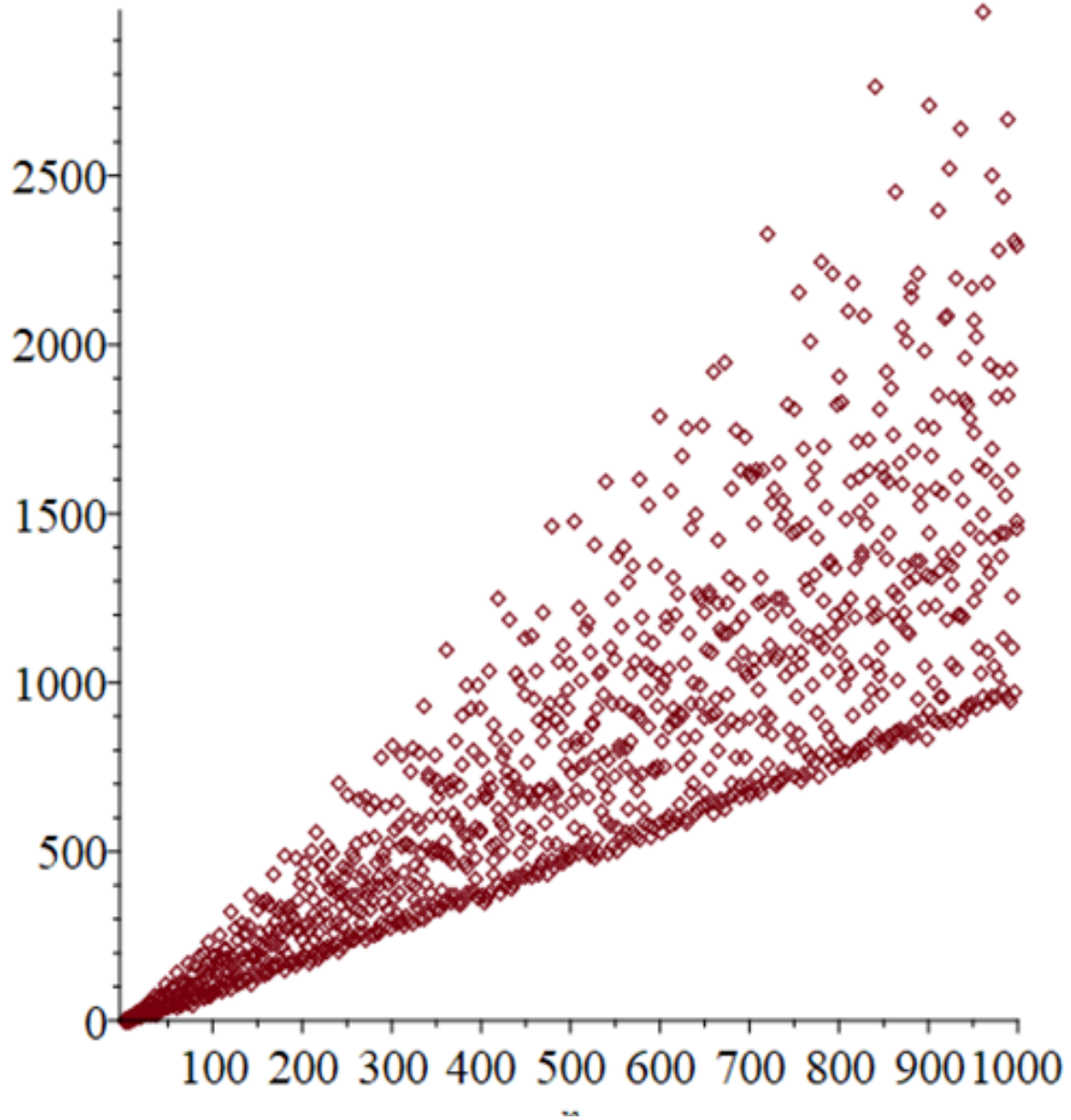
- $E_5 = \{6, 11, 14, 15, 35\}$,
- $E_7 = \{6, 31, 33, 35\}$,
- $E_{11} = \{6, 11, 14, 31\}$
- $E_{31} = \{6, 14, 33, 77\}$.

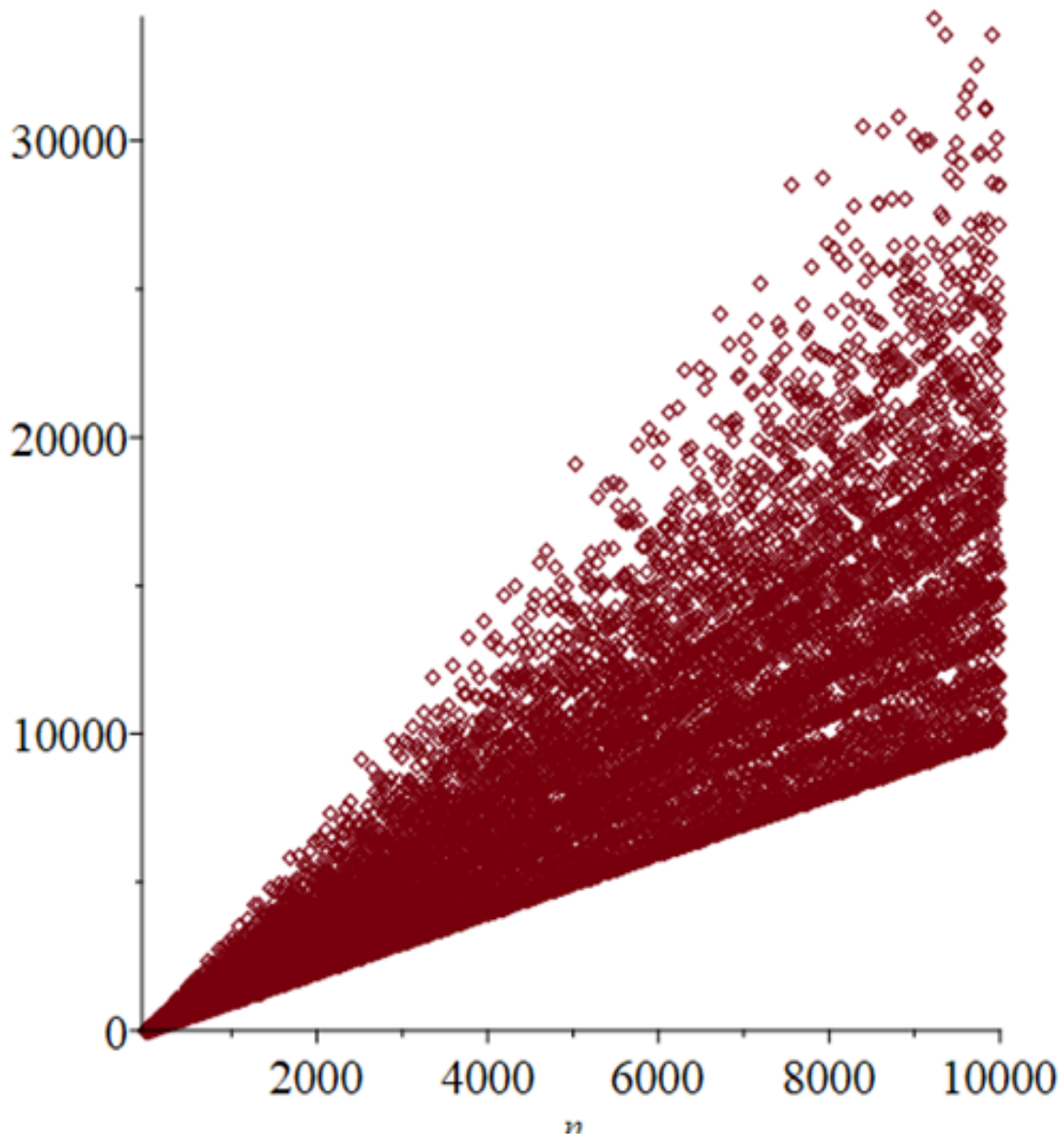
Moreover, from the proof of Theorem 5.4, for any prime $p \geq 23$ we deduce that

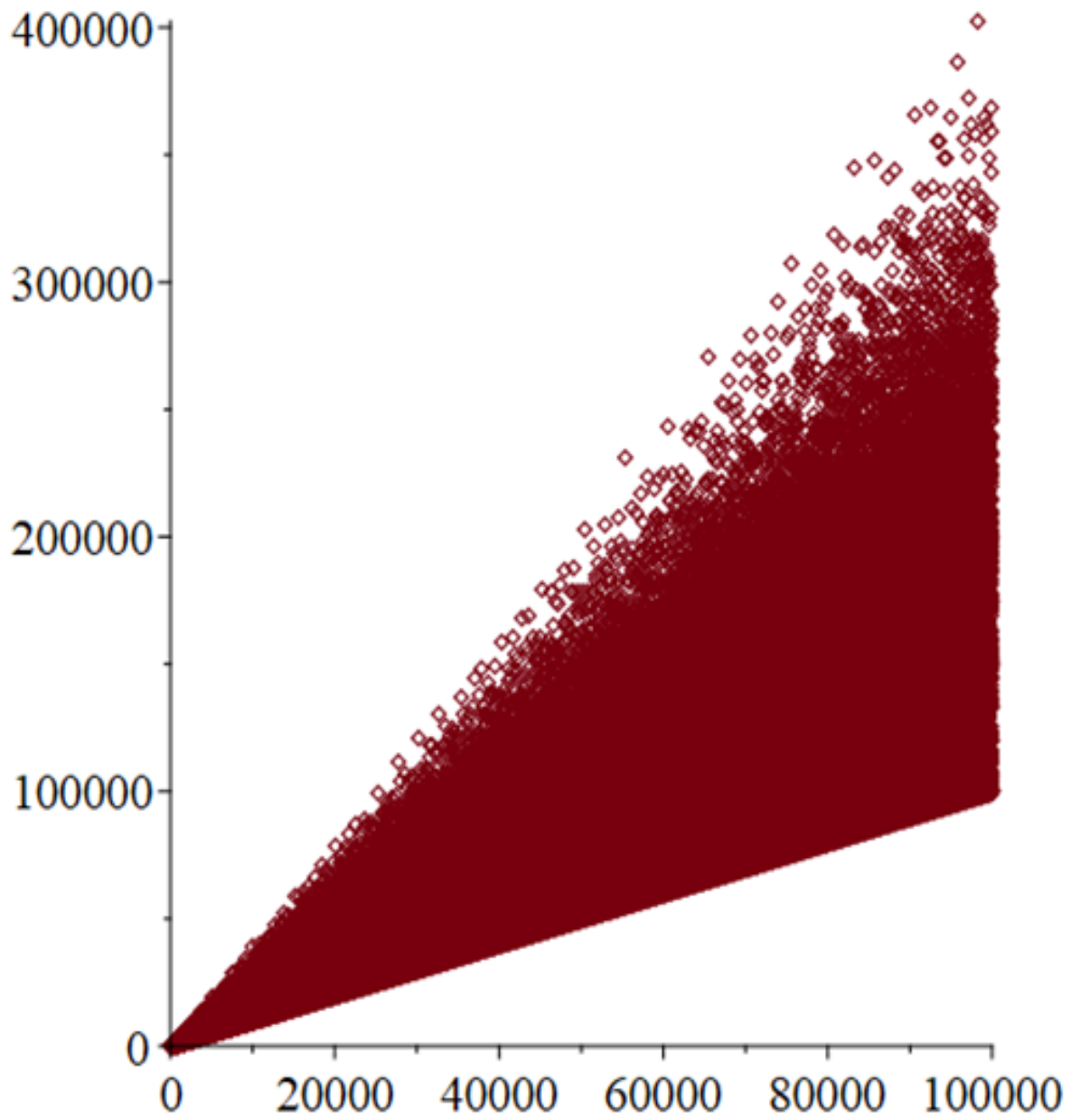
$$L_p = \{1, 2, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 21, 35\}.$$

Representing the sequence $n \mapsto \sigma(n) - \tau(\varphi(n^2))$ for n from 1 to 100000

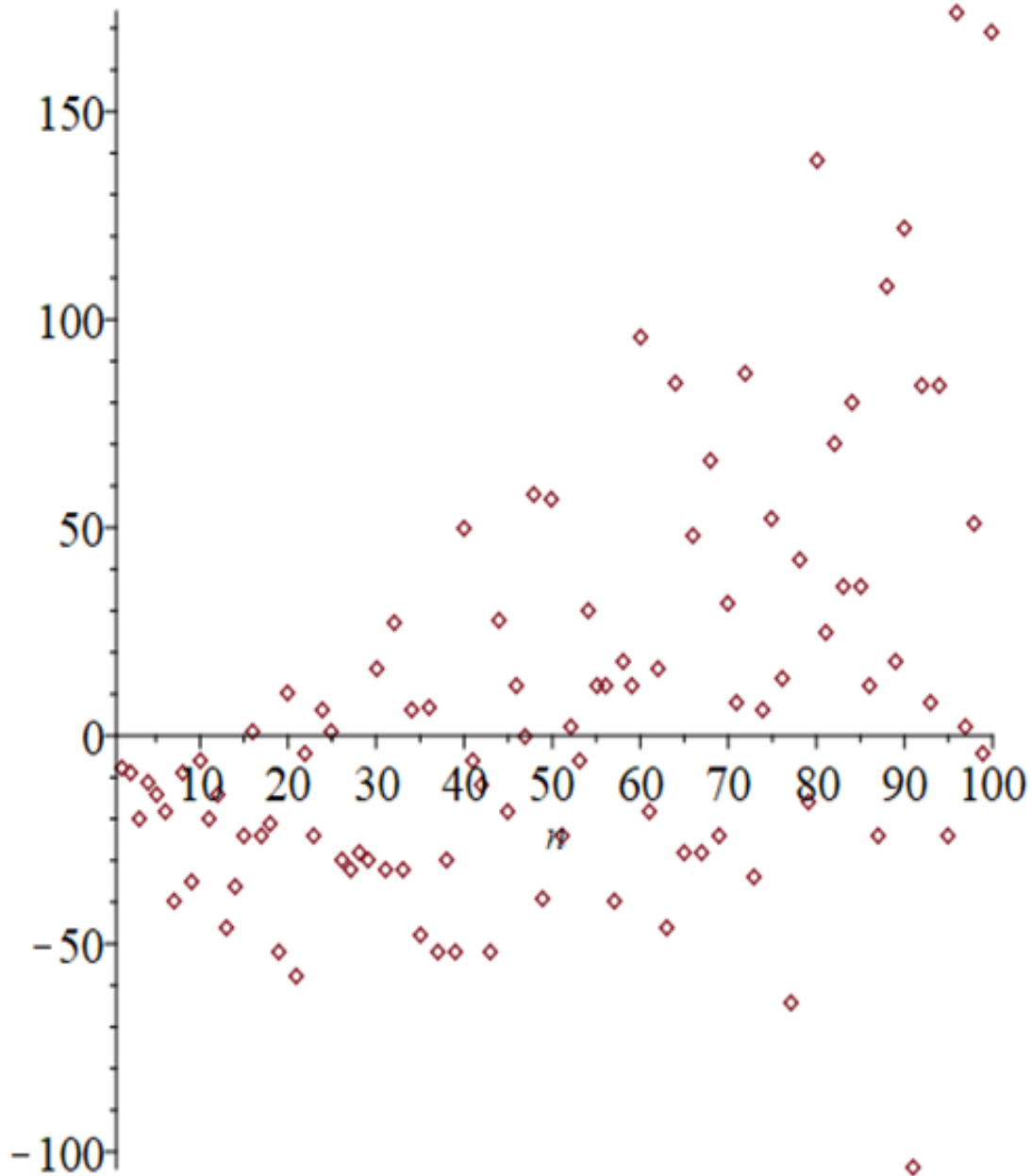




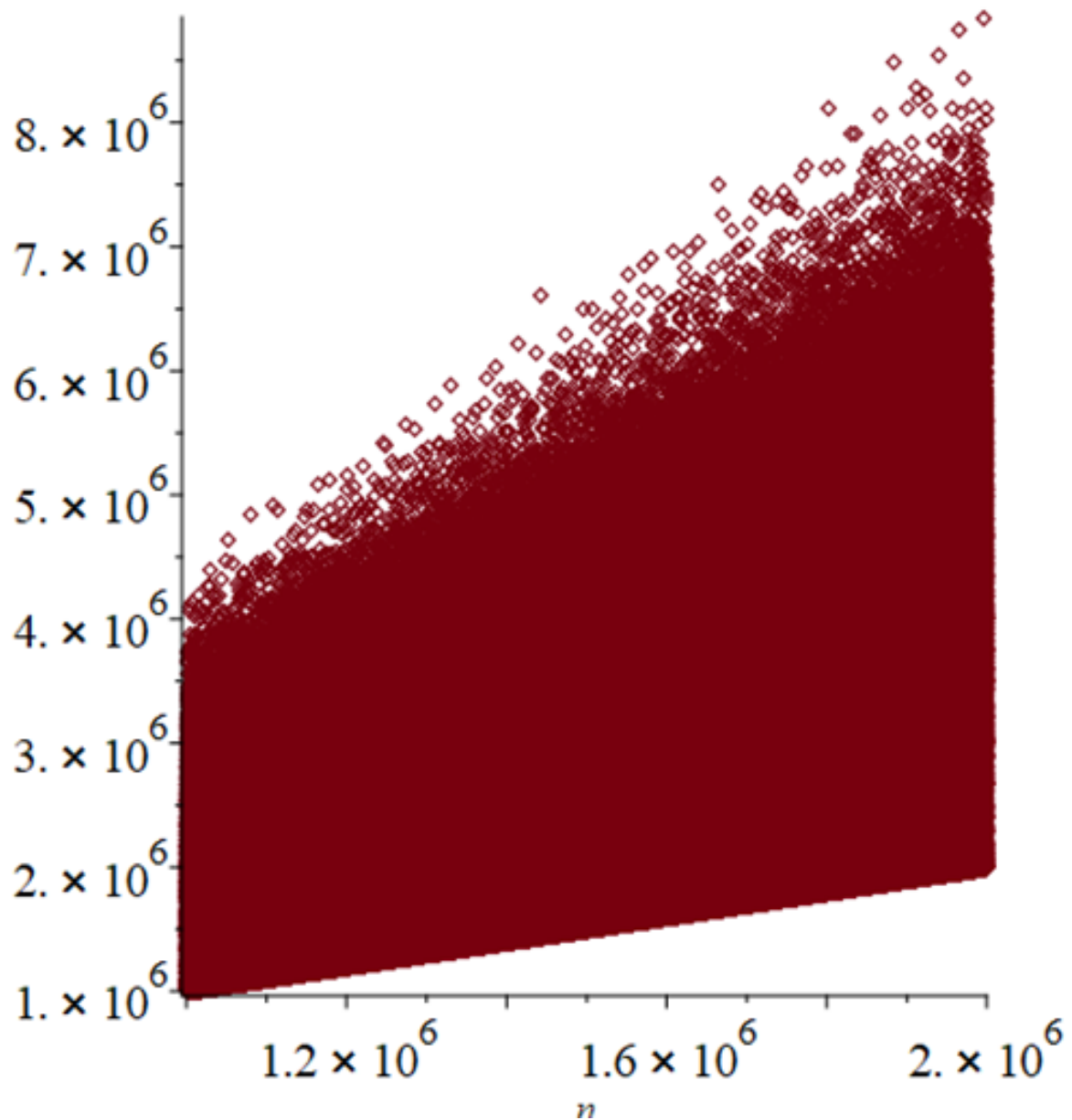




Representing the sequence $n \mapsto \sigma(n) - \tau(100n\varphi(n))$ for n from 1 to 100



Representing the sequence $n \mapsto \sigma(n) - \tau(\varphi(n^2))$ for n from 1000000 to 2000000



Conclusion and open problems

The work contained in this thesis is essentially composed of two main parts. The first part consists of preliminary concepts of Diophantine equations and arithmetic functions, in addition to detailing some of the most important similar modern work.

In the second part, we present the results and proofs we have obtained in solving some Diophantine equations that contain arithmetic functions in the last three chapters.

The different results that we have proved in the chapter 03 give rise to diophantine equations that deserve to be studied. Here, we give some examples.

1. In Theorem 3.1, we need to solve the system:

$$\begin{cases} 2 \cdot 3^t + 1 \text{ is prime,} \\ ab + 2a + 2b + 1 = 3bt, \end{cases}$$

where a, b, t are non-negative integers.

2. In Proposition 3.4, we need to solve the system:

$$\begin{cases} p & \text{is prime,} \\ 2^t p^4 + 1 & \text{is prime,} \\ t & \text{positive integer.} \end{cases}$$

3. In Proposition 3.6, we need to solve the system:

$$\begin{cases} p & \text{is prime,} \\ 2^s p^2 + 1 & \text{is prime,} \\ ab + 2a + 2b + 1 = 3bs, \end{cases}$$

where a, b, s are positive integers.

In the chapter 04, we investigated the solutions n of the equation (i) $\tau(kn) = \varphi(\varphi(n))$ and also the respective solutions of the corresponding inequalities (ii) $\tau(kn) < \varphi(\varphi(n))$ and (iii) $\tau(kn) > \varphi(\varphi(n))$. Since the positive integers are naturally partitioned into 3 subsets according to (i), (ii) and (iii), we have also characterized the elements of the sets $\mathbb{E}_1, \mathbb{L}_1, \mathbb{G}_1, \mathbb{E}_p, \mathbb{L}_p$ and \mathbb{G}_p whenever p is prime. As a conclusion, we gave the relation between the sizes of these sets, where $\cap \mathbb{E}_p = \{\emptyset\}$ as p runs through the sequence of all primes. The same technique is applied to characterize the elements $\mathbb{E}_k, \mathbb{L}_k$ and \mathbb{G}_k whenever k is composite.

In the chapter 05, it seems that the most Diophantine equations of the form $\sigma(n) = F(n)$, where F is an arithmetic function formed by τ and φ , have only finite number of

positive solutions. In this sense we have already seen in [15], the author conjectured that $n = 2$ is the only known solution of $\sigma(n) = n + \varphi(n)$.

Similarly, in the present work we prove that each of the equations $\sigma(n) = \tau(kn \cdot \varphi(n))$ has finitely many solutions. Even though it may seem that any finite set of consecutive integers includes in L_k for some $k \geq 1$. For example, $\{1, 2, 3\} \subset L_3$, $\{1, 2, 3, 4, 5\} \subset L_5$, $\{1, 2, 3, 4, 5, 6, 7\} \subset L_{10}$ and $\{1, 2, \dots, 11\} \subset L_{12}$. In the same context and for further research we propose the following question: When does $\{1, 2, \dots, k\} \subset L_{k+1}$? There are also questions on the set E_k ; we ask whether E_k includes one element for infinitely many k or not. For example, we can prove that $E_{21} = \{51\}$, $E_{32} = \{10\}$, $E_{38} = \{23\}$, $E_{46} = \{95\}$ and $E_{48} = \{47\}$. In particular, when does $E_k = \{p\}$ with p is prime? On the other hand, we have $E_{27} = E_{51} = \{\emptyset\}$. Are there some others k with this property, are there infinitely many?

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ملخص

الهدف الرئيسي من هذه الاطروحة هو إيجاد حلول وخصائص بعض المعادلات الديوفونتية التي تحوي دوال حسابية. لذلك تم دراسة عدة معادلات من هذا الشكل منها التي هي ذات حلول منتهية تم إيجادها كلها ومنها الغير منتهية تم تمثيل جزء منها وإثبات أنها غير منتهية.

كلمات مفتاحية: المعادلات الديوفانتية. الدوال الحسابية. مجموع القواسم. عدد القواسم. دالة أولر.

Abstract

The main aim of this thesis is to find solutions and properties to some Diophantine equations containing arithmetic functions.

To this end, several equations of this form have been studied, among those containing finite solutions, all have been found, and among those containing infinite solutions, a part has been represented and proved infinite.

Key words: Diophantine equations, arithmetic functions, number of positive divisors, sum of positive divisors, Euler's phi function.

Résumé

Le but principal de cette thèse est de trouver des solutions et des propriétés aux quelques équations diophantiennes contenant des fonctions arithmétiques.

A cette fin, plusieurs équations de cette forme ont été étudiées, parmi celles contenant des solutions finies, toutes ont été trouvées, et parmi celles contenant des solutions infinies, une partie a été représentée et prouvée infinie.

Mots clés : Equations diophantiennes, fonctions arithmétiques, la somme de diviseur positive, le nombre de diviseur positive, fonction d'Euler.

