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Linear Fredholm integro-differential equations, High-order equations, Pell-Lucas polynomials, Numerical methods

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ملخص

تقدم هده الأطروحة نهجا عددا لحل المعادلات التفاضلية التكاملية من نوع فريدهولم الخطية دات الرتب العالية بناء على كثيرات حدود بيل -لوكاس . تم تقديم العديد من الأمثلة العددية الإثبات دقة وكفاءة وموثوقية الطريقة الحسابية المقترحة.

كلمات مفتاحية: المعادلات التفاضلية التكاملية لفريدهولم الخطية؛ معادلات دات رتب عالية ؛ كثيرات حدود بيل-لوكاس؛ الطرق العددية.

Abstract

This thesis presents a numerical approach for solving high-order linear Fredholm integro-differential equations based on Pell-Lucas polynomials. Several numerical examples are provided to demonstrate the accuracy, efficiency, and reliability of the proposed computational method.

Keywords :
Linear Fredholm integro-differential equations; High-order equations; Pell–Lucas polynomials; Numerical methods.

Résumé

Cette thèse présente une approche numérique pour résoudre les équations intégro-différentielles de Fredholm linéaires d'ordre élevé basées sur les polynomes de Pell-Lucas. Plusieurs exemples numériques sont fournis pour démontrer l'exactitude, l'efficacité et la fiabilité de la méthode de calcul proposée.

Mots-clés : Equations intégro-différentielles linéaires de Fredholm ; Equations d'ordre élevé ; polynomes de Pell-Lucas ; Méthodes numériques.

Appreciation

First and foremost, I would like to express my deepest gratitude to Almighty God for giving me the strength, health, and patience to complete this graduation thesis.

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Dedication

I dedicate this modest work and the fruit of my perseverance:

*To my dear parents,**Mohamed Brikel** and **Laamria Noui**,*
whose endless love, prayers, and profound sacrifices have always been my guiding light.

*To my beloved husband,**Younes Brikel**,*
for his unwavering support, patience, and encouragement throughout this academic journey.

*To the heartbeat of my life, my sweet little daughter,**Rahil**,*
who brought pure joy into my world and made every challenge worth fighting for.

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Contents

Appreciation & Dedication **2**

Chapter 1 Preliminary **7**

1.1 Functional spaces 7

1.1.1 Normed Vector Spaces 7

1.1.2 Banach Space 8

1.1.3 Hilbert Space 8

1.2 Fundamental concepts of Linear Operators 9

1.2.1 Linear Operators 9

1.2.2 Bounded Linear Operators 10

1.2.3 Compact Linear Operators 10

1.3 Integral Operators 11

2. 1 Classification of Integral Equations 12

2.1. 1 Fredholm Integral Equation 12

2.1. 2 Volterra Integral Equations 13

2.1. 3 Volterra-Fredholm Integral Equations 13

2.2 Classification of integro-differential equations 14

2.2. 1 Fredholm Integro-Differential Equations 14

2.2. 2 Volterra Integro-Differential Equations 14

2.2. 3 Voltera-Fredholm Integro-Differential Equations 15

2.3 Pell-Lucas polynomials 15

2.4 Properties of Pell-Lucas Polynomials 16

2.4. 1 Generating Function 16

2.4. 2 Orthogonality and weight-function 16

2.4. 3 Shifted Pell-Lucas Polynomials . . . 17
3.1 Numerical Solution Techniques for Fredholm Integro-Differential Equations . . . 19
3.1. 1 Matrix Representation of the Integral Part . . . 21
3.1. 2 Collocation Points and Boundary Conditions . . . 22
3.2 Numerical Results . . . 25
Conclusion . . . 31

Introduction

Integro -differential equations arise naturally in many problems of applied mathematics,physics,and engineering.In particular,linear Fredholm integro-differential equations of the form

$$\sum_{k=0}^{m} P_k(x)u^{(k)}(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \quad x \in [a,b], \tag{1}$$

play an imortant role in the modeling of systems with memory effects and boundary constraints. In general,exact analytical solutions of high-order linear Fredholm integro-differential equations are difficult to obtain.Therefore, numerical methods are required to construct accurate approximate solutions.Various numerical approaches have been proposed, such as collocation methods, spectral methods,and polynomial-based approximation techniques.

In this work,a numerical method based on **Pell–Lucas polynomials** is proposed for solving high-order linear Fredholm integro-differential equations.the unknown solution $u(x)$ is approximated by a finite series

$$U_N(x) = \sum_{n=0}^{N} c_n PL_n(x), \tag{2}$$

where $PL_n(x)$ denotes the Pell–Lucas polynomials and c_n are unknown coefficients. By substituting this approximation into the original equation and applying a suitable numerical scheme , the problem is reduced to a system of algebraic equations for the coefficients c_n. Several numerical examples are presented to illustrate the accuracy and efficiency of the proposed method .the numerical results confirm that the Pell–Lucas polynomial approach provides reliable approximations and can be effectively applied to linear Fredholm integro-differential equations.

Organization of the thesis

the organization of this thesis is as follows:

- **Chapter 1:** presents the necessary preliminaries and basic concepts of functional spaces and linear operators.
- **chapter 2:** Is devoted to the study of integral equations and the properties of Pell–Lucas polynomials.
- **chapter 3:** Describes the proposed numerical method and its mathematical implementation.

Chapter 1

Preliminary

In this chapter we introduce the essential concepts, notations and functional spaces required for the study.

1.1 Functional spaces

1.1.1 Normed Vector Spaces

Let F be a vector space over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A **norm** on F is a function $\|\cdot\| : F \to \mathbb{R}_+$ that satisfies the following axioms for all $u, v \in F$ and $\lambda \in \mathbb{K}$:

(i) Positivity: $\|u\| \geq 0$, and $\|u\| = 0$ if and only if $u = 0$.

(ii) Homogeneity: $\|\lambda u\| = |\lambda| \|u\|$.

(iii) Triangle Inequality: $\|u + v\| \leq \|u\| + \|v\|$.

the pair $(F, \|\cdot\|)$ is then called a **normed vector space**.

Examples:

1. **The Space $\mathbb{R}^n$:** for an element $u = (u_1, \dots, u_n)$, we can define:

$$\|u\|_2 = \left(\sum_{i=1}^{n} |u_i|^2 \right)^{1/2}, \quad \|u\|_\infty = \max_{1 \leq i \leq n} |u_i|$$

2. **The Space $C[a, b]$:** the space of continuous functions on $[a, b]$ is commonly equipped with:

   - **Maximum Norm:** $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$, convergence in this norm corresponds to **uniform convergence**.
   - **L^2 Norm:** $\|f\|_2 = \left(\int_a^b |f(x)|^2 dx \right)^{1/2}$, convergence here is called **mean square convergence**.

1.1.2 Banach Space

A sequence $(u_n)_{n\in\mathbb{N}}$ in a normed space $(F, \|\cdot\|)$ is called a **Cauchy sequence** if for every $\epsilon > 0$, there exists an integer $N_\epsilon \in \mathbb{N}$ such that for all $p, q \geq N_\epsilon$:

$$\|u_p - u_q\| < \epsilon \tag{1.3}$$

A normed vector space $(F, \|\cdot\|)$ is said to be **complete** if every Cauchy Sequence (u_n) in F converges to an element $u \in F$ in the norm $\|\cdot\|$, which means:

$$\lim_{n\to\infty} \|u_n - u\| = 0 \tag{1.4}$$

Consequently, a complete normed vector space is defined as a **Banach space**

As a prime example, the space $C[a, b]$ of continuous functions forms a complete space when it is equipped with the **maximum norm**,which is defined for any $f \in C[a, b]$ as:

$$\|f\|_\infty = \max_{t\in[a,b]} |f(t)| \tag{1.5}$$

Every finite-dimensional normed vector space is automatically a Banach space.

$$\dim F < \infty \tag{1.6}$$

1.1.3 Hilbert Space

Let H be a vector space. An inner product on H is a function $\langle\cdot,\cdot\rangle : H^2 \to \mathbb{K}$ that associates each pair of vectors $(u, v) \in H^2$ with a scalar, satisfying the following properties for all $u, v, f \in H$ and $\alpha, \beta \in \mathbb{K}$:

(i) Linearity:
$\langle \alpha u + \beta v, f\rangle = \alpha\langle u, f\rangle + \beta\langle v, f\rangle$.

(ii) Symmetry:
$\langle u, v\rangle = \overline{\langle v, u\rangle}$.

(iii) Positivity:
$\langle u, u\rangle \geq 0$, and $\langle u, u\rangle = 0 \iff u = 0$.

A normed vector space H equipped with an **inner product**. If this space is complete with respect to the norm induced by the inner product:

$$\|u\| = \sqrt{\langle u, u\rangle} \tag{1.7}$$

it is called a **Hilbert Space**.

Example in $L^2[a,b]$

The space $L^2[a,b]$ is the Hilbert space in the context of integral equations. For any two functions$u, v \in L^2[a,b]$, the inner product is defined as:

$$\langle u, v\rangle = \int_a^b u(x)v(x)dx \tag{1.8}$$

In this case ,we say that u and v are **orthogonal** if $\langle u, v\rangle = 0$.

1.2 Fundamental concepts of Linear Operators

1.2.1 Linear Operators

Let X and Y be two normed spaces, An operator $B : X \longrightarrow Y$ is called a linear operator if for all $u, v \in X$ and for all scalars $\alpha, \beta(\mathbb{R}$ or$\mathbb{C})$:

$$B(\alpha u + \beta v) = \alpha B(u) + \beta B(v) \tag{1.9}$$

Linearity ensures that the operator preserves vector addition and scalar multiplicatin.
Furthermore, we say that the linear operator B is **continuous** at a point $u_0 \in X$ if for every sequence (u_n) in X that converges to u_0, the sequence $B(u_n)$ converges to $B(u_0)$. In other words:

$$\lim_{n\to\infty} B(u_n) = B\left(\lim_{n\to\infty} u_n\right) = B(u_0). \tag{1.10}$$

We say that the operator B is continuous on X if it is continuous at every point of X.

1.2.2 Bounded Linear Operators

B linear Operator $B : X \longrightarrow Y$ is said to be bounded if there exists a constant $C > 0$ such that:

$$\|Bu\|_Y \leq C\|u\|_X, \quad \forall u \in X \tag{1.11}$$

the smallest constant C satisfying this inequality represrnts the bound of the Operator B. Since a linear operator is bounded if and only if it is continuous, we can introduce its norm. For a bounded linear operator $B : X \longrightarrow Y$,the operator norm is defined by :

$$\|B\| = \sup_{u \neq 0} \frac{\|Bu\|_Y}{\|u\|_X}$$

.

Equivalently:

$$\|B\| = \sup_{u \neq 1} \|B(u)\|_Y = \sup_{\substack{u \in X \\ u \neq 0, \|u\| \leq 1}} \|B(u)\|_Y. \tag{1.12}$$

$$B \text{ continuity linear operator } \iff \quad B \text{ boundedn}$$

1.2.3 Compact Linear Operators

Let X and Y be Banach Spaces, B linear operator $B : X \rightarrow Y$is called compact if it maps every bounded subset into a set with a compact closure in Y, meaning that $\overline{B(\Omega)}$ is compact. Equivalently, B is compact if for every bounded sequence (φ_n) in X, there exists a subsequence (φ_{n_k}) such that the corresponding sequence of images $B\varphi_{n_k}$converges in Y.

For any compact operators B_1 and B_2 and any scalars α and β,, their linear combination

$$B = \alpha B_1 + \beta B_2$$

is also compact, since the sequence of images is expressed as

$$B\varphi_n = \alpha B_1 \varphi_n + \beta B_2 \varphi_n$$

and necessarily contains a convergent subsequence.

The composition of two linear operators $DB : X \to Z$ is compact provided that either B or D is compact, and the limit B of a norm-convergent sequence of compact operators $B(B_n)$ satisfying:

$$\|B_n - B\| \to 0$$

is also compact.

In addition, if the domain space X is finite-dimensional ($\dim X < \infty$), any bounded linear operator is automatically compact because its range is finite-dimensional and satisfies

$$\dim B(X) \leq \dim X$$

. Consequently, the identity operator I on X is compact if and only if the space X is finite-dimensional.
This shows that while every compact operator is fundamentally bounded and continuous, the converse holds only in infinite-dimensional spaces, as the identity operator on an infinite-dimensional spaces remains bounded but is not compact.

1.3 Integral Operators

Let $G \subset \mathbb{R}^n$ be a bounded domain. An integral operator B is a classic class of linear mappings defined on a suitable function space, which transforms a given function u on G, the operator B is defined by:

$$(Bu)(x) = \int_G k(x, y)u(t)dt \tag{1.13}$$

where the function $k(x, y)$ is termed the kernel of the integral operator, defined on the product set G.

If the kernel $k(x, y)$ satisfies appropriate integrability conditions, the operator B defines a bounded linear mapping:

$$B : L^p(G) \to L^p(G), \quad \text{for} 1 \leq p \leq p \leq \infty$$

Consequently, its operator norm $\|B\|_p$ is finite, and for every $u \in L^p(G)$ we have:

$$\|Bu\|_p \leq \|B\|_p \|u\|_p \tag{1.14}$$

Chapter 2

Integral Equations

In this chapter ,we discuss integral equations and some basic concepts related to thrm we also present a classification of these equations gives a genaral overview that helps in understanding the main types of integral equations.

2. 1 Classification of Integral Equations

2.1. 1 Fredholm Integral Equation

A fredholm integral equation is an integral equation in which the limits of integration are constants ,the general form is given by:

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt \tag{1.15}$$

were $u(x)$ is the unknown function ,$f(x)$ is a given function,$k(x,t)$ is called the kernel of the equation ,and λ is a constant parameter.
Fredholm integral equations are classified into two types:

- **First Kind:**

  $$f(x) = \int_a^b K(x,t)u(t)\,dt \tag{1.16}$$

  In this case ,the unknown function appears only unded the integral sign.

- **Second Kind:**

  $$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt \tag{1.17}$$

  Here, the unknown function appears both inside and outside the integral sing.

Example

$$u(x) = 1 + \int_0^1 (x+t)u(t)\,dt \tag{1.18}$$

since the limits of integration are constants from 0 to 1, this equation is a fredholm integral equation.

2.1. 2 Volterra Integral Equations

Volterra integral equations are integral equations in which at least one limit of integration is a variable they are divided into two types:

- **First Kind:**the unknowo function $u(x)$ appears only inside the integral.

$$f(x) = \int_0^x k(x,t)u(t)\,dt \tag{1.19}$$

Example:

$$xe^{-x} = \int_0^x e^{t-x}u(t)\,dt \tag{1.20}$$

- **Second Kind:** the unknown function $u(x)$ appears inside and outside the integral .

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)\,dt \tag{1.21}$$

Example:

$$u(x) = x + \int_0^x (x-t)u(t)\,dt \tag{1.22}$$

2.1. 3 Volterra-Fredholm Integral Equations

Volterra-Fredholm integral equations are a type of integral equations that combine volterra integral equations and fredholm integral equations ,they appear in many applications such as mathematical ,physical and biological models
the general form is given by :

$$u(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)u(t)\,dt + \lambda_2 \int_a^b k_2(x,t)u(t)\,dt \tag{1.23}$$

where $u(x)$ is the unknown function , $f(x)$ is a known function ,and $k(x,t)$ is called the kernel function the unknown function appears inside and outside the integral.

Example:

$$u(x) = 6x + 3x^2 + 2 - \int_0^x xU(t)\,dt - \int_0^1 tU(t)\,dt \tag{1.24}$$

2.2 Classification of integro-differential equations

integro-differentiol equations contain both integral operators and differential operators ,and the derivatives of the unknown function may appear in any order these equations are classified according to the type of integral.

2.2. 1 Fredholm Integro-Differential Equations

Fredholm integro-differential equations appear in the general form:

$$u^{(k)}(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt \tag{1.25}$$

these equations are pravided with initial or boundary conditions to ensure a unique and well-defined solution.

Examples: $u(a) = \alpha$, $u(b) = \beta$

- **First Kind:** the unknown function appears inside the integral only .

$$\begin{cases} u'(x) = x + \int_0^1 (x+t)u(t)dt \\ u(0) = 1 \end{cases} \tag{1.26}$$

- **Second Kind:** the unknown function appears inside and outside the integral.

$$\begin{cases} u''(x) = 6x + \int_0^1 (x+t)u(t)dt \\ u'(0) = 1, \quad u(0) = 0 \end{cases} \tag{1.27}$$

2.2. 2 Volterra Integro-Differential Equations

Volteraa integro-differential equations arise when initial value problems transformed into integral equations to obtain a particular solution initial conditions must be specified.
the general form of a volterra integro-differential equation is given by :

$$u^{(k)}(x) = f(x) + \lambda \int_0^x k(x,t)u(t)\,dt \tag{1.28}$$

2.2. 3 Voltera-Fredholm Integro-Differential Equations

Volterra-Fredholm integro -differential equations arise in the same way as Volterra -Fredholm integral equations but they also contain ordinary derivatives of the unknown function in addition to the integral terms.
the general form is :

$$u^{(k)}(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)u(t)\,dt + \lambda_2 \int_a^b k_2(x,t)u(t)\,dt \tag{1.29}$$

where $u(x)$ is the unknown function, $u^{(n)}(x)$ is the k the derivative, k_1, k_2 are kernel functions, $f(x)$ is a given functions, λ_1, λ_2 is a constant.

Examples

$$\begin{cases} u''(x) = x + \int_0^1 (1+x-t)u(t)dt \\ u'(0) = 0, \quad u(0) = 1 \end{cases} \tag{1.30}$$

it is a Volterra -Fredholm-differential equation of Second Order.

2.3 Pell-Lucas polynomials

for $n \geq 1$, the Pell-Lucas polynomials $PL_n(x)$ of degree n can be expressed in the following explicit form:

$$PL_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} (2x)^{n-2k} \tag{1.31}$$

$$\begin{aligned} PL_3(x) &= 8x^3 + 6x \\ PL_4(x) &= 16x^4 + 16x^2 + 2 \\ PL_5(x) &= 32x^5 + 40x^3 + 10x \\ PL_6(x) &= 64x^6 + 96x^4 + 36x^2 + 2 \end{aligned} \tag{1.32}$$

The Pell-Lucas polynomials $PL_n(x)$ can also be defined by the following recurrence relation:

$$PL_n(x) = 2xPL_{n-1}(x) + PL_{n-2}(x), \quad n = 2, 3, \ldots n \tag{1.33}$$

where the initial terms are given by :

$$\begin{aligned} PL_0(x) &= 2 \\ PL_1(x) &= 2x \\ PL_2(x) &= 4x^2 + 2 \end{aligned} \tag{1.34}$$

2.4 Properties of Pell-Lucas Polynomials

2.4. 1 Generating Function

The sequences of these polynomials are generated by the Following generating function:

$$W(x,t) = \sum_{n=0}^{\infty} PL_{n+1}(x)t^n = \frac{(2x+2t)}{1-2xt-t^2} \tag{1.35}$$

The Pell-Lucas polynomial satisfy the following recurrence relation:

$$PL_n(x) = 2xPL_{n-1}(x) + PL_{n-2}(x) \quad n \geq 2 \tag{1.36}$$

By using the generator function, it can also be clearly expressed as:

$$PL_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} \tag{1.37}$$

2.4. 2 Orthogonality and weight-function

The Pell-Lucas polynomials are orthogonal, with respect to the inner product:

$$\langle PL_n(x), PL_m(x) \rangle = \int_{-1}^{1} \frac{1}{\sqrt{x^2+1}} PL_n(x)PL_m(x)dx \tag{1.38}$$

where $y(x) = \frac{1}{\sqrt{x^2+1}}$ is the weight function. this integral evaluates to:

$$\int_{-1}^{1} \frac{1}{\sqrt{x^2+1}} PL_m(x)PL_n(x)dx = \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & ifm = n = 0 \\ \pi & \text{if } m = n \neq 0 \end{cases} \tag{1.39}$$

The Pell-Lucas polynomials $PL_n(x)$ satisfy the following second -order linear differential equation:

$$(x^2+1)PL_n''(x) + xPL_n'(x) - n^2 PL_n(x) = 0 \quad n \in \mathbb{N} \tag{1.40}$$

2.4. 3 Shifted Pell-Lucas Polynomials

For the treatment of linear Fredholm integro-differential equations on the interval $[0,1]$, it is necessary to transform the Pell-Lucas polynomials from the interval $[-1,1]$ to the interval $[0,1]$. This is achieved by introducing the change of variable $x = 2t-1$.

The shifted Pell-Lucas polynomials are denoted by $PL_n^*(t)$ and are defined as:

$$PL_n^*(t) = PL_n(2t-1) \tag{1.41}$$

The shifted Pell-Lucas polynomials satisfy the following recurrence relation:

$$PL_n^*(t) = 2(2t-1)PL_{n-1}^*(t) + PL_{n-2}^*(t) \quad n \geq 2 \tag{1.42}$$

with initial conditions $PL_0^*(t) = 2$ and $PL_1^*(t) = 4t-2$.

The explicit form of $PL_n^*(t)$ is given by:

$$PL_n^*(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} 2^{n-2k}(2t-1)^{n-2k} \tag{1.43}$$

consequently , the first few shifted Pell-Lucas polynomials are constructed as follows:

$$\begin{aligned}
PL_0^*(t) &= 2 \\
PL_1^*(t) &= 4t-2 \\
PL_2^*(t) &= 16t^2 - 16t + 6 \\
PL_3^*(t) &= 64t^3 - 96t^2 + 60t - 14
\end{aligned} \tag{1.44}$$

The weight function $y^*(t)$ for the shifted Pell-Lucas polynomials on the interval $[0,1]$ is expressed as:

$$y^*(t) = \frac{1}{\sqrt{(2t-1)^2+1}} \tag{1.45}$$

As a result , the orthogonality condition for the shifted Pell-Lucas polynomials is written as:

$$\int_0^1 PL_n^*(t)PL_m^*(t)y^*(t)dt = \begin{cases} 0 & n \neq m \\ 2\pi & n = m = 0 \\ \pi & n = m \neq 0 \end{cases} \tag{1.46}$$

Chapter 3

Numerical Solution

This chapter presents the numerical methods used to solve Fredholm integro-differential equations. To demonstrate the validity and efficiency of the proposed approach, some illustrative numerical examples are also provided.

3.1 Numerical Solution Techniques for Fredholm Integro-Differential Equations

We consider the linear Fredholm integro-differential equation of high-order:

$$\sum_{k=0}^{m} P_k(x)u^{(k)}(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \tag{1.47}$$

with the following initial conditions:

$$u^{(k)}(0) = \mu_k, \quad k = 0, 1, \dots, m-1 \tag{1.48}$$

we assume that the approximate solution $u_N(x)$ of the integro -differential equation can be expressed in the form of the shifted Pell-Lucas polynomials expansion as follows:

$$u_N(x) = \sum_{i=0}^{n} C_i PL_i^*(x) = PL^*(x)C \tag{1.49}$$

where $PL^*(x)$ is the shifted Pell-Lucas vector defined by:

$$PL^*(x) = [PL_0^*(x), PL_1^*(x), \dots, PL_n^*(x)] \tag{1.50}$$

and C is the unknown coefficients vector given by:

$$C = [c_0, c_1, \dots, c_n]^T \tag{1.51}$$

To approximate the derivatives of the solution $u_N(x)$ in (**??**), we first determine the derivatives of the shifted Pell-Lucas basis vector $PL^*(x)$. By introducing the operational matrix of differentiation M, the derivatives of the basis vector can be established systematically as follows:

- **First derivative:**
$$\frac{d}{dx}PL^*(x) = PL^*(x)M \tag{1.52}$$
- **Second derivative:**
$$\frac{d^2}{dx^2}PL^*(x) = PL^*(x)M^2 \tag{1.53}$$
- **k-th order derivative:**
$$\frac{d^k}{dx^k}PL^*(x) = PL^*(x)M^k, \quad k = 1, 2, \ldots \tag{1.54}$$

By utilizing the operational matrix of differentiation M, we can represent the derivatives of $U_N(x)$ in the following matrix form:

$$u_N^{(k)}(x) = PL^*(x) \cdot M^k \cdot C \tag{1.55}$$

where M is the $(N+1)(N+1)$ operational matrix of differentiation. where if N is odd:

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & -3 & 0 & 5 & \ldots & (-1)^{\frac{N-1}{2}}N \\ 0 & 0 & 4 & 0 & -8 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 6 & 0 & -10 & \ldots & (-1)^{\frac{N-3}{2}}2N \\ 0 & 0 & 0 & 0 & 8 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & \ldots & (-1)^{\frac{N-5}{2}}2N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 2N \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{(N+1)(N+1)} \tag{1.56}$$

and if N is even

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & -3 & 0 & 5 & \ldots & 0 \\ 0 & 0 & 4 & 0 & -8 & 0 & \ldots & (-1)^{\frac{N-2}{2}}2N \\ 0 & 0 & 0 & 6 & 0 & -10 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & \ldots & (-1)^{\frac{N-4}{2}}2N \\ 0 & 0 & 0 & 0 & 0 & 10 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{(N+1)(N+1)} \tag{1.57}$$

3.1. 1 Matrix Representation of the Integral Part

To transform the Fredholm integral part into a matrix form , we consider the integral:

$$I(x) = \lambda \int_0^1 K(x,t)u(t)dt \tag{1.58}$$

The kernel function $K(x,t)$ is approximated by employing the Taylor series expansion as follows:

$$K(x,t) \approx \sum_{m=0}^{N}\sum_{n=0}^{N} K_{mn}x^m t^n \tag{1.59}$$

Which is written in matrix form using the standard basis vectors:

$$K(x,t) \approx T(x)K_T T^T(t) \tag{1.60}$$

where $T(x) = [1, x, x^2, \ldots, x^n]$ represents the standard basis vector, and $K_T = (K_{mn})$ is the corresponding kernel matrix in the standard basis.

The coefficients K_{mn} of the kernel matrix are given by the following partial derivatives relation:

$$K_{mn} = \frac{1}{m!n!}\frac{\partial^{m+n}K(0,0)}{\partial x^m \partial t^n} \tag{1.61}$$

For consistency with the Pell-Lucas approximation, we transform the kernel into the shifted Pell-Lucas basis $PL^*(x)$ as follows:

$$K(x,t) = PL^*(x)K_{PL}(PL^*(t))^T \tag{1.62}$$

where $K_{PL} = (K_{ij})$ is the transformed kernel matrix relative to the Pell-Lucas basis, obtained by utilizing the inverse of the basis vectors, which represents:

$$K_{PL} = (PL^*)^{-1}(x)T(x)K_T T^t(t)((PL^*)^t)^{-1}(t) \tag{1.63}$$

Finally , substituting both the kernel matrix $K(x,t)$ and the approximate solution $u_N(t) = PL^*(t)C$ into the integral, we obtain:

$$I(x) = \lambda \int_0^1 \left[PL^*(x)K_{PL}(PL^*(t))^T\right]\left[PL^*(t)C\right]dt \tag{1.64}$$

Since the vector $PL^*(x)$, the matrix K_{PL}, and the coefficient vector $\mathbf{C}$ are independent of the integration variable t, they can be moved outside the integral:

$$I(x) = \lambda PL^*(x)K_{PL}ZC \tag{1.65}$$

By defining the product matrix Z as follows:

$$Z = \int_0^1 (PL^*(t))^T PL^*(t)dt \tag{1.66}$$

3.1. 2 Collocation Points and Boundary Conditions

To find the numerical solution, the matrix equation (39) is evaluated at a set of collocation points $\{x_i\}_{i=0}^N$. these points are typically chosen as equally spaced nodes in the interval $[a, b]$, defined by:

$$x_i = a + \frac{b-a}{N}i, \quad i = 0, 1, \ldots, N \tag{1.67}$$

The approximate solution $u_N(x)$ is expressed using the shifted Pell-Lucas basis $PL^*(x)$.

$$u_N(x) = PL^*(x)C \tag{1.68}$$

By substituting the matrix relations for the derivatives and the integral part into the main integro-differential equation, we obtain the following expression:

$$\sum_{k=0}^{m} P_k(x)PL^*(x)M^kC + \lambda PL^*(x)K_{PL}ZC = f(x) \tag{1.69}$$

If the collocation points x_i are used instead of x in the expression, it becomes:

$$\sum_{k=0}^{m} P_k(x_i)PL^*(x_i)M^kC + \lambda PL^*(x_i)K_{PL}ZC = f(x_i) \tag{1.70}$$

where :

$$P_k(x_i) = \begin{bmatrix} P_k(x_0) & 0 & \ldots & 0 \\ 0 & P_k(x_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & P_k(x_N) \end{bmatrix}, \quad F = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} \tag{1.71}$$

$$PL^*(x_i) = \begin{bmatrix} PL_0^*(x_0) & PL_1^*(x_0) & \dots & PL_N^*(x_0) \\ PL_0^*(x_1) & PL_1^*(x_1) & \dots & PL_N^*(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ PL_0^*(x_N) & PL_1^*(x_N) & \dots & PL_N^*(x_N) \end{bmatrix} \tag{1.72}$$

$$(K_{Pl})_{ij} = \begin{bmatrix} k_{0,0} & k_{0,1} & \dots & k_{0,N} \\ k_{1,0} & k_{1,1} & \dots & k_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N,0} & k_{N,1} & \dots & k_{N,N} \end{bmatrix} \tag{1.73}$$

$$Z = \begin{bmatrix} z_{0,0} & z_{0,1} & \cdots & z_{0,N} \\ z_{1,0} & z_{1,1} & \cdots & z_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ z_{N,0} & z_{N,1} & \cdots & z_{N,N} \end{bmatrix} \tag{1.74}$$

by assembling the aforementioned matrices for both differential and integral parts, the sesulting system can be written in a compact matrix form as follows:

$$\mathbf{GC} = \mathbf{F} \quad \text{or} \quad [\mathbf{G}; \mathbf{F}] \tag{1.75}$$

where :

$$\mathbf{G} = \sum_{k=0}^{m} P_k(x_i) PL^*(x_i) M^k + \lambda PL^*(x_i) K_{PL} Z \tag{1.76}$$

The fundamental matrix $\mathbf{G}$ is obtained by substituting the collocation points x_i into the general matrix form as follows:

$$\mathbf{G} = \begin{bmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,N} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N-m,0} & g_{N-m,1} & \cdots & g_{N-m,N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N,0} & g_{N,1} & \cdots & g_{N,N} \end{bmatrix} \tag{1.77}$$

Since the integro-differential equation is of order m, the matrix $\mathbf{G}$ alone is not sufficient to obtain a unique solution for the problem Therefore, we must incorporate m initial conditions into the system

This process is carried out by replacing the last m rows of the matrix $\mathbf{G}$ with new rows representing these conditions Similarly,we modify the right-hand side vector $\mathbf{F}$ by replacing its last m entries with the given initial condition values, to finally obtain the modified matrix $\tilde{\mathbf{G}}$ and the modified vector $\tilde{\mathbf{F}}$

$$[\tilde{\mathbf{G}};\tilde{\mathbf{F}}] = \left[\begin{array}{cccc|c} g_{0,0} & g_{0,1} & \cdots & g_{0,N} & f(x_0) \\ g_{1,0} & g_{1,1} & \cdots & g_{1,N} & f(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{N-m,0} & g_{N-m,1} & \cdots & g_{N-m,N} & f(x_{N-m}) \\ \psi_{0,0} & \psi_{0,1} & \cdots & \psi_{0,N} & \mu_0 \\ \psi_{1,0} & \psi_{1,1} & \cdots & \psi_{1,N} & \mu_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_{m-1,0} & \psi_{m-1,1} & \cdots & \psi_{m-1,N} & \mu_{m-1} \end{array}\right] \tag{1.78}$$

If the rank of $[\tilde{\mathbf{G}};\tilde{\mathbf{F}}]$ matrice is equal to $N+1$, then the linear system has a unique solution; consequently, we can uniquely determine the coefficients matrix $\mathbf{C}$ as follows:

$$\mathbf{C} = (\widetilde{\mathbf{G}})^{-1}\widetilde{\mathbf{F}} \tag{1.79}$$

Consequently, the final approximate solution $u_N(x) = PL^*(x)C$ is obtained by substituting the calculated coefficients C.

3.2 Numerical Results

Example 01: Consider the following fredholm integro-differential equation:

$$\begin{cases} u'''(x) + xu'(x) + xu(x) = -e^{-x} - e + 1 + e\int_0^1 u(t)dt, \quad x \in [0,1] \\ u(0) = 1, \quad u'(0) = -1, \quad u''(0) = 1 \end{cases} \tag{1.80}$$

The exact solution for this problem is given by

$$u_{ex}(x) = e^{-x}$$

Table 1.1: These numerical results and absolute errors are obtained by applying the high-order Pell-Lucas collocation method with $N = 10$.

Val of x	Exact Solution (u_{ex})	Approximate Solution (u_N)	Absolute Error ($\|u_{ex} - u_N\|$)
0	1.0000e+00	1.0000e+00	2.2204e-16
1.0000e-01	9.0484e-01	9.0484e-01	1.3212e-14
2.0000e-01	8.1873e-01	8.1873e-01	8.2045e-14
3.0000e-01	7.4082e-01	7.4082e-01	2.0106e-13
4.0000e-01	6.7032e-01	6.7032e-01	3.6904e-13
5.0000e-01	6.0653e-01	6.0653e-01	5.8065e-13
6.0000e-01	5.4881e-01	5.4881e-01	8.3378e-13
7.0000e-01	4.9659e-01	4.9659e-01	1.1209e-12
8.0000e-01	4.4933e-01	4.4933e-01	1.3824e-12
9.0000e-01	4.0657e-01	4.0657e-01	1.9730e-12
1.0000e+00	3.6788e-01	3.6788e-01	4.2854e-11

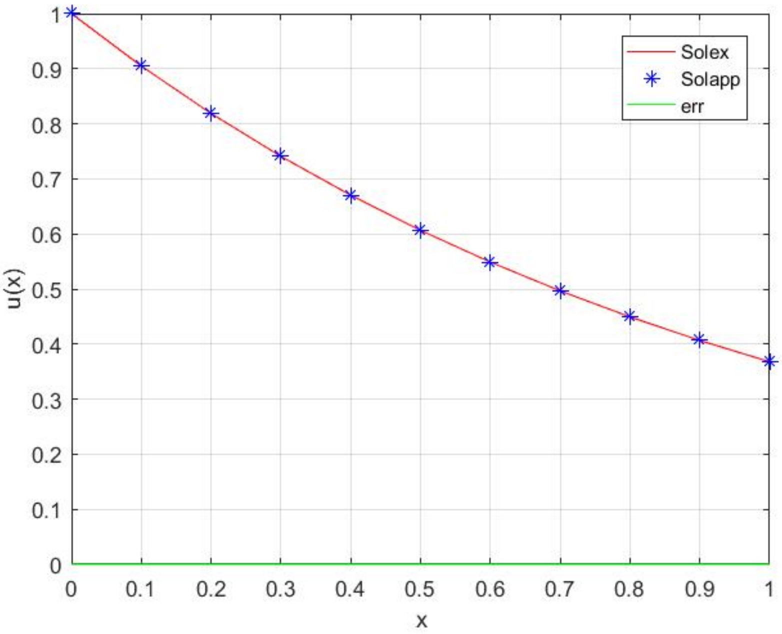

Figure 1.1: The comparison between the exact solution and the approximate solution for Example 01.

Example 02:Consider the following linear Fredholm integro-differential equation:

$$\begin{cases} u''(x) + xu'(x) + \pi^2 u(x) = \pi x \cos(\pi x) - \frac{2x+1}{\pi} + \int_0^1 (x+t)u(t)\,dt, \quad x \in [0,1] \\ u(0) = 0, \quad u'(0) = \pi \end{cases} \tag{1.81}$$

The exact solution for this problem is given by:

$$u_{ex}(x) = \sin(\pi x)$$

Table 1.2: These numerical results and absolute errors are obtained by applying the high-order Pell-Lucas collocation method with $N = 10$.

Val of x	Exact Solution (u_{ex})	Approximate Solution (u_N)	Absolute Error ($\lvert u_{ex} - u_N \rvert$)
0	0	-4.0771e-16	4.0771e-16
1.0000e-01	3.0902e-01	3.0902e-01	5.6846e-09
2.0000e-01	5.8779e-01	5.8779e-01	1.3911e-08
3.0000e-01	8.0902e-01	8.0902e-01	2.0681e-08
4.0000e-01	9.5106e-01	9.5106e-01	2.5829e-08
5.0000e-01	1.0000e+00	1.0000e+00	2.8816e-08
6.0000e-01	9.5106e-01	9.5106e-01	2.9426e-08
7.0000e-01	8.0902e-01	8.0902e-01	2.8034e-08
8.0000e-01	5.8779e-01	5.8779e-01	2.3419e-08
9.0000e-01	3.0902e-01	3.0902e-01	4.8314e-08
1.0000e+00	1.2246e-16	7.3549e-07	7.3549e-07

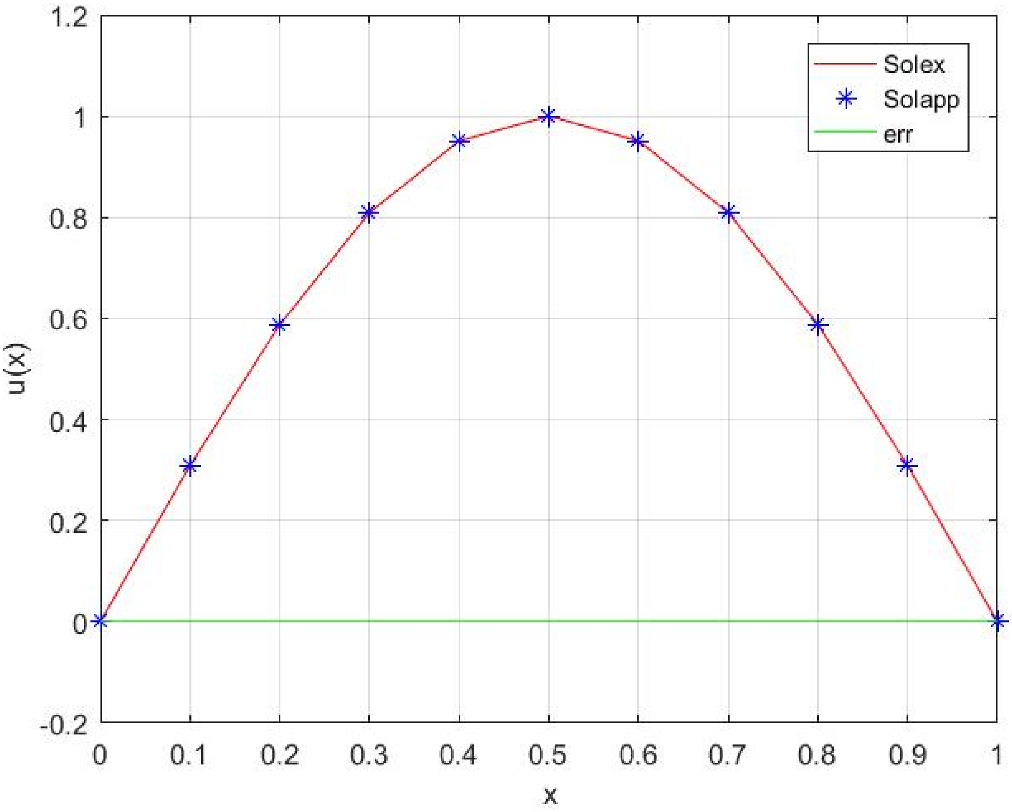

Figure 1.2: The comparison between the exact solution and the approximate solution for Example 02.

Example 03:Consider the following linear Fredholm integro-differential equation:

$$\begin{cases} u''(x) - u'(x) = f(x) + \int_0^1 xt\, u(t)\, dt, \quad x \in [0,1] \\ u(0) = 0, \quad u'(0) = -1 \end{cases} \tag{1.82}$$

where the continuous function $f(x)$ is given by:

$$f(x) = \left(\frac{1}{x^2 - x + 1}\right)^2 \cdot \left[2x^7 - 13x^6 + 31x^5 - 46x^4 + 41x^3 - 25x^2 + 8x + 1\right] + \frac{121}{120}x - \frac{1}{6}\sqrt{3}\,\pi x$$

The exact solution for this problem is given by:

$$u_{ex}(x) = \ln(x^2 - x + 1) - \frac{1}{2}(x^2 - x)^2$$

Table 1.3: These numerical results and absolute errors are obtained by applying the high-order Pell-Lucas collocation method with $N = 16$.

Val of x	Exact Solution (u_{ex})	Approximate Solution (u_N)	Absolute Error ($\lvert u_{ex} - u_N\rvert$)
0	0	-6.0255e-17	6.0255e-17
6.2500e-02	-6.2097e-02	-6.2097e-02	1.6061e-09
1.2500e-01	-1.2181e-01	-1.2181e-01	4.1708e-09
1.8750e-01	-1.7688e-01	-1.7688e-01	6.8323e-09
2.5000e-01	-2.2522e-01	-2.2522e-01	9.7041e-08
3.1250e-01	-2.6495e-01	-2.6495e-01	1.2789e-08
3.7500e-01	-2.9453e-01	-2.9453e-01	1.6114e-08
4.3750e-01	-3.1277e-01	-3.1277e-01	1.9700e-08
5.0000e-01	-3.1893e-01	-3.1893e-01	2.3572e-08
5.6250e-01	-3.1277e-01	-3.1277e-01	2.7759e-08
6.2500e-01	-2.9453e-01	-2.9453e-01	3.2285e-08
6.8750e-01	-2.6495e-01	-2.6495e-01	3.7185e-08
7.5000e-01	-2.2522e-01	-2.2522e-01	4.2479e-08
8.1250e-01	-1.7688e-01	-1.7688e-01	4.8261e-08
8.7500e-01	-1.2181e-01	-1.2181e-01	5.4036e-08
9.3550e-01	-6.2097e-02	-6.2097e-02	7.9002e-08
1.0000e+00	0	7.1856e-07	7.1856e-07

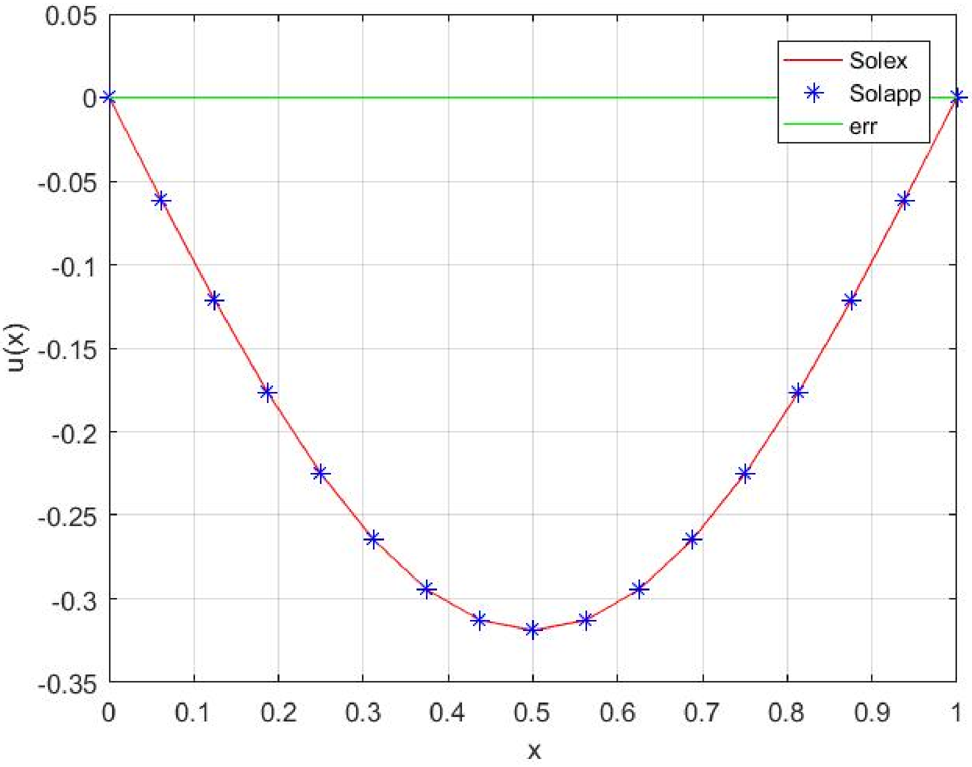

Figure 1.3: The comparison between the exact solution and the approximate solution for Example 03.

Conclusion

In this thesis we present a numerical method based on Pell-Lucas polynomials for solving high-order linear Fredholm integro-differential equations. This technique transforms high-order Fredholm linear differential-integral equations into a system of matrix equations, which can then be efficiently solved to obtain approximate solutions. The numerical examples presented in this work confirm that the Pell-Lucas matrix method is a powerful and reliable tool. The computational results demonstrate rapid convergence across a variety of test cases, including problems with known exact solutions involving exponential, logarithmic, and trigonometric functions.

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