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Representation Theorems For Some Ordered Algebraic Structures

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Dedication

To the cherished memory of my late professor, Dr. Oumhani Ali,
whose wisdom and guidance illuminated my academic path.
I dedicate this humble work to him, and

I sincerely hope that everyone who reads this dissertation will remember him with compassion and pray for his soul, asking Allah to grant him mercy and forgiveness.

إلى ذكرى أستاذي الراحل الدكتور أمهاني علي، الذي أنار بعلمه وتوجيهه مسيرتي العلمية في هذا المسار الأكاديمي. أهدي إليه هذا

العمل المتواضع، و أرجو من كل من يقرأ هذه الرسالة أن يترحّم عليه و يدعو له بالرحمة و المغفرة، سائلين الله أن يتغمّده بواسع رحمته و رضوانه.

Table of contents

List of Figures

List of Tables

List of Symbols

Symbol	Name of Symbol	Page
(Ω, Σ)	Topological space	6
Ω	Set	6
Σ	Topology	6
$\emptyset$	Empty set	6
$\mathfrak{B}$	Basis for a topology	6
$(\mathcal{P}, \leqslant)$	Partially ordered set (poset)	9
$\leqslant$	Order relation	9
$\wedge$	Meet, infimum or intersection	10
$\vee$	Join, supremum or union	10
$(\mathcal{L}, \leqslant, \wedge, \vee)$	Lattice	10
$0, 1$	Least and greatest element of a bounded lattice	11
$\mathcal{I}$	Ideal	11
$\mathcal{F}$	Filter	11
ϕ	Homomorphism or morphism	12
$(\Lambda, \trianglelefteq)$	Partially ordered set	13
$\trianglelefteq$	Pointwise order relation	14
θ	Topology	14
$\mathfrak{P}(A)$	Priestley dual space of a bounded distributive lattice	14
$f \trianglelefteq g$	Pointwise order on homomorphisms	14
$L(\Sigma)$	Lattice of clopen increasing sets of a Priestley space	14
$\mathscr{P}^*(H)$	Set of all non-empty subsets of H	16
$\cdot$	Hyperoperation	16
$\sqcup$	Hyper-join or hyperunion	17
$\mathcal{A} \wedge \mathcal{B}$	Setwise meet of hyperoperations	17

Continued on next page

Symbol	Name of Symbol	Page
$\mathcal{A} \sqcup \mathcal{B}$	Setwise hyper-join	17
$\leqslant$	Partial order on a hyperlattice	17
$\langle S \rangle$	Filter generated by a set S	19
$\downarrow a$	Principal ideal generated by a	19
$\pi_{\mathfrak{P}}$	Characteristic function of a prime filter	21
$\mathfrak{S}(\mathcal{A})$	Spectral dual space of a hyperlattice	24
$\leqslant$	Pointwise order on morphisms	24
$\Phi_{\mathcal{H}}$	Canonical isomorphism between a hyperlattice and its bidual	25
$\mathcal{T}(f)$	Priestley space morphism induced by a homomorphism f	26
$G_{\mathcal{S}}$	Canonical isomorphism between a Priestley space and its bidual	26
$\mathcal{L}(h)$	Hyperlattice homomorphism induced by a Priestley space morphism h	27
$(S, \star, \preceq)$	Negatively ordered monoid or ordered semigroup	33
$\diamond$	Commutative monoid operation	33
$\bot$	Least element (bottom)	34
$\top$	Greatest element (top)	34
Υ	Generic set	34
$[\sigma)$	Filter of a negatively ordered commutative monoid generated by a hyperlattice filter	37
Φ	Surjective homomorphism to the two-element Boolean algebra	39
$\mathfrak{T}(\mathcal{H})$	Dual representation space of a hyperlattice with monoid	41
$*$	Semigroup operation on the dual Priestley space	41

Continued on next page

Symbol	Name of Symbol	Page
π_P	Canonical isomorphism between a Priestley space and its bidual (with semigroup)	47
D_{12}	Divisors of 12 for an example	49
$\circledast$	Hyperoperation of product	57
$\mathfrak{I}$	Hyperideal	57
$\mathfrak{F}$	Hyperfilter	58
$[\delta)$	Hyperfilter generated by a hyperlattice hyperfilter	59
$\mathfrak{D}(\mathcal{E})$	Dual representation space of a hyperlattice with hypermonoid	62
$\Xi_{\mathcal{P}}$	Canonical isomorphism between a Priestley space (semihypergroup) and its bidual	67

Introduction

The origins of hyperstructure theory can be traced back to Frédéric Marty's seminal 1934 paper [34], where he proposed a generalization of classical group theory into a broader algebraic framework. Although Marty's early passing limited his direct contributions to only two publications, his ideas established the foundation of hypergroups (or multigroups). Since then, hyperstructures have evolved into an independent algebraic discipline, with applications spanning algebra, combinatorics, topology, computer science, and logic. The study of these foundations has been enriched by several contributions, and some applications can also be found in [13], [14], [16], [25].

At the core to this theory lies the notion of a hyperoperation, which extends the classical concept of an operation by allowing multiple outputs instead of a single value. Algebraic hyperstructures are defined as sets equipped with one or more hyperoperations, encompassing systems such as hypergroups, hyperrings, and hyperlattices. In logic, multialgebras have served as semantic models, and Avron with collaborators introduced non-deterministic matrices (Nmatrices) to analyze paraconsistent logics, particularly those in the class of Logics of Formal Inconsistency (LFIs) [9], [24].

The study of hyperlattices enriched with hyperoperations was initiated by Benado [10] under the name "multistructure" and later refined by Morgado [35]. Subsequent contributions by Koguep et al. [32] and Konstantinidou and Mittas [33] established the theoretical framework for ideals and filters in hyperlattice systems. Ameri et al. [2] advanced the analysis of prime ideals and filters, while Rasouli and Davvaz [40], [41] formalized hyperlattices and demonstrated their connections to classical lattice theory, including the construction of topological spaces derived from prime ideal collections in distributive hyperlattices. Modern mathematicians have established this as an independent algebraic discipline, actively employed in cross-disciplinary studies by researchers worldwide [1], [2], [32], [33], [39], [40], [41], Tang et al. [44], and Kehayopulu [28], [29], [30] [31], and books such as [15], [17], [45].

Representation theory has played a pivotal role in the development of lattice-based structures. Stone's 1937 theorem [43] established an isomorphism between Boolean algebras and their prime ideals, initiating extensive advances in distributive lattice representation. Birkhoff [11] later proved that every finite distributive lattice is isomorphic to the lattice of ideals generated by its join-irreducible elements. Priestley [37], [38] enriched this field by introducing a duality framework for bounded distributive lattices. Canonical extensions, studied by Gehrke et al. [21], [22], [23] and Jonsson-Tarski [26], [27]. Topological methods applied to distributive lattices [42] and Boolean algebras [43] play a pivotal role in investigating logical semantics for propositional systems, while subsequent developments further deepened these connections by introducing algebraic perspectives that enriched the semantic analysis.

further deepened these connections by offering algebraic perspectives on logical semantics.

Building on these developments, recent works [3], [4], [5], [6], [7], [8], [37], [38], [46] and [47] have expanded representation theory into broader categorical frameworks, highlighting the potential of hyperlattices in unifying algebraic and topological perspectives. This thesis contributes to this line of research by establishing a fundamental representation theorem for bounded distributive hyperlattices endowed with negatively ordered commutative monoids and hypermonoid structures, thereby situating hyperlattice theory within the wider landscape of duality and categorical representation.

Research Motivation. Despite significant progress in hyperlattice theory, the integration of negatively ordered commutative monoids and hypermonoids into Priestley-style duality remains underexplored. This thesis addresses this gap by extending representation theory to a broader categorical framework, contributing to the unification of algebraic and topological perspectives.

Potential Applications. Beyond its theoretical value, the results of this work have potential applications in mathematical logic and theoretical computer science. The categorical duality developed here provides tools for analyzing formal specification languages and computational models, offering both deeper theoretical insights and practical advantages in decidability and structural analysis.

This study aims to:

1. Establish a categorical duality between Priestley-style spaces and bounded distributive

hyperlattices with negative ordering, generalizing classical duality results[1].

2. Analyze homomorphisms and morphisms in negatively ordered hyperlattice-based monoids, and explore their role in categorical equivalence[1].

3. Develop a unified framework that connects algebraic hyperstructures with topological duality, enhancing the structural understanding of distributive systems [1].

4. Extend the representation theory to hyperlattices with negatively ordered commutative hypermonoids structures[2].

5. Introduce and formalize the notion of negatively ordered commutative semihypergroups within the context of hyperlattice theory[2].

6. Provide illustrative constructions and counterexamples to clarify the behavior of prime filters, ideals, and dual spaces in the proposed setting[2].

7. Demonstrate the applicability of the developed theory to logical semantics and algebraic specification, particularly in systems involving non-deterministic operations[2].

This thesis is structured as follows:

The thesis is organized into four chapters, each building upon the previous to develop a comprehensive framework of representation and duality in hyperlattice-ordered commutative monoids and hypermonoids. Chapter 1 provides the foundational background, introducing topological spaces, distributive lattices, Priestley duality, and hyperlattices, thereby establishing the theoretical basis for the subsequent developments. Chapter 2 focuses on representation and duality in bounded distributive hyperlattices, drawing on the published work of A. Amroune and A. Oumhani, and highlights the role of prime ideals, filters, and categorical connections. Chapter 3 extends Priestley duality to bounded distributive hyperlattices equipped with negatively ordered commutative monoids. It begins with the study of hyperlattices arising from negatively ordered commutative monoids, then moves to the analysis of ideals and filters within these structures, followed by an examination of homomorphisms and morphisms, and culminates in the development of a Priestley-style representation framework with key theorems and applications. Chapter 4 further generalizes the representation approach to distributive hyperlattices endowed with ordered commutative

[1]This part is published in QRS Journal of Quasigroups and Related Systems [19].
[2]This part is accepted in JDMSC Journal of Discrete Mathematical Sciences and Cryptography [20].

hypermonoid structures. It opens with the foundations of negatively ordered semihypergroups, proceeds to the study of structural properties of hyperlattice-ordered commutative hypermonoids, continues with an exploration of hyperideals and hyperfilters, advances to the analysis of morphisms in hyperlattice-ordered commutative hypermonoids, and concludes with Priestley-style representation and biduality, thereby completing the theoretical framework and demonstrating its applicability to logical semantics and algebraic specification.

The majority of the results presented in this thesis have been published in international journals. The results included in Chapter 3 and Chapter 4 have been described in [19] and [20], respectively.

CHAPTER 1

FOUNDATIONS FRAMEWORK: TOPOLOGICAL SPACES, DISTRIBUTIVE LATTICES, PRIESTLEY DUALITY AND HYPERLATTICES

The purpose of this chapter is to establish the theoretical foundations that support the development of the thesis. We begin with an overview of topological spaces, emphasizing their role in providing the structural background for continuity, compactness, and separation axioms. This framework naturally leads to the study of distributive lattices, which constitute the algebraic backbone of our investigation. Building upon these notions, we present Priestley duality for bounded distributive lattices, a fundamental correspondence that bridges algebraic and topological perspectives and serves as a powerful tool in representation theory. Finally, we introduce the concept of hyperlattices, extending classical lattice theory to a broader setting. This extension not only enriches the algebraic landscape but also prepares the ground for the original contributions that will be developed in the subsequent chapters.

1.1 Topological Spaces and Basic Properties

We start by recalling the basic notions of topology, including bases, subspace topologies, products of topological spaces, and compact spaces. These ideas provide the analytical background for later developments. At the same time, topological spaces represent one of the most fundamental frameworks in modern mathematics. They generalize the concepts of continuity and convergence beyond metric spaces, and they serve as a foundation for studying compactness, connectedness, and product structures. This preliminary discussion will provide the essential tools needed for the study of ordered structures and duality theory in the chapters that follow, thereby establishing the groundwork for the original contributions of this thesis.

Formally, a topological space is a pair (Ω, Σ), where Ω is any set, and Σ is a collection of subsets of Ω called the topology. The collection Σ must satisfy the following axioms:

1. $\emptyset$ and Ω belong to Σ.

2. Any union of elements of Σ is also in Σ.

3. Any finite intersection of elements of Σ belongs to Σ.

The sets in Σ are referred to as open sets. If a set is open and closed in a topological space, then it is called open-and-closed or closed-and-open (or clopen for short).

Let $\mathcal{C}$ be the family of closed sets of the topological space $(\Omega, \mathcal{C})$. Then $\mathcal{C}$ satisfies the following conditions:

(F1) $\emptyset \in \mathcal{C}$ and $\Omega \in \mathcal{C}$,

(F2) If $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n \in \mathcal{C} \Rightarrow \bigcup_{i=1}^{n} \mathcal{C}_i \in \mathcal{C}$,

(F3) If $\{\mathcal{C}_i\}_{i \in I} \subseteq \mathcal{C} \Rightarrow \bigcap_{i \in I} \mathcal{C}_i \in \mathcal{C}$.

Example 1.1. Let Ω be a non-empty set. The collection $\{\emptyset, \Omega\}$ is a topology on Ω called the trivial topology or (indiscrete topology).

1.1.1 Basis sub-basis for a Topological Space

Definition 1.1. A basis for a topology on a set Ω is a collection $\mathfrak{B}$ of subsets of Ω that satisfies the following conditions:

1. For every point $\omega \in \Omega$, there exists at least one basis element $B^* \in \mathfrak{B}$ such that $\omega \in B^*$.

2. If a point ω lies in the intersection $B_1 \cap B_2$ of two basis elements $B_1, B_2 \in \mathfrak{B}$, then there exists another basis element $B^\star \in \mathfrak{B}$ such that

$$\omega \in B^\star \subseteq B_1 \cap B_2$$

Every open set in Σ can be formed as a union of basis elements from $\mathfrak{B}$.

A subset B of Σ is called a sub-basis of the topology if the collection B' obtained by taking all finite intersections of elements of B is a base of the topology.

1.1.2 Subspace Topology

Definition 1.2. Given a topological space (Ω, Σ) and a subset $\Lambda \subseteq \Omega$, the subspace topology on Λ is defined by

$$\Sigma_\Lambda = \{U \cap \Lambda \mid U \in \Sigma\}$$

A set is open in Λ if it is the intersection of an open set of Ω with Λ.

Lemma 1.1. *If $\mathcal{B}$ is a basis for the topology Σ, then the collection:*

$$\mathcal{B}_Y = \{Y \cap B \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Σ.

Example 1.2. Let $\Omega = \{x, y, z, w\}$ and $\Sigma = \{\varnothing, \{x, y\}, \{x, y, z, w\}\}$. For $\Lambda = \{x, y, w\}$, the subspace topology is $\Sigma_\Lambda = \{\varnothing, \{x, y\}, \{x, y, w\}\}$.

1.1.3 Products of Topological Spaces

Let (Ω_1, Σ_1) and (Ω_2, Σ_2) be two topological spaces, and consider the cartesian product $\Omega_1 \times \Omega_2$.

Definition 1.3. The product topology on $\Omega_1 \times \Omega_2$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $\mathcal{U} \times \mathcal{V}$, where $\mathcal{U}$ is an open set of Ω_1 and $\mathcal{V}$ is an open set of Ω_2.

Theorem 1.1. *If $\mathcal{B}$ is a basis for the topology Σ_1 and $\mathcal{C}$ is a basis for the topology Σ_2, then the collection*

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the product topology on $\Omega_1 \times \Omega_2$.

1.1.4 Compact Spaces

Definition 1.4. Let $\mathcal{A}$ be a subset of the topological space (Ω, Σ). An open cover of $\mathcal{A}$ is a collection $\mathcal{O}$ of open sets whose union contains $\mathcal{A}$. A subcover derived from the open cover

$\mathcal{O}$ is a subcollection $\mathcal{O}'$ of $\mathcal{O}$ whose union contains $\mathcal{A}$.

Definition 1.5. A topological space (Ω, Σ) is compact if every open cover has a finite subcover. That is, if

$$\Omega = \bigcup_{\alpha \in \mathcal{A}} U_\alpha, \quad U_\alpha \in \Sigma,$$

then there exists a finite set $\mathcal{A}_0 \subseteq \mathcal{A}$ such that

$$\Omega = \bigcup_{\alpha \in \mathcal{A}_0} U_\alpha$$

Definition 1.6. A subspace A of Ω is compact if and only if every open cover of A by open sets in Ω has a finite subcover.

Theorem 1.2. *Each closed subset of a compact space is compact.*

Example 1.3. (i) The closed interval $[0, 1]$ in $\mathbb{R}$ (with the standard topology) is compact.

(ii) Any finite topological space is compact.

1.2 Distributive Lattices: Structure and Examples

Distributive lattices constitute one of the central algebraic structures in this thesis. They arise naturally in the study of ordered sets and provide a rich framework for connecting algebraic and topological perspectives. In this section, we recall the fundamental definitions and properties of lattices, focusing on the distributive case. Special attention is given to filters and ideals, which play a crucial role in representation theory, as well as to lattice homomorphisms, which preserve the algebraic structure and allow the construction of categorical correspondences. We also present illustrative examples of distributive lattices, highlighting their relevance in both pure and applied mathematics. This discussion sets the stage for the introduction of Priestley duality in the next subsection.

A binary operation on a non-empty set $\mathcal{X}$ is any function f from the Cartesian product $\mathcal{X} \times \mathcal{X}$ into $\mathcal{X}$, i.e., $f : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$. A binary operation f on $\mathcal{X}$ is called: (i) commutative, if $f(x, y) = f(y, x)$, for any $x, y \in \mathcal{X}$, (ii) associative, if $f(f(x, y), z) = f(x, f(y, z))$ for any $x, y, z \in \mathcal{X}$. An element $e \in \mathcal{X}$ is called: (i) a neutral element of f if $f(x, e) = f(e, x) = x$, for any $x \in \mathcal{X}$, (ii) an absorbing element of f, if $f(e, x) = f(x, e) = e$, for any $x \in \mathcal{X}$.

Example 1.4. The addition $(+)$ and the multiplication $(\times)$ on the set of real numbers $\mathbb{R}$ are commutative and associative binary operations. Where, 0 (resp. 1) is the neural element of $(+)$ (resp. $(\times)$). Also, 0 is the absorbing element of $(\times)$.

Definition 1.7. A binary relation on a set $\mathcal{X}$ is a subset of the Cartesian product $\mathcal{X} \times \mathcal{X}$.

Definition 1.8. An order relation (or partial order) $\leqslant$ is a binary relation on a set $\mathcal{P}$ which is reflexive (i.e., $x \leqslant x$, for any $x \in \mathcal{P}$), antisymmetric (i.e., $x \leqslant y$ and $y \leqslant x$ imply $x = y$, for any $x, y \in \mathcal{P}$) and transitive (i.e.,$x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$, for any $x, y, z \in \mathcal{P}$).

A set $\mathcal{P}$ equipped with an order relation $\leqslant$ on $\mathcal{P}$ is said to be a partially ordered set (poset, for short) and denoted by $(\mathcal{P}, \leqslant)$.

A poset $(\mathcal{P}, \leqslant)$ is totally ordered if every $x, y \in \mathcal{P}$ are comparable (i.e., $x \leq y$ or $y \leq x$).

Definition 1.9. Let $(\mathcal{P}, \leqslant)$ be a poset and let A be a subset of $\mathcal{P}$. An element $l \in \mathcal{P}$ is called a lower bound of A if $l \leqslant a$, for any $a \in A$. l_0 is called the greatest lower bound (or the infimum) of A if l_0 is a lower bound of A and $m \leqslant l_0$, for any lower bound m of A. Upper bound and least upper bound (or supremum) are defined dually.

Example 1.5. Let $D(30)$ be the set of positive divisors of 30, and $|$ be the divisibility relation. One can easily verify that the divisibility relation $|$ is reflexive, antisymmetric and transitive on $D(30)$. Thus, $|$ is an order relation (a partial order on $D(30)$, i.e., the structure $(D(30), |)$ is a poset. The poset $(D(30), |)$ has 1 as the least element and 30 as the greatest element. Indeed, 1 divides all the elements of $D(30)$, and any element of $D(30)$ divides 30. Thus, the structure $(D(30), |, 1, 30)$ is a bounded poset. The corresponding Hasse diagram is illustrated in Figure 1.1

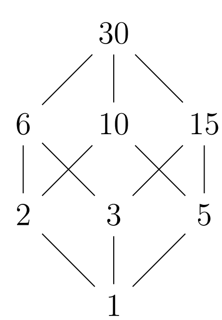

Figure 1.1: Hasse Diagram of $(D(30), |)$

Lemma 1.2. *(Zorn's lemma) Let $\mathcal{P}$ be a poset. If every totally ordered subset of $\mathcal{P}$ has an upper bound, then $\mathcal{P}$ contains a maximal element.*

Definition 1.10. Let $(\mathcal{P}_1, \leqslant_1)$ and $(\mathcal{P}_2, \leqslant_2)$ be two posets. A mapping φ from $\mathcal{P}_1$ into $\mathcal{P}_2$ is called an order isomorphism if it is surjective and satisfies

$$x \leqslant_1 y \text{ if and only if } \varphi(x) \leqslant_2 \varphi(y), \text{for any } x, y \in \mathcal{P}_1.$$

1.2.1 Lattices

In this subsection, we define a lattice as a poset in which every pair of elements has a meet and a join. We highlight their algebraic significance and provide some examples.

Definition 1.11. *[18]* A poset $(\mathcal{P}, \leqslant)$ is called a meet-semilattice (resp. join-semilattice) if any two elements x and y of $\mathcal{P}$ have a greatest lower bound (resp. least upper bound), denoted by $x \wedge y$ (resp. $x \vee y$) and called the meet (infimum) (resp. join (supremum)) of x and y.

A poset $(\mathcal{L}, \leqslant)$ is called a lattice if it is both a meet- and a join-semilattice.

We recall that a lattice can also be defined as an algebraic structure. A set $\mathcal{L}$ equipped with two binary operations $\wedge$ and $\vee$ that are commutative (i.e.,$x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ for any $x, y \in \mathcal{L}$), associative (i.e., $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ for any $x, y, z \in \mathcal{L}$) and idempotent (i.e., $x \wedge x = x$ and $x \vee x = x$, for any $x \in \mathcal{L}$) and they satisfy the absorption laws (i.e., $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$, for any $x, y \in \mathcal{L}$). In this case, the order relation $\leqslant$ associated with $\mathcal{L}$ is related to the meet and the join operations as follows: $x \wedge y = x$ if and only if $x \leqslant y$ if and only if $x \vee y = y$, for any $x, y \in \mathcal{L}$.

Usually, the structure $(\mathcal{L}, \leqslant, \wedge, \vee)$ is used to describe a lattice.

Example 1.6. (i) The set of positive integers N^* equipped with the divisibility order $|$ has a structure of a lattice, where gcd (i.e., the greatest common divisor) and lcm (i.e., the least common multiple) are its meet and joint operations, respectively;

(ii) The bounded poset given in Example 1.5 is a lattice.

Definition 1.12. *[18]* A poset $(\mathcal{L}, \leqslant)$ is called a complete lattice if every subset S of $\mathcal{L}$ has both a greatest lower bound denoted by $\bigwedge S$ and called the infimum of S, and a least upper bound denoted by $\bigvee S$ and called the supremum of S.

Example 1.7. The $([0, 1], \leqslant, \max, \min)$ is a complete lattice.

Definition 1.13. Let $\mathcal{L}$ be a lattice and let M be a non-empty subset of $\mathcal{L}$. M is a sublattice of $\mathcal{L}$ if for any $x, y \in M$, it holds that $x \wedge y \in M$ and $x \vee y \in M$.

1.2.2 Distributive and Bounded Lattice

This subsection focuses on distributive lattice, where meet and join operations distribute over eachother, and bounded lattices, which contain least and greatest elements. These properties are crucial for duality and representation theorems.

Definition 1.14. *[18]* A lattice $(\mathcal{L}, \leqslant, \wedge, \vee)$ is called distributive if one of the following two equivalent conditions holds:

(D1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, for any $x, y, z \in \mathcal{L}$;

(D2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, for any $x, y, z \in \mathcal{L}$.

Definition 1.15. A lattice $(\mathcal{L}, \leqslant, \wedge, \vee)$ is called bounded if it has a least and a greatest element denoted by 0 and 1 respectively. Then the structure $(\mathcal{L}, \leqslant, \wedge, \vee, 0, 1)$ is used to describe a bounded lattice.

Example 1.8. (i) The lattice $(D(30), |, gcd, lcm, 1, 30)$ is bounded and distributive;

(ii) The lattice of real numbers $(R, \leqslant, min, max)$ is distributive, but is not bounded.

1.2.3 Ideals and Filters

Ideals and filters are special subsets of lattices that capture upward and downward closure properties. They play a central role in distributive lattices and areessential for constructing dualities.

Definition 1.16. [12, 18] A nonempty subset $\mathcal{I}$ (resp. $\mathcal{F}$) of $\mathcal{L}$ is called an ideal (a filter) provided for every $x, y \in \mathcal{L}$,

(i) $x \in \mathcal{I}$ and $y \leqslant x$, then $y \in \mathcal{I}$;

(ii) $x, y \in \mathcal{I}$ implies $x \vee y \in \mathcal{I}$.

(for every $x, y \in \mathcal{L}$,

(i) $x \in \mathcal{F}$ and $x \leqslant y$, then $y \in \mathcal{F}$;

(ii) $x, y \in \mathcal{F}$ implies $x \wedge y \in \mathcal{F}$).

A proper ideal $\mathcal{I}$ (proper filter $\mathcal{F}$) is called a prime ideal (prime filter) if it satisfies: $x \wedge y \in \mathcal{I}$ implies $x \in \mathcal{I}$ or $y \in \mathcal{I}$ ($x \vee y \in \mathcal{F}$ implies $x \in \mathcal{F}$ or $y \in \mathcal{F}$), for any $x, y \in \mathcal{L}$.

1.2.4 Lattices Isomorphisms

This subsection introduces lattice homomorphisms *[18]*, which preserve meet and join operations, and lattice isomorphisms, which establish structural equivalence between lattices. These concepts are vital for understanding algebraic frameworks and categorical connections.

Let Γ and $\Gamma^{'}$ be two lattices. A mapping $\phi : \Gamma \to \Gamma^{'}$ is called:

(i) a $\wedge$-homomorphism (resp. $\vee$-homomorphism) if it satisfies $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ (resp. $\phi(x \vee y) = \phi(x) \vee \phi(y)$), for any $x, y \in \Gamma$;

(ii) a lattice homomorphism if it is both a $\wedge$- and a $\vee$-homomorphism;

(iii) a $\wedge$-monomorphism (resp. $\vee$-monomorphism) if it is an injective $\wedge$-homomorphism (resp. $\vee$-homomorphism);

(iv) a $\wedge$-epimorphism (resp. $\vee$-epimorphism) if it is a surjective $\wedge$-homomorphism (resp. $\vee$-homomorphism);

(v) a lattice isomorphism if it is a bijective lattice homomorphism. If $\Gamma = \Gamma^{'}$, a lattice isomorphism $\wedge : \Gamma \to \Gamma$ is called a lattice automorphism.

Definition 1.17. If Γ and $\Gamma^{'}$ are bounded lattices, a lattice homomorphism $\phi : \Gamma \to \Gamma^{'}$ is said to be a bounded-lattice homomorphism if it preserves 0 and 1, i.e., $\phi(0) = 0$ and $\phi(1) = 1$.

The following proposition shows that an order isomorphism between two lattices is equivalent to a lattice isomorphism.

Proposition 1.1. *Let ϕ be a mapping from Γ into $\Gamma^{'}$. The following statements are equivalent:*

(i) *ϕ is an order isomorphism;*

(ii) *ϕ is a lattice isomorphism.*

Proposition 1.2. *([[12], Proposition 1.3.9]) Let F be a subset of Γ. The following conditions are equivalent,*

(i) *F is a prime filter;*

(ii) *$\Gamma - F$ is a prime ideal;*

(iii) *There is a surjective lattice homomorphism $\phi : \Gamma \to \{0, 1\}$ such that $F = \phi^{-1}(\{1\})$.*

Corollary 1.1. ([[12], Corollary 1.3.13]), Let Γ be a distributive lattice. If $\alpha, \beta \in \Gamma$ are such that $\alpha \nleq \beta$, there is a prime filter F such that $\alpha \in F$ and $\beta \notin F$.

1.3 Priestley Duality for Bounded Distributive Lattices

Priestley duality provides a fundamental correspondence between bounded distributive lattices and certain ordered topological spaces, known as Priestley spaces. This duality not only establishes a deep connection between algebra and topology but also serves as a powerful tool in representation theory. In what follows, we present the basic definitions of Priestley spaces and examine their key properties, such as compactness and order-separation. We then present the dual equivalence between bounded distributive lattices and Priestley spaces, highlighting how algebraic operations correspond to topological constructions *[12]*, *[18]*. This framework will later be extended to hyperlattices, thereby preparing the ground for the original contributions of the thesis.

1.3.1 Priestley Spaces and Their Properties

Here, we present the basic definitions of Priestley spaces and examine their essential features. Particular emphasis is placed on compactness and order-separation, which ensure the robustness of these ordered topological structures *[37]*. These properties form the foundation for establishing the categorical correspondence with bounded distributive lattices *[18]*.

Definition 1.18. Let $(\Lambda, \trianglelefteq)$ be a partially ordered set.

- A subset $E \subseteq \Lambda$ is increasing if $x \in E$ and $x \trianglelefteq y$ imply $y \in E$.
- A subset $E \subseteq \Lambda$ is descending if $x \in E$ and $y \trianglelefteq x$ imply $y \in E$.

Definition 1.19. An ordered topological space is a triple $(\Lambda, \theta, \trianglelefteq)$, where (Λ, θ) is a topological space and $(\Lambda, \trianglelefteq)$ is a poset. A set that is both open and closed is called θ-clopen. Such a space is said to be totally disconnected with respect to the order if, for every distinct $x, y \in X$, there exist:

- an increasing θ-clopen set U containing x,
- a decreasing θ-clopen set V containing y,

with $U \cap V = \varnothing$.

Definition 1.20. A Priestley space is a compact, totally disconnected ordered topological space.

Definition 1.21. Let $(\Lambda, \theta, \trianglelefteq)$ and $(X, \sigma, \sqsubseteq)$ be Priestley spaces.

- A function $f : \Lambda \to X$ is increasing if $x \trianglelefteq y \Rightarrow f(x) \sqsubseteq f(y)$.
- If f is increasing and continuous, it is a homomorphism of Priestley spaces.
- If f is bijective, it is an isomorphism of Priestley spaces.

1.3.2 Duality between Lattices and Priestley Spaces

In what follows, we highlight the dual equivalence between bounded distributive lattices and Priestley spaces. This correspondence illustrates how algebraic operations translate into topological constructions, thereby revealing the deep interplay between algebra and topology *[37, 43]*. Such a framework provides the necessary background for extending duality principles to hyperlattices in the subsequent chapter *[8]*.

Definition 1.22. Let A be a bounded distributive lattice. Its dual Priestley space is defined as:

$$\mathfrak{P}(A) = (\Lambda, \theta, \trianglelefteq),$$

where:

- Λ is the set of lattice homomorphisms $f : A \to \{0, 1\}$ preserving bounds;
- θ is the product topology inherited from $\{0, 1\}^A$;

- the order $\trianglelefteq$ is defined pointwise: $f \trianglelefteq g \iff f(a) \trianglelefteq g(a)$, for all $a \in A$.

Thus, $\mathfrak{P}(A)$ is a Priestley space.

Lemma 1.3.
- *If $\Sigma = (\Lambda, \theta, \trianglelefteq)$ is a Priestley space, then $L(\Sigma) = \{U \subseteq \Lambda \mid U$ is increasing and θ-clopen$\}$, forms a bounded distributive lattice under intersection and union.*
- *For a bounded distributive lattice A, the mapping $F_A : A \to L(\mathfrak{P}(A)), \quad F_A(a) = \{f \in \Lambda \mid f(a) = 1\}$ is is a lattice isomorphism.*
- *If $f : A_1 \to A_2$ is a lattice homomorphism, then $\mathfrak{P}(f) : \mathfrak{P}(A_2) \to \mathfrak{P}(A_1), \quad \mathfrak{P}(f)(g) = g \circ f$ is a homomorphism of Priestley spaces, i.e., a continuous and increasing map.*
- *If $\Sigma = (\Lambda, \theta, \trianglelefteq)$ is a Priestley space, then $G_\Sigma : \Sigma \to \mathfrak{P}(L(\Sigma)), \quad G_\Sigma(x)(U) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{otherwise,} \end{cases}$ for all $U \in L(\Sigma)$is an isomorphism of Priestley spaces, i.e., a bijection, continuous and increasing map..*
- *If $h : \Sigma_1 \to \Sigma_2$ is a homomorphism of Priestley spaces, then $L(h) : L(\Sigma_2) \to L(\Sigma_1), \quad L(h)(U) = h^{-1}(U)$, for every $U \in L(\Sigma_2)$ is a lattice homomorphism.*
- *If $\psi : A_1 \to A_2$ is a lattice homomorphism, then $L(\mathfrak{P}(\psi)) \circ F_{A_1} = F_{A_2} \circ \psi$.*
- *If $\gamma : \Sigma_1 \to \Sigma_2$ is an homomorphism of Priestley space, then $T(L(\gamma)) \circ G_{\Sigma_1} = G_{\Sigma_2} \circ \gamma$.*

Theorem 1.3 (Priestley Duality)**.** *The category of bounded distributive lattices is dually equivalent to the category of Priestley spaces.*

Figure 1.2: Simplified diagram of Priestley duality: lattice $\leftrightarrow$ Priestley space.

1.4 On Hyperlattices and Their Algebraic Features

Hyperlattices represent a natural extension of classical lattice theory, where operations may yield sets of outcomes rather than single elements. This generalization enriches the algebraic framework and opens new perspectives for studying distributive structures [15,45]. In this section, we introduce the basic definition of hyperlattices and highlight their fundamental properties. Special attention is given to ideals and filters, which capture the order-theoretic and algebraic behavior of hyperlattices, and to hyperlattice isomorphisms, which preserve structural features and enable categorical correspondences. Together, these notions form the algebraic foundation upon which the duality theory for hyperlattices and the original contributions of this thesis will be developed [8].

Hyperoperations

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. More exactly, if H is a nonempty set and $\mathscr{P}^*(H)$ is the set of all nonempty subsets of H.

Definition 1.23. *[34,45]* Let H be a nonempty set and $\mathscr{P}^*(H)$ denotes the set of all nonempty subsets of H. A binary operation $\cdot$ of the following type:

$$\cdot : H \times H \longrightarrow \mathscr{P}^*(H)$$

is said to be a hyperoperation.

- The $(\cdot)$ in H is called associative (resp. commutative) if $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all x, y, z in H ($x \cdot y = y \cdot x$ for all x, y in H).
- An element $e \in H$ is called an identity (or unit) if for all $x \in H$, $x \in e \cdot x$ and $x \in x \cdot e$.
- An element $x^{-1} \in H$ is called an inverse of $x \in H$ if there is an identity $e \in \mathscr{P}^*(H)$, such that $e \in x \cdot x^{-1}$ and $e \in x^{-1} \cdot x$.

Example 1.9. Let $H = \{a, b, c\}$ be a set. Define the hyperoperations $\circ$ on H by the following Table 1.1:

$\circ$	a	b	c
a	$\{a\}$	$\{a,c\}$	$\{a,b\}$
b	$\{a,c\}$	$\{a,b\}$	$\{a\}$
c	$\{a,b\}$	$\{a\}$	$\{a,c\}$

Table 1.1: Hyperoperation structure on H

1.4.1 Hyperlattices: Definitions and Basic Concepts

Definition 1.24. *[8]*Let H be a nonempty set, $\wedge$ be a binary operation and $\sqcup$ be a hyperoperation on H. H is called a hyperlattice if for all a,b,$c \in H$ the following conditions hold

(i) $a \in a \sqcup a$, and $a \wedge a = a$ (Identity and idempotent);

(ii) $a \sqcup b = b \sqcup a$, and $a \wedge b = b \wedge a$ (Commutative);

(iii) $a \in [a \wedge (a \sqcup b)] \cap [a \sqcup (a \wedge b)]$ (Absorption compatibility);

(iv) $a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$, and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (Associativity);

(v) $a \in a \sqcup b \Rightarrow a \wedge b = b$ (Weak cancellation to meet).

For all nonempty subsets $\mathcal{A}$ and $\mathcal{B}$ of H, we define

$$\mathcal{A} \wedge \mathcal{B} = \{a \wedge b \mid a \in \mathcal{A}, b \in \mathcal{B}\} \text{ and } \mathcal{A} \sqcup \mathcal{B} = \cup\{a \sqcup b \mid a \in \mathcal{A}, b \in \mathcal{B}\}.$$

The converse of condition $(\mathbf{v})$ in Definition 1.24 is true. Indeed, using $(\mathbf{iii})$ in Definition 1.24, we obtain $a \in a \sqcup b$ by taking $b = a \wedge b$. Hence, we can define a partial order on H by:

$$a \leqslant b \Leftrightarrow b \in a \sqcup b \Leftrightarrow a \wedge b = a.$$

Definition 1.25. *[1]*A hyperlattice H is called bounded if there exists $0, 1 \in H$ such that for all $a \in L$, $0 \leqslant a \leqslant 1$. We say that 0 is the least element and 1 is the greatest element.

Definition 1.26. *[2]*Let H be a hyperlattice. H is called distributive if for all $a, b, c \in H$,

$$a \wedge (b \sqcup c) = (a \wedge b) \sqcup (a \wedge c)$$

A hyperlattice H with the property

$$a \sqcup (b \wedge c) = (a \sqcup b) \wedge (a \sqcup c)$$

is called dual distributive.

Example 1.10. Let $H = \{0, a, 1\}$. Consider the following Cayley tables

$\wedge$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

$\sqcup$	0	a	1
0	H	$\{a,1\}$	$\{1\}$
a	$\{a,1\}$	$\{a\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$

Table 1.2: Cayley table for the operations $\wedge$ and $\sqcup$ of $H = \{0, a, 1\}$

Then $(H, \wedge, \sqcup, 0, 1)$ is a distributive hyperlattice.

Consider a lattice $(H, \wedge, \vee)$. We define the Nakano hyperoperation $\sqcup$ on H by $x \sqcup y = \{z \in H \mid z \vee x = z \vee y = x \vee y\}$, for all $x, y \in H$. To the best of our knowledge, the $\sqcup$ hyperoperation was first introduced by Nakano in *[36]*, which is an investigation of hyperrings.

Lemma 1.4. *[8]If $(H, \wedge, \vee)$ is a distributive lattice, then $(H, \wedge, \sqcup)$ is a distributive hyperlattice where $a \sqcup b = \{x \in H \mid a \vee b = a \vee x = b \vee x\}$ for all $a, b \in H$.*

Proof. Straightforward. □

Example 1.11. Let $H = \{0, a, b, 1\}$ be a distributive lattice, where $\wedge$ and $\vee$ is defined as in the following Cayley tables

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

Table 1.3: Cayley table of distributive lattice $H = \{0, a, b, 1\}$

We use the Nakano hyperoperation, and we obtain the following Cayley table.

$\sqcup$	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$
a	$\{a\}$	$\{0,a\}$	$\{1\}$	$\{b,1\}$
b	$\{b\}$	$\{1\}$	$\{0,b\}$	$\{a,1\}$
1	$\{1\}$	$\{b,1\}$	$\{a,1\}$	H

Table 1.4: Nakano hyperoperation of $H = \{0, a, b, 1\}$

Then $(H, \wedge, \sqcup)$ is a distributive hyperlattice.

Lemma 1.5. *[8]Let $H = \{0,1\}$. Then $(H, \wedge, \sqcup)$ is a bounded distributive hyperlattice, where the operations and hyperoprations are defined as follows:*

$\wedge$	0	1
0	0	0
1	0	1

$\sqcup$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	H

Table 1.5: Cayley table for the operations $\wedge$ and $\sqcup$ of $H = \{0,1\}$

Proof. Straightforward. □

1.4.2 Ideals and Filters: Initial Lemmas and Propositions

Definition 1.27. *[2, 39]*A nonempty subset $\mathcal{I}$ of a hyperlattice H is called an ideal if the following conditions hold

(i) Downward closure: $i_1 \in \mathcal{I}$, $i_2 \leqslant i_1$ and $i_2 \in H$, then $i_2 \in \mathcal{I}$;

(ii) Closure under hyperoperation: $i_1, i_2 \in \mathcal{I}$, then $i_1 \sqcup i_2 \subseteq \mathcal{I}$.

A proper Ideal $\mathcal{I}$ is called a prime ideal if $i_1 \wedge i_2 \in \mathcal{I}$, then $i_1 \in \mathcal{I}$ or $i_2 \in \mathcal{I}$ for all $i_1, i_2 \in H$.

Definition 1.28. *[2, 39]* A nonempty subset $\mathcal{F}$ of a hyperlattice H is called a filter if the following conditions hold

(i) Upward closure: $f_1 \in \mathcal{F}$, $f_1 \leqslant f_2$ and $f_2 \in H$, then $f_2 \in \mathcal{F}$;

(ii) Closure under meet: $f_1, f_2 \in \mathcal{F}$, then $f_1 \wedge f_2 \in \mathcal{F}$.

A proper filter $\mathcal{F}$ is called prime if for all $f_1, f_2 \in H$ $(f_1 \sqcup f_2) \cap \mathcal{F} \neq \emptyset$ implies $f_1 \in \mathcal{F}$ or $f_2 \in \mathcal{F}$.

Theorem 1.4. *[8] If $\mathcal{I}$ is a prime ideal of H, then $H \backslash \mathcal{I}$ is a prime filter of H. Conversely, if $\mathcal{F}$ is a prime filter of H, then $H \backslash \mathcal{F}$ is a prime ideal of H.*

Proof. First, we prove that $H \backslash \mathcal{I}$ is a prime filter of H. Let $x \in H \backslash \mathcal{I}$, $y \in H$ such that $x \leqslant y$, we show that $y \in H \backslash \mathcal{I}$. It is clear that $x \notin \mathcal{I}$. Suppose $y \notin H \backslash \mathcal{I}$, so $y \in \mathcal{I}$. We have $x \leq y$ and $\mathcal{I}$ is an ideal, therefore $x \in \mathcal{I}$ that is a contradiction. So $y \in H \backslash \mathcal{I}$. Assume $x, y \in H \backslash \mathcal{I}$. We show that $x \wedge y \in H \backslash \mathcal{I}$. It is clear that $x, y \notin \mathcal{I}$. If $x \wedge y \in \mathcal{I}$, then $x \in \mathcal{I}$ or $y \in \mathcal{I}$ because I is a prime ideal, which is a contradiction. So $x \wedge y \in H\backslash\mathcal{I}$. So $H\backslash\mathcal{I}$ is a filter. It is enough to show that $H\backslash\mathcal{I}$ is a prime filter. Suppose $x, y \in H$ and $(x \sqcup y) \cap (H\backslash\mathcal{I}) \neq \emptyset$. So there exists $z \in H$ such that $z \in x \sqcup y$ and $z \in H\backslash\mathcal{I}$. If $x, y \notin H\backslash\mathcal{I}$, then $x, y \in \mathcal{I}$, therefore $x \sqcup y \subseteq \mathcal{I}$ because I is an ideal. Hence $(x \sqcup y) \cap (H\backslash\mathcal{I}) = \emptyset$, which is a contradiction. So $x \in H\backslash\mathcal{I}$ or $y \in H\backslash\mathcal{I}$.

Similarly, we prove that $H\backslash\mathcal{F}$ is a prime ideal of H. □

Example 1.12. In Example 1.10. Consider $\mathcal{F} = \{1\}$, F is a filter of H, but it is not prime. We have $H\backslash\mathcal{F} = \{0, a\}$, $0 \sqcup a = \{a, 1\}$, and $\{a, 1\} \nsubseteq H\backslash\mathcal{F}$, so $H\backslash\mathcal{F}$ is not an ideal of H.

Proposition 1.3. *[8] [Filter generated] For a nonempty $S \subseteq H$, the smallest filter containing S is*

$$\langle S \rangle = \{\, x \in H \mid s_1 \wedge \cdots \wedge s_n \leqslant x \text{ for some } s_1, \ldots, s_n \in S \,\}.$$

In particular, for $a \in H$, the principal filter generated by a is $\langle a \rangle = \{x \in H \mid a \leqslant x\}$

Proof.

1. Non-emptiness: for $s \in S$, clearly $s \leqslant s$, hence $s \in \langle S \rangle$.
2. Upward closure: if $x \in \langle S \rangle$ and $x \leqslant y$, then $y \in \langle S \rangle$.
3. Closure under meet: if $x, y \in \langle S \rangle$, then $x \wedge y \in \langle S \rangle$.
4. Minimality: any filter containing S must also contain $\langle S \rangle$.

□

Proposition 1.4. *[32,39]Let $(H, \wedge, \sqcup)$ be a distributive hyperlattice. If $a \in H$, then $\downarrow a = \{x \in H \mid x \leqslant a\}$ is an ideal.*

Proof. First, note that $\downarrow a \neq \varnothing$ since $a \leqslant a$, hence $a \in \downarrow a$.

Next, suppose $x \in \downarrow a$ and $y \in H$ with $y \leqslant x$. Then $y \leqslant x \leqslant a$, so $y \in \downarrow a$. Thus, $\downarrow a$ is downward closed.

Now, let $x, y \in \downarrow a$. Then $x \leqslant a$ and $y \leqslant a$. In a distributive hyperlattice, we have

$$a \wedge (x \sqcup y) = (a \wedge x) \sqcup (a \wedge y).$$

Since $x \leqslant a$ and $y \leqslant a$, it follows that $a \wedge x = x$ and $a \wedge y = y$. Hence, $a \wedge (x \sqcup y) = x \sqcup y$.

For any $z \in x \sqcup y$, we obtain $a \wedge z = z$, which implies $z \leqslant a$. Therefore, $z \in \downarrow a$, and so $x \sqcup y \subseteq \downarrow a$.

Thus, $\downarrow a$ is nonempty, downward closed, and closed under the hyperjoin $\sqcup$. Consequently, $\downarrow a$ is an ideal of H. □

Theorem 1.5. *[8] Let $(H, \wedge, \sqcup)$ be a distributive hyperlattice, $\mathfrak{F}$ a filter, and $\mathcal{I}$ an ideal with $\mathfrak{F} \cap \mathcal{I} = \varnothing$. Then there exists a prime filter $\mathfrak{P}$ such that $\mathfrak{F} \subseteq \mathfrak{P}$ and $\mathfrak{P} \cap \mathcal{I} = \varnothing$.*

Proof.

1. Define
$$\mathcal{G} = \{ \mathfrak{F}' \supseteq \mathfrak{F} \mid \mathfrak{F}' \text{ is a filter and } \mathfrak{F}' \cap \mathcal{I} = \varnothing \}.$$
Clearly, $\mathcal{G}$ is nonempty.
2. By Zorn's Lemma, $\mathcal{G}$ has a maximal element $\mathfrak{P}$.
3. Suppose $\mathfrak{P}$ is not prime. Then there exist $x, y \in H$ with $(x \sqcup y) \cap \mathfrak{P} \neq \varnothing$ but $x, y \notin \mathfrak{P}$. Extending $\mathfrak{P}$ by x (or y) would yield a larger filter still disjoint from $\mathcal{I}$, contradicting the maximality of $\mathfrak{P}$.
4. Therefore, $\mathfrak{P}$ is prime, contains $\mathfrak{F}$, and is disjoint from $\mathcal{I}$.

□

Corollary 1.2. *[8]* If $\mathcal{I}$ is an ideal and $a \notin \mathcal{I}$, then there exists a prime filter $\mathfrak{P}$ with $a \in \mathfrak{P}$ and $\mathfrak{P} \cap \mathcal{I} = \varnothing$.

Proof.

1. Consider the principal filter $\langle a \rangle = \{ x \in H \mid a \leqslant x \}$.
2. Since $a \notin \mathcal{I}$, we have $\langle a \rangle \cap \mathcal{I} = \varnothing$.
3. By Theorem 1.5, there exists a prime filter $\mathfrak{P}$ such that $\langle a \rangle \subseteq \mathfrak{P}$ and $\mathfrak{P} \cap \mathcal{I} = \varnothing$.
4. Hence $a \in \mathfrak{P}$ and the claim follows.

□

Corollary 1.3. *[8]* Let H be a distributive hyperlattice. If $a, b \in H$ are such that $a \nleqslant b$, there is a prime filter $\mathfrak{F}$ such that $a \notin \mathfrak{F}$ and $b \in \mathfrak{F}$.

Proof. Take $\mathfrak{I} = \downarrow b$ in Corollary 1.2 . □

1.4.3 Hyperlattices Isomorphisms

Definition 1.29. *[8]*Let H and H'be two hyperlattices and $\phi : H \to H'$be a mapping.

1. ϕ is said to be a hyperlattice homomorphism if $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ and $\phi(x \sqcup y) \subseteq \phi(x) \sqcup \phi(y)$, for all $x, y \in H$.

2. ϕ is said to be a strong homomorphism of hyperlattices if $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ and $\phi(x \sqcup y) = \phi(x) \sqcup \phi(y)$, for all $x, y \in H$.

If ϕ is a bijection, then ϕ is said to be a hyperlattices isomorphism (strong isomorphism).

Example 1.13. Let H be the hyperlattice in Example 1.11, and $\phi : H \to H$ is defined by: $\phi(0) = 0, \phi(a) = a$, and $\phi(1) = a$. Then ϕ is a homomorphism.

Proposition 1.5. *Let $(H, \wedge, \sqcup)$ be a hyperlattice and $\mathfrak{P}$ a prime filter. Define $\pi_{\mathfrak{P}} : H \to \{0, 1\}$ by*

$$\pi_{\mathfrak{P}}(x) = \begin{cases} 1, & x \in \mathfrak{P}, \\ 0, & x \notin \mathfrak{P}. \end{cases}$$

Then $\pi_{\mathfrak{P}}$ is a surjective hyperlattice homomorphism and $\mathfrak{P} = \pi_{\mathfrak{P}}^{-1}(\{1\})$.

Proof.

1. **Meet preservation.** If $x \wedge y \in \mathfrak{P}$ then $x, y \in \mathfrak{P}$, hence $\pi_{\mathfrak{P}}(x \wedge y) = 1 = \pi_{\mathfrak{P}}(x) \wedge \pi_{\mathfrak{P}}(y)$. If $x \wedge y \notin \mathfrak{P}$ then at least one of $x, y \notin \mathfrak{P}$, so $\pi_{\mathfrak{P}}(x \wedge y) = 0 = \pi_{\mathfrak{P}}(x) \wedge \pi_{\mathfrak{P}}(y)$.

2. **Hyperjoin compatibility.** Let $\sqcup$ be the hyperoperation on $\{0, 1\}$. If $(x \sqcup y) \cap \mathfrak{P} \neq \varnothing$, primality yields $x \in \mathfrak{P}$ or $y \in \mathfrak{P}$, so $\pi_{\mathfrak{P}}(x \sqcup y) = \{1\} \subseteq \pi_{\mathfrak{P}}(x) \sqcup \pi_{\mathfrak{P}}(y)$. Otherwise, $\pi_{\mathfrak{P}}(x \sqcup y) = \{0\} \subseteq \pi_{\mathfrak{P}}(x) \sqcup \pi_{\mathfrak{P}}(y)$. Hence $\pi_{\mathfrak{P}}(x \sqcup y) \subseteq \pi_{\mathfrak{P}}(x) \sqcup \pi_{\mathfrak{P}}(y)$.

□

Chapter 2

Representation and Duality in Bounded Distributive Hyperlattices

In this chapter, we recall and reformulate some published results concerning the representation of bounded distributive hyperlattices and their connection with Priestley duality. Representation and duality theories play a central role in the study of algebraic structures, and distributive hyperlattices provide a natural extension of classical lattice theory.

The results presented here, originally established by (A. Amroune and A. Oumhani) *[8]*, provide the theoretical foundation for understanding the categorical equivalence between bounded distributive hyperlattices and Priestley spaces. For completeness, we present these results in a unified manner, with consistent notation and explanatory remarks.

Our aim here is not to introduce new findings, but rather to integrate these results into the framework of the thesis, ensuring that the representation theorem and the duality principles are clearly stated, rigorously explained, and illustrated with examples. This chapter thus serves as a bridge between the classical background introduced in Chapter One and the original contributions that will be developed in Chapter Three.

2.1 Dual Spaces of Bounded Distributive Hyperlattices

The present section is devoted to the study of dual spaces associated with bounded distributive hyperlattices. By extending the framework of Priestley duality, we define the order-topological structures that correspond to hyperlattices and analyze their fundamental properties. This construction provides a representation of hyperlattices through clopen increasing subsets and establishes the bidirectional link between algebraic and topological perspectives. The results obtained here form the basis for the correspondence and isomorphism theorems developed in the subsequent sections.

Definition 2.1. *[8]*Let $\mathcal{A} = (\mathfrak{A}, \wedge, \sqcup, 0, 1)$ be a bounded distributive hyperlattice. The spectral dual of $\mathcal{A}$, denoted $\mathfrak{S}(\mathcal{A})$, consists of:

- The carrier set $\mathfrak{P}$ of all nontrivial hyperlattice homomorphisms $\phi : \mathcal{A} \to \{\mathbf{0}, \mathbf{1}\}$, where $\{\mathbf{0}, \mathbf{1}\} = (\{0, 1\}, \wedge, \sqcup, 0, 1))$ is the two-element Boolean algebra (structure).
- A point-by-point order $\leqslant$ on $\mathfrak{P}$, where $\chi \leqslant \psi$ iff $\chi(\varepsilon) \leqslant \psi(\varepsilon)$ for all $\varepsilon \in \mathfrak{A}$.
- The initial topology θ inherited from the product space $\{\mathbf{0}, \mathbf{1}\}^{\mathcal{A}}$ equipped with the discrete topology.

Theorem 2.1. *Let $\mathscr{H} = (H, \wedge, \sqcup, \perp, \top)$ be a bounded distributive hyperlattice. Then the spectral representation space $\mathfrak{S}(H)$, consisting of:*

- *The set of all proper hyperlattice morphisms to $\{\mathbf{0}, \mathbf{1}\}$*
- *Equipped with the pointwise order and spectral topology*

forms a Priestley space in the sense of [8].

2.2 Isomorphism in Priestley Spaces

The present section focuses on the role of isomorphisms within Priestley spaces and their connection to bounded distributive hyperlattices. By examining morphisms and the constructions they induce, we establish the structural correspondence between hyperlattices and their dual spaces. The results presented here highlight how the duality framework ensures that algebraic and topological perspectives are faithfully mirrored through isomorphisms

Definition 2.2. *[8]* Let $(S, \rho, \trianglelefteq)$ and $(T, \upsilon, \sqsubseteq)$ be two Priestley spaces. A mapping $\psi : S \to T$ is called a morphism of Priestley spaces if it satisfies the following conditions:

1. **Order preservation:** For all $x, y \in S$, whenever $x \trianglelefteq y$, it follows that $\psi(x) \sqsubseteq \psi(y)$.
2. **Topological continuity:** The map ψ is continuous with respect to the topologies ρ and υ; that is, for every open set $U \subseteq T$, the preimage $\psi^{-1}(U)$ is open in S.

If, in addition, ψ is bijective, then it is called an isomorphism of Priestley spaces.

Lemma 2.1. *[8] Let $\mathcal{Y} = (Y, \sigma, \sqsubseteq)$ be a Priestley space. Define*

$$L(\mathcal{Y}) = \{\, U \subseteq Y \mid U \text{ is } \sigma\text{-clopen and increasing} \,\}.$$

Then there exists a hyperoperation $\veebar$ on $L(\mathcal{Y})$, such that $\big(L(\mathcal{Y}), \cap, \veebar, \varnothing, Y\big)$ forms a bounded distributive hyperlattice.

Lemma 2.2. *[8] Let $\mathcal{H}$ be a bounded distributive hyperlattice and let $\mathcal{T}(\mathcal{H}) = (\mathcal{X}, \tau, \preceq)$ denote its Priestley dual space. Define*

$$\Phi_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathcal{L}(\mathcal{T}(\mathcal{H})), \qquad \Phi_{\mathcal{H}}(\alpha) = \{\, \varphi \in \mathcal{X} \mid \varphi(\alpha) = 1 \,\},$$

for each $\alpha \in \mathcal{H}$, where $\mathcal{L}(\mathcal{T}(\mathcal{H}))$ is the set of all increasing τ-clopen subsets of $\mathcal{X}$. Then, $\Phi_{\mathcal{H}}$ is a hyperlattice isomorphism.

Proof. **Preservation of meet ($\curlywedge$):** For any $\alpha, \beta \in \mathcal{H}$, $\Phi_{\mathcal{H}}(\alpha \curlywedge \beta) = \Phi_{\mathcal{H}}(\alpha) \cap \Phi_{\mathcal{H}}(\beta)$, since every homomorphism preserves finite meets.

Compatibility with the hyperoperation ($\blacktriangledown$): For the hyperoperation, $\Phi_{\mathcal{H}}(\alpha \blacktriangledown \beta) \subseteq \Phi_{\mathcal{H}}(\alpha) \cup \Phi_{\mathcal{H}}(\beta)$. Thus, the homomorphism condition is satisfied.

Injectivity. Suppose $\Phi_{\mathcal{H}}(\alpha) = \Phi_{\mathcal{H}}(\beta)$ with $\alpha \neq \beta$. By the prime filter separation property, there exists a homomorphism φ such that $\varphi(\alpha) = 1$ and $\varphi(\beta) = 0$. Hence $\varphi \in \Phi_{\mathcal{H}}(\alpha) \setminus \Phi_{\mathcal{H}}(\beta)$, contradiction. Therefore, $\alpha = \beta$.

Surjectivity. Let $U \in \mathcal{L}(\mathcal{T}(\mathcal{H}))$. For each $\varphi \in U$ and $\psi \notin U$, the Priestley separation property provides an element $\gamma \in \mathcal{H}$ with $\varphi(\gamma) = 1$, $\psi(\gamma) = 0$. Compactness ensures a finite family $\{\gamma_1, \ldots, \gamma_n\}$ suffices, so $U = \bigcup_{i=1}^{n} \Phi_{\mathcal{H}}(\gamma_i)$.

By distributivity, this finite union can be represented by a single element $\Gamma \in \mathcal{H}$ such that $U = \Phi_{\mathcal{H}}(\Gamma)$.

Conclusion. The mapping $\Phi_{\mathcal{H}}$ preserves the operations, is injective, and surjective. Therefore, it is a hyperlattice isomorphism between $\mathcal{H}$ and $\mathcal{L}(\mathcal{T}(\mathcal{H}))$. $\square$

2.3 Correspondence between Bounded Distributive Hyperlattices and Priestley Spaces

The present section establishes the correspondence between bounded distributive hyperlattices and Priestley spaces. By analyzing how homomorphisms in one framework induce morphisms in the other, we demonstrate the mutual translation between algebraic and topological structures. The results confirm that the duality is coherent and bidirectional, ensuring that each homomorphism or isomorphism has a precise counterpart in the opposite category.

Lemma 2.3. *[8] Let $f : \mathcal{A}_1 \to \mathcal{A}_2$ be a hyperlattice homomorphism. Then the map*

$$\mathcal{T}(f) : \mathcal{T}(\mathcal{A}_2) \longrightarrow \mathcal{T}(\mathcal{A}_1), \qquad \mathcal{T}(f)(g) = g \circ f,$$

for each $g \in \mathcal{T}(\mathcal{A}_2)$, is a homomorphism of Priestley spaces.

Proof. **Order preservation.** If $g_1 \preceq g_2$ in $\mathcal{T}(\mathcal{A}_2)$, then for every $\alpha \in \mathcal{A}_1$, $g_1(\alpha) \preceq g_2(\alpha)$.

Thus, for each α, $(g_1 \circ f)(\alpha) \preceq (g_2 \circ f)(\alpha)$, which shows $\mathcal{T}(f)(g_1) \preceq \mathcal{T}(f)(g_2)$. Hence, $\mathcal{T}(f)$ is increasing.

Continuity. For any $\alpha \in \mathcal{A}_1$, consider the basic clopen set

$$F_{\mathcal{A}_1}(\alpha) = \{g \in \mathcal{T}(\mathcal{A}_1) \mid g(\alpha) = 1\}.$$

Then

$$\mathcal{T}(f)^{-1}(F_{\mathcal{A}_1}(\alpha)) = \{g \in \mathcal{T}(\mathcal{A}_2) \mid (g \circ f)(\alpha) = 1\} = \{g \in \mathcal{T}(\mathcal{A}_2) \mid g(f(\alpha)) = 1\} = F_{\mathcal{A}_2}(f(\alpha)).$$

Since $F_{\mathcal{A}_2}(f(\alpha))$ is clopen, the preimage is clopen, proving continuity.

Conclusion. Therefore, $\mathcal{T}(f)$ is a Priestley space homomorphism. □

Lemma 2.4. *[8] Let $\mathcal{S} = (X, \tau, \preceq)$ be a Priestley space. Then the map $G_{\mathcal{S}}$ is given by*

$$G_{\mathcal{S}} : X \longrightarrow \mathcal{T}(\mathcal{L}(\mathcal{S})) \qquad G_{\mathcal{S}}(x)(Y) = \begin{cases} 1 & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y, \end{cases}$$

for all $Y \in \mathcal{L}(\mathcal{S})$, is an isomorphism of Priestley spaces.

Proof. **Surjectivity.** For each $f \in \mathcal{T}(\mathcal{L}(\mathcal{S}))$, define

$$U = \{Y \in \mathcal{L}(\mathcal{S}) \mid f(Y) = 1\}, \quad V = \{Z \in \mathcal{L}(\mathcal{S}) \mid f(Z) = 0\}.$$

By compactness and separation in Priestley spaces, there exists $x \in X$ such that $G_{\mathcal{S}}(x) = f$. Hence, $G_{\mathcal{S}}$ is surjective.

Injectivity. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since Priestley spaces are totally order-disconnected, there exists a clopen increasing set $Y \in \mathcal{L}(\mathcal{S})$ such that $x_1 \in Y$ and $x_2 \notin Y$. Then $G_{\mathcal{S}}(x_1)(Y) = 1$ while $G_{\mathcal{S}}(x_2)(Y) = 0$, so $G_{\mathcal{S}}(x_1) \neq G_{\mathcal{S}}(x_2)$. Thus, $G_{\mathcal{S}}$ is injective.

Continuity. For any clopen set $Y \in \mathcal{L}(\mathcal{S})$, $G_{\mathcal{S}}^{-1}(F_{\mathcal{L}(\mathcal{S})}(Y)) = Y$, which is clopen in X. Hence $G_{\mathcal{S}}$ is continuous.

Order-preservation. Since all $Y \in \mathcal{L}(\mathcal{S})$ are increasing, $x \preceq y$ implies $G_{\mathcal{S}}(x) \preceq G_{\mathcal{S}}(y)$.

Conclusion. Therefore, $G_{\mathcal{S}}$ is a Priestley space isomorphism. □

Lemma 2.5. *[8] Let $h : \mathcal{S}_1 \to \mathcal{S}_2$ be a homomorphism of Priestley spaces. Then the map*

$$\mathcal{L}(h) : \mathcal{L}(\mathcal{S}_2) \longrightarrow \mathcal{L}(\mathcal{S}_1), \qquad \mathcal{L}(h)(Y) = h^{-1}(Y),$$

for every $Y \in \mathcal{L}(\mathcal{S}_2)$, is a hyperlattice homomorphism.

Proof. **Preservation of meet ($\curlywedge$).** For all $Y, Z \in \mathcal{L}(\mathcal{S}_2)$,

$$\mathcal{L}(h)(Y \curlywedge Z) = h^{-1}(Y \cap Z) = h^{-1}(Y) \cap h^{-1}(Z) = \mathcal{L}(h)(Y) \curlywedge \mathcal{L}(h)(Z).$$

Compatibility with the hyperoperation ($\blacktriangledown$). Since inverse images commute with unions, for all $Y, Z \in \mathcal{L}(\mathcal{S}_2)$, $\mathcal{L}(h)(Y \blacktriangledown Z) \subseteq \mathcal{L}(h)(Y) \blacktriangledown \mathcal{L}(h)(Z)$. Thus, the hyperoperation condition is satisfied.

Conclusion. Therefore, $\mathcal{L}(h)$ is a hyperlattice homomorphism. □

2.4 Main Theorems of Duality

This section presents the fundamental results that complete the duality between bounded distributive hyperlattices and Priestley spaces. The following theorems establish the consistency of the canonical constructions and prove the categorical equivalence of the two frameworks.

Theorem 2.2. *[8] For any hyperlattice homomorphism $f : \mathcal{A}_1 \to \mathcal{A}_2$, the equality*

$$\mathcal{L}(\mathcal{T}(f)) \circ \Phi_{\mathcal{A}_1} \;=\; \Phi_{\mathcal{A}_2} \circ f$$

holds.

Proof. Take any $\alpha \in \mathcal{A}_1$. Then $(\mathcal{L}(\mathcal{T}(f)) \circ \Phi_{\mathcal{A}_1})(\alpha) = \mathcal{L}(\mathcal{T}(f))(\Phi_{\mathcal{A}_1}(\alpha))$. By definition, $\Phi_{\mathcal{A}_1}(\alpha) = \{g \in \mathcal{T}(\mathcal{A}_1) \mid g(\alpha) = 1\}$. Hence, $\mathcal{L}(\mathcal{T}(f))(\Phi_{\mathcal{A}_1}(\alpha)) = \{h \in \mathcal{T}(\mathcal{A}_2) \mid (h \circ f)(\alpha) = 1\}$.

But this is exactly $\{h \in \mathcal{T}(\mathcal{A}_2) \mid h(f(\alpha)) = 1\} = \Phi_{\mathcal{A}_2}(f(\alpha))$.

Therefore, the equality holds. □

Theorem 2.3. *[8] For any homomorphism $h : \mathcal{S}_1 \to \mathcal{S}_2$ of Priestley spaces, we have*

$$\mathcal{T}(\mathcal{L}(h)) \circ G_{\mathcal{S}_1} \;=\; G_{\mathcal{S}_2} \circ h.$$

Proof. Take any $x \in \mathcal{S}_1$. Then $(\mathcal{T}(\mathcal{L}(h)) \circ G_{\mathcal{S}_1})(x) = \mathcal{T}(\mathcal{L}(h))(G_{\mathcal{S}_1}(x))$.

By definition,

$$G_{\mathcal{S}_1}(x)(Y) = \begin{cases} 1 & \text{if } x \in Y, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } Y \in \mathcal{L}(\mathcal{S}_1).$$

Thus, $(\mathcal{T}(\mathcal{L}(h))(G_{\mathcal{S}_1}(x)))(Z) = G_{\mathcal{S}_1}(x)(h^{-1}(Z)) = \begin{cases} 1 & \text{if } x \in h^{-1}(Z), \\ 0 & \text{otherwise.} \end{cases}$

But $x \in h^{-1}(Z)$ iff $h(x) \in Z$. Hence

$$(\mathcal{T}(\mathcal{L}(h)) \circ G_{\mathcal{S}_1})(x)(Z) = G_{\mathcal{S}_2}(h(x))(Z).$$

Therefore the equality holds. □

Theorem 2.4. *[8] For every bounded distributive hyperlattice $\mathcal{H}$ and every Priestley space $\mathcal{S}$, the following equalities hold:*

$$\mathcal{L}(G_{\mathcal{S}}) \circ \Phi_{\mathcal{L}(\mathcal{S})} \;=\; \mathrm{id}_{\mathcal{L}(\mathcal{S})}, \qquad \mathcal{T}(\Phi_{\mathcal{H}}) \circ G_{\mathcal{T}(\mathcal{H})} \;=\; \mathrm{id}_{\mathcal{T}(\mathcal{H})}.$$

Proof. **First equality.** Take any $Y \in \mathcal{L}(\mathcal{S})$. Then $(\mathcal{L}(G_{\mathcal{S}}) \circ \Phi_{\mathcal{L}(\mathcal{S})})(Y) = \mathcal{L}(G_{\mathcal{S}})(\Phi_{\mathcal{L}(\mathcal{S})}(Y))$.

By definition,

$$\Phi_{\mathcal{L}(\mathcal{S})}(Y) = \{g \in \mathcal{T}(\mathcal{L}(\mathcal{S})) \mid g(Y) = 1\}.$$

Hence

$$\mathcal{L}(G_{\mathcal{S}})(\Phi_{\mathcal{L}(\mathcal{S})}(Y)) = \{x \in \mathcal{S} \mid G_{\mathcal{S}}(x)(Y) = 1\} = \{x \in \mathcal{S} \mid x \in Y\} = Y.$$

Thus the first equality holds.

Second equality. Take any $\varphi \in \mathcal{T}(\mathcal{H})$. Then $(\mathcal{T}(\Phi_{\mathcal{H}}) \circ G_{\mathcal{T}(\mathcal{H})})(\varphi) = \mathcal{T}(\Phi_{\mathcal{H}})(G_{\mathcal{T}(\mathcal{H})}(\varphi))$.
By definition,

$$G_{\mathcal{T}(\mathcal{H})}(\varphi)(\alpha) = \begin{cases} 1 & \text{if } \varphi \in \Phi_{\mathcal{H}}(\alpha), \\ 0 & \text{otherwise}, \end{cases} \quad \text{for all } \alpha \in \mathcal{H}.$$

Thus,

$$(\mathcal{T}(\Phi_{\mathcal{H}})(G_{\mathcal{T}(\mathcal{H})}(\varphi)))(\alpha) = G_{\mathcal{T}(\mathcal{H})}(\varphi)(\Phi_{\mathcal{H}}(\alpha)) = \begin{cases} 1 & \text{if } \varphi(\alpha) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence this map coincides with φ. Therefore the second equality holds.

Conclusion. Both equalities are satisfied, showing the natural equivalence between the constructions. □

Theorem 2.5. *The dual of the category of Priestley spaces is equivalent to the category of bounded distributive hyperlattices.*

Proof. By Lemma2.2, each bounded distributive hyperlattice $\mathcal{H}$ is naturally isomorphic to $\mathcal{L}(\mathcal{T}(\mathcal{H}))$.

By Lemma 2.4, each Priestley space S is naturally isomorphic to $\mathcal{T}(\mathcal{L}(S))$.

Together with Theorems2.2 and 2.3, these results establish the desired dual equivalence. □

2.5 Illustrative Examples of Representation in a Bounded Distributive Hyperlattice

This example illustrates the that there is a isomorphism between a bounded distributive hyperlattice and their bidual.

Example 2.1. Let $B = \{\theta, \alpha, \beta, \omega\}$. Consider the following Cayley tables:

$\curlywedge$	θ	α	β	ω
θ	θ	θ	θ	θ
α	θ	α	θ	α
β	θ	θ	β	β
ω	θ	α	β	ω

$\curlyvee$	θ	α	β	ω
θ	$\{\theta\}$	$\{\alpha\}$	$\{\beta\}$	$\{\omega\}$
α	$\{\alpha\}$	$\{\theta,\alpha\}$	$\{\omega\}$	$\{\beta,\omega\}$
β	$\{\beta\}$	$\{\omega\}$	$\{\theta,\beta\}$	$\{\alpha,\omega\}$
ω	$\{\omega\}$	$\{\beta,\omega\}$	$\{\alpha,\omega\}$	$\{\omega\}$

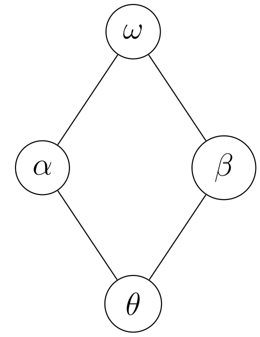

Figure 2.1: Cayley tables and Hasse diagram of $(B, \curlywedge, \curlyvee, \theta, \omega)$.

Then, $(B, \curlywedge, \curlyvee, \theta, \omega)$ is a bounded distributive hyperlattice. Let $T(B)$ be the set of homomorphisms from B onto $\{0,1\} = \{g_1, g_2\}$, where

$g_1:$	θ	α	β	ω
	0	0	1	1

$g_2:$	θ	α	β	ω
	0	1	0	1

Table 2.1: The dual space of $B = \{\theta, \alpha, \beta, \omega\}$

The bidual structure is recovered as: $U(T(B)) = \{\emptyset, \{g_1\}, \{g_2\}, \{g_1, g_2\}\}$. Define $Y = \{g_1, g_2\}$. Then, $(U(T(B)), \cap, \cup, \emptyset, Y)$ is a bounded distributive hyperlattice.

Finally, $A_B : B \longrightarrow U(T(B))$ is given by

$$A_B(\theta) = \emptyset, \quad A_B(\alpha) = \{g_2\}, \quad A_B(\beta) = \{g_1\}, \quad A_B(\omega) = \{g_1, g_2\}.$$

is a bounded distributive hyperlattice isomorphism.

Example 2.2. Let $(X, \rho, \preceq)$ be a Priestley space, where $X = \{\alpha, \beta, \gamma\}$, and the partial order $\preceq$ is given by:

$\preceq$	α	β	γ
α	1	0	1
β	0	1	1
γ	0	0	1

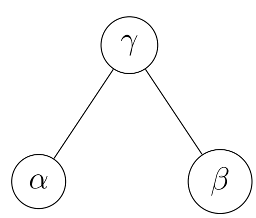

Figure 2.2: Partial order table and Hasse diagram of $(X, \preceq)$.

The associated distributive hyperlattice is: $\mathscr{L}(X) = \{\emptyset, \{\gamma\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, X\}$.

The hyperoperation $\curlyvee$ on $\mathscr{L}(X)$ is defined by:

$$\Lambda_1 \curlyvee \Lambda_2 = \{\Lambda \in (\mathscr{L}(X) \mid \Lambda_1 \cup \Lambda_2 = \Lambda \cup \Lambda_1 = \Lambda \cup \Lambda_2\}.$$

The dual space $T(\mathscr{L}(X))$ is given by: $T(\mathscr{L}(X)) = \{g_1, g_2, g_3\}$, with evaluations:

$\mathscr{L}(X)$	$\emptyset$	$\{\gamma\}$	$\{\alpha,\gamma\}$	$\{\beta,\gamma\}$	X
g_1	0	0	1	0	1
g_2	0	0	0	1	1
g_3	0	1	1	1	1

Table 2.2: Evaluations in the dual space $T(\mathscr{L}(X))$

Finally, the isomorphism $G_X : X \longrightarrow T(\mathscr{L}(X))$ is defined by:

$$G_X(\alpha) = g_1, \quad G_X(\beta) = g_2, \quad G_X(\gamma) = g_3.$$

Chapter Conclusion

This chapter has presented the representation of bounded distributive hyperlattices and emphasized the role of Priestley duality in establishing a coherent framework for their study. The results are based on the published article by [8], which rigorously established the equivalence between the category of Priestley spaces and the dual of the category of bounded distributive hyperlattices. By incorporating these findings into the thesis, this chapter serves as a bridge between established literature and the original contributions that will be developed in the subsequent chapters.

Chapter 3

Priestley Duality for Bounded Distributive Hyperlattices Negatively Ordered Commutative Monoids

The results presented in this chapter were published in the journal *QRS [19]*. They constitute the first step toward establishing a duality framework for bounded distributive hyperlattices equipped with negatively ordered commutative monoids. In what follows, we present the main contributions of this work, which extends the classical theory of Priestley duality to a broader algebraic context, where hyperlattice structures interact with monoidal operations under negative ordering constraints.

The motivation behind this study lies in bridging the gap between distributive hyperlattices and duality theory, thereby providing new insights into the representation of algebraic structures that generalize bounded distributive lattices. By establishing a dual equivalence in this enriched environment, we highlight both the theoretical depth and the potential applications of such constructions.

Furthermore, this chapter emphasizes the originality of the results, as they represent the first systematic attempt to develop a duality theory for hyperlattices combined with commutative monoids. The exposition follows the structure of the published article, while integrating additional clarifications and examples to ensure coherence within the thesis framework.

In summary, this chapter is devoted to the study of Priestley duality within the framework of bounded distributive hyperlattices negatively ordered commutative monoids. It begins with the construction of hyperlattices associated with such monoids, followed by the analysis of ideals and filters. The discussion then proceeds to homomorphisms of hyperlattice-ordered commutative monoids, and culminates in the Priestley-style representation. Finally, the chapter concludes with applications to the main representation theorem, thereby consolidating the theoretical results with illustrative examples.

3.1 Hyperlattice of Negatively Ordered Commutative Monoids

In this section, we recall the notion of negatively ordered commutative monoids within the framework of hyperlattices. The presentation includes the essential definitions and emphasizes some fundamental properties through selected theorems. To make the discussion more concrete, representative examples are provided, illustrating how these abstract structures operate in practice.

Let $\mathcal{G} = (S, \star, \preceq)$ be a structure where:

- $(S, \star)$ is a semigroup, i.e., a set S equipped with an associative binary operation $\star$;
- $\preceq$ is a partial order on S, making $(S, \preceq)$ a poset;
- The operation $\star$ is order-preserving (or compatible) if for all $x, y, z \in S$, we have:

$$x \preceq y \ \Rightarrow \ z \star x \preceq z \star y \quad \text{and} \quad x \star z \preceq y \star z.$$

Then $\mathcal{G}$ is called an ordered semigroup.

Such a structure is called *positively ordered* if $x \preceq x \star y$ and $y \preceq x \star y$ for all $x, y \in S$, and *negatively ordered* if the reverse inequalities hold.

Definition 3.1. Let Υ be a non-empty set, equipped with two binary operations $\sqcap$ and $\diamond$, and a hyperoperation $\Cup$. The structure Υ is called a bounded hyperlattice ordered commutative monoid if the following conditions hold:

1. $(\Upsilon, \sqcap, \Cup)$ is a hyperlattice;
2. $(\Upsilon, \diamond, e)$ is a commutative monoid with identity element e;
3. For all $x, y, z \in \Upsilon$: $x \diamond (y \sqcap z) \preceq (x \diamond y) \sqcap (x \diamond z), \quad x \diamond (y \Cup z) \subseteq (x \diamond y) \Cup (x \diamond z)$.

   These imply that if $y \preceq z$, then $y \diamond x \preceq z \diamond x$, for all $x, y, z \in \Upsilon$.

The structure is called distributive when it additionally satisfies $x \sqcap (y \Cup z) \subseteq (x \sqcap y) \Cup (x \sqcap z)$.

A bounded structure contains universal bounds $\perp$ and $\top$ such that $\perp \preceq w_1 \preceq \top$ for all $w_1 \in \Upsilon$. If $(\Upsilon, \sqcap, \Cup, \diamond, \perp, \top)$ is a bounded hyperlattice positively (negatively) ordered commutative monoid then $e = \perp$ $(e = \top)$.

Example 3.1. Let $\Upsilon = \{0, \vartheta, \varsigma, \varpi, 1\}$. Its Hasse diagram is:

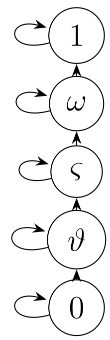

Figure 3.1: Hasse diagram of the linearly ordered set Υ.

$(\Upsilon, \leqslant, 0, 1)$ is a bounded distributive lattice. Define on Υ the hyperoperation $\Cup$ by $\alpha \Cup \beta = \{t \in \Upsilon \mid t \geq \alpha \vee \beta\}$ for all $\alpha, \beta \in \Upsilon$ and the binary operation $\diamond$ by the following Table 3.1,

$\diamond$	0	ϑ	ς	ω	1
0	0	ϑ	ς	ω	1
ϑ	ϑ	ς	ω	1	0
ς	ς	ω	1	0	ϑ
ω	ω	1	0	ϑ	ς
1	1	0	ϑ	ς	ω

Table 3.1: Cayley table for the binary operation on Υ.

Then $(\Upsilon, \sqcap, \Cup, \diamond, 0, 1)$ is a bounded distributive hyperlattice ordered commutative monoid.

Theorem 3.1. *Consider the bounded distributive lattice $\mp = (\Upsilon, \sqcap, \vee, \perp, \top)$. The following characterizations hold:*

1. *The structure $\mp = (\Upsilon, \sqcap, \Cup, \diamond, \perp, \top)$ can be endowed with the form of a bounded distributive hyperlattice equipped with a positively ordered commutative monoid. The operations are defined as:*

   - *Hyperjoin:* $w_1 \Cup u_2 = \{z \in \Upsilon \mid w_1 \vee u_2 = w_1 \vee z = u_2 \vee z\}$.

- *Operation law:* $w_1 \diamond u_2 = \begin{cases} w_1 \vee u_2 & \textit{if } w_1 = \bot \textit{ or } u_2 = \bot; \\ \top & \textit{otherwise}. \end{cases}$

2. *The same lattice* $\mp = (\Upsilon, \sqcap, \Cup, \diamond, \bot, \top)$ *can also be interpreted as a bounded distributive hyperlattice endowed with a negatively ordered commutative monoid. The operations are given by:*

   - *Hyperjoin:* $w_1 \Cup u_2 = \{z \in \Upsilon \mid w_1 \vee u_2 = w_1 \vee z = u_2 \vee z\}$.
   - *Operation law:* $w_1 \diamond u_2 = \begin{cases} w_1 \sqcap u_2 & \textit{if } w_1 = \top \textit{ or } u_2 = \top; \\ \top & \textit{otherwise}. \end{cases}$

3. *Finally, the lattice* $\mp = (\Upsilon, \sqcap, \Cup, \diamond, \bot, \top)$ *can be structured as a bounded distributive hyperlattice equipped with a positively ordered commutative monoid, where:*

   - *Hyperjoin:* $w_1 \Cup u_2 = \{z \in \Upsilon \mid z \geq w_1 \vee u_2\}$.
   - *Operation law:* $w_1 \diamond u_2 = \begin{cases} w_1 \vee u_2 & \textit{if } w_1 = \bot \textit{ or } u_2 = \bot; \\ \top & \textit{otherwise}. \end{cases}$

Proof. The proof follows directly from Definition 3.1. □

Corollary 3.1. Let $\mathcal{B} = \{0, 1\}$ be the Boolean algebra. Then $(\mathcal{B}, \sqcap, \Cup, \diamond, 0, 1)$ is a bounded distributive hyperlattice negatively ordered commutative monoid, with operations defined by Table 3.2:

$\sqcap$	0	1
0	0	0
1	0	1

$\Cup$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0, 1\}$

$\diamond$	0	1
0	0	0
1	0	1

Table 3.2: Operations tables for $\sqcap, \Cup, \diamond$ on $\{0, 1\}$

3.2 Ideals and Filters within Hyperlattice-Ordered Commutative Monoids

In this section, we introduce the concepts of ideals and filters in the framework of hyperlattice-ordered commutative monoids. Particular attention is devoted to prime ideals and prime filters, which serve as key instruments for analyzing the internal structure of these algebraic systems. The discussion emphasizes how these notions interact with the hyperlattice ordering, thereby providing a clearer perspective on the interplay between algebraic and order-theoretic properties.

Definition 3.2. Consider a structure $(\Upsilon, \sqcap, \Cup, \diamond)$ where Υ is a hyperlattice equipped with a commutative monoid operation. A subset $\mathcal{I} \sqsubseteq \Upsilon$, with $\mathcal{I} \neq \emptyset$, is termed an ideal if the following conditions are fulfilled :

1. Whenever $x \in \mathcal{I}$ and $y \preceq x$, then y also belongs to $\mathcal{I}$.

2. The hyperjoin of any two elements in $\mathcal{I}$ remains within $\mathcal{I}$; that is, $x, y \in \mathcal{I}$ implies $x \Cup y \sqsubseteq \mathcal{I}$.

3. The product of any element in $\mathcal{I}$ with an arbitrary element of Υ lies in $\mathcal{I}$; i.e., $x \in \mathcal{I}$, $y \in \Upsilon$ implies $x \diamond y \in \mathcal{I}$.

Furthermore, a proper ideal $\mathcal{I}$ is designated as prime if

1. For all $x, y \in \Upsilon$, the inclusion $x \diamond y \in \mathcal{I}$ necessitates that at least one of x or y belongs to $\mathcal{I}$.

2. Whenever $x \sqcap y \in \mathcal{I}$, then at least one of x or y is in $\mathcal{I}$.

Definition 3.3. Let $(\Upsilon, \sqcap, \Cup, \diamond)$ be a hyperlattice endowed with a commutative monoid structure. A non-empty subset $\mathcal{F} \subseteq \Upsilon$ is called a filter if it satisfies the following properties:

1. If $x \in \mathcal{F}$ and $x \preceq y$, then $y \in \mathcal{F}$.

2. The meet of any two elements in $\mathcal{F}$ is also in $\mathcal{F}$; that is, $x, y \in \mathcal{F}$ implies $x \sqcap y \in \mathcal{F}$.

3. The product of any two elements in $\mathcal{F}$ remains in $\mathcal{F}$; i.e., $x, y \in \mathcal{F}$ implies $x \diamond y \in F$.

A proper filter $\mathcal{F}$ is referred to as prime if

1. Whenever a non-empty intersection $(x \Cup y) \cap \mathcal{F} \neq \emptyset$, then at least one of x or y is in $\mathcal{F}$;

2. If the product $x \diamond y$ in $\mathcal{F}$, then either $x \in \mathcal{F}$ or $y \in \mathcal{F}$.

The following theorem shows that the complement of a prime filter (ideal) is a prime ideal (filter).

Theorem 3.2. *Let $(\mathcal{H}, \curlywedge, \curlyvee, \circ)$ be a commutative monoid structured as a hyperlattice and negatively ordered. Then the following duality principle holds:*

- *If $\mathcal{I}$ is a prime ideal of $\mathcal{H}$, then the complement $\mathcal{H} \setminus \mathcal{I}$ is a prime filter of $\mathcal{H}$.*
- *Conversely, if $\mathcal{F}$ is a prime filter of $\mathcal{H}$, then the complement $\mathcal{H} \setminus \mathcal{F}$ is a prime ideal of $\mathcal{H}$.*

Proof. We first prove that $\mathcal{H} \setminus \mathcal{I}$ is a prime filter.

Upward closure: Let $\mu \in \mathcal{H} \setminus \mathcal{I}$ and $\nu \in \mathcal{H}$ such that $\mu \preceq \nu$. If $\nu \in \mathcal{I}$, then $\mu \in \mathcal{I}$, contradiction. Hence, $\nu \in \mathcal{H} \setminus \mathcal{I}$.

Closure under meet: Suppose $\mu, \nu \in \mathcal{H} \setminus \mathcal{I}$. If $\mu \curlywedge \nu \in \mathcal{I}$, then by primality of $\mathcal{I}$, either $\mu \in \mathcal{I}$ or $\nu \in \mathcal{I}$, contradiction. Thus $\mu \curlywedge \nu \in \mathcal{H} \setminus \mathcal{I}$.

Closure under product (using negative order): Let $\mu, \nu \in \mathcal{H} \setminus \mathcal{I}$. Since $\mathcal{H}$ is negatively ordered, if $\mu \circ \nu \in \mathcal{I}$, then by primality of $\mathcal{I}$, either $\mu \in \mathcal{I}$ or $\nu \in \mathcal{I}$, contradiction. Therefore $\mu \circ \nu \in \mathcal{H} \setminus \mathcal{I}$.

Hence, $\mathcal{H} \setminus \mathcal{I}$ is a filter.

Primality conditions: - Let $\mu, \nu \in \mathcal{H}$ such that $(\mu \curlyvee \nu) \cap (\mathcal{H} \setminus \mathcal{I}) \neq \emptyset$. Then there exists $\theta \in \mu \curlyvee \nu$ with $\theta \notin \mathcal{I}$. If both $\mu, \nu \in \mathcal{I}$, then $\mu \curlyvee \nu \subseteq \mathcal{I}$, contradiction. Thus, $\mu \in \mathcal{H} \setminus \mathcal{I}$ or $\nu \in \mathcal{H} \setminus \mathcal{I}$.

- Let $\mu, \nu \in \mathcal{H}$ such that $\mu \circ \nu \in \mathcal{H} \setminus \mathcal{I}$. If $\mu, \nu \in \mathcal{I}$, then since $\mathcal{I}$ is an ideal, $\mu \circ \nu \in \mathcal{I}$, contradiction. Hence $\mu \in \mathcal{H} \setminus \mathcal{I}$ or $\nu \in \mathcal{H} \setminus \mathcal{I}$.

Thus $\mathcal{H} \setminus \mathcal{I}$ is a prime filter.

The converse: By symmetry, if $\mathcal{F}$ is a prime filter of $\mathcal{H}$, then its complement $\mathcal{H} \setminus \mathcal{F}$ satisfies the axioms of a prime ideal. □

Proposition 3.1. *Let σ be a hyperlattice filter of a distributive hyperlattice negatively ordered commutative monoid $(\mathcal{H}, \curlywedge, \curlyvee, \circ)$. Then the least hyperlattice-ordered commutative monoid filter containing σ is given by*

$$[\sigma) \;=\; \{\, x \in \mathcal{H} \mid a_1 \circ a_2 \circ \cdots \circ a_n \preceq x \textit{ for some } a_1, \ldots, a_n \in \sigma \,\}.$$

Proof. First, we show that $[\sigma)$ is non-empty. Indeed, for any $a \in \sigma$, since $a \preceq a$, it follows that $a \in [\sigma)$. Hence $[\sigma) \neq \emptyset$.

Next, we prove that $[\sigma)$ is a filter of $(\mathcal{H}, \curlywedge, \curlyvee, \circ)$. Let $x \in [\sigma)$ and $y \in \mathcal{H}$ with $x \preceq y$. Then there exist $a_1, \ldots, a_n \in \sigma$ such that $a_1 \circ \cdots \circ a_n \preceq x \preceq y$, which implies $y \in [\sigma)$.

Moreover, if $x, y \in [\sigma)$, then there exist $a_1, \ldots, a_n, b_1, \ldots, b_m \in \sigma$ such that $a_1 \circ \cdots \circ a_n \preceq x$, $b_1 \circ \cdots \circ b_m \preceq y$. Consequently, $(a_1 \circ \cdots \circ a_n) \curlywedge (b_1 \circ \cdots \circ b_m) \preceq x \curlywedge y$. Since $\mathcal{H}$ is negatively ordered, we deduce that $x \curlywedge y \in [\sigma)$.

For the product, let $x, y \in [\sigma)$. Then there exist $a_1, \ldots, a_n, b_1, \ldots, b_m \in \sigma$ such that $a_1 \circ \cdots \circ a_n \preceq x$, $b_1 \circ \cdots \circ b_m \preceq y$.

It follows that $(a_1 \circ \cdots \circ a_n) \circ y \preceq x \circ y$, and $(a_1 \circ \cdots \circ a_n) \circ (b_1 \circ \cdots \circ b_m) \preceq (a_1 \circ \cdots \circ a_n) \circ y$, and hence, $(a_1 \circ \cdots \circ a_n) \circ (b_1 \circ \cdots \circ b_m) \preceq x \circ y$. Therefore $x \circ y \in [\sigma)$.

Now, observe that for any $a \in \sigma$, we have $a \preceq a$, so $a \in [\sigma)$. Thus $\sigma \subseteq [\sigma)$.

Finally, let F be a filter of $(\mathcal{H}, \curlywedge, \curlyvee, \circ)$ with $\sigma \subseteq F$. For any $x \in [\sigma)$, there exist $a_1, \ldots, a_n \in \sigma$ such that $a_1 \circ \cdots \circ a_n \preceq x$.

Since F is a filter containing σ, it follows that $x \in F$. Hence $[\sigma) \subseteq F$.

This completes the proof. $\square$

Theorem 3.3. *Let $(\mathcal{H}, \wedge, \sqcup, \circ)$ be a distributive hyperlattice endowed with a negatively ordered commutative monoid structure. Let $\mathcal{F}$ be a filter and $\mathcal{I}$ an ideal of $\mathcal{H}$. If $\mathcal{F} \cap \mathcal{I} = \emptyset$, then there exists a prime filter $\mathcal{P}$ such that $\mathcal{F} \subseteq \mathcal{P}$ and $\mathcal{P} \cap \mathcal{I} = \emptyset$.*

Proof. We show that such a prime filter $\mathcal{P}$ can be constructed.

Consider the primality condition: if $(\mu \sqcup \nu) \cap \mathcal{P} \neq \emptyset$, then necessarily $\mu \in \mathcal{P}$ or $\nu \in \mathcal{P}$ (see [8]). This ensures that $\mathcal{P}$ respects the prime property with respect to the hyper-join.

Now suppose $\mu \circ \nu \in \mathcal{P}$. Since $\mathcal{H}$ is a distributive hyperlattice and negatively ordered commutative monoid, it follows that both $\mu \in \mathcal{P}$ and $\nu \in \mathcal{P}$. This guarantees closure under the product operation in accordance with the prime filter axioms.

Thus, starting from $\mathcal{F}$ disjoint from $\mathcal{I}$, one can extend $\mathcal{F}$ to a prime filter $\mathcal{P}$ such that $\mathcal{F} \subseteq \mathcal{P}$ and $\mathcal{P} \cap \mathcal{I} = \varnothing$. $\square$

Corollary 3.2. Let $(\mathcal{H}, \wedge, \sqcup, \circ)$ be a distributive hyperlattice negatively ordered commutative monoid. If Π is an ideal of $\mathcal{H}$ and $\alpha \in \mathcal{H} \setminus \Pi$, then there exists a prime filter Φ of $\mathcal{H}$ such that

$$\alpha \in \Phi \quad \text{and} \quad \Phi \cap \Pi = \varnothing.$$

Proof. Let Π be an ideal of $\mathcal{H}$ and define the set $\Gamma = \{\, \beta \in \mathcal{H} \setminus \Pi \mid \alpha \circ \cdots \circ \alpha \preceq \beta \,\}$.

Clearly, $\Gamma \cap \Pi = \varnothing$. It is straightforward to verify that Γ is a filter of $(\mathcal{H}, \wedge, \sqcup, \circ)$. By Theorem 3.3, there exists a prime filter Φ such that $\Gamma \subseteq \Phi$ and $\Phi \cap \Pi = \varnothing$.

Since $\alpha \in \Gamma$, it follows that $\alpha \in \Phi$. $\square$

3.3 Homomorphisms of Hyperlattice-Ordered Commutative Monoids

In this section, we study homomorphisms between hyperlattice-ordered commutative monoids. The focus is on how these mappings preserve both the algebraic and order-theoretic structures, and on the consequences that arise from such preservation. By examining the behavior of homomorphisms in this framework, we gain insight into structural compatibility and open the way for further developments related to representation and duality.

Definition 3.4. Let $(\mathcal{H}, \wedge, \sqcup, \cdot)$ and $(\mathcal{H}', \curlywedge, \curlyvee, \diamond)$ be two hyperlattices equipped with commutative monoid structures. A mapping $\varphi : \mathcal{H} \longrightarrow \mathcal{H}'$ is called a homomorphism of hyperlattice-ordered commutative monoids provided that, for all $x, y \in \mathcal{H}$, the following conditions are satisfied:

- **Exact preservation of meets:** $\varphi(x \wedge y) = \varphi(x) \curlywedge \varphi(y)$,
- **Inclusive mapping of joins:** $\varphi(x \sqcup y) \subseteq \varphi(x) \curlyvee \varphi(y)$,
- **Order-respecting multiplication:** $\varphi(x \cdot y) \preceq \varphi(x) \diamond \varphi(y)$.

If, in addition, φ is bijective, then it is called a *hyperlattice-ordered commutative monoid isomorphism*.

Proposition 3.2. *Let $\mathcal{H} = (X, \curlywedge, \curlyvee, *)$ be a hyperlattice endowed with a commutative monoid structure, and let $\mathbf{B} = (\{0,1\}, \wedge, \vee, \cdot, 0, 1)$ denote the bounded hyperlattice negatively ordered commutative monoid introduced in Corollary 3.1. Suppose $G \subseteq X$. If G is a prime filter, then there exists a surjective homomorphism of negatively ordered commutative hyperlattices $\Phi : \mathcal{H} \longrightarrow \mathbf{B}$, such that $G = \Phi^{-1}(\{1\})$.*

Proof. Define the mapping $\Phi : \mathcal{H} \longrightarrow \mathbf{B}$ by $\Phi(x) = \begin{cases} 1 & \text{if } x \in G, \\ 0 & \text{otherwise.} \end{cases}$
We verify the homomorphism conditions: For the conditions $\Phi(x \curlywedge y) = \Phi(x) \wedge \Phi(y)$ and $\Phi(x \curlyvee y) \subseteq \Phi(x) \vee \Phi(y)$, the verification follows directly from the properties established in [8, Proposition 2.12].

For the third homomorphism axiom, consider the product: We have $\Phi(x) \cdot \Phi(y) = 1 \Leftrightarrow (x \in G \text{ and } y \in G)$. Since G is a filter, this is equivalent to $x * y \in G$, hence $\Phi(x * y) = 1$. Therefore, $\Phi(x) \cdot \Phi(y) = \Phi(x * y)$. Thus, Φ satisfies all homomorphism axioms. Moreover, Φ is surjective, and by construction we have $G = \Phi^{-1}(\{1\})$. $\square$

Corollary 3.3. Let $\mathcal{K} = (Y, \triangle, \triangledown, \circ)$ be a distributive hyperlattice negatively ordered commutative monoid. For any $u, v \in Y$ with $u \ntrianglelefteq v$, there exists a prime filter $\mathcal{F} \subseteq Y$ such that $u \notin \mathcal{F}$ and $v \in \mathcal{F}$.

Proof. Consider the principal ideal $\downarrow v = \{y \in Y \mid y \trianglelefteq v\}$, which, by Corollary 3.2, is an ideal of the distributive hyperlattice negatively ordered commutative monoid $\mathcal{K}$. Since $u \ntrianglelefteq v$, it follows that $u \notin \downarrow v$. By the separation property between prime filters and ideals, there exists a prime filter $\mathcal{F}$ disjoint from $\downarrow v$ such that $v \in \mathcal{F}$ while $u \notin \mathcal{F}$. This completes the proof. $\square$

3.4 Priestley-Style Representation in Hyperlattices-Ordered Commutative Monoids

This section develops the Priestley-style representation within hyperlattice-ordered negatively ordered commutative monoids. We begin with the foundations of Priestley spaces and their homomorphisms, proceed to the construction of dual spaces, and then examine biduality and structural equivalence. The discussion culminates in the main representation theorem, which establishes a rigorous correspondence between algebraic structures and their Priestley-style dual spaces.

3.4.1 Priestley Spaces Negatively Ordered Commutative Semigroups

Inspired by the notion of Priestley spaces (see, [37] and [38]), in the following, we give some definitions and properties of Priestley space equipped with a negatively ordered commutative semigroup structure.

Definition 3.5. A Priestley-ordered commutative semigroup is characterized as a structure $\mathfrak{E} = (\mathcal{E}, \tau, \preccurlyeq, \circledast)$ satisfying:

- $(\mathcal{E}, \tau, \preccurlyeq)$ is a Priestley space, i.e., a compact and totally disconnected ordered topological space.
- $(\mathcal{E}, \preccurlyeq, \circledast)$ is a commutative semigroup with a negative order, meaning that for all $\varepsilon_1, \varepsilon_2 \in \mathcal{E}$,

$$\varepsilon_1 \circledast \varepsilon_2 \preccurlyeq \varepsilon_1 \quad \text{and} \quad \varepsilon_1 \circledast \varepsilon_2 \preccurlyeq \varepsilon_2.$$

Definition 3.6. A topological-algebraic structure $\mathfrak{E} = (\mathcal{E}, \tau, \preccurlyeq, \circledast)$ is classified as a Priestley-ordered commutative semigroup when:

- The triple $(\mathcal{E}, \tau, \preccurlyeq)$ forms a space of Priestley,
- The algebraic structure $(\mathcal{E}, \preccurlyeq, \circledast)$ constitutes a commutative semigroup with negative ordering.

Definition 3.7. Given a bounded distributive hyperlattice endowed with a negatively ordered commutative monoid structure $\mathcal{H} = (H, \sqcap, \sqcup, \odot, \bot, \top)$. Its dual representation space is defined as $\mathfrak{T}(\mathcal{H}) = (\mathcal{X}, \varsigma, \preccurlyeq, *)$, where:

- $\mathcal{X}$ is the set of all homomorphisms $\psi : \mathcal{H} \to \{\bot, \top\}$ preserving $\bot$ and $\top$.
- The order $\preccurlyeq$ is pointwise: $\psi \preccurlyeq \varphi$ iff $\psi(d) \leq \varphi(d)$ for every $d \in \mathcal{H}$.
- ς is the product topology inherited from $\{\bot, \top\}^{\mathcal{H}}$.
- The semigroup operation $*$ on $\mathcal{X}$ is given by

$$(\psi * \varphi)(d) = \begin{cases} \psi(d) \sqcap \varphi(d), & \text{if } \psi(d) \sqcup \varphi(d) = \top, \\ \bot, & \text{otherwise}. \end{cases}$$

Proposition 3.3. *Let $\mathscr{H} = (E, \wedge, \veebar, \boxdot, 0, 1)$ be a bounded distributive hyperlattice endowed with the structure of a negatively ordered commutative monoid. Then the dual space $\mathscr{T}(\mathscr{H}) = (Z, \mathcal{T}, \preccurlyeq, \diamond)$ is a Priestley space equipped with a negatively ordered commutative semigroup structure.*

Proof. Consider the dual space $\mathscr{T}(\mathscr{H}) = (Z, \mathcal{T}, \preccurlyeq, \diamond)$ associated with $\mathscr{H} = (E, \wedge, \veebar, \boxdot, 0, 1)$. It is already established (see [8]) that $(Z, \mathcal{T}, \preccurlyeq)$ forms a Priestley space. Thus, it remains to verify that the binary operation $\diamond$ on Z endows $(Z, \preccurlyeq, \diamond)$ with the structure of a negatively ordered commutative semigroup.

Commutativity and associativity: For any $\phi_1, \phi_2 \in Z$, a straightforward computation shows that $\phi_1 \diamond \phi_2$ is a homomorphism. Moreover,

$$\phi_1 \diamond \phi_2 = \phi_2 \diamond \phi_1, \quad \text{and} \quad (\phi_1 \diamond \phi_2) \diamond \phi_3 = \phi_1 \diamond (\phi_2 \diamond \phi_3),$$

which ensures that $(Z, \diamond)$ is a commutative semigroup.

Order preservation: $(Z, \preccurlyeq)$ is partially ordered by the relation $\phi_1 \preccurlyeq \phi_2 \Leftrightarrow \phi_1(a) \preccurlyeq \phi_2(a)$.

Let $\phi_1, \phi_2, \phi_3 \in Z$ with $\phi_1 \preccurlyeq \phi_2$. Then, for all $a \in E$, $\phi_1(a) \wedge \phi_3(a) \preccurlyeq \phi_2(a) \wedge \phi_3(a)$, and $\phi_3(a) \wedge \phi_1(a) \preccurlyeq \phi_3(a) \wedge \phi_2(a)$.

Hence, $\phi_1 \diamond \phi_3 \preccurlyeq \phi_2 \diamond \phi_3$ and $\phi_3 \diamond \phi_1 \preccurlyeq \phi_3 \diamond \phi_2$, which shows that $(Z, \preccurlyeq, \diamond)$ is a partially ordered semigroup.

Negative ordering: Finally, for all $\phi_1, \phi_2 \in Z$, $(\phi_1 \diamond \phi_2)(a) = \phi_1(a) \wedge \phi_2(a) \preccurlyeq \phi_1(a)$ and $(\phi_1 \diamond \phi_2)(a) \preccurlyeq \phi_2(a)$. Thus, $\phi_1 \diamond \phi_2 \preccurlyeq \phi_1$ and $\phi_1 \diamond \phi_2 \preccurlyeq \phi_2$, which confirms that $(Z, \preccurlyeq, \diamond)$ is negatively ordered.

Therefore, we conclude that $\mathscr{T}(\mathscr{H}) = (Z, \mathcal{T}, \preccurlyeq, \diamond)$ is indeed a Priestley space endowed with the structure of a negatively ordered commutative semigroup. $\square$

3.4.2 Homomorphism in Negatively Ordered Structures

In what follows, we introduce the notion of a Priestley space negatively ordered commutative semigroup homomorphism.

Definition 3.8. Let $\mathscr{X} = (\mathfrak{X}, \tau, \preccurlyeq, \star)$ and $\mathscr{Y} = (\mathfrak{Y}, \tau', \sqsubseteq, \diamond)$ be two Priestley spaces endowed with negatively ordered commutative semigroup structures. A mapping $\psi : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is called a Priestley space negatively ordered commutative semigroup homomorphism if it satisfies the following conditions:

- **Monotonicity (Order preservation):** $\xi_1 \preccurlyeq \xi_2 \Longrightarrow \psi(\xi_1) \sqsubseteq \psi(\xi_2), \quad \forall \xi_1, \xi_2 \in \mathfrak{X}$.
- **Topological continuity:** ψ is continuous with respect to the topologies τ and τ'.
- **Submultiplicativity:** $\psi(\xi_1 \star \xi_2) \sqsubseteq \psi(\xi_1) \diamond \psi(\xi_2), \quad \forall \xi_1, \xi_2 \in \mathfrak{X}$.

If ψ is bijective, then f is called a Priestley space negatively ordered commutative semigroup isomorphism.

3.4.3 Dual Construction of Priestley Space

In what follows, we establish necessary conditions ensuring that the dual of a Priestley space, negatively ordered commutative semigroup, admits the structure of a bounded distributive hyperlattice negatively ordered commutative monoid.

Lemma 3.1. *Let $\Delta = (\mathfrak{X}, \tau, \preccurlyeq, \diamond)$ be a Priestley space endowed with a negatively ordered commutative semigroup. Then there exists a binary hyperoperation $\curlyvee$ and a binary operation $\boxtimes$ such that*

$(\mathcal{L}(\Delta), \curlywedge, \curlyvee, \boxtimes, \emptyset, \mathfrak{X})$ *is a bounded distributive hyperlattice negatively ordered commutative monoid, where* $\mathcal{L}(\Delta) = \{Y \subseteq \mathfrak{X} \mid Y$ *is* τ*-clopen and* $\preccurlyeq$ *-increasing*$\}$. *The operations are defined by,*

1. *The hyperjoin* $\curlyvee$ *is given by:*

$$U_1 \curlyvee U_2 = \{Q \in \mathcal{L}(\Delta) \mid U_1 \cup U_2 \subseteq Q \text{ and } U_1 \cup Q = U_2 \cup Q\}.$$

2. *The product* $\boxtimes$ *is defined as:*

$$U_1 \boxtimes U_2 = \begin{cases} U_1 \curlywedge U_2 & \text{if } U_1 = \mathfrak{X} \text{ or } U_2 = \mathfrak{X}, \\ \emptyset & \text{otherwise}. \end{cases}$$

for all clopen increasing sets $U_1, U_2 \in \mathcal{L}(\Delta)$.

Proof. A straightforward verification shows that the algebra $\big(\mathcal{L}(\Delta), \curlywedge, \cup, \emptyset, \mathfrak{X}\big)$ constitutes a bounded distributive lattice, as referenced in [37, 38]. According to Definition 3.1, we conclude that the algebra $(\mathcal{L}(\Delta), \curlywedge, \curlyvee, \boxtimes, \emptyset, \mathfrak{X})$ is a bounded distributive hyperlattice that is negatively ordered and a commutative monoid.

□

3.4.4 Biduality in Bounded Distributive Hyperlattice Negatively Ordered Commutative Monoids

In the following lemma, we show that the bidual of a bounded distributive hyperlattice negatively ordered commutative monoid is a bounded distributive hyperlattice negatively ordered commutative monoid.

Lemma 3.2. *Let* $(\mathcal{H}, \wedge, \uplus, \circledast, 0, 1)$ *be a bounded distributive hyperlattice equipped with the operations of a negatively ordered commutative monoid. Then* $(\mathcal{L}(\mathcal{T}(\mathcal{H})), \cap, \uplus, \diamond, \emptyset, X)$ *is also a bounded distributive hyperlattice equipped with the operations of a negatively ordered commutative monoid.*

Proof. By construction, we have $\mathcal{L}(\mathcal{T}(\mathcal{H})) = \{\, Y \subseteq X \mid Y$ is $\preceq$-increasing and τ-clopen$\,\}$.

The operations are defined as follows:

$$U \ \uplus \ V = \{\, W \in \mathcal{L}(\mathcal{T}(\mathcal{H})) \mid U \cup V = U \cup W = V \cup W \,\},$$

$$U \diamond V = \begin{cases} U \cap V, & \text{if } U = X \text{ or } V = X, \\ \emptyset, & \text{otherwise.} \end{cases}$$

for all $U, V \in \mathcal{L}(\mathcal{T}(\mathcal{H}))$. Hence, by Definition3.1, the required properties follow and the structure is established. □

3.4.5 Biduality and Isomorphisms in Hyperlattice-Ordered Monoids

Lemma 3.3. *For any bounded distributive hyperlattice* $\mathscr{H} = (\mathcal{H}, \curlywedge, \curlyvee, \circledast, 0, 1)$ *equipped with negatively ordered commutative monoid. Define the mapping*

$$\Phi_{\mathscr{H}} : \mathscr{H} \hookrightarrow \mathcal{L}(\mathscr{T}(\mathscr{H})), \qquad \Phi_{\mathscr{H}}(\alpha) = \{\, \phi \in \mathcal{Y} \mid \phi(\alpha) = 1 \,\}.$$

Then $\Phi_{\mathcal{H}}$ *is an isomorphism of hyperlattices negatively ordered commutative monoids.*

Proof. Since $\mathcal{H}$ is a hyperlattice, then $\Phi_{\mathscr{H}}$ is a hyperlattice isomorphism (see [8]). Moreover, by Lemma3.2, $\mathcal{L}(\mathscr{T}(\mathscr{H}))$ is itself a distributive hyperlattice endowed with a negative order and a commutative monoidal structure.

To verify the monoidal compatibility, let $\alpha, \beta \in \mathcal{H}$ and $g \in \mathcal{Y}$. If $g \in \Phi_{\mathscr{H}}(\alpha \circledast \beta)$, then $g(\alpha \circledast \beta) = 1 \Rightarrow g(\alpha) \circledast g(\beta) = 1$. Thus $g(\alpha) = 1$ and $g(\beta) = 1$, which implies $g \in \Phi_{\mathscr{H}}(\alpha) \cap \Phi_{\mathscr{H}}(\beta) \Rightarrow g \in \Phi_{\mathscr{H}}(\alpha) \diamond \Phi_{\mathscr{H}}(\beta)$. Therefore, $\Phi_{\mathscr{H}}(\alpha \circledast \beta) \subseteq \Phi_{\mathscr{H}}(\alpha) \diamond \Phi_{\mathscr{H}}(\beta)$.

Consequently, $\Phi_{\mathscr{H}}$ establishes an isomorphism from $\mathscr{H}$ onto $\mathcal{L}(\mathscr{T}(\mathscr{H}))$ as a distributive hyperlattice with negative order and commutative monoidal structure. □

Lemma 3.4. *Given a structure-preserving map* $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ *between hyperlattice negatively ordered commutative monoid homomorphism, the induced mapping*

$$\mathcal{D}(\phi) : \mathcal{D}(\mathcal{H}_2) \to \mathcal{D}(\mathcal{H}_1), \quad \psi \mapsto \psi \circ \phi$$

constitutes a morphism of a Priestley space negatively ordered commutative semigroup.

Proof. We establish the required properties:

1. **Monotonicity and Continuity**: It has been demonstrated that the mapping $\mathcal{D}(\phi)$ preserves both order-theoretic properties and topological consistency, as documented in prior contribution [4], [5], [6], [7] [8], [37], [38].

2. **Operation Compatibility**: For any pair $\varrho_1, \varrho_2 \in \mathcal{D}(\mathcal{H}_2)$ and element $h \in H_1$:

1. When $\varrho_1(\phi(h)) = 1$, the composition satisfies:

$$(\mathcal{D}(\phi)(\varrho_1) \diamond \mathcal{D}(\phi)(\varrho_2))(h) = \varrho_2(\phi(h))$$

   The relation $\varrho_1 \star \varrho_2 \preceq \varrho_2$ ensures:

$$\mathcal{D}(\phi)(\varrho_1 \star \varrho_2)(h) = (\varrho_1 \star \varrho_2)(\phi(h)) \leq (\mathcal{D}(\phi)(\varrho_1) \diamond \mathcal{D}(\phi)(\varrho_2))(h)$$

2. The symmetric case $\varrho_2(\phi(h)) = 1$ follows similarly.

3. When $\varrho_1(\phi(h)) = \varrho_2(\phi(h)) = 0$, we obtain:

$$\mathcal{D}(\phi)(\varrho_1) \diamond \mathcal{D}(\phi)(\varrho_2) = \mathcal{D}(\phi)(\varrho_1 \star \varrho_2) = 0$$

Thus, $\mathcal{D}(\phi)$ preserves the Priestley structure and acts as a homomorphism of negatively ordered commutative semigroups. □

3.4.6 Biduality and Structural Equivalence in Priestley Space

Lemma 3.5. *Assume that $\mathscr{P}_1 = (P_1, \mathcal{T}, \trianglelefteq, *)$ and $\mathscr{P}_1 = (P_2, \mathcal{T}', \preceq, \diamond)$ are Priestley spaces endowed with negatively ordered commutative semigroups. If $\mu : \mathscr{P}_1 \to \mathscr{P}_2$ is a homomorphism between such structures, then the inverse image mapping*

$$L(\mu) : \gamma \mapsto \mu^{-1}(\gamma)$$

establishes a surjective homomorphism from $L(\mathscr{P}_2)$ onto $L(\mathscr{P}_1)$, thereby ensuring the correspondence between the underlying algebraic structures.

Proof. Define $L(\mathscr{P}_1) = \{Z \subseteq X \mid Z \text{ is increasing and } \tau\text{-clopen}\}$ and
$L(\mathscr{P}_2) = \{Z \subseteq Y \mid Z \text{ is increasing and } \tau\text{-clopen}\}$.

For every $y \in L(\mathscr{P}_2)$, we have $L(\mu)(\gamma) \in L(\mathscr{P}_1)$.

Step 1: Preservation of Set-Theoretic Operations. For all $\gamma, \varepsilon \in L(\mathscr{P}_2)$, since μ^{-1} commutes with set-theoretical operations, we obtain

$$L(\mu)(\gamma \cap \varepsilon) = L(\mu)(\gamma) \cap L(\mu)(\varepsilon), \qquad L(\mu)(\gamma \curlyvee \varepsilon) \subseteq L(\mu)(\gamma) \curlyvee L(\mu)(\varepsilon) \quad \text{(see [4], [5], [6], [7] [8]).}$$

Step 2: Case Analysis for the Third Homomorphism Axiom. We distinguish three cases:

1. If $\gamma = Y$, then $L(\mu)(\gamma \diamond \varepsilon) = L(\mu)(\varepsilon) = X * L(h)(\varepsilon) = L(\mu)(\gamma) * L(\mu)(\varepsilon)$.

2. If $\varepsilon = Y$, the argument is analogous to case (1).

3. If $\gamma, \varepsilon \neq Y$, then $L(\mu)(\gamma \diamond \varepsilon) = \emptyset \preceq L(\mu)(\gamma) * L(\mu)(\varepsilon)$.

From the above, we conclude that $L(\mu)$ is a homomorphism of hyperlattice-ordered commutative monoids. □

Theorem 3.4. *Let $f : \mathscr{E}_1 \to \mathscr{E}_2$ be a hyperlattice negatively ordered commutative monoid homomorphism. Then*

$$L(\mathscr{T}(f)) \circ \varphi_{\mathscr{E}_1} \;=\; \varphi_{\mathscr{E}_2} \circ f.$$

Which can be expressed by the following commutative diagram

$$\begin{tikzcd}
\mathscr{E}_1 \arrow[r, "f"] \arrow[d, "\varphi_{\mathscr{E}_1}"'] & \mathscr{E}_2 \arrow[d, "\varphi_{\mathscr{E}_2}"] \\
L(\mathscr{T}(\mathscr{E}_1)) \arrow[r, "L(\mathscr{T}(f))"'] & L(\mathscr{T}(\mathscr{E}_2))
\end{tikzcd}$$

Proof. We observe that for any $a \in \mathscr{E}_1$, the expression $(\varphi_{\mathscr{E}_2} \circ f)(a)$ evaluates to:

$$\begin{aligned}
(\varphi_{\mathscr{E}_2}(f(a)) &= \{g \in \mathscr{T}(\mathscr{E}_2) \mid g(f(a)) = 1\} \\
&= \{g \in \mathscr{T}(\mathscr{E}_2) \mid (g \circ f)(a) = 1\} \\
&= \{g \in \mathscr{T}(\mathscr{E}_2) \mid g \circ f \in \varphi_{\mathscr{E}_2}(a)\} \\
&= \{g \in \mathscr{T}(\mathscr{E}_2) \mid \mathscr{T}(f)(g) \in \varphi_{\mathscr{E}_1}\} \\
&= L(\mathscr{T}(f))(\varphi_{\mathscr{E}_1}(a)) \\
&= ((L(\mathscr{T}(f)) \circ \varphi_{\mathscr{E}_1})(a)).
\end{aligned}$$

Hence, we conclude that $(\varphi_{\mathscr{E}_2} \circ f)(a) = ((L(\mathscr{T}(f)) \circ \varphi_{\mathscr{E}_1})(a)$ for all $a \in \mathscr{E}_1$, which completes the proof. This proof adapts the arguments from [12, Lemma 3.11] to our generalized framework. □

Lemma 3.6. *For any $P = (X, \tau, \preccurlyeq, \circledast)$ be a Priestley space endowed with a negative order and a* *commutative semigroup operation. Define the mapping*

$$\pi_P : P \longrightarrow \mathscr{T}(L(P)), \quad \pi_P(\rho)(Y) = \begin{cases} 1 & \textit{if } \rho \in Y, \\ 0 & \textit{if } \rho \notin Y, \end{cases}$$

for every $Y \in L(P)$. Then π_P is an isomorphism of negatively ordered commutative semigroups between P and $\mathscr{T}(L(P))$.

Proof. We proceed step by we verify the necessary conditions:

Structural background. By assumption, $P = (X, \tau, \preccurlyeq, \circledast)$ is a Priestley space with a negatively ordered commutative semigroup structure. From Lemma3.1, the lattice $L(P)$ is a bounded distributive hyperlattice equipped with a negatively ordered commutative monoid structure. From Lemma 3.2, the space $\mathscr{T}(L(P))$ inherits the structure of a Priestley space negatively ordered commutative semigroup.

Properties of morphism π_P. The map π_P is monotone (order-preserving), continuous, and bijective (see [4], [5], [6], [7] [8], [37], and [38]). Hence, it is a candidate for an isomorphism between the two structures.

Compatibility with the semigroup operation. Take arbitrary $\rho, \sigma \in X$ and consider any $Y \in L(P)$.

Case 1: If $\pi_P(\rho \circledast \sigma)(Y) = 0$, then clearly $\pi_P(\rho \circledast \sigma)(Y) \leq \pi_P(\rho)(Y) \diamond \pi_P(\sigma)(Y)$.

Case 2: If $\pi_P(\rho \circledast \sigma)(Y) = 1$, then $\rho \circledast \sigma \in Y$. Since Y is an increasing τ-clopen subset and P is a negatively ordered commutative semigroup, it follows that $\rho \in Y$ and $\sigma \in Y$. Therefore, $\pi_P(\rho)(Y) = 1$ and $\pi_P(\sigma)(Y) = 1$. Consequently,

$$\pi_P(\rho)(Y) \diamond \pi_P(\sigma)(Y) = 1,$$

which implies

$$\pi_P(\rho \circledast \sigma)(Y) \leq \pi_P(\rho)(Y) \diamond \pi_P(\sigma)(Y).$$

Conclusion. In both cases, the compatibility condition holds. Thus, π_P preserves the semigroup operation and is an isomorphism of Priestley spaces negatively ordered commutative semigroups. □

Theorem 3.5. *Let $\delta : \mathscr{P}_1 \to \mathscr{P}_2$ be a Priestley space negatively ordered commutative semigroup isomorphism. Then $\mathscr{T}(L(\delta)) \circ \pi_{\mathscr{P}1} = \pi_{\mathscr{P}2} \circ \delta$.*

$$\begin{tikzcd}
\mathscr{P}_1 \arrow[r, "\delta"] \arrow[d, "\pi_{\mathscr{P}_1}"'] & \mathscr{P}_2 \arrow[d, "\pi_{\mathscr{P}_2}"] \\
\mathscr{T}(L(\mathscr{P}_1)) \arrow[r, "\mathscr{T}(L(\delta))"'] & \mathscr{T}(L(\mathscr{P}_2))
\end{tikzcd}$$

Proof. Similarly to [12, Lemma 3.12]. □

3.4.7 Main Representation Theorem in Priestley-Style Duality

In the following result, we give a Priestley duality for a bounded distributive hyperlattices structures that are endowed with compatible negatively ordered commutative monoids operations.

Our following theorem establishes a duality correspondence in the Priestley style for algebraic structures combining:

- Bounded distributive hyperlattice properties,
- Commutative monoid operations,
- Negative order relations.

Theorem 3.6 (Duality for Hyperlattice Structures)**.** *There exists a dual equivalence between:*

($\mathcal{D}1$) *The category of bounded distributive hyperlattices with negatively ordered commutative monoid structures.*

($\mathcal{D}2$) *The category of Priestley spaces carrying compatible negatively ordered commutative semigroup operations.*

Proof. The conclusions in in Lemmas 3.3 and 3.6, as well as Theorems3.4 and 3.5, immediately dictate this. □

Theorem 3.7. *A contravariant equivalence exists between:*

(i) *The category $\mathcal{BDHL}$ consists of bounded distributive hyperlattice structures that are endowed with compatible negatively ordered commutative monoid operations.*

(ii) *The category $\mathcal{PS}$ consists of Priestley spaces equipped with negatively ordered commutative semigroup operations.*

Proof. The verification follows by synthesizing the essential components from:

(ii) The structural characterization in Lemma 3.3.

(iii) The representation theorem in Lemma 3.6 .

(iv) The categorical framework established in Theorems 3.4 and 3.5.

□

3.5 Application to Representation Theorems in Hyperlattice Negatively Ordered Commutative Monoid

The following example illustrates the fact that there is a isomorphism between a bounded hyperlattice negatively commutative monoid and their bidual.

Example 3.2. Let $D_{12} = \{1, 2, 3, 4, 6, 12\}$ denote the set of positive divisors of 12. Equipped with the operations $\curlywedge = \gcd$ and $\curlyvee = \mathrm{lcm}$, the structure $(D_{12}, \curlywedge, \curlyvee, 1, 12)$ forms a bounded lattice. Its Hasse diagram is:

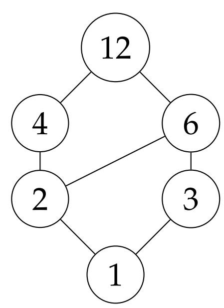

Figure 3.2: The Hasse diagram of the lattice $(D_{12}, \curlywedge, \curlyvee, 1, 12)$

We define on D_{12} a hyperoperation $\Cup$ and a binary operation $\circledast$ as follows:

$$d_1 \Cup d_2 := \{z \in D_{12} \mid \curlyvee(z, d_1) = \curlyvee(z, d_2) = \curlyvee(d_1, d_2)\},$$

$$d_1 \circledast d_2 := \begin{cases} \curlywedge(d_1, d_2) & \text{if } \curlyvee(d_1, d_2) = 12, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by invoking Theorem 3.1(ii), the algebraic system $(D_{12}, \curlywedge, \Cup, \circledast, 1, 12)$ constitutes a bounded distributive hyperlattice negatively ordered commutative monoid.

The dual space $T(D_{12})$ consists of all homomorphisms from D_{12} to $\{0, 1\}$, and is explicitly given by Table 3.3:

D_{12}	1	2	3	4	6	12
σ_1	0	0	0	1	0	1
σ_2	0	1	0	1	1	1
σ_3	0	0	1	0	1	1

Table 3.3: The dual space $T(D_{12})$

The bidual lattice $L(T(D_{12}))$ is composed of the subsets:

$$\{\emptyset, \{\sigma_2\}, \{\sigma_3\}, \{\sigma_1, \sigma_2\}, \{\sigma_2, \sigma_3\}, X\}, \quad \text{where } X = \{\sigma_1, \sigma_2, \sigma_3\}.$$

This yields the bounded distributive hyperlattice equipped with negatively ordered commutative monoid $(L(T(D_{12})), \curlywedge, \Cup, \circledast, \emptyset, X)$, where the operations are defined by:

$$\mathcal{A} \Cup \mathcal{B} := \{\mathcal{C} \in L(T(D_{12})) \mid \mathcal{C} \curlyvee \mathcal{A} = \mathcal{C} \curlyvee \mathcal{B} = \mathcal{A} \curlyvee \mathcal{B}\},$$

$$\mathcal{A} \circledast \mathcal{B} := \begin{cases} \mathcal{A} \curlywedge \mathcal{B} & \text{if } \mathcal{A} = X \text{ or } \mathcal{B} = X, \\ \emptyset & \text{otherwise.} \end{cases}$$

The canonical isomorphism $\Phi_{D_{12}} : D_{12} \to L(T(D_{12}))$ is given by Table 3.4:

D_{12}	1	2	3	4	6	12
$\Phi_{D_{12}}$	$\emptyset$	$\{\sigma_2\}$	$\{\sigma_3\}$	$\{\sigma_1, \sigma_2\}$	$\{\sigma_2, \sigma_3\}$	X

Table 3.4: The canonical isomorphism

Thus, $\Phi_{D_{12}}$ establishes an isomorphism between the original bounded hyperlattice ordered commutative monoid and its bidual.

The following example shows that there is an isomorphism between a Priestley space negatively ordered commutative semigroup and its bidual.

Example 3.3. (Priestley-style Bidual Isomorphism): Consider the structure $(\mathcal{X}, \tau, \trianglelefteq, \circledast)$, where $\mathcal{X} = \{\xi, \eta, \theta, \zeta\}$. The order relation $\trianglelefteq$ is illustrated in Figure 3.3, and the binary operation $\circledast$ is specified by the operation table given in Table 3.5

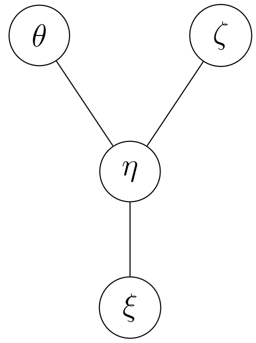

Figure 3.3: The Hasse diagram of the partially ordered set $(\mathcal{X}, \trianglelefteq)$

$\circledast$	ξ	η	θ	ζ
ξ	ξ	ξ	ξ	ξ
η	ξ	η	η	η
θ	ξ	η	θ	η
ζ	ξ	η	η	ζ

Table 3.5: The binary operation $(\mathcal{X}, \circledast)$

The family of clopen downsets is: $L(\mathcal{X}) = \{\emptyset, \{\theta\}, \{\zeta\}, \{\theta, \zeta\}, \{\eta, \theta, \zeta\}, \mathcal{X}\}$. We introduce two algebraic operations on $L(\mathcal{X})$:

1. The hyperjoin $\veebar$ is given by: $\Lambda_1 \veebar \Lambda_2 = \{\Lambda \in (L(\mathcal{X}) \mid \Lambda_1 \cup \Lambda_2 \subseteq \Lambda$ and $\Lambda_1 \cup \Lambda = \Lambda_2 \cup \Lambda\}$.
2. The binary operation $\diamond$ is defined as: $\Lambda_1 \boxtimes \Lambda_2 = \begin{cases} \Lambda_1 \cap \Lambda_2 & \text{if } \Lambda_1 = \mathcal{X} \text{ or } \Lambda_2 = \mathcal{X}, \\ \emptyset & \text{otherwise.} \end{cases}$

   for all $\Lambda_1, \Lambda_2 \in L(\mathcal{X})$

By using Lemma 3.1, we derive that $(L(\mathcal{X}), \cap, \veebar, \boxtimes, \emptyset, \mathcal{X})$ constitutes a bounded hyperlattice possessing negatively ordered commutative monoid.

The set of homomorphisms is: $T(L(\mathcal{X})) = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$, with the following values:

$L(\mathcal{X})$	$\emptyset$	$\{\theta\}$	$\{\zeta\}$	$\{\theta, \zeta\}$	$\{\eta, \theta, \zeta\}$	$\mathcal{X}$
φ_1	0	0	0	0	0	1
φ_2	0	0	0	0	1	1
φ_3	0	0	1	1	1	1
φ_4	0	1	0	1	1	1

Table 3.6: The set of homomorphisms $T(L(\mathcal{X}))$

Finally, the canonical map

$$\pi_{\mathcal{X}} : \mathcal{X} \longrightarrow T(L(\mathcal{X})), \qquad \pi_{\mathcal{X}}(\xi) = \varphi_1,\ \pi_{\mathcal{X}}(\eta) = \varphi_2,\ \pi_{\mathcal{X}}(\theta) = \varphi_3,\ \pi_{\mathcal{X}}(\zeta) = \varphi_4$$

is an isomorphism of Priestley spaces negatively ordered commutative semigroups.

Chapter Conclusion

Chapter 3 has developed the theoretical framework of hyperlattice-ordered negatively ordered commutative monoids. The discussion began with the foundations of Priestley spaces and their role in capturing the duality between algebraic and topological structures. Subsequent sections introduced homomorphisms and dual constructions, leading to a systematic study of biduality in bounded distributive hyperlattices. Through a sequence of lemmas and theorems, structural equivalence was established, culminating in the main representation theorem. This result provides a rigorous Priestley-style correspondence, thereby linking the algebraic properties of hyperlattice-ordered monoids with their dual topological spaces. Overall, the chapter consolidates the duality principles and sets the stage for further applications in representation theory.

CHAPTER 4

PRIESTLEY-STYLE DUALITY FOR DISTRIBUTIVE HYPERATTICE AND NEGATIVELY ORDERED HYPERMONOIDS

This chapter is devoted to establishing a rigorous Priestley-type duality that connects bounded distributive hyperlattices endowed with negatively ordered commutative hypermonoid structures to the category of Priestley spaces associated with negatively ordered commutative semihypergroups. Unlike the preceding chapters, which primarily focused on algebraic foundations and partial dualities, the present discussion advances the theory by constructing a categorical correspondence that unifies distributivity, negative order, and hypermonoidal operations within a coherent dual framework.

To achieve this, we introduce and formalize several fundamental notions such as negatively ordered semihypergroups, hyperlattice-ordered hypermonoids, hyperideals, and hyperfilters that serve as the building blocks of the duality. These concepts are not treated in isolation; rather, they are systematically integrated to demonstrate how algebraic structures can be faithfully mirrored in their associated Priestley spaces. The chapter proceeds through carefully designed definitions, illustrative examples, and intermediate results that culminate in the main representation theorem. This theorem confirms that distributive hyperlattices under negatively ordered hypermonoids admit a precise Priestley-style duality, thereby reinforcing the deep interplay between algebra and topology.

Beyond its immediate technical contributions, this chapter highlights the broader significance of the duality framework. By situating distributive hyperlattices within negatively ordered hypermonoids and linking them to Priestley spaces, the results open pathways for further exploration in categorical representation theory and ordered algebraic systems. In this way, the chapter not only consolidates the theoretical foundations but also positions the duality as a cornerstone for future developments in the study of algebraic-topological correspondences.

4.1 Foundations of Negatively Ordered Semihypergroups

This section lays the groundwork for negatively ordered semihypergroups, introducing them as independent algebraic structures that extend the classical notion of semigroups. The focus is on formal definitions, the role of negative order in shaping algebraic operations, and illustrative examples that clarify their behavior. By establishing these foundations, we prepare the conceptual framework necessary for embedding hyperlattices into negatively ordered commutative hypermonoids.

Let Υ be a non-empty set and $P^*(\Upsilon)$ denotes the set of all non-empty subsets of Υ. Maps $\boxtimes : \Upsilon \times \Upsilon \to P^*(\Upsilon)$, are called hyperoperations see [34].

Definition 4.1. *[15, 17, 45]* An hypergroupoid is a non-empty set Υ with an hyperoperation $\boxtimes$ on Υ. For subsets $P, T \subseteq \Upsilon$, define $P \boxtimes T = \bigcup_{(p,t)\in P\times T}(p \boxtimes t)$.

Definition 4.2. *[17]* A semihypergroup is a hypergroupoid $(\Upsilon, \boxtimes)$ when the hyperoperation is associative such that for all w_1, w_2 and w_3 in Υ we have $(w_1 \boxtimes w_2) \boxtimes w_3 = w_1 \boxtimes (w_2 \boxtimes w_3)$.

A semihypergroup Υ with hyperoperation $\boxtimes$ is commutative if the symmetric property:

$$w_1 \boxtimes w_2 = w_2 \boxtimes w_1 \text{ holds for all } w_1, w_2 \in \Upsilon.$$

An element e of Υ is qualified,

- *Weak identity* if $w \in e \boxtimes w \cap w \boxtimes e$ for all $w \in \Upsilon$.
- *Strong identity* if $\{w\} = e \boxtimes w = w \boxtimes e$ for all $w \in \Upsilon$.

Definition 4.3. A partially ordered semihypergroup $(\Upsilon, \boxtimes, \preccurlyeq)$ satisfies:

1. $(\Upsilon, \boxtimes)$ is a semihypergroup;
2. $(\Upsilon, \preccurlyeq)$ is a poset;
3. Order compatibility: $\xi_1 \preccurlyeq \xi_2$ implies $\xi_1 \boxtimes q \preccurlyeq^+ \xi_2 \boxtimes q$ and $q \boxtimes \xi_1 \preccurlyeq^+ q \boxtimes \xi_2$ for all $q \in \Upsilon$.

where $\preccurlyeq^+$ denotes the natural extension of $\preccurlyeq$ to non-empty subsets.

Example 4.1. Let $\Pi = \{a, b, c\}$ with $a \star b = \{a, b\}$, $b \star c = \{b, c\}$, and $a \star c = \{a, c\}$. Define $a \preceq b$, $b \preceq c$ and $a \preceq c$.

Then, $(\Pi, \star, \preceq)$ is a partially ordered semihypergroup.

4.2 Structural Properties of Hyperlattice-Ordered Commutative Hypermonoids

This section examines the structural properties of hyperlattice-ordered commutative hypermonoids. Special attention is given to distributivity, boundary elements, and the interaction between hypermonoidal operations and negative order. Preliminary propositions are introduced to highlight the algebraic coherence of these systems, ensuring that the framework is robust enough to support duality constructions.

Definition 4.4. Let $\mathcal{U} = (\Upsilon, \curlywedge, \Cup, \circledast)$ be an algebraic structure where:

- $(\Upsilon, \curlywedge, \Cup)$ forms a hyperlattice;
- $(\Upsilon, \circledast, \epsilon)$ constitutes a commutative hypermonoid.

We call Υ a hyperlattice-ordered commutative hypermonoid if it satisfies the following distributive laws:

1. $w_1 \circledast (u_2 \curlywedge z) \preceq (w_1 \circledast u_2) \curlywedge (w_1 \circledast z)$;
2. $w_1 \circledast (u_2 \Cup z) \subseteq (w_1 \circledast u_2) \Cup (w_1 \circledast z)$ for all $w_1, u_2, z \in \Upsilon$.

These conditions imply the order-preserving property: $w_1 \preceq u_2 \Rightarrow w_1 \circledast z \preceq u_2 \circledast z$.

The structure $\mathcal{U} = (\Upsilon, \curlywedge, \Cup, \circledast)$ is called distributive when it additionally satisfies

$$w_1 \curlywedge (u_2 \Cup z) \subseteq (w_1 \curlywedge u_2) \Cup (w_1 \curlywedge z).$$

A bounded structure contains universal bounds 0 and 1 such that $0 \preceq w_1 \preceq 1$ for all $w_1 \in \Upsilon$. In positively (negatively) ordered cases, the identity element ε coincides with 0 (respectively 1).

Example 4.2. Let us examine the divisor lattice $\mathcal{D}_{18} = (D(18), \gcd, \mathrm{lcm}, 1, 18)$. Whose Hasse diagram is illustrated in the following Figure 4.1:

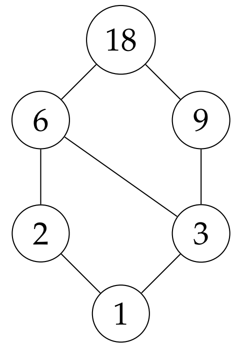

Figure 4.1: Hasse diagram of the divisor lattice $\mathcal{D}_{18}$

Define on $\mathcal{D}$ the hyperoperations:

- $w_1 \Cup u_2 = \{k \in D(18) \mid \mathrm{lcm}(k, w_1) = \mathrm{lcm}(k, u_2) = \mathrm{lcm}(w_1, u_2)\}$.
- $w_1 \circledast u_2 = \begin{cases} \downarrow (\gcd(w_1, u_2)) & \text{if } w_1 = 18 \text{ or } u_2 = 18; \\ \{1\} & \text{otherwise.} \end{cases}$

Then, by routine calculations $(\mathcal{D}_{18}, \gcd, \Cup, \circledast, 1, 18)$ is a bounded distributive hyperlattice negatively ordered commutative hypermonoid.

Theorem 4.1. *Each bounded distributive lattice $\mathcal{L} = (L, \curlywedge, \vee, 0, 1)$ admits the construction of three distinct hypermonoid structures:*

1. Negative hypermonoid I *with operations:*

$$w_1 \Cup u_2 = \{z \in L \mid w_1 \vee u_2 = w_1 \vee z = u_2 \vee z\}$$
$$w_1 \circledast u_2 = \begin{cases} \downarrow (w_1 \curlywedge u_2) & \textit{if } w_1 = 1 \textit{ or } u_2 = 1; \\ \{0\} & \textit{otherwise}. \end{cases}$$

2. Negative hypermonoid II *with operations:*

$$w_1 \Cup u_2 = \{z \in L \mid z \geq w_1 \vee u_2\}.$$
$$w_1 \circledast u_2 = \begin{cases} \{t \in L \mid t < w_1 \curlywedge u_2\} & \textit{if } w_1 = 1 \textit{ or } u_2 = 1; \\ \{0\} & \textit{otherwise}. \end{cases}$$

3. Positive hypermonoid *with operations:*

$$w_1 \Cup u_2 = \{z \in L \mid w_1 \vee u_2 = w_1 \vee z = u_2 \vee z\}.$$

$$w_1 \circledast u_2 = \begin{cases} \uparrow (w_1 \vee u_2) & \textit{if } w_1 = 0 \textit{ or } u_2 = 0; \\ \{1\} & \textit{otherwise}. \end{cases}$$

Proof. The assertion is established directly from Definition 4.4via standard algebraic procedures. □

Corollary 4.1. The Boolean algebra $\mathcal{B} = \{0, 1\}$ constitutes a bounded distributive hyperlattice, structured as a negatively ordered hypermonoid with hyperoperations defined by Table 4.1:

$\Cup$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0,1\}$

$\circledast$	0	1
0	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{0,1\}$

Table 4.1: Hyperoperations tables on $\{0, 1\}$

4.3 Hyperideals and Hyperfilters: Dual Constructions

This section introduces hyperideals and hyperfilters as central tools for dual constructions. These structures capture the distributive and order-theoretic aspects of hyperlattice-ordered commutative hypermonoids, and serve as the bridge between algebraic definitions and topological representations. Through formal definitions and examples, we demonstrate how hyperideals and hyperfilters contribute to the construction of Priestley spaces, thereby linking algebraic operations with their dual topological counterparts.

Definition 4.5. Take $\mathcal{H} = (\Upsilon, \wedge, \sqcup, \circledast)$ be a commutative hypermonoid with hyperlattice ordering. A subset $\mathfrak{I} \subseteq \Upsilon$ is said to be a hyperideal when it satisfies these axioms:

1. **Order Compatibility**: For $\xi_1 \in \mathfrak{I}$ and $\xi_2 \preceq \xi_1$, we have $\xi_2 \in \mathfrak{I}$;

2. **Join Closure**: Given $\xi_1, \xi_2 \in \mathfrak{I}$, their hyperjoin satisfies $\xi_1 \sqcup \xi_2 \subset \mathfrak{I}$;

3. **Product Absorption**: If either ξ_1 or ξ_2 lies in $\mathfrak{I}$, then their hyperproduct $\xi_1 \circledast \xi_2$ is contained in $\mathfrak{I}$ for all $\xi_1, \xi_2 \in \Upsilon$.

Definition 4.6. A proper hyperideal $\mathfrak{I}$ is a prime hyperideal if:

1. **Prime Condition**: when $\xi_1 \wedge \xi_2 \in \mathfrak{I}$ necessitates that at least one of ξ_1 or ξ_2 belongs to $\mathfrak{I}$;

2. **Intersection Condition**: For arbitrary elements $\xi_1, \xi_2 \in \Upsilon$, non-empty intersection $(\xi_1 \circledast \xi_2) \cap \mathfrak{I} \neq \emptyset$ implies $\xi_1 \in \mathfrak{I}$ or $\xi_2 \in \mathfrak{I}$.

Definition 4.7. Let $\mathscr{A} = (U, \wedge, \sqcup, \star)$ be an algebraic structure in which $(U, \wedge, \sqcup)$ is a hyperlattice and $(U, \star)$ is a commutative hypermonoid endowed with a compatible order. A non-void subset $\mathfrak{F} \subset U$ qualifies as a hyperfilter provided it fulfills the following criteria:

1. **Upward Closure**: For any elements $u_1 \in \mathfrak{F}$ and $u_2 \in U$, if $u_1 \preccurlyeq u_2$, then u_2 must belong to $\mathfrak{F}$;

2. **Meet Stability**: For two elements $u_1, u_2 \in \mathfrak{F}$, their meet $u_1 \wedge u_2$ is necessarily contained in $\mathfrak{F}$;

3. **Product Condition**: Given $u_1 \in \mathfrak{F}$ and $u_2 \in U$, if their hyperproduct $u_1 \star u_2$ intersects $\mathfrak{F}$ non-emptily, then u_2 must be in $\mathfrak{F}$.

Definition 4.8. A proper hyperfilters $\mathfrak{F}$ is a prime hyperfilters if:

1. **Prime Property**: whenever $u_1 \sqcup u_2$ meets $\mathfrak{F}$, then either $u_1 \in \mathfrak{F}$ or $u_2 \in \mathfrak{F}$;

2. **Strong Absorption**: If the hyperproduct $u_1 \star u_2$ has non-empty intersection with $\mathfrak{F}$, then both u_1 and u_2 must belong to $\mathfrak{F}$.

Theorem 4.2. *Let $\mathcal{H} = (U, \wedge, \sqcup, \odot)$ be a commutative hypermonoid with hyperlattice structure. There exists a complement duality between:*

1. *Prime hyperideals $\mathfrak{I} \subseteq U$ and their set complements $U \setminus \mathfrak{I}$ which form prime hyperfilters;*

2. *Prime hyperfilters $\mathfrak{F} \subseteq U$ and their set complements $U \setminus \mathfrak{F}$ which form prime hyperideals.*

Proof. We establish both directions of the correspondence, relying on the duality principle:

Part (i): Let $\mathfrak{I}$ be a prime hyperideal. We show that $U \setminus \mathfrak{I}$ is a prime hyperfilter:

(α) Suppose $u_1 \in U \setminus \mathfrak{J}$ and $u_2 \succeq u_1$, if $u_2 \in \mathfrak{J}$ then by the hyperideal property it follows that u_1 belongs necesarely to $\mathfrak{J}$, contradicting the assumption $u_1 \in U \setminus \mathfrak{J}$. Thus, $u_2 \in U \setminus \mathfrak{J}$.

(β) For $u_1, u_2 \in U \setminus \mathfrak{J}$, if $u_1 \wedge u_2$ must lie in $\mathfrak{J}$. By th primality, this would imply that both u_1 and u_2 are elements of $\mathfrak{J}$. Which contradiction. Hence, $u_1 \wedge u_2 \in U \setminus \mathfrak{J}$.

(γ) If the hyperproduct $u_1 \odot u_2$ meets $U \setminus \mathfrak{J}$ and $u_1 \in U \setminus \mathfrak{J}$, but $u_2 \in \mathfrak{J}$, then $u_1 \odot u_2 \subseteq \mathfrak{J}$ by the hyperideal property, contradicting the non-empty intersection.

Next, we show that $U \setminus \mathfrak{J}$ is prime. Suppose that $u_1, u_2 \in U$ and $(u_1 \sqcup u_2) \cap (U \setminus \mathfrak{J}) \neq \emptyset$.

If $u_1, u_2 \notin U \setminus \mathfrak{J}$, then $u_1, u_2 \in \mathfrak{J}$, therefore $u_1 \sqcup u_2 \subseteq \mathfrak{J}$, from the fact that $\mathfrak{J}$ is a hyperideal. Hence $(u_1 \sqcup u_2) \cap (U \setminus \mathfrak{J}) = \emptyset$, which is a contradiction. So $u_1 \in U \setminus \mathfrak{J}$ or $u_2 \in U \setminus \mathfrak{J}$.

Next, suppose that $(u_1 \odot u_2) \cap (U \setminus \mathfrak{J}) \neq \emptyset$.

If $u_1 \notin U \setminus \mathfrak{J}$ or $u_2 \notin U \setminus \mathfrak{J}$, then $u_1 \in \mathfrak{J}$ or $u_2 \in \mathfrak{J}$, therefore $u_1 \odot u_2 \subseteq \mathfrak{J}$, from the fact that $\mathfrak{J}$ is a hyperideal. Hence, $(u_1 \odot u_2) \cap (U \setminus \mathfrak{J}) = \emptyset$, which is a contradiction. So $u_1, u_2 \in U \setminus \mathfrak{J}$. Thus, $U \setminus \mathfrak{J}$ is a prime hyperfilter.

Part (ii): Assume that $\mathfrak{Q}$ is a prime hyperfilter, then its complement $U \setminus \mathfrak{Q}$ satisfies, by duality of the definitions:

- It is downward closed, as the logical counterpart of the upward closure of $\mathfrak{Q}$,
- It enjoys join absorption, inherited from the meet stability of $\mathfrak{Q}$,
- It preserves primality, directly induced by the prime hyperfilter conditions.

Therefore, $U \setminus \mathfrak{Q}$ constitutes a prime hyperideal. □

Proposition 4.1. *[8] If $\Phi \neq \emptyset$ of a hyperlattice ordered commutative hypermonoid $(\Upsilon, \wedge, \sqcup, \circledast)$, then the minimal hyperlattice filter encompassing Φ is expressed as:*

$$\langle \Phi \rangle = \{\mathfrak{a} \in \Upsilon \mid \zeta_1 \wedge \ldots \wedge \zeta_n \preceq \mathfrak{a}, \textit{ for certain elements } \zeta_1, .., \zeta_n \in \Phi\}.$$

If $\Phi = \{\zeta\}$, we write $\langle \Phi \rangle = \uparrow \zeta = \{w \in \Upsilon \mid \zeta \preceq w\}$.

Proposition 4.2. *If δ is a hyperlattice filter of a distributive hyperlattice negatively ordered commutative hypermonoid $(\Upsilon, \wedge, \sqcup, \circledast)$, then the smallest hyperlattice ordered commutative hypermonoid hyperfilter encompassing δ admits the representation,*

$$[\delta) = \{w \in \Upsilon \mid \zeta_1 \circledast \ldots \circledast \zeta_n \preceq w, \textit{ for certain elements } \zeta_1, .., \zeta_n \in \delta\}.$$

Proof. Take $a \in \delta$. The reflexivity of $\preceq$ gives $a \preceq a$, and we obtain a belongs to $[\delta)$. Hence, the interval $[\delta)$ is non-empty. To prove that the set $[\delta)$ is a hyperfilter,

consider $q, o \in \Upsilon$ verifying $q \preceq o$, then $\xi_1 \circledast ... \circledast \xi_n \preceq q \preceq o$, so $o \in [\delta)$. On the other hand, for every $q, o \in [\delta)$, there exists $\xi_1, \xi_2, ..., \xi_n, o_1, o_2, ..., o_m$ such that $\xi_1 \circledast \xi_2 \circledast ... \circledast \xi_n \preceq q$ and $o_1 \circledast o_2 \circledast ... \circledast o_m \preceq o$. Then $(\xi_1 \circledast \xi_2 \circledast ... \circledast \xi_n) \wedge (o_1 \circledast o_2 \circledast ... \circledast o_m) \preceq q \wedge o$. Since Υ is negatively ordered so, $q \wedge o \in [\delta)$. Now let $q \in [\delta)$ and $(q \circledast o) \cap [\delta) \neq \emptyset$, then there exist $w \in (q \circledast o) \cap [\delta)$. Since $w \in [\delta)$, by the definition of $[\delta)$, there exist $\zeta_1, \ldots, \zeta_n \in [\delta)$ such that $\zeta_1 \circledast \cdots \circledast \zeta_n \preceq w$. But $w \in q \circledast o$. As $q \in [\delta)$, closure of $[\delta)$ under hyperproducts implies that o must also belong to $[\delta)$, otherwise, $(q \circledast o)$ could not yield an element of $[\delta)$. Hence $o \in [\delta)$.

Next, consider any element $e \in \delta$. By reflexivity of the order relation, we have $e \preceq e$, which immediately implies $e \in [\delta)$. This establishes the inclusion $\delta \subseteq [\delta)$. Now, let $\mathcal{F}$ be an arbitrary hyperfilter containing $\delta \subseteq \mathcal{F}$. Subsequently, for any $q \in [\delta)$,there exists a finite collection of elements $\{\xi_i\}_{i=1}^n \subseteq \delta$ such that the hyperproduct $\circledast_{i=1}^n \xi_i \preceq q$, then $q \in \mathcal{F}$. Therefore $[\delta) \subseteq \mathcal{F}$ □

Theorem 4.3. *Assume that $(\Upsilon, \wedge, \sqcup, \circledast)$ is a distributive hyperlattice with negatively ordered commutative hypermonoid structure. Let $\mathcal{F}$ be a hyperfilter and Γ a hyperideal of Υ. Whenever $\mathcal{F}$ and Γ do not intersect (disjoint), then there exists a prime hyperfilter $\mathcal{P}$ that contains $\mathcal{F}$ and remains disjoint from Γ.*

Proof. Similarly to [8, Theorem 2.9]. □

Corollary 4.2. Consider a distributive hyperlattice negatively ordered commutative hypermonoid $(\Upsilon, \wedge, \sqcup, \circledast)$. If Γ is a hyperideal and $\lambda \notin \Gamma$, then there exists a prime hyperfilter $\mathcal{P}$ containing λ and disjoint with Γ.

Proof. Let Γ be a hyperideal. Define

$$\mathcal{F} = \{x \in \Upsilon \backslash \Gamma \mid \lambda \circledast ... \circledast \lambda \preceq x\}.$$

Clearly, $\mathcal{F}$ and Γ are disjoint. One readily verifies that $\mathcal{F}$ is a hyperfilter. According to Theorem 4.3, a prime hyperfilter $\mathcal{P}$ exists that contains $\mathcal{F}$ and is disjoint from Γ. □

4.4 Morphisms and Categorical Equivalence

We now examine structure-preserving mappings between hyperlattice-ordered commutative hypermonoids. By analyzing the preservation of distributive and order-theoretic properties under morphisms, we establish the categorical framework necessary for duality. The results presented here clarify how morphisms act as structural maps that ensure consistency between algebraic and topological perspectives, paving the way for the main representation theorem.

Definition 4.9. Given two algebraic structures $(\Upsilon, \{\wedge, \Cup, \circledast\})$ and $(\Upsilon', \{\curlywedge, \Cap, \circledcirc\})$ combining the axioms of hypermonoid and hyperlattice, a function $\alpha : \Upsilon \to \Upsilon'$ qualifies as a morphism whenever:

- Meets are preserved exactly: $\alpha(r \wedge s) = \alpha(r) \curlywedge \alpha(s)$,
- Joins are mapped inclusively: $\alpha(r \Cup s) \subseteq \alpha(r) \Cap \alpha(s)$,
- Multiplication is order-respecting: $\alpha(r \circledast s) \preceq \alpha(r) \circledcirc \alpha(s)$.

A bijective morphism, we call it a structure isomorphism between these algebraic systems.

Proposition 4.3. *Let $(\Lambda, \wedge, \sqcup, \circledast)$ be a hyperlattice-ordered commutative hypermonoid and $(\mathbb{B}, \wedge, \sqcup, \diamond, 0, 1)$ the bounded structure from Corollary 4.1 with negative order. Every prime hyperfilter $\mathcal{J} \subset \Lambda$ induces a surjective structure-preserving map $\phi : \Lambda \twoheadrightarrow \{0, 1\}$ characterized by $\mathcal{J} = \phi^{-1}(1)$.*

Proof. Put $\phi(\Theta) = \begin{cases} 1, & \text{if } \Theta \subseteq \mathcal{J}, \\ 0, & \text{otherwise.} \end{cases}$

For the conditions $\phi(r \wedge s) = \phi(r) \wedge \phi(s)$ and $\phi(r \sqcup s) \subseteq \phi(r) \sqcup \phi(s)$ similarly to [8, Proposition 2.12].

For the third homomorphism axiom:

We have $\phi(r \circledast s) = 1 \Rightarrow r \circledast s \subseteq \mathcal{J} \Rightarrow (r \in \mathcal{J} \text{ and } s \in \mathcal{J}) \Rightarrow (\phi(r) = 1 \text{ and } \phi(s) = 1) \Rightarrow \phi(r) \diamond \phi(s) = \{0, 1\}$. Hence, $\phi(r \circledast s) \preceq \phi(r) \diamond \phi(s)$. ☐

Corollary 4.3. For a distributive hyperlattice negatively ordered commutative hypermonoid Λ. If $\varrho, \nu \in \Lambda$ are such that $\varrho \not\leq \nu$, there exists a prime hyperfilter $\mathcal{J}$ containing ν but not containing ϱ.

Proof. Taking $\downarrow \nu$ in Corollary 4.2, which is a hyperideal of a distributive hyperlattice negatively ordered commutative hypermonoid and $\varrho \notin \downarrow \nu$. ☐

4.5 Priestley-Style Representation and Biduality

This section culminates in the Priestley-style representation theorem, which establishes a faithful duality between bounded distributive hyperlattices with negatively ordered commutative hypermonoid structures and Priestley spaces associated with negatively ordered semihypergroups. Biduality and structural equivalence are emphasized, confirming that algebraic and topological perspectives remain in precise correspondence. This theorem represents the central achievement of the chapter.

4.5.1 Priestley-Ordered Commutative Semihypergroup and Homomorphism

This part introduces Priestley-ordered commutative semihypergroup, presents their basic properties through a key proposition, and defines homomorphisms as structure-preserving maps within this framework.

Definition 4.10. A Priestley-ordered commutative semihypergroup is characterized as a structure $\mathfrak{Y} = (\mathcal{Y}, \tau, \leq, *)$, in which,

- The structure $(\mathcal{Y}, \tau, \leq)$ is a space of Priestley.
- The algebraic structure $(\mathcal{Y}, \leq, *)$ constitutes a commutative semi hypergroup with negative ordering.

Definition 4.11. A topological-algebraic structure $\mathfrak{X} = (\mathcal{X}, \tau, \leq, *)$ is defined as a Priestley-ordered commutative semihypergroup if and only if:

- The triple $(\mathcal{X}, \tau, \leq)$ forms a Priestley space,
- The algebraic structure $(\mathcal{X}, \leq, *)$ constitutes a commutative semihypergroup with negative ordering compatible with the Priestly framework.

Definition 4.12. Given a bounded distributive hyperlattice endowed with a negatively ordered commutative hypermonoid structure $\mathcal{E} = (E, \wedge, \sqcup, \circledcirc, \perp, \top)$. Its dual representation space $\mathfrak{D}(\mathcal{E})$ is the quadruple $(\mathcal{Y}, \mathcal{T}, \preccurlyeq, *)$ where:

(i) $\mathcal{Y}$ denotes the collection of all structure-preserving morphisms $\phi : \mathcal{E} \to \mathbf{2}$, where $\mathbf{2} = (\{\perp, \top\}, \wedge, \sqcup, \diamond, \perp, \top)$ is the two-element Boolean algebra (structure)

(ii) The partial order $\preccurlyeq$ on $\mathcal{Y}$ is characterized by the following definition:

$$\phi \preccurlyeq \psi \iff \forall q \in E, \phi(q) \leq \psi(q).$$

(iii) The initial topology $\mathcal{T}$ is induced by the product topology on $\mathbf{2}^E$.

(iv) The hyperoperation $* : \mathcal{Y} \times \mathcal{Y} \to \mathcal{P}^*(\mathcal{Y})$ is defined in a pointwise manner:

$$(\phi * \psi)(q) = \begin{cases} \downarrow [\phi(q) \wedge \psi(q)] & \text{if } \phi(q) \sqcup \psi(q) = \top, \\ \{\bot\} & \text{otherwise.} \end{cases}$$

Proposition 4.4. *Given a bounded distributive hyperlattice $\mathscr{H} = (L, \wedge, \vee, \odot, 0, 1)$ equipped with a negatively ordered commutative hypermonoid structure, its dual space $\mathscr{P}(\mathscr{H}) = (Y, \mathcal{O}, \preccurlyeq, \diamond)$ forms a Priestley space with a compatible negatively ordered commutative hypergroupoid structure.*

Proof. Let $\mathscr{H}^* = (Y, \mathcal{O}, \preccurlyeq)$ denote the Priestley space associated with $\mathscr{H}$ as established in prior work. We verify that the hyperoperation $\diamond$ endows $(Y, \preccurlyeq)$ with the required algebraic structure:

1. **Commutativity**: For any $\phi_1, \phi_2 \in Y$, the equality $\phi_1 \diamond \phi_2 = \phi_2 \diamond \phi_1$ follows immediately from the commutativity of $\odot$.

2. **Associativity**: The relation $(\phi_1 \diamond \phi_2) \diamond \phi_3 = \phi_1 \diamond (\phi_2 \diamond \phi_3)$ holds through direct computation.

3. **Order Compatibility**: For $\phi_1 \preccurlyeq \phi_2$, we observe:

$$\begin{aligned} (\phi_1 \diamond \phi_3)(a) &= \inf\{\phi_1(b) \wedge \phi_3(c) \mid b \odot c \leq a\}, \\ &\leq \inf\{\phi_2(b) \wedge \phi_3(c) \mid b \odot c \leq a\}, \\ &= (\phi_2 \diamond \phi_3)(a). \end{aligned}$$

   establishing $\phi_1 \diamond \phi_3 \preccurlyeq \phi_2 \diamond \phi_3$.

4. **Negative Ordering**: The inequalities $\phi_1 \diamond \phi_2 \preccurlyeq \phi_1$ and $\phi_1 \diamond \phi_2 \preccurlyeq \phi_2$ follow from:

$$(\phi_1 \diamond \phi_2)(a) \leq (\phi_1 \wedge \phi_2)(a) \leq \phi_i(a) \quad (i = 1, 2).$$

Thus, $\mathscr{P}(\mathscr{H})$ satisfies all axioms of a negatively ordered commutative hypergroupoid in the category of Priestley spaces. □

In the following, we give a definition of Priestley space negatively ordered commutative semi hypergroup homomorphism.

Definition 4.13. Consider be two a Priestley spaces negatively ordered commutative semi-hypergroups. $\mathscr{Q} = (\mathfrak{Q}, \mathcal{T}, \sqsubseteq, *)$, and $\mathscr{Q}' = (\mathfrak{Q}', \mathcal{T}', \precsim, \diamond)$. A map $\phi : \mathfrak{Q} \to \mathfrak{Q}'$ is called:

(HP1) **Monotonic** if for each $\xi_1, \xi_2 \in \mathfrak{Q}$, $\xi_1 \sqsubseteq \xi_2 \Rightarrow \phi(\xi_1) \precsim \phi(\xi_2)$;

(HP2) **Structure-preserving** if it satisfies:

- Monotonicity (HP1);
- Topological continuity;
- Submultiplicativity: $\phi(u_1 * u_2) \precsim \phi(u_1) \diamond \phi(u_2)$ for all $u_1, u_2 \in \mathfrak{Q}$.

A bijective structure-preserving map ϕ whose inverse also preserves structure is called a structural isomorphism between these spaces.

4.5.2 Duality and Biduality in Priestley-Style Structures

This unit provides the necessary conditions under which the dual of a Priestley space negatively ordered commutative semihypergroup becomes a bounded distributive hyperlattice negatively ordered commutative hypermonoid. It further establishes, through a lemma, that the bidual of such a structure preserves the same distributive and ordered properties, thereby confirming the stability of duality within this framework.

Lemma 4.1. *Let $\mathscr{P} = (\mathfrak{Y}, \mathcal{T}, \sqsubseteq)$ be a space of Priestley.*

There are canonical hyperoperations $\curlyvee$ with $\boxtimes$ endowing the collection

$$\mathcal{L}(\mathscr{P}) = \{Y \subseteq \mathfrak{Y} \mid Y \text{ is } \mathcal{T}\text{-clopen and } \sqsubseteq\text{-increasing}\},$$

with a structure of a bounded distributive hyperlattice with negatively ordered commutative hypermonoid operations $(\mathcal{L}(\mathscr{P}), \cap, \curlyvee, \boxtimes, \varnothing, \mathfrak{Y})$, where:

1. *The hyperjoin $\curlyvee$ is given by:*

$$V_1 \curlyvee V_2 = \{Q \in \mathcal{L}(\mathscr{P}) \mid V_1 \cup V_2 \subseteq Q \text{ and } V_1 \cup Q = V_2 \cup Q\}.$$

2. *The hyperproduct $\boxtimes$ is defined as:*

$$V_1 \boxtimes V_2 = \begin{cases} \downarrow (V_1 \cap V_2) & \textit{if } V_1 = \mathfrak{Y} \textit{ or } V_2 = \mathfrak{Y}, \\ \{\min(\mathcal{L}(\mathscr{P}))\} & \textit{otherwise}. \end{cases}$$

for all clopen increasing sets $V_1, V_2 \in \mathcal{L}(\mathscr{P})$.

Proof. A direct verification establishes that the algebra $(\mathcal{L}(\mathscr{P}), \cap, \cup, \varnothing, \mathfrak{Y})$ constitutes a bounded distributive lattice, as documented in referenced [37,38]. Furthermore, according to Definition 4.4, the algebra $(\mathcal{L}(\mathscr{P}), \cap, \curlyvee, \boxtimes, \varnothing, \mathfrak{Y})$ is characterized as a bounded distributive hyperlattice, negatively ordered, and endowed with the structure of a commutative hypermonoid. □

Lemma 4.2. *Let $\mathfrak{E} = (E, \wedge, \sqcup, \circledcirc, 0, 1)$ be a bounded distributive hyperlattice negatively ordered commutative hypermonoid. Then, $(L(\mathfrak{D}(\mathfrak{E})), \cap, \sqcup, \odot, \phi, X)$ a bounded distributive hyperlattice negatively ordered commutative hypermonoid.*

Proof. We have $L(\mathfrak{D}(\mathfrak{E})) = \{Y \subseteq X \mid Y \text{ is increasing and } \tau\text{-clopen}\}$. $\sqcup, \circledcirc$ are defined by

$$A \sqcup B = \{C \in L(\mathfrak{D}(\mathfrak{E})) \mid A \cup B = A \cup C = B \cup C\},$$

$$A \odot B = \begin{cases} \downarrow (A \cap B) \text{ if } A = X \text{ or } B = X; \\ \{\phi\} \text{ otherwise.} \end{cases}$$

for all $A, B \in L(\mathfrak{D}(\mathfrak{E}))$. Hence, by Definition 4.4 $(L(\mathfrak{D}(\mathfrak{E})), \cap, \sqcup, \odot, \phi, X)$ a bounded distributive hyperlattice negatively ordered commutative hypermonoid. □

4.5.3 Bidual Representation and Isomorphisms of Hyperlattice-Ordered Hypermonoids

This part presents the bidual framework of bounded distributive hyperlattices-ordered hypermonoids. It shows how Priestley spaces can be represented as hyperlattices, extends the correspondence to morphisms, and confirms the natural commutativity of the associated diagrams. These results prepare the ground for the categorical equivalence formalized in Theorem 4.6.

Lemma 4.3. *For any bounded distributive hyperlattice $\mathscr{E} = (E, \curlywedge, \curlyvee, \circledast, \bot, \top)$ equipped with negatively ordered commutative hypermonoid structure, the canonical embedding*

$$\Phi_{\mathscr{E}} : \mathscr{E} \hookrightarrow \mathcal{L}(\mathfrak{T}(\mathscr{E}))$$

given by

$$\Phi_{\mathscr{E}}(e) = \{\phi \in \mathrm{Spec}(\mathscr{E}) \mid \phi(e) = \top\}$$

establishes an isomorphism in the category of hyperlattice-ordered commutative hypermonoids.

Proof. Since E hyperlattice then $\Phi_{\mathscr{E}}$ is an isomorphism of hyperlattices see [8]. $\mathcal{L}(\mathfrak{T}(\mathscr{E}))$ is hyperlattice negatively ordered commutative hypermonoid by Lemma 4.1. For the condition $\Phi_E(t \circledast s) \subseteq \Phi_E(t) \circledcirc \Phi_E(s)$, for all $t, s \in E$ and $\phi \in X$; $\phi \in \Phi_E(t \circledast s) \Rightarrow \phi(t \circledast s) = 1 \Rightarrow \phi(t) \diamond \phi(s) = \{0, 1\} \Rightarrow (\phi(t) = 1$ and $\phi(s) = 1) \Rightarrow \phi \in \Phi_E(t) \cap \Phi_E(s) \Rightarrow \phi \in \Phi_E(t) \circledcirc \Phi_E(s) =\downarrow (\Phi_E(t) \cap \Phi_E(s))$. So $\Phi_E(t \circledast s) \preceq \Phi_E(t) \circledcirc \Phi_E(s)$. Therefore, $\Phi_{\mathscr{E}}$ is an isomorphism of E onto $\mathcal{L}(\mathfrak{T}(\mathscr{E}))$. $\square$

Lemma 4.4. *Given a structure-preserving map $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ between hyperlattice negatively ordered commutative hypermonoid homomorphism, the induced mapping*

$$\mathcal{D}(\phi) : \mathcal{D}(\mathcal{H}_2) \to \mathcal{D}(\mathcal{H}_1), \quad \psi \mapsto \psi \circ \phi$$

constitutes a morphism of ordered topological semi hypergroups.

Proof. We establish the required properties:

1. **Order and Continuity**: The mapping $\mathcal{D}(\phi)$ preserves both the specialization order and topological structure, as shown in prior work [4], [5], [6], [7] [8], [37], [38] .

2. **Operation Compatibility**: For any pair $\varrho_1, \varrho_2 \in \mathcal{D}(\mathcal{H}_2)$ and element $h \in \mathcal{H}_1$:

1. When $\varrho_1(\phi(h)) = 1$, the composition satisfies:

$$(\mathcal{D}(\phi)(\varrho_1) \diamond \mathcal{D}(\phi)(\varrho_2))(h) = \varrho_2(\phi(h))$$

The relation $\varrho_1 \star \varrho_2 \preceq \varrho_2$ ensures:

$$\mathcal{D}(\phi)(\varrho_1 \star \varrho_2)(h) \leq (\mathcal{D}(\phi)(\varrho_1) \diamond \mathcal{D}(\phi)(\varrho_2))(h)$$

2. The symmetric case $\varrho_2(\phi(h)) = 1$ follows similarly.

3. When $\varrho_1(\phi(h)) = \varrho_2(\phi(h)) = 0$, we obtain:

$$\mathcal{D}(\phi)(\varrho_1) \diamond \mathcal{D}(\phi)(\varrho_2) = \mathcal{D}(\phi)(\varrho_1 \star \varrho_2) = 0$$

Thus, $\mathcal{D}(\phi)$ preserves all necessary algebraic and topological structure. □

4.5.4 Bidual Representation and Isomorphisms of Priestley Space Ordered Commutative Semi hypergroup

This subsection develops the biduality and isomorphism framework within Priestley spaces associated with negatively ordered commutative semihypergroups. It shows how algebraic structures can be represented through dual spaces, extends the correspondence to morphisms, and confirms the natural commutativity of the related diagrams. These results provide the foundation for categorical equivalence and strengthen the dual representation theory.

Lemma 4.5. *For any Priestley space $\mathscr{P} = (X, \tau, \preceq, \star)$ endowed with a negatively ordered commutative semihypergroup structure, the evaluation map*

$$\Xi_{\mathscr{P}} : \mathscr{P} \to \mathfrak{D}(\mathcal{L}(\mathscr{P}))$$

defined by

$$\Xi_{\mathscr{P}}(\rho)(\mathcal{Y}) = \begin{cases} \top & \text{if } \rho \in \mathcal{Y}; \\ \perp & \text{otherwise}. \end{cases}$$

for all clopen increasing sets $Y \in \mathcal{L}(\mathscr{P})$, establishes an isomorphism in the category of such structures.

Proof. We verify the necessary conditions:

1. **Well-definedness**: By Lemma 4.1, $\mathcal{L}(\mathscr{P})$ forms a bounded distributive hyperlattice, and by Lemma 4.2, its dual $\mathfrak{D}(\mathcal{L}(\mathscr{P}))$ is again a Priestley space with semi hypergroup structure.

2. **Morphism Properties**:

- Order-preservation and continuity follow from standard results: [4], [5], [6], [7] [8], [37], and [38].

- Bijectivity is established through Stone-type duality arguments.

3. **Operation Compatibility**: For $\rho, \sigma \in X$ and $\mathcal{Y} \in \mathcal{L}(\mathscr{P})$:

- When $\rho \star \sigma \notin \mathcal{Y}$, the inequality holds trivially,

- When $\rho \star \sigma \in \mathcal{Y}$, the increasing property ensures $\{\rho, \sigma\} \subseteq \mathcal{Y}$, making $\Xi_{\mathscr{P}}(\rho)(\mathcal{Y}) \diamond \Xi_{\mathscr{P}}(\sigma)(\mathcal{Y}) = \{\bot, \top\}$ dominate $\Xi_{\mathscr{P}}(\rho \star \sigma)(\mathcal{Y})$.

Thus $\Xi_{\mathscr{P}}$ preserves all required structure. □

Lemma 4.6. *Consider two Priestley spaces $\mathscr{P}_1 = (X, \mathcal{T}, \leq, \star)$ and $\mathscr{P}_2 = (Y, \mathcal{T}', \preceq, \diamond)$ endowed with negatively ordered commutative semihypergroup structures. For any structure-preserving map $\omega : \mathscr{P}_1 \to \mathscr{P}_2$, the inverse image mapping*

$$\mathcal{L}(\omega) : \mathcal{L}(\mathscr{P}_2) \to \mathcal{L}(\mathscr{P}_1), \quad V \mapsto \omega^{-1}(V)$$

constitutes a surjective homomorphism between the corresponding algebraic structures.

Proof. We verify the homomorphism conditions:

1. **Well-definedness**: The preimage $\omega^{-1}(V)$ of any increasing $\mathcal{T}'$-clopen set $V \subseteq Y$ remains increasing and $\mathcal{T}$-clopen in X, as established in [4], [5], [6], [7] [8].

2. **Operation Preservation**:

- For meets: $\mathcal{L}(\omega)(V \cap W) = \mathcal{L}(\omega)(V) \cap \mathcal{L}(\omega)(W)$.

- For hyperjoins: $\mathcal{L}(\omega)(V \curlyvee W) \subseteq \mathcal{L}(\omega)(V) \curlyvee \mathcal{L}(\omega)(W)$.

3. **Hyperproduct Compatibility**:

1. When $V = Y$: $\mathcal{L}(\omega)(V \diamond W) = \mathcal{L}(\omega)(\downarrow W) = \downarrow \mathcal{L}(\omega)(W) = X \star \mathcal{L}(\omega)(W)$.

2. The symmetric case $W = Y$ follows similarly.

3. For $V, W \neq Y$: $\mathcal{L}(\omega)(V \diamond W) = \emptyset \preceq \mathcal{L}(\omega)(V) \star \mathcal{L}(\omega)(W)$.

Thus, $\mathcal{L}(\omega)$ preserves all required operations. □

Theorem 4.4. *For any homomorphism $\varphi : \mathscr{H}_1 \to \mathscr{H}_2$ between negatively ordered commutative hypermonoids, there exists a canonical correspondence making the following square commutative:*

$$\Psi_{\mathscr{H}_2} \circ \varphi = \widetilde{\varphi} \circ \Psi_{\mathscr{H}_1}$$

where:

1. $\Psi_{\mathscr{H}_i} : \mathscr{H}_i \to \mathcal{L}(\mathfrak{Spec}(\mathscr{H}_i))$ *are the spectral embeddings;*

2. $\widetilde{\varphi} = \mathcal{L}(T(\varphi))$ *corresponds to the induced dual morphism;*

3. *The map* $T(\varphi) : \mathfrak{Spec}(\mathscr{H}_2) \to \mathfrak{Spec}(\mathscr{H}_1)$ *is given by the composition* $T(\varphi)(\psi) = \psi \circ \varphi$.

In summary, this commutative diagram provides a precise visualization of the categorical equivalence described above.

$$\begin{tikzcd}
\mathscr{H}_1 \arrow[r, dashed, "\varphi"] \arrow[d, dashed, "\Psi_{\mathscr{H}_1}"'] & \mathscr{H}_2 \arrow[d, dashed, "\Psi_{\mathscr{H}_2}"] \\
\mathcal{L}(\mathfrak{Spec}(\mathscr{H}_1)) \arrow[r, dashed, "\widetilde{\varphi}"'] & \mathcal{L}(\mathfrak{Spec}(\mathscr{H}_2))
\end{tikzcd}$$

Proof. The equality follows by direct computation for any $x \in \mathscr{H}_1$:

$$\begin{aligned}
(\Psi_{\mathscr{H}_2} \circ \varphi)(x) &= \{\psi \in \mathfrak{Spec}(\mathscr{H}_2) \mid \psi(\varphi(x)) = 1\} \\
&= \{\psi \circ \varphi \in \mathfrak{Spec}(\mathscr{H}_1) \mid \psi(\varphi(x)) = 1\} \\
&= (\widetilde{\varphi} \circ \Psi_{\mathscr{H}_1}(x).
\end{aligned}$$

This proof adapts the arguments from [12, Lemma 3.11] to our generalized framework. □

Theorem 4.5. *Let* $\gamma : \mathscr{S}_1 \to \mathscr{S}_2$ *be a Priestley space negatively ordered commutative semi hypergroup isomorphism. Then,*

$$\mathcal{T}(\mathfrak{L}(\gamma)) \circ \pi_{\mathscr{S}_1} = \pi_{\mathscr{S}_2} \circ \gamma.$$

where:

1. $\pi_{\mathscr{S}_i} : \mathscr{S}_i \to \mathcal{T}(\mathfrak{L}(\mathscr{S}_i))$ *they correspond to the spectral embeddings;*

2. $\widetilde{\gamma} = \mathcal{T}(\mathfrak{L}(\gamma))$ *corresponds to the induced dual morphism;*

3. *The map* $\mathfrak{L}(\gamma) : \mathfrak{L}(\mathscr{P}_2) \to \mathfrak{L}(\mathscr{P}_1)$ *is given by a homomorphism* $\mathfrak{L}(\gamma)(y) = \gamma^{-1}(y)$.

This diagram serves as a clear illustration of the categorical equivalence discussed above.

$$\begin{tikzcd}
\mathscr{S}_1 \arrow[r, dashed, "\gamma"] \arrow[d, dashed, "\pi_{\mathscr{S}_1}"'] & \mathscr{S}_2 \arrow[d, dashed, "\pi_{\mathscr{S}_1}"] \\
\mathcal{T}(\mathfrak{L}(\mathscr{S}_1)) \arrow[r, dashed, "\mathcal{T}(\mathfrak{L}(\gamma))"'] & \mathcal{T}(\mathfrak{L}(\mathscr{S}_2))
\end{tikzcd}$$

Proof. Similarly to [12, Lemma 3.12]. □

4.5.5 Duality Correspondence in Priestley-Style for Algebraic Structures

In the following result, we give a Priestley duality for a bounded distributive hyperlattices negatively ordered commutative hypermonoids.

Theorem 4.6. *The dual of the category of a Priestley spaces negatively ordered commutative semi-hypergroups is equivalent to the category of bounded distributive hyperlattices negatively ordered commutative hypermonoids.*

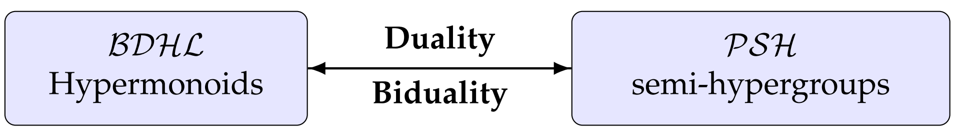

Figure 4.2: Simplified diagram of Priestley duality: $\mathcal{BDHL} \Leftrightarrow \mathcal{PSH}$.

Proof. The verification follows by synthesizing the essential components from:

(ii) The structural characterization in Lemma 4.3.

(iii) The representation theorem in Lemma 4.5 .

(iv) The categorical framework established in Theorems 4.4 and 4.5.

□

4.6 Applications in Representation Theory

The following examples demonstrates that there is an isomorphism both a between hyperlattice-ordered hypermonoids and their bidual, and between Priestley space negatively ordered commutative semihypergroups and their bidual.

Example 4.3. Let us consider the bounded distributive lattice $\mathcal{E} = (\mathbb{E}, \curlywedge, \curlyvee, \perp, \top)$, whose Hasse diagram is depicted in Figure 4.3. The set $\mathbb{E}$ consists of the elements $\{\perp, c, a, b, d, \top\}$.

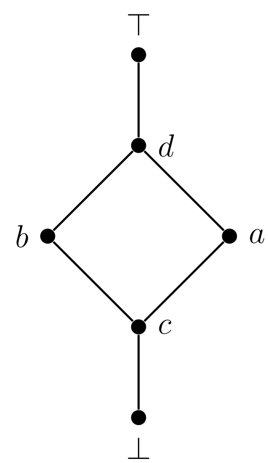

Figure 4.3: Hasse diagram representation of $\mathcal{E}$

We enrich $\mathbb{E}$ with two hyperoperations defined as follows:

- **Hyperjoin:** For any $\omega_1, \omega_2 \in \mathbb{E}$,

$$\omega_1 \curlyvee \omega_2 = \{r \in \mathbb{E} \mid r \curlyvee \omega_1 = r \curlyvee \omega_2 = \omega_1 \curlyvee \omega_2\}.$$

- **Hypermeet:**

$$\omega_1 \circledcirc \omega_2 = \begin{cases} \downarrow (\omega_1 \curlywedge \omega_2) & \text{if } \omega_1 = \top \text{ or } \omega_2 = \top, \\ \{\perp\} & \text{otherwise}. \end{cases}$$

According to Theorem 4.1, the structure $(\mathbb{E}, \curlywedge, \curlyvee, \circledcirc, \perp, \top)$, equipped with the above hyperoperations, forms a bounded distributive hyperlattice endowed with a negatively ordered commutative hypermonoid structure.

Let $T(\mathcal{E})$ denote the set of all lattice homomorphisms from $\mathbb{E}$ to the two-element lattice $\{\perp, \top\}$. This yields four distinct mappings: $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, whose values are summarized in Table 4.2

$\mathbb{E}$	$\bot$	c	a	b	d	$\top$
σ_1	0	0	0	0	0	1
σ_2	0	0	0	1	1	1
σ_3	0	0	1	0	1	1
σ_4	0	1	1	1	1	1

Table 4.2: Complete list of lattice homomorphisms

The bidual lattice $L(T(\mathcal{E}))$ consists of the subsets $\{\emptyset, \{\sigma_4\}, \{\sigma_3, \sigma_4\}, \{\sigma_2, \sigma_4\}, \{\sigma_2, \sigma_3, \sigma_4\}, X\}$, where $X = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$. This yields a bounded distributive hyperlattice negatively ordered commutative hypermonoid $(L(T(\mathcal{E})), \cap, \cup, \curlywedge, \curlyvee)$ with the following hyperoperations:

- For all $A, B \in L(T(\mathcal{E}))$,

$$A \curlyvee B = \{C \in L(T(\mathcal{E})) \mid A \cup B = C \cup A = C \cup B\},$$

$$A \circledcirc B = \begin{cases} \downarrow (A \cap B) & \text{if } A = X \text{ or } B = X, \\ \emptyset & \text{otherwise.} \end{cases}$$

The canonical homomorphism $\Phi_{\mathcal{E}} : \mathcal{E} \to L(T(\mathcal{E}))$ is defined by the mapping in Table 4.3

$\mathbb{E}$	$\bot$	c	a	b	d	$\top$
$\Phi_{\mathbb{E}}$	$\emptyset$	$\{\sigma_4\}$	$\{\sigma_3, \sigma_4\}$	$\{\sigma_2, \sigma_4\}$	$\{\sigma_2, \sigma_3, \sigma_4\}$	X

Table 4.3: The homomorphism $\Phi_{\mathbb{E}}$

This mapping establishes an isomorphism between $\mathcal{E}$ and its bidual $(L(T(\mathcal{E})), \curlywedge, \curlyvee, \circledcirc, \emptyset, X)$.

Example 4.4. Consider the Priestley space negatively ordered commutative semihypergroups $\mathcal{P} = (P, \tau, \preceq, \diamond)$ consisting of four elements $P = \{p, q, r, s\}$, where the partial order $\preceq$ is depicted in Figure 4.4. The hyperoperation $\diamond$ is defined by: For all $x, y \in P$, $x \diamond y = \inf\{x, y\}$.

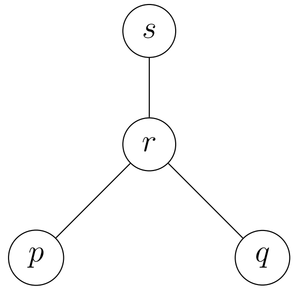

Figure 4.4: Inverted poset structure of $\mathcal{P} = \{p, q, r, s\}$.

The set of clopen increasing subsets of $\mathcal{P}$ forms the lattice:

$$\mathcal{C}(P) = \{\emptyset, \{s\}, \{r, s\}, \{q, r, s\}, \{p, r, s\}, P\}.$$

We define two algebraic hyperoperations on $\mathcal{C}(P)$ as follows:

- The *hyperjoin* $\curlyvee$ is given by:

$$\Lambda_1 \curlyvee \Lambda_2 = \{\Lambda \in \mathcal{C}(P) \,|\, \Lambda_1 \cup \Lambda_2 \subseteq \Lambda \text{ and } \Lambda_1 \cup \Lambda = \Lambda_2 \cup \Lambda\}\,.$$

- The *hyperproduct* $\circledast$ is defined by:

$$\Lambda_1 \circledast \Lambda_2 = \begin{cases} \downarrow (\Lambda_1 \cap \Lambda_2) & \text{if } \Lambda_1 = P \text{ or } \Lambda_2 = P, \\ \{\emptyset\} & \text{otherwise.} \end{cases}$$

By invoking Lemma4.1, we conclude that the structure $(\mathcal{C}(P), \cap, \curlyvee, \circledast, \emptyset, P)$ constitutes a bounded hyperlattice endowed with a negatively ordered commutative hypermonoid structure.

The set of structure-preserving maps $\mathcal{H}(\mathcal{C}(P)) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ is characterized by the following Table4.4:

$\mathcal{C}(P)$	$\emptyset$	$\{s\}$	$\{r, s\}$	$\{q, r, s\}$	$\{p, r, s\}$	P
γ_1	0	0	0	1	0	1
γ_2	0	0	0	0	1	1
γ_3	0	0	1	1	1	1
γ_4	0	1	1	1	1	1

Table 4.4: Structure-preserving mappings

We define the natural correspondence $\vartheta_P : P \to \mathcal{H}(\mathcal{C}(P))$ by:

$$\vartheta_P(q) = \gamma_1, \quad \vartheta_P(p) = \gamma_2, \quad \vartheta_P(r) = \gamma_3, \quad \vartheta_P(s) = \gamma_4.$$

This mapping establishes an isomorphism in the category of Priestley spaces equipped with negatively ordered commutative semihypergroup operations.

Chapter Conclusion

This chapter has established a Priestley-type duality between bounded distributive hyperlattices equipped with negatively ordered commutative hypermonoid structures and Priestley spaces associated with negatively ordered commutative semihypergroups. Through the introduction and formalization of key notions including negatively ordered semihypergroups, hyperlattice-ordered monoids, hyperideals, and hyperfilters the theoretical foundation of the duality framework has been consolidated. The results presented here highlight the structural correspondence between algebraic and topological perspectives, thereby reinforcing the significance of negatively ordered hyperstructures within the broader context of duality theory.

Conclusion

In this thesis, we have established a representation framework for certain ordered algebraic structures, focusing on negatively ordered commutative monoids and hyperlattice-ordered commutative hypermonoids. By constructing categorical representations and exploring Priestley-style dualities, we demonstrated deep correspondences between algebraic systems and their dual topological spaces. This work extends Priestley's classical results and situates them within the broader context of hyperstructures, thereby contributing to the enrichment of duality theory in algebra and topology.

Specifically, we proved that the dual of the category of Priestley spaces enriched with negatively ordered commutative semigroups is equivalent to the category of bounded distributive hyperlattices equipped with negatively ordered commutative monoid structures. Furthermore, we established a categorical duality between bounded distributive hyperlattices with negatively ordered commutative hypermonoid structures and Priestley spaces enriched with negatively ordered commutative semihypergroup structures, in order to prove their categorical equivalence. These results highlight the robustness of Priestley-style representation in capturing the structural essence of ordered algebraic systems.

Beyond its technical contributions, this thesis emphasizes the theoretical significance of categorical dualities as a unifying tool for algebraic and topological frameworks. The established correspondences not only deepen our understanding of ordered hyperstructures but also open perspectives for applications in mathematical logic, symbolic systems, and ordered data structures. The categorical approach adopted here provides a fertile ground for extending representation theorems to broader classes of hyperstructures, including non-commutative and non-distributive settings, and for investigating their relevance in computer science, information systems, and formal reasoning.

Future research may explore the refinement of these dualities, the development of analogous frameworks for other algebraic categories, and the integration of hyperlattice-based methods into computational models. Such directions would further consolidate the role of representation theorems as a cornerstone in the study of ordered algebraic structures.

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ملخّص: في هذه الأطروحة، قمنا بتوسيع نتائج بريستلي الكلاسيكية من خلال إثبات نظرية تمثيل ضمن إطار المشبكات الفائقة التوزيعية المحدودة و المونويدات التبادلية المرتبة سلبيا. و قد أثبتنا وجود تكافؤات تصنيفية بين فضاءات بريستلي المزودة بالمجموعات شبه التبادلية المرتبة سلبيا و بين فئة المشبكات الفائقة التوزيعية المحدودة المزودة ببنى المونويدات التبادلية المرتبة سلبيا. علاوة على ذالك، أقامت هذه الدراسة ثنائية من نوع بريستلي بين فئة المشبكات الفائقة التوزيعية المحدودة المزودة ببنية المونويد الفائق التبادلي المرتب سلبيا و بين فئة فضاءات بريستلي المرتبطة بالمجموعات شبه الفائقة التبادلية المرتبة سلبيا، و ذالك لإثبات تكافؤها التصنيفي.
الكلمات المفتاحية: نظرية التمثيل، البنى الجبرية المرتبة، المشبكات الفائقة، فضاءات بريستل، الثنائية.

Abstract: In this thesis, we extended Priestley's classical results by proving a representation theorem within the framework for algebraic structures of bounded distributive hyperlattice and negatively ordered commutative monoids. We establish categorical equivalence between the dual of Priestley spaces equipped with negatively ordered commutative semigroups and the category of bounded distributive hyperlattice with negatively ordered commutative monoids. Furthermore, this research establishes a Priestley-style duality between the category of bounded distributive hyperlattices endowed with a negatively ordered commutative hypermonoid structure and the category of Priestley spaces associated with negatively ordered commutative semihypergroups, in order to establish categorical equivalence between them.
Keywords: Representation theorem, Ordered algebraic structures, Hyperlattices, Priestley space, Priestly duality.

Résumé : Dans cette thèse, nous étendons les résultats classiques de Priestley en démontrant un théorème de représentations dans un cadre de dualité pour les structures algébriques du type hypertreillis distributifs bornés et des monoïdes commutatifs ordonné négativement. Nous établissons des équivalence catégoriques entre les espaces de Priestley munis de semi-groupes commutatifs ordonnés négativement et la catégorie des hypertreillis distributifs bornés munis de structures des monoïdes commutatifs ordonnés négativement. De plus, cette thèse établit une dualité de type Priestley entre la catégorie des hypertreillis distributifs bornés dotés d'une structure de d'hypermonoïde commutatif ordonné négativement et la catégorie des espaces de Priestley associés aux semi-hypergroupes commutatifs ordonnés négativement, afin d'établir l'équivalence catégorique entre elles.
Mots-clés: Théorème de représentations, Structurees algébriques ordonnées, Hypertreillis, hypermonoide, Espaces de Priestley, Dualité de Priestley.