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Theme

Chebyshev Wavelets Method for Solving Volterra Integral Equations

Presented by :

M^r ***Fatima Medjnah***

In front of the jury composed of :

Noui Djaidja	M.C.A,	University of Msila	**President**.
Amina Khirani	M.C.A,	University of Msila	**Supervisor.**
Nabila Seghiri	M.C.B,	University of Msila	**Examiner.**

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Contents

Introduction

Volterra integral equations of the second kind arise in various fields such as physics, engineering, biology, and mathematical modeling. Owing to their integral structure and the presence of a variable upper limit, these equations are often difficult to solve analytically. This difficulty has motivated the development of efficient and reliable numerical methods.

In recent years, a wide variety of numerical techniques has been proposed for solving Volterra integral equations of the second kind. Among the most widely used are quadrature methods, which approximate the integral using classical rules such as the Trapezoidal and Simpson rules. These approaches are extensively discussed in [1, 2].

Iterative methods, such as Picard iteration, construct the solution through successive approximations. These techniques are presented in detail in [3]. Another important approach is the collocation method, which enforces the integral equation at a finite set of selected points known as collocation points. This method is discussed in [4] as an efficient strategy for solving integral equations.

In recent years, spectral methods have attracted considerable attention for the numerical treatment of integral and differential equations due to their high accuracy and rapid convergence. Among these methods, the collocation approach is particularly notable for its flexibility and strong theoretical foundation.

This study focuses on applying the collocation method using Chebyshev wavelets as basis functions to approximate the solutions of linear Volterra integral equations of the second kind. Chebyshev wavelets are specifically chosen due to their outstanding mathematical and computational properties. Notably, their orthogonality simplifies the resulting system of equations, while their compact support leads to sparse matrices that significantly reduce computational cost and memory usage. Furthermore, when combined with Chebyshev collocation points, they provide exceptionally fast convergence and eliminate Runge's phenomenon, making them highly efficient in approximating smooth functions.

The objective of this work is to derive the Chebyshev wavelets collocation method, implement it on a set of test problems, and evaluate its performance by comparing the obtained results with exact solutions and other numerical techniques.

Below is a summary of each chapter in this thesis:

Chapter 1: Basic Concepts This chapter lays the essential theoretical groundwork for the thesis. It begins with the definition of contraction mappings and presents the Banach-Picard Fixed-Point Theorem, ensuring the existence and uniqueness of solutions for specific equation classes. The chapter then introduces Chebyshev polynomials, covering their definitions, key properties such as orthogonality and recurrence relations, and their applications in function approximation. It concludes with a detailed construction of Chebyshev wavelets, which serve as the fundamental tool for function approximation and discretization in the subsequent chapters.

Chapter 2: Volterra Integral Equations This chapter introduces the core concepts and classifications of integral equations, focusing particularly on Volterra integral equations. It differentiates between Fredholm and Volterra equations, as well as between first-kind and second-kind equations. The existence and uniqueness of solutions for Volterra equations are thoroughly discussed. Furthermore, the chapter surveys alternative numerical and analytical methods commonly used for solving Volterra equations, including the Adomian decomposition method and the method of successive approximations.

Chapter 3: Solving Volterra Equations via the Chebyshev Collocation Method This chapter is devoted to implementing the Chebyshev wavelet collocation method to solve linear Volterra integral equations of the second kind. It details the general framework, emphasizing the use of Chebyshev wavelets as basis functions to transform the integral equation into a linear algebraic system of the form $AC = F$. Several illustrative examples are provided to demonstrate the step-by-step construction of approximate solutions. Finally, numerical experiments assess the accuracy and efficiency of the method, with results compared against exact solutions to showcase its strengths and fast convergence.

Chapter 1

Basic Concepts

In this chapter, we study the Chebyshev polynomials, focusing on their main properties such as orthogonality, recurrence relations, and explicit formulas. These elements are essential for constructing the collection method, which will be used later to solve Volterra integral equations.

1.1 Contraction Mappings

Let (X, d) and (Y, δ) be two metric spaces, and let $k \in [0, 1[$. A function $f : X \to Y$ is said to be a contraction with constant k if:

$$\forall (x, y) \in X \times X, \quad \delta(f(x), f(y)) \leq kd(x, y).$$

Such a function f is also called a k-contraction.

1.2 Banach-Picard Fixed-Point Theorem

Let (X, d) be a complete metric space, and let $f : X \to X$ be a contraction mapping, meaning that there exists $k \in [0, 1[$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq kd(x, y). \tag{1.1}$$

Then, f has a unique fixed point $\ell \in E$ such that $f(\ell) = \ell$.
Moreover, for any initial point $u_0 \in X$, the sequence (u_n) defined by

$$u_0 \in X,$$

$$u_{n+1} = f(u_n), \quad n \geq 0,$$

converges to ℓ, and the following estimates hold for all $n \in \mathbb{N}$:

$$d(u_n, \ell) \leq k^n d(u_0, \ell), \tag{1.2}$$

and

$$d(u_n, \ell) \leq \frac{k}{1-k} d(u_n, u_{n-1}). \tag{1.3}$$

The fixed-point theorem is a fundamental tool in analysis and is used to prove other important results such as the local inversion theorem and the Cauchy-Lipschitz theorem.

Théorème 1. *Let A be an operator on a space X, such that A^n is a contraction operator. Then, the equation:*

$$Au = u$$

admits a unique solution in X.

Proof. Since A^n is a contraction, it has a unique fixed point denoted by u_0. Therefore, we have:

$$\|A(u_0) - u_0\| = \|A^n(A(u_0)) - A^n(u_0)\| \leq k\|A(u_0) - u_0\|,$$

which implies that $A(u_0) = u_0$, since $0 < k < 1$. The uniqueness of the fixed point of A follows from the fact that it is also a fixed point of A^n. □

1.3 Definition of Chebyshev polynomials

The Chebyshev polynomials are obtained by expanding the formula [10]:

$$T_n(x) = \cos(n \arccos(x)),$$

and satisfy the following recurrence relation:

$$\begin{cases} T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \\ T_1(x) = x, \quad T_0(x) = 1, \quad \text{for } n \geq 1. \end{cases}$$

1.3.1 Orthogonality Property

These polynomials are orthogonal with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ on $[-1, 1]$, and

$$\int_{-1}^{1} w(x)T_n(x)T_m(x)\,dx = \begin{cases} \pi, & n = m = 0, \\ \pi/2, & n = m \neq 0, \\ 0, & n \neq m. \end{cases}$$

1.3.2 The First Chebyshev polynomials

The first few Chebyshev polynomials are:

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1, \\ T_5(x) &= 16x^5 - 20x^3 + 5x. \end{aligned}$$

1.4 Chebyshev wavelets

1.4.1 Definition of Chebyshev Wavelets

The wavelets are stretched versions of the original wavelet T with the same basic shape but at a different scale and frequency[10]. We construct them from the translation parameter b and the scaling parameter a, then we have the following family of wavelets:

$$\theta_{a,b}(x) = \frac{1}{\sqrt{|a|}} T\left(\frac{x-b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0.$$

So, the Chebyshev wavelets are defined as:

$$\theta_{i,j}(x) = \begin{cases} 2^{\frac{k}{2}} \tilde{T}_j(2^k x - 2i + 1), & \frac{i-1}{2^{k-1}} \leq x \leq \frac{i}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\tilde{T}_j(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & j = 0, \\ \sqrt{\frac{2}{\pi}} T_j(t), & j > 0, \end{cases}$$

where k is a positive integer number, $j = 0, 1, 2, \ldots, n-1$, $i = 1, 2, \ldots, 2^{k-1}$ and T_j is the Chebyshev polynomial of degree j. However, the family of Chebyshev wavelets $\{\theta_{i,j}\}$ defines an orthonormal basis for $L^2_{w_1}([0,1])$ such that:

$$w_k(x) = \begin{cases} w_{1,k}(x), & 0 \leq x < \frac{1}{2^{k-1}}, \\ w_{2,k}(x), & \frac{1}{2^{k-1}} \leq x < \frac{2}{2^{k-1}}, \\ \vdots & \\ w_{2^{k-1},k}(x), & \frac{2^{k-1}-1}{2^{k-1}} \leq x \leq 1, \end{cases}$$

where $w_{i,k}(x) = w\left(2^k x - 2i + 1\right)$. The Chebyshev wavelets charts are proven for $k = 1$ and $n = 1, 2, 3, 4$ in Figure 1.

1.4.2 Function Approximation

For any function $\phi(x)$ in $L^2_{w_k}([0,1])$ can be expanded as:

$$\phi(x) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{i,j} \theta_{i,j}(x), \tag{1.4}$$

where $c_{i,j} = \langle \phi, \theta_{i,j} \rangle$, such that $\langle \cdot, \cdot \rangle$ is the inner product in $L^2_{w_1}([0,1])$. So we approximate the function $\phi(x)$ by truncating the infinite series (2):

$$\phi_n(x) = \sum_{i=1}^{2^{k-1}} \sum_{j=0}^{n-1} c_{i,j}\theta_{i,j}(x) = C^T P(x), \tag{1.5}$$

where, C^T and $P(x)$ are $2^{k-1}n \times 1$ matrices:

$$C^T = [c_{1,0}, c_{1,1}, \ldots, c_{1,n-1}, c_{2,0}, c_{2,1}, \ldots, c_{2,n-1}, \ldots, c_{2^{k-1},0}, \ldots, c_{2^{k-1},n-1}],$$

and

$$P(x) = [\theta_{1,0}, \theta_{1,1}, \ldots, \theta_{1,n-1}, \theta_{2,0}, \theta_{2,1}, \ldots, \theta_{2,n-1}, \ldots, \theta_{2^{k-1},0}, \theta_{2^{k-1},1}, \ldots, \theta_{2^{k-1},n-1}]^T.$$

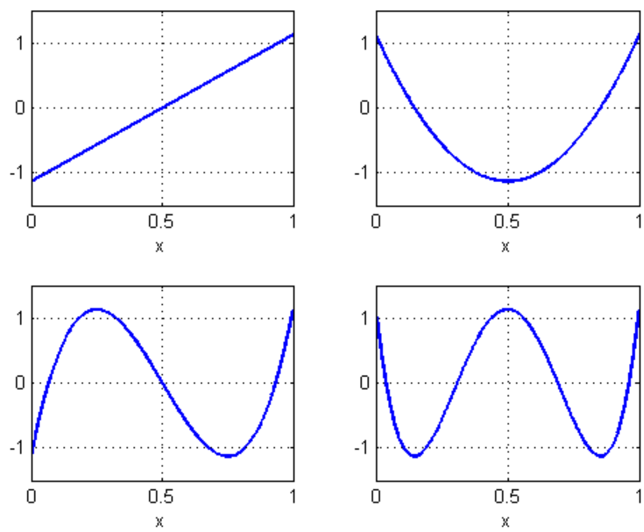

Figure 1.1: Chebyshev wavelets for$k = 1$

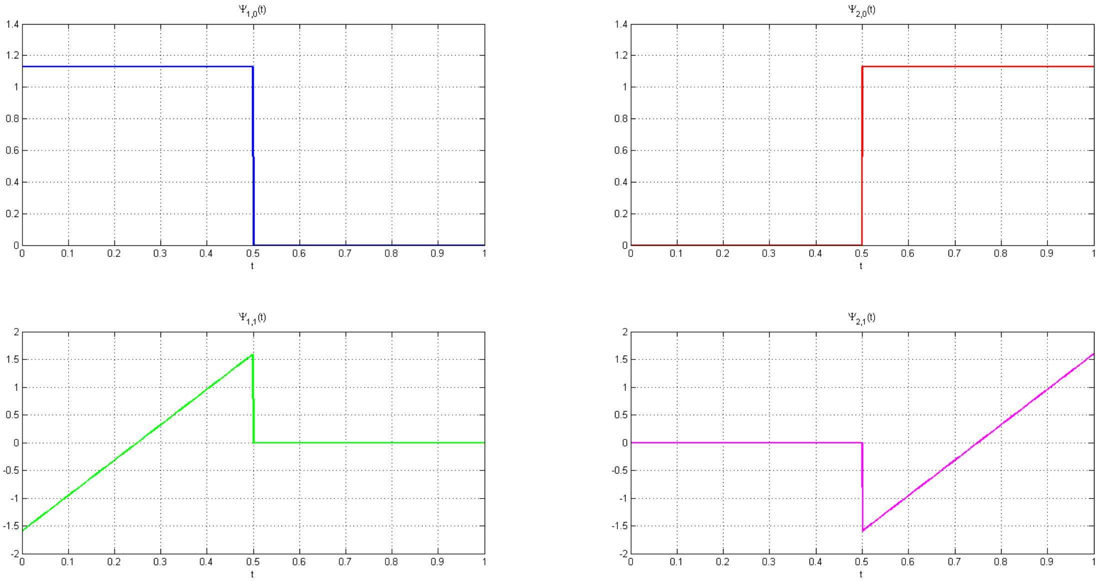

Figure 1.2

1.5 Collocation Method

The principle of the collocation method is based on searching for an approximate solution to a mathematical problem by satisfying the conditions at a set of specific points called collocation points.

The main steps of the collocation method principle:

1- Choice of collocation points:

The solution space is discretized using a set of collocation points. These points can be chosen in different ways, such as equally spaced points, Gauss-Lobatto points, Chebyshev points, etc. The choice of collocation points can influence the accuracy and convergence of the method.

2- Formulation of collocation conditions:

The collocation conditions are formulated by evaluating the integral equations, differential equation, or partial differential equations, as well as the boundary conditions, at these collocation points. This amounts to requiring that the approximate solution satisfies the problem's equations at these specific points.

3- Approximation of the solution:

The solution is approximated by a linear combination of basis functions, often polynomials, defined over the collocation points. The basis functions must be chosen in a way that respects the properties and requirements of the problem. The coefficients of this linear combination represent the unknowns of the method, which need to be determined.

4- Solving the system of equations:

By substituting the solution approximation into the collocation conditions, a system of algebraic equations is obtained. This system can be linear or non-linear, depending on the nature of the problem. Solving this system allows determining the values of the basis function coefficients, which represent the approximation of the solution.

5- Evaluation of the solution:

Once the coefficients of the basis functions are obtained, the approximate solution can be evaluated using this expression. This provides a numerical representation of the solution for the considered mathematical problem.

Application of the collocation method

We consider the equation:

$$u(x) = f(x) + \int_a^x k(x,t)u(t)dt, \quad x \in [a,b] \tag{1.6}$$

where $f(x)$ and $K(x,t)$ are given continuous functions defined on $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$, respectively, and u is the unknown function to be determined. The collocation method applied to Equation (1.6) consists in seeking an approximate solution in a finite-dimensional subspace, by requiring that Equation (1.6) be verified only on a finite number of points called collocation points.

In practice, we choose a sequence of subspaces $X_n \subset X$, $n \geq 1$ of finite dimension, generally subspaces of $C([a,b])$ or $L^2([a,b])$. Let $\{\psi_1, \psi_2, \dots, \psi_n\}$ be a basis of X_n. We seek a function $u_n \in X_n$ such that:

$$u_n(x) = \sum_{j=0}^{n} c_j \psi_j(x) \tag{1.7}$$

To determine the coefficients (c_j), by substituting this function into equation (1.6), and requiring that the equation be exact in the sense that the residual:

$$\begin{aligned} r_n(x) &= u_n(x) - \int_a^x k(x,t) u_n(t) dt - f(x), \quad x \in [a,b] \\ &= \sum_{j=0}^{n} c_j \psi_j(x) - \int_a^x k(x,t) \left(\sum_{j=0}^{n} c_j \psi_j(t) \right) dt - f(x) \\ &= \sum_{j=0}^{n} c_j \left[\psi_j(x) - \int_a^x k(x,t) \psi_j(t) dt \right] - f(x) \end{aligned} \tag{1.8}$$

is zero on a system of nodes $x_1, \dots, x_n \in [a,b]$ (i.e., at the collocation points), which leads to solving the linear system:

$$\sum_{j=0}^{n} c_j \left[\psi_j(x_i) - \int_a^b k(x_i,t) \psi_j(t) dt \right] = f(x_i), \quad i = 1, \dots, n \tag{1.9}$$

Equation (1.9) represents a linear system of (n) unknowns that can be written in the form:

$$AC = F \tag{1.10}$$

where:

$$\begin{aligned} A &= \left[\psi_j(x_i) - \int_a^b k(x_i,t) \psi_j(t) dt \right], \quad i,j = 1, \dots, n \\ C &= [c_1, \dots, c_n]^T \\ F &= [f(x_1), \dots, f(x_n)]^T \end{aligned}$$

$$C = A^{-1}F \tag{1.11}$$

Chapter 2

Introduction to Volterra Integral Equations

[6] This chapter introduces the basic concepts, classifications, and fundamental properties of integral equations, with a particular focus on Volterra integral equations. Moreover, it discusses both the exact and approximate solutions commonly used in the numerical treatment of such equations.

2.1 Definition of Integral Equations

An integral equation is an equation in which the unknown function $u(x)$ appears under an integral sign. A general form of an integral equation is given by:

$$h(x)u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)\,dt, \tag{2.1}$$

where $g(x)$ and $h(x)$ define the limits of integration. λ is a constant parameter, and $K(x,t)$ is a function of two variables x and t, referred to as the **kernel** of the integral equation.

The unknown function $u(x)$ appears both inside and outside the integral. The functions $f(x)$ and $K(x,t)$ are given in advance. It is important to note that the limits of integration, $g(x)$ and $h(x)$, may be variables, constants, or a combination of both.

2.2 Classification of Integral Equations

Integral equations are divided into several types, mainly depending on the limits of integration and the kernel of the equation. In this text, we focus on the following main types of integral equations:

2.2.1 Fredholm Integral Equations

For Fredholm integral equations, the limits of integration are fixed. Then, the unknown function $u(x)$ may appear only inside the integral equation by the form:

$$f(x) = \int_a^b K(x,t)u(t)\,dt. \tag{2.2}$$

This is called Fredholm integral equation of the first kind. But, for Fredholm integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral. The second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt. \tag{2.3}$$

2.2.2 Volterra Integral Equations

In Volterra integral equations, at least one of the limits of integration is variable. For the first kind of Volterra integral equation, the unknown function $u(x)$ appears only inside the integral in the form:

$$f(x) = \int_a^x K(x,t)u(t)\,dt. \tag{2.4}$$

However, in Volterra integral equations of the second kind, the unknown function $u(x)$ appears both inside and outside the integral. The second kind is given by the form:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)\,dt. \tag{2.5}$$

2.2.3 Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equations arise from parabolic boundary value problems, from the mathematical modelling of the spatio-temporal development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely:

$$u(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)\,dt + \lambda_2 \int_a^b K_2(x,t)u(t)\,dt, \tag{2.6}$$

and

$$u(x,t) = f(x,t) + \lambda \int_a^t \int_\Omega F(x,t,\xi,\tau,u(\xi,\tau))\,d\xi\,d\tau, \quad (x,t) \in \Omega \times [0,T], \tag{2.7}$$

where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on $D = \Omega \times [0,T]$, and Ω is a closed subset of $\mathbb{R}^n$, $n = 1,2,3$. It is interesting to note that Equation 2.6 contains separate Volterra and Fredholm integral equations, whereas Equation 2.7 contains mixed Volterra and Fredholm integral equations. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear

only inside the integral, the resulting equations are of the first kind, but will not be examined in this text.

2.2.4 Singular Integral Equations

Volterra integral equations of the first kind is given by:

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)\,dt, \tag{2.8}$$

or of the second kind

$$u(x) = f(x) + \int_{g(x)}^{h(x)} K(x,t)u(t)\,dt, \tag{2.9}$$

are called singular if one of the limits of integration $g(x)$, $h(x)$ or both are infinite. Moreover, the previous two equations are called singular if the kernel $K(x,t)$ becomes unbounded at one or more points in the interval of integration. In this text we will focus our attention on equations of the form:

$$f(x) = \int_a^x \frac{1}{(x-t)^\alpha} u(t)\,dt, \quad 0 < \alpha < 1,$$

or of the second kind:

$$u(x) = f(x) + \int_a^x \frac{1}{(x-t)^\alpha} u(t)\, dt, \quad 0 < \alpha < 1.$$

The last two standard forms are called generalized Abel's integral equation and weakly singular integral equations respectively. For $\alpha = \frac{1}{2}$, the equation:

$$f(x) = \int_a^x \frac{1}{\sqrt{x-t}}\, dt$$

is called Abel's singular integral equation. It is to be noted that the kernel in each equation becomes infinite at the upper limit $t = x$.

2.3 Linearity and Nonlinearity

Linear Volterra Equations

If the exponent of the unknown function $u(x)$ inside the integral sign is one, the integral equation is called *linear*. To explain this concept, we consider the equations:
Example 2.1.1 Consider the linear Volterra integral equation of the second kind:

$$u(x) = 1 + x + \int_0^x (x-t)u(t)dt \tag{1}$$

with the exact solution $u(x) = \exp(x)$.
Example 2.1.2

$$u(x) = 1 - \int_0^x (x-t)u(t)dt \tag{2}$$

Nonlinear Volterra Equations

The nonlinear Volterra integral equation of the second kind is represented by the form:

$$u(x) = f(x) + \int_0^x K(x,t)F(u(t))dt \tag{3}$$

However, the nonlinear Volterra integral equations of the *first kind* contains the nonlinear function $F(u(x))$ inside the integral sign. The nonlinear Volterra integral equation of the first kind is expressed in the form:

$$f(x) = \int_0^x K(x,t)F(u(t))dt \tag{4}$$

For these two kinds of equations, the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $F(u(x))$ is a nonlinear function of $u(x)$ such as $u^2(x)$, $\sin(u(x))$, and $e^{u(x)}$.

Example 2.2.1 Consider the nonlinear Volterra integral equation of the second kind:

$$u(x) = x + \int_0^x [u(t)]^2 dt \tag{5}$$

with the exact solution $u(x) = \tan(x)$.

Example 2.2.2

$$u(x) = 1 + \int_0^x (1 + x - t)(u(t))^4 dt \tag{6}$$

is nonlinear integral equation.

2.4 Homogeneous and inhomogeneous Integral Equations

Integral equations of the second kind are classified as *homogeneous* or *inhomogeneous*. If the function $f(x)$ in the second kind of Volterra integral equations is identically zero, the equation is called homogeneous. Otherwise, it is called inhomogeneous. Notice that this property holds for equations of the second kind only. To clarify this concept, we consider the following equations:

$$u(x) = \sin x + \int_0^x xtu(t)dt \tag{7}$$

$$u(x) = x + \int_0^1 (x-t)^2 u(t)dt \tag{8}$$

$$u(x) = \int_0^x (1 + x - t)u^4(t)dt \tag{9}$$

The first two equations are inhomogeneous because $f(x) = \sin x$ and $f(x) = x$, whereas the last two equations are homogeneous because $f(x) = 0$ for each equation. We usually use specific approaches for homogeneous equations, and other methods are used for inhomogeneous equations.

2.5 Existence and Uniqueness of solution of Volterra Integral Equation

Theorem 2.1[7] *Let $K(x,t)$ be a continuous function for $x, t \in [a,b]$, such that*

$$|K(x,t)| \leq M, \quad M > 0$$

then the Volterra equation:

$$u(x) - \lambda \int_a^x K(x,t)u(t)\,dt = f(x) \tag{2.10}$$

It admits a unique solution $u(x)$ for every $f(x)$ in $L_2([a,b])$ and every $\lambda \in \mathbb{R}$.

Proof. For the Volterra integral equation, we consider the operator:

$$Tu = f(x) + \lambda Au = f(x) + \lambda \int_a^x K(x,t)u(t)\,dt$$
$$Au = \int_a^x K(x,t)u(t)\,dt$$

and we show that the operator T^n is a contraction for $n \in \mathbb{N}$, so T has a fixed point.

$$\begin{aligned}
Tu &= f + \lambda Au \\
T^2u &= T(Tu) = T(f + \lambda Au) = f + \lambda A(f + \lambda Au) = f + \lambda Af + \lambda^2 A^2 \\
&\vdots \\
&\vdots \\
&\vdots \\
T^n u &= f + \lambda Af + \lambda^2 A^2 f + \cdots + \lambda^{n-1}A^{n-1}f + \lambda^n A^n u
\end{aligned}$$

so that :

$$\|T^n(u_1) - T^n(u_2)\| = \|\lambda^n A^n u_1 - \lambda^n A^n u_2\| = |\lambda|^n \|A^n u_1 - A^n u_2\|$$

$$\|T^n(u_1) - T^n(u_2)\| = |\lambda|^n \left\| \int_a^x K_n(x,t)(u_1(t) - u_2(t))\,dt \right\|$$

To determine $K_n(x,t)$, we use the relation:

$$\begin{aligned} Au &= \int_a^x K(x,t)u(t)\,dt \\ A^2u &= \int_a^x K(x,t)\int_a^t K(t,z)u(z)\,dz\,dt \\ &= \int_a^x u(z)\,dz\int_z^x K(x,t)K(t,z)\,dt \\ &= \int_a^x K_2(x,z)u(z)\,dz. \end{aligned}$$

Thus,

$$K_2(x,t) = \int_z^x K(x,z)K(z,t)\,dt$$

by recurrence we obtain:

$K_1(x,t) = K(x,t)$,

$K_2(x,t) = \int_t^x K(x,z)K(z,t)\,dz$,

$K_3(x,t) = \int_t^x K_2(x,z)K(z,t)\,dz$,

$K_n(x,t) = \int_t^x K_{n-1}(x,z)K(z,t)\,dz$,

since by the hypothesis we have $|K(x,t)| \leq M$, then

$$|K_n(x,t)| \leq \frac{M^n(x-t)^{n-1}}{(n-1)!}, \quad a \leq t \leq z \leq b \tag{2.11}$$

for $n=1$ the expression 2.11 is true, we assume that it is true for $m \in \mathbb{N}$,

$$|K_m(x,t)| \leq \frac{M^m(x-t)^{m-1}}{(m-1)!}$$

so :

$$\begin{aligned} |K_{m+1}(x,t)| &= \left|\int_t^x K_m(x,z)K(z,t)\,dz\right| \\ &\leq \int_t^x |K_m(x,z)K(z,t)|\,dz \\ &\leq \frac{M^{m+1}}{m-1}\int_t^x (x-z)^{m-1}\,dz \\ &\leq \frac{M^{m+1}}{m}(x-z)^m \end{aligned}$$

thus

$$\|T^n(u_1) - T^n(u_2)\| \leq \frac{|\lambda|^n M^n}{(n-1)!}\|u_1 - u_2\|$$

for $n \in \mathbb{N}$ large enough , we obtain :

$$\frac{|\lambda|^n M^n}{(n-1)!} < 1$$

Since T^n is a contraction, T admits a unique fixed point, according to **Theorem(1.3)**

$$Tu = u \Leftrightarrow u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)\,dt.$$

this fixed point is the solution of the equation 2.10. □ □

2.6 Solution Methods for Volterra Integral Equations

2.6.1 The Adomian Decomposition Method

[6] The Adomian Decomposition Method involves expressing the unknown function $u(x)$ as an infinite series composed of components, each determined through a recursive decomposition process.

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{2.12}$$

Or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \dots \tag{2.13}$$

where the components $u_n(x)$, for $n \geq 0$, are determined recursive manner. The decomposition method concerns itself with finding the components We consider the components $u_0, u_1, u_2, \dots$ individually. As will be seen through the text, the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated.

To establish the recurrence relation, we substitute 2.10 into the Volterra integral equation 2.5 to obtain:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^x K(x,t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt \tag{2.14}$$

or equivalently,

$$u_0(x)+u_1(x)+u_2(x)+\cdots = f(x)+\lambda\int_a^x K(x,t)[u_0(t)+u_1(t)+u_2(t)+\dots]\,dt. \quad (2.15)$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, the components $u_j(x)$, $j \geq 1$, of the unknown function $u(x)$ are completely determined by setting the recurrence relation:

It is clearly seen that the decomposition method transforms the integral equation into an elegant process of determining computable components. It has been formally shown by many researchers that if an exact solution exists for the problem, then the obtained series converges very rapidly to that

solution. The convergence of the decomposition series has been thoroughly studied to confirm its rapid convergence.

However, for concrete problems where a closed-form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. The more components we include, the higher the accuracy we can achieve.

Example 2.2 *Solve the following Volterra integral equation:*

$$u(x) = 1 - \int_0^x u(t)\,dt. \quad (2.16)$$

We note that $f(x) = 1$, $\lambda = -1$, and the kernel is $K(x,t) = 1$. Recall that the solution $u(x)$ is assumed to have a series form as given in 2.12. Substituting the decomposition series 2.12 into both sides of equation 2.16 gives:

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 - \int_0^x [u_0(t) + u_1(t) + u_2(t) + \dots]\,dt. \quad (2.17)$$

We identify the zeroth component by all terms that are not under the integral sign. *Therefore, we obtain the following recurrence relation:*

$$u_0(x) = 1, \quad u_{k+1}(x) = -\int_0^x u_k(t)\,dt, \quad k \geq 0. \quad (2.18)$$

Thus, we get:

$$
\begin{aligned}
u_0(x) &= 1, \\
u_1(x) &= -\int_0^x u_0(t)\,dt = -\int_0^x 1\,dt = -x, \\
u_2(x) &= -\int_0^x u_1(t)\,dt = -\int_0^x (-t)\,dt = \frac{1}{2!}x^2, \\
u_3(x) &= -\int_0^x u_2(t)\,dt = -\int_0^x \frac{1}{2!}t^2\,dt = -\frac{1}{3!}x^3, \\
u_4(x) &= -\int_0^x u_3(t)\,dt = -\int_0^x \left(-\frac{1}{3!}t^3\right)\,dt = \frac{1}{4!}x^4,
\end{aligned}
$$

and so on. Using 2.12 gives the series solution:

$$u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots, \tag{2.19}$$

which converges to the closed-form solution:

$$u(x) = e^{-x}.$$

2.6.2 Successive Approximations Method

[6]

The successive approximations method, also known as the Picard iteration method, provides a scheme that can be used to solve initial value problems or integral equations. This method finds successive approximations to the solution by starting with an initial guess, called the zeroth approximation.

As will be seen, the zeroth approximation is any chosen real-valued function that is used in a recurrence relation to determine the subsequent approximations.

Consider the linear Volterra integral equation of the second kind:

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)\,dt \tag{2.20}$$

where $u(x)$ is the unknown function to be determined, $K(x,t)$ is the kernel, and λ is a parameter.

The successive approximations method introduces the recurrence relation:

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)\,dt, \quad n \geq 1 \tag{2.21}$$

The zeroth approximation $u_0(x)$ can be any selected real-valued function. We always start with an initial guess for $u_0(x)$, most commonly, we select 0, 1, x, for $u_0(x)$.

and by using the recurrence relation from equation 2.20, several successive approximations $u_k(x)$, $k \geq 1$, can be determined as follows:

$$\begin{aligned}
u_1(x) &= f(x) + \lambda \int_0^x K(x,t)u_0(t)\,dt \\
u_2(x) &= f(x) + \lambda \int_0^x K(x,t)u_1(t)\,dt \\
u_3(x) &= f(x) + \lambda \int_0^x K(x,t)u_2(t)\,dt \\
&\vdots \\
u_n(x) &= f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)\,dt
\end{aligned}$$

The question of convergence of the sequence $u_n(x)$ is justified by the following theorem.

Théorème 2. *If the function $f(x)$ in equation 2.10 is continuous on the interval $0 \leq x \leq a$, and the kernel $K(x,t)$ is also continuous in the triangle $0 \leq t \leq x \leq a$, then the sequence of successive approximations $u_n(x)$, $n \geq 0$ converges to the exact solution $u(x)$ of the Volterra integral equation under consideration.*

Exemple 3. *Solve the following Volterra integral equation using the method of successive approximations:*

$$u(x) = 1 - \int_a^x (x-t)u(t)\,dt. \tag{2.22}$$

For the zeroth approximation, we choose:

$$u_0(x) = 1. \tag{2.23}$$

The successive approximation method admits the following iteration formula:

$$u_{n+1}(x) = 1 - \int_a^x (x-t)u_n(t)\,dt, \quad n \geq 0. \tag{2.24}$$

Substituting $u_0(x)$ *into the iteration formula gives:*

$$\begin{aligned}
u_1(x) &= 1 - \int_a^x (x-t)(1)\,dt = 1 - \frac{1}{2!}x^2,\\
u_2(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4,\\
u_3(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6,\\
u_4(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8,
\end{aligned}$$

Consequently, we obtain:

$$u_{n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!},$$

the solution of equation 2.22:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \cos(x).$$

Chapter 3

Solving Volterra Integral Equations via the Chebyshev Wavelet Collocation Method

In this chapter, we present the Collocation method as an efficient numerical technique for solving Volterra integral equations, particularly those of the second kind. The main idea is to approximate the unknown solution using a finite sum of chebyshev wavelets polynomials.

3.1 General Approach chebyshev wavelets polynomials using Collocation Method

Consider the general Volterra integral equation of the second kind:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)\,dt$$

where $f(x)$ and $K(x,t)$ are known functions and λ is a given constant.

The Collocation method proceeds as follows:

1. **Approximation:** Assume the approximate solution $u_n(x)$ is a finite sum of

chebyshev wavelet polynomials:

$$u_n(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \Psi_{nm}(x), \tag{3.1}$$

where $\Psi_{nm}(x)$ denotes the n-the Chebyshev wavelet polynomial and c_{nm} are unknown coefficients to be determined.

2. **Substitution:** Substitute $u_n(x)$,

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \Psi_{nm}(x) = f(x) + \lambda \int_a^x K(x,t) \left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \Psi_{nm}(x) \right) dt$$

3. **Collocation method:** By evaluating this equation in $2^{k-1}M$ points $\{x_i\}_{i=1}^{2^{k-1}M}$ in the interval $[0,1]$ we have a system of linear equations:

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \Psi_{nm}(x_i) = f(x_i) + \lambda \int_0^{x_i} K(x_i,t) \left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \Psi_{nm}(x_i) \right) dt$$

4. **Linear System:** We obtain the linear system:

$$AC = F,$$

where

- C and F are $2^{k-1}M$ vectors given by:

  $C = [c_{1,0}, c_{1,1}, \ldots, c_{1,M-1}, c_{2,0}, c_{2,1}, \ldots, c_{2,M-1}, \ldots, c_{2^{k-1},0}, \ldots, c_{2^{k-1},M-1}]^T,$

  $F = [F_1, F_1, \ldots, F_{2^{k-1}M}]^T$

- A is a matrix of size $(2^{k-1}M) \times (2^{k-1}M)$, such that

$$A_{i,(n,m)} = \Psi_{nm}(x_i) - \int_0^{x_i} K(x_i,t)\, \Psi_{nm}(t)\, dt, \quad \begin{cases} i = 1, \ldots, 2^{k-1}M \\ n = 1, \ldots, 2^{k-1}, \\ m = 0, \ldots, M-1. \end{cases}$$

Once this system is solved for C_{nm}, the approximate solution is:

$$u_n(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \Psi_{nm}(x)$$

3.2 Illustrative Examples

To demonstrate the effectiveness of the collection method, let us consider the following example [11]:

Exemple 4. *[11] Consider the following Volterra integral equation of the second kind, for $k = 2, M = 2$:*

$$u(x) = e^x + \int_0^x u(t)\,dt, \quad 0 \leq x \leq 1$$

where the exact solution is given by:

$$u_{exact}(x) = (1 + x)e^x.$$

We aim to solve this equation using the chebyshev wevelet collection method with $n = 2$, by approximating the solution as:

$$u_2(x) = \sum_{n=1}^{2} \sum_{m=0}^{1} c_{nm}\Psi_{nm}(x)$$

Substituting this approximation into the original equation leads to:

$$\sum_{n=1}^{2} \sum_{m=0}^{1} c_{nm}\Psi_{nm}(x_i) - \int_0^x \left(\sum_{n=1}^{2} \sum_{m=0}^{1} c_{nm}\Psi_{nm}(x_i) \right) dt = e^x \tag{3.2}$$

for $k = 1$ and $M = 2$. In this case, the Four basis functions are given by

$$\left.\begin{aligned} \Psi_{1,0} &= \frac{2}{\sqrt{\pi}}, \\ \Psi_{1,1} &= 2\sqrt{\frac{2}{\pi}}(4t - 1), \end{aligned}\right\} \text{for } t \in [0, 1/2],$$

$$\left.\begin{aligned} \Psi_{2,0} &= \frac{2}{\sqrt{\pi}}, \\ \Psi_{2,1} &= 2\sqrt{\frac{2}{\pi}}(4t - 3), \end{aligned}\right\} \text{for } t \in [1/2, 1].$$

We multiply both sides by each polynomial $\Psi_{n,m}(x)$ for $n = 1, 2$, and integrate over $[0, 1]$. This yields four equations:

we choose four values in the interval[0,1],$x = 0.2; 0.4, 0.6, 0.8$

Case $x = 0.2$

$$\begin{aligned} &C_{1,0}\Psi_{1,0}(0.2) - \int_0^{0.2} C_{1,0}\Psi_{1,0}(t)\,dt + C_{1,1}\Psi_{1,1}(0.2) - \int_0^{0.2} C_{1,1}\Psi_{1,1}(t)\,dt \\ &+C_{2,0}\Psi_{2,0}(0.2) - \int_0^{0.2} C_{2,0}\Psi_{2,0}(t)\,dt + C_{2,1}\Psi_{2,1}(0.2) - \int_0^{0.2} C_{2,1}\Psi_{2,1}(t)\,dt = e^{0.2} \end{aligned}$$

Case $x = 0.4$

$$\begin{aligned} &C_{1,0}\Psi_{1,0}(0.4) - \int_0^{0.4} C_{1,0}\Psi_{1,0}(t)\,dt + C_{1,1}\Psi_{1,1}(0.4) - \int_0^{0.4} C_{1,1}\Psi_{1,1}(t)\,dt \\ &+C_{2,0}\Psi_{2,0}(0.4) - \int_0^{0.4} C_{2,0}\Psi_{2,0}(t)\,dt + C_{2,1}\Psi_{2,1}(0.4) - \int_0^{0.4} C_{2,1}\Psi_{2,1}(t)\,dt = e^{0.4} \end{aligned}$$

Case $x = 0.6$

$$\begin{aligned} &C_{1,0}\Psi_{1,0}(0.6) - \int_0^{0.6} C_{1,0}\Psi_{1,0}(t)\,dt + C_{1,1}\Psi_{1,1}(0.6) - \int_0^{0.6} C_{1,1}\Psi_{1,1}(t)\,dt \\ &+C_{2,0}\Psi_{2,0}(0.6) - \int_0^{0.6} C_{2,0}\Psi_{2,0}(t)\,dt + C_{2,1}\Psi_{2,1}(0.6) - \int_0^{0.6} C_{2,1}\Psi_{2,1}(t)\,dt = e^{0.6} \end{aligned}$$

Case $x = 0.8$

$$\begin{aligned} &C_{1,0}\Psi_{1,0}(0.8) - \int_0^{0.8} C_{1,0}\Psi_{1,0}(t)\,dt + C_{1,1}\Psi_{1,1}(0.8) - \int_0^{0.8} C_{1,1}\Psi_{1,1}(t)\,dt \\ &+C_{2,0}\Psi_{2,0}(0.8) - \int_0^{0.8} C_{2,0}\Psi_{2,0}(t)\,dt + C_{2,1}\Psi_{2,1}(0.8) - \int_0^{0.8} C_{2,1}\Psi_{2,1}(t)\,dt = e^{0.8} \end{aligned}$$

Thus, we obtain the following matrix system:

$$AC = F$$

where

$$A = \begin{bmatrix} 0.4514 & -0.0638 & 0 & 0 \\ 0.3385 & 0.5426 & 0 & 0 \\ -0.2821 & 0.0000 & 0.5078 & -0.4149 \\ -0.2821 & -0.0000 & 0.3949 & 0.2553 \end{bmatrix}$$

$$F = \begin{bmatrix} 1.2215 \\ 1.4919 \\ 1.8222 \\ 2.2256 \end{bmatrix}$$

To solve the given linear system of equations, we can represent it in a matrix form as $A \cdot c = F$, where A is the coefficient matrix, c is the vector of unknown variables, and F is the constant vector. We can isolate the unknown vector c by multiplying both sides of the equation by the inverse of A. This mathematical operation yields the final solution: $c = A^{-1} \cdot F$. By performing matrix multiplication between A^{-1} and F, the values of the vector c are uniquely determined.

we find C.

$$C = \begin{bmatrix} 2.8440 \\ 0.9752 \\ 6.5632 \\ 1.7069 \end{bmatrix}$$

Hence

$$u_2(x) = \sum_{n=1}^{2} \sum_{m=0}^{1} C_{nm} \psi_{nm}(x)$$

By substituting the coefficients C and the wavelets ψ, we obtain the approximate solution:

$$u_2(x) = C_{10}\psi_{10}(x) + C_{11}\psi_{11}(x) + C_{20}\psi_{20}(x) + C_{21}\psi_{21}(x) \tag{3.3}$$

$$u_2(x) = 2\psi_{10}(x) + C_{11}\psi_{11}(x) + C_{20}\psi_{20}(x) + C_{21}\psi_{21}(x) \tag{3.4}$$

$$u_2(x) = \begin{cases} 2.8440\left(\dfrac{2}{\sqrt{\pi}}\right) + 0.9152\left(\dfrac{2}{\sqrt{\pi}}\right)(4x-1), & 0 \le x < \dfrac{1}{2} \\ \\ 6.5632\left(\dfrac{2}{\sqrt{\pi}}\right) + 1.7069 \times 2\sqrt{\dfrac{2}{\pi}}(4x-3), & \dfrac{1}{2} \le x \le 1 \end{cases}$$

$$u_2(x) = \begin{cases} 3.2091 + 1.0327(4x-1), & 0 \le x < \dfrac{1}{2} \\ \\ 7.4058 + 2.7238(4x-3), & \dfrac{1}{2} \le x \le 1 \end{cases}$$

3.3 Numerical examples

In this section, we employ the Chebyshev wavelet collocation method with $k = 2$ to approximate the solutions of several examples for different values of M, thereby illustrating the accuracy, efficiency, and reliability of the proposed method.

We note that the absolute error at a given point x is defined as

$$|u_{\text{exact}}(x) - u_{\text{approx}}(x)|.$$

Example 3.1 Consider the following Volterra integral equation of the second kind,

$$u(x) = 1 + \int_0^x tu(t)\, dt, \quad 0 \le x \le 1 \tag{3.5}$$

where the exact solution is given by:

$$u_{\text{exact}}(x) = e^{\frac{x^2}{2}}. \tag{3.6}$$

We apply the Chebyshev wavelet collocation method with to approximate the solution $u(x)$, using three different values of n, namely $n = 2$, $n = 4$, and $n = 8$. In this comparison, we focus on the approximate solutions, and the corresponding absolute errors for each value of n. The results are presented in Table (3.1) and Figure (3.1):

x	Error($n = 2$)	Error($n = 4$)	Error($n = 8$)
0	0.043331	0.00052376	1.9258e-10
0.2	0.00022385	9.2144e-06	2.3421e-11
0.4	6.2986e-07	2.3295e-05	5.7923e-11
0.6	0.0010656	4.6439e-05	5.9524e-08
0.8	0.00026261	1.7406e-05	1.2643e-07
1	0.091143	0.0014483	1.0341e-05

Table 3.1: Results for Example 3.1

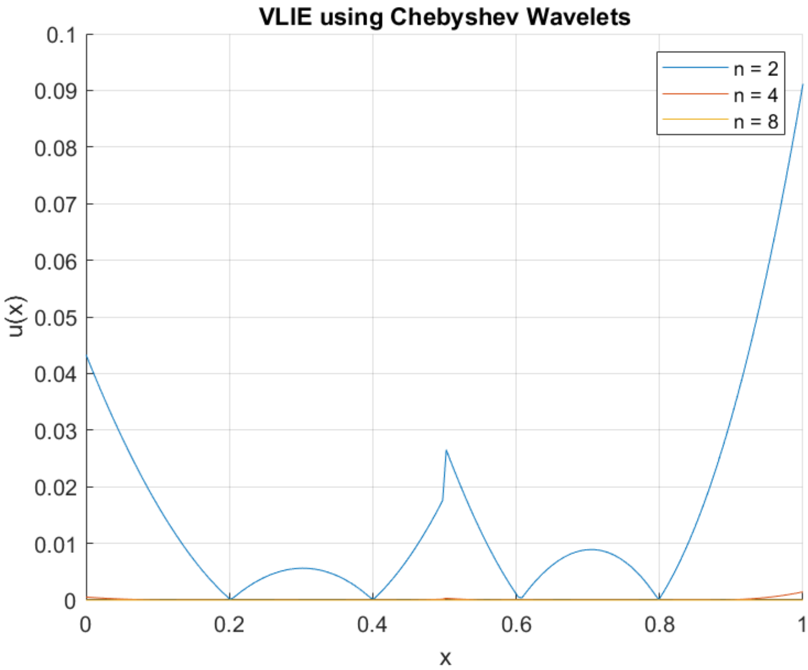

Figure 3.1: Absolute Errors for Example 3.1

Example 3.2 Consider the following Volterra integral equation of the second kind:

$$u(x) = e^x + \int_0^x u(t)\, dt, \quad 0 \leq x \leq 1 \tag{3.7}$$

Assume that the exact solution is known and given by:

$$u_{\text{exact}}(x) = (1 + x)e^x. \tag{3.8}$$

We apply the Chebyshev wavelet collocation method to approximate the solution $u(x)$, using three different values of n, namely $n = 2$, $n = 4$, and $n = 8$.

In this comparison, we focus on the approximate solutions, for each value of n. The results are presented in Table (3.2) and Figure (3.2):

x	Error($n=2$)	Error($n=4$)	Error($n=8$)
0	0.17351	0.0010545	1.7278e-09
0.2	0.016737	2.6761e-05	3.2822e-11
0.4	0.017151	1.1008e-05	4.7505e-11
0.6	0.029655	8.5255e-06	5.7205e-07
0.8	0.030718	6.8804e-05	1.9803e-07
1	0.37179	0.0020123	5.1506e-05

Table 3.2: Results for Example 3.2

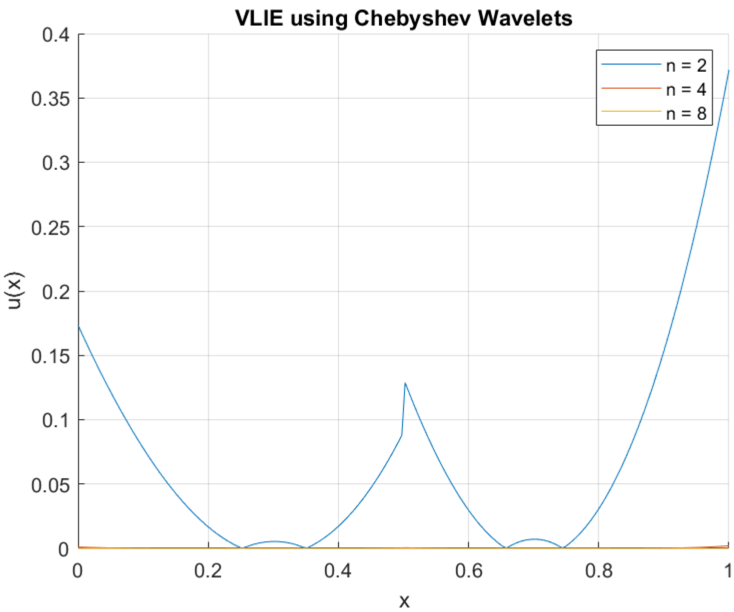

Figure 3.2: Absolute Errors for Example 3.2

Example 3.3 Let us consider the following Volterra integral equation of the second kind:

$$u(x) = 1 + \int_0^x (t-x)u(t)\,dt, \tag{3.9}$$

where that the exact solution is given by:

$$u_{\text{exact}}(x) = cos(x). \tag{3.10}$$

We apply the Chebyshev wavelet collocation method to approximate the solution. To assess the accuracy of our method, we computed the error for $n = 2$, $n = 4$, and $n = 8$. A detailed comparison is presented in the following table.

x	Error($n = 2$)	Error($n = 4$)	Error($n = 8$)
0	0.039185	0.00014852	1.393e-10
0.2	0.00045313	3.5396e-06	6.2861e-13
0.4	0.0010186	8.0389e-06	5.3868e-13
0.6	0.0015985	7.8046e-06	1.3062e-08
0.8	0.0021755	6.3074e-06	5.7673e-08
1	0.025023	0.00010371	7.2155e-07

Table 3.3: Results for Example 3.3

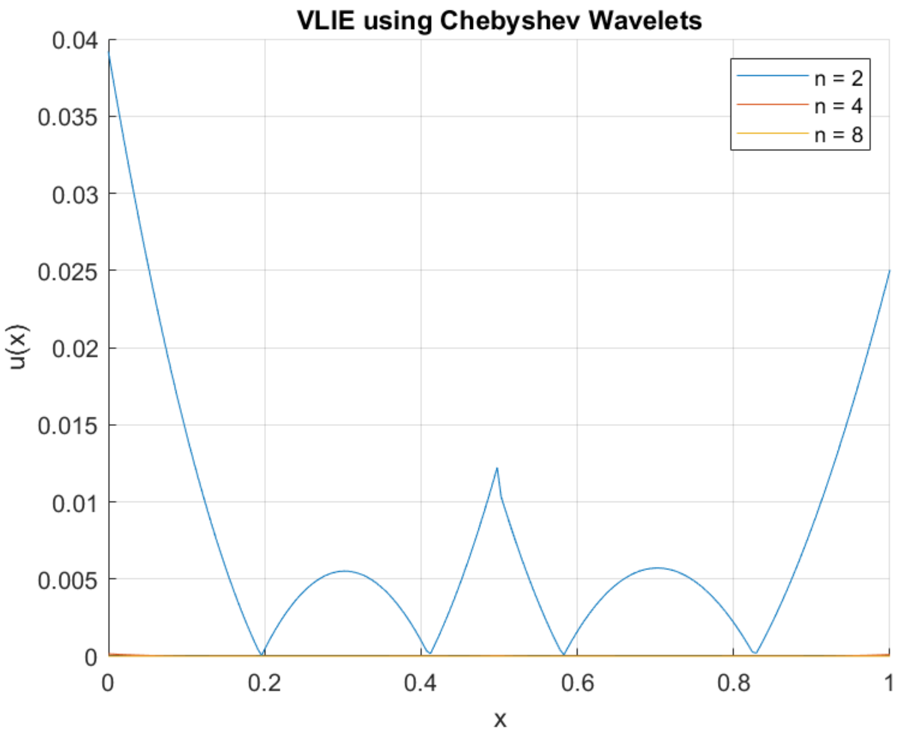

Figure 3.3: Absolute Errors for Example 3.3

Example 3.3

We consider here the Volterra integral equation:

$$u(x) = 5x^3 - x^5 + \int_0^x tu(t)\,dt. \tag{3.11}$$

where that the exact solution is given by:

$$u_{\text{exact}}(x) = 5x^3, \tag{3.12}$$

We apply the Chebyshev wavelet collocation method to approximate the solution $u(x)$, using three different values of n, namely $n = 2$, $n = 4$, and $n = 8$. The results are presented in the following table(3.4) and figure(3.4).

x	Error($n = 2$)	Error($n = 4$)	Error($n = 8$)
0	0.24321	8.3267e-17	1.2548e-15
0.2	0.0014271	6.9389e-18	6.9389e-18
0.4	0.00036016	0	5.5511e-17
0.6	0.010486	1.2766e-07	5.5658e-09
0.8	0.0014811	3.5257e-07	2.1078e-07
1	0.95248	3.1808e-06	1.7683e-05

Table 3.4: Results for Example 3.4

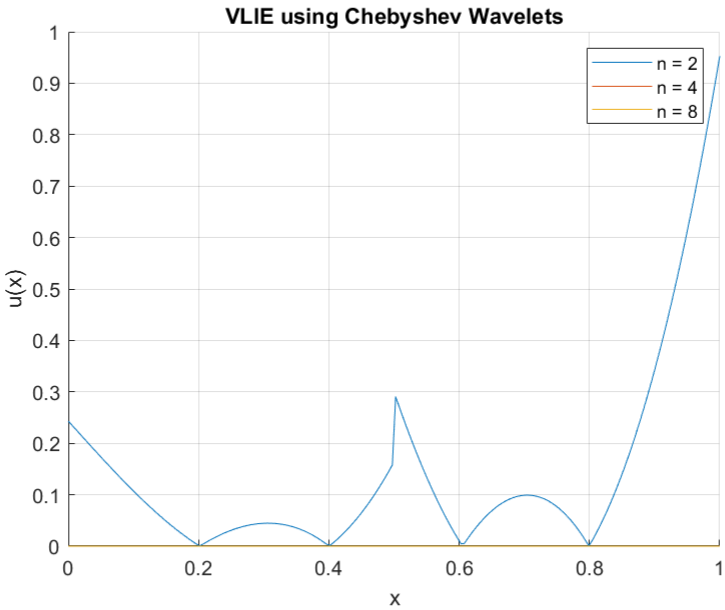

Figure 3.4: Absolute Errors for Example 3.4

Conclusion

In this work, we have successfully studied the numerical solution of linear Volterra integral equations of the second kind using the **Chebyshev wavelet collocation method** by employing **Chebyshev polynomials** as basis functions.

The method was applied to several illustrative examples **The errors were computed for different values of** n, and it was observed that as n increases, the error decreases significantly. This confirms the efficiency and high accuracy of the method, even for relatively small values of n.

This study highlights the potential of the Chebyshev-based approach in solving integral equations. Future work could focus on extending this approach to more complex problems, such as nonlinear or multi-dimensional integral equations, or combining it with adaptive techniques to improve convergence and reduce computational cost.

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ملخص

يهدف هذا العمل إلى دراسة والحل العددي لمعادلات فولترا التكاملية من النوع الثاني. تعتمد الطريقة المقترحة على تقريب الدالة المجهولة باستخدام **مويجات تشيبيشيف** وتطبيق **طريقة التجميع** لتحويل المعادلة التكاملية المعقدة إلى نظام من المعادلات الجبرية الخطية يسهل حله كمبيوترياً. وقد تم إدراج عدة أمثلة عددية لتوضيح مدى دقة وفعالية هذه التقنية ومقارنتها بالحلول الدقيقة.

كلمات مفتاحية

معادلات فولترا التكاملية، طريقة التجميع، مويجات تشيبيشيف، التقريب العددي، النظام الجبري الخطي.

Abstract

This work focuses on the study and numerical solution of second-kind Volterra integral equations. The proposed approach is based on approximating the unknown function using **Chebyshev wavelets**, combined with the **collocation method** to transform the complex integral equation into a system of linear algebraic equations that can be easily solved. Several numerical examples are presented to demonstrate the accuracy and efficiency of this technique compared with exact solutions.

Key words

Volterra integral equations, Collocation method, Chebyshev wavelets, Numerical approximation, Linear algebraic system.

Résumé

Ce travail porte sur l'étude et la résolution numérique des équations intégrales de Volterra du deuxième genre. La méthode proposée repose sur l'approximation de la solution inconnue par les **ondelettes de Chebyshev**, combinée avec la **méthode de collocation** afin de transformer l'équation intégrale complexe en un système d'équations algébriques linéaires facile à résoudre. Plusieurs exemples numériques sont présentés pour illustrer la précision et l'efficacité de cette technique en la comparant aux solutions exactes.

Mot-clés

Équations intégrales de Volterra, méthode de collocation, ondelettes de Chebyshev, approximation numérique, système algébrique linéaire.