بسم الله الرحمن الرحيم

PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

University Mohamed Boudiaf-M'sila
Department of Mathematics
Faculty of Mathematics and Computer Science

Master of Mathematics

Domain : Mathematics and Computer Science
Sector : Mathematics
Option : Functional Analysis

Theme

Free Banach Spaces

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University year 2018/2019

Dedication

To my Parents

Acknowledgements

In The Name of Allah The Beneficent, The Mo

I first thank my God who gave me the strength to complete this modest work. I would like to thank my supervisor Pr. MEZRAG Lahcene for what he has done to me by presenting this subject, his encouragement and his patience. Pr. MOUSSAI Madani, Pr. ACHOUR Dahman, and Pr. DRIHEM Douadi, and all teachers, whom I have the honor of studying, I also shared their observations and suggestions that enabled me to distort this modest work. I would like to them my gratitude and deep gratitude. I would also to thank all members of the jury for their honor by agreeing to judge this work. I can close my thanks without resorting to the beings that are my dearest, my family that has a fundamental and continuing role in my life. Thank you.

❖ *Bouatia Salim*
M'sila June 20, 2019

Notations

$\mathrm{Lip}_0(X) = X^{\#}$	Space of all Lipschitz functions from X into $\mathbb{R}$ which vanish at e
$\mathrm{Lip}(X)$	Space of bounded Lipschitz functions from X into $\mathbb{R}$
(X, ρ, e)	Metric space marked
$\mathcal{M}_0(X)$	The class of complete pointed metric spaces
Lip(.)	Norm of Lip_0 space
$\mathcal{F}(X)$	Free Lipshitz space
Æ(X)	Arens Eells space
$B(X)$	Free Banach space

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Introduction

If X is a pointed metric space, i.e., it is equipped with a distinguished "base point" e, then we define $\mathrm{Lip}_0(X)$ to be the space of all (possibly unbounded) Lipschitz functions from X into $\mathbb{R}$ which vanish at e, with norm $\mathrm{Lip}(.)$. The closed unit ball of this space being compact for the topology of pointwise convergence on X, $\mathrm{Lip}_0(X)$ has a canonical predual. Every $\mathrm{Lip}_0(X)$ space is a dual Banach space. This was first shown by Arens-Eells $Æ(X)$; but was later rediscovered in varying degrees of generality by a number of authors. We know of two essentially different methods of constructing the predual. We termed the natural preduals "Arens-Eells spaces", but in their celebrated paper [GK03] Godefroy and Kalton renamed them "Lipschitz-free spaces" $\mathcal{F}(X)$. As a result these spaces now go by two different names in the literature.

The goal of the present memory is to show that $Æ(X)$ is a isometric to $\mathcal{F}(X)$ for any X in $\mathcal{M}_0$ and we recall of the Pestove's theorem for uniqueness.

We now describe the contents of this thesis. Chapter 1 gathers general results about metric marked space and Lipschitz spaces. The main result in this chapter is the theorem 1.2 (Nonlinear Hahn-Banach theorem), and we refer a correspondence between linear and nonlinear case. Chapter 2 content constriction of the predual space Arens-Eells $Æ(X)$ by completion of the molecule space, and we show the theorem of Dixmier-Ng (theorem 2.2), that ensures a presence of conjugate space. In chapter 3 we begin by definition of Dirac map, and we refer the span of $\delta(X)$ that result free Lipschitz space $\mathcal{F}(X)$, and we ended by chapter 4 which contains theorem of Pestov and proprieties of free Banach space.

This memory was based on the work of Godefroy-Kalton, and the excellent book of Nik weaver.

CHAPTER 1

The Space $\mathbf{Lip}_0(X)$

Let X be a metric space and let e be a distinguished element of X. Then the Lipschitz space $\text{Lip}_0(X)$ is the set of all scalar-valued Lipschitz functions on X which vanish at e. With the norm of such a function being its Lipschitz number. The space $\text{Lip}(X)$ of all bounded scalar-valued Lipschitz functions is also available; with an appropriate norm this too is a Banach space.

1.1 Lipschitz Functions

Metric Spaces. The notion of metric spaces was formalized by Maurice Fréchet in his thesis "Doctorat d'Etat" in 1906 (see, "Sur quelques points du calcul fonctionnel", Rendic. Circ. Mat. Palermo 22 (1906) 1 74) and was among the first who used the word space. A good reference for this is the book of weaver [Wea99].

Definition 1.1. *Let X be a non empty set. We say that ρ is a distance on X if ρ is an application from X^2 into $\mathbb{R}$ such that for all x, y, z in X, we have*

1. $\rho(x,y) = 0 \Leftrightarrow x = y \quad (separation)$,
2. $\rho(x,y) = \rho(y,x) \quad (symmetry)$,
3. $\rho(x,z) \leq \rho(x,y) + \rho(y,z) \quad (triangular \quad inequality)$.

The space X equipped with ρ is called metric space (X, ρ).

Definition 1.2. *A metric space (X, ρ) is called discrete if there exists a constant $\delta > 0$ such that*

$$\forall x_1, x_2\,(x_1 \neq x_2) \in X, \quad \rho\,(x_1, x_2) \geq \delta.$$

A discrete metric space X is called locally finite if for every $a \in X$ and every $r > 0$ the set $\{x \in X : \rho(x,a) \leq r\}$ is finite.

Example 1.1. *The set $\mathbb{N}$ with the distance $\rho\,(n,m) = |n - m|$ is discrete.The set $\mathbb{N}$ withe discrete distance is not locally finite.*

Let $(X, \rho, e\ )$ be a pointed metric space, i.e., a metric space (*X*,*ρ*) with a distinguished or neutral element *e* (a fixed point in *X* which is taken to be the zero element if *X* is a normed space). We denote by $\mathcal{M}_0$ the class of complete pointed metric spaces.

Definition 1.3. *A metric space (X, ρ) is called metrically convex if for all $x_1, x_2 \in X$ and $0 < t < 1$, there is $x_t \in X$ such that*

$$\rho(x_0, x_t) = t\rho(x_1, x_2) \quad and \quad \rho(x_1, x_t) = (1-t)\rho(x_1, x_2).$$

1.1.1 Lipschitz functions

The natural morphism between metric spaces are Lipschitz functions like linear operators between Banach spaces. In mathematical analysis, Lipschitz continuity, named after Rudolf Lipschitz, is a strong form of uniform continuity for functions.

Definition 1.4. *A map $f : (X,\rho_X) \longrightarrow (Y,\rho_Y)$ between two metric spaces is called Lipschitz if there is a positive constant C such that*

$$\forall x, y \in X, \quad \rho_Y(f(x), f(y)) \leq C\rho_X(x, y). \tag{1.1}$$

If $C = 1$, the map is called nonexpansive (and contraction if $C < 1$).
For a Lipschitz map f, we define its Lipschitz constant by

$$\|f\|_{Lip} = Lip(f) := \sup_{x \neq y} \frac{\rho_Y(f(x), f(y))}{\rho_X(x, y)} = \inf \{C : C \text{ verifying (1.1)}\}.$$

Let $(X,e_X,\rho_X),(Y,e_Y,\rho_Y)$ be pointed metric spaces. We say a map $f : (X,e_X,\rho_X) \longrightarrow (Y,e_Y,\rho_Y)$ preserves distinguished point if $f(e_X) = e_Y$.

Definition 1.5. *Let $(X,\rho_X),(Y,\rho_Y)$ be two metric spaces. A map $f : (X,\rho_X) \longrightarrow (Y,\rho_Y)$ is called bi-Lipschitz or quasi-isometry, if f is bijective (one-to-one and onto) and both f, f^{-1} are Lipschitz.*

In this case X and Y are called

1. Lipschitz isomorphic or Lipschitz homeomorphic (Nigel Kalton) or
2. Quasi-isometric (Nik Weaver).

A bi-Lipschitz function f is an isometry if

$$\forall x, y \in X, \rho_Y(f(x), f(y)) = \rho_X(x, y).$$

Let X, Y be (finite) metric spaces, ($|X| = |Y|$). The Lipschitz distance between X, Y is $\rho(X, Y) = inf\,\{\text{Lip}(f)\text{Lip}(f^{-1}), f \text{ bi-Lipschitz } X \text{ onto } Y\}$ where $|X|$ denotes cardinal of X. It is not a distance like the usual distance but $ln\,(\rho)$ is a distance.

In the theory of the nonlinear geometry of Banach spaces, the linear isomorphisms are replaced by bi-Lipschitz maps, the isometric isomorphism correspond exactly isometric and the Banach-Mazur distance by the Lipschitz distance or distortion.

Definition 1.6. *Let $(X,\rho),(Y,\rho')$ be two metric spaces. We call Lipschitz embedding of X into Y any application f from X into Y such that there are constants $C_1, C_2 > 0$ that for any x, y in X, we have*

$$\frac{1}{C_2}\rho(x, y) \leq \rho'(f(x), f(y)) \leq C_1\rho(x, y).$$

The smallest constant C_1 is called expansion or Lipschitz constant and the smallest constant C_2 is called contraction. The distortion of the embedding is C_1C_2. If $C_1 = C_2 = 1$ the embedding is an isometry.

Proposition 1.1

Let X, Y and Z be metric spaces and let $f : (X,\rho_X) \longrightarrow (Y,\rho_Y), g : (Y,\rho_Y) \longrightarrow (Z,\rho_Z)$ be Lipschitz maps. Then $g \circ f : (X,\rho_X) \longrightarrow (Z,\rho_Z)$ is Lipschitz and $\text{Lip}(g \circ f) \leq \text{Lip}(g)\text{Lip}(f)$.

Proof. For x, y in X, we have

$$\begin{aligned}\rho_Z\,(g \circ f(x), g \circ f(y) &\leq \text{Lip}(g)\rho_Y\,(f(x), f(y)) \\ &\leq \text{Lip}(g)\text{Lip}(f)\rho_X\,(x, y)\end{aligned}$$

and this shows the proposition. ■

Theorem 1.1

Let X_0, Y_0 be metric spaces and let X, Y be their completions. Let $f_0 \colon X_0 \longrightarrow Y_0$ be Lipschitz. Then f has a unique Lipschitz extension $f \colon X \longrightarrow Y$ such that $\text{Lip}(f) = \text{Lip}(f_0)$.

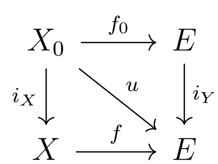
Proof. Since Lipschitz functions are continuous and X_0 is dense in X, there is at most one Lipschitz extension. Consider x in $X \diagdown X_0$ and put

$$f(x) = \lim_n f_0(x_n)$$

where x is a Cauchy sequence in X_0 such that $x_n \longrightarrow x$. We have Lip (f) = Lip (f_0). Indeed

$$\begin{aligned}\rho_Y(f(x), f(y)) &= \rho_Y(\lim_n f_0(x_n), \lim_n f_0(y_n)) \\ &= \lim_n \rho_Y(f_0(x_n), f_0(y_n)) \\ &\leq \lim_n \text{Lip}(f_0)\rho_X(x_n, y_n) \\ &\leq \text{Lip}(f_0)\rho_X(x, y).\end{aligned}$$

This implies that $\text{Lip}(f) \leq \text{Lip}(f_0)$. For the converse, consider the following diagram.



and we have in the first part

$$\begin{aligned}\text{Lip}(i_Y \circ f_0) &= \sup_{x \neq y} \frac{\rho_Y(i_Y \circ f_0(x), i_Y \circ f_0(y))}{\rho_{X_0}(x,y)} \\ &= \sup_{x \neq y} \frac{\rho_{Y_0}(f(x), f(y))}{\rho_{X_0}(x,y)} \\ &= \text{Lip}(f_0)\end{aligned}$$

and in the second part

$$\text{Lip}(i_Y \circ f_0) = \text{Lip}(f \circ i_X) \leq \text{Lip}(f).$$

This implies that $\text{Lip}(f_0) \leq \text{Lip}(f)$ and this completes the proof. ■

Proposition 1.2

Let (X, ρ) be metric space. For Lipschitz functions $f, g\colon (X, \rho) \longrightarrow \mathbb{R}$ and scalar $a \in \mathbb{R}$, the Lipschitz constant has the properties

(a) $\text{Lip}(f + g) \leq \text{Lip}(f) + \text{Lip}(g)$

(b) $\text{Lip}(af) = |a|\text{Lip}(f)$

(c) $\text{Lip}(\min(f, g) or \max(f, g)) \leq \max(\text{Lip}(f), \text{Lip}(g))$

where $\min(f, g)$ (resp. $\max(f, g)$) denotes the pointwise minimum (resp. maximum) of the functions f and g.

Proof. (a) and (b) are obvious. For (c) let $h = max(f, g)$ and fix x, y in X.
Let $C = max(\text{Lip}(f), \text{Lip}(g))$. Without loss of generality suppose $h(x) \geq h(y)$ and $h(x) = f(x)$. Then

$$h(x) - h(y) \leq f(x) - f(y) \leq C\rho(x, y).$$

Taking the sup over x, y in X we obtain $\text{Lip}(g) \leq C$. From the formula
$\min(f, g) = -\max(-f, -g)$ we get the second inequality. ■

Proposition 1.3

Let X, Y be metric spaces and let f and $\{f_n\}_{n \in \mathbb{N}}$ be Lipschitz functions from X to Y. Suppose that $f_n \longrightarrow f$ pointwise. Then

$$\text{Lip}(f) \leq \sup_n \text{Lip}(f_n).$$

Proof. Let x, y be in X. We have

$$\begin{aligned}
\rho_Y(f(x), f(y)) &= \lim_{n\to+\infty} \rho_Y(f_n(x), f_n(y)) \\
\frac{\rho_Y(f(x), f(y))}{\rho_X(x,y)} &= \lim_{n\to+\infty} \frac{d_Y(f_n(x), f_n(y))}{d_X(x,y)} \\
\sup_{x\neq y}\frac{\rho_Y(f(x), f(y))}{\rho_X(x,y)} &= \sup_{x\neq y}\lim_{n\to+\infty} \frac{\rho_Y(f_n(x), f_n(y))}{\rho_X(x,y)} \\
&\leq \sup_{x\neq y}\sup_{n} \frac{\rho_Y(f_n(x), f_n(y))}{\rho_X(x,y)}
\end{aligned}$$

by permitting the sup, we obtain the result. ■

Corollary 1.1

$\sum_{n\geq 0} f_n$ converges poitwise then $\text{Lip}\left(\sum_{n\geq 0} f_n\right) \leq \sum_{n\geq 0} \text{Lip}\,(f_n)$.

Proof. Let $g_n = \sum_{i=1}^{n} f_i$ and $f = \sum_{n\geq 0} f_n$. Then $g_n \longrightarrow f$ pointwise and $\text{Lip}\,(g_n) \leq \sum_{i=1}^{n} \text{Lip}\,(f_i)$. So by Proposition 1.3 we have

$$\begin{aligned}
\text{Lip}(f) &\leq \sup \text{Lip}(g_n) \\
&\leq \sum_{i=1}^{\infty} \text{Lip}\,(f_i)
\end{aligned}$$

and this ends the proof. ■

Proposition 1.4

Let X be a metric space and Let $f, g\colon X \longrightarrow \mathbb{R}$ be a Lipschitz maps. Then

(a) $\text{Lip}(fg) \leq \|f\|_\infty \text{Lip}(g) + \|f\|_\infty \text{Lip}(g)$.

(b) $\text{Lip}(\frac{1}{f}) \leq \dfrac{\text{Lip}(f)}{\epsilon^2}\ ,\ if\ \ |f(x)| \geq \epsilon > 0$ for all $x \in X$.

If $diam(X) < \infty$, then the product of any two scalar valued Lipschitz functions is Lipschitz .

Proof. (a) For all $x, y \in X$, we have

$$\begin{aligned}
|fg(x) - fg(y)| &\leq |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)| \\
&\leq \|f\|_\infty \text{Lip}(g) + \|g\|_\infty \text{Lip}(f).
\end{aligned}$$

(b) For all $x, y \in X$, we have

$$\begin{aligned}\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| &= \frac{|f(x) - f(y)|}{|f(x)f(y)|} \\ &\leq \frac{1}{\epsilon^2}\text{Lip}(f)\rho(x, y).\end{aligned}$$

Then Lip $\left(\frac{1}{f}\right) \leq \frac{\text{Lip}(f)}{\epsilon^2}$. ■

Theorem 1.2. Nonlinear Hahn-Banach theorem, McShane-Whitney extension theorem

Let E be a subset of a metric space (X, ρ) and let $f\colon E \longrightarrow l_\infty(I)$ be a Lipschitz function . Then f can be extended to a Lipschitz function $\tilde{f}\colon X \longrightarrow l_\infty(I)$ with the some Lipschitz constant (we say that $l_\infty(I)$ is 1-injective).

Proof. By considering each coordinate separately, it suffices to prove that for $\mathbb{R}$ instead of $l_\infty(I)$. Fix z in $X - E$. We must find a value for $\tilde{f}(z)$ such that for all $x \in E$

$$\left|\tilde{f}(z) - f(x)\right| \leq \text{Lip}(f)\rho(x, y), \qquad \forall x \in E$$

or equivalently

$$f(y) - \text{Lip}(f)\rho(y, z) \leq \tilde{f}(z) \leq f(x) + \text{Lip}(f)\rho(x, z), \qquad \forall x \in E$$

hence

$$\sup_{y \in E}\left(f(y) - \text{Lip}(f)\rho(y, z) \leq \tilde{f}(z)\right) \leq \inf_{x \in E}\left(f(x) + \text{Lip}(f)\rho(x, z)\right)$$

it is possible because for all x, yin E, we have

$$f(x) - f(y) \leq \text{Lip}(f)\rho(x, y) \leq \text{Lip}(f)\left(\rho(x, z) + \rho(y, z)\right)$$

we put

$$\tilde{f}(z) = \inf_{x \in E}\left(f(x) + \text{Lip}(f)\rho(x, z)\right)$$

and Zorn's lemma end the proof.

Direct proof see [Nao15]. Define the function $\tilde{f}\colon X \longrightarrow \mathbb{R}$ by the formula

$$\tilde{f}(z) = \inf_{x \in E}\left(f(x) + \text{Lip}(f)\rho(x, z)\right), \qquad z \in X.$$

To see that this function satisfies the results, fix any arbitrary $x_0 \in E$, then for any $x \in E$

$$f(x_0) - f(x) \leq \text{Lip}(f)\rho(x, x_0)$$

$$\leq \text{Lip}(f)\rho(x_0, z) + \rho(x, z).$$

This implies (that $f(x) + \text{Lip}(f)\rho(x, z)$ is bounded bellow).

$$f(x_0) - \text{Lip}(f)\rho(x_0, z) \leq f(x) + \text{Lip}(f)\rho(x, z).$$

So $\tilde{f}(z)$ is well defined. Also, if $z \in E$ the above shows that $\tilde{f}(z) = f(z)$. Finally (by the definition of the inf), for $x, y \in X$ and $\epsilon > 0$, choose $x_z \in E$ such that

$$\tilde{f}(z) \geq f(x_z) + \text{Lip}(f)\rho(x_z, z) - \epsilon$$

$$-\tilde{f}(z) \geq -f(x_z) - \text{Lip}(f)\rho(x_z, z) + \epsilon.$$

Then

$$\tilde{f}(y) - \tilde{f}(z) \leq f(x_z) + \text{Lip}(f)\rho(x_z, y) - f(x_z) - \text{Lip}(f)\rho(x_z, z) + \epsilon$$
$$\leq \text{Lip}(f)\rho(y, z) + \epsilon.$$

Thus, we see that $\tilde{f}$ is indeed Lip(f)-Lipschitz. ■

Remark 1.1. *The correspondence between linear and nonlinear case*

Linear	*Lipschitz*
Banach space	*metric space*
isometric	*isomorphism isometric*
topological	*isomrphism bi-Lipschitz or quasi-isometric*
Banach-Mazur distance	*Lipschitz distance or distortion*

1.1.2 Retract spaces

The notion of Lipschitz retract in metric spaces is like the linear projection in Banach spaces .

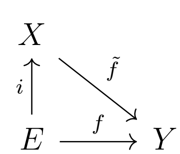
Definition 1.7. *Let X be a metric space and let E be a subspace of X. A Lipschitz map $p\colon X \longrightarrow E$ is called a Lipschitz retraction if $p \backslash E = Id$. In this case, we say that E is a Lipschitz retract of X. A metric space E is called an absolute Lipschitz retract if it is a Lipschitz retract of every metric space containing it.*

Proposition 1.5

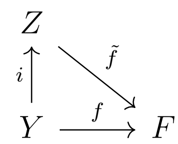
Let Y be a metric space. Then, the following properties are equivalent.

(i) The space Y is an absolute retract space.

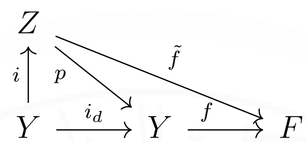
(ii) For every metric space X , for every subset $E \subset X$ and for every Lipschitz function $f\colon E \longrightarrow Y$ can be extended to a Lipschitz function $\tilde{f}\colon X \longrightarrow Y$



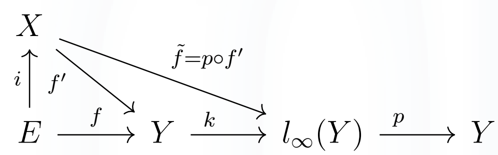
(iii) For every metric space Z containing Y and for every metric space F, then every Lipschitz function $f\colon Y \longrightarrow F$ can be extended to a Lipschitz function $\tilde{f}\colon Z \longrightarrow F$



Proof. (iii) or $(ii) \implies (i)$ We take $F = Y$ and $f = id_Y$ or $E = Y$ and $f = id_Y$
and $(i) \implies (iii)$ $\tilde{f} = f \circ p$ is the extension by the following diagram



$(i) \implies (ii)$ Y can be regarded as a subspace of $l_\infty(Y)$. Hence there is a Lipschitz retraction $p\colon l_\infty(Y) \longrightarrow Y$.
Let $k \circ f\colon E \longrightarrow l_\infty(Y)$ be a Lipschitz function .By the previous proposition, there is a Lipschitz extension $\tilde{f}\colon X \longrightarrow l_\infty(Y)$. If we take $\tilde{f} = p \circ f'$, we prove this implication.



and we end the proof of the proposition. ∎

1.2 Lipschitz Spaces

Definition 1.8. *[Wea99] Let (X, ρ) be a metric space. Then Lip (X) is the space of all scalar valued Lipschitz functions on X with the norm*

$$\|f\|_L = \max\{\|f\|_\infty, Lip(f)\}.$$

Let now (X, ρ, e) be a pointed metric space with a distinguished "base point" e which is fixed in advance. We denote by $Lip_0(X)$ the space of all scalar valued Lipschitz mappings on X; vanishing at e with the norm

$$Lip(f) := \sup_{x \neq y} \frac{\rho_Y(f(x), f(y))}{\rho_X(x, y)}.$$

The spaces $Lip(X)$ and $Lip_0(X, Y)$ become Banach spaces. We put

$$X^{\#} = Lip_0(X) = Lip_0(X, \mathbb{R}).$$

This Banach space of Lipschitz functions is called also Lipschitz dual. It has been used by various mathematicians as a framework to extend results from linear functional analysis to the nonlinear case.

Example 1.2.

$$l_\infty(X) = \left\{ f\colon X \longrightarrow \mathbb{K} \quad such \quad that \quad \sup_{x\in X}|f(x)| < \infty \right\}.$$

Let X be a pointed metric space of finite diameter, i.e., $\sup_{x,y\in X} \rho(x,y) < \infty$.

We show that $Lip_0(X) \subset l_\infty(X)$*. We have* $Lip_0(f) := \sup_{x\neq y} \frac{|f(x) - f(y)|}{\rho_X(x,y)}$*. This implies that by taking* $y = 0, |f(x)| \leq Lip_0(f)\rho(x,0)$*. Consequently* $f \in l_\infty(X)$*.*

Remark 1.2.

1. *Lip (.) is only a seminorm, not a norm on* $Lip(X)$ *.*
2. *Consider the set of all real-valued Lipschitz functions modulo the set of constant functions. Lip (.) descends to a norm on this quotient space and it is not hard to see that the result is isometrically isomorphic to* $Lip_0(X)$ *(regardless of the choice of base point). With this procedure there is no good way to define products or a partial order on the quotient.*
3. *The space* $Lip_0(X)$ *does not depend on the choice of base point. If* x_0 *and* $x_0^{'}$ *are two different distinguished elements, then the linear map*

$$\varphi\colon Lip_0(X, x_0) \longrightarrow Lip_0(X, x_0^{'})$$

$$f \longmapsto f - f(x_0^{'})$$

is a surjective isometry. However, in general φ *is not compatible with products and fails to preserve the partial order.*

Example 1.3. *For any* $f \in L_\infty([0,1], dx)$ *define*

$$F(t) = \int_0^t f(x)dx.$$

Then for any $a, b, (a \leq b) \in [0,1]$*, we have*

$$|F(b) - F(a)| = \left| \int_a^b f(x)dx \right| \leq \|f\|_\infty (b-a).$$

Thus F *is Lipschitz and* $Lip(F) \leq \|f\|_\infty$*. Moreover* $F(0) = 0$ *and hence* $F \in Lip_0([0,1])$*. Conversely, every Lipschitz function on* $[0,1]$ *is absolutely continuous and there for both differentiable and up to a* *constant, equal to the integral of its derivative. Furthermore, if f is the derivative of F then it is easy to see from the definition of the derivative that $\|f\|_\infty \leq Lip(F)$. This means that the map*

$$\phi\colon L_\infty([0,1],dx) \longrightarrow Lip_0([0,1])$$

$$f \longmapsto F$$

is an isometric isomorphism. This map preserves the partial order in the sense that $f \leq g$ implies $F \leq G$, but not conversely. It is not compatible with produce.

Definition 1.9. *Consider X, Y in $\mathcal{M}_0$ and let $T\colon X \longrightarrow Y$ be a Lipschitz map which preserve base point. We define $T^{\#}\colon Lip_0(X) \longrightarrow Lip_0(Y)$ by*

$$T^{\#}(g)(x) = (g \circ T)(x) = g(T(x)).$$

The definition make sens by the property of composition maps.

Proposition 1.6

Consider X, Y in $\mathcal{M}_0$ and let $T\colon X \longrightarrow Y$ be a Lipschitz map which preserve base point. Then $T^{\#}$ is a bounded linear map and $\|T^{\#}\| = \text{Lip}(T)$. The map $T^{\#}$ is compatible with products and preserves order .

Proof. We have

$$\text{Lip}\left(T^{\#}(g)\right) = \text{Lip}\left((g \circ T)\right) \leq \text{Lip}(g)\text{Lip}(T)$$

so $\|T^{\#}\| \leq \text{Lip}(T)$. For the converse inequality fix $p, q \in Y$ let $g = \rho_Y(., p) - \rho_Y(e_Y, q)$, then $\text{Lip}(g) = 1$ and

$$\begin{aligned}
\|T^{\#}\| &\geq \text{Lip}\left(T^{\#}(g)\right) \\
&\geq \frac{\left|T^{\#}(g)(x) - T^{\#}(g)(y)\right|}{\rho_X(x,y)} \\
&\geq \frac{|gT(x) - gT(y)|}{\rho_X(x,y)} \\
&\geq \frac{|gT(x) - gT(y)|}{\rho_Y\left(T(x), T(y)\right)} \frac{\rho_Y\left(T(x), T(y)\right)}{\rho_X(x,y)}.
\end{aligned}$$

Taking the supremum over x and y, we find $\|T^{\#}\| \geq \text{Lip}(T)$. ■

Remark 1.3. *We have $(T_1 T_2)^{\#} = T_1^{\#} T_2^{\#}$ and if $T_1 \leq T_2$ implies $T_1^{\#} \leq T_2^{\#}$.*

Proposition 1.7

Consider X, Y in $\mathcal{M}_0$ and let $T\colon\ X \longrightarrow Y$ be a Lipschitz map which preserve base point.Then

1. $T^{\#}$ is surjective if, and only if, $T\colon\ X \longrightarrow T(X)$ is a quasi isometry.
2. $T^{\#}$ is injective if, and only if, $T(X)$ is dense in X.
3. $T^{\#}$ is isomorphism if, and only if, T is a quasi isometry.

CHAPTER 2

Arens-Eells space

We shall present first the construction of Arens and Eells [AE56] (see also[Wea99, p. 38]) of the space for which $\mathrm{Lip}_0(X)$ is the dual space. Remark that another, less explicit, realization of $\mathrm{Lip}_0(X)$ as a dual space was given by de Leeuw [Lee61] (see also [[Wea99], p. 33]). It was shown by Arens and Eells[AE56] (see also [Wea99]) that $\mathrm{Lip}_0(X)$ is even a dual Banach space (but not reflexive if X is infinite and does not have constant functions in general), i.e., there exists a Banach space Z such that $\mathrm{Lip}_0(X)$ is isometrically isomorphic to Z . This canonical space is known as the Arens-Eells space in [Wea99] and the Lipschitz-free space on X in [Kal04]. It well be noted as in by $\mathcal{F}(X, \rho_X)$. This chapter is based on the book of Nik Weaver.

2.1 Construction of this space

De Leeuw's map. Let $\mathcal{M}_0$ be the class of (complete) pointed metric spaces. For X in $\mathcal{M}_0$, $\mathrm{Lip}_0(X)$ is not a simple Banach space but, it is a dual Banach space.

Definition 2.1 (De Leeuw's map). *[Lee61] Let (X,ρ) be a metric space. Let $\tilde{X}$ be the set*

$$\left\{(x,y) \in X^2 \;:\; x \neq y\right\} = X^2 \setminus D_X.$$

For any $f : X \longrightarrow \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), define

$$(\phi f)(x,y) = \frac{f(x) - f(y)}{\rho(x,y)}.$$

De Leeuw's map is the map

$$Lip_0(X) \longrightarrow l_\infty(\tilde{X})$$
$$f \longrightarrow \phi f$$

and we have $Lip(f) = \|\phi f\|_{l_\infty(\tilde{X})}$.

Lemma 2.1

Let X be in $\mathcal{M}_0$ and $f, f_i \in \mathrm{Lip}_0(X)$ $(i \in I)$. Then $f_i \longrightarrow f$ pointwise if, and only if, $\phi f_i \longrightarrow \phi f$ pointwise.

Proof. Let (x,y) be in $\tilde{X}$. We have

$$(\phi f_i)(x,y) = \frac{f_i(x) - f_i(y)}{\rho(x,y)} \longrightarrow \frac{f(x) - f(y)}{\rho(x,y)} = (\phi f)(x,y).$$

For the converse, consider x in $\tilde{X}$. We have

$$f_i\,(x) = \rho(x,e)(\phi f_i)(x,e) \longrightarrow \rho(x,e)(\phi f)(x,e) = f(x)$$

and this end the proof. ■

Theorem 2.1

Let X be in $\mathcal{M}_0$.

1. De Leeuw's map is a linear isometry of Lip_0 into $l_\infty(\tilde{X})$.
2. We have $\phi(fg) = f\phi(g) + \phi(f)g$ for all f, g in $\mathrm{Lip}_0(X) \cap l_\infty(X)$.

3. The image $\phi(\text{Lip}_0(X))$ is weak* closed in $l_\infty(\tilde{X})$ (i.e., $\sigma\left(l_\infty(\tilde{X}), (l_1(\tilde{X})\right)$).

Proof. (a) It is trivial.
(b) Note that $diam(X) < 1$ implies $\text{Lip}_0(X) \subset l_\infty(X)$. In any case, for $x, y \in X$, we have

$$\begin{aligned}\phi(fg)(x,y) =& \frac{(fg)(x) - (fg)(y)}{\rho(x,y)} \\ =& \frac{f(x)(g(x) - g(y))}{\rho(x,y)} + \frac{g(y)(f(x) - f(y))}{\rho(x,y)} \\ =& f(x)\phi(g)(x,y) + g(x)\phi(f)(x,y).\end{aligned}$$

(c) The proof of this part uses the Krein-Smulian theorem, which states that if F is a linear subspace of a dual Banach space E^*, and $F \cap \mathcal{B}_{E^*}$ is weak*-closed, then $F$is weak*-closed is E^*. Thus, to show that a linear subspace of a dual Banach space is weak*-closed, it is sufficient to verify closure under weak*-limits of bounded nets. The weak*-topology on $l_\infty(\tilde{X})$ is actually very simple on bounded sets.It is just the of pointwise convergence. To see this, let $A = \mathcal{B}_{l_\infty(\tilde{X})}$ equipped with the relative weak*-topology, let $B = \mathcal{B}_{l_\infty(\tilde{X})}$ be the same set equipped with the topology of pointwise convergence, and let $\phi : A \longrightarrow B$ be the identity map. Thus, a net in A converges if summation against anything in $l_1(\tilde{X})$ converges in $\mathbb{F}$.Because the characteristic function of any single element of $\tilde{X}$ belongs to $l_1(\tilde{X})$ it follows that convergence in A implies convergence in B. That is,ϕ is continuous. But $A$is compact and B is Hausdorff, so ϕ must be a homeomorphism. Thus, on the unit ball of $l_\infty(\tilde{X})$ (and hence on any bounded set) the weak*-topology agrees with the topology of pointwise convergence.
Now let $(\Phi(f_i))$ be a bounded net in $\Phi(\text{Lip}_0(X))$ which is weak convergent, hence pointwise convergent. Since Φ is an isometry, (f_i) is bounded in $\text{Lip}_0(X)$, and (f_i) converges pointwise by Lemma 2.1, so its pointwise limit f also belongs to $\text{Lip}_0(X)$ by Proposition 1.3. Clearly $\Phi(f) = \lim \Phi(f_i)$, so we see that $\Phi(\text{Lip}_0(X))$ is closed under pointwise convergence of bounded nets, as desired. ■

2.1.1 Properties and Characterizations of Conjugate Spaces

For more details on this subsection, we can consult, [Hol75] (Richard B. Holmes. Geometric Functional Analysis and its Applications, Springer Verlag New York Heidelberg Berlin, 1975).
Let E be a Banach space. To say that E is a conjugate space if there exists a Banach space B such that E is isometrically isometric to B^* ($E \equiv B^*$). We shall begin by presenting a simple condition sufficient to guarantee that such a space B exists. This result will be called the "Dixmier-Ng theorem".

Theorem 2.2. Dixmier-Ng theorem

Let E be a Banach space. Suppose that there is a Hausdorff locally convex topology σ on E such that $\mathcal{B}_E$ is σ-compact. Then E is a conjugate space.

Proof. Let $B = \left\{\varphi \in E^{'} : \varphi|_{\mathcal{B}_E}\ is\ \sigma - continuous\right\}$ ($E^{'}$ = algebraic conjugate space). Then B is a closed linear subspace of E^*, and is therefore a Banach space. (To see that $B \subset E^*$ observe that for any $\varphi \in B$ the image $\varphi(\mathcal{B}_E)$ is a compact hence bounded set of scalars, that is $\|\varphi\|$ finite and so $\varphi \in E^*$. B is closed in E^* because convergence in E^* entails uniform convergence on $\mathcal{B}_E$.) We now bring in the (canonical embedding) operator $J_{E,B}\ : E \longrightarrow B^*$ defined by

$$< \varphi, J_{E,B}(x) >= \varphi(x).$$

This operator assigns to each $x \in X$ the functional "evaluation at x" in B^* . We clearly have $\|J_{E,B}(x)\| \leq 1$.The proof will be completed by showing that $J_{E,B}(x)$ is an isomorphic isometry between E and B^*. We do this by showing that $J_{E,B}(x)$ is injective and that it maps $\mathcal{B}_E$ onto $\mathcal{B}_{B^*}$.The first assertion follows because B is total. Indeed, B contains the dual space E, which certainly separates the points of E. The second assertion follows from the fact (evident by definition of B) that $J_{E,B}$ is continuous from the σ-topology on E into the weak*-topology on B^*.This means in particular that $J_{E,B}(\mathcal{B}_E)$ is w^*-compact in B^*. But, by the Goldstine-Weston density lemma , this image is also weak*-dense in $\mathcal{B}_{B^*}$. ■

Remark 2.1. *Any weak*-closed linear subspace F of a conjugate space E^* is itself a conjugate space. This follows from the observation that $\mathcal{B}_F$. is compact in the (relative) weak*-topology.*

We now give an example.

Example 2.1. *Consider the space Lip$(X, \rho, \mathbb{R})$ of bounded Lipschitz functions defined on the metric space (X, ρ) and normed by $\|.\|_L = max\left\{\|.\|_\infty, Lip(.)\right\}$. Let be σ the topology of pointwise convergence on X, which we denote by $\sigma(Lip(X, \rho, \mathbb{R}), X)$. Then $\mathcal{B}_x$ is certainly a $\sigma(Lip(X, \rho, \mathbb{R}), X)$-closed subset of X. We have*

$$\mathcal{B}_{Lip(X,\rho,\mathbb{R})} \subset [-1, 1]^X .$$

Since $[-1, 1]$ is compact by Tychonov's theorem we have $[-1, 1]^X$. Consequently, $\mathcal{B}_{Lip(X,\rho,\mathbb{R})}$ is $\sigma(Lip(X, \rho, \mathbb{R}), X)$-compact and so X is a conjugate space. Let $X^\# = Lip_0(X)$. Let $\mathcal{T}$ be the topology of pointwise convergence on $X^\#$,i.e., $\mathcal{T} = \sigma(x^\#, X)$. Then $\mathcal{B}_{X^\#}$ is certainly a $\mathcal{T}$-closed subset of $X^\#$. In addition, $\mathcal{B}_{X^\#}$ is contained in the product B^X, where $B \equiv \{\lambda \in F : |\lambda| \leq 1\}$. Consequently, $\mathcal{B}_{X^\#}$ is $\mathcal{T}$-compact and so $X^\#$ is a conjugate space.

Remark 2.2. *Do not know about $\mathcal{B}_{X^{\#}}$ everything.*

Theorem 2.3. Dixmier-Goldberg-Ruston

Let E be a Banach space. E is a conjugate space if, and only if, there is a total subspace V of E^* such that B_E is $\sigma(E, V)$-compact.

Proof. The condition for E to be a conjugate space is an immediate consequence of Alaoglu's theorem and the Dixmier-Ng theorem. ■

Lemma 2.2

Let E and F be Banach spaces and suppose that $T : E \longrightarrow F^*$ is an isometric (resp. an isomorphism) between E and F^* . Then there exists a subspace B of E^* such that $J_B : E \longrightarrow B^*$ is an isometry (resp. an isomorphism).

Proof. Let B be the range of $T^* \circ J_F$. For any $y \in F$ set $b = T^*(J_F(y))$. Then for any $x \in E$

$$\begin{aligned} < y, T(x) > &=< T(x), J_F(y) > \\ &=< x, b > \\ &=< b, J_B(x) > \\ &=< T^*(J_F(y)), J_B(x) > \\ &=< y, (T^* \circ J_F)^* \circ J_B(x) > . \end{aligned}$$

This proves that T= $(T^* \circ J_F)^* \circ J_B$ and consequently that

$$J_B = ((T^* \circ J_F)^*)^{-1} \circ T.$$

Since J_F is always an isometry, the last equality exhibits J_B as a composite of isometries (resp. of isomorphisms). Further, range (J_B) is all of B^* because $T^* \circ J_F \colon F \longrightarrow B$ is surjective, and hence so is $((T^* \circ J_F)^*)^{-1} : F^* \longrightarrow B^*$. ■

This lemma makes it clear that any reflexive space E has a unique predual, namely E^* . A closed linear subspace B of E^* is said to be minimal if it is total and no proper subspace of B is both total and closed. Also, B is said to be duxial (or norming determining) if

$$\sup \{|< x, b >| : b \in \mathcal{B}_B\} = \|x\| , \; x \in E.$$

That is, B is duxial exactly when J$_B$ is an isometry.

2.2 Arens Eells space

Theorem 2.4. [Wea99]

$\text{Lip}_0(X)$ is a dual space, for every $X \in \mathcal{M}_0$. On bounded sets the weak* topology agrees with the topologie of pointwise convergence.

Let (X, e, ρ) be pointed a metric space. A molecule on X is a real valued function m on X with finite support (i.e., the set where m has non-zero values) and satisfies

$$\sum_{x \in supp(m)} m(x) = 0$$

Denote by $\mathcal{M}(x)$ the real linear space of molecules on X. We can write

$$\begin{aligned} m &= \sum_{x \in supp(m)} m(x)\chi(x) \\ &= \sum_{i=1}^{n} m(x_i)\chi(x_i) \end{aligned}$$

where $supp(m) = \{x_1, \ldots, x_n\}$ and $\chi(x)$ denotes the characteristic function of the set $\{x\}$. For $x, y \in X$ we define the basic molecule $m_{xy} = \chi(x) - \chi(y)$(with $x, y \in X$ are called atoms). It is easy to see that every molecule m can be written as a (non unique) finite linear combination of basic molecule (the condition $\sum_{i=1}^{n} m(x_i) = 0$ insures that such representations of m exist). We have

$$\begin{aligned} m &= \sum_{j=1}^{l} \lambda_j \left(\chi(x_j) - \chi(x'_j)\right) \\ &= \sum_{j=1}^{l} \lambda_j m(x_j x'_j). \end{aligned}$$

Example 2.2. *Consider $m : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$\begin{cases} m(0) = -4 \\ m(1) = 1 \\ m(2) = 3 \\ 0 \quad otherwise \end{cases}$$

$$\begin{aligned} m &= -4\chi(0) + \chi(1) + 3\chi(2) \\ &= -3\chi(0) - 1\chi(0) + \chi(1) + 3\chi(2) \\ &= 1.\left(\chi(1) - \chi(0)\right) + 3\left(\chi(2) - \chi(0)\right) \end{aligned}$$

Put now

$$\|m\|_{\mathcal{M}(X)} = \inf\left\{\sum_{j=1}^{l} |\lambda_j|\, \rho_X\left(x_j, x_j'\right)\right\}$$

over all representation of $m = \sum_{j=1}^{l} \lambda_j \left(\chi(x_j) - \chi(x_j')\right)$.
It follows that $\|.\|_{\mathcal{M}(X)}$ is a norm on the vector space $\mathcal{M}(X)$. Denote by Æ(X, ρ_X) the completion of the normed space $(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)})$. This space was first introduced by Arens and Eells [AE56] in 1956. Originally, the basic idea goes back to Kantorovich [Kan42]. The terminology Arens-Eells space Æ(X, ρ) is due to Weaver [Wea99]. A different notation and appellation was used in [GK03] by Godefroy and Kalton. It is the Lipschitz-free space denoted by $\mathcal{F}(X)$ which we briefly introduce in the sequel. Similar space was introduced by Pestov in [Pes86] under the name free Banach space.

Remark 2.3. *Every molecule m is uniquely expressible in the form*

$$m = \sum_{j=1}^{l} \lambda_j \left(\chi(x_j) - \chi(e)\right)$$

where the points x_j are all distinct and none equals to e. Indeed, suppose that there is two representations

$$\sum_{j=1}^{l} \lambda_j \left(\chi(x_j) - \chi(x_e)\right) = \sum_{j=1}^{l} \alpha_j \left(\chi(y_j) - \chi(e)\right)$$

where $x_i \neq x_j \neq e$ and $y_i \neq y_{j_1} \neq e$. We have

$$\begin{aligned} \lambda_1 &= \alpha_{i_1} \quad x_1 = y_{j_1} \\ \lambda_2 &= \alpha_{i_2} \quad x_2 = y_{j_2} \\ \vdots &= \vdots \qquad \vdots = \vdots \\ \lambda_l &= \alpha_{i_l} \quad x_l = y_{j_l} \end{aligned}$$

We now prove that $(\text{Æ}(X))^* \overset{isometrically}{\equiv} \text{Lip}_0(X)$.

Theorem 2.5. [Wea99]

($Æ(X))^*$ is isometrically isomorphic to $\text{Lip}_0(X)$.

Proof. Define $S : Æ(X,\rho)^* \longrightarrow \text{Lip}_0(X)$ by

$$(S\varphi)(x) = \varphi\left((\chi(x) - \chi(e))\right).$$

Since $\|\chi(x) - \chi(x')\|_{Æ(X,\rho)} = \rho(x,x')$ for all $x, x' \in X$, we have

$$\begin{aligned} \|(S\varphi)(x) - (S\varphi)(x')\| &= |\varphi\left(\chi(x) - \chi(e)\right) - \varphi\left(\chi(x') - \chi(e)\right)| \\ &= |\varphi\left(\chi(x) - \chi(x')\right)| \\ &\leq \|\varphi\| \, \rho(x,x'). \end{aligned}$$

Also $(S\varphi)(e) = \varphi(0)$, so indeed $S\varphi \in \text{Lip}_0(X)$. It follows that S is a nonexpansive linear mapping from $Æ^*(X,\rho)$ to $\text{Lip}_0(X)$. Define now $R : \text{Lip}_0(X) \longrightarrow Æ^*(X,\rho)$ by

$$(Rf)(m) = \sum_x m(x) f(x)$$

for $f \in \text{Lip}_0(x)$ and m a molecule. If $m = \sum_{j=1}^{l} \lambda_j \left(\chi(x_j) - \chi(x'_j)\right)$, we have

$$\begin{aligned} |(Rf)(m)| &= \left|\sum_x m(x) f(x)\right| \\ &\leq \sum_{j=1}^{l} |\lambda_j| \left|f(x_j) - f(x'_j)\right| \\ &\leq \text{Lip}(f) \sum_{j=1}^{l} |\lambda_j| \, \rho\left(x_j, x'_j\right). \end{aligned}$$

Hence $|(Rf)(m)| \leq \text{Lip}(f) \, \|m\|_{\mathcal{M}(X)}$, which uniquely extends to a continuous linear functional on the completion $Æ(X,\rho)$ of $\mathcal{M}(x)$, denoted by the same symbol Rf. Thus $Rf \in Æ(X,\rho)$ and $\|Rf\| \leq \text{Lip}(f)$. Straightforward calculations show that R and S are inverses. Indeed, for all $x \in X$

$$\begin{aligned} (SR)(f)(x) &= S(R(f))(x) \\ &= R(f)\left(\chi(x) - \chi(e)\right) \\ &= f(x) \end{aligned}$$

and for all $m \in \mathcal{M}(X)$

$$\begin{aligned}(R \circ S)(\varphi)(m) &= R(S(\varphi))(m) \\ &= \sum_{x} m(x) S(\varphi)(x) \\ &= \sum_{j=1}^{l} \lambda_j \left(S(\varphi)(x_j) - S(\varphi)(x'_j) \right) \\ &= \sum_{j=1}^{l} \lambda_j \varphi \left(\chi(x_j) - \chi(x'_j) \right) \\ &= \varphi(m).\end{aligned}$$

The operators R, S are nonexpansive and $R \circ S = S \circ R = Id$, so S is isometric ($\|x\| = \|(R \circ S)(x)\| \leq \|R\| \, \|S(x)\| \leq \|S(x)\|$) and hence $\text{Lip}_0(X)$ is isometrically isomorphic to Æ$^*(X, \rho)$ (for more information see [Cob03]). ■

2.3 Properties

Proposition 2.1

Let (X, e, ρ) be a pointed metric space.

1. For any molecule m we have

$$\|m\|_{Æ(X,\rho_X)} = \sup \left\{ |< m, f >| = \left| \sum_{x \in X} m(x) f(x) \right| : f \in \mathcal{B}_{X^\#} \right\}$$

   and there exists $f \in \mathcal{B}_{X^\#}$ such that $< m, f >= \|m\|_{Æ(X,\rho_X)}$.

2. $\|m\|_{Æ(X,\rho_X)}$ is a norm on $\mathcal{M}(X)$ and $\|\chi(x) - \chi(y)\|_{Æ} = \rho(x, y)$ for all x, y in X.
3. $\|m\|_{Æ(X,\rho_X)}$ is the largest seminorm on $\mathcal{M}(X)$ which satisfies for all x, y in X, $\|\chi(x) - \chi(y)\|_{Æ} = \rho(x, y)$.

Proof. (1) This follows from the identification of $\text{Lip}_0(X, \rho)$ with Æ$(X, \rho_X)^*$ and the Hahn-Banach theorem.

(2) The inequality $\|\chi(x) - \chi(y)\|_{Æ} \leq \rho(x, y)$ follows from the definition. Conversly, fix x in X and define

$$f_x(y) = \rho(x, y) - \rho(x, e).$$

We have $f_x \in B_{Lip_0(X,\rho)}$ because $f_x(e) = 0$ and $\text{Lip}(f_x) = 1$. Indeed,

$$\begin{aligned}\text{Lip}(f_x) &= \sup_{y_1 \neq y_2} \frac{|f_x(y_1) - f_x(y_2)|}{\rho(y_1, y_2)} \\ &\geq \sup_{x \neq y} \frac{|f_x(y) - f_x(x)|}{\rho(y, x)} \\ &\geq \frac{\rho(x, y)}{\rho(x, y)} = 1\end{aligned}$$

and

$$\begin{aligned}\text{Lip}(f_x) &= \sup_{y_1 \neq y_2} \frac{|f_x(y_1) - f_x(y_2)|}{\rho(y_1, y_2)} \\ &\leq \sup_{y_1 \neq y_2} \frac{|\rho(x, y_1) - \rho(x, y_2)|}{\rho(y_1, y_2)} \\ &\leq \frac{\rho(y_1, y_2)}{\rho(y_1, y_2)} = 1.\end{aligned}$$

By part (1), we have

$$\begin{aligned}\|\chi(x) - \chi(y)\|_{Æ} &\geq |< m_{xy}, f_x >| \\ &\geq |m(x)f_x(x) + m(y)f_x(y)| \\ &\geq |-m(x)\rho(x, e) + m(y)\rho(x, y) + m(y)\rho(x, e)| \\ &\geq |m(y)\rho(x, y)| \\ &\geq \rho(x, y).\end{aligned}$$

(3) Let $\|.\|_0$ be any semi norm such that

$$\|\chi(x) - \chi(y)\|_0 \leq \rho(x, y)$$

for all $x, y \in X$. Let $m = \sum_{i=1}^n a_i m_{x_i y_i}$ be a molecule. We have

$$\begin{aligned}\|m\|_0 &= \left\| \sum_{i=1}^n a_i m_{x_i y_i} \right\|_0 \\ &\leq \sum_{i=1}^n |a_i| \, \|m_{x_i y_i}\|_0 \\ &\leq \sum_{i=1}^n |a_i| \, \rho(x_i, y_i).\end{aligned}$$

Taking the infimum of all such representation of m yields $\|m\|_0 \leq \|m\|_{Æ}$. ■

Proposition 2.2

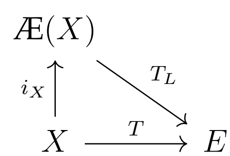
If X_0 is a subset of a metric space X containing the base point, then $Æ(X_0)$ can be identified naturally and isometrically as a linear subspace of $Æ(X)$.

Proof. Consequence of Hahn-Banach Theorem. ■

The following theorem was proved independently in [Pes86, Theorem 1] for free Banach space. It is known as the linearization of Lipschitz operators.

Theorem 2.6. [Wea99, Theorem 2.2.4]

Let (X, ρ_X, e) be a pointed metric space. Let E be a Banach space and let $T : X \longrightarrow E$ be a Lipschitz map which preserves base point (i. e., $T(e) = 0$). Then there is a unique bounded linear operator $u : Æ(X, \rho_X) \longrightarrow E$ (noted T_L) such that $T = T_L \circ i$ and $\|T_L\| =$ Lip(T) $(i_X : X \longrightarrow Æ(X, \rho_X))$.



Proof. Every molecule m is uniquely expressible in the form

$$m = \sum_{j=1}^{l} \lambda_j \left(\chi(x_j) - \chi(e)\right)$$

where the points x_j are all distinct and none equals to e. We then define T_L by

$$T_L(m) = \sum_{j=1}^{l} \lambda_j T(x_j)$$

Since T_L is essentially an extension of T that is $T = T_L \circ i$ and we automatically have $\|T_L\| \geq$ Lip(T). For the rest it will suffice to show that $\|T_L\| \leq$ Lip(T) (in particular, this implies that T_L is bounded and hence it extends to all $Æ(X, \rho_X)$). Define a semi norm $\|.\|_0$ on the space of molecules by setting

$$\|m\|_0 = \frac{\|T_L(m)\|}{\text{Lip}(T)}.$$

Then

$$\begin{aligned} \|\chi(x) - \chi(y)\|_0 =& \frac{\|T(x) - T(y)\|}{\text{Lip}(T)} \\ \leq& \rho(x, y) \end{aligned}$$

for all $x, y \in X$. This implies that $\|.\|_0 \leq \|.\|_{Æ}$ (because $\|m\|_{Æ} = \sup \left\{|< m, T >| : T \in \mathcal{B}_{\text{Lip}_0(X)}\right\}$ and the sup is attained). Thus $\|T_L(m)\| \leq \text{Lip}(T)\, \|m\|_{Æ}$, which shows that $\|u\| \leq \text{Lip}(T)$ as desired. ■

Proposition 2.3

The weak* $\left(\text{Lip}_0(X), \sigma\left(\text{Lip}_0(X), Æ(X, \rho_X)\right)\right)$ topology agrees with the topology of pointwise convergence on bounded subset of $\text{Lip}_0(X)$.

Proof. Let T_i, T be in $\text{Lip}_0(X)$ such that

$$T_i \longrightarrow T;\ \sigma\left(\text{Lip}_0(X), Æ(X, \rho_X)\right).$$

Then, for all x in X we have

$$T_i(x) = T_{iL}\left(\chi(x) - \chi(e)\right) \longrightarrow T_L\left(\chi(x) - \chi(e)\right).$$

For the converse, it is a classical result. ■

Corollary 2.1

The application $i_X : X \longrightarrow Æ(X, \rho)$ defined by $i_X(x) = \chi(x) - \chi(e) = m_{xe}$ is an isometric embedding of X into $Æ(X, \rho)$.

Proof. We have by (2) in Proposition 2.1 $\|i_X(x) - i_X(y)\| = \|\chi(x) - \chi(y)\|_{Æ} = \rho(x, y)$ for all $x, y \in X$. So i_X is an isometry. ■

Remark 2.4. *On bounded subsets of Lip*$_0(X)$ *its weak* topology (i.e.,* $\left(X^{\#}, \sigma\left(X^{\#}, , Æ(X, \rho_X)\right)\right)$ *agrees with the topology of pointwise convergence. Indeed, If* $T_i \longrightarrow T$ *for the weak*topology in Lip*$_0(X)$*, then*

$$T_i(x) = T_{iL}\left(\chi(x) - \chi(e)\right) \longrightarrow T_L\left(\chi(x) - \chi(e)\right)$$

for all x *in* X*. Hence weak*convergence implies pointwise convergence.*

Suppose (X, ρ) is a compact metric space and for $0 < \theta < 1$ we consider the metric $\rho_\theta(x, y) = \rho(x, y)_\theta$ on X. We will denote (X, ρ_θ) by X_θ; of course, Lipschitz functions on X_θ are simply θ-Hölder functions on X. In this case we can define a subspace $\text{lip}_0(X_\theta)$ to be the set of functions $f \in \text{Lip}_0(X_\theta)$ so that

$$\lim_{\tau \to 0} \sup \left\{ \frac{f(y) - f(x)}{\rho_\theta(x, y)} : 0 < \rho_\theta(x, y) < \tau \right\} = 0$$

It then follow that [Wea99].

Theorem 2.7. [Kal04]

If $0 < \theta < 1$ and X is a compact subset of a finite-dimensional normed space we have that $\text{lip}_0(X_\theta)$ is isomorphic to c_0 and Æ(X_θ) is isomorphic to l_1.

Theorem 2.8. [Kal04]

Let X be a compact convex subset of a Hilbert space containing the origin. Then Æ(X_θ) is isomorphic to l_1 if and only if X is finite-dimensional.

Theorem 2.9. [Kal04]

If K is a compact metric space Æ$(C(K))$ is isomorphic to Æ(c_0).

CHAPTER 3

Lipschitz free space

We shall refer to $\mathcal{F}(X)$ as the *Lipschitz free space over* X. We note that such a notion has been investigated before in the frame of topological vector spaces. Our purpose in considering the free spaces is to investigate the following general problem: if X and Y are Lipschitz isomorphic Banach spaces, that is, if there exists a bijective and bi-Lipschitz map $F\colon X \longrightarrow Y$, are X and Y linearly isomorphic? It is known that the answer to this question is negative in full generality, but it remains an important open problem in the separable case. We refer to the authoritative book [M. Bell and W. Marciszewski, On scattered Eberlein compact spaces, preprint, 2001] for this topic and related matters.

3.1 Definition

Introduction J.-A. Johnson in [Joh70], proved without any reference to molecules that the closed linear subspace of $(X^{\#})^{*}$ spanned by the evaluation functions $\delta_x : X^{\#} \longrightarrow \mathbb{K}$, given by

$$\delta_x(f) = f(x);\; x \in X$$

(we note that any weak$^{*}-$closed linear subspace X of a conjugate space Y is itself a conjugate space. This follows from the observation that $\mathcal{B}_X$ is compact in the (relative) weak topology) is a predual of $X^{\#}$. This space was called Lipschitz free and denoted $\mathcal{F}(X)$ by Godefroy and Kalton in [GK03]. Define

$$\delta : X \longrightarrow (X^{\#})^{*}$$
$$x \longrightarrow \delta_x$$

where $\delta_x(f) = f(x)$ .The application δ is an isometry. For $x_1, x_2 \in X$, we have in the first part

$$\begin{aligned}\|\delta_{x_1} - \delta_{x_2}\| &= \sup_{\text{Lip}(f)=1} \|\delta_{x_1}(f) - \delta_{x_2}(f)\| \\ &= \sup_{\text{Lip}(f)=1} \|f(x_1) - f(x_2)\| \\ &\leq \rho(x_1, x_2).\end{aligned}$$

In the second part, for a fixed $x_0 \in X$. let defined by

$$g(x) = \rho(x, x_1) - \rho(x_0, x_2).$$

We have

$$\begin{aligned}\|\delta_{x_1} - \delta_{x_2}\| &\geq g(x_1) - g(x_2) \\ &\geq \rho(x_1, x_2).\end{aligned}$$

The subset $\delta(X)$ is linearly independent in $(X^{\#})^{*}$ see [Mic64]. Indeed, let $x_1; \ldots; x_n; x_{n+1}$ be distinct elements of X, then $\delta_{x_{n+1}}$ cannot be a linear combination of
$\delta_{x_1}; \ldots; \delta_{x_n}$. If $g(x) = \rho\left(x, \{x_1; \ldots; x_n\}\right)$ for $x \in X$, then $g \in X^{\#}$ and

$$\begin{aligned}\delta_{x_i}(g) \; &= g(x_i)\; for 1 \leq i \leq n \\ \delta_{x_{n+1}}(g) &= g(x_{n+1}).\end{aligned}$$

This implies that $\delta_{x_{n+1}}$ cannot be a linear combination of $\delta_{x_1}; \ldots; \delta_{x_n}$ and consequently $\delta(X)$ is linearly independent in $(X^{\#})^{*}$.

We note that any weak$^*-$closed linear subspace B of a conjugate space E^* is itself a conjugate space. This follows from the observation that $\mathcal{B}_B$ is compact in the (relative) weak$^*-$topology.

$$(\overline{\text{span}(X)})^* = \mathcal{F}(X)^* = X^{\#}$$

The application $\delta_X : X \longrightarrow \mathcal{F}(X, \rho_X)$ is an isometric embedding. We can see $\mathcal{F}(X, \rho_X)$ as the completion of the set of all measures of finite support under the norm

$$\|\mu\| = \sup_{\|f\|_{\text{Lip}} \leq 1} \int f d\mu.$$

3.2 Properties

Theorem 3.1

Let $X \in \mathcal{M}_0$ be infinite. Then $\text{Lip}_0(X)$ is not reflexive.

The following proposition is due to Lindenstrauss [Lin64] when X is a Banach.

Proposition 3.1

If X is a Banach space then there is a norm one projection p from $\text{Lip}_0(X)$ onto its subspace X^* .

Proposition 3.2. [GK03]

Let $(X_1, \rho_{X_1}); (X_2, \rho_{X_2})$ be two pointed metric spaces.
Let $T : (X_1, \rho_{X_1}) \longrightarrow (X_2, \rho_{X_2})$ be a Lipschitz map such that $T(0) = 0$. Then, there is a unique map, i. e., $\hat{T} : \mathcal{F}(X_1, \rho_{X_1}) \longrightarrow \mathcal{F}(X_2, \rho_{X_2})$ such that $\hat{T}\delta_{X_1} = \delta_{X_2}T$ the following diagram commutes.

$$\begin{array}{ccc} (X_1, \rho_{X_1}) & \xrightarrow{T} & (X_2, \rho_{X_2}) \\ \delta_{X_1}\big\downarrow & & \big\downarrow\delta_{X_2} \\ \mathcal{F}(X_1, \rho_{X_1}) & \xrightarrow{\hat{T}} & \mathcal{F}(X_2, \rho_{X_2}) \end{array}$$

and $\left\|\hat{T}\right\| = \text{Lip}(T)$.

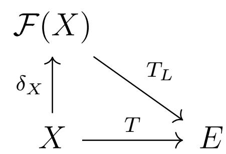
Proof. The linear map $T^{\#} : \text{Lip}_0(X_2) \longrightarrow \text{Lip}_0(X_1)$ defined by $T^{\#}(F) = F \circ T$ is the pointwise-to-pointwise continuous,hence there is a linear map $\hat{T}$ between the preduals such that $\hat{T}^* = T^{\#}$. It is clear that $\|T^{\#}\| = \text{Lip}(T)$ and $\left\|\hat{T}\right\| = \left\|\hat{T}^*\right\| = \|T^{\#}\|$.

■

The Banach space $\mathcal{F}(X, \rho_X)$ has some remarkable properties, from which we mention the following one.

Theorem 3.2. [Pet18] fundamental linearisation property

Let (X, ρ_X, e) be a pointed metric space and let E be a Banach space. Let $T : X \longrightarrow E$ be a Lipschitz map such that $T(e) = 0$. Then, there is a unique linear map $T_L : \mathcal{F}(X, \rho_X) \longrightarrow E$ such that $T_L \circ \delta_X = T$ and $\|T_L\| = \text{Lip}(T)$.



Moreover, the linear isometry $\Phi : T \in \text{Lip}_0(X, E) \longrightarrow T_L \in \mathcal{L}(\mathcal{F}(X), E)$ is into.

Proof. Let us fix a Banach space E. We start by proving the first part of the proposition. That is, we show that Φ is a linear isometry. Let us fix a Lipschitz map $T \in \text{Lip}_0(X, E)$. Let $\bar{T}$ be the map defined on span$\{\delta_x : x \in X\}$ by $\bar{T}(\sum_{i=1}^n a_i \delta(x_i)) = \sum_{i=1}^n a_i T(x_i) \in E$. Using Hahn-Banach theorem we have the following estimate for every $\gamma = \sum_{i=1}^n a_i \delta(x_i)$

$$\begin{aligned} \left\|\bar{T}\gamma\right\|_E = \left\|\sum_{i=1}^n a_i T(x_i)\right\|_E &= \sup\left\{x^*(\sum_{i=1}^n a_i T(x_i)) : x^* \in B_{E^*}\right\} \\ &= \sup\left\{\sum_{i=1}^n a_i (x^* \circ T)(x_i) : x^* \in B_{E^*}\right\} \\ &\leq= \sup\left\{\sum_{i=1}^n a_i f(x_i) : f \in \text{Lip}(T) B_{\text{Lip}_0(X)}\right\} \\ &= \text{Lip}(T)\, \|\gamma\|_{\mathcal{F}(X)}. \end{aligned}$$

Thus $\left\|\bar{T}\right\| \leq \text{Lip}(T)$. Now we want to prove the reverse inequality. To this end, let us fix $\epsilon > 0$ and consider $x \neq y \in X$ such that $\|T_x - T_y\|_E \geq (\text{Lip}(T) - \epsilon)\rho(x, y)$. Then, le us define $m_{xy} := \dfrac{\delta(x) - \delta(y)}{\rho(x, y)}$.

Clearly $\|m_{xy}\| = 1$, and $\|T m_{xy}\|_E = \dfrac{\|T_x - T_y\|_E}{\rho(x, y)} \geq (\text{Lip}(T) - \epsilon)\rho(x, y)$. Since ϵ was chosen arbitrarily, we actually get $\left\|\bar{T}\right\| \geq \text{Lip}(T)$ and so $\left\|\bar{T}\right\| = \text{Lip}(T)$.

To finish, we extend $\bar{T}$ to $\mathcal{F}(X)$ and we denote T_L this unique continuous extension (which has the same norm).

It remains to show that the linear isometry $\Phi : T \in \text{Lip}_0(X, E) \longrightarrow T_L \in \mathcal{L}(\mathcal{F}(X), E)$ is onto. To this end, consider $L \in \mathcal{L}(\mathcal{F}(X), E)$. Then, define T on X by $Tx = L\delta(x)$ for evry $x \in X$. The map T is clearly Lipschitz and satisfies $\Phi T = L$. ∎

CHAPTER 4

Free Banach space

WE introduce below the notion of free Banach space $B(X)$ over a metric space X with marked point e. Let $X = (X, \rho)$ be a metric space and e a point of X. We define a free Banach space over $\mathbb{K}$ of the metric space X, as a Banach space with a fixed isometric embedding of X in $B(X)$, which takes e into 0, such that $B(X)$ is the closed linear span of X; and for every Banach space E and every nonexpanding map $f\colon X \longrightarrow E$ [i.e., such that $\|f(x) - f(y)\| \leq \rho(x, y)$ for all $x, y \in X$] taking e into 0_E, there is a linear operator $\tilde{f}\colon B(X) \longrightarrow E$ of norm not exceeding 1, whose restriction to X is f. This space is unique to an isometry meadows.

The following results was independently proved by Flood in [Flo75], Pestov in [Pes86],M. Dubeia, E.D. Tymchatynb, A. Zagorodnyuka,c in [DTZ09] and Weaver in [Wea99, p.41].

4.1 Pestov's Theorem

Theorem 4.1. [DTZ09]

Let (X, ρ, e) be a pointed metric space. Then there exists a unique, up to an isometric isomorphism, Banach space $B(X)$ over the field $\mathbb{F}$ and an isometric embedding $i_X : X \longrightarrow B(X)$ such that

1. The linear span of i_X is dense in $B(X)$.
2. Every map T in $\text{Lip}_0(X, E)$ can be extended to a continuous linear operator $T_L : B(X) \longrightarrow E$ such that $\|T_L\| = \text{Lip}(T)$ for any arbitrary normed space.

Corollary 4.1

Every normed space X can be isometrically embedded into $B(X)^{**}$ (the second dual space to $B(X)$) such that the linear span of its range is weakly dense in $B(X)^{**}$ and the space $\text{Lip}_0(X)$ is isometrically isomorphic to $B(X)^*$.

Proof. Let $I : B(X) \longrightarrow B(X)^{**}$ be the canonical isometrical embedding, and let $v : X \longrightarrow B(X)$ be a Lipschitz map with $\text{Lip}(v) = 1$. Then $I \circ v$ is the requested embedding from X into $B(X)^{**}$. ■

4.2 Properties

Proposition 4.1

Let X_0 be a normed subspace of a normed space X. Then the point $y \in X$ belongs to $B(X_0)$ if, and only if, $y \in \bar{X}_0$, where $y \in \bar{X}_0$ is the closure of $X_0 \subset X$.

Proof. If $y \in \bar{X}_0$, then evidently, $y \in B(X_0)$. Suppose that $y \in X, y \in B(X_0)$ and $y \neq X_0$. Thus $\rho(y, X_0) = d > 0$. Let $z_n \in spanX_0$ and $z_n \longrightarrow y$. Then for every $f \in \text{Lip}_0(X)$, $f(z_n) \longrightarrow f(y)$. Let us construct a function $f_0 : X \longrightarrow \mathbb{R}$ such that $f_0(y) = d$, for every $x \in X$ such that $\rho(y, x) < d$ put $f_0(x) = d - \rho(x, y)$ and if $\rho(x, y) \geq d$, put $f_0(x) = 0$. The map f_0 is a Lipschitz map which vanishes on X_0. Since $e \in X_0$, $f_0 \in \text{Lip}_0(X)$. But $f_0(z_n) = 0$ while $f_0(y) = d > 0$. This a contradiction. ■

By Proposition 4.1 the free Banach space over a normed space coincides with the free Banach space over the completion of that normed space.

Corollary 4.2

If X_1 and X_2 are closed normed subspace of a complete normed set Y such that $X_1 \cap X_2 = \{e\}$, then $B(X_1 \cup X_2)$ is isomorphic to $B(X_1) \oplus B(X_2)$.

Proof. The natural projection of $B(X_1 \cup X_2)$ to $B(X_1)$ has kernel $B(X_2)$ by Proposition 4.1. ■

4.3 Lipschitz retractions and the structure of free Banach spaces

Proposition 4.2

If a normed subspace X_0 of the normed space X is a λ-Lipschitz retract of X, then $B(X_0)$ is λ-complemented in $B(X)$.

Proof. For the proof we can consult [DTZ09]. ■

Proposition 4.3

Let X be a Banach space. Then X is a 1-Lipschitz retract of $B(X)$.

Proof. For the proof we can consult [DTZ09]. ■

Conclusion. The concept of a free Banach space $B(X)$ over a metric space X such that every Lipschitz map from X to a normed space E can be extended to a continuous linear operator from $B(X)$ to E was introduced in [Pes86]. On the other hand the same concept was introduced by R.-F. Arens and J. Eells in [AE56], and a detailed study of Lipschitz-free spaces is made in Weaver's book [Wea99] where they are called Arens-Eells spaces. The name of Lispchitz-free spaces and the notation $\mathcal{F}(M)$ are due to Godefroy and Kalton in [GK03], this paper contains major results in this area and has popularized its study, we can conclude now by Pestov's theorem [Pes86], the two space are the predual of Lipschitz space $\text{Lip}_0(X)$.

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Abstract

ليكن X فضاءا متريا موسوما و ليكن $\mathrm{Lip}_0(X)$ فضاءليبتشز. نستنتج من نظرية ديكسمير ان الفضاء $\mathrm{Lip}_0(X) = X^{\#}$ هو فضاء ثنوي. ننشئ فضاء أرنس آل $Æ(X)$ و الذي هو فضاء ماقبل ثنوي للفضاء $X^{\#}$. ثم نهتم بالفضاء الأخر الذي أنشأه جونسون , والذي سمي فيما بعد ب $\mathcal{F}(X)$ علي يد غودفروا و كالتون. وننهي المذكرة باثبات أن $Æ(X) \equiv \mathcal{F}(X)$ وذلك بفضل نظرية بيستوف.

<u>الكلمات المفتاحية:</u>الفضاءالمتري الموسوم,فضاء أرانس و أل ,فضاء ليبتشز , فضاء ليبتشز الحر.

Soit X un espace métrique pointé et $\mathrm{Lip}_0(X)$ l'espace de Lipschitz. D'après le théorème de Diximier $\mathrm{Lip}_0(X) = X^{\#}$ est un dual. On construit l'espace de Arens Eells $Æ(X)$ qui est le predual de $X^{\#}$. Puis on s'intéresse à l'autre construction due à Johnson qui est noté par Godfroy et Kalton par $\mathcal{F}(X)$. On termine ce mémoire par montrer que $Æ(X) \equiv \mathcal{F}(X)$ par le théorème de Pestov .

<u>Mots-Clés:</u> Espace metrique pointé, Espace de Arens Eells, Espace de Lipschitz , Espace de Lipschitz libre.

Let X a pointed metric space, and Lip(X) the Lipschitz space. By the theorem of Diximer $\mathrm{Lip}_0(X) = X^{\#}$ is a dual. We are built the space of Arens Eells $Æ(X)$ which is the predual of $X^{\#}$. Then we are interested in the other construction due to Johnson who is noted by Godfroy and Kalton by $\mathcal{F}(X)$. We ending this memory by proof that Æ(X) $\equiv \mathcal{F}(\mathcal{X})$ by Pestov's theorem.

<u>Keywords:</u> Metric pointed, Arens Eells space, Lipschitz space , free Lipschitz space.