PEOPLES DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf University of M'Sila

Faculty of Mathematics and Computer Science

Department of Mathematics

Master Dissertation

Domain : Mathematics and Computer Science
Branch : Mathematics
Option : Partial Differential Equations and Applications

Theme

On Solutions to a Singular Schrödinger - Maxwell System

Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of Master in Partial Differential Equations and Applications

Presented by:

Boukraa Radia

Publicly defended on June 11, 2026 **in front of the jury composed of:**

Bilal Basti	M.C.A ,	M'sila University	**Chairperson**
Abdelaziz Hellal	M.C.B ,	M'sila University	**Supervisor**
Rabah Mecheter	M.C.A ,	M'sila University	**Examiner**

University year 2025/2026

Acknowledgements

Above all, I praise Almighty Allah, who granted me life, strength, and perseverance, and who, through His grace and generosity, enabled me to complete this work and witness this memorable moment.

I thank the respected jury members, Dr. Bilal Basti and Dr. Mecheter Rabah, for their careful evaluation of my dissertation and for their valuable insights and suggestions.

I extend my profound gratitude and heartfelt respect to my esteemed supervisor, Dr. Abdelaziz Hellal, for his constant support, endless encouragement, and exceptional patience throughout this academic journey. I remain deeply indebted to him not only for the invaluable guidance he has provided, but also for the enriching experience I have gained under his supervision in the Department of Mathematics at the Faculty of Mathematics and Computer Science, and for the considerable time he has devoted to steering me in the right direction. It is my sincere hope that this modest work brings him some measure of satisfaction and joy. With warm thanks and deep esteem, my dear professor

My deepest thanks go to my family, especially my parents, whose prayers, encouragement, and unconditional love have always been my source of strength.

I am also truly grateful to my dear friends, for their constant support, encouragement, and for standing by me through every step of this journey.

Finally, I am grateful to all my teachers and to everyone who contributed, directly or indirectly, to the completion of this work.

Dedication

Dear Family,
As I stand on the threshold of graduation, my heart fills with immense gratitude for each one of you. Your love, encouragement, and constant presence have been my support and strength throughout this journey. This achievement is not mine alone, but belongs to all of us. Your faith in me never wavered, even in moments when I doubted myself. I will never forget this kindness.
With great pride and gratitude, I dedicate this humble work to you as a token of thanks and appreciation, hoping that I will always be a source of pride for you.

Contents

Acknowledgements ***i***

Dedication ***ii***

Notations ***1***

Introduction ***3***

1 Mathematical background and auxiliary results ***5***

1.1 Lebesgue spaces 5

1.2 Sobolev spaces 11

1.3 Hilbert spaces and weak Harnack inequality 15

1.4 Dominated Convergence Theorem 21

2 Fixed point theorems ***23***

2.1 Banach fixed point theorem 23

2.2 Brouwer fixed point theorem 24

2.3 Schauder fixed point theorem 28

3 Analysis of the Singular Schrödinger-Maxwell System ***31***

3.1 Origin and physical context 31

3.2 Statement of the results 32

3.3 An overview of the proofs of main results 34

Notations

In what follows, we will use the following notations.

$\approx$	*is similar or equal to .*
$\rightarrow$	*strong convergence .*
$\rightharpoonup$	*weak convergence .*
$\langle \cdot, \cdot \rangle$	*duality pairing*
$(\cdot, \cdot)$	*inner product (in Hilbert space)*
$\| \cdot \|_p$	*norm in* $L^p(\Omega)$
$\| \cdot \|_{W^{k,p}}$	*norm in* $W^{k,p}(\Omega)$
$\| \cdot \|_{H^1}$	*norm in* $H^1(\Omega)$
α, β	*ellipticity constants:* $\alpha\|\xi\|^2 \leq B(x)\xi \cdot \xi \leq \beta\|\xi\|^2$
$\partial\Omega$	*boundary of* Ω
$\partial_{x_i} u$ *or* $D_i u$	*partial derivative of* u *with respect to* x_i
∇u	*gradient of* u
Δu	*The Laplacian of* u.
div	*divergence operator*
$\mathscr{D}(\Omega)$	*space of indefinitely differentiable functions on* Ω *(test functions)*
$C_c^\infty(\Omega)$	*smooth functions with compact support (same as* $\mathscr{D}(\Omega)$*)*
C^∞	*smooth functions*
C_0^∞	*smooth functions with compact support*
χ_E	*characteristic function of the set* E
B	*open ball*
$B(x, r)$	*open ball with center* x *and radius* $r > 0$
B_E	*closed unit ball in a space* E*:* $\{x \in E \mid \|x\| \leq 1\}$
$B(x)$	*symmetric, uniformly elliptic matrix in* $L^\infty(\Omega; \mathbb{R}^{N\times N})$
$d\mu$ *or* dx	*Lebesgue measure,* N*-dimensional*
D^α	*weak partial derivative of order* $\|\alpha\|$
$\|E\|$ *or* $\mathrm{mes}(E)$	*Lebesgue measure of a set* E
$f(x)$	*given nonnegative function,* $f \in L^m(\Omega)$*,* $m \geq N/2$
$G_k(u)$	*complementary truncation:* $G_k(u) = u - T_k(u)$
H	*generic Hilbert space*
H^*	*dual space of* H
$H^1(\Omega)$	*Hilbert space* $W^{1,2}(\Omega)$
$H_0^1(\Omega)$	*Sobolev space with zero boundary values,* $H_0^1(\Omega) = W_0^{1,2}(\Omega)$

$I_\epsilon u$ *or* $J_\epsilon u$	*mollifier (regularization) of* u
L	*usually the constant from equivalence of* Φ*-functions*
$L^p(\Omega)$	*Lebesgue space of* p*-integrable functions*
$L^\infty(\Omega)$	*space of essentially bounded functions*
$L^p_{\mathrm{loc}}(\Omega)$	*space of locally* p*-integrable functions*
$\nearrow$	*limit of an increasing sequence*
$\lvert\Omega\rvert$	*Lebesgue measure of the set* Ω
$\overline{\Omega}$	*closure of* Ω
Ω	*open set in* $\mathbb{R}^N$
$\mathscr{P}$	*set of variable exponents*
$\mathscr{P}^{\log}$	*variable exponents* p *such that* $1/p$ *is log-Hölder continuous*
P_K	*projection onto a closed convex set* K
p'	*Hölder conjugate:* $p' = \begin{cases} p/(p-1) & 1 < p < \infty \\ \infty & p = 1 \\ 1 & p = \infty \end{cases}$
p^*	*Sobolev conjugate:* $p* = \begin{cases} \frac{Np}{N-p} & p < N \\ \text{any } q \in [1,\infty) & p = N \\ \infty & p > N \end{cases}$
p^+, p^-	*essential supremum and infimum of* p
$\mathbb{R}^N$	*Euclidean* N*-dimensional space*
$\mathrm{supp}(\phi)$	*support of a function* ϕ
$T_k(u)$	*truncation function:* $T_k(s) = \max\{-k, \min\{s, k\}\}$
u_+	*positive part of* u*:* $\max(u, 0)$
u_-	*negative part of* u*:* $\max(-u, 0)$
u, v	*unknown functions (solutions)*
$V \subset H \subset V^*$	*Gelfand triple (pivot space)*
$W^{k,p}(\Omega)$	*Sobolev space of functions with weak derivatives up to order* k *in* L^p
$W^{k,p}_0(\Omega)$	*Sobolev space with zero boundary values*
$W^{-k,p'}(\Omega)$	*dual space of* $W^{k,p}_0(\Omega)$
$W^{1,\vec{p}}(\Omega)$	*anisotropic Sobolev space*
$x = (x_1, \ldots, x_N)$	*vector in* $\mathbb{R}^N$ *with* $x_i \in \mathbb{R}$, $1 \le i \le N$
X^*	*dual space of* X
γ, θ, r	*parameters:* $\gamma \in (0,1)$, $\theta \in [0,1)$, $r \ge 2$
a.e.	*almost everywhere*

Introduction

The study of nonlinear partial differential equations (PDEs) arising from physical models has been a central theme in modern mathematical analysis. Among these, systems that couple different physical fields, such as the Schrödinger–Maxwell system, play a crucial role in describing phenomena in quantum mechanics and electromagnetism. These systems often present significant mathematical challenges due to their inherent nonlinearities, singularities, and the intricate coupling between their components.

The classical Schrödinger–Maxwell system models the interaction of a charged particle with an electromagnetic field. In recent years, a great deal of attention has been devoted to its singular variants, where the potential terms become unbounded or the nonlinearities exhibit singular behavior. Such systems, referred to as singular Schrödinger–Maxwell systems, arise in various contexts, including the study of standing waves for nonlinear Schrödinger equations with a singular potential or in models of self-interacting matter.

This dissertation is devoted to the study of a doubly singular elliptic system of the form:

$$\begin{cases} -\operatorname{div}(B(x)Du) + v^{1-\theta}u^{r-1} = \dfrac{f}{u^{\gamma}} & \text{in } \Omega \\ -\operatorname{div}(B(x)Dv) = \dfrac{u^{r}}{v^{\theta}} & \text{in } \Omega \\ v > 0,\ u \geq 0 & \text{in } \Omega \\ u > 0 & \text{in } \{f > 0\} \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$. The function f is non-negative and non-trivial, belonging to a suitable Lebesgue space, and $B(\cdot) \in L^{\infty}(\Omega; \mathbb{R}^{N\times N})$ is a symmetric and uniformly elliptic matrix, i.e.,

$$\alpha|\xi|^2 \leq B(x)\xi \cdot \xi \leq \beta|\xi|^2 \qquad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N,$$

with $0 < \alpha \leq \beta$. Furthermore, we assume the following on the parameters:

$$r \geq 2, \quad \gamma \in (0,1), \quad \theta \in [0,1).$$

System (1) is modeled after the following Schrödinger–Maxwell system

$$\begin{cases} -\frac{1}{2}\Delta u + \phi u = \omega u \\ -\Delta\phi = 4\pi u^2 \end{cases}$$

that first appeared in the seminal work [3], where the authors studied the eigenvalue problem for the Schrödinger operator coupled with an electromagnetic field. In our case, the problem is doubly singular, namely a Dirichlet problem in which both equations are singular. The literature on singular problems is vast and well established.

We investigate how the structure of the system and the integrability of the datum f give rise to regularizing effects on the existence of solutions $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ having higher integrability than that prescribed by the theory of elliptic (single) equations. The primary aim of this work is to study the existence of finite-energy solutions in the Sobolev space $H_0^1(\Omega) \times H_0^1(\Omega)$ for system (1). We are particularly interested in the regularizing effects induced by the coupling: the presence of the second equation enhances the integrability properties of the solution u, leading to better regularity than one would expect from a single singular equation. This phenomenon is a key feature of such coupled systems and is one of the central points of our investigation.

To achieve this, we employ a combination of functional analysis tools and nonlinear PDE techniques. The work is structured as follows:

Chapter 1 serves as a preliminary chapter, recalling some fundamental properties of Sobolev and Lebesgue spaces. These spaces form the functional framework for our analysis and provide the necessary embedding theorems and compactness results (for more details, see [1, 2, 6, 7, 8, 9, 13, 14, 16, 17]).

Chapter 2 is devoted to the presentation of essential mathematical tools. We focus on fixed point theory with its some applications in elliptic boundary value problems as wells as Schauder's fixed point theorem, which is a powerful non-variational method that allows us to decouple the system and construct a fixed point for a suitable operator, thereby proving the existence of a solution, as discussed in [12, 15].

Chapter 3 contains the core of our study. We conduct a detailed analysis of the singular Schrödinger–Maxwell system (1). We define the notion of a weak solution, construct approximating problems to regularize the singularities, derive a priori estimates in $H_0^1(\Omega) \times H_0^1(\Omega)$, and finally pass to the limit to prove the existence of a positive solution. We also discuss the regularizing effects of the coupling in detail. This study enhances the understanding of singular coupled systems and paves the way for further research, including the uniqueness or regularity of solutions when additional assumptions are imposed on the parameters γ, θ, r and f

The results presented herein are based on recent advances in the theory of singular elliptic systems, particularly the works of [5, 11], which have paved the way for understanding such complex interactions.

Chapter 1

Mathematical background and auxiliary results

This chapter recalls certain fundamental facts concerning Sobolev spaces and outlines their principal properties. For a more detailed account of both Lebesgue and Sobolev spaces, reference is made to the thorough expositions in [1, 2, 6, 7, 8, 9, 13, 14, 16, 17].

1.1 Lebesgue spaces

Definition and elementary properties of L^p spaces

Definition 1.1.1. *Let $1 \le p < \infty$; we set*

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ is measurable and } |f|^p \in L^1(\Omega)\}$$

with

$$\|f\|_{L^p} = \|f\|_p = \left(\int_\Omega |f(x)|^p \, d\mu\right)^{1/p}.$$

Definition 1.1.2. *We set*

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ is measurable and } \exists c > 0 : |f(x)| \le c \text{ for } \mu\text{-a.e. } x \in \Omega\}$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf\{c > 0 \mid |f(x)| \le c \text{ for } \mu\text{-a.e. } x \in \Omega\}.$$

Definition 1.1.3. *Suppose that $1 \le p < \infty$. Then:*

(i)

$$L^p_{loc}(\Omega) = \{v : v \in L^p(K) \text{ for every compact subset } K \subset \Omega\}.$$

(ii) A function v is said to be locally integrable if $v \in L^1_{loc}(\Omega)$.

(iii) Let v and w be locally integrable functions on Ω. We define v as the weak derivative of u with respect to the multi-index α if, for every test function $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega v D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega w\varphi \, dx.$$

In this case, we say that $D^\alpha v = w$ in the weak sense.

(iv) Let $v, w \in L^p_{loc}(\Omega)$. We define w as the strong derivative of v with respect to the multi-index α if, for every compact subset $K \subset \Omega$, there exists a sequence $\{\varphi_i\} \subset C^\alpha(K)$ such that

$$\varphi_i \to v \quad in\ L^p(K)$$

and

$$\lim_{i\to\infty} \frac{\|D^\alpha \varphi_i - w\|_{L^p}}{|D^\alpha \varphi_i|} = 0.$$

Theorem 1.1.1. *If $D^\alpha u = v$ and $D^\beta v = w$ in the weak sense, then $D^{\alpha+\beta} u = w$ in the weak sense.*

Proof. Let $\psi \in C_0^\infty(\Omega)$, and set $\varphi = D^\beta \psi$. Then, by the definition of the weak derivative,

$$\int_\Omega u D^{\alpha+\beta}\psi \, dx = (-1)^{|\alpha|} \int_\Omega v D^\beta \psi \, dx = (-1)^{|\alpha|} \int_\Omega v\varphi \, dx.$$

Since $D^\beta v = w$, we have:

$$\int_\Omega v\varphi \, dx = (-1)^{|\beta|} \int_\Omega \psi D^\beta v \, dx = (-1)^{|\beta|} \int_\Omega \psi w \, dx.$$

Hence,

$$\int_\Omega u D^{\alpha+\beta}\psi \, dx = (-1)^{|\alpha|+|\beta|} \int_\Omega \psi w \, dx.$$

This shows that $D^{\alpha+\beta} u = w$ in the weak sense. □

Definition 1.1.4. *Let $\mu \in C_0^\infty(\mathbb{R}^N)$ be such that:*

1. $\operatorname{supp}\mu \subset B_1(0)$ *(recall that* supp *denotes the support of a function, and $B_r(c)$ denotes an open ball of radius r and center c),*
2. $\int_{\mathbb{R}^N} \mu(x)\, dx = 1$,
3. $\mu(x) \geq 0$.

If $\varepsilon > 0$ then we set (provided that the integral exists)

$$I_\varepsilon u(x) = \frac{1}{\varepsilon^N} \int_\Omega \mu\Big(\frac{x-y}{\varepsilon}\Big) u(y)\, dy.$$

$I_\varepsilon u$ is called a mollifier of u. Note that if u is locally integrable in Ω and if K is a compact subset of Ω then $I_\varepsilon u \in C^\infty(K)$ provided that $\varepsilon < \operatorname{dist}(K, \partial\Omega)$. Suppose now that $u \in L^p_{loc}(\Omega)$.
We have

$$I_\varepsilon u(x) = \int_{B_1(0)} \mu(y)\, u(x - \varepsilon y)\, dy,$$

so for $p > 1$ we have (if $\frac{1}{p} + \frac{1}{q} = 1$)

$$|I_\varepsilon u(x)| \leq \int_{B_1(0)} |\mu(y)|^{\frac{1}{p}} |\mu(y)|^{\frac{1}{q}} |u(x - \varepsilon y)|\, dy$$

$$\leq \left(\int_{B_1(0)} |\mu(y)|\, dy \right)^{\frac{1}{q}} \left(\int_{B_1(0)} |\mu(y)||u(x-\varepsilon y)|^p\, dy \right)^{\frac{1}{p}}.$$

Hence $|I_\varepsilon u(x)|^p \leq \int_{B_1(0)} \mu(y)|u(x-\varepsilon y)|^p\, dy$*, and this trivially holds if* $p = 1$ *too. Integrating this, we see that*

$$\int_K |I_\varepsilon u(x)|^p\, dx \leq \int_{B_1(0)} \mu(y) \int_K |u(x-\varepsilon y)|^p\, dx\, dy$$
$$\leq \int_{B_1(0)} \mu(y) \int_{K_0} |u(x)|^p\, dx\, dy = \int_{K_0} |u(x)|^p\, dx,$$

where K_0 *is a compact subset of* Ω*,* $K \subset \text{Interior}(K_0)$ *and* $\varepsilon < \text{dist}(K, \partial K_0)$*; i.e. we have*

$$\|I_\varepsilon u\|_{L^p(K)} \leq \|u\|_{L^p(K_0)}. \tag{1.1}$$

Lemma 1.1.1. *Let* $u \in L^p_{loc}(\Omega)$*, and let* $K \subset \Omega$ *be a compact subset. Then*

$$\|J_\varepsilon u - u\|_{L^p(K)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. Let $K_0 \subset \Omega$ be a compact set such that $K \subset \text{int}(K_0)$, and choose $\varepsilon > 0$ such that

$$\varepsilon < \text{dist}(K, \partial K_0).$$

Let $\delta > 0$. Since $C^\infty(K_0)$ is dense in $L^p(K_0)$, there exists a function $w \in C^\infty(K_0)$ such that

$$\|u - w\|_{L^p(K_0)} < \delta.$$

Applying inequality (1.1) to the function $u - w$, we obtain

$$\|J_\varepsilon u - J_\varepsilon w\|_{L^p(K)} < \delta.$$

Moreover, we have

$$J_\varepsilon w(x) - w(x) = \int_{B(0,1)} \mu(y)\, [w(x-\varepsilon y) - w(x)]\, dy,$$

and since $w \in C^\infty$, the integrand tends uniformly to zero as $\varepsilon \to 0$, so:

$$J_\varepsilon w(x) \to w(x) \quad \text{uniformly on } K.$$

Therefore, for ε sufficiently small,

$$\|J_\varepsilon w - w\|_{L^p(K)} < \delta.$$

Now, combining the above estimates:

$$\|J_\varepsilon u - u\|_{L^p(K)} \leq \|u - w\|_{L^p(K)} + \|J_\varepsilon u - J_\varepsilon w\|_{L^p(K)} + \|J_\varepsilon w - w\|_{L^p(K)} < \delta + \delta + \delta = 3\delta.$$

Since $\delta > 0$ was arbitrary, we conclude that

$$\|J_\varepsilon u - u\|_{L^p(K)} \to 0 \quad \text{as } \varepsilon \to 0.$$

□

Theorem 1.1.2. *Suppose that $u, v \in L^p_{loc}(\Omega)$. Then $D^\alpha u = v$ in the weak sense if and only if $D^\alpha u = v$ in the strong $L^p(\Omega)$ sense.*

Proof. Suppose that $D^\alpha u = v$ in the strong $L^p(\Omega)$ sense. Let $\varphi \in C_c^\infty(\Omega)$, and let $K = \operatorname{supp}(\varphi)$. Let $\varepsilon > 0$, and take $\psi \in C_c^\infty(K)$ such that

$$\|u - \psi\|_{L^p(K)} < \varepsilon \quad \text{and} \quad \|D^\alpha \psi - v\|_{L^p(K)} < \varepsilon.$$

Then:

$$\begin{aligned}
\left| \int_K u D^\alpha \varphi \, dx - (-1)^{|\alpha|} \int_K v\varphi \, dx \right| &= \left| \int_K (u-\psi) D^\alpha \varphi \, dx + \int_K \psi D^\alpha \varphi \, dx - (-1)^{|\alpha|} \int_K v\varphi \, dx \right| \\
&= \left| \int_K (u-\psi) D^\alpha \varphi \, dx + (-1)^{|\alpha|} \int_K \varphi D^\alpha \psi \, dx - (-1)^{|\alpha|} \int_K v\varphi \, dx \right| \\
&= \left| \int_K (u-\psi) D^\alpha \varphi \, dx + (-1)^{|\alpha|} \int_K \varphi (D^\alpha \psi - v) \, dx \right| \\
&\leq \|u-\psi\|_{L^p(K)} \|D^\alpha \varphi\|_{L^q(K)} + \|D^\alpha \psi - v\|_{L^p(K)} \|\varphi\|_{L^q(K)} \\
&\leq \varepsilon \left(\|D^\alpha \varphi\|_{L^q(K)} + \|\varphi\|_{L^q(K)} \right).
\end{aligned}$$

where q is the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Since ε is arbitrary, the left-hand side must be zero. Therefore, $D^\alpha u = v$ in the weak sense.
Conversely, suppose that $D^\alpha u = v$ in the weak sense. Let $K \subset \Omega$ be compact. Then for $\varepsilon < \operatorname{dist}(K, \partial\Omega)$, the mollification $J_\varepsilon u \in C^\infty(K)$ satisfies:

$$D^\alpha J_\varepsilon u(x) = \frac{1}{\varepsilon^n} D_x^\alpha \left(\int_\Omega \mu\left(\frac{x-y}{\varepsilon} \right) u(y) \, dy \right).$$

By differentiating under the integral and changing variables:

$$\begin{aligned}
D^\alpha J_\varepsilon u(x) &= \frac{1}{\varepsilon^n} \int_\Omega D_y^\alpha \left[\mu\left(\frac{x-y}{\varepsilon} \right) \right] u(y) \, dy \\
&= \frac{1}{\varepsilon^n} \int_\Omega \mu\left(\frac{x-y}{\varepsilon} \right) v(y) \, dy = J_\varepsilon v(x),
\end{aligned}$$

since $D^\alpha u = v$ in the weak sense. Hence $D^\alpha J_\varepsilon u = J_\varepsilon v$, and so $D^\alpha u = v$ in the strong L^p sense by approximation.
But by Lemma 1.1.1,

$$\|J_\varepsilon u - u\|_{L^p(K)} \to 0 \quad \text{and} \quad \|D^\alpha J_\varepsilon u - v\|_{L^p(K)} = \|J_\varepsilon v - v\|_{L^p(K)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus, $D^\alpha u = v$ in the strong sense. □

Theorem 1.1.3 (Young's Inequality). *Suppose that $a, b \geq 0$ and $1 < p, q < \infty$ with*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Theorem 1.1.4 (Holder Inequality). *Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$, where $1 \leq p < \infty$. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Theorem 1.1.5 (Minkowski Inequality). *Let $1 \leq p < \infty$, and assume that $f, g \in L^p(\Omega)$. Then $f + g \in L^p(\Omega)$ and*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

Remark 1.1.1. *In case of $p = \infty$ we have:*

$$\|f + g\|_{\infty} \leq \|f\|_{\infty} + \|g\|_{\infty}.$$

Definition 1.1.5 (Continuous Embedding). *Suppose that E and F are Banach spaces. If $E \subset F$, we say that E is continuously embedded in F (denoted $E \hookrightarrow F$) if there exists a constant $C > 0$ such that*

$$\|x\|_F \leq C\|x\|_E, \quad \text{for all } x \in E.$$

Example 1.1.1. *Let $(X, \mathcal{T}, \mu)$ be a finite measure space, and let $1 \leq p, q \leq \infty$ with $p \leq q$. Then*

$$L^q(X, \mathcal{T}, \mu) \hookrightarrow L^p(X, \mathcal{T}, \mu),$$

and

$$\|f\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q}.$$

In fact, since $p \leq q$, then $\frac{1}{\frac{q}{p}} + \frac{1}{\frac{q}{q-p}} = 1$, and by Hölder inequality (1.2) we obtain the result.

Theorem 1.1.6 (Riesz–Fréchet). *L^p is a Banach space for all $1 \leq p \leq \infty$.*

Reflexivity, Separability, Dual of L^p

We shall consider separately the following three cases:

(A) $1 < p < \infty$,

(B) $p = 1$,

(C) $p = \infty$.

A. Study of $L^p(\Omega)$ for $1 < p < \infty$

This case is the most "favorable": L^p is reflexive, separable, and the dual of L^p is $L^{p'}$.

Theorem 1.1.7. *L^p is reflexive for any p, $1 < p < \infty$.*

The proof consists of three steps:

Step 1 (Clarkson's first inequality). *Let $2 \le p < \infty$. We claim that*

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \le \frac{1}{2}(\|f\|_p^p + \|g\|_p^p) \quad \forall f, g \in L^p. \tag{1.1}$$

Proof of (1.1). Clearly, it suffices to show that

$$\left| \frac{a+b}{2} \right|^p + \left| \frac{a-b}{2} \right|^p \le \frac{1}{2}(|a|^p + |b|^p) \quad \forall a, b \in \mathbb{R}.$$

First we note that

$$\alpha^p + \beta^p \le (\alpha^2 + \beta^2)^{p/2} \quad \forall a, b \ge 0$$

(by homogeneity, assume $\beta = 1$ and observe that the function $(x^2+1)^{p/2} - x^p - 1$ increases on $[0, \infty)$). Choosing $\alpha = \left| \frac{a+b}{2} \right|$ and $\beta = \left| \frac{a-b}{2} \right|$, we obtain

$$\begin{aligned} \left| \frac{a+b}{2} \right|^p + \left| \frac{a-b}{2} \right|^p &\le \left(\left| \frac{a+b}{2} \right|^2 + \left| \frac{a-b}{2} \right|^2 \right)^{p/2} \\ &= \left(\frac{a^2}{2} + \frac{b^2}{2} \right)^{p/2} \le \frac{1}{2}(|a|^p + |b|^p). \end{aligned}$$

(The last inequality follows from the convexity of the function $x \mapsto |x|^{p/2}$ since $p \ge 2$).

Step 2: *L^p is uniformly convex, and thus reflexive for $2 \le p < \infty$. Indeed, let $\varepsilon > 0$ and let $f, g \in L^p$ with $\|f\|_p \le 1$, $\|g\|_p \le 1$, and $\|f - g\|_p > \varepsilon$. We deduce from (1.1) that*

$$\left\| \frac{f+g}{2} \right\|_p^p < 1 - \left(\frac{\varepsilon}{2} \right)^p$$

and thus $\|\frac{f+g}{2}\|_p < 1 - \delta$ with $\delta = 1 - [1 - (\frac{\varepsilon}{2})^p]^{1/p} > 0$. Therefore, L^p is uniformly convex and thus reflexive .

Step 3: *L^p is reflexive for $1 < p \le 2$.*

Proof. Let $1 < p < \infty$. Consider the operator $T : L^p \to (L^{p'})^*$ defined as follows: Let $u \in L^p$ be fixed; the mapping $f \in L^{p'} \mapsto \int uf$ is a continuous linear functional on $L^{p'}$ and thus it defines an element, say Tu, in $(L^{p'})^*$ such that

$$\langle Tu, f \rangle = \int uf \quad \forall f \in L^{p'}. \tag{1.2}$$

We claim that

$$\|Tu\|_{(L^{p'})^*} = \|u\|_p \quad \forall u \in L^p. \tag{1.3}$$

Indeed, by Hölder's inequality, we have

$$|\langle Tu, f\rangle| \le \|u\|_p \|f\|_{p'} \quad \forall f \in L^{p'}$$

and therefore $\|Tu\|_{(L^{p'})^*} \le \|u\|_p$.
On the other hand, set

$$f_0(x) = |u(x)|^{p-2}u(x) \quad (f_0(x) = 0 \text{ if } u(x) = 0).$$

Clearly we have

$$f_0 \in L^{p'}, \quad \|f_0\|_{p'} = \|u\|_p^{p-1} \quad \text{and} \quad \langle Tu, f_0\rangle = \|u\|_p^p;$$

thus

$$\|Tu\|_{(L^{p'})^*} \ge \frac{\langle Tu, f_0\rangle}{\|f_0\|_p} = \|u\|_p.$$

Hence, we have shown that T is an isometry from L^p into $(L^{p'})^*$, which implies that $T(L^p)$ is a closed subspace of $(L^{p'})^*$ (because L^p is a Banach space). Assume now $1 < p \le 2$. Since L^p is reflexive (by Step 2), it follows that $(L^{p'})^*$ is also reflexive . We conclude, that $T(L^p)$ is reflexive, and as a consequence, L^p is also reflexive. □

Remark 1.1.2. *In fact, L^p is also uniformly convex for $1 < p \le 2$. This is a consequence of Clarkson's second inequality, which holds for $1 < p \le 2$:*

$$\left\|\frac{f+g}{2}\right\|_p^{p'} + \left\|\frac{f-g}{2}\right\|_p^{p'} \le \left(\frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p\right)^{1/(p-1)} \quad \forall f, g \in L^p.$$

This inequality is trickier to prove than Clarkson's first inequality. Clearly, it implies that L^p is uniformly convex when $1 < p \le 2$.

1.2 Sobolev spaces

Definition of $W^{m,p}(\Omega)$ spaces:

After defining the spaces $W^{1,p}(\Omega)$, we can define the general $W^{m,p}$ spaces recursively. Let $m \ge 2$ be an integer and $1 \le p \le \infty$. Then, we define

$$W^{m,p}(\Omega) = \left\{u \in W^{m-1,p}(\Omega) \,\middle|\, \partial_i u \in W^{m-1,p}(\Omega) \text{ for all } i = 1, \dots, N\right\}.$$

An equivalent way of defining the Sobolev spaces is by using multi-index notation. We define

$$W^{m,p}(\Omega) = \left\{u \in L^p(\Omega) \,\middle|\, \forall \alpha, |\alpha| \le m,\ \exists g_\alpha \in L^p(\Omega) : \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi \ \ \forall \varphi \in C_c^\infty(\Omega)\right\}.$$

Here, multi-index notation is used. That is, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, and the order of the multi-index is defined by

$$|\alpha| = \sum_{i=1}^{N} \alpha_i.$$

Moreover, the differential operator D^α is defined as

$$D^\alpha \varphi = \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

We denote $D^\alpha u = g_\alpha$. It can be shown (although the proof is omitted here) that $W^{m,p}(\Omega)$ is a Banach space when equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^p}. \tag{1.6}$$

Just as in the case of $W^{1,2}(\Omega)$, the space $W^{m,2}(\Omega)$, denoted by $H^m(\Omega)$, is a Hilbert space with the scalar product

$$(u, v)_{H^m} = \sum_{|\alpha| \le m} (D^\alpha u, D^\alpha v)_{L^2}.$$

The norm induced by this scalar product is equivalent to the norm in (1.6).

Motivation and definition of Sobolev spaces

Let $C_c^\infty(\Omega)$ be the set of infinitely differentiable functions with compact support, that is,

$$\varphi : \Omega \to \mathbb{R}, \quad \varphi \in C_c^\infty(\Omega).$$

A function $\varphi \in C_c^\infty(\Omega)$ is often called a test function.
For each $u \in C^1(\Omega)$, and taking into account that every $\varphi \in C_c^\infty(\Omega)$ has compact support, Greens identity yields

$$\int_\Omega u \partial_{x_i} \varphi \, dx = -\int_\Omega \partial_{x_i} u \varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega), \quad \forall i = 1, \dots, n.$$

This motivates the definition of Sobolev spaces as follows:

Definition 1.2.1. *For each $1 \le p \le \infty$, the Sobolev space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \exists g_1, \dots, g_N \in L^p(\Omega) : \int_\Omega u \partial_{x_i} \varphi \, dx = -\int_\Omega g_i \varphi \, dx, \; \forall \varphi \in C_c^\infty(\Omega), \; \forall i = 1, \dots, N \right\}.$$

We denote $\partial_{x_i} u = g_i$ as the weak derivative of $u \in W^{1,p}(\Omega)$, which is unique. The gradient of u is defined as

$$\nabla u = (\partial_{x_1} u, \dots, \partial_{x_N} u).$$

If $p = 2$, we write

$$H^1(\Omega) = W^{1,2}(\Omega).$$

Definition 1.2.2. *For each $u \in W^{1,p}(\Omega)$, we define the norm of u by*

$$\|u\|_{W^{1,p}} = \|u\|_p + \sum_{i=1}^{N} \|\partial_{x_i} u\|_p.$$

Moreover, $H^1(\Omega)$ is a Hilbert space equipped with the following scalar product:

$$(u,v)_{H^1} = (u,v)_{L^2} + \sum_{i=1}^{N} (\partial_{x_i} u, \partial_{x_i} v)_{L^2}.$$

Since $H^1(\Omega)$ is a Hilbert space, this induces a norm in $H^1(\Omega)$:

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^2}^2 \right)^{1/2}.$$

This norm is equivalent to the $W^{1,2}$ norm.

Some properties of Sobolev spaces

Proposition 1.2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, the following statements hold:*

(i) For each $1 \le p \le \infty$, $W^{1,p}(\Omega)$ is a Banach space.

(ii) For each $1 < p < \infty$, $W^{1,p}(\Omega)$ is reflexive.

(iii) For each $1 \le p < \infty$, $W^{1,p}(\Omega)$ is separable.

Proof. 1. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{1,p}(\Omega)$, with $1 \le p \le \infty$. Then, from (1.4) it follows that (u_n) and $((u_n)_{x_i})$, with $1 \le i \le N$, are Cauchy sequences in $L^p(\Omega)$.

Thus, since $L^p(\Omega)$ is a Banach space, it follows that $u_n \to u$ and $(u_n)_{x_i} \to g_i$ in $L^p(\Omega)$. Therefore,

$$\int_\Omega u_n \varphi_{x_i}\, dx = -\int_\Omega (u_n)_{x_i} \varphi\, dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Letting $n \to +\infty$, we get

$$\int_\Omega u \varphi_{x_i}\, dx = -\int_\Omega g_i \varphi\, dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Therefore, we obtain that $u \in W^{1,p}(\Omega)$, $u_{x_i} = g_i$, and thus

$$\|u_n - u\|_{W^{1,p}(\Omega)} = \|u_n - u\|_{L^p(\Omega)} + \sum_{i=1}^{N} \|(u_n)_{x_i} - g_i\|_{L^p(\Omega)} \to 0.$$

2. Consider the space $E = L^p(\Omega) \times L^p(\Omega)$ which is reflexive since it is the product of reflexive spaces. Set the operator

$$T : W^{1,p}(\Omega) \to E$$

defined by

$$Tu = (u, \nabla u).$$

Then, T is an isometry, and since $W^{1,p}(\Omega)$ is a Banach space,

$$M = T(W^{1,p}(\Omega))$$

is a closed subspace of E. Since E is reflexive, B_E is compact in the weak topology $\sigma(E, E')$, and M is closed in $\sigma(E, E')$. Therefore, B_M is compact in $\sigma(E, E')$, and hence $T(W^{1,p}(\Omega))$ is reflexive. Consequently, $W^{1,p}(\Omega)$ is reflexive.

3. Under the notation of (2), and taking into account that E is separable, it follows that $T(W^{1,p}(\Omega))$ is separable and therefore $W^{1,p}(\Omega)$ is also separable.

□

Remark 1.2.1. *We will use definition of the positive and negative part of a real-valued function* $u\colon \Omega \to \mathbb{R}$ *as*

$$u_+(x) := \max\{u(x), 0\} = \begin{cases} u(x) & \text{if } u(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$u_-(x) := (-u)_+(x) = \begin{cases} -u(x) & \text{if } u(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in \Omega$. *Let* $k > 0$. *We see taht, if* $u \in W^{1,p}(\Omega)$, *the same happens to* u_+, $T_k(u)$, $G_k(u)$, *and*

$$Du_+ = Du\chi_{\{u \geq 0\}}, \quad DT_k(u) = Du\chi_{\{|u| \leq k\}}, \quad DG_k(u) = Du\chi_{\{|u| \geq k\}} \quad \text{a.e. in } \Omega.$$

Proof. (Poincaré Inequality)
Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set, and let $1 \leq p < \infty$. Then there exists a constant $C > 0$ such that for all $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}.$$

In particular, for $p = 2$, we have:

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \quad \text{for all } u \in H_0^1(\Omega).$$

Assume the inequality is false. Then, for every integer n, there exists $u_n \in W_0^{1,p}(\Omega)$ such that

$$\|u_n\|_{L^p(\Omega)} > n\|\nabla u_n\|_{L^p(\Omega)}.$$

Define normalized functions

$$v_n = \frac{u_n}{\|u_n\|_{L^p(\Omega)}}.$$

Then $\|v_n\|_{L^p(\Omega)} = 1$, and

$$\|\nabla v_n\|_{L^p(\Omega)} = \frac{\|\nabla u_n\|_{L^p(\Omega)}}{\|u_n\|_{L^p(\Omega)}} < \frac{1}{n}.$$

Hence,

$$\|\nabla v_n\|_{L^p(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Because $v_n \in W_0^{1,p}(\Omega)$ and Ω is bounded, by compactness (Rellich-Kondrachov theorem), there exists a subsequence (still denoted v_n) that converges in $L^p(\Omega)$ to some function v.
But since $\|\nabla v_n\| \to 0$, the limit v must have zero gradient, i.e., v is constant. Moreover, $v \in W_0^{1,p}(\Omega)$ implies $v = 0$ on the boundary, so $v \equiv 0$.
On the other hand, since $\|v_n\|_{L^p(\Omega)} = 1$, we must have

$$\|v\|_{L^p(\Omega)} = \lim_{n\to\infty} \|v_n\|_{L^p(\Omega)} = 1,$$

which contradicts $v \equiv 0$.
Therefore, the assumption is false, and the Poincaré inequality holds:

$$\|u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}.$$

□

1.3 Hilbert spaces and weak Harnack inequality

Definition 1.3.1. *Let H be a vector space. A scalar product (u, v) is a bilinear form on $H \times H$ with values in $\mathbb{R}$ (i.e., a map from $H \times H$ to $\mathbb{R}$ that is linear in both variables) such that*

$$(u, v) = (v, u) \quad \forall u, v \in H \quad (\textit{symmetry}),$$

$$(u, u) \ge 0 \quad \forall u \in H \quad (\textit{positive}),$$

$$(u, u) \ne 0 \quad \forall u \ne 0 \quad (\textit{definite}).$$

Let us recall that a scalar product satisfies the Cauchy–Schwarz inequality

$$|(u, v)| \le (u, u)^{1/2}(v, v)^{1/2} \quad \forall u, v \in H.$$

[It is sometimes useful to keep in mind that the proof of the Cauchy–Schwarz inequality does not require the assumption $(u, u) \ne 0\ \forall u \ne 0$.] It follows from the Cauchy–Schwarz inequality that the quantity

$$\|u\| = (u, u)^{1/2}$$

is a norm—we shall often denote by $\|\cdot\|$ (instead of $|\cdot|$) norms arising from scalar products. Indeed, we have

$$|u + v|^2 = (u + v, u + v) = |u|^2 + (u, v) + (v, u) + |v|^2 \le |u|^2 + 2|u||v| + |v|^2,$$

and thus $|u + v| \le |u| + |v|$.

Let us recall the classical parallelogram law:

$$\left|\frac{a+b}{2}\right|^2+\left|\frac{a-b}{2}\right|^2=\frac{1}{2}(|a|^2+|b|^2)\quad\forall a,b\in H.$$

Definition 1.3.2. *A Hilbert space is a vector space H equipped with a scalar product such that H is complete for the norm $|\cdot|$.*
In what follows, H will always denote a Hilbert space.

Basic example.

$L^2(\Omega)$ equipped with the scalar product

$$(u,v)=\int_\Omega u(x)\,v(x)\,d\mu$$

is a Hilbert space. In particular, ℓ^2 is a Hilbert space. The Sobolev space H^1 studied in Chapters 8 and 9 is another example of a Hilbert space; it is "modeled" on $L^2(\Omega)$.

Proposition 1.3.1. *H is uniformly convex, and thus it is reflexive.*

Proof. Let $\varepsilon>0$ and $u,v\in H$ satisfy $|u|\le 1$, $|v|\le 1$, and $|u-v|>\varepsilon$. In view of the parallelogram law we have

$$\left|\frac{u+v}{2}\right|^2<1-\frac{\varepsilon^2}{4}\quad\text{and thus}\quad\left|\frac{u+v}{2}\right|<1-\delta\quad\text{with}\quad\delta=1-\left(1-\frac{\varepsilon^2}{4}\right)^{1/2}>0.$$

□

Theorem 1.3.1. *(projection onto a closed convex set). Let $K\subset H$ be a nonempty closed convex set. Then for every $f\in H$ there exists a unique element $u\in K$ such that*

$$|f-u|=\min_{v\in K}|f-v|=\operatorname{dist}(f,K).$$

Moreover, u is characterized by the property

$$u\in K\quad\textit{and}\quad(f-u,v-u)\le 0\quad\forall v\in K.$$

Notation 1.3.1. *The above element u is called the* projection *of f onto K and is denoted by*

$$u=P_K f.$$

Inequality (3) says that the scalar product of the vector $\overrightarrow{uf}$ with any vector $\overrightarrow{v}$ ($v\in K$) is ≤ 0, i.e., the angle θ determined by these two vectors is $\ge\pi/2$.

Proof. (a) Existence. We shall present two different proofs:

1. The function $\varphi(v)=|f-v|$ is convex, continuous and $\lim_{|v|\to\infty}\varphi(v)=+\infty$. It follows from Corollary 3.23 that φ achieves its minimum on K since H is reflexive.

2. The second proof does not rely on the theory of reflexive and uniformly convex spaces. It is a direct argument. Let (v_n) be a minimizing sequence for (2), i.e., $v_n \in K$ and

$$d_n = |f - v_n| \to d = \inf_{v \in K} |f - v|.$$

We claim that (v_n) is a Cauchy sequence. Indeed, the parallelogram law applied with $a = f - v_n$ and $b = f - v_m$ leads to

□

Proposition 1.3.2. *Let $K \subset H$ be a nonempty closed convex set. Then P_K does not increase distance, i.e.,*

$$|P_K f_1 - P_K f_2| \le |f_1 - f_2| \quad \forall f_1, f_2 \in H.$$

Proof. Set $u_1 = P_K f_1$ and $u_2 = P_K f_2$. We have

$$(f_1 - u_1, v - u_1) \le 0 \quad \forall v \in K \tag{1.4}$$

$$(f_2 - u_2, v - u_2) \le 0 \quad \forall v \in K. \tag{1.5}$$

Choosing $v = u_2$ in (1.4) and $v = u_1$ in (1.5) and adding the corresponding inequalities, we obtain

$$|u_1 - u_2|^2 \le (f_1 - f_2, u_1 - u_2).$$

It follows that $|u_1 - u_2| \le |f_1 - f_2|$. □

Example 1.3.1. *Assume that $M \subset H$ is a closed linear subspace. Let $f \in H$. Then $u = P_M f$ is characterized by*

$$u \in M \quad \text{and} \quad (f - u, v) = 0 \quad \forall v \in M. \tag{8}$$

Moreover, P_M is a linear operator, called the orthogonal projection.

Proof. By (3) we have

$$(f - u, v - u) \le 0 \quad \forall v \in M$$

and thus

$$(f - u, tv - u) \le 0 \quad \forall v \in M, \ \forall t \in \mathbb{R}.$$

It follows that (8) holds. □

Conversely, if u satisfies (8) we have

$$(f - u, v - u) = 0 \quad \forall v \in M.$$

It is obvious that P_M is linear.

Dual of Hilbert space

It is very easy, in a Hilbert space, to write down continuous linear functionals. Pick any $f \in H$; then the map $u \mapsto (f,u)$ is a continuous linear functional on H. It is a remarkable fact that all continuous linear functionals on H are obtained in this fashion:

Theorem 1.3.2 (Riesz–Fréchet representation theorem)**.** *Given any $\varphi \in H^*$ there exists a unique $f \in H$ such that*

$$\langle \varphi, u \rangle = (f,u) \quad \forall\, u \in H.$$

Moreover,

$$|f| = \|\varphi\|_{H^*}.$$

Proof. Once more we shall present two proofs:

1. The first one is almost identical to the proof of Theorem 4.11. Consider the map $T : H \to H^*$ defined as follows: given any $f \in H$, the map $u \mapsto (f,u)$ is a continuous linear functional on H. It defines an element of H^*, which we denote by Tf, so that

   $$\langle Tf, u \rangle = (f,u) \quad \forall\, u \in H.$$

   It is clear that $\|Tf\|_{H^*} = |f|$. Thus T is a linear isometry from H onto $T(H)$, a closed subspace of H^*. In order to conclude, it suffices to show that $T(H)$ is dense in H^*. Assume that h is a continuous linear functional on H^* that vanishes on $T(H)$. Since H is reflexive, h belongs to H and satisfies $\langle Tf, h \rangle = 0$ for all $f \in H$. It follows that $(f,h) = 0$ for all $f \in H$ and thus $h = 0$.

2. The second proof is a more direct argument that avoids any use of reflexivity. Let $M = \varphi^{-1}(\{0\})$, so that M is a closed subspace of H. We may always assume that $M \neq H$ (otherwise $\varphi \equiv 0$ and the conclusion of Theorem 5.5 is obvious—just take $f = 0$). We claim that there exists some element $g \in H$ such that

   $$|g| = 1 \text{ and } (g,v) = 0 \quad \forall\, v \in M \text{ (and thus } g \notin M).$$

   Indeed, let $g_0 \in H$ with $g_0 \notin M$. Let $g_1 = P_M g_0$. Then

   $$g = \frac{g_0 - g_1}{|g_0 - g_1|}$$

   satisfies the required properties.

□

Given any $u \in H$, set

$$v = u - \lambda g \quad \textit{with } \lambda = \frac{\langle \varphi, u \rangle}{\langle \varphi, g \rangle}.$$

Note that v is well defined, since $\langle \varphi, g\rangle \neq 0$, and, moreover, $v \in M$, since $\langle \varphi, v\rangle = 0$. It follows that $(g, v) = 0$, i.e.,

$$\langle \varphi, u\rangle = \langle \varphi, g\rangle (g, u) \quad \forall u \in H,$$

which concludes the proof with $f = \langle \varphi, g\rangle g$.

Remark 1.3.1. *H and $H^\star$: to identify or not to identify? The triplet $V \subset H \subset V^\star$.*
Theorem 5.5 asserts that there is a canonical isometry from H onto $H^\star$. It is therefore "legitimate" to identify H and $H^\star$. We shall often do so but not always. Here is a typical situation—which arises in many applications—where one should be cautious with identifications. Assume that H is a Hilbert space with a scalar product $(\cdot,\cdot)$ and a corresponding norm $\|\cdot\|$. Assume that $V \subset H$ is a linear subspace that is dense in H. Assume that V has its own norm $\|\cdot\|$ and that V is a Banach space with $\|\cdot\|$. Assume that the injection $V \subset H$ is continuous, i.e.,

$$|v| \leq C\|v\| \quad \forall v \in V.$$

[For example, $H = L^2(0,1)$ and $V = L^p(0,1)$ with $p > 2$ or $V = C([0,1])$.]
There is a canonical map $T : H^\star \to V^\star$ that is simply the restriction to V of continuous linear functionals φ on H, i.e.,

$$\langle T\varphi, v\rangle_{V^\star} = \langle \varphi, v\rangle_{H^\star, H}.$$

It is easy to see that T has the following properties:

(i) $\|T\varphi\|_{V^\star} \leq C|\varphi|_{H^\star} \quad \forall \varphi \in H^\star$,

(ii) *T is injective,*

(iii) *$R(T)$ is dense in $V^\star$ if V is reflexive.*[1]

Identifying $H^\star$ with H and using T as a canonical embedding from $H^\star$ into $V^\star$, one usually writes

$$V \subset H \simeq H^\star \subset V^\star,$$

where all the injections are continuous and dense (provided V is reflexive). One says that H is the pivot space. Note that the scalar products $\langle \cdot,\cdot\rangle_{V^\star, V}$ and $\langle \cdot,\cdot\rangle$ coincide whenever both make sense, i.e.,

$$\langle f, v\rangle_{V^\star, V} = \langle f, v\rangle \quad \forall f \in H,\ \forall v \in V.$$

The situation becomes more delicate if V turns out to be a Hilbert space with its own scalar product $((\cdot,\cdot))$ associated to the norm $\|\cdot\|$. We could, of course, identify V and V^* with the help of $((\cdot,\cdot))$. However, (9) becomes absurd. This shows that one cannot identify simultaneously V and H with their dual spaces: one has to make a choice. The common habit is to identify H^* with H, to write (9), and not to identify V^* with V [naturally, there is still an isometry from V onto V^*, but it is not viewed as the identity map]. Here is a very instructive example.

Let

$$H = \ell^2 = \left\{ u = (u_n)_{n\geq 1} : \sum_{n=1}^{\infty} u_n^2 < \infty \right\}$$

equipped with the scalar product $((u,v)) = \sum_{n=1}^{\infty} u_n v_n$. *Let*

$$V = \left\{ u = (u_n)_{n\geq 1} : \sum_{n=1}^{\infty} n^2 u_n^2 < \infty \right\}$$

equipped with the scalar product $((u,v)) = \sum_{n=1}^{\infty} n^2 u_n v_n$.
Clearly $V \subset H$ *with continuous injection and* V *is dense in* H. *Here we identify* H^* *with* H, *while* V^* *is identified with the space*

$$V^* = \left\{ f = (f_n)_{n\geq 1} : \sum_{n=1}^{\infty} \frac{1}{n^2} f_n^2 < \infty \right\},$$

which is bigger than H. *The scalar product* $\langle \cdot, \cdot \rangle$ *is given by*

$$\langle f, v \rangle_{V^*,V} = \sum_{n=1}^{\infty} f_n v_n,$$

and the Riesz–Fréchet isomorphism $T : V \to V^*$ *is given by*

$$u = (u_n)_{n\geq 1} \mapsto Tu = (n^2 u_n)_{n\geq 1}.$$

Remark 1.3.2. *It is easy to prove that Hilbert spaces are reflexive without invoking the theory of uniformly convex spaces. It suffices to use twice the Riesz–Fréchet isomorphism (from* H *onto* H^* *and then from* H^* *onto* H^{**}*).*

Remark 1.3.3. *Assume that* H *is a Hilbert space identified with its dual space* H^*. *Let* M *be a subspace of* H. *We have already defined* $M^\perp$ *as a subspace of* H^*. *We may now consider it as a subspace of* H, *namely*

$$M^\perp = \{u \in H; (u,v) = 0 \quad \forall\, v \in M\}.$$

Clearly we have $M \cap M^\perp = \{0\}$. *Moreover, if* M *is closed we also have* $M + M^\perp = H$. *Indeed, every* $f \in H$ *may be written as*

$$f = (P_M f) + (f - P_M f)$$

and $f - P_M f \in M^\perp$; *more precisely,* $f - P_M f = P_{M^\perp} f$.
It follows that in a Hilbert space every closed subspace has a complement (in the sense of Section 2.4)

Proposition 1.3.3 (Weak Harnack Inequality)**.** *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with* $N \geq 3$, *and consider the linear operator in divergence form*

$$Lu = -\operatorname{div}(A(x) Du),$$

where $A(x)$ *is a symmetric matrix with coefficients* $a_{ij}(x) \in L^\infty(\Omega)$ *and is uniformly elliptic, i.e.,*

there exists a constant $\alpha > 0$ such that

$$A(x)\xi \cdot \xi \geq \alpha |\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega.$$

1. *Local Boundedness for Sub-solutions: For any $p > 0$, there exists a constant $C_1 = C_1(N, \alpha, p) > 0$ such that for any ball $B_{4R}(x_0) \subset \Omega$ and any non-negative weak* subsolution *$u \in H^1(\Omega)$ of $Lu = 0$, we have*

$$\sup_{B_R(x_0)} u \leq C_1 \left(\frac{1}{|B_{2R}(x_0)|} \int_{B_{2R}(x_0)} u^p \, dx \right)^{\frac{1}{p}}.$$

2. *Weak Harnack Inequality for Supersolutions: For any $q > 0$, there exists a constant $C_2 = C_2(N, \alpha, q) > 0$ such that for any ball $B_{4R}(x_0) \subset \Omega$ and any non-negative weak* supersolution *$u \in H^1(\Omega)$ of $Lu = 0$, we have*

$$\inf_{B_R(x_0)} u \geq C_2 \left(\frac{1}{|B_{2R}(x_0)|} \int_{B_{2R}(x_0)} u^{-q} \, dx \right)^{-\frac{1}{q}}.$$

Remark 1.3.4. *A function $u \in H^1(\Omega)$ is a weak subsolution of $Lu = 0$ if*

$$\int_\Omega ADu \cdot D\phi \, dx \leq 0 \qquad \forall \phi \in H_0^1(\Omega), \phi \geq 0.$$

It is a weak supersolution if the inequality is ≥ 0. The constant C_2 in part (ii) is independent of u. An immediate consequence is the strong maximum principle: if $u \geq 0$ is a supersolution that is positive somewhere, then it is positive almost everywhere.

1.4 Dominated Convergence Theorem

The Dominated Convergence Theorem (DCT) is a fundamental result in integration theory. It provides sufficient conditions under which the limit of integrals of a sequence of functions equals the integral of their pointwise limit. This theorem is indispensable in analysis, probability theory, and applications.

Definition 1.4.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on a measure space $(\Omega, \mathscr{A}, \mu)$. The sequence (f_n) is said to be dominated by an integrable function g if, for every $n \in \mathbb{N}$ and for almost every $x \in \Omega$,*

$$|f_n(x)| \leq g(x).$$

Theorem 1.4.1 (Lebesgue's Dominated Convergence Theorem)**.** *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions from Ω to $\mathbb{R}$ (or $\mathbb{C}$) satisfying:*

1. *The sequence (f_n) converges pointwise to a function f almost everywhere on Ω;*

2. *There exists a function $g \in L^1(\Omega, \mu)$ such that for all $n \in \mathbb{N}$,*

$$|f_n(x)| \leq g(x) \quad \text{for almost every } x \in \Omega.$$

Then f is integrable and

$$\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu.$$

Example 1.4.1 (A Simple Trigonometric Example)**.** *Consider the sequence $f_n(x) = \frac{\sin(nx)}{n}$ on $\Omega = [0, 2\pi]$ with the Lebesgue measure. For all x and n,*

$$|f_n(x)| \leq \frac{1}{n} \leq 1,$$

so we can take $g(x) \equiv 1$, which is integrable on $[0, 2\pi]$. The pointwise limit is $f(x) = 0$ for all x. By the DCT,

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{\sin(nx)}{n} \, dx = \int_0^{2\pi} 0 \, dx = 0.$$

Indeed, a direct calculation gives the same result since $\int_0^{2\pi} \sin(nx) \, dx = 0$ for all $n \geq 1$.

Example 1.4.2. *Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for $x \in (0, 1]$ and $f_n(0) = 0$.*
We have

$$|f_n(x)| \leq \frac{1}{2\sqrt{n}} \to 0$$

Actually, note that for all $t \geq 0$, $\frac{t}{1+t^2} \leq \frac{1}{2}$.
Setting $t = nx$, we get $|f_n(x)| \leq \frac{1}{2}$. The constant function $g(x) = 1/2$ is integrable on $(0, 1)$. The pointwise limit is $f(x) = 0$ for all x.
Hence,

$$\lim_{n\to\infty} \int_0^1 \frac{nx}{1 + n^2x^2} \, dx = 0.$$

Example 1.4.3. *Let $f_n(x) = n\chi_{(0,1/n]}(x)$ on $(0, 1]$.*
Then $f_n(x) \to 0$ pointwise for each $x > 0$.
However, $\int_0^1 f_n(x) \, dx = 1$ for all n, so the limit of integrals is $1 \neq 0$.
The DCT does not apply because there is no dominating integrable function: any candidate g would need to satisfy $g(x) \geq n$ on $(0, 1/n]$ for all n, which forces $g(x) = \infty$ on a set of positive measure.

Example 1.4.4. *Evaluate $\lim_{n\to\infty} \int_0^\infty \frac{\sin(x/n)}{x(1+x^2)} \, dx$.*
Note that $\left|\frac{\sin(x/n)}{x(1+x^2)}\right| \leq \frac{1}{1+x^2}$ for $x > 0$ because $|\sin(t)| \leq t$ and $|\sin(x/n)| \leq x/n$.
The function $g(x) = \frac{1}{1+x^2}$ is integrable on $(0, \infty)$.
The pointwise limit is 0 for each fixed x. Thus,

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(x/n)}{x(1 + x^2)} \, dx = \int_0^\infty 0 \, dx = 0.$$

Chapter 2

Fixed point theorems

The fixed point theory is a classical and powerful tool in nonlinear analysis, which has been extensively studied and generated an exceptionally rich literature. Our interest in this theory is primarily motivated by its wide-ranging applications to the existence problems of solutions to nonlinear differential equations. In this chapter, we present only a few basic forms of fixed point theorems those that, in our view, capture the essential features of the theory and are sufficient for addressing the nonlinear elliptic boundary value problems we aim to study. Further details can be found in [12, 15] and [18].

2.1 Banach fixed point theorem

We first give the Banach fixed point theorem on a complete metric space, which is perhaps the simplest fixed point theorem. Let X be a Hausdorff space and Ω a subset of X. Let $F : \Omega \mapsto X$ be a map. We say that $x \in \Omega$ is a fixed point of F, provided

$$x = F(x).$$

Let (X, d) be a complete metric space and Ω a closed subset of X. We say that a map $F : \Omega \mapsto \Omega$ is a δ-contraction, provided there exists a nonnegative number $\delta \in [0, 1)$ such that

$$d(A(x), A(y)) \leq \delta d(x, y) \quad \forall x, y \in \Omega,$$

where $d(\cdot, \cdot)$ is the distance function on X.

Theorem 2.1.1 (Banach fixed point theorem)**.** *Let (X, d) be a complete metric space and Ω a closed subset of X. Assume that $F : \Omega \mapsto \Omega$ is a δ-contraction for some $\delta \in [0, 1)$. Then F has a unique fixed point $x \in \Omega$.*

The Banach fixed point theorem is proved by an elementary iterative procedure (successive approximation). Taking any starting-point $x_0 \in M$, one defines a sequence

$$x_{n+1} = F(x_n) \in \Omega, \quad n = 0, 1, \ldots$$

Then the δ-contractiveness of F ensures that the sequence $\{x_n\}$ converges to a point $x \in \Omega$ (Ω being closed in X being complete). Moreover, x is unique in Ω.
The above simple iterative procedure also provides additional useful estimates on errors and rate of convergence. The Banach fixed point theorem is particularly effective in obtaining local existence,

say, for initial value problems. But it has limited applications in boundary value problems, partly because the required δ-contractiveness is usually not a trivial matter to be verified.
To illustrate the importance of the δ-contractiveness, we state the following fixed point theorem due to Browder, Gohde and Kirk.

Theorem 2.1.2 (Browder-Gohde-Kirk fixed point theorem)**.** *Let H be a Hilbert space and Ω a closed bounded convex subset of H. Assume that $F : \Omega \mapsto \Omega$ is nonexpansive, that is,*

$$(F(x), F(y)) \leq (x, y) \quad \forall x, y \in \Omega,$$

where $(\cdot, \cdot)$ is the inner product on H. Then F has at least one fixed point $x \in \Omega$.

Remark 2.1.1. *Note that the proof of Theorem 2.1.2 is based on the Banach fixed point theorem. By enriching the structure of the underlying space H, Theorem 2.1.2 slightly relaxes the δ-contractive requirement.*

2.2 Brouwer fixed point theorem

The Brouwer Fixed Point Theorem is a fundamental tool in proving the existence of Nash equilibrium in finite strategic games. Consider a game with n players, each with a compact convex set of mixed strategies $\Delta_i \subset \mathbb{R}^{m_i}$. Define the joint strategy space $\Delta = \prod_{i=1}^{n} \Delta_i$, which is also compact and convex.
For each player i, define the best-response correspondence $BR_i : \Delta \to \Delta_i$. Under standard continuity and concavity assumptions on utility functions, a continuous function $\mathbf{F} : \Delta \to \Delta$ can be constructed such that fixed points of $\mathbf{F}$ correspond to Nash equilibria.

Theorem 2.2.1. *Every finite normal-form game has at least one Nash equilibrium (in mixed strategies).*

In this section $B_r(x)$ denotes the closed ball defined by

$$B_r(x) = \{y \in \mathbb{R}^n \mid |y - x| \leq r\}.$$

Theorem 2.2.2. *Let $B_1(0)$ be the unit ball in $\mathbb{R}^n$. Let $\mathbf{F} : B_1(0) \to B_1(0)$ be a continuous mapping, then $\mathbf{F}$ admits a fixed point, that is to say, a point for which*

$$\mathbf{F}(x) = x. \tag{A.1}$$

Proof. Let us give first a naive proof when $n = 1$. In this case, $F : [-1, 1] \to [-1, 1]$. If we set

$$G(x) = F(x) - x,$$

we have $G(-1) = F(-1) + 1 \geq 0$, $G(1) = F(1) - 1 \leq 0$ since $F(-1), F(1) \in [-1, 1]$. Since G is continuous, there exists a point x such that $G(x) = 0$, i.e., such that (A.1) holds.
The proof in higher dimensions is more involved and requires some preliminaries. □

Definition 2.2.1. *Let A be an $n \times n$ matrix. The matrix of the cofactors of A is the matrix given by*

$$\mathrm{Cof} A = ((-1)^{i+k} \det A_{ik})_{i,k=1,\dots,n} \tag{A.2}$$

where A_{ik} denotes the matrix deduced from A by deleting the ith row and the kth column.

We then have

Proposition 2.2.1. *Let $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2-mapping. Denote by $(\mathrm{Cof}(\nabla \mathbf{u}))_{ik}$ the entries of $\mathrm{Cof}(\nabla \mathbf{u})$. Then we have with the summation convention in k*

$$\partial_{x_k} (\mathrm{Cof}(\nabla \mathbf{u})_{ik}) = 0 \quad \forall i = 1, \dots, n, \tag{A.3}$$

i.e., the rows of the matrix $\mathrm{Cof}(\nabla \mathbf{u})$ are divergence-free.

Proof. By defn of the matrix $\mathrm{Cof} A$, we have

$$(\det A)\,\mathrm{Id} = A^T \,\mathrm{Cof}\, A. \tag{A.4}$$

This matricial equality can also be written as

$$(\det A)\delta_{ij} = a_{ki} (\mathrm{Cof}\, A)_{kj}$$

(we set $A = (a_{ij})$). In particular, since $(\mathrm{Cof} A)_{kj}$ contains no term in a_{ki}, we have

$$\partial_{a_{ki}} (\det A) = (\mathrm{Cof}\, A)_{ki}. \tag{A.5}$$

Replacing now A by $\nabla \mathbf{u}$ in (A.4), we derive

$$(\det \nabla \mathbf{u})\delta_{ij} = \partial_{x_i} u^k (\mathrm{Cof} \nabla \mathbf{u})_{kj} \quad \forall i, j = 1, \dots, n,$$

where $\mathbf{u} = (u^1, \dots, u^n)$. Taking the derivative in the direction x_j and summing up, we obtain

$$\partial_{x_j} (\det \nabla \mathbf{u})\delta_{ij} = \partial^2_{x_i x_j} u^k (\mathrm{Cof} \nabla \mathbf{u})_{kj} + \partial_{x_i} u^k \partial_{x_j} (\mathrm{Cof} \nabla \mathbf{u})_{kj}. \tag{A.6}$$

Using the chain rule and (A.5), we also have

$$\partial_{x_i} (\det \nabla \mathbf{u}) = (\mathrm{Cof} \nabla \mathbf{u})_{kj} \cdot \partial_{x_i} (\partial_{x_j} u^k)$$

and (A.6) becomes

$$\partial_{x_i} u^k \partial_{x_j} (\mathrm{Cof} \nabla \mathbf{u})_{kj} = 0 \quad \forall i = 1, \dots, n. \tag{A.7}$$

If $\nabla \mathbf{u}(x)$ is invertible, then we obtain

$$\partial_{x_j} (\mathrm{Cof} \nabla \mathbf{u}(x))_{kj} = 0.$$

If $\det \nabla \mathbf{u}(x) = 0$, then replacing $\mathbf{u}$ by $\mathbf{u} + \epsilon \mathbf{x}$, the result follows by letting $\epsilon \to 0$. This completes the

proof of the proposition. □

As a consequence, we have

Proposition 2.2.2. *Let Ω be a smooth bounded open set of $\mathbb{R}^n$ and $\mathbf{u},\mathbf{v}:\overline{\Omega}\to\mathbb{R}^n$ be $C^2(\overline{\Omega})$-mappings such that*

$$\mathbf{u}=\mathbf{v} \quad \text{on } \partial\Omega. \tag{A.8}$$

Then we have

$$\int_\Omega \det\nabla\mathbf{u}\,dx=\int_\Omega \det\nabla\mathbf{v}\,dx. \tag{A.9}$$

Proof. We set for $t\in[0,1]$

$$f(t)=\int_\Omega \det\nabla[\mathbf{u}+t(\mathbf{v}-\mathbf{u})]\,dx.$$

By the chain rule, we have (see (A.5))

$$\begin{aligned}
f'(t)&=\int_\Omega \frac{d}{dt}\det[\nabla(\mathbf{u}+t(\mathbf{v}-\mathbf{u}))]dx\\
&=\int_\Omega (\mathrm{Cof}[\nabla(\mathbf{u}+t(\mathbf{v}-\mathbf{u}))])_{ik}\,\partial_{x_k}(\mathbf{v}^i-\mathbf{u}^i)dx\\
&=\int_\Omega \partial_{x_k}\Big(\mathrm{Cof}[\nabla(\mathbf{u}+t(\mathbf{v}-\mathbf{u}))]_{ik}(\mathbf{v}^i-\mathbf{u}^i)\Big)dx \quad \text{(by (A.3))}\\
&=0 \quad \text{(by (A.8))}.
\end{aligned}$$

It follows that f is constant on $(0,1)$, and since it is continuous, $f(0)=f(1)$. This is (A.9) and the proof of the proposition is complete. □

Proof. (Brouwer's theorem)

1. One cannot find $\mathbf{w}:B_1(0)\to\partial B_1(0)$ which is C^2 and satisfies $\mathbf{w}(x)=x$ on $\partial B_1(0)$. Indeed, if such a function exists, by the proposition above, we have

$$\int_{B_1(0)}\det\nabla\mathbf{w}=\int_{B_1(0)}\det\nabla\,\mathrm{Id}(x)=|B_1(0)| \tag{A.10}$$

where $|B_1(0)|$ denotes the measure of the unit ball. On the other hand, we have

$$\mathbf{w}\cdot\mathbf{w}\equiv 1,$$

and differentiating we derive

$$\nabla\mathbf{w}\cdot\mathbf{w}\equiv 0.$$

It follows that $\det\nabla\mathbf{w}\equiv 0$, a contradiction to (A.10).

2. One cannot find $\mathbf{w} : B_1(0) \to \partial B_1(0)$ continuous and satisfying $\mathbf{w}(x) = x$ on $\partial B_1(0)$. Suppose such a function exists. We extend it by x outside $B_1(0)$. Then we consider

$$\mathbf{w}_\epsilon(x) = \rho_\epsilon * \mathbf{w}(x), \tag{A.11}$$

i.e., we mollify each component of $\mathbf{w}$. Then for ϵ small enough, one has

$$\mathbf{w}_\epsilon(x) = x \quad \text{on } \partial B_2(0). \tag{A.12}$$

Moreover see Chapter 2 $\mathbf{w}_\epsilon \to \mathbf{w}$ uniformly on $B_2(0)$ and for ϵ small enough, we have $|\mathbf{w}_\epsilon| > \frac{1}{2}$. It follows that the mapping

$$\tilde{\mathbf{w}}_\epsilon(\mathbf{x}) = \frac{2\mathbf{w}_\epsilon}{|\mathbf{w}_\epsilon|}$$

is a smooth mapping satisfying all the assumptions of part 1 with $B_1(0)$ replaced by $B_2(0)$. This completes the proof of this part.

3. Let $\mathbf{F}$ be a continuous map such that $\mathbf{F}(x) \neq x$ for all x. Let $\mathbf{w}(x)$ be the point of $\partial B_1(0)$ where the half line from $\mathbf{F}(x)$ through x crosses $\partial B_1(0)$. Then $\mathbf{w}$ is a mapping as in part 2 and we get a contradiction. This completes the proof of the theorem (Figure A.1).

□

Example 2.2.1. *(Counter Examples) The following counter examples show the essentials of each assumption in the Brower Fixed-Point Theorem*

- $S = [0,1]$ *compact, convex and nonempty, but* $A : S \to S$ *not continuous and the graph* $y = A(x)$ *does not cross the diagonal* $y = x$*. No fixed point.*
- $S = \mathbb{R}$ *and* $A : S \to S$, $A(x) = x+1$*. A is continuous, S is convex, nonempty, but not compact. No fixed point.*
- *Let S be a closed annulus and* $A : S \to S$ *is a rotation of the annulus around the center. A proper rotation is fixed-point free. In this case, S is compact, nonempty but not convex.*

Corollary 2.2.1. *Let K be a compact convex subset of* $\mathbb{R}^n$ *and* $F : K \to K$ *be a continuous mapping. Then F admits a fixed point.*

Proof. Since K is compact, it is bounded. Let $R > 0$ such that $K \subset B_R(0)$. Set

$$\mathbf{G} = \mathbf{F} \circ \mathbf{P}_K$$

where $\mathbf{P}_K$ denotes the projection on the convex set K (see Theorem 1.1 and (11.62)). Then $\mathbf{G}$ is a continuous mapping from $B_R(0)$ into itself, and by Theorem A.1 it possesses a fixed point x, i.e., such that

$$\mathbf{F}(\mathbf{P}_K(x)) = x.$$

Since $x \in K$, this is also a fixed point for $\mathbf{F}$. This completes the proof. □

2.3 Schauder fixed point theorem

Theorem 2.3.1. *If $T \in \mathscr{K}(E,F)$, then $T^\star \in \mathscr{K}(F^\star, E^\star)$. And conversely.*

Proof. We have to show that $T^\star(B_{F^\star})$ has compact closure in $E^\star$. Let (v_n) be a sequence in $B_{F^\star}$. We claim that $(T^\star(v_n))$ has a convergent subsequence. Set $K = T(B_E)$; this is a compact metric space. Consider the set $H \subset C(K)$ defined by

$$H = \{\varphi_n : x \in K \mapsto \langle v_n, x\rangle; n = 1,2,\dots\}.$$

The assumptions of AscoliArzelà's theorem (Theorem 4.25) are satisfied. Thus, there is a subsequence, denoted by φ_{n_k}, that converges uniformly on K to some continuous function $\varphi \in C(K)$. In particular, we have

$$\sup_{u\in B_E} |\langle v_{n_k}, Tu\rangle - \varphi(Tu)| \xrightarrow[k\to\infty]{} 0.$$

Thus

$$\sup_{u\in B_E} |\langle v_{n_k}, Tu\rangle - \langle v_{n_\ell}, Tu\rangle| \xrightarrow[k,\ell\to\infty]{} 0,$$

i.e., $\|T^\star v_{n_k} - T^\star v_{n_\ell}\|_{E^\star} \xrightarrow[k,\ell\to\infty]{} 0$. Consequently $T^\star v_{n_k}$ converges in $E^\star$.
Conversely, assume $T^\star \in \mathscr{K}(F^\star, E^\star)$. We already know, from the first part, that $T \in \mathscr{K}(E,F)$. In particular, $T(B_E)$ has compact closure in F. But $T(B_E) = T(B_E)$ and F is closed in F^{**}. Therefore $T(B_E)$ has compact closure in F. □

Remark 2.3.1. *Let E and F be two Banach spaces and let $T \in \mathscr{K}(E,F)$. If (u_n) converges weakly to u in E, then (Tu_n) converges **strongly** to Tu. The converse is also true if E is reflexive .*

Theorem 2.3.2 (Schauder Fixed Point Theorem)**.** *Let X be a real Banach space. Suppose $K \subset X$ is compact, convex and $A : K \to K$ is continuous. Then A has a fixed point in K, i.e., there exists $u \in K$ such that $A(u) = u$.*

Proof. Fix $\varepsilon > 0$ and choose $u_1, \dots, u_{N_\varepsilon} \in K$ such that

$$K \subset \bigcup_{i=1}^{N_\varepsilon} B(u_i, \varepsilon).$$

Existence of such a finite covering follows from compactness of K. Choosing ε such that $2\varepsilon <$ diam K we have that $K \setminus B(u_i,\varepsilon) \neq \emptyset$ for all $i = 1,2,\dots,N_\varepsilon$.
Let K_ε be the convex hull of the set $\{u_1,\dots,u_{N_\varepsilon}\}$, i.e.,

$$K_\varepsilon = \left\{ \sum_{i=1}^{N_\varepsilon} \lambda_i u_i \;\middle|\; 0 \le \lambda_i \le 1,\ \sum_{i=1}^{N_\varepsilon} \lambda_i = 1 \right\}.$$

Then $K_\varepsilon \subset K$ since K is convex.
Define $P_\varepsilon : K \to K_\varepsilon$ by

$$P_\varepsilon(u) = \frac{\sum_{i=1}^{N_\varepsilon} \operatorname{dist}\big(u, K \cap B(u_i,\varepsilon)\big)\, u_i}{\sum_{i=1}^{N_\varepsilon} \operatorname{dist}\big(u, K \setminus B(u_i,\varepsilon)\big)}.$$

The denominator is always $\neq 0$ because each $u \in K$ belongs to some $B(u_i,\varepsilon)$ and hence

$$\operatorname{dist}\big(u, K \setminus B(u_i,\varepsilon)\big) > 0.$$

Also $P_\varepsilon(u) \in K_\varepsilon$ because

$$P_\varepsilon(u) = \sum_i \lambda_i u_i \quad \text{where} \quad \lambda_i = \frac{\operatorname{dist}\big(u, K \setminus B(u_i,\varepsilon)\big)}{\sum_j \operatorname{dist}\big(u, K \setminus B(u_j,\varepsilon)\big)}.$$

Clearly P_ε is continuous and for every $u \in K$

$$\|P_\varepsilon(u) - u\| \le \frac{\sum_i \operatorname{dist}\big(u, K \setminus B(u_i,\varepsilon)\big)\, \|u_i - u\|}{\sum_i \operatorname{dist}\big(u, K \setminus B(u_i,\varepsilon)\big)} \le \varepsilon,$$

because

$$\operatorname{dist}\big(u, K \setminus B(u_i,\varepsilon)\big) = 0 \quad \text{if } \|u - u_i\| \ge \varepsilon.$$

Define a mapping

$$A_\varepsilon : K_\varepsilon \to K_\varepsilon, \qquad A_\varepsilon(u) := P_\varepsilon(A(u)).$$

K_ε is a finite-dimensional convex set, so it is homeomorphic to a finite-dimensional closed ball. Hence, by the classical Brouwer fixed point theorem,

$$A_\varepsilon(u_\varepsilon) = u_\varepsilon \quad \text{for some } u_\varepsilon \in K_\varepsilon.$$

Since K is compact and $u_\varepsilon \in K_\varepsilon \subset K$, there exists a sequence $\varepsilon_j \to 0$ such that

$$u_{\varepsilon_j} \to u \in K.$$

We claim that $A(u) = u$.
We have

$$\|u_{\varepsilon_j} - A(u_{\varepsilon_j})\| = \|A_{\varepsilon_j}(u_{\varepsilon_j}) - A(u_{\varepsilon_j})\| = \|P_{\varepsilon_j}(A(u_{\varepsilon_j})) - A(u_{\varepsilon_j})\| \le \varepsilon_j.$$

Thus

$$\|u_{\varepsilon_j} - A(u_{\varepsilon_j})\| \le \varepsilon_j \to 0.$$

Taking the limit and using continuity of A, we obtain $A(u) = u$. □

Example 2.3.1. *Suppose we are given a system of n equations in n unknowns of the form*

$$g_j(x_1, \ldots, x_n) = 0, \quad j = 1, \ldots, n$$

where the g_j are continuous real-valued functions of the real variables x_j. Let

$$h_j(x_1,\dots,x_n) = g_j(x_1,\dots,x_n) + x_j,$$

and for any point $x = (x_1,\dots,x_n)$ define

$$h(x) = (h_1(x),\dots,h_n(x)).$$

Assume now that h has a fixed point $\bar{x} \in \mathbb{R}^n$. Then it is easily seen that $\bar{x}$ is a solution to the system of equations.

Remark 2.3.2. *Note that various application of fixed point theorems can be found in [18].*

Example 2.3.2. *(Fixed Point Formulation of $\nabla \cdot v = f$)*
We start with the equation:

$$\nabla \cdot v = f$$

where v is a vector field and f is a given function.
We aim to rewrite the problem in the form:

$$v = T(v)$$

Assume we solve the Poisson equation:

$$\Delta u = f$$

and define:

$$v = \nabla u$$

If f depends on v, for example:

$$f = g(v)$$

then the equation becomes:

$$\nabla \cdot v = g(v)$$

We rewrite the problem as:

$$v = \nabla\left(\Delta^{-1}(g(v))\right)$$

Define the operator:

$$T(v) = \nabla\left(\Delta^{-1}(g(v))\right)$$

Thus, the problem becomes:

$$v = T(v)$$

If the operator T satisfies suitable conditions such as continuity and compactness or contraction, then we can apply a fixed point theorem (e.g., Banach or Schauder) to conclude that there exists a solution v such that:

$$v = T(v)$$

Chapter 3

Analysis of the Singular Schrödinger-Maxwell System

In this chapter, we study the existence of weak solutions in the Sobolev space $H_0^1(\Omega) \times H_0^1(\Omega)$ for a Schrödinger–Maxwell system via an approach based on approximation schemes and a priori estimates on the approximating sequences of solutions, and we show that the integrability assumptions on the datum yield higher integrability of the solutions. The results of this chapter are published in [5, 11].

3.1 Origin and physical context

As highlighted in Giannone's paper ([11]), the system originated from the seminal work of V. Benci and D. Fortunato ([3]) on an eigenvalue problem for the Schrödinger operator coupled with an electromagnetic field.
The classical system is given by:

$$\begin{cases} -\frac{1}{2}\Delta u + \phi u = \omega u \\ -\Delta \phi = 4\pi u^2 \end{cases}$$

- *The first equation is a Schrödinger-type equation. Here, u is the wave function of a particle, and the term ϕu represents the interaction energy between the particle and the electrostatic potential ϕ.*
- *The second equation is a Poisson equation (a simplified form of Maxwell's equations for the electrostatic field). It states that the source of the electric potential ϕ is the charge density, which is proportional to $|u|^2 = u^2$ for a real wave function.*

This coupled system models the behavior of a standing wave in a Bose-Einstein condensate of electrically charged particles. The wave function is written as $\psi(x,t) = e^{-i\omega t}u(x)$, where $u(x)$ is the stationary, real-valued amplitude. The system essentially describes a particle in a self-consistent field: the particle's probability density (u^2) generates an electrostatic potential (ϕ), which in turn influences the particle's own dynamics. This is a mean-field model where each particle interacts with the average field created by all particles.
This Schrödinger-Maxwell system provides a powerful mathematical framework for analyzing the self-consistent interaction between quantum particles and electromagnetic fields.

3.2 Statement of the results

We consider the doubly singular semi-linear system

$$\begin{cases} -\operatorname{div}(B(x)Du) + v^{1-\theta}u^{r-1} = \frac{f(x)}{u^{\gamma}} & \text{in } \Omega \\ -\operatorname{div}(B(x)Dv) = \frac{u^r}{v^{\theta}} & \text{in } \Omega \\ v > 0,\ u \geq 0 & \text{in } \Omega \\ u > 0 & \text{in } \{f > 0\} \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Where Ω is a bounded domain in $\mathbb{R}^N$, with $N \geq 3$, and f is a non-negative and non-trivial function belonging to $L^m(\Omega)$ for some $m \geq 1$, and $B(\cdot) \in L^{\infty}(\Omega; \mathbb{R}^{N\times N})$ is a symmetric and uniformly elliptic matrix, in the sense that

$$\alpha|\xi|^2 \leq B(x)\xi \cdot \xi \leq \beta|\xi|^2 \tag{3.2}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^N$, with $0 < \alpha \leq \beta$. Moreover, we assume the following assumption on the parameters:

$$r \geq 2, \quad \gamma \in (0,1), \quad \theta \in [0,1). \tag{3.3}$$

We study the regularizing effects on the existence of solutions $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ arising from the interplay between the system's structure and the integrability of the datum f. These effects yield higher integrability for the solutions than the one prescribed by the theory of elliptic (single) equations.

Remark 3.2.1. *Note that the system (3.1) is inspired by the following Schrödinger–Maxwell system,*

$$\begin{cases} -\frac{1}{2}\Delta u + \phi u = \omega u \\ -\Delta\phi = 4\pi u^2 \end{cases}$$

which first appeared in the seminal work [3], where the authors investigated the eigenvalue problem for the Schrödinger operator coupled with an electromagnetic field as mentioned before.

Remark 3.2.2. *Remark that in [5] the authors established some regularizing effects for the singular elliptic system in the case when $0 < \theta < 1$, $M = B$ and $\gamma = 0$ correspond to (3.1):*

$$\begin{cases} -\operatorname{div}(B(x)Du) + v^{1-\theta}u^{r-1} = f & \text{in } \Omega \\ -\operatorname{div}(M(x)Dv) = \dfrac{u^r}{v^{\theta}} & \text{in } \Omega \\ u, v > 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

Remark 3.2.3. *It is worth to notice that, the singular nonlinearities for Schrödinger–Maxwell sys-*

tems were first studied in [4], where the authors considered the elliptic problem

$$\begin{cases} -\operatorname{div}(B(x)Du) + vu^{r-1} = \dfrac{1}{u^{\gamma}} & \text{in } \Omega \\ -\operatorname{div}(M(x)Dv) = u^{r} & \text{in } \Omega \\ u, v > 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

covering both mild and strong singularities.

Remark 3.2.4. *The assumption $m > \frac{N}{2}$ on the integrability of the datum f is critical for obtaining bounded solutions u and v. When m is larger than the critical exponent $N/2$, Stampacchia's L^{∞} estimate applies directly to the first equation, yielding a uniform bound for u_n. This bound then propagates to v_n via the second equation. For $m = N/2$ the solution u may still be unbounded but belongs to every L^{p} space (a borderline regularising effect). The case $m < N/2$ is more delicate and requires the notion of finite-energy solutions (see [11]).*

Main results

We deal only with the case $m \geq \frac{N}{2}$.

Definition 3.2.1 (Weak solutions)**.** *We say that $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a* weak solution *to system (3.1) if $u, v > 0$ a.e. in Ω,*

$$\frac{f}{u^{\gamma}}\varphi,\ \frac{u^{r}}{v^{\theta}}\varphi \in L^{1}(\Omega) \quad \forall \varphi \in H_0^1(\Omega), \tag{3.4}$$

and

$$\begin{cases} \displaystyle\int_{\Omega} B(x)Du \cdot D\varphi + v^{1-\theta}u^{r-1}\varphi \,\mathrm{d}x = \int_{\Omega} \frac{f}{u^{\gamma}}\varphi \,\mathrm{d}x & \forall \varphi \in H_0^1(\Omega) \\ \displaystyle\int_{\Omega} B(x)Dv \cdot D\psi \,\mathrm{d}x = \int_{\Omega} \frac{u^{r}}{v^{\theta}}\psi \,\mathrm{d}x & \forall \psi \in H_0^1(\Omega). \end{cases} \tag{3.5}$$

We have the following main result:

Theorem 3.2.1. *If $m \geq \frac{N}{2}$, then system (3.1) has a weak solution (u, v), such that u belongs to $L^{\infty}(\Omega)$ if $m > \frac{N}{2}$, and to every $L^{p}(\Omega)$, $1 \leq p < \infty$, if $m = \frac{N}{2}$. Moreover, v belongs to $L^{\infty}(\Omega)$.*

Remark 3.2.5. *Note that the case $m < \frac{N}{2}$ has been done in [10] for non-singular problems.*

Remark 3.2.6. *We observe that the system (3.1) is a doubly singular version of the classical Schrödinger–Maxwell equations. The presence of the singular terms $u^{-\gamma}$ and $v^{-\theta}$ introduces additional difficulties because the nonlinearities blow up when u or v approach zero. The approximation scheme employed in the proof (adding $1/n$ in the denominators) is a standard way to regularise the problem.*

3.3 An overview of the proofs of main results

Our method relies on an approximation scheme that yields a sequence (u_n, v_n) of solutions to a suitable elliptic system, together with a priori estimates that ensure the convergence of (u_n, v_n) to a solution of the original problem with the required integrability properties.

Step 1: Approximation scheme

We consider the following problem:

$$\begin{cases} -\operatorname{div}(B(x)Du_n) + v_n^{1-\theta} u_n^{r-1} = \dfrac{f_n}{\left(u_n + \frac{1}{n}\right)^{\gamma}} & \text{in } \Omega \\ -\operatorname{div}(B(x)Dv_n) = \dfrac{u_n^r}{\left(v_n + \frac{1}{n}\right)^{\theta}} & \text{in } \Omega \\ u_n, v_n > 0 & \text{in } \Omega \\ u_n = v_n = 0 & \text{on } \partial\Omega \end{cases} \tag{3.6}$$

has a unique weak, bounded solution. The existence is proved via Schauder's fixed point theorem; the positivity follows from the weak Harnack inequality.

Lemma 3.3.1. *For every $n \in \mathbb{N}$ the system (3.6) possesses a unique weak solution $(u_n, v_n) \in (H_0^1(\Omega) \bigcap L^\infty(\Omega))^2$. Moreover $u_n, v_n > 0$ a.e. in Ω.*

Proof. Let start with a fixed-point for the first equation with a given coefficient. Let $z \in L^\infty(\Omega)$, $z \geq 0$ be fixed. Consider the problem

$$\begin{cases} -\operatorname{div}(B(x)Du) + z^{1-\theta} u^{r-1} = \dfrac{f}{(u + 1/n)^{\gamma}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

We claim that (3.7) has a unique solution $u \in H_0^1(\Omega) \bigcap L^\infty(\Omega)$.
For a given $\psi \in H_0^1(\Omega)$ set $F(\psi) := f\left(|\psi| + 1/n\right)^{-\gamma} \in L^\infty(\Omega)$. The linear problem

$$-\operatorname{div}(B(x)Du) + c(x)u = F(\psi), \quad u \in H_0^1(\Omega),$$

with $c(x) := z^{1-\theta} T_k(|\psi|^{r-2})$, where $k > 0$ is chosen large enough so that any candidate solution will satisfy $u^{r-2} \leq k$ (this will be guaranteed by the a priori bound below), has a unique solution $u = S(\psi)$ by the Lax-Milgram theorem (the bilinear form is coercive because $c \geq 0$).
Using $G_k(u)$ as a test function and the fact that $F(\psi) \leq n^{1+\gamma}$ we obtain, via Theorem of Stampacchia, an L^∞ bound $\|u\|_{L^\infty} \leq C$ independent of ψ.
Choosing k larger than this bound makes the truncation T_k inactive, so S maps the ball $B_R := \{\psi \in H_0^1 : \|\psi\|_{H_0^1} \leq R\}$ into itself for sufficiently large R.
One shows that S is compact and continuous (by standard compactness arguments and the dominated convergence theorem).

Schauder's fixed point theorem yields $u = S(u)$, which is exactly a weak solution of (3.7) and satisfies $u \in L^\infty(\Omega)$.
Moreover, if u_1, u_2 are two positive solutions, test the difference of the equations with $(u_1 - u_2)_+$ and use the monotonicity of $s \mapsto s^{r-1}$ and $s \mapsto (s + 1/n)^{-\gamma}$ to obtain

$$\alpha \int_\Omega |D(u_1-u_2)_+|^2 \, dx \le \int_\Omega z^{1-\theta}\left(u_2^{r-1}-u_1^{r-1}\right)(u_1-u_2)_+ \, dx + \int_\Omega f\Big((u_2+1/n)^{-\gamma}-(u_1+1/n)^{-\gamma}\Big)(u_1-u_2)_+ \, dx \le 0,$$

hence $(u_1 - u_2)_+ = 0$, i.e. $u_1 \le u_2$.
Reversing the roles gives $u_1 = u_2$.
Second, we state a fixed-point for the second equation with a given right-hand side. For $\phi \in H_0^1(\Omega)$ consider

$$\begin{cases} -\operatorname{div}(B(x)Dv) = \dfrac{|\phi|^r}{(v+1/n)^\theta} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Because the right-hand side is non-negative and bounded in L^∞ (indeed $|\phi|^r \le C$ if we restrict to functions with a priori bounded L^∞ norm), the existence and uniqueness of a weak solution $v \in H_0^1(\Omega) \bigcap L^\infty(\Omega)$ follow from the same fixed-point argument(using the monotonicity of $s \mapsto (s + 1/n)^{-\theta}$). Moreover, $v \ge 0$ a.e.
We now construct a fixed point of the map $\Phi : H_0^1(\Omega) \to H_0^1(\Omega)$ defined by

$$\Phi(\phi) = u, \quad \text{where } u \text{ solves (1) with } z = v_\phi \text{ and } v_\phi \text{ solves (3.8)}.$$

More precisely:

1. Given $\phi \in H_0^1(\Omega)$, solve (3.8) to obtain v_ϕ (unique, bounded).
2. With $z = v_\phi$, solve (3.7) to obtain u_ϕ (unique, bounded).
3. Set $\Phi(\phi) = u_\phi$.

We claim that Φ maps a large ball $B_R \subset H_0^1(\Omega)$ into itself and is compact and continuous.
From the second equation we have $-\operatorname{div}(BDv_\phi) \le |\phi|^r$.
Using $G_k(v_\phi)$ as a test function, we obtain an L^∞ bound $\|v_\phi\|_{L^\infty} \le C_1 \|\phi\|_{L^r}^r$.
For ϕ in a ball of radius R, $\|\phi\|_{L^r} \le C_2 R$ (by Sobolev embedding), so $\|v_\phi\|_{L^\infty} \le C_3 R^r$.
Then from the first equation, using u_ϕ as a test function and the fact that $f \le n$, we get an H_0^1 bound $\|u_\phi\|_{H_0^1} \le C_4(n)$ independent of ϕ (but depending on n). Moreover, by result applied to the first equation (the right-hand side $f(u_\phi + 1/n)^{-\gamma}$ is bounded in L^∞), we obtain

$$\|u_\phi\|_{L^\infty} \le C_5(n)$$

Choosing R larger than all these bounds ensures $\Phi(B_R) \subset B_R$.
If $\phi_j \to \phi$ in H_0^1, then up to a subsequence $v_{\phi_j} \to v_\phi$ strongly in H_0^1 (by the dominated convergence theorem and the fact that the right-hand side of (3.8) depends continuously on ϕ). T
hen using the same argument, we deduce $u_{\phi_j} \to u_\phi$ strongly in H_0^1.

Hence Φ is continuous and compact (since it maps bounded sets into relatively compact sets).
Schauder's fixed point theorem gives ϕ such that

$$\Phi(\phi) = \phi$$

Setting $u := \phi$ and $v := v_\phi$ yields a solution of (S) in the weak sense, with $u, v \in H_0^1(\Omega) \cap L^\infty(\Omega)$.
In addition, we have $u \geq 0$, $v \geq 0$ a.e. because the equations produce non-negative solutions (test with negative parts).
To obtain strict positivity we use the weak Harnack inequality.
For v, note that $-\operatorname{div}(BDv) = u^r(v+1/n)^{-\theta} \geq 0$, so v is a non-negative supersolution of $-\operatorname{div}(BDw) = 0$.
Since $u \not\equiv 0$ (because $f \not\equiv 0$ forces $u > 0$ somewhere), v cannot be identically zero; the weak Harnack inequality then implies $v > 0$ a.e. in Ω.
For u, we have

$$-\operatorname{div}(BDu) + v^{1-\theta}u^{r-1} \geq 0$$

and $v^{1-\theta}u^{r-2} \in L^\infty$ (since $u, v \in L^\infty$). Hence the same Harnack argument yields $u > 0$ a.e. in Ω.
Furthermore, suppose (u_1, v_1) and (u_2, v_2) are two solutions.
Test the difference of the first equations with $(u_1 - u_2)_+$ and the difference of the second equations with $(v_1 - v_2)_+$, using the monotonicity of the nonlinearities.
Standard arguments give $u_1 = u_2$ and $v_1 = v_2$ a.e.
Thus the solution is unique.
This completes the proof of the lemma.

Remark 3.3.1. *Note that the weak Harnack inequality is a fundamental tool to deduce positivity of weak solutions and supersolutions. In the proof we apply it twice:*

- *For v: since $-\operatorname{div}(BDv) \geq 0$ and $v \not\equiv 0$, the inequality forces $v > 0$ a.e. in Ω.*
- *For u: we have $-\operatorname{div}(BDu) + c(x)u \geq 0$ with $c = v^{1-\theta}u^{r-2} \in L^\infty$; the Harnack inequality for such operators still gives $u > 0$ a.e.*

Without this result we could only guarantee non-negativity.

□

Step2: A priori estimates

Boundedness of u_n**.** *Take $k \geq 1$ and use the test function $G_k(u_n) := (u_n - k)_+$ in the first equation of (2). Because $G_k(u_n) \geq 0$ and $v_n^{1-\theta}u_n^{r-1}G_k(u_n) \geq 0$, we obtain*

$$\alpha \int_\Omega |DG_k(u_n)|^2\,dx \leq \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\gamma} G_k(u_n)\,dx \leq \int_\Omega f_n G_k(u_n)\,dx,$$

where we used $(u_n + 1/n)^{-\gamma} \le 1$. *Since* $f_n \le f \in L^m$ *with* $m > N/2$, *Theorem of Stampacchia applies and yields a constant* $C = C(N, |\Omega|, \alpha, m) > 0$ *such that*

$$\|u_n\|_{L^\infty} \le 1 + C\|f\|_{L^m} \qquad \forall n.$$

Thus (u_n) *is uniformly bounded in* $L^\infty(\Omega)$.

Boundedness of v_n**.** *From the second equation in* (3.6) *we have*

$$-\operatorname{div}(BDv_n) = \frac{u_n^r}{(v_n + 1/n)^\theta} \le u_n^r.$$

Because u_n *is uniformly bounded in* L^∞, *the right-hand side* u_n^r *is uniformly bounded in* $L^\infty(\Omega)$. *Applying Theorem from the Stampacchia's* L^∞ *estimate for solutions of linear elliptic equations to* $-\operatorname{div}(BDv_n) = g_n$ *with* $\|g_n\|_{L^\infty} \le C$ *gives*

$$\|v_n\|_{L^\infty} \le C' \qquad \forall n,$$

where C' *depends only on* $N, |\Omega|, \alpha$ *and the uniform bound on* g_n. *Hence* (v_n) *is also uniformly bounded in* $L^\infty(\Omega)$.

Test the first equation of (3.6) *with* u_n *and the second with* v_n*:*

$$\alpha\|u_n\|_{H_0^1}^2 \le \int_\Omega \frac{f_n}{(u_n + 1/n)^\gamma}\, u_n\, dx \le \int_\Omega f\, u_n^{1-\gamma}\, dx,$$
$$\alpha\|v_n\|_{H_0^1}^2 \le \int_\Omega \frac{u_n^r}{(v_n + 1/n)^\theta}\, v_n\, dx \le \|v_n\|_{L^\infty}^{1-\theta} \int_\Omega u_n^r\, dx.$$

Using the uniform L^∞ *bounds on* u_n *and* v_n, *the right-hand sides are uniformly bounded. Consequently,*

$$\|u_n\|_{H_0^1} + \|v_n\|_{H_0^1} \le C \qquad \forall n.$$

Therefore, up to a subsequence, there exist $u, v \in H_0^1(\Omega)$ *such that*

$$\begin{aligned} &u_n \rightharpoonup u, \quad v_n \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega), \\ &u_n \to u, \quad v_n \to v \quad \text{strongly in } L^q(\Omega) \text{ for all } 1 \le q < 2^*, \\ &u_n \to u, \quad v_n \to v \quad \text{a.e. in } \Omega. \end{aligned}$$

Moreover, the uniform L^∞ *bounds imply* $u, v \in L^\infty(\Omega)$ *with*

$$\|u\|_{L^\infty} \le \liminf \|u_n\|_{L^\infty}, \qquad \|v\|_{L^\infty} \le \liminf \|v_n\|_{L^\infty}.$$

Next, define the function

$$g(x) := \begin{cases} \dfrac{f(x)}{u(x)^\gamma} & \text{if } f(x) > 0 \text{ and } u(x) > 0, \\ 0 & \text{if } f(x) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

For any $\phi \in H_0^1(\Omega) \bigcap L^\infty(\Omega)$ with $\phi \geq 0$, we consider the first equation in (2). Because $u_n \to u$ a.e. and u_n is uniformly bounded, we have

$$\frac{f_n}{(u_n + 1/n)^\gamma} \to \frac{f}{u^\gamma} \quad \text{a.e. on } \{u > 0\},$$

and the sequence is bounded by an integrable function (e.g. by $f\,(u/2)^{-\gamma}$ for large n where $u_n \geq u/2 > 0$). Fatous lemma gives

$$\int_\Omega BDu \cdot D\phi + v^{1-\theta} u^{r-1} \phi \, dx \geq \int_\Omega g\,\phi \, dx.$$

Hence $g\phi \in L^1(\Omega)$ for every such ϕ, which forces g to be finite a.e.; therefore $u > 0$ a.e. in $\{f > 0\}$. Moreover,

$$-\operatorname{div}(BDu) + v^{1-\theta} u^{r-1} \geq 0 \quad \text{in the sense of distributions.}$$

The coefficient $v^{1-\theta} u^{r-2}$ belongs to $L^\infty(\Omega)$ because $u, v \in L^\infty$ and $r \geq 2$.
By the weak Harnack inequality, every non-negative weak supersolution of $-\operatorname{div}(BDw) + cw \geq 0$ with $c \geq 0$ and $c \in L^\infty$ is either identically zero or strictly positive a.e.
Since u is not identically zero (because $f \not\equiv 0$ and the right-hand side forces $u > 0$ where $f > 0$), we conclude $u > 0$ a.e. in Ω. Consequently $g = f u^{-\gamma}$ a.e.
For v, define

$$h(x) := \begin{cases} \dfrac{u(x)^r}{v(x)^\theta} & \text{if } v(x) > 0, \\ +\infty & \text{if } v(x) = 0. \end{cases}$$

Using the second equation of (2) and the same Fatou argument (note that $u_n^r \to u^r$ a.e. and is uniformly bounded, while $(v_n + 1/n)^{-\theta}$ converges a.e. to $v^{-\theta}$ on $\{v > 0\}$), we obtain for every $\psi \in H_0^1(\Omega) \bigcap L^\infty(\Omega)$, $\psi \geq 0$:

$$\int_\Omega BDv \cdot D\psi \, dx \geq \int_\Omega h\,\psi \, dx.$$

Hence h is finite a.e., so $v > 0$ a.e. in Ω.
The weak Harnack inequality applied to the linear operator $-\operatorname{div}(BD\cdot)$ (since v satisfies $-\operatorname{div}(BDv) \geq 0$) then yields $v > 0$ a.e.

Step 3: Passing to the limit

To obtain equality in the equations, we need to show that the approximate solutions converge strongly enough to pass the nonlinear terms to the limit.

The uniform L^∞ bounds and the strong convergence in L^2 (which follows from the compact embedding $H_0^1 \hookrightarrow L^2$) imply

$$v_n^{1-\theta} u_n^{r-1} \to v^{1-\theta} u^{r-1} \quad \text{strongly in } L^2(\Omega),$$

because the product of uniformly bounded sequences converging a.e. converges in every L^p with $p < \infty$ (by the dominated convergence theorem).
Similarly,

$$\frac{u_n^r}{(v_n + 1/n)^\theta} \to \frac{u^r}{v^\theta}$$

a.e. and is uniformly bounded, hence strongly in L^2.
Moreover, it is easy to see that the gradients converge almost everywhere: there exists a subsequence such that

$$Du_n \to Du, \qquad Dv_n \to Dv \quad \text{a.e. in } \Omega.$$

(This is because the right-hand sides $\frac{f_n}{(u_n+1/n)^\gamma} - v_n^{1-\theta} u_n^{r-1}$ and $\frac{u_n^r}{(v_n+1/n)^\theta}$ are bounded in the space of Radon measures, as shown in the paper using the uniform energy bounds.)
Now take a test function $\phi \in H_0^1(\Omega) \bigcap L^\infty(\Omega)$ (the restriction to bounded test functions is harmless; the extension to all $\phi \in H_0^1(\Omega)$ follows by truncation).
In the first equation of (3.6) we have

$$\int_\Omega BDu_n \cdot D\phi \, dx = \int_\Omega \frac{f_n}{(u_n + 1/n)^\gamma} \phi \, dx - \int_\Omega v_n^{1-\theta} u_n^{r-1} \phi \, dx.$$

Because $Du_n \to Du$ a.e. and BDu_n is bounded in L^2, the left-hand side converges to $\int_\Omega BDu \cdot D\phi \, dx$ (by the dominated convergence theorem, using the uniform bound and the a.e. convergence).
The right-hand side converges to

$$\int_\Omega f u^{-\gamma} \phi \, dx - \int_\Omega v^{1-\theta} u^{r-1} \phi \, dx$$

by the strong convergence of the integrands. Thus

$$\int_\Omega BDu \cdot D\phi + v^{1-\theta} u^{r-1} \phi \, dx = \int_\Omega \frac{f}{u^\gamma} \phi \, dx \qquad \forall \phi \in H_0^1(\Omega) \bigcap L^\infty(\Omega).$$

Extending to all $\phi \in H_0^1(\Omega)$ is done by replacing ϕ with $T_k(\phi)$ (truncation at level k), using the equation for $T_k(\phi)$, and letting $k \to \infty$ (monotone convergence).
The second equation is treated identically, yielding

$$\int_\Omega ADv \cdot D\psi \, dx = \int_\Omega \frac{u^r}{v^\theta} \psi \, dx \qquad \forall \psi \in H_0^1(\Omega).$$

Therefore (u, v) is a weak solution of (3.1) in the sense of Definition 3.2.1. Moreover, $u, v \in L^\infty(\Omega)$.
This completes the proof of Theorem 3.2.1 for the case $m > N/2$.

Remark 3.3.2. *Observe that one can reveal that several questions remain open for the system (3.1):*

- *Uniqueness of weak solutions is not known for general f outside the dual space.*

- *The optimal regularity of u and v when m is exactly $N/2$ (only L^p for all $p < \infty$, not L^∞) is this sharp?*
- *The case of a sign-changing datum f or a measure-valued datum is largely unexplored.*

Conclusion

In this work, we studied a doubly singular elliptic system with singular terms $u^{-\gamma}$ and $v^{-\theta}$. Despite the singular nature of the problem, the coupling between the equations produces a regularizing effect that enhances the integrability of the solutions beyond what is typically expected from the singular theory for a single equation. By applying Schauder's fixed point theorem together with standard properties of Sobolev and Lebesgue spaces, we studied the existence of finite-energy solutions in the product space $H_0^1(\Omega) \times H_0^1(\Omega)$. This result contributes to the understanding of singular coupled systems and opens the door to further investigations, such as the uniqueness or regularity of solutions under additional assumptions on the parameters γ, θ, r and f .

Bibliography

[1] *E. Akkoyunly, Variable exponent Lebesgue and sobolev spaces and some Applications, All sciences Academy design, March 2025.*

[2] *I.p.Arribas, Sobolev Spaces, Final Degree Dissertation Degree in Mathematics,(2016), 1-17.*

[3] *V. Benci, D. Fortunato, An eigenvalue problem for the Schrdinger-Maxwell equations, Topol. Methods Nonlinear Anal. 11 (1998), 283-293.*

[4] *L. Boccardo, S. Buccheri, C.A. dos Santos, An elliptic system with singular nonlinearities: existence via non variational arguments, J. Math. Appl. 516 (2022) 126-490.*

[5] *L. Boccardo, L. Orsina, A singular system of Schrdinger-Maxwell equations, Mediterr. J. Math. 21 (2024).*

[6] *V.I. Bogachev, Measure Theory, Springer-Verlag, Berlin (2007).*

[7] *H.Brezis, Cours Analyse focntionelle Téorie et Application, Masson,paris (1983).*

[8] *H.Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer Science . Business Media. LLC. 233, New York, USA . (2010).*

[9] *A.I. Ghehioueche, On Lebesgue and Sobolev Spaces with Variable Exponents, Master memory, Functional Anal., Dep. Maths, University of M'sila ,(2019),1-50.*

[10] *R. Durastanti, Regularizing effect for some p-Laplacian systems, Nonlinear Anal. 188 (2019) 425-438.*

[11] *G. Giannone, Regularizing effects for an elliptic system of singular equations, J. Math. Anal. Appl. 554 (2026) 129-950.*

[12] *H. Le Dret. Nonlinear Elliptic Partial Differential Equations: An introduction, Universitext, Springer, Berlin, (2018).*

[13] *H. Lakehal. Some remarks to anisotropic variable exponent Sobolev spaces, Master memory. Functional Anal., Dep. Maths., University of M'sila,(2021),1-93*

[14] *A. Naidji. Some Existence and Regularity Results for Unilateral Problems with Degenerate Coercivity, PDEs. Dep. Maths, University of M'sila, (2023),1-66.*

[15] N.S. Papageorgiou, P. Winkert, Applied Nonlinear Functional Analysis: An Introduction, De Gruyter, Berlin, (2018).

[16] R. Bousbaa. On Generalized Orlicz Spaces. Master memory, Functional Anal., Dep. Maths, University of M'sila,(2020), 1-108.

[17] F. Zarouki. On Weak Lebesgue (Marcinkiewicz) Spaces with Variable Exponents, Functional Anal., Dep. Maths, University of M'sila, (2023), 1-50.

[18] H. Zou, Fixed Point Theory and Elliptic Boundary Value Problems, Chapter, Dep. Maths, University of Alabama at Birminglam. USA. (2008).

Abstract

In this work, we study a doubly singular elliptic system of the form:

$$\begin{cases} -div(B(x)\nabla u) + v^{1-\theta}u^{r-1} = \dfrac{f}{u^{\gamma}} & in\ \Omega \\ -div(B(x)\nabla v) = \dfrac{u^{r}}{v^{\theta}} & in\ \Omega \\ u = v = 0 & on\ \partial\Omega \end{cases}$$

where Ω is an open subset of $\mathbb{R}^N$ with $N \geq 3$, and the parameters $\gamma \in (0,1)$, $\theta \in [0,1)$, $r \geq 2$, and $f \geq 0$. The aim of this work is to study the existence of finite-energy solutions for such singular problems. The solutions we seek belong to the Sobolev space $H_0^1(\Omega) \times H_0^1(\Omega)$. Since the structure of the coupled system generates regularizing effects that give the solutions higher integrability than that provided by the singular theory of a single equation, our study relies on the use of Schauder's fixed point theorem and some basic properties of Sobolev and Lebesgue spaces.

Keywords: *Schrödinger-Maxwell system, singular problem, Sobolev spaces, existence of solution, regularizing effects, Schauder's fixed point theorem.*

Résumé

Dans ce travail, nous étudions un système elliptique singulier de la forme:

$$\begin{cases} -\operatorname{div}(B(x)\nabla u) + v^{1-\theta}u^{r-1} = \dfrac{f}{u^{\gamma}} & dans\,\Omega \\ -\operatorname{div}(B(x)\nabla v) = \dfrac{u^{r}}{v^{\theta}} & in\,\Omega \\ u = v = 0 & sur\,\partial\Omega \end{cases}$$

oú Ω *est un ouvert de* $\mathbb{R}^N$ *avec* $N \geq 3$, *et les paramètres* $\gamma \in (0,1)$, $\theta \in [0,1)$, $r \geq 2$, *et* $f \geq 0$.
L'objectif de ce travail est d'étudier l'existence de solutions à énergie finie pour de tels problèmes singuliers. Les solutions recherchées appartiennent l'espace de Sobolev $H_0^1(\Omega) \times H_0^1(\Omega)$. *Étant donnè que la structure du système couplè génère des effets régularisants qui confèrent aux solutions une intégrabilité supérieure à celle fournie par la théorie singulière d'une équation unique, notre étude basé sur l'utilisation du théorème du point fixe de Schauder et de certaines propriétés de base des espaces de Sobolev et de Lebesgue.*

Mots-clés : *Système de Schrödinger-Maxwell, problème singulier, espaces de Sobolev, existence de solution, régularisants, théorème du point fixe de Schauder.*

الملخص

في هذا العمل، ندرس نظامًا ناقصيا مزدوج و شاذ من الشكل :

$$\begin{cases} -\operatorname{div}(B(x)\nabla u) + v^{1-\theta}u^{r-1} = \dfrac{f}{u^{\gamma}} & \Omega \\ -\operatorname{div}(B(x)\nabla v) = \dfrac{u^{r}}{v^{\theta}} & \Omega \\ u = v = 0 & \partial\Omega \end{cases}$$

حيث Ω مجموعة مفتوحة من $\mathbb{R}^N$ مع $N \geq 3$ ، والمعاملات $\gamma \in (0,1)$ ، $\theta \in [0,1)$ ، $r \geq 2$ ، و $f \geq 0$. الهدف من هذا العمل هو دراسة وجود حلول ذات طاقة متنهية لمثل هذه المسائل الشاذة. الحلول التي نبحث عنها تنتمي إلى فضاء $Sobolev$ $H_0^1(\Omega) \times H_0^1(\Omega)$. بما أن بنية النظام المزدوج تولد تأثيرات منتظمة تمنح الحلول قابلية تكامل أعلى من تلك التي توفرها النظرية الشاذة لمعادلة واحدة، فإن دراستنا تعتمد على استخدام نظرية النقطة الصامدة ل $Schauder$ وبعض الخصائص الأساسية لفضاءات $Sobolev$ و $Lebesgue$.

الكلمات المفتاحية : نظام $Schrөdinger-Maxwell$ ، مسألة شاذة، فضاءات $Sobolev$ ، وجود الحل، تأثيرات منتظمة، نظرية النقطة الصامدة ل $Schauder$.