ANISOTROPIC SINGULAR PERTURBATIONS OF VARIATIONAL INEQUALITIES

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Abstract. We consider variational inequalities involving the $p-$ Laplace operator with anisotropic singular perturbations where the convex set, on which the problem is defined, is also subject to perturbations. This leads to introduce a new convergence of sets, in some suitable sense, conceived from the Mosco convergence and matching well to the anisotropic singular perturbations. Convergence results and their rates are established. In order to illustrate the introduced convergence sets, obstacle and elasto-plastic perturbed problems are dealt with. This allows to go deeper in the analysis of the suggested convergence on concrete sets in Sobolev spaces.

Contents

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$ with a sufficiently smooth boundary. We denote by

$$x = (x_1, \dots, x_n) = (X_1, X_2)$$

the points in $\mathbb{R}^n$ where $X_1 = (x_1, \dots, x_q)$ and $X_2 = (x_{q+1}, \dots, x_n)$, n and q are integers. We also denote

$$\nabla u := \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix} = \begin{pmatrix} (\partial_{x_1} u, \dots, \partial_{x_q} u)^T \\ (\partial_{x_{q+1}} u, \dots, \partial_{x_n} u)^T \end{pmatrix}$$

and for $\varepsilon > 0$,

$$\nabla^\varepsilon u := \begin{pmatrix} \varepsilon \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix}, \quad \nabla^0 u := \begin{pmatrix} 0 \\ \nabla_{X_2} u \end{pmatrix}.$$

The parameter of perturbation ε appears only with the $X_1 -$ *direction* of the gradient as well as in the convex set, on which the problem is defined below. For this reason we refer to this perturbation as anisotropic.

For $1 < p < +\infty$, we consider as a model problem the following variational inequality, involving a perturbed $p-$Laplace operator,

$$\begin{cases} \displaystyle\int_\Omega |\nabla^\varepsilon u|^{p-2} \nabla^\varepsilon u \cdot \nabla^\varepsilon (v-u)\, dx \geq \langle f, v-u \rangle_{W_0^{1,p}(\Omega)}, & \forall v \in K_\varepsilon, \\ u \in K_\varepsilon, \end{cases} \tag{1.1}$$

where $\langle \cdot \rangle_V$ denotes the duality brackets between a space V and its dual V', $K_\varepsilon \neq \varnothing$ is a closed convex subset of $W_0^{1,p}(\Omega)$ for all $\varepsilon > 0$. Assuming $f \in W^{-1,p'}(\Omega)$, where p' is the conjugate of p, the above problem has a unique solution $u_\varepsilon \in K_\varepsilon$. Many works were recently taking care of singular *anisotropic* perturbations for different type of problems, see for instance [4, 7, 9–12, 14–16]. In particular, in [12], an abstract approach of variational inequalities is elaborated and illustrated by some applications to show that the theory covers the singular perturbations of anisotropic type as well as the isotropic ones. Although, the study, given in [12], is as general as possible, it does not include problems as (1.1) for two reasons. The perturbed operator in [12] has the form $\varepsilon A + B$, with nonlinear operators A and B defined on different Banach spaces. That is to say that the perturbed and the unperturbed parts of the operator are entirely distinct which is not the case of course in Problem (1.1). However, the main difference is coming from the convex sets which are also subject to perturbations. Thus some convergence definitions, fitting with the anisotropic nature of the above perturbation, have to be considered on the sequence of the convex sets (K_ε) when $\varepsilon \to 0$. The convergence of sets, related to perturbations, is dealt with in numerous works since the fundamental papers of Mosco [24, 25] where perturbations of variational inequalities of linear and nonlinear operators, with the convex sets also subject to perturbations, are considered. (See also Attouch [1] and the references therein).

In the next section we establish a priori estimates and convergences of u_ε, solution to quasi-linear variational inequalities, provided that some boundedness assumptions and convergences of the sequence (K_ε) hold. In fact without these types of assumptions there is no chance to envisage any boundedness or convergence of the solutions. We can see this clearly if for example the sets K_ε are parallel hyperplanes such $dist\,(0, K_\varepsilon) = \frac{1}{\varepsilon}$. This of course leads to introduce the convergence notion on the convex sets, derived from the Mosco convergence and adapted to the present type of perturbations.

In the third and forth sections, we apply the above results to some important problems. The first one is a problem with constraints on the state where the convex set is determined by perturbed obstacles. We give sufficient conditions on the convergence of the obstacles to guarantee the suitable convergence of the convex sets. In the case of (isotropic) perturbations, an abundant literature has been devoted to this subject, (see for instance Attouch and Picard [2], Boccardo and Murat [5], Dal Maso [13], Mosco [24, 25] and related works). The second example is an elasto-plastic torsion problem where the convex set is determined by constraints on the gradient of the solution. There are some works about the isotropic case that give sufficient conditions on the constraints to insure the convenient convergence of the convex sets. (See for instance Azevedo and Santos [3], Kunze and Rodrigues [19], Lagnese [20]). To ensure the convergence of the sequence (K_ε), we are led to show some density results, then we establish the convergence of the sequence (K_ε) in the case of cylindrical and some noncylindrical domains. The fifth section is devoted to investigate the rate of the convergence far from the boundary layer for cylindrical domains, i.e. $\Omega = \omega_1 \times \omega_2$.

2. Convergence of convex sets and of solutions

2.1. **Anisotropic Sobolev-type spaces.** Throughout this paper the orthogonal projections of Ω onto the space $X_2 = 0$ and $X_1 = 0$ are denoted by Π_1 and Π_2 respectively. For any $X_1 \in \Pi_1$ we denote by Ω_{X_1} the section of Ω above X_1 i.e.

$$\Omega_{X_1} = \{\, X_2 \in \mathbb{R}^{n-q} \mid (X_1, X_2) \in \Omega \,\}.$$

Then consider the following anisotropic Sobolev space

$$\mathcal{W}\,(\Omega) := \left\{ u \in L^p\,(\Omega) \;\mid \nabla_{X_2} u \in [L^p\,(\Omega)]^{n-q} \right\},$$

equipped with the norm

$$v \to \left(|v|^p_{L^p(\Omega)} + |\nabla_{X_2} v|^p_{L^p(\Omega)} \right)^{1/p}.$$

It is clear that $W^{1,p}(\Omega)$ is a subspace of $\mathcal{W}(\Omega)$. We denote by $\mathcal{W}_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$, the space of C^∞ functions with a compact support in Ω, in $\mathcal{W}(\Omega)$, i.e.

$$\mathcal{W}_0(\Omega) := \overline{\mathcal{D}(\Omega)}^{\mathcal{W}(\Omega)}.$$

Since Ω is bounded, the following Poincaré inequality

$$|v|_{L^p(\Omega)} \le C_p \, |\nabla_{X_2} v|_{L^p(\Omega)}, \qquad \forall v \in \mathcal{W}_0(\Omega)$$

holds for some constant C_p depending on Ω. Thus, the map

$$v \to |\nabla_{X_2} v|_{L^p(\Omega)}.$$

define a norm on $\mathcal{W}_0(\Omega)$. One can check that

$$W^{1,p}_0(\Omega) \subset \mathcal{W}_0(\Omega) \subset L^p(\Omega) \quad \text{and} \quad L^{p'}(\Omega) \subset \mathcal{W}'_0(\Omega) \subset W^{-1,p'}(\Omega).$$

We can easily show that the dual space $\mathcal{W}'_0(\Omega)$ can be identified with the set of distributions such as

$$f \in \mathcal{W}'_0(\Omega) \quad \Leftrightarrow \quad \exists f_0, f_i \in L^{p'}(\Omega), i = q+1, \cdots, n, \text{ such that } f = f_0 + \sum_{i=q+1}^{n} \partial_{x_i} f_i.$$

More characterizations of the elements of $\mathcal{W}_0(\Omega)$ will be given at the end of this section.

2.2. **A perturbed variational inequality.** To deal with the above model problem and more general variational inequalities, let

$$f \in \mathcal{W}'_0(\Omega). \tag{2.1}$$

and consider the following nonlinear elliptic problem defined as

$$\begin{cases} \displaystyle\int_\Omega a(x, \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon (v_\varepsilon - u_\varepsilon) \, dx \ge \langle f, v_\varepsilon - u_\varepsilon \rangle_{W^{1,p}_0(\Omega)}, & \forall v_\varepsilon \in K_\varepsilon, \\ u_\varepsilon \in K_\varepsilon \end{cases} \tag{2.2}$$

where $K_\varepsilon \neq \varnothing$ is a closed convex subset of $W^{1,p}_0(\Omega)$ depending on $\varepsilon > 0$. The function $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function satisfying the following standard assumptions
Growth condition. For $p > 1$, there exists a constant M such that

$$|a(x, \xi)| \le M \left(g(x) + |\xi|^{p-1} \right), \forall \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega. \tag{2.3}$$

where $g \in L^{p'}(\Omega)$ and $| \cdot |$ is the usual Euclidean norm.
Monotonicity. For all $\xi, \eta \in \mathbb{R}^n$ and a.e. $x \in \Omega$, we have

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) \ge 0, \tag{2.4}$$

where " $\cdot$ " is the scalar product in $\mathbb{R}^n$.
Coercivity. For a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$, there exist a constant $\alpha > 0$ such that

$$a(x, \xi) \cdot \xi \ge \alpha \, |\xi|^p. \tag{2.5}$$

Under these assumptions, Problem (2.2) has a solution $u_\varepsilon \in K_\varepsilon$, which is unique if a is strictly monotone, i.e. the inequality (2.4) is strict for $\eta \neq \xi$ (see Chipot [6], Kinderlehrer and Stampacchia [18]).

2.3. **A priori estimates.** The first theorem below shows that the a priori estimate here looks like estimates for elementary problems as linear elliptic ones (see [10]) provided that a sequence $w_\varepsilon \in K_\varepsilon$, satisfying the same estimate, exists. In fact we are speaking about a necessary and sufficient conditions.

Theorem 1. *Under the assumption* (2.1), *assume in addition that there exists a sequence* $w_\varepsilon \in K_\varepsilon$ *for all* $\varepsilon > 0$, *such that*

$$\varepsilon \nabla_{X_1} w_\varepsilon \quad \textit{and} \quad \nabla_{X_2} w_\varepsilon \quad \textit{are bounded in } L^p(\Omega) \tag{2.6}$$

independently of ε, *then*

$$u_\varepsilon, \quad \varepsilon \nabla_{X_1} u_\varepsilon \quad \textit{and} \quad \nabla_{X_2} u_\varepsilon \quad \textit{are bounded in } L^p(\Omega) \tag{2.7}$$

and

$$a(\cdot, \nabla^\varepsilon u_\varepsilon) \quad \textit{is bounded in } L^{p'}(\Omega). \tag{2.8}$$

Proof. Taking $v_\varepsilon = w_\varepsilon$ in (2.2), it follows that

$$\begin{aligned} \int_\Omega a(x, \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)\, dx &\leq \langle f, u_\varepsilon - w_\varepsilon \rangle_{\mathcal{W}_0(\Omega)} \\ &\leq |f|_{\mathcal{W}_0'(\Omega)} |\nabla_{X_2}(u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \\ &\leq |f|_{\mathcal{W}_0'(\Omega)} |\nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)}, \end{aligned}$$

then

$$\begin{aligned} \alpha |\nabla^\varepsilon u_\varepsilon|^p_{L^p(\Omega)} &\leq \int_\Omega a(x, \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon u_\varepsilon dx \\ &\leq |f|_{\mathcal{W}_0'(\Omega)} \left(|\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)} + |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)} \right) + \int_\Omega a(x, \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon w_\varepsilon dx. \end{aligned} \tag{2.9}$$

Using Hölder's inequality and (2.3), the last integral can be estimated as follows

$$\begin{aligned} \int_\Omega a(x, \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon w_\varepsilon dx &\leq |a(x, \nabla^\varepsilon u_\varepsilon)|_{L^{p'}(\Omega)} |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)} \\ &\leq M \left| g + |\nabla^\varepsilon u_\varepsilon|^{(p-1)} \right|_{L^{p'}(\Omega)} |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)} \\ &= C \left(|g|_{L^{p'}(\Omega)} |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)} + |\nabla^\varepsilon u_\varepsilon|^{p/p'}_{L^p(\Omega)} |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)} \right) \end{aligned}$$

since $(p-1)p' = p$. Throughout this paper, the positive constant C is independent of ε and may take different values at different occurrences. Then by Young's inequality, it comes

$$\int_\Omega a(x, \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon w_\varepsilon dx \leq \frac{\alpha}{4} |\nabla^\varepsilon u_\varepsilon|^p_{L^p(\Omega)} + C \left(|g|^{p'}_{L^{p'}(\Omega)} + |\nabla^\varepsilon w_\varepsilon|^p_{L^p(\Omega)} \right).$$

A similar inequality yields

$$|f|_{\mathcal{W}_0'(\Omega)} \left(|\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)} + |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)} \right) \leq \frac{\alpha}{4} |\nabla^\varepsilon u_\varepsilon|^p_{L^p(\Omega)} + C \left(|f|^{p'}_{\mathcal{W}_0'(\Omega)} + |\nabla^\varepsilon w_\varepsilon|^p_{L^p(\Omega)} \right).$$

Going back to (2.9), we deduce

$$\frac{\alpha}{2}\left|\nabla^{\varepsilon}u_{\varepsilon}\right|_{L^{p}(\Omega)}^{p}\leq C\left(|f|_{\mathcal{W}_0'(\Omega)}^{p'}+|g|_{L^{p'}(\Omega)}^{p'}+\left|\nabla^{\varepsilon}w_{\varepsilon}\right|_{L^{p}(\Omega)}^{p}\right). \tag{2.10}$$

This means that $|\nabla^{\varepsilon}u_{\varepsilon}|_{L^p(\Omega)}$ is bounded since $|\nabla^{\varepsilon}w_{\varepsilon}|_{L^p(\Omega)}$ is assumed to be bounded and that

$$\varepsilon\left|\nabla_{X_1}u_{\varepsilon}\right|, \quad \left|\nabla_{X_2}u_{\varepsilon}\right| \quad \text{and} \quad u_{\varepsilon} \quad \text{are bounded in } L^p(\Omega).$$

The boundedness of u_ε follows from L^p-Poincaré's inequality in the X_2-*direction*. For the last estimate (2.8), one has

$$|a(x,\nabla^{\varepsilon}u_{\varepsilon})|_{L^{p'}(\Omega)}^{p'}\leq C\int_{\Omega}\left(g+|\nabla^{\varepsilon}u_{\varepsilon}|^{(p-1)}\right)^{p'}dx\leq C\left(|g|_{L^{p'}(\Omega)}^{p'}+|\nabla^{\varepsilon}u_{\varepsilon}|_{L^{p}(\Omega)}^{p/p'}\right)$$

which ends the proof of the theorem. □

Remark 1.

i) Using the continuous injection $L^p(\Omega)\subset\mathcal{D}'(\Omega)$, with the continuity of the derivative operator in $\mathcal{D}'(\Omega)$, we can check that

$$\varepsilon\nabla_{X_1}u_{\varepsilon}\rightharpoonup 0 \quad \text{in } L^p(\Omega).$$

Here and in the following, the vectorial convergence means the convergence component by component.

ii) Let $\gamma\geq 0$, then we infer from (2.10) that

$$|\nabla^{\varepsilon}w_{\varepsilon}|_{L^p(\Omega)}=O\left(\varepsilon^{-\gamma}\right) \;\Rightarrow\; |\nabla^{\varepsilon}u_{\varepsilon}|_{L^p(\Omega)}=O\left(\varepsilon^{-\gamma}\right) \tag{2.11}$$

iii) In particular the assumption (2.6) holds if $\underline{\lim}_{\varepsilon\to 0}K_\varepsilon\neq\varnothing$, i.e. $\cap_{\varepsilon<\varepsilon_0}K_\varepsilon\neq\varnothing$ for some $\varepsilon_0>0$. For instance, this is the case of a monotone sequence of sets (K_ε) (in the inclusion sense). Thereby, it suffices to fix $w_\varepsilon=w_0\in\cap_{\varepsilon<\varepsilon_0}K_\varepsilon$.

2.4. **Convergence of convex sets.** The existence of a sequence as w_ε cannot give more than the weak convergence of subsequence of u_ε without any identification of the limits. Then we are led to introduce a convergence of the closed convex sets fitting with the anisotropic singular perturbations. This beforehand serves to define the limit problem and then go deeper in the behaviour investigation of u_ε.

Let (K_ε) be a sequence of closed convex subsets of $W_0^{1,p}(\Omega)$, we shall denote by[1]

$$as-\underline{\lim}_{\varepsilon\to 0}K_{\varepsilon} \tag{2.12}$$

the set of all w in $\mathcal{W}_0(\Omega)$, such that the strong convergence

$$\begin{pmatrix}\varepsilon\nabla_{X_1}w_{\varepsilon}\\ \nabla_{X_2}(w_{\varepsilon}-w)\end{pmatrix}\to 0 \quad \text{in } L^p(\Omega), \quad \text{as } \varepsilon\to 0, \tag{2.13}$$

holds for some sequence $w_\varepsilon\in K_\varepsilon$. We shall also denote by

$$aw-\overline{\lim}_{\varepsilon\to 0}K_{\varepsilon} \tag{2.14}$$

[1] a stands for anisotropic, s for strong and w for weak.

the set of all w in $\mathcal{W}_0(\Omega)$, such that the weak convergence

$$\begin{pmatrix} \varepsilon' \nabla_{X_1} w_{\varepsilon'} \\ \nabla_{X_2} (w_{\varepsilon'} - w) \end{pmatrix} \rightharpoonup 0 \quad in\ L^p(\Omega), \quad \text{as } \varepsilon' \to 0, \tag{2.15}$$

holds at least for a subsequence $w_{\varepsilon'} \in K_{\varepsilon'}$.

Remark 2.

i) *It is clear that*

$$as - \underline{\lim}_{\varepsilon \to 0} K_\varepsilon \subset aw - \overline{\lim}_{\varepsilon \to 0} K_\varepsilon. \tag{2.16}$$

ii) *If* (2.13) *holds, it follows that*

$$a(\cdot, \nabla^\varepsilon w_\varepsilon) \to a(\cdot, \nabla^0 w) \quad in\ L^{p'}(\Omega), \tag{2.17}$$

due to the continuity of the function a in the second variable.

The limit $as - \underline{\lim}_{\varepsilon \to 0} K_\varepsilon$ inherit the following proprieties.

Lemma 1. *The set $as - \underline{\lim}_{\varepsilon \to 0} K_\varepsilon$ is convex and closed in $\mathcal{W}_0(\Omega)$.*

Proof. Let $v_n \in as - \underline{\lim}_{\varepsilon \to 0} K_\varepsilon$ a sequence such that $v_n \to v$ in $\mathcal{W}_0(\Omega)$, i.e.

$$\nabla_{X_2} (v_n - v) \to 0 \quad in\ L^p(\Omega), \quad \text{as } n \to \infty.$$

To show the closeness, one has to show that $v \in as - \underline{\lim}_{\varepsilon \to 0} K_\varepsilon$. From the definition of $as - \underline{\lim}_{\varepsilon \to 0} K_\varepsilon$ there exists a "sequence" $v_n^\varepsilon \in K_\varepsilon$ such that

$$\begin{pmatrix} \varepsilon \nabla_{X_1} v_n^\varepsilon \\ \nabla_{X_2} (v_n^\varepsilon - v_n) \end{pmatrix} \to 0 \quad in\ L^p(\Omega), \quad \text{as } \varepsilon \to 0. \tag{2.18}$$

Consider then for every n a $\varepsilon(n) > 0$ such that for every $\varepsilon \leq \varepsilon(n)$ it holds that

$$\left| \begin{matrix} \varepsilon \nabla_{X_1} v_n^\varepsilon \\ \nabla_{X_2} (v_n^\varepsilon - v_n) \end{matrix} \right|_{L^p(\Omega)} \leq \frac{1}{n}. \tag{2.19}$$

By (2.18) such an $\varepsilon(n)$ exists and without loss of generality one can assume that it is chosen strictly decreasing towards 0. Let $\varepsilon \leq \varepsilon(1)$. Denote by N_ε the integer n satisfying

$$\varepsilon(N_\varepsilon + 1) < \varepsilon \leq \varepsilon(N_\varepsilon).$$

Such $N_\varepsilon \to \infty$ as $\varepsilon \to 0$. One has $v_{N_\varepsilon}^\varepsilon \in K_\varepsilon$ and by (2.19)

$$\begin{aligned} \left| \begin{matrix} \varepsilon \nabla_{X_1} v_{N_\varepsilon}^\varepsilon \\ \nabla_{X_2} \left(v_{N_\varepsilon}^\varepsilon - v\right) \end{matrix} \right|_{L^p(\Omega)} &= \left| \begin{matrix} \varepsilon \nabla_{X_1} v_{N_\varepsilon}^\varepsilon \\ \nabla_{X_2} \left(v_{N_\varepsilon}^\varepsilon - v_{N_\varepsilon} + v_{N_\varepsilon} - v\right) \end{matrix} \right|_{L^p(\Omega)} \\ &\leq \left| \begin{matrix} \varepsilon \nabla_{X_1} v_{N_\varepsilon}^\varepsilon \\ \nabla_{X_2} \left(v_{N_\varepsilon}^\varepsilon - v_{N_\varepsilon}\right) \end{matrix} \right|_{L^p(\Omega)} + \left| \nabla_{X_2} (v_{N_\varepsilon} - v) \right|_{L^p(\Omega)} \\ &\leq \frac{1}{N_\varepsilon} + \left| \nabla_{X_2} (v_{N_\varepsilon} - v) \right|_{L^p(\Omega)} \to 0, \end{aligned}$$

when $\varepsilon \to 0$. This shows that $v \in as - \underline{\lim}_{\varepsilon \to 0} K_\varepsilon$.

To check that $as-\underline{\lim}_{\varepsilon\to 0}K_\varepsilon$ is convex, let $v^1,v^2\in as-\underline{\lim}_{\varepsilon\to 0}K_\varepsilon$ then for some sequences $v_\varepsilon^1,v_\varepsilon^2\in K_\varepsilon$ one has

$$\left|\begin{array}{c}\varepsilon\nabla_{X_1}v_\varepsilon^i\\ \nabla_{X_2}\left(v_\varepsilon^i-v^i\right)\end{array}\right|_{L^p(\Omega)}\to 0,\quad\text{for } i=1,2.$$

It follows that for every $\alpha\in[0,1]$

$$\left|\begin{array}{c}\varepsilon\nabla_{X_1}\left(\alpha v_\varepsilon^1+(1-\alpha)\,v_\varepsilon^2\right)\\ \nabla_{X_2}\left(\left(\alpha v_\varepsilon^1+(1-\alpha)\,v_\varepsilon^2\right)-\left(\alpha v^1+(1-\alpha)\,v^2\right)\right)\end{array}\right|_{L^p(\Omega)}\to 0,\quad\text{for } i=1,2$$

and $\alpha v^1+(1-\alpha)\,v^2\in as-\underline{\lim}_{\varepsilon\to 0}K_\varepsilon$. This ends the proof of the lemma. □

Now we introduce the desired limit set and its convergence sense.

Definition 1. *A sequence (K_ε) of subsets of $W_0^{1,p}(\Omega)$ converges to a nonempty set $\mathcal{K}\subset\mathcal{W}_0(\Omega)$ iff*

$$aw-\overline{\lim}_{\varepsilon\to 0}K_\varepsilon=as-\underline{\lim}_{\varepsilon\to 0}K_\varepsilon=\mathcal{K},$$

and we denote $K_\varepsilon\overset{a}{\to}\mathcal{K}$ or $a-\lim_{\varepsilon\to 0}K_\varepsilon=\mathcal{K}$.

We have to mention that the perturbation here is singular, i.e. K_ε and $\mathcal{K}$ are not in the same space as it is the case in Mosco [24, 25], and is anisotropic since the perturbation affects only the X_1-direction.

Remark 3. *In practice, taking into account (2.16), the above convergence holds iff*

$$\begin{cases} i) & \mathcal{K}\subset as-\underline{\lim}_{\varepsilon\to 0}K_\varepsilon,\\ ii) & aw-\overline{\lim}_{\varepsilon\to 0}K_\varepsilon\subset\mathcal{K}.\end{cases}\tag{2.20}$$

The following lemma may simplify the verification of (2.20–i).

Lemma 2. *Let D be a dense subset in $\mathcal{K}$, then*

$$\mathcal{K}\subset as-\underline{\lim}_{\varepsilon\to 0}K_\varepsilon\iff D\subset as-\underline{\lim}_{\varepsilon\to 0}K_\varepsilon.$$

Proof. It suffices to note that if $D\subset as-\underline{\lim}_{\varepsilon\to 0}K_\varepsilon$ then, due to Lemma 1, it follows that $\mathcal{K}=\overline{D}^{\mathcal{W}(\Omega)}\subset\overline{as-\underline{\lim}_{\varepsilon\to 0}K_\varepsilon}^{\mathcal{W}(\Omega)}=as-\underline{\lim}_{\varepsilon\to 0}K_\varepsilon$. □

Let us now summarize some basic properties of the above convergence in the following lemma.

Lemma 3.

i) *Let $K_{\varepsilon'}$ be a "subsequence" of K_ε, then*

$$as-\underline{\lim}_{\varepsilon\to 0}K_\varepsilon\quad\subset\quad as-\underline{\lim}_{\varepsilon'\to 0}K_{\varepsilon'},\tag{2.21}$$

$$aw-\overline{\lim}_{\varepsilon'\to 0}K_{\varepsilon'}\quad\subset\quad aw-\overline{\lim}_{\varepsilon\to 0}K_\varepsilon.\tag{2.22}$$

In particular if K_ε converges then $K_{\varepsilon'}$ also converges and we have

$$a-\lim_{\varepsilon\to 0}K_\varepsilon=a-\lim_{\varepsilon\to 0}K_{\varepsilon'}.\tag{2.23}$$

ii) *If the sequence* K_ε *is constant, i.e.* $K_\varepsilon = K$, $\forall \varepsilon > 0$ *(or it is constant for* ε *small), then*

$$K \overset{a}{\to} \mathcal{K} = \overline{K}^{\mathcal{W}(\Omega)}, \tag{2.24}$$

and in particular, if $K_\varepsilon = W_0^{1,p}(\Omega)$

$$W_0^{1,p}(\Omega) \overset{a}{\to} \mathcal{W}_0(\Omega). \tag{2.25}$$

Proof. The two first inclusions are immediate since a sequence or a subsequence of $K_{\varepsilon'}$ is also a subsequence of K_ε and the equality (2.23) can be easily deduced if we notice that

$$as - \underline{\lim}_{\varepsilon\to 0} K_\varepsilon \subset as - \underline{\lim}_{\varepsilon'\to 0} K_{\varepsilon'} \subset aw - \overline{\lim}_{\varepsilon'\to 0} K_{\varepsilon'} \subset aw - \overline{\lim}_{\varepsilon\to 0} K_\varepsilon.$$

For the next point, the inclusion (2.20–i) follows by taking $D = K$ in Lemma 2, and the inclusion (2.20–ii) holds since $K_\varepsilon \subset \mathcal{K}$, $\forall \varepsilon > 0$, and the convex $\mathcal{K}$ is also weakly closed. This shows (2.24), (2.25) and completes the proof. □

2.5. **Convergence of solutions.** Assuming that $K_\varepsilon \overset{a}{\to} \mathcal{K}$ ($\mathcal{K} \neq \phi$), we shall show some convergence results for the solution u_ε when $\varepsilon \to 0$ and identify its limit. First, the candidate limit $\tilde{u}$ will be defined as a solution of the following problem

$$\begin{cases} \displaystyle\int_\Omega a(x,\nabla^0 u)\cdot\nabla^0(v-u)\,dx \geq \langle f, v-u\rangle_{\mathcal{W}_0(\Omega)}, \quad \forall v \in \mathcal{K}, \\ u \in \mathcal{K} \end{cases} \tag{2.26}$$

It is clear that the operator $u \to -\nabla^0\cdot a(x,\nabla^0 u)$ is coercive, bounded, hemicontinuous and monotone on $\mathcal{W}_0(\Omega)$ and thanks to Lemma 1, the set $\mathcal{K}$ is convex and closed in $\mathcal{W}_0(\Omega)$. Thus Problem (2.26) has a solution $\tilde{u} \in \mathcal{K}$.

To prove the next theorem we need the Minty Lemma (see Chipot [8]).

Lemma 4. *Let* X *be a Banach space,* T *be a monotone hemicontinuous operator from a closed convex set* K *in* X *into* X' *and* $f \in X'$, *then* $u_0 \in K$ *satisfies*

$$\langle Tu_0, v-u_0\rangle_X \geq \langle f, v-u_0\rangle_X, \quad \forall v \in K,$$

if and only if

$$\langle Tv, v-u_0\rangle_X \geq \langle f, v-u_0\rangle_X, \quad \forall v \in K.$$

Now we have the following convergence results.

Theorem 2. *Under the hypotheses of Theorem 1, assume in addition that* $K_\varepsilon \overset{a}{\to} \mathcal{K}$ *as* $\varepsilon \to 0$, *then -up to a subsequence- we have*

$$u_\varepsilon \rightharpoonup \tilde{u} \quad , \quad \varepsilon\nabla_{X_1}u_\varepsilon \rightharpoonup 0 \quad \text{and} \quad \nabla_{X_2}u_\varepsilon \rightharpoonup \nabla_{X_2}\tilde{u} \quad \text{in } L^p(\Omega), \tag{2.27}$$

where $\tilde{u}$ *is a solution to the variational inequality* (2.26). *Moreover:*

- *if the function* a *is strictly monotone then the above weak convergences hold for the whole sequence.*

- *if the function a is strongly monotone in the sense that, for some constants $c > 0$,*

$$(a(x, \eta) - a(x, \xi)) \cdot (\eta - \xi) \geq c\,|\eta - \xi|^p, \qquad \forall \eta, \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega, \tag{2.28}$$

then we have the strong convergences

$$u_\varepsilon \to \tilde{u} \quad , \quad \varepsilon \nabla_{X_1} u_\varepsilon \to 0 \quad \text{and} \quad \nabla_{X_2} u_\varepsilon \to \nabla_{X_2} \tilde{u} \quad \text{in } L^p(\Omega). \tag{2.29}$$

Proof. First, it is clear that the assumption $K_\varepsilon \xrightarrow{a} \mathcal{K}$ implies (2.6). Then, thanks to Theorem 1, there exists a subsequence of u_ε -still labelled u_ε- such that

$$u_\varepsilon \rightharpoonup \tilde{u}, \quad \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} \tilde{u}, \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0 \quad \text{in} \quad L^p(\Omega). \tag{2.30}$$

Such $\tilde{u}$ is necessarily in the set $aw - \overline{\lim}_{\varepsilon\to 0} K_\varepsilon = \mathcal{K}$ by (2.15). Next, we choose an arbitrary $w \in as - \underline{\lim}_{\varepsilon\to 0} K_\varepsilon = \mathcal{K}$, and let (w_ε) be a sequence satisfying (2.13). By the monotonicity assumption, we rewrite (2.2) as

$$\int_\Omega a(x, \nabla^\varepsilon w_\varepsilon) \cdot \nabla^\varepsilon (w_\varepsilon - u_\varepsilon)\, dx \geq \langle f, w_\varepsilon - u_\varepsilon \rangle_{\mathcal{W}_0(\Omega)}. \tag{2.31}$$

Passing to the limit and using the convergences (2.13), (2.17), (2.30), it comes

$$\int_\Omega a(x, \nabla^0 w) \cdot \nabla^0 (w - \tilde{u})\, dx \geq \langle f, w - \tilde{u} \rangle_{\mathcal{W}_0(\Omega)},$$

for every $w \in \mathcal{K}$. Thanks to Minty's lemma, it follows that

$$\int_\Omega a(x, \nabla^0 \tilde{u}) \cdot \nabla^0 (w - \tilde{u})\, dx \geq \langle f, w - \tilde{u} \rangle_{\mathcal{W}_0(\Omega)}, \qquad \forall w \in \mathcal{K},$$

i.e. $\tilde{u}$ is a solution to (2.26).

If the function a is strictly monotone then the solution $\tilde{u}$ of (2.26) is unique and the weak convergences (2.27) hold for the whole sequence.

If now (2.28) holds, and since $\tilde{u} \in \mathcal{K}$, we can take $w = \tilde{u}$ in (2.13), i.e.

$$\varepsilon \nabla_{X_1} w_\varepsilon \to 0 \quad \text{and} \quad \nabla_{X_2} w_\varepsilon \to \nabla_{X_2} \tilde{u} \quad \text{in } L^p(\Omega),$$

for some sequence $w_\varepsilon \in K_\varepsilon$ and it follows that

$$\begin{aligned} c\,|\nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|^p_{L^p(\Omega)} &\leq \int_\Omega (a(x, \nabla^\varepsilon u_\varepsilon) - a(x, \nabla^\varepsilon w_\varepsilon)) \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)\, dx \\ &\leq \langle f, u_\varepsilon - w_\varepsilon \rangle_{\mathcal{W}_0(\Omega)} - \int_\Omega a(x, \nabla^\varepsilon w_\varepsilon) \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)\, dx \to 0. \end{aligned}$$

Whence

$$\varepsilon \nabla_{X_1} u_\varepsilon \to 0 \quad \text{and} \quad \nabla_{X_2} u_\varepsilon \to \nabla_{X_2} \tilde{u} \quad \text{in } L^p(\Omega).$$

The strong convergence $u_\varepsilon \to \tilde{u}$ in $L^p(\Omega)$ follows by Poincaré's inequality in the X_2-direction. This ends the proof. □

Remark 4. *When $\mathcal{K}$ is a closed linear subspace of $\mathcal{W}_0(\Omega)$ the variational inequality (2.26) is reduced to the integral identity*

$$\int_\Omega a(x, \nabla^0 \tilde{u}) \cdot \nabla^0 w\, dx = \langle f, w \rangle_{\mathcal{W}_0(\Omega)}, \qquad \forall w \in \mathcal{K}.$$

2.6. **Perturbed $p-$Laplace operator.** A common example of the function a is given by

$$a(x,\xi) = |\xi|^{p-2}\,\xi, \qquad \forall \xi \in \mathbb{R}^n \text{ and } p > 1. \tag{2.32}$$

This corresponds to the perturbed $p-$Laplace operator considered in Problem (1.1). We show here strong convergences, even if the $p-$Laplacian is not strongly monotone for $1 < p \leq 2$.

The following Lemma summarizes some inequalities related to the $p-$Laplacian, required here and later in the last section. The proof can be found for instance in [8].

Lemma 5. *For all $p > 1$ and $\xi, \eta \in \mathbb{R}^n$, it holds that for some constants C_i, $i = 1, \cdots, 4$ depending on p*

$$C_1 \{|\xi| + |\eta|\}^{p-2} |\xi - \eta|^2 \leq \left(|\xi|^{p-2}\,\xi - |\eta|^{p-2}\,\eta\right) \cdot (\xi - \eta) \tag{2.33}$$

$$\left||\xi|^{p-2}\,\xi - |\eta|^{p-2}\,\eta\right| \leq C_2 \{|\xi| + |\eta|\}^{p-2} |\xi - \eta|\,. \tag{2.34}$$

If $p \geq 2$, then

$$\left(|\xi|^{p-2}\,\xi - |\eta|^{p-2}\,\eta\right) \cdot (\xi - \eta) \geq C_3 |\xi - \eta|^p\,. \tag{2.35}$$

If $1 < p \leq 2$ then

$$\left||\xi|^{p-2}\,\xi - |\eta|^{p-2}\,\eta\right| \leq C_4 |\xi - \eta|^{p-1}\,. \tag{2.36}$$

The limit of u_ε, solution of Problem (2.2), is the unique solution of the problem

$$\begin{cases} \displaystyle\int_\Omega |\nabla_{X_2} u|^{p-2}\,\nabla_{X_2} u \cdot \nabla_{X_2}(v-u)\,dx \geq \langle f, v-u\rangle_{\mathcal{W}_0(\Omega)}\,, \quad \forall v \in \mathcal{K}, \\ u \in \mathcal{K}. \end{cases} \tag{2.37}$$

The $p-$Laplacian is strictly monotone, and due to the inequality (2.35), it is strongly monotone for $p \geq 2$, then the strong convergences (2.29) are ensured by Theorem 2. For $p > 1$ arbitrary, we have the following theorem.

Theorem 3. *Let $p > 1$ and u_ε be the unique solution of Problem (1.1). Assume that $K_\varepsilon \overset{a}{\to} \mathcal{K}$ as $\varepsilon \to 0$, then we have*

$$u_\varepsilon \to \tilde{u}, \qquad \varepsilon \nabla_{X_1} u_\varepsilon \to 0 \quad \text{and} \quad \nabla_{X_2} u_\varepsilon \to \nabla_{X_2} \tilde{u} \quad \text{in } L^p(\Omega)\,,$$

where $\tilde{u}$ is the solution to the variational inequality (2.37).

Proof. Thanks to the first assertion of Theorem 2 we have

$$\begin{pmatrix} \varepsilon \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2} u_\varepsilon \end{pmatrix} \rightharpoonup \begin{pmatrix} 0 \\ \nabla_{X_2} \tilde{u} \end{pmatrix} \text{ in } L^p(\Omega)\,, \tag{2.38}$$

for the whole sequence and $p > 1$ arbitrary. To get the strong convergence, one note that

$$\begin{aligned} &\int_\Omega \left(|\nabla^\varepsilon u_\varepsilon|^{p-2}\,\nabla^\varepsilon u_\varepsilon - |\nabla^\varepsilon w_\varepsilon|^{p-2}\,\nabla^\varepsilon w_\varepsilon\right) \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)\,dx \\ &= |\nabla^\varepsilon u_\varepsilon|^p_{L^p(\Omega)} + |\nabla^\varepsilon w_\varepsilon|^p_{L^p(\Omega)} - \int_\Omega |\nabla^\varepsilon u_\varepsilon|^{p-2}\,\nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon w_\varepsilon dx - \int_\Omega |\nabla^\varepsilon w_\varepsilon|^{p-2}\,\nabla^\varepsilon w_\varepsilon \cdot \nabla^\varepsilon u_\varepsilon dx \\ &\geq |\nabla^\varepsilon u_\varepsilon|^p_{L^p(\Omega)} + |\nabla^\varepsilon w_\varepsilon|^p_{L^p(\Omega)} - |\nabla^\varepsilon u_\varepsilon|^{p-1}_{L^p(\Omega)} \times |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)} - |\nabla^\varepsilon w_\varepsilon|^{p-1}_{L^p(\Omega)} \times |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)}\,. \end{aligned}$$

Hölder's inequality is used to obtain the two last terms. The above inequality can also be written as

$$\begin{aligned} 0 &\le \left(|\nabla^{\varepsilon} u_{\varepsilon}|_{L^p(\Omega)}^{p-1} - |\nabla^{\varepsilon} w_{\varepsilon}|_{L^p(\Omega)}^{p-1} \right) \times \left(|\nabla^{\varepsilon} u_{\varepsilon}|_{L^p(\Omega)} - |\nabla^{\varepsilon} w_{\varepsilon}|_{L^p(\Omega)} \right) \\ &\le \int_{\Omega} \left(|\nabla^{\varepsilon} u_{\varepsilon}|^{p-2} \nabla^{\varepsilon} u_{\varepsilon} - |\nabla^{\varepsilon} w_{\varepsilon}|^{p-2} \nabla^{\varepsilon} w_{\varepsilon} \right) \cdot \nabla^{\varepsilon} (u_{\varepsilon} - w_{\varepsilon})\, dx \\ &\le \langle f, u_{\varepsilon} - w_{\varepsilon} \rangle_{\mathcal{W}_0(\Omega)} - \int_{\Omega} |\nabla^{\varepsilon} w_{\varepsilon}|^{p-2} \nabla^{\varepsilon} w_{\varepsilon} \cdot \nabla^{\varepsilon} (u_{\varepsilon} - w_{\varepsilon})\, dx. \end{aligned}$$

Since $\tilde{u} \in \mathcal{K}$ choosing $w_{\varepsilon} \in K_{\varepsilon}$ such that

$$\varepsilon \nabla_{X_1} w_{\varepsilon} \to 0 \quad \text{and} \quad \nabla_{X_2} w_{\varepsilon} \to \nabla_{X_2} \tilde{u} \quad \text{in } L^p(\Omega),$$

and passing to the limit in the above inequality we end up with

$$|\nabla^{\varepsilon} u_{\varepsilon}|_{L^p(\Omega)} \to |\nabla_{X_2} \tilde{u}|_{L^p(\Omega)}, \quad \text{as } \varepsilon \to 0.$$

Taking into account (2.38) the strong convergence follows since $L^p(\Omega)$ is uniformly convex for $p > 1$. This ends the proof. □

The above theorem can be used to give an equivalent definition of the space $\mathcal{W}_0(\Omega)$.

Corollary 4. *Let Ω be a bounded open domain of $\mathbb{R}^n$. Then, it holds that*

$$u \in \mathcal{W}_0(\Omega) \ \textit{iff } u \in \mathcal{W}(\Omega) \ \textit{and } u(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1}), \quad a.e. \ X_1 \in \Pi_1.$$

Proof. For $u \in \mathcal{W}_0(\Omega)$ there exists a sequence $(u_n)_n \subset \mathcal{D}(\Omega)$ such that $u_n \to u$ in $\mathcal{W}(\Omega)$. In particular we have

$$|\nabla_{X_2}(u_n - u)|_{L^p(\Omega)} \to 0.$$

By the Lebesgue theorem we get - up to a subsequence -

$$|\nabla_{X_2}(u_n(X_1, \cdot) - \nabla_{X_2} u(X_1, \cdot))|_{L^p(\Omega_{X_1})} \to 0, \text{ for a.e. } X_1 \in \Pi_1.$$

Since $v \to |\nabla_{X_2} v|_{L^p(\Omega_{X_1})}$ is a norm on $W_0^{1,p}(\Omega_{X_1})$, we infer that $u(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1})$, for a.e. $X_1 \in \Pi_1$, and

$$\mathcal{W}_0(\Omega) \subset \left\{ u \in L^p(\Omega) \,\middle|\, \nabla_{X_2} u \in [L^p(\Omega)]^{n-q}, u(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1}), \text{ a.e. } X_1 \in \Pi_1 \right\}. \quad (2.39)$$

For the converse inclusion, we use an anisotropic perturbation argument. Let

$$u \in \left\{ u \in L^p(\Omega) \,\middle|\, \nabla_{X_2} u \in [L^p(\Omega)]^{n-q}, u(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1}), \text{ a.e. } X_1 \in \Pi_1 \right\}. \quad (2.40)$$

Since $\nabla_{X_2} u \in [L^p(\Omega)]^{n-q}$, then we can take $\nabla_{X_2}\left(|\nabla_{X_2} u|^{p-2} \nabla_{X_2} u \right) \in \mathcal{W}_0'(\Omega)$ as a source term in the following quasilinear problem

$$\begin{aligned} \int_{\Omega} |\nabla^{\varepsilon} v_{\varepsilon}|^{p-2} \nabla^{\varepsilon} v_{\varepsilon} \cdot \nabla^{\varepsilon} v dx &= \left\langle \nabla_{X_2}\left(|\nabla_{X_2} u|^{p-2} \nabla_{X_2} u \right), v \right\rangle_{\mathcal{W}_0(\Omega)} \\ &= \int_{\Omega} |\nabla_{X_2} u|^{p-2} \nabla_{X_2} u \cdot \nabla_{X_2} v dx, \quad \forall v \in W_0^{1,p}(\Omega) \end{aligned}$$

where the unique solution v_ε is in $W_0^{1,p}(\Omega)$. Choosing $K_\varepsilon = W_0^{1,p}(\Omega)$ and $\mathcal{K} = \mathcal{W}_0(\Omega)$, then due to Lemma 3 and Theorem 3, we have

$$\nabla^\varepsilon v_\varepsilon \to \nabla_{X_2}\tilde{u} \text{ in } L^p(\Omega)$$

where $\tilde{u} \in \mathcal{W}_0(\Omega)$ is the solution of problem

$$\int_\Omega |\nabla_{X_2}\tilde{u}|^{p-2}\nabla_{X_2}\tilde{u}\cdot\nabla_{X_2}v dx = \int_\Omega |\nabla_{X_2}u|^{p-2}\nabla_{X_2}u\cdot\nabla_{X_2}v dx, \tag{2.41}$$

for every $v \in W_0^{1,p}(\Omega)$. It remain to check that $u = \tilde{u}$ in Ω. We give here a proof for cylindrical domains, i.e.

$$\Omega = \omega_1 \times \omega_2, \quad \text{where } \omega_1 \subset \mathbb{R}^q \text{ and } \omega_2 \subset \mathbb{R}^{n-q}. \tag{2.42}$$

For general domains, i.e. not necessarily cylindrical, we can argue as in [10]. Let $\varphi_1 \in \mathcal{D}(\omega_1)$ and $\varphi_2 \in W_0^{1,p}(\omega_2)$, then we derive from (2.41)

$$\int_{\omega_1}\varphi_1(X_1)\int_{\omega_2}|\nabla_{X_2}\tilde{u}|^{p-2}\nabla_{X_2}\tilde{u}(X_1,X_2)\cdot\nabla_{X_2}\varphi(X_2)\,dX_2\,dX_1$$
$$= \int_{\omega_1}\varphi_1(X_1)\int_{\omega_2}|\nabla_{X_2}u|^{p-2}\nabla_{X_2}u(X_1,X_2)\cdot\nabla_{X_2}\varphi(X_2)\;dX_2\;dX_1 \quad \forall\varphi_1 \in \mathcal{D}(\omega_2),$$

since $\varphi_1\varphi_2 \in W_0^{1,p}(\Omega)$. Thus, for a.e. $X_1 \in \omega_1$, we have

$$\int_{\omega_2}|\nabla_{X_2}\tilde{u}|^{p-2}\nabla_{X_2}\tilde{u}(X_1,X_2)\cdot\nabla_{X_2}\varphi_2(X_2)\,dX_2$$
$$= \int_{\omega_2}|\nabla_{X_2}u|^{p-2}\nabla_{X_2}u(X_1,X_2)\cdot\nabla_{X_2}\varphi_2(X_2)\,dX_2, \quad \forall\varphi_2 \in W_0^{1,p}(\omega_2). \tag{2.43}$$

By (2.39) and assumption (2.40) we have $\tilde{u}(X_1,\cdot) \in W_0^{1,p}(\omega_2)$ and $u(X_1,\cdot) \in W_0^{1,p}(\omega_2)$, for a.e. $X_1 \in \Pi_1$. Thus we can take $\tilde{u}(X_1,\cdot) - u(X_1,\cdot)$ as a test function in (2.43) and it comes that

$$\int_{\omega_2}\Big(|\nabla_{X_2}\tilde{u}|^{p-2}\nabla_{X_2}\tilde{u}(X_1,X_2) - |\nabla_{X_2}u|^{p-2}\nabla_{X_2}u(X_1,X_2)\Big)\cdot\nabla_{X_2}(\tilde{u}-u)(X_1,X_2)\,dX_2 = 0.$$

Due to the strict monotonicity, which follows from (2.33) by taking $\xi = \nabla^0\tilde{u}$ and $\eta = \nabla^0 u$, we infer that

$$(\tilde{u}-u)(X_1,\cdot) = 0 \;\text{ a.e. in } \omega_2,$$

for a.e. $X_1 \in \omega_1$. This implies that $\tilde{u} = u$ a.e. in Ω and the corollary follows. □

3. Convex sets with perturbed obstacle constraints

In this section and in the next one, we give some examples of convex sets where the convergence in the sense of Definition 1 can be ensured. We refer the reader to Chipot [6], Kinderlehrer and Stampacchia [18] for applications and more mathematical background about related problems.

For $\varepsilon > 0$, we consider a sequence of obstacles $\psi_\varepsilon \in W^{1,p}(\Omega)$ and the associated sequence of convex sets

$$K_\varepsilon = K_{\psi_\varepsilon} := \left\{v \in W_0^{1,p}(\Omega) \mid v \geq \psi_\varepsilon \text{ a.e. in } \Omega\right\}.$$

The set K_ε is not empty provided that $\psi_\varepsilon^+ := \max\{\psi_\varepsilon, 0\} \in W_0^{1,p}(\Omega)$. Assuming that, in some sense, ψ_ε converges to some $\psi_0 \in \mathcal{W}(\Omega)$ such that $\psi_0^+ \in \mathcal{W}_0(\Omega)$, one expects that the limit of $K_{\psi_\varepsilon}$is

$$\mathcal{K} = \mathcal{K}_{\psi_0} := \{v \in \mathcal{W}_0(\Omega) \mid v \geq \psi_0 \text{ a.e. in } \Omega\}.$$

To be more precise we have the following result.

Theorem 5. *Under the above assumptions, if*

$$\psi_\varepsilon \rightharpoonup \psi_0, \quad \text{in } L^p(\Omega) \text{ as } \varepsilon \to 0, \tag{3.1}$$

then we have $aw - \overline{\lim}_{\varepsilon\to 0} K_{\psi_\varepsilon} \subset \mathcal{K}_{\psi_0}$.

Moreover if

$$\nabla^\varepsilon \psi_\varepsilon \to \nabla^0 \psi_0 \quad \text{in } L^p(\Omega) \text{ as } \varepsilon \to 0, \tag{3.2}$$

then it holds that $a - \lim_{\varepsilon\to 0} K_{\psi_\varepsilon} = \mathcal{K}_{\psi_0}$.

Proof. Let $v \in aw - \overline{\lim}_{\varepsilon\to 0} K_{\psi_\varepsilon}$. Then there exists a subsequence $v_\varepsilon \in K_{\psi_\varepsilon}$, such that $\nabla^\varepsilon v_\varepsilon \rightharpoonup \nabla^0 v$ in $L^p(\Omega)$. Since $v_\varepsilon \geq \psi_\varepsilon$ a.e. in Ω, it comes that

$$\int_\Omega (v_\varepsilon - \psi_\varepsilon)\,\varphi\, dx \geq 0, \qquad \forall \varphi \in \mathcal{D}(\Omega), \varphi \geq 0.$$

Due to Poincaré's inequality, in the $X_2 - direction$, v_ε is bounded in $L^p(\Omega)$. We have then –up to a new subsequence– $v_\varepsilon \rightharpoonup v$ in $L^p(\Omega)$ and using (3.1), it comes that

$$\int_\Omega (v_\varepsilon - \psi_\varepsilon)\,\varphi\, dx \to \int_\Omega (v - \psi_0)\,\varphi\, dx \geq 0, \qquad \forall \varphi \in \mathcal{D}(\Omega), \varphi \geq 0.$$

Thus we have $v \geq \psi_0$, a.e. in Ω, i.e. $v \in \mathcal{K}_{\psi_0}$.

To establish the last assertion, we have just to show that $\mathcal{K}_{\psi_0} \subset as - \underline{\lim}_{\varepsilon\to 0} K_{\psi_\varepsilon}$. Let $v \in \mathcal{K}_{\psi_0}$ then, due to the last assertion of Lemma 3, there exists a sequence $w_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\nabla^\varepsilon w_\varepsilon \to \nabla^0 v, \quad \text{in } L^p(\Omega) \text{ as } \varepsilon \to 0. \tag{3.3}$$

But w_ε may not belong to K_ε, hence we consider the sequence

$$v_\varepsilon = \max\{\psi_\varepsilon, w_\varepsilon\} = (\psi_\varepsilon - w_\varepsilon)^+ + w_\varepsilon \in W_0^{1,p}(\Omega). \tag{3.4}$$

It is clear that $v_\varepsilon \geq \psi_\varepsilon$, i.e. $v_\varepsilon \in K_{\psi_\varepsilon}$. This sequence converges to v in the sense of (2.13). Indeed, due to (3.2), (3.3) and the boundedness of the positive part as an operator on $W^{1,p}(\Omega)$ (see for instance Heinonen et al. [17, Lemma 1.22]) we have

$$\nabla^\varepsilon\left((\psi_\varepsilon - w_\varepsilon)^+\right) \to \nabla^0\left((\psi_0 - v)^+\right) \quad \text{in } L^p(\Omega)$$

and it follows that

$$\nabla^\varepsilon v_\varepsilon \to \nabla^0\left((\psi_0 - v)^+ + v\right) = \nabla^0 v \quad \text{in } L^p(\Omega) \text{ as } \varepsilon \to 0.$$

This means that $v \in as - \underline{\lim}_{\varepsilon\to 0} K_{\psi_\varepsilon}$ and the theorem is proved. □

4. Convex sets with perturbed gradient constraints

In this section, the convex set is determined by a constraint on the gradient of the solution. Consider a sequence of nonnegative functions $\beta_\varepsilon \in L^\infty(\Omega)$ and consider the set

$$K_\varepsilon = K_{\beta_\varepsilon} := \left\{ v \in W_0^{1,p}(\Omega) \mid \ |\nabla^\varepsilon v| \le \beta_\varepsilon \text{ a.e. in } \Omega \right\},$$

which is a nonempty closed convex set of $W_0^{1,p}(\Omega)$. The problem (2.2) with the constraint set K_{β_ε} admits a solution $u_\varepsilon \in K_{\beta_\varepsilon}, \forall \varepsilon > 0$.

Assuming that β_ε converges to some nonnegative function β_0 in $L^\infty(\Omega)$, i.e.

$$|\beta_\varepsilon - \beta_0|_{L^\infty(\Omega)} \to 0, \text{ as } \varepsilon \to 0, \tag{4.1}$$

one expects that the limit of K_{β_ε} is

$$\mathcal{K} = \mathcal{K}_{\beta_0} := \{ v \in \mathcal{W}_0(\Omega) \mid |\nabla_{X_2} v| \le \beta_0 \text{ a.e. in } \Omega \}.$$

4.1. Preliminary results.

Let us start by the first inclusion in (2.20).

Theorem 6. *Assume that (4.1) holds, then* $aw - \overline{\lim}_{\varepsilon\to 0} K_{\beta_\varepsilon} \subset \mathcal{K}_{\beta_0}$.

Proof. For $\delta > 0$, let

$$M_\delta := \{ v \in \mathcal{W}_0(\Omega) \mid |\nabla_{X_2} v| \le \beta_0 + \delta \text{ a.e. in } \Omega \},$$

which is a convex and closed set in $\mathcal{W}_0(\Omega)$, hence it is also a weakly closed set. Let $v \in aw - \overline{\lim}_{\varepsilon\to 0} K_{\beta_\varepsilon}$. Then there exists a subsequence $v_\varepsilon \in K_{\beta_\varepsilon}$ (still labeled v_ε) such that $\nabla^\varepsilon v_\varepsilon \rightharpoonup \nabla^0 v$ in $L^p(\Omega)$. As $\beta_\varepsilon \to \beta_0$ in $L^\infty(\Omega)$, then $\beta_\varepsilon \le \beta_0 + \delta$, *a.e.* in Ω, for ε small enough, and it comes that

$$|\nabla_{X_2} v_\varepsilon| \le |\nabla^\varepsilon v_\varepsilon| \le \beta_\varepsilon \le \beta_0 + \delta, \text{ a.e. in } \Omega.$$

Thus $v_\varepsilon \in M_\delta$, for ε small enough, which implies that its weak limit v also belongs to M_δ. Since this holds for arbitrary δ we deduce that $v \in \bigcap_{\delta>0} M_\delta = \mathcal{K}_{\beta_0}$. □

To show the other inclusion, i.e. $\mathcal{K}_{\beta_0} \subset as - \underline{\lim}_{\varepsilon\to 0} K_{\beta_\varepsilon}$, we have first to check that $\mathcal{K}_{\beta_0} \subset L^\infty(\Omega)$.

Lemma 6. *Let Ω be a domain of $\mathbb{R}^n$, bounded in the X_2-direction, then*

$$\{ v \mid v \in \mathcal{W}_0(\Omega), |\nabla_{X_2} v| \in L^\infty(\Omega) \} \subset L^\infty(\Omega).$$

Proof. As $\mathcal{D}(\Omega)$ is dense in $\mathcal{W}_0(\Omega)$ by definition, the canonical extension $u \to \tilde{u}(x)$, where

$$\tilde{u}(x) := \begin{cases} u(x), & x \in \Omega \\ 0, & x \notin \Omega \end{cases}$$

define a continuous mapping from $\mathcal{W}_0(\Omega)$ to $\mathcal{W}(\mathbb{R}^n)$ and it holds, in the distributional sense, that

$$\nabla_{X_2} \tilde{u} = \nabla_{X_2} u \ \chi_\Omega. \tag{4.2}$$

Let $u \in \mathcal{W}_0(\Omega)$ such that $|\nabla_{X_2} u| \in L^\infty(\Omega)$. We consider the ball $B_R \subset \mathbb{R}^{n-q}$ such that $\Omega \subset\subset \mathbb{R}^q \times B_R$ and we denote by u_ε the convolution of $\tilde{u}$ defined as

$$u_\varepsilon(x) = \rho_\varepsilon * \tilde{u}(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(y)\, \tilde{u}(x-y)\, dy \tag{4.3}$$

for a.e. $x \in \mathbb{R}^n$, where ρ_ε is the usual mollifier in $\mathbb{R}^n$. We can show that

$$u_\varepsilon \to \tilde{u} \text{ in } L^p(\mathbb{R}^n) \text{ and } |u_\varepsilon|_{L^p(\mathbb{R})} \le |\tilde{u}|_{L^p(\mathbb{R})}.$$

In particular

$$u_\varepsilon \to \tilde{u} \text{ in } L^p(\mathbb{R}^q \times B_R). \tag{4.4}$$

Going back to (4.3), one has

$$\partial_{x_i} u_\varepsilon(x) = \int_{\mathbb{R}} \rho_\varepsilon(y)\, \partial_{x_i}\tilde{u}(x-y)\, dy, \text{ for } i = q+1, \cdots n$$

and thus

$$|\nabla_{X_2} u_\varepsilon(x)|^2 = \sum_{i=q+1}^{n} |\partial_{x_i} u_\varepsilon(x)|^2 \le \int_{\mathbb{R}^n} \rho_\varepsilon(y)\, |\nabla_{X_2}\tilde{u}(x-y)|^2\, dy$$

by Jensen's inequality. Thanks to (4.2) and $|\nabla_{X_2} u| \in L^\infty(\Omega)$, it holds that $|\nabla_{X_2}\tilde{u}| \in L^\infty(\mathbb{R}^n)$ and thus

$$|\nabla_{X_2} u_\varepsilon| \in L^\infty(\mathbb{R}^n).$$

For ε small enough, $\operatorname{supp}(u_\varepsilon) \subset \mathbb{R}^q \times B_R$ and by the mean value theorem, applied for the smooth function u_ε, it follows that

$$|u_\varepsilon| \le CR \text{ a.e. in } \mathbb{R}^q \times B_R,$$

for some constant C. Since the convergence (4.4) preserves boundedness in $L^\infty(\mathbb{R}^q \times B_R)$, we infer that

$$|\tilde{u}| \le CR \text{ a.e. in } \mathbb{R}^q \times B_R,$$

which means that $u \in L^\infty(\Omega)$ as claimed. □

Remark 5. *If we assume $p > n-q$, the above lemma follows by using Sobolev's injections. In fact, for a.e. X_1 we have $u(X_1,\cdot) \in W_0^{1,p}(\Omega_{X_1}) \subset L^\infty(\Omega_{X_1})$ and*

$$\begin{aligned} |u(X_1,\cdot)|_{L^\infty(\Omega_{X_1})} &\le C\, |\nabla_{X_2} u(X_1,\cdot)|_{L^p(\Omega_{X_1})} \\ &\le C\, |\nabla_{X_2} u(X_1,\cdot)|_{L^\infty(\Omega_{X_1})} \end{aligned}$$

since Ω is bounded in the X_2-direction, whence

$$|u(X_1,\cdot)|_{L^\infty(\Omega_{X_1})} \le C\, |\nabla_{X_2} u|_{L^\infty(\Omega)}, \quad \textit{for a.e. } X_1,$$

where the last C is independent of X_1. Thus $|u|_{L^\infty(\Omega)} \le C\, |\nabla_{X_2} u|_{L^\infty(\Omega)}$.

4.2. **Convergence for cylindrical domains.** Let now $v \in \mathcal{K}_{\beta_0}$, due to the last assertion of Lemma 3, there exists a sequence $v_\varepsilon \in W_0^{1,p}(\Omega)$ such that $\nabla^\varepsilon v_\varepsilon \to \nabla^0 v$ in $L^p(\Omega)$. To insure that $v \in as - \underline{\lim}_{\varepsilon\to 0} K_{\beta_\varepsilon}$, each v_ε must be chosen in K_{β_ε}, i.e. v_ε satisfies a constraint on the gradient $|\nabla^\varepsilon v_\varepsilon| \le \beta_\varepsilon$ a.e. in Ω. Such a sequence v_ε can be explicitly constructed for some types of domains as it will be described in the remainder of this section.

We assume in this subsection that Ω is cylindrical, i.e.

$$\Omega = \omega_1 \times \omega_2, \quad \omega_1 \subset \mathbb{R}^q \text{ and } \omega_2 \subset \mathbb{R}^{n-q} \tag{4.5}$$

where ω_1 is a regular bounded domain.

Theorem 7. *Assume that (4.1) and (4.5) hold. In addition suppose that*

$$\beta_0 \in C\left(\bar{\Omega}\right) \text{ and } \beta_\varepsilon \ge \sigma > 0 \text{ a.e. in } \Omega, \tag{4.6}$$

for some constant σ. Then $\mathcal{K}_{\beta_0} \subset as - \underline{\lim}_{\varepsilon\to 0} K_{\beta_\varepsilon}$, i.e. $\mathcal{K}_{\beta_0} = a - \lim_{\varepsilon\to 0} K_{\beta_\varepsilon}$.

($C\left(\bar{\Omega}\right)$ is the space of restrictions of $C(\mathbb{R}^n)$ –functions to Ω).

Proof. First, we proceed by truncation to show that the set

$$\mathcal{D}_{\beta_0} := \left\{ v \in \mathcal{K}_{\beta_0} \mid \operatorname{supp}(v) \subset \omega_1 \times \bar{\omega}_2 \right\} \tag{4.7}$$

is dense in $\mathcal{K}_{\beta_0}$. Then the support of the functions of $\mathcal{D}_{\beta_0}$ can be kept in $\omega_1 \times \bar{\omega}_2$ after a partial regularization in X_1. This leads to define, for each function in $\mathcal{D}_{\beta_0}$, a converging sequence in K_{β_ε} towards this function. Due to Lemma 2, this is sufficient to conclude that $\mathcal{K}_{\beta_0} \subset as - \underline{\lim}_{\varepsilon\to 0} K_{\beta_\varepsilon}$.

i) Truncation. Let $v \in \mathcal{K}_{\beta_0}$ and for a small $d_1 > 0$, consider the set

$$\omega_1' := \left\{ X_1 \in \omega_1 \mid \operatorname{dist}(X_1, \mathbb{R}^q \backslash \omega_1) > d_1 \right\},$$

and the truncated function

$$v' := v \chi_{\omega_1'}$$

where $\chi_{\omega_1'}$ is the indicator function of the set ω_1'. We still have

$$\left|\nabla_{X_2} v'\right| \le \beta_0 \text{ a.e. in } \Omega,$$

i.e. $v' \in \mathcal{K}_{\beta_0}$. Moreover, for any $\delta > 0$, there exists $d_1 > 0$, small enough, such that

$$\begin{aligned} \left|\nabla_{X_2}\left(v - v'\right)\right|_{L^p(\Omega)} &= \left|\nabla_{X_2} v \chi_{\omega_1 \backslash \omega_1'}\right|_{L^p(\Omega)} \\ &\le |\beta_0|_\infty \left|\chi_{\omega_1 \backslash \omega_1'}\right|_{L^p(\Omega)} \\ &= |\beta_0|_\infty \left[meas\left(\left(\omega_1 \backslash \omega_1'\right) \times \omega_2\right)\right]^{\frac{1}{p}} \le \delta, \end{aligned}$$

which is nothing else than the density of the set of functions, supported far from the boundary $\partial\omega_1 \times \omega_2$, in $\mathcal{K}_{\beta_0}$.

ii) Regularization in the X_1-direction. For $\alpha \in (0,1)$, consider the (positive) $\varepsilon^{\alpha}-$mollifier sequence

$$\rho_{\varepsilon}(X_1) := \frac{1}{\varepsilon^{q\alpha}} \rho\left(\frac{X_1}{\varepsilon^{\alpha}}\right). \tag{4.8}$$

Let $v \in \mathcal{D}_{\beta_0}$, then for ε small enough we still have $\operatorname{supp}(\rho_{\varepsilon} * v) \subset \omega_1 \times \bar{\omega}_2$, which in particular implies that

$$\rho_{\varepsilon} * v \in W_0^{1,p}(\Omega).$$

Extending v by 0 outside Ω, we can show that for a.e. $X_2 \in \omega_2$

$$\begin{aligned} \rho_{\varepsilon} * v(\cdot, X_2) &\to v(\cdot, X_2) \quad \text{in } L^p(\mathbb{R}^q) \\ \nabla_{X_2}(\rho_{\varepsilon} * v(\cdot, X_2)) &= \rho_{\varepsilon} * \nabla_{X_2} v(\cdot, X_2) \to \nabla_{X_2} v \quad \text{in } L^p(\mathbb{R}^q). \end{aligned}$$

Since

$$|\nabla_{X_2}(\rho_{\varepsilon} * v(\cdot, X_2))|_{L^p(\mathbb{R}^q)} \leq |\rho|_{L^1(\mathbb{R}^q)} |\nabla_{X_2} v(\cdot, X_2)|_{L^p(\mathbb{R}^q)} = |\nabla_{X_2} v(\cdot, X_2)|_{L^p(\mathbb{R}^q)},$$

by Lebesgue's theorem in $\mathbb{R}^{n-q}$ we derive

$$\nabla_{X_2}(\rho_{\varepsilon} * v) \to \nabla_{X_2} v \quad \text{in } L^p(\mathbb{R}^n). \tag{4.9}$$

On the other hand, we have

$$\begin{aligned} \partial_{x_i}(\rho_{\varepsilon} * v)(x) &= (\partial_{x_i}\rho_{\varepsilon}) * v(x) \\ &= \int_{\mathbb{R}^q} \partial_{x_i}\rho_{\varepsilon}(Y_1)\, v(X_1 - Y_1, X_2)\, dY_1 \\ &= \frac{1}{\varepsilon^{\alpha}} \int_{\mathbb{R}^q} \frac{1}{\varepsilon^{q\alpha}} \partial_{x_i}\rho\left(\frac{Y_1}{\varepsilon^{\alpha}}\right) v(X_1 - Y_1, X_2)\, dY_1, \quad \text{for } i = 1, \cdots, q. \end{aligned}$$

Then by the property of the convolution product in $L^p(\mathbb{R}^q)$ one gets

$$\begin{aligned} |\varepsilon \partial_{x_i}(\rho_{\varepsilon} * v)(\cdot, X_2)|_{L^p(\mathbb{R}^q)} &\leq \varepsilon^{1-\alpha} |\partial_{x_i}\rho|_{L^1(\mathbb{R}^q)} |v(\cdot, X_2)|_{L^p(\mathbb{R}^q)} \\ &\leq C\varepsilon^{1-\alpha} |v(\cdot, X_2)|_{L^p(\mathbb{R}^q)}. \end{aligned}$$

Elevating to the power p and integrating on $\mathbb{R}^{n-q}$, it comes

$$|\varepsilon \partial_{x_i}(\rho_{\varepsilon} * v)|_{L^p(\mathbb{R}^n)} \leq C\varepsilon^{1-\alpha} |v|_{L^p(\mathbb{R}^n)}, \ \text{for } i = 1, \cdots, q,$$

i.e.,

$$|\varepsilon \nabla_{X_1}(\rho_{\varepsilon} * v)|_{L^p(\Omega)} \to 0 \text{ as } \varepsilon \to 0.$$

Thus, taking into account (4.9), we deduce that

$$\nabla^{\varepsilon}(\rho_{\varepsilon} * v) \to \nabla^0 v \quad \text{in } L^p(\Omega) \ \text{ as } \varepsilon \to 0. \tag{4.10}$$

Until now, the convenient sequence is not yet defined since $\rho_\varepsilon * v$ may not belong to K_ε. Due to Lemma 6 it holds that $\mathcal{K}_{\beta_0} \subset L^\infty(\Omega)$ and thus

$$\begin{aligned}
|\nabla^\varepsilon(\rho_\varepsilon * v)| &\leq |\varepsilon(\nabla_{X_1}\rho_\varepsilon) * v| + |\rho_\varepsilon * \nabla_{X_2}v| \\
&\leq C\varepsilon^{1-\alpha}|v|_\infty + \rho_\varepsilon * |\nabla_{X_2}v| \\
&\leq C\varepsilon^{1-\alpha}|v|_\infty + \rho_\varepsilon * \beta_0 \\
&\leq C\varepsilon^{1-\alpha}|v|_\infty + |\rho_\varepsilon * \beta_0 - \beta_0| + |\beta_0 - \beta_\varepsilon| + \beta_\varepsilon,
\end{aligned}$$

hence

$$|\nabla^\varepsilon(\rho_\varepsilon * v)| \leq C_\varepsilon + \beta_\varepsilon \tag{4.11}$$

where $C_\varepsilon \to 0$, as $\varepsilon \to 0$. This implies that (recall that $\beta_\varepsilon \geq \sigma$)

$$|\nabla^\varepsilon(\rho_\varepsilon * v)| \leq \frac{C_\varepsilon}{\sigma}\sigma + \beta_\varepsilon \leq (1 + C_\varepsilon/\sigma)\beta_\varepsilon,$$

i.e.

$$v_\varepsilon := \frac{1}{1 + C_\varepsilon/\sigma}\,\rho_\varepsilon * v \in K_{\beta_\varepsilon}$$

and

$$\nabla^\varepsilon v_\varepsilon \to \nabla^0 v \quad \text{in } L^p(\Omega) \text{ as } \varepsilon \to 0.$$

(In the above estimates we used the fact $\rho_\varepsilon * \beta_0, \beta_\varepsilon \to \beta_0$ uniformly in $\mathbb{R}^q \times \omega_2$, $\beta_0(\cdot, X_2)$ assumed to be extended outside ω_1 for a.e. $X_2 \in \omega_2$). This completes the proof. □

4.3. **Convergence for some noncylindrical domains.** We consider now two classes of domains satisfying some star-shapeness type properties.

4.3.1. *Star shaped domains in the X_2−direction.* We assume that each section Ω_{X_1} is regular and intersects the hyperplane $X_2 = 0$. The domain Ω is said to be a star shaped domain in the X_2−direction if

$$\Omega_\lambda \subset \Omega, \quad \forall \lambda > 1, \tag{4.12}$$

where

$$\Omega_\lambda := \{(X_1, X_2) \in \mathbb{R}^n \mid (X_1, \lambda X_2) \in \Omega\}.$$

For $v \in \mathcal{W}_0(\Omega)$ (resp. $v \in W_0^{1,p}(\Omega)$), we set

$$v_\lambda(X_1, X_2) := v(X_1, \lambda X_2), \tag{4.13}$$

and it is clear that $v_\lambda \in \mathcal{W}_0(\Omega_\lambda)$ (resp. $v_\lambda \in W_0^{1,p}(\Omega_\lambda)$), for $\lambda > 1$, and so it can be extended by 0 to have $v_\lambda \in \mathcal{W}_0(\Omega)$ (resp. $v_\lambda \in W_0^{1,p}(\Omega)$ or $v_\lambda \in W^{1,p}(\mathbb{R}^n)$). It is also easy to check that

$$\nabla_{X_2}v_\lambda = \lambda(\nabla_{X_2}v)_\lambda.$$

Due to the mean continuity of the L^p−functions, (see Nečas [26, Page 51]), we have

$$\lim_{\lambda\to 1} v_\lambda = v \text{ and } \lim_{\lambda\to 1}\nabla_{X_2}v_\lambda = \nabla_{X_2}v \text{ in } L^p(\Omega).$$

Moreover, we can easily see that $|v_\lambda|_\infty = |v|_\infty$ and if $v \in \mathcal{K}_{\beta_0}$ then

$$I_\lambda v_\lambda \in \mathcal{K}_{\beta_0} \text{ where } I_\lambda = \frac{\sigma}{\lambda\left(\sigma + 2\left|(\beta_0)_\lambda - \beta_0\right|_\infty\right)}.$$

Indeed, without loss of generality we can assume $\beta_0 \geq \sigma/2$ and for $v \in \mathcal{K}_{\beta_0}$ we can write

$$\begin{aligned} |\nabla_{X_2} v_\lambda| = \lambda\left|(\nabla_{X_2} v)_\lambda\right| &\leq \lambda\,(\beta_0)_\lambda \\ &\leq \lambda\left(\left|(\beta_0)_\lambda - \beta_0\right| + \beta_0\right) \\ &\leq \lambda\beta_0\left(\frac{\left|(\beta_0)_\lambda - \beta_0\right|}{\beta_0} + 1\right) \leq \beta_0/I_\lambda, \quad \text{a.e. in } \Omega_\lambda. \end{aligned}$$

So we end up with $I_\lambda v_\lambda \in \mathcal{K}_{\beta_0}$. We have to mention that the function $I_\lambda v_\lambda$ will play an essential role in the following to show that $a - \lim_{\varepsilon\to 0} K_{\beta_\varepsilon} = \mathcal{K}_{\beta_0}$ since

$$\lim_{\lambda\to 1} \nabla_{X_2}(I_\lambda v_\lambda) = \nabla_{X_2} v \text{ in } L^p(\Omega). \tag{4.14}$$

The last limit is an immediate consequence of $\lim_{\lambda\to 1} I_\lambda = 1$ which is fulfilled thanks to the uniform continuity of β_0 on $\bar{\Omega}$.

On the other hand, if $dist\,(\text{supp}\,(v), \partial\Omega) = 0$, we may also have $dist\,(\text{supp}\,(v_\lambda), \partial\Omega) = 0$. The set of points that keeps $\text{supp}(v_\lambda)$ and $\partial\Omega$ adhesive, is included in $\partial\Omega \cap \partial\Omega_\lambda$, for λ sufficiently close to 1, see Figure 1. To be more precise, we consider the set $\mathcal{M}$ defined as follows

$$\begin{aligned} (X_1, X_2) \in \mathcal{M} \Leftrightarrow\ & \exists\lambda_0(X_1) \geq 1, \text{ such that } (X_1, X_2) \in \partial\Omega_\lambda, \forall\lambda \in [1, \lambda_0(X_1)], \\ \Leftrightarrow\ & \exists\lambda_0(X_1) \geq 1, \text{ such that } (X_1, \lambda X_2) \in \partial\Omega, \forall\lambda \in [1, \lambda_0(X_1)]. \end{aligned} \tag{4.15}$$

The set $\mathcal{M}$ is not empty. In fact, since the intersection of every section Ω_{X_1} with the hyperplane $X_2 = 0$ is not empty, then

$$\forall (X_1, 0) \in \partial\Omega, \text{ we have } (X_1, 0) \in \partial\Omega_\lambda, \forall\lambda \geq 1,$$

i.e. $(X_1, 0) \in \mathcal{M}$. Note that (4.15) implies, for a fixed X_1, that the whole segment $\{(X_1, \lambda X_2) \mid \forall\lambda \in [1, \lambda_0(X_1)]\}$ is included in $\partial\Omega$.

Define M as the projection of $\mathcal{M}$ on the hyperplane $X_2 = 0$ and for $s > 0$ we denote by M_s its $s-neighborhood$ (see Figure 1), then we have the following theorem.

Theorem 8. *Assume that* (4.1) *and* (4.6) *hold. In addition suppose that* Ω *satisfies* (4.12) *and*

$$meas\,(M_s) \to 0, \quad as \quad s \to 0, \tag{4.16}$$

then $\mathcal{K}_{\beta_0} \subset as - \underline{\lim}_{\varepsilon\to 0} K_{\beta_\varepsilon}$, *i.e.* $\mathcal{K}_{\beta_0} = a - \lim_{\varepsilon\to 0} K_{\beta_\varepsilon}$.

Proof. Let $v \in \mathcal{K}_{\beta_0}$. As above, we proceed by truncation (which is more delicate in this case) and regularization in the $X_1 - direction$ of the function v_λ, defined by (4.15).

i) Truncation. The function v_λ is not necessarily supported inside Ω because of the points of the set $\mathcal{M}$. To recover this property and ensure it certainly, we need to avoid this kind of points by considering the truncated function

$$\hat{v} := v\ \chi_{\Omega\setminus(M_s\times\mathbb{R}^{n-q})},$$

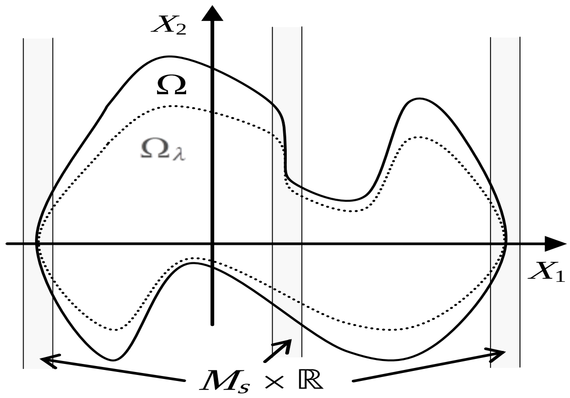

Figure 1. A star-shaped domain in the X_2–direction

for which we have $I_\lambda \hat{v}_\lambda \in \mathcal{K}_{\beta_0}$, with a support strictly included in Ω. Then under the assumption (4.16), we have

$$\begin{aligned} |\nabla_{X_2}(I_\lambda v_\lambda - I_\lambda \hat{v}_\lambda)|_{L^p(\Omega)} &= I_\lambda \left|(\nabla_{X_2} v_\lambda)\ \chi_{\Omega\cap(M_s\times\mathbb{R}^{n-q})}\right|_{L^p(\Omega)} \\ &\leq |\beta_0|_\infty \left|\chi_{\Omega\cap(M_s\times\mathbb{R}^{n-q})}\right|_{L^p(\Omega)} \\ &\leq C\,|\beta_0|_\infty \,(meas\,(M_s))^{\frac{1}{p}} \to 0 \quad (\text{as } s\to 0). \end{aligned}$$

Thanks to Lemma 2, it is sufficient, in the following, to consider functions v_λ such that $dist\,(\operatorname{supp}(v_\lambda), \partial\Omega) > 0$.

ii) Regularization in the X_1 – direction. Let $\rho_\varepsilon(X_1)$ be the mollifier sequence defined in (4.8) and for every $X_1 \in \Pi_1$ extend v_λ by 0 outside Ω_{X_1}. Note that for ε small enough, it holds that $\rho_\varepsilon * v_\lambda \in W_0^{1,p}(\Omega)$. Since $|I_\lambda v_\lambda|_\infty \leq |v_\lambda|_\infty = |v|_\infty$ then arguing as above, this time with the function $I_\lambda v_\lambda$, we deduce

$$\nabla^\varepsilon(\rho_\varepsilon * I_\lambda v_\lambda) \to I_\lambda \nabla^0 v_\lambda \quad \text{in } L^p(\Omega)\,, \text{ as } \ \varepsilon\to 0. \tag{4.17}$$

The desired sequence is given by

$$v_{\varepsilon,\lambda} := \frac{1}{1+C_\varepsilon/\sigma}\rho_\varepsilon * (I_\lambda v_\lambda) \in W_0^{1,p}(\Omega)\,, \tag{4.18}$$

where C_ε is still defined by (4.11). It holds that $|\nabla^\varepsilon v_{\varepsilon,\lambda}| \leq \beta_\varepsilon$ a.e. in Ω, which means that $v_{\varepsilon,\lambda} \in K_{\beta_\varepsilon}$. Due to (4.17), it follows that

$$\left|\nabla^\varepsilon v_{\varepsilon,\lambda} - I_\lambda \nabla^0 v_\lambda\right|_{L^p(\Omega)} \to 0, \quad \text{as } \varepsilon \to 0$$

and thus $I_\lambda v_\lambda \in as - \underline{\lim}_{\varepsilon\to 0} K_{\beta_\varepsilon}$. Taking now (4.14) with Lemma 1 into account, we infer that $v \in as - \underline{\lim}_{\varepsilon\to 0} K_{\beta_\varepsilon}$, and the theorem is proved. □

4.3.2. *Star-shaped domains.* We consider now that Ω is a star shaped domain, i.e.

$$\Omega_\lambda \subset \Omega, \quad \forall\lambda > 1, \tag{4.19}$$

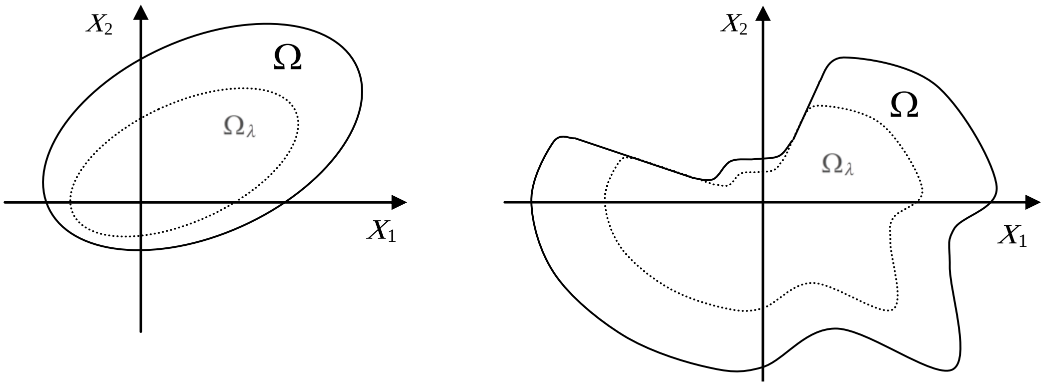

FIGURE 2. A strictly and non-strictly strar-shaped domains

where this time $\Omega_\lambda := \{x \in \mathbb{R}^n \mid \lambda x \in \Omega\}$. The star-shapeness is strict if the above inclusion is strict, i.e. $\Omega_\lambda \subset\subset \Omega$, $\forall \lambda > 1$.

Remark 6. *Star-shapeness in X_2-direction does not imply star-shapeness (in all directions), or vice versa. This can be easily verified in dimension two (see Figures 1 and 2).*

For $v \in \mathcal{W}_0(\Omega)$ (resp. $\in W_0^{1,p}(\Omega)$), we set this time $v_\lambda(x) := v(\lambda x) \in \mathcal{W}_0(\Omega_\lambda)$ (resp. $\in W_0^{1,p}(\Omega_\lambda)$) and as in the previous case we can check that,

$$\lim_{\lambda \to 1} I_\lambda v_\lambda = v \;\text{ and }\; \lim_{\lambda \to 1} \nabla_{X_2} I_\lambda v_\lambda = \nabla_{X_2} v \;\text{ in } L^p(\Omega).$$

Moreover if $v \in \mathcal{K}_{\beta_0}$, then $I_\lambda v_\lambda \in \mathcal{K}_{\beta_0}$. In this case we have also

Theorem 9. *Assume that (4.1) and (4.6) holds and that Ω is strictly star-shaped, then $\mathcal{K}_{\beta_0} \subset as - \lim_{\varepsilon \to 0} K_{\beta_\varepsilon}$, i.e. $\mathcal{K}_{\beta_0} = a - \lim_{\varepsilon \to 0} K_{\beta_\varepsilon}$.*

Proof. We follow the same argument as in the proof of Theorem 8. The truncation step is not needed. □

Remark 7.

i) It is also possible to assume that Ω is a strictly star-shaped domain with respect to another point a in Ω else the origin, i.e. $\Omega - a$ is a strictly star-shaped domain. Of course assuming this allows to keep the result of the above theorem.

ii) If the star-shapeness is not strict, then the argument of truncation (in the X_1-direction), to avoid the points of $\mathcal{M}$ defined above, may not work. In dimension two, this is illustrated in Figure 2 where the right domain is not strictly star-shaped. The assumption (4.16) cannot hold because meas $(M) > 0$.

5. PROBLEMS WITH CONVEX SETS DEFINED ON THE SECTIONS

In this section we suppose that Ω is cylindrical namely

$$\Omega = \omega_1 \times \omega_2,$$

where ω_1, ω_2 are bounded open subsets in $\mathbb{R}^p$ and $\mathbb{R}^{n-p}$ respectively.

5.1. **Problems setting.** For $\varepsilon > 0$, let $K_\varepsilon(X_1)$ be a family of closed convex sets of $W_0^{1,p}(\omega_2)$ depending on X_1. Then we consider two problems having constraints involving this family.

5.1.1. *Problem with perturbed operator.* We consider the set

$$G_\varepsilon := \left\{ v \in W_0^{1,p}(\Omega) \mid v(X_1,\cdot) \in K_\varepsilon(X_1),\ \text{for } a.e. X_1 \in \omega_1 \right\}$$

for which we have the following assertion.

Proposition 1. *The set G_ε is closed in $W_0^{1,p}(\Omega)$ and convex.*

Proof. It is easy to see that G_ε defined above is convex. Let then $v_n \in G_\varepsilon$ be a converging sequence in $W_0^{1,p}(\Omega)$. Denote v its limit. Applying the inverse Lebesgue theorem in $L^p(\omega_1)$ we deduce that -up to a subsequence-

$$\int_{\omega_2} |v_n(X_1,\cdot) - v(X_1,\cdot)|^p \, dX_2 \ \to 0 \ \text{ a.e. in } \omega_1$$

and thus $v_n(X_1,\cdot) \to v(X_1,\cdot)$ a.e. in ω_2 (up to a subsequence). This means that $v(X_1,\cdot) \in K_\varepsilon(X_1)$, for a.e. $X_1 \in \omega_1$, since $K_\varepsilon(X_1)$ is closed which ends the proof. □

By consequence, for $f \in L^{p'}(\Omega)$, there exists a unique u_ε in G_ε, solution to the following perturbed problem

$$\begin{cases} \displaystyle\int_\Omega |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon (v - u_\varepsilon) \, dx \geq \int_\Omega f(v - u_\varepsilon) \, dx, \qquad \forall v \in G_\varepsilon, \\ u_\varepsilon \in G_\varepsilon. \end{cases} \tag{5.1}$$

5.1.2. *Problem with unperturbed operator.* In order to see how the behaviour of u_ε, solution to (5.1), looks like as $\varepsilon \to 0$, we will consider two problems with unperturbed operators defined as natural limits of the perturbed $p-$Laplacian defined in (5.1). In fact since there is a reduction in the dimension we are, of course, led to consider a limit problem defined on ω_2 and for technical reasons a possible equivalent one defined on the whole Ω. For a.e. X_1 in ω_1, let us consider

$$\begin{cases} \displaystyle\int_{\omega_2} |\nabla_{X_2} w(X_1,\cdot)|^{p-2} \nabla_{X_2} w(X_1,\cdot) \cdot \nabla_{X_2}(v - w(X_1,\cdot)) \, dX_2 \\ \qquad\qquad \displaystyle\geq \int_{\omega_2} f(v - w(X_1,\cdot)) \, dX_2, \quad \forall v \in K_\varepsilon(X_1), \\ w(X_1,\cdot) \in K_\varepsilon(X_1). \end{cases} \tag{5.2}$$

The solution w of the above problem exists and is unique since the operator is strictly monotone and it can be considered as a function on the whole Ω. Then set

$$\mathcal{G}_\varepsilon := \{ v \in \mathcal{W}_0(\Omega) \mid v(X_1,\cdot) \in K_\varepsilon(X_1) \ \text{ for a.e. } X_1 \in \omega_1 \},$$

where the $\mathcal{W}_0(\Omega)$ is defined above and can be written here as

$$\mathcal{W}_0(\Omega) = L^p\left(\omega_1; W_0^{1,p}(w_2)\right).$$

Arguing as in Proposition 1, we can show that

$$\text{the set } \mathcal{G}_\varepsilon \text{ is closed in } \mathcal{W}_0(\Omega) \text{ and convex.} \tag{5.3}$$

Thus the second problem, having $\mathcal{G}_\varepsilon$ as constraints sets,

$$\begin{cases} \displaystyle\int_\Omega |\nabla_{X_2} w_\varepsilon|^{p-2} \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2}(v - w_\varepsilon)\, dx \ \geq \int_\Omega f(v - w_\varepsilon)\, dx, \quad \forall v \in \mathcal{G}_\varepsilon, \\ w_\varepsilon \in \mathcal{G}_\varepsilon. \end{cases} \tag{5.4}$$

has a unique solution w_ε.

Remark 8.

i) It is clear that $G_\varepsilon \subset \mathcal{G}_\varepsilon$.

ii) The regularity of w, solution to Problem (5.2), in X_1-direction depends on the regularity of f and on the convex sets G_ε.

In the sequel we need the following convexity type lemma.

Lemma 7. *Let φ be a smooth function in $C^\infty(\bar{\omega}_1)$ such that*

$$0 \leq \varphi(X_1) \leq 1, \ \forall X_1 \in \omega_1.$$

If $v_1, v_2 \in \mathcal{G}_\varepsilon$ then $\varphi v_1 + (1-\varphi) v_2 \in \mathcal{G}_\varepsilon$.

Proof. If $v_1, v_2 \in \mathcal{G}_\varepsilon$ then for a.e. X_1, $v_1(X_1,\cdot), v_2(X_1,\cdot) \in K_\varepsilon(X_1)$ and

$$\varphi(X_1) v_1(X_1,\cdot) + (1 - \varphi(X_1)) v_2(X_1,\cdot) \in K_\varepsilon(X_1).$$

This completes the proof since $\varphi v_1 + (1-\varphi) v_2 \in \mathcal{W}_0(\Omega)$. □

Proposition 2. *Assume, for a.e. $X_1 \in \omega_1$, that*

$$\textit{the set of restrictions of functions from } \mathcal{G}_\varepsilon \textit{ on } \Omega_{X_1} \textit{ is dense in } K_\varepsilon(X_1), \tag{5.5}$$

then Problem (5.2) and Problem (5.4) have the same unique solution.

Proof. Thanks to the uniqueness of the solution for the two problems, it is enough to show that w_ε, solution to (5.4), is also a solution to Problem (5.2). Taking as a test function in (5.4), thanks to Lemma 7 and for $v \in \mathcal{G}_\varepsilon$, $w_\varepsilon + \frac{\varphi}{\|\varphi\|_\infty}(v - w_\varepsilon)$ where $\varphi \gneqq 0$, $\varphi \in \mathcal{D}(\omega_1)$, we get

$$\int_{\omega_1} \varphi \int_{\omega_2} |\nabla_{X_2} w_\varepsilon|^{p-2} \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2}(v - w_\varepsilon)\, dX_2\, dX_1 \geq \int_{\omega_1} \varphi \int_{\omega_2} f(v - w_\varepsilon)\, dX_2\, dX_1.$$

Since $\varphi \geq 0$ is arbitrary in $\mathcal{D}(\omega_1)$ we can rewrite the above inequality, for a.e. X_1, as

$$\int_{\omega_2} |\nabla_{X_2} w_\varepsilon|^{p-2} \nabla_{X_2} w_\varepsilon(X_1,\cdot) \cdot \nabla_{X_2}(v(X_1,\cdot) - w_\varepsilon(X_1,\cdot))\, dX_2$$
$$\geq \int_{\omega_2} f(X_1,\cdot)(v(X_1,\cdot) - w_\varepsilon(X_1,\cdot))\, dX_2, \quad \forall v \in \mathcal{G}_\varepsilon.$$

By the density assumption (5.5), this means that $w_\varepsilon(X_1,\cdot)$ is also a solution to (5.2) for a.e. X_1. This ends the proof. □

Remark 9. *Here is some cases where the assumption* (5.5) *holds.*

i) The set of restrictions of functions from $\mathcal{G}_\varepsilon$ *on* Ω_{X_1} *is equal to* K_ε *if the convex set* $K_\varepsilon(X_1) = K_\varepsilon$ *is independent of* X_1. *To see this, we can consider* K_ε *as a subset of* $\mathcal{G}_\varepsilon$, *since for all* $v_2 = v_2(X_2) \in K_\varepsilon$ *we have* $v(x) = v_2(X_2) \in \mathcal{G}_\varepsilon$.

ii) We get the same conclusion if the family of convex sets $K_\varepsilon(X_1)$ *is defined by an obstacle, i.e.* $K_\varepsilon(X_1) = K_{\psi_\varepsilon(X_1,\cdot)}$ *where* $\psi_\varepsilon \in \mathcal{W}(\Omega)$, *such that* $\psi_\varepsilon^+ \in \mathcal{W}_0(\Omega)$. *Indeed, for a fixed* $X_1 \in \omega_1$ *and any* $v_2 \in K_\varepsilon(X_1)$, *i.e.* $v_2 \geq \psi_\varepsilon(X_1,\cdot)$, *we consider the function* $v \in \mathcal{W}_0(\Omega)$ *defined as*

$$v(X_1,X_2) = \max\{\psi_\varepsilon(X_1,X_2), v_2(X_2)\}.$$

Clearly $v(X_1,\cdot) = v_2$ *and* $v \in \mathcal{G}_\varepsilon$ *since* $v(X_1,\cdot) \geq \psi_\varepsilon(X_1,\cdot)$ *for a.e.* $X_1 \in \omega_1$.

5.2. **Rate of convergence.** Let $\omega_1' \subset\subset \omega_1$ and denote

$$\Omega' = \omega_1' \times \omega_2.$$

The following theorem ties the behaviour of the solution of problem (5.1) to the behaviour of the solution of problem (5.4).

Theorem 10. *Assume that there exists a sequence* $v_\varepsilon \in G_\varepsilon$ *for all* $\varepsilon > 0$, *satisfying* (2.6). *Then*

$$\nabla^\varepsilon u_\varepsilon \text{ and } \nabla_{X_2} w_\varepsilon \text{ are bounded in } L^p(\Omega) \tag{5.6}$$

where u_ε (*resp.* w_ε) *is the solution of problem* (5.1) (*resp.* (5.4)).

Assume in addition that $w_\varepsilon \in W^{1,p}(\Omega)$, *then :*

- *If* $p \geq 2$

$$|u_\varepsilon - w_\varepsilon|_{L^p(\Omega')}, \quad |\nabla^\varepsilon(u_\varepsilon - w_\varepsilon)|_{L^p(\Omega')}$$
$$\leq C\varepsilon^{\frac{1}{(p-1)}} \left\{ \left(|\varepsilon \nabla_{X_1} w_\varepsilon|_{L^p(\Omega)} + 1 \right)^{p-2} |\nabla_{X_1} w_\varepsilon|_{L^p(\Omega)} + 1 \right\}^{\frac{1}{(p-1)}}.$$

- *If* $1 < p < 2$

$$|u_\varepsilon - w_\varepsilon|_{L^p(\Omega')}, \quad |\nabla^\varepsilon(u_\varepsilon - w_\varepsilon)|_{L^p(\Omega')}$$
$$\leq C\varepsilon^{p-1} \left(|\nabla_{X_1} w_\varepsilon|^{p-1}_{L^p(\Omega)} + \varepsilon^{2-p} \right) \left(|\varepsilon \nabla_{X_1} w_\varepsilon|^p_{L^p(\Omega)} + 1 \right)^{\frac{(2-p)}{p}}.$$

- *In particular if* $p = 2$ *then*

$$|u_\varepsilon - w_\varepsilon|_{L^2(\Omega')}, \quad |\nabla^\varepsilon(u_\varepsilon - w_\varepsilon)|_{L^2(\Omega')} \leq C\varepsilon \left(|\nabla_{X_1} w_\varepsilon|_{L^2(\Omega)} + 1 \right).$$

Proof. First, the boundedness of $\nabla^\varepsilon u_\varepsilon$ follows directly from Theorem 1, and the boundedness of $\nabla_{X_2} w_\varepsilon$ follows also by the same argument used to show Theorem 1.

In the sequel of the proof, we shall use the following smooth cut-off functions depending on X_1 and satisfying

$$0 \leq \rho \leq 1, \quad \operatorname{supp}(\rho) \subset \omega_1, \rho = 1 \text{ on } \omega_1'.$$

Let $\alpha = \max\{2, p\}$. Then we test Problems (5.1) and (5.4) by

$$v = u_\varepsilon + \rho^\alpha (w_\varepsilon - u_\varepsilon) \in G_\varepsilon, \qquad v = w_\varepsilon + \rho^\alpha (u_\varepsilon - w_\varepsilon) \in \mathcal{G}_\varepsilon$$

respectively we get

$$\int_\Omega |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon (\rho^\alpha (u_\varepsilon - w_\varepsilon))\, dx \le \int_\Omega f \rho^\alpha (u_\varepsilon - w_\varepsilon)\, dx,$$
$$\int_\Omega |\nabla_{X_2} w_\varepsilon|^{p-2} \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2} (\rho^\alpha (u_\varepsilon - w_\varepsilon))\, dx \ge \int_\Omega f \rho^\alpha (u_\varepsilon - w_\varepsilon)\, dx.$$

Combining the above inequalities yields

$$\int_\Omega |\nabla^\varepsilon u_\varepsilon|^{p-2} \begin{pmatrix} \varepsilon \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2} u_\varepsilon \end{pmatrix} \cdot \begin{pmatrix} \varepsilon \nabla_{X_1} (\rho^\alpha (u_\varepsilon - w_\varepsilon)) \\ \rho^\alpha \nabla_{X_2} (u_\varepsilon - w_\varepsilon) \end{pmatrix} dx$$
$$\le \int_\Omega \rho^\alpha |\nabla_{X_2} w_\varepsilon|^{p-2} \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2} (u_\varepsilon - w_\varepsilon)\, dx,$$

then

$$\int_\Omega \rho^\alpha |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)\, dx \le \int_\Omega \rho^\alpha |\nabla_{X_2} w_\varepsilon|^{p-2} \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2} (u_\varepsilon - w_\varepsilon)\, dx$$
$$- \alpha \varepsilon^2 \int_\Omega \rho^{\alpha-1} (u_\varepsilon - w_\varepsilon) |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} \rho dx.$$

Setting

$$I_\varepsilon := \int_\Omega \rho^\alpha \left\{ |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla^\varepsilon u_\varepsilon - |\nabla^\varepsilon w_\varepsilon|^{p-2} \nabla^\varepsilon w_\varepsilon \right\} \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)\, dx,$$

then the above inequality can be written as

$$\begin{aligned} I_\varepsilon \le & \int_\Omega \rho^\alpha \left\{ \left|\nabla^0 w_\varepsilon\right|^{p-2} \nabla^0 w_\varepsilon - |\nabla^\varepsilon w_\varepsilon|^{p-2} \nabla^\varepsilon w_\varepsilon \right\} \cdot \nabla^0 (u_\varepsilon - w_\varepsilon)\, dx \\ & - \varepsilon^2 \int_\Omega \rho^\alpha |\nabla^\varepsilon w_\varepsilon|^{p-2} \nabla_{X_1} w_\varepsilon \cdot \nabla_{X_1} (u_\varepsilon - w_\varepsilon)\, dx \\ & - \alpha \varepsilon^2 \int_\Omega \rho^{\alpha-1} (u_\varepsilon - w_\varepsilon) |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} \rho dx. \end{aligned} \tag{5.7}$$

Applying inequality (2.34) we get

$$\begin{aligned} I_\varepsilon \le & C \int_\Omega \rho^\alpha \{|\nabla^\varepsilon w_\varepsilon| + |\nabla_{X_2} w_\varepsilon|\}^{p-2} \left|\nabla^0 w_\varepsilon - \nabla^\varepsilon w_\varepsilon\right| \left|\nabla^0 (u_\varepsilon - w_\varepsilon)\right| dx \\ & + \varepsilon^2 \int_\Omega \rho^\alpha |\nabla^\varepsilon w_\varepsilon|^{p-2} |\nabla_{X_1} w_\varepsilon| \, |\nabla_{X_1} (u_\varepsilon - w_\varepsilon)|\, dx \\ & + \alpha \varepsilon^2 \int_\Omega \rho^{\alpha-1} |\nabla_{X_1} \rho| \, |u_\varepsilon - w_\varepsilon| \, |\nabla^\varepsilon u_\varepsilon|^{p-2} |\nabla_{X_1} u_\varepsilon|\, dx, \end{aligned} \tag{5.8}$$

and by consequence we derive

$$\begin{aligned} I_\varepsilon \le \; & C\varepsilon \int_\Omega |\nabla^\varepsilon w_\varepsilon|^{p-2} |\nabla_{X_1} w_\varepsilon| \; |\rho \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|\, dx \\ & + C\varepsilon^2 \int_\Omega |\nabla^\varepsilon u_\varepsilon|^{p-2} |\nabla_{X_1} u_\varepsilon| \, |\rho (u_\varepsilon - w_\varepsilon)|\, dx. \end{aligned} \tag{5.9}$$

Then we distinguish two cases according to the values of p.

• If $p \geq 2$, then thanks to the Hölder inequality (where $\frac{p-2}{p}+\frac{1}{p}+\frac{1}{p}=1$ and $\frac{p-1}{p}+\frac{1}{p}=1$), it follows that

$$I_\varepsilon \leq C\varepsilon \left|\nabla^\varepsilon w_\varepsilon\right|_{L^p(\Omega)}^{p-2} \left|\nabla_{X_1} w_\varepsilon\right|_{L^p(\Omega)} \left|\rho\nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)} + C\varepsilon \left|\nabla^\varepsilon u_\varepsilon\right|_{L^p(\Omega)}^{p-1} \left|\rho\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)}.$$

The L^p-Poincaré inequality on ω_2 in the last term yields

$$\left|\rho\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)} \leq C\left|\rho\nabla^0\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)} \leq C\left|\rho\nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)}.$$

Thus

$$I_\varepsilon \leq C\varepsilon \left\{\left|\nabla^\varepsilon w_\varepsilon\right|_{L^p(\Omega)}^{p-2} \left|\nabla_{X_1} w_\varepsilon\right|_{L^p(\Omega)} + \left|\nabla^\varepsilon u_\varepsilon\right|_{L^p(\Omega)}^{p-1}\right\} \left|\rho\nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)}. \tag{5.10}$$

Due to the uniform monotonicity of the $p-Laplacian$ (2.35), we derive since $\alpha = p$

$$\begin{aligned} C_3\left|\rho\nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)}^p &\leq I_\varepsilon \\ &\leq C\varepsilon \left\{\left|\nabla^\varepsilon w_\varepsilon\right|_{L^p(\Omega)}^{p-2} \left|\nabla_{X_1} w_\varepsilon\right|_{L^p(\Omega)} + \left|\nabla^\varepsilon u_\varepsilon\right|_{L^p(\Omega)}^{p-1}\right\} \left|\rho\nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)}, \end{aligned}$$

i.e.

$$\left|\rho\nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)} \leq C\varepsilon^{1/(p-1)} \left\{\left|\nabla^\varepsilon w_\varepsilon\right|_{L^p(\Omega)}^{p-2} \left|\nabla_{X_1} w_\varepsilon\right|_{L^p(\Omega)} + \left|\nabla^\varepsilon u_\varepsilon\right|_{L^p(\Omega)}^{p-1}\right\}^{1/(p-1)} \tag{5.11}$$

and taking into account (5.6), we end up with

$$\left|\rho\nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)} \leq C\varepsilon^{1/(p-1)} \left\{\left(\left|\varepsilon\nabla_{X_1} w_\varepsilon\right|_{L^p(\Omega)} + 1\right)^{p-2} \left|\nabla_{X_1} w_\varepsilon\right|_{L^p(\Omega)} + 1\right\}^{1/(p-1)}.$$

• If $1 < p < 2$, then the inequalities (5.9) and $\left|\nabla^\varepsilon \xi\right|^{p-2} \leq \left|\varepsilon\nabla_{X_1}\xi\right|^{p-2}$imply

$$I_\varepsilon \leq C\varepsilon^{p-1} \int_\Omega \left|\nabla_{X_1} w_\varepsilon\right|^{p-1} \left|\rho\nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right| dx + C\varepsilon \int_\Omega \left|\varepsilon\nabla_{X_1} u_\varepsilon\right|^{p-1} \left|\rho\left(u_\varepsilon - w_\varepsilon\right)\right| dx.$$

Thanks to the Hölder inequality (where $\frac{p-1}{p}+\frac{1}{p}=1$), it follows that

$$I_\varepsilon \leq C\varepsilon^{p-1} \left|\nabla_{X_1} w_\varepsilon\right|_{L^p(\Omega)}^{p-1} \left|\rho\nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)} + C\varepsilon \left|\varepsilon\nabla_{X_1} u_\varepsilon\right|_{L^p(\Omega)}^{p-1} \left|\rho\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)}$$

and applying Poincaré's inequality on ω_2 we get

$$I_\varepsilon \leq C\varepsilon^{p-1} \left\{\left|\nabla_{X_1} w_\varepsilon\right|_{L^p(\Omega)}^{p-1} + \varepsilon^{2-p} \left|\varepsilon\nabla_{X_1} u_\varepsilon\right|_{L^p(\Omega)}^{p-1}\right\} \left|\rho\nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)}. \tag{5.12}$$

Since $\alpha = 2$, we rewrite (2.33) for $\xi = \nabla^\varepsilon u_\varepsilon$ and $\eta = \nabla^\varepsilon w_\varepsilon$ as

$$\begin{aligned} \left|\rho\nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right|^p \leq C &\left\{\rho^2\left(\left|\nabla^\varepsilon u_\varepsilon\right|^{p-2}\nabla^\varepsilon u_\varepsilon - \left|\nabla^\varepsilon w_\varepsilon\right|^{p-2}\nabla^\varepsilon w_\varepsilon\right) \cdot \nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right\}^{\frac{p}{2}} \\ &\times \left\{\left|\nabla^\varepsilon u_\varepsilon\right| + \left|\nabla^\varepsilon w_\varepsilon\right|\right\}^{\frac{p(2-p)}{2}}. \end{aligned}$$

Integrating on Ω and applying Hölder's inequality (where $\frac{p}{2}+\frac{2-p}{2}=1$) in the right hand side, we get

$$\left|\rho\nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)}^p \leq C\left(I_\varepsilon\right)^{\frac{p}{2}} \times \left\{\int_\Omega \left(\left|\nabla^\varepsilon u_\varepsilon\right| + \left|\nabla^\varepsilon w_\varepsilon\right|\right)^p dx\right\}^{\frac{(2-p)}{2}}.$$

Taking into account (5.12), it follows that

$$|\rho\nabla^{\varepsilon}(u_{\varepsilon}-w_{\varepsilon})|_{L^p(\Omega)}^{\frac{p}{2}} \leq C\varepsilon^{\frac{p}{2}(p-1)}\left\{|\nabla_{X_1}w_{\varepsilon}|_{L^p(\Omega)}^{p-1}+\varepsilon^{2-p}|\varepsilon\nabla_{X_1}u_{\varepsilon}|_{L^p(\Omega)}^{p-1}\right\}^{\frac{p}{2}} \times\left\{|\nabla^{\varepsilon}u_{\varepsilon}|_{L^p(\Omega)}^{p}+|\nabla^{\varepsilon}w_{\varepsilon}|_{L^p(\Omega)}^{p}\right\}^{\frac{(2-p)}{2}},$$

i.e.,

$$|\rho\nabla^{\varepsilon}(u_{\varepsilon}-w_{\varepsilon})|_{L^p(\Omega)} \leq C\varepsilon^{p-1}\left\{|\nabla_{X_1}w_{\varepsilon}|_{L^p(\Omega)}^{p-1}+\varepsilon^{2-p}|\varepsilon\nabla_{X_1}u_{\varepsilon}|_{L^p(\Omega)}^{p-1}\right\} \times\left\{|\nabla^{\varepsilon}u_{\varepsilon}|_{L^p(\Omega)}^{p}+|\nabla^{\varepsilon}w_{\varepsilon}|_{L^p(\Omega)}^{p}\right\}^{\frac{(2-p)}{p}}. \tag{5.13}$$

Taking into account (5.6) and $|\nabla^{\varepsilon}w_{\varepsilon}|_{L^p(\Omega)}^{p} \leq 2^{p-1}\left\{|\varepsilon\nabla_{X_1}w_{\varepsilon}|_{L^p(\Omega)}^{p}+|\nabla_{X_2}w_{\varepsilon}|_{L^p(\Omega)}^{p}\right\}$, we end up with

$$|\rho\nabla^{\varepsilon}(u_{\varepsilon}-w_{\varepsilon})|_{L^p(\Omega)} \leq C\varepsilon^{p-1}\left\{|\nabla_{X_1}w_{\varepsilon}|_{L^p(\Omega)}^{p-1}+\varepsilon^{2-p}\right\}\times\left\{|\varepsilon\nabla_{X_1}w_{\varepsilon}|_{L^p(\Omega)}^{p}+1\right\}^{\frac{(2-p)}{p}}.$$

This ends the proof of the theorem. □

The behaviour of u_{ε} depends essentially on the behaviour of $\nabla_{X_1}w_{\varepsilon}$. This is more emphasized in the following corollary.

Corollary 11. *Under the assumption of Theorem 10 we have*

Assuming that $\varepsilon\nabla_{X_1}w_{\varepsilon}$ *is bounded in* $L^p(\Omega)$, $1<p<+\infty$, *then*

$$|u_{\varepsilon}-w_{\varepsilon}|_{L^p(\Omega')},\ \ |\nabla^{\varepsilon}(u_{\varepsilon}-w_{\varepsilon})|_{L^p(\Omega')}\ \ \text{are bounded.} \tag{5.14}$$

Assuming that $\nabla_{X_1}w_{\varepsilon}$ *is bounded in* $L^p(\Omega)$ *then :*

- *If* $p\geq 2$

$$|u_{\varepsilon}-w_{\varepsilon}|_{L^p(\Omega')},\ \ |\nabla^{\varepsilon}(u_{\varepsilon}-w_{\varepsilon})|_{L^p(\Omega')}\leq C\varepsilon^{\frac{1}{(p-1)}}.$$

- *If* $1<p<2$

$$|u_{\varepsilon}-w_{\varepsilon}|_{L^p(\Omega')},\ \ |\nabla^{\varepsilon}(u_{\varepsilon}-w_{\varepsilon})|_{L^p(\Omega')}\leq C\varepsilon^{p-1}.$$

- *In particular if* $p=2$ *then*

$$|\nabla_{X_1}(u_{\varepsilon}-w_{\varepsilon})|_{L^2(\Omega')}\leq C \quad \text{and} \quad (u_{\varepsilon}-w_{\varepsilon})\rightharpoonup 0 \quad \text{in } H^1(\Omega').$$

Proof. We need only to show the last weak convergence. If $p=2$ then $\nabla(u_{\varepsilon}-w_{\varepsilon})$ is bounded in $L^2(\Omega')$ and we can extract a weakly converging sequence in $L^2(\Omega')$ that should have the same distributional limit, which is 0 since $(u_{\varepsilon}-w_{\varepsilon})\to 0$ in $L^2(\Omega')$. By the uniqueness of the limit, the whole sequence $(u_{\varepsilon}-w_{\varepsilon})\rightharpoonup 0$ in $H^1(\Omega')$. This ends the proof. □

In accordance with Remark 1–ii, we can state the following generalization of Theorem 10.

Theorem 12. *Assume that there exists a sequence* $v_\varepsilon \in G_\varepsilon$ *for all* $\varepsilon > 0$, *such that* $|\nabla^\varepsilon v_\varepsilon|_{L^p(\Omega)} = O(\varepsilon^{-\gamma})$, *for some* $\gamma \geq 0$, *then*

$$|\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)} = O\left(\varepsilon^{-\gamma}\right) \text{ and } |\nabla_{X_2} w_\varepsilon|_{L^p(\Omega)} = O\left(\varepsilon^{-\gamma}\right). \tag{5.15}$$

Assume in addition that $w_\varepsilon \in W^{1,p}(\Omega)$, *then :*

- *If* $p \geq 2$,

$$|\nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega')} \leq C\left\{\left(|\varepsilon\nabla_{X_1} w_\varepsilon|_{L^p(\Omega)} + \varepsilon^{-\gamma}\right)^{p-2} |\varepsilon\nabla_{X_1} w_\varepsilon|_{L^p(\Omega)} + \varepsilon^{1-\gamma(p-1)}\right\}^{\frac{1}{p-1}}. \tag{5.16}$$

- *If* $1 < p < 2$,

$$|\nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega')} \leq C\left(|\varepsilon\nabla_{X_1} w_\varepsilon|_{L^p(\Omega)}^{p-1} + \varepsilon^{1-\gamma(p-1)}\right)\left(|\varepsilon\nabla_{X_1} w_\varepsilon|_{L^p(\Omega)}^{p} + \varepsilon^{-\gamma p}\right)^{\frac{(2-p)}{p}}. \tag{5.17}$$

- *In particular if* $p = 2$ *then*

$$|\nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^2(\Omega')} \leq C\left(|\varepsilon\nabla_{X_1} w_\varepsilon|_{L^2(\Omega)} + \varepsilon^{1-\gamma}\right). \tag{5.18}$$

Proof. The estimation of $\nabla^\varepsilon u_\varepsilon$ is already stated in Remark 1–ii and the estimation of $\nabla_{X_2} w_\varepsilon$ can be obtained by an analogous argument. The inequalities (5.16) and (5.17) are direct consequences of (5.11) and (5.13) respectively. □

Example. The above theorem extends earlier results, obtained in [10], to nonlinear problems and variational inequalities. Let us give another example illustrating the above results. Let $\phi_\varepsilon \in W_0^{1,p}(\Omega)$, $\varphi_\varepsilon \in W_0^{1,p}(\omega_1)$ be smooth functions such that

$$\phi_\varepsilon^+ \in W_0^{1,p}(\Omega), \quad \varphi_\varepsilon > 0 \text{ on } \Omega \quad \text{and} \quad \psi_\varepsilon = \frac{\phi_\varepsilon}{\varphi_\varepsilon} \in W_0^{1,p}(\Omega).$$

Then consider the following non empty convex set related to φ_ε

$$\mathcal{G}_\varepsilon = G_\varepsilon = \left\{\varphi_\varepsilon\varphi \in W_0^{1,p}(\Omega) \mid \varphi \in W_0^{1,p}(\omega_2),\ \varphi_\varepsilon(X_1)\varphi(X_2) \geq \phi_\varepsilon(x) \text{ a.e. } x \in \Omega\right\}.$$

For $X_1 \in \omega_1$, we define the convex set $K_\varepsilon(X_1)$ as

$$K_\varepsilon(X_1) = K_{\psi_\varepsilon(X_1,\cdot)} = \left\{\varphi_\varepsilon(X_1)\varphi \in W_0^{1,p}(\omega_2) \mid \varphi(X_2) \geq \psi_\varepsilon(X_1, X_2) \text{ a.e. } X_2 \in \omega_2\right\}.$$

It is clear that $K_\varepsilon(X_1)$ is not empty since $\psi_\varepsilon^+(X_1,\cdot) = \frac{\phi_\varepsilon^+(X_1,\cdot)}{\varphi_\varepsilon(X_1)} \in W_0^{1,p}(\omega_1)$.

Thanks to Remark 9-*ii*, the hypothesis (5.5) holds and thus Problems (5.2) and (5.4) have the same solution in this case. In particular, the convergences of Theorems 10, 12 and Corollary 11 depend on the behaviour of $|\nabla\varphi_\varepsilon|_{L^p(\omega_1)}$, as $\varepsilon \to 0$.

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