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Study of an impulsive second-order mixed boundary value problem with a parameter using variational methods.

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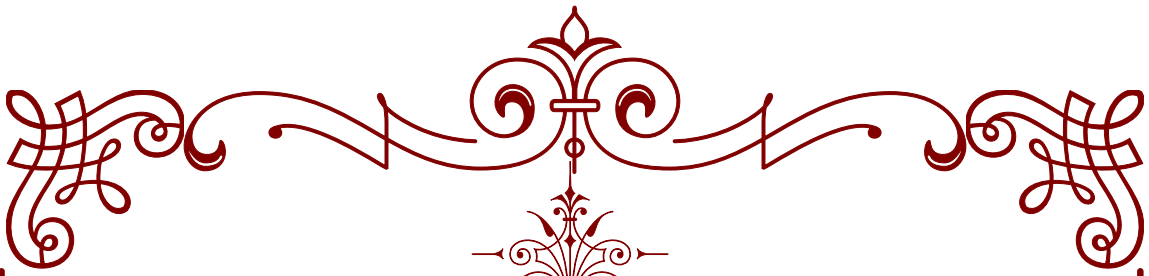
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We thank God

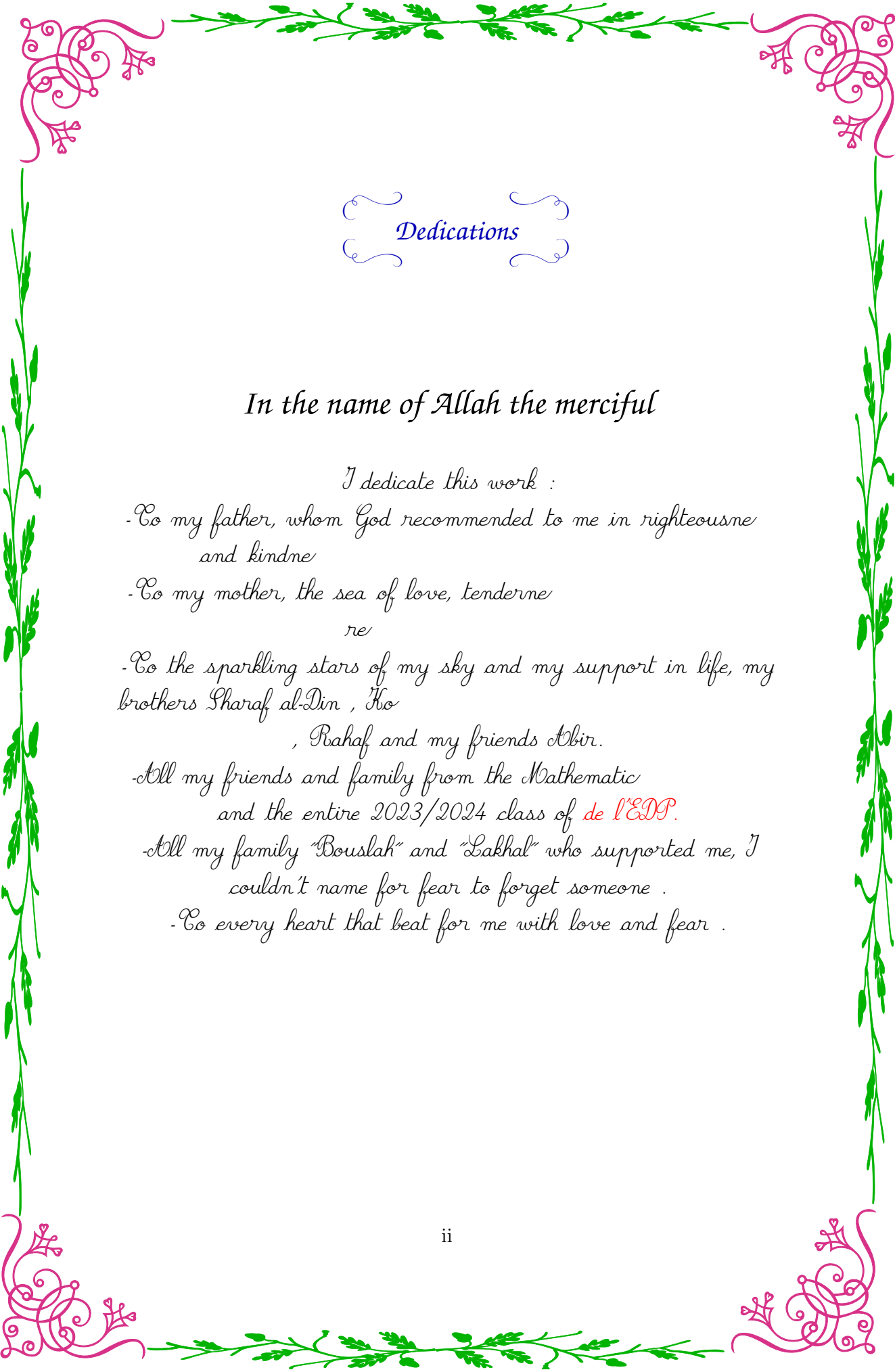
Almighty for giving

us health. and the desire to start and finish this dissertation. First of all, this work would not be as rich and would not have could have had the day without the help and supervision of MR. Dahmane BOUAFIA, we thank him for the quality of his supervision exceptional, for his patience, his rigor and his availability during our preparation of this dissertation. I thank my very dear parents who have always been there for me and all my family and friends. Their unconditional support and their encouragement was a great help. I thank the jury members, each in their own name, for having agreed to review my work. Our thanks also go to all our teachers. the faculty of mathematics and computer science, the department of mathematics for their generosity and great patience which they were able to demonstrate despite their academic responsibilities and professionals.



❖ Feryal





Dedications

In the name of Allah the merciful

I dedicate this work :

- To my father, whom God recommended to me in righteousness and kindness*
- To my mother, the sea of love, tenderness*
- To the sparkling stars of my sky and my support in life, my brothers Sharaf al-Din , Ho , Rahaf and my friends Abir.*
- All my friends and family from the Mathematic and the entire 2023/2024 class of **de l'EDP**.*
- All my family "Bousslah" and "Lakhal" who supported me, I couldn't name for fear to forget someone .*
- To every heart that beat for me with love and fear .*

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Notation

We introduce the necessary notations and definitions that are used later on.

$$L^p(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable} : |u|^p \in L^1(\Omega) \right\} \text{ with } 1 \leq p < \infty.$$

$$\|u\|_{L^p} = \left(\int_{\Omega} |u(t)|^p dt \right)^{\frac{1}{p}}.$$

$$L^\infty(\Omega) = \{ u : \Omega \rightarrow \mathbb{R} \text{ measurable and } \exists C : |u(t)| \leq C \text{ a.e on } \Omega \}.$$

$$\|u\|_{L^\infty} = \inf \{ C : |u(t)| \leq C \text{ a.e on } \Omega \}.$$

$$C(\Omega) : \text{space of continuous functions on } \Omega.$$

$$C_0^\infty(0, +\infty) \text{ the space of infinitely differentiable functions with compact support in } (0, +\infty).$$

$$H : \text{Hilbert space.}$$

$$\langle \cdot, \cdot \rangle : \text{duality bracket.}$$

$$(\cdot, \cdot) : \text{inner product.}$$

$$H^m(0, 1) = \{ u \in L^2(0, 1) : |u^{(i)}| \in L^2(0, 1) \quad \forall i \leq m \}.$$

$$H_0^m(0, 1) : \text{the completion of } \mathcal{D}(0, 1) \text{ in } H^m(0, 1).$$

$$C[0, T] := \left\{ u : [0, T] \rightarrow \mathbb{R} \text{ is continuous} \right\}.$$

$$\|u\|_\infty = \sup_{t \in [0, T]} |u(t)|.$$

$$X^* : \text{dual topology.}$$

$$D(A) : \text{domain of definition of an unbounded operator } A.$$

$$R(A) : \text{image of } A \text{ which is also noted by } Im A$$

$$\mathcal{L}(X, Y) : \text{set of continuous linear applications.}$$

$$X \hookrightarrow Y : \text{we write } X \hookrightarrow Y \text{ to mean that } X \text{ is included in } Y \text{ and} \\ \text{the canonical injection of } X \text{ into } Y \text{ is continuous.}$$

$$X \hookrightarrow\hookrightarrow Y : \text{to imply that } X \text{ is included in } Y \text{ and that the canonical injection of } X \\ \text{into } Y \text{ is compact.}$$

$$u_n \rightharpoonup u_0 : u_n \text{ converge weakly to } u_0.$$

$$i.e. \quad \text{that is to say.}$$

$$a.e. \quad \text{almost everywhere.}$$

dF	derivative in the sense of Frechet which is also noted by F' .
d_GF	derivative in the sense of Gateaux which sometimes is also noted by F' .
$\overline{B_\rho(0)}$	the closed ball with center 0 and radius ρ .
$(P.S)$	Palais-Smale condition.
$diam F$	Diameter of a set F .
$l.s.c$	lower Semi continuous .
$u.s.c$	upper Semi continuous .
$w.l.s.c$	Weakly lower semi continuous .
$w.l.s.c$	Weakly upper Semi continuous .
$S^{K-1} = S(O, a)$	sphere of space R^K

Introduction

Variational methods are commonly used to solve mixed boundary value problems for impulsive differential equations with a parameter . These methods involve optimizing functionals to find solutions that satisfy both the differential equation and the boundary conditions. By applying variational techniques, one can derive conditions that lead to the oscillation of solutions in second-order impulsive differential systems of mixed type. The study of these problems involves understanding the behavior of solutions subjected to sudden changes and is crucial in various fields like mechanical systems, biology, population dynamics, and control theory. The use of variational methods allows for a deeper analysis of the oscillatory behavior of solutions in impulsive differential systems, providing insights into complex real-world phenomena. This memory is on the screen of a device that is available to others Jun Wang and Weigao Ge [1]. The work consists of the existence and multiplicity of solutions for an impulsive second-order to mixed boundary problem on the closed bounded interval $[0, T]$. The proof of the main result is based on critical point theory, Col's lemma and the saddle point lemma . This dissertation includes three chapters :

The first chapter is preliminaries which contains some basic tools which we frequently use later . We have divided the chapter by sections , the first section contains some notions about Banach operators such as Continuity of operators , Lower semi continuous functions , Differentiability in Banach spaces and Lebesgue and Sobolev spaces . We have presented in the second section an overview of the Critical point theory such as Mountain Pass Theorem . In section three a reminder about the Fourier series such as Fourier series of 2π periodic functions and Fourier series of an arbitrary period function.

Then, in the second chapter, we discuss the existence of a solution for a problem to the fol-

lowing impulsive second order limits:

$$\begin{cases} -u''(t) = \lambda u(t) + f(t, u(t)), & t \neq t_i, t \in [0, T], \\ -\Delta u'(t_i) = I_i(u(t_i)), & i = 1, 2, \dots, l, \\ u'(0) = 0, u(T) = 0, \end{cases} \quad (1)$$

where $\lambda \in \mathbb{R}$, $0 = t_0 < t_1 < \dots < t_l < t_{l+1} = T$, $\Delta u'(t_i) = u'(t_i^+) - u'(t_i^-)$, (where $u'(t_i^+)$ (resp. $u'(t_i^-)$)) denotes the right limit (resp. left limit) of $u'(t)$ at $t = t_i$, $u'(t_i^-) = u'(t_i)$, $I_i \in C(\mathbb{R}, \mathbb{R})$, $i = 1, 2, \dots, l$, $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$.

Finally, in the last chapter we studied the multiplicity of solutions for the same problem which we studied in the previous chapter.

To obtain the results, we used the following conditions and assumptions:

(H1) $f(u, t) = o(|u|)$, $I_i(u) = o(|u|)$ as $|u| \rightarrow 0$, $i = 1, 2, \dots, l$

(H2) There exist constants $\mu > 2$ and $r \geq 0$ such that for $|\epsilon| \geq r$,

$$0 < \mu F(t, u) \leq u f(t, u), \quad 0 < \mu \int_0^u I_i(s) ds \leq u I_i(u), \quad i = 1, 2, \dots, l$$

(H3) $f(t, u)$, $I_i(u)$ are odd in u .

(H4) There exist $x_1 > 0$, $r > \lambda_k$, λ_k is the k -th eigenvalue of 2.4 such that

$$r x_1 + f(t, x_1) \leq 0, t \in [0, T], I_i(x_1) \leq 0, \quad i = 1, 2, \dots, l$$

SOME PRELIMINARIES

In this chapter we give a series of notions that will be useful for the rest of this memory. We recall some definitions and properities used in the remainder of the these thesis , this to allow a more rapid assimilation of our work . These consist essentially of functional analysis results . We thus find some properities for weak convergence , continuity and differentiability of operators in Banch space , critical points theory and Ekeland's variational principls .

1.1 Operators on Banach spaces.

Let X and Y be two normed Banach spaces.

Definition 1.1. (Linear Bounded Operator)[22] Let A be a linear operator such that $D(A) = X$ et $R(A) \subset Y$. We say that A is bounded if it is bounded on the unit ball $\overline{B}(0, 1)$, i.e. If the set

$$\left\{ \|Ax\|_Y : x \in X, \|x\|_X \leq 1 \right\}$$

is bounded.

According to this definition, if A is bounded, there exists a constant $c > 0$ such that for all x where $\|x\|_X \leq 1$, we have the inequality

$$\|Ax\|_Y \leq c.$$

Theorem 1.1. [22] *A is bounded if and only if there exists a constant $c > 0$ such that*

$$\|Ax\|_Y \leq c\|x\|_X, \quad x \in X$$

for all $x \in X$

Definition 1.2. (Dual space)[22] The set of continuous linear functionals, defined on a normed vector space , constitutes a vector space . It is called dual of the space X and we denote X^* .

Equipped X^* with the norm

$$\|f\|_{X^*} = \sup_{x \in X, \|x\|_X \leq 1} |\langle f, x \rangle|, \quad \forall f \in X^*$$

X provided with this norm is a Banach space and we have the inequality

$$|\langle f, x \rangle| \leq \|f\|_{X^*} \|x\|_X, \quad \forall f \in X^*, \quad \forall x \in X.$$

Definition 1.3. (Weak convergence) It is said that a sequence $(x_n) \in X$ converges weakly to x , if

$$\forall f \in X^*, \langle f, x_n \rangle \longrightarrow \langle f, x \rangle$$

and we write $x_n \rightharpoonup x$.

Proposition 1.1. [23] Let X a Banach spaces and (x_n) be a sequence of X . We have

1. If $x_n \longrightarrow x$, so $x_n \rightharpoonup x$.
2. If $x_n \rightharpoonup x$, so (x_n) is bounded and $\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$.

Proposition 1.2. [23] When X is of finite dimension, a sequence (x_n) converges weakly if and only if it converges strongly.

Theorem 1.2. [23] A Banach space X is reflexive if and only if its closed ball is weakly compact.

Corollary 1.1. [23] If X is a reflexive Banach space then any bounded sequence $\{x_n\} \subset X$ with $\|x_n\| \leq M$, contains a subsequence which converges weakly to an element $x \in X$ satisfying $\|x\| \leq M$.

1.1.1 Continuity of operators

We will consider operators T from X into Y and we will give a definition concerning the properties of the continuity of T .

The easy notion is the following

Definition 1.4. The operator $T : X \longrightarrow Y$ is said to be continuous at x , if for any sequence $\{x_n\} \subset X$ which converges to x , $(T(x_n))_n$ Converges to $T(x)$.

T is called continuous on $\Omega \subset X$ if T is continuous at any point $x \in \Omega$.

Definition 1.5. An operator $T : X \longrightarrow Y$ is said to be compact if it is continuous and has the property, for any (x_n) bounded in X , the sequence $(T(x_n))$ admits a convergent subsequence.

Definition 1.6. Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ two Banach spaces. An operator $T : X \rightarrow Y$ is said to be completely continuous if it is continuous, and the image of any bounded set of X is relatively compact in Y .

1.1.2 Lower semi continuous functions

Definition 1.7. 1. The functional $f : X \rightarrow \mathbb{R}$ is said to be weakly lower semi-continuous (*w.l.s.c.*) at the point x_0 , if whenever sequence $\{x_n\} \subset X$ such that $x_n \rightharpoonup x_0$, it follows

$$f(x_0) \leq \liminf_{n \rightarrow \infty} f(x_n).$$

Example 1.1. Let the functional J be defined on a Hilbert space H as follows

$$J : u \rightarrow \|u\|^2,$$

then J is weakly lower semi-continuous (*w.l.s.c.*).

Indeed, let (u_n) , such that $u_n \rightharpoonup u$, show that

$$\|u\|^2 \leq \liminf_{n \rightarrow +\infty} \|u_n\|^2.$$

We obtained

$$\begin{aligned} \|u_n - u\|^2 &= (u_n - u, u_n - u) \\ &= \|u_n\|^2 - 2(u_n, u) + \|u\|^2 \geq 0, \end{aligned}$$

We know that $H^* = H$, and $u_n \rightharpoonup u$, so

$$\forall u \in H : (u_n, u) \rightarrow (u, u) = \|u\|^2.$$

It follows that

$$\|u\|^2 \leq \liminf_{n \rightarrow +\infty} \|u_n\|^2.$$

Remark 1.1. 1. It is easy to see that if f is continuous in x_0 , then f is (*u.s.c.*) and (*l.s.c.*) and conversely.

2. It is easy to see that if f is weakly continuous in x_0 , then f is (*w.u.s.c.*) and (*w.l.s.c.*) and conversely.

1.1.3 Differentiability in Banach spaces

Definitions and fundamental properties

1.1.3.1 Fréchet differentiability

Let X and Y be two Banach spaces, U an open of X .

Definition 1.8. Let $u \in U$, and $J : U \rightarrow Y$. We say that J is differentiable at the point u if there exists a continuous linear mapping $A \in \mathcal{L}(X, Y)$ such that

$$R(h) = J(u + h) - J(u) - A.h, \quad \text{for } h \in X, \text{ is small,}$$

then

$$\frac{R(h)}{\|h\|} \rightarrow 0 \quad \text{when } \|h\| \rightarrow 0 \quad \text{i.e.}$$

$$\forall \varepsilon > 0, \exists \delta > 0 \quad \text{such that if } \|h\|_X \leq \delta, \text{ then } \|R(h)\|_Y \leq \varepsilon \|h\|_X.$$

If such an application $A \in \mathcal{L}(X, Y)$ exists, it is necessarily unique. We note by

$$A = dJ(u) \text{ or } A' = J'.$$

It is called differential (in the sense of Fréchet) from J in u , or a linear application tangent to J in u . In the absence of differentiable additional precision will mean in the following differentiable in the sense of Fréchet.

Examples 1.1. 1. If $J(u) = c$. Then J Fréchet differentiable and $dJ(u) = 0, \forall u$.

2. If $A \in \mathcal{L}(X, Y)$ $A(u + h) - A(u) = A.h$ and so $dA(u) = A, \forall u \in X$.

3. If $X = H$ is a space of Hilbert and $J(u) = \|u\|^2 = \langle u, u \rangle$ then $dE(u)h = 2\langle u, h \rangle$.

Therefore we have all the classical properties.

1.1.3.2 Gâteaux differentiability

Let us begin by defining the notion of directional derivative

Definition 1.9. Let $J : U \rightarrow Y, u \in U, v \in X$ and $v \neq 0$.

We call the directional derivative in u of J in the direction v , denoted $\partial_v E(u)$, the limit where it exists

$$\partial_v J(u) = \lim_{t \rightarrow 0} \frac{J(u + tv) - J(u)}{t}.$$

The notion of directional derivative is thus an extension of the notion of partial derivative. If J is Fréchet differentiable, then for all $v \in X$ the directional derivative in the direction v is given by

$$\partial_v J(u) = dJ(u)v.$$

Indeed $J(u + tv) = J(u) + dJ(u)(tv) + R(tv)$.

Then

$$\frac{J(u + tv) - J(u)}{t} = dJ(u)(v) + \frac{R(tv)}{t}, \quad \frac{R(tv)}{t} \rightarrow 0, \text{ when } t \rightarrow 0$$

Definition 1.10. We say that $J : U \rightarrow Y$ is Gâteaux differentiable in u (G-differentiable in u), if there exists a continuous linear mapping A from X to Y , ($A \in \mathcal{L}(X, Y)$) such that for any $v \in X$, the directional derivative of J in u in the direction v exists and is equal to $A(v)$, i.e.

$$\partial_v J(u) = \lim_{t \rightarrow 0} \frac{J(u + tv) - J(u)}{t} = A(v), \quad t \rightarrow 0, \forall u \in X.$$

It is then verified that, if such an application A exists, it is unique. We notice

$$A = d_G J(u).$$

Proposition 1.3. [24] *If J is differentiable Fréchet in u , it is Gâteaux differentiable in u and*

$$d_G J(u) = dJ(u).$$

The reciprocal is false in general, yet in finite dimension.

Example 1.2. Let $J : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\begin{cases} J(x, y) = [\frac{x^2 y}{x^4 + y^2}]^2, & \text{if } y \neq 0 \\ J(x, 0) = 0 \end{cases}$$

G-differentiability does not imply the continuity of J ! (while of course differentiability in the sense of Fréchet implies).

On the other hand, if we know that J is C^1 in the sense of Gâteaux, then J is Fréchet differentiable on U and the two notions coincide and we have

Theorem 1.3. ([25], [24]) *If $J : U \rightarrow Y$ is Gâteaux differentiable in an open neighborhood $U \subset X$, of u and $d_G J : U \rightarrow X^*$ is continuous at u , then J is Fréchet differentiable at u and the two derivatives at u coincide, i.e.,*

$$J'(v) = d_G J(v), \quad \forall v \in U.$$

Proof. Let $r \in (0, 1]$ be such that $B_r(u) \subset U$. Since $d_G J : U \rightarrow X^*$ is continuous at u , for every $\varepsilon > 0$ there exists a $\delta \in (0, 1]$ such that

$$\|d_G J(u + tv) - d_G J(u)\| < \varepsilon \quad \text{for } \|v\| < \delta, \quad |t| < 1. \quad (1.1)$$

On the other hand, for each $v \in B_r(0)$ the function

$$t \in [0, 1] \mapsto (d_G J(u + tv), v),$$

is the derivative of the function

$$g(t) = J(u + tv) \quad t \in [0, 1].$$

Consequently

$$\int_0^1 (d_G J(u + tv), v) dt = g(1) - g(0) = J(u + v) - J(u).$$

Hence

$$J(u + v) - J(u) - (d_G J(u), v) = \int_0^1 (d_G J(u + tv) - d_G J(u), v) dt. \quad (1.2)$$

Let

$$R(u, v) = \int_0^1 (d_G J(u + tv) - d_G J(u), v) dt.$$

From (1.1), for all $v \in B_\delta(0)$ we have

$$\begin{aligned} \|R(u, v)\| &\leq \int_0^1 \|d_G J(u + tv) - d_G J(u)\| \|v\| dt \\ &\leq \varepsilon \|v\|. \end{aligned}$$

Hence $R(u, v) = o(\|v\|)$ when $\|v\| \rightarrow 0$. Then, (1.2) shows that $d_G J(u)$ is the Fréchet derivative of J at u . $\square$

Remark 1.2. In application, it is easier to verify the differentiability in the sense of Gâteaux. If one wants to prove that J is C^1 , it is enough to prove that it is Gâteaux differentiable, then to verify that the differentiable $d_G J$ is continuous.

Definition 1.11. Let $u_0 \in X$. We say that u_0 is a local minimum for J if there exists $\delta > 0$ such that

$$J(u_0) \leq J(v), \quad \forall v \in B(u_0, \delta), \quad v \neq u_0.$$

Then we have

Proposition 1.4. ([25], [24]) *Let $u_0 \in U \subset X$, a local minimum (or maximum) of J . Then if J is Gâteaux differentiable in u_0 , then*

$$d_G J(u_0) = 0.$$

Proof. Under the hypothesis of this theorem, suppose without loss of generality that u_0 is a minimum point (otherwise consider the function $-J$ instead of J).

Since $u_0 \in U$ and U is open, there exists a positive real number r such that the open ball $B_r(u_0)$ is contained in U .

Now let $h \in X \setminus \{0\}$. Then for every t such that

$$|t| \leq \frac{r}{\|h\|}, \quad \text{we have,} \quad J(u_0 + th) \geq J(u_0),$$

and so by the Gâteaux differentiability of J at u_0 , we have

$$d_G J(u_0)(h) = \lim_{t \rightarrow 0^+} \frac{1}{t} [J(u_0 + th) - J(u_0)] \geq 0.$$

It also follows that $d_G J(u_0)(-h) \geq 0$, i.e., $d_G J(u_0)(h) \leq 0$, by linearity of $d_G J(u_0)$.

Therefore $d_G J(u_0)(h) = 0$, for all $h \in X$ indeed. Thus $d_G J(u_0) = 0$. □

Proposition 1.5. ([25], [24]) *Let X be a Banach space, and J be an application from X to $\mathbb{R}$, be $u \in X$. Assume that*

$$J(u) = \inf_{v \in X} J(v), \quad \text{i.e., } J(u) \leq E(v), \quad \forall v \in X.$$

Then, if J is Gâteaux differentiable in u , we have

$$d_G J(u) = 0.$$

Definition 1.12. Let $J : X \rightarrow \mathbb{R} \cup \{+\infty\}$. We say that J is coercive if

$$J(u) \rightarrow +\infty, \quad \text{when } \|u\| \rightarrow +\infty; \quad \text{i.e.}$$

$$\forall A > 0, \exists B > 0, \text{ such that } \|u\| \geq B \text{ leads } J(u) \geq A.$$

Definition 1.13. A Hilbert space is an inner product space which is complete as a metric space, and denoted by H .

1.1.4 Lebesgue and Sobolev spaces

Let Ω an open of $\mathbb{R}$, and $(\Omega, \mathcal{M}, dx)$ denote a measure space, where $\mathcal{M}$ is a σ -algebra in Ω .

We assume that $f : \Omega \rightarrow \mathbb{R}$ is a measurable function and integrable function. We shall often write $\int f$ instead of $\int_{\Omega} f$, and we shall also use the notation

$$\|f\|_{L^1(\Omega)} = \int_{\Omega} |f(x)| dx, \text{ or } \|f\|_{L^1} = \int |f|.$$

Theorem 1.4. (Lebesgue's dominated convergence theorem)[23] Let (f_n) be a sequence of functions in L^1 that satisfy

(a) $(f_n(x)) \rightarrow f(x)$ a.e. on Ω .

(b) There is a function $g \in L^1(\Omega)$ such that for all n , $|f_n(x)| \leq g(x)$, a.e. on Ω .

Then

$$f \in L^1(\Omega) \quad \text{and} \quad \|f_n - f\|_{L^1} \rightarrow 0, \quad n \rightarrow \infty.$$

Definition 1.14. Let $p \in \mathbb{R}$ with $1 \leq p < \infty$, we put

$$L^p(\Omega) = \{f : \Omega \rightarrow \mathbb{R}, \text{ measurable and } |f|^p \in L^1(\Omega)\}.$$

We notice $\|f\|_{L^p} = \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}}$, and $\|\cdot\|_{L^p}$ is a norm.

Definition 1.15. We put

$$L^{\infty}(\Omega) = \{f : \Omega \rightarrow \mathbb{R}, \text{ measurable and } \exists C : |f(x)| \leq C, \text{ a.e. on } \Omega\}.$$

We notice

$$\|f\|_{L^{\infty}} = \inf\{C : |f(x)| \leq C, \text{ a.e. on } \Omega\}$$

Theorem 1.5. [23] The space L^p is reflexive for $1 < p < \infty$, and its dual is L^q such that $\frac{1}{p} + \frac{1}{q} = 1$

Theorem 1.6. (Cauchy-Schwarz inequality)[23] Let f and $g \in L^2(\Omega)$. Then we have

$$\int_{\Omega} |fg| \leq \left(\int_{\Omega} |f|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |g|^2 \right)^{\frac{1}{2}}.$$

Theorem 1.7. (Hölder's inequality)[23] Let $1 \leq p \leq +\infty$; we denote by q the conjugate exponent, $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p \leq +\infty$. Then

(i) $fg \in L^1(\Omega)$ and

$$(ii) \int_{\Omega} |fg| \leq \left(\int_{\Omega} |f|^p \right)^{\frac{1}{p}} \left(\int_{\Omega} |g|^q \right)^{\frac{1}{q}} = \|f\|_p \|g\|_q.$$

Lemma 1.1. (Fatou's lemma)[23] Let (f_n) be a sequence of functions in L^1 that satisfy

1. for all n , $f_n \geq 0$ a.e. on Ω

$$2. \sup_n \int f_n < +\infty.$$

For almost all $x \in \Omega$ we set $f(x) = \liminf_{n \rightarrow \infty} f_n(x) \leq +\infty$. Then $f \in L^1$ and

$$\int f \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Lemma 1.2. [23] Let $g \in L^1_{loc}(\Omega)$, for y_0 fixed in Ω , set

$$v(x) = \int_{y_0}^x g(t) dt, \quad x \in \Omega.$$

Then $v \in C(\Omega)$ and

$$\int_{\Omega} v \varphi' = - \int_{\Omega} g \varphi, \quad \forall \varphi \in C_c^1(\Omega).$$

Lemma 1.3. [23] Let $f \in L^1_{loc}(\Omega)$ be such that

$$\int_{\Omega} f \varphi' = 0, \quad \forall \varphi \in C_c^1(\Omega).$$

Then there exists a constant C such that $f = C$, a.e. on Ω .

1.2 Critical point theory

Definition 1.16. Let J be a differentiable functional from X to $\mathbb{R}$. A point $u \in X$ is said to be critical for J if and only if

$$dJ(u) = 0.$$

Definition 1.17. Let the functional J , of class C^1 be defined on X . A critical value of J , is a number $c \in \mathbb{R}$, such that there exists $u \in X$, with

$$J(u) = c, \quad dJ(u) = 0.$$

If we are able to determine a critical value, we have proved the existence of a critical point, in some situations it is useful to introduce the following definitions and theorems.

Definition 1.18. (Minimizing sequence) A minimizing sequence of a functional $J : X \rightarrow]-\infty, +\infty]$ is a sequence (u_n) such that

$$\lim_{n \rightarrow +\infty} J(u_n) = \inf_{u \in X} J(u)$$

Theorem 1.8. ([22], [27]) Let J be a functional defined on a reflexive Banach space X . Assume

- (i) J is coercive, i.e., $J(u) \rightarrow +\infty$ for $\|u\| \rightarrow +\infty$,
- (ii) J is weakly lower semi-continuous.

Then J is bounded from below on X , and achieved its infimum at some point u_0 . If, moreover, J is Gâteaux-differentiable at u_0 , then $J'(u_0) = 0$.

Proof. Suppose that J is not bounded below, then there is a sequence (u_n) such that

$$\lim_{n \rightarrow +\infty} J(u_n) = -\infty.$$

According to (i), the sequence (u_n) is uniformly bounded, so it admits a subsequence (u_{n_k}) such that

$$u_{n_k} \rightharpoonup u_0, \quad \text{with} \quad \lim_{k \rightarrow +\infty} J(u_{n_k}) = -\infty.$$

From (ii), we have

$$J(u_0) \leq \liminf_{k \rightarrow +\infty} J(u_{n_k}) = \lim_{k \rightarrow +\infty} J(u_{n_k}) = -\infty.$$

Thus, we obtain an absurd. Then J is bounded below.

Now we show that J is achieved its minimum. Let $\alpha = \inf_{v \in X} J(v)$ and let (u_n) a minimizing sequence of J i.e.,

$$\lim_{n \rightarrow +\infty} J(u_n) = \alpha,$$

as above, (u_n) admits a subsequence (u_{n_k}) such that $u_{n_k} \rightharpoonup u_0 \in X$, consequently

$$J(u_0) \leq \liminf_{k \rightarrow +\infty} J(u_{n_k}) = \lim_{k \rightarrow +\infty} J(u_{n_k}) = \alpha.$$

According to $J(u_0) \geq \alpha$ and definition of α then $J(u_0) = \alpha$. □

Palais-Smale's sequence and condition

Definition 1.19. [28] Let X be a Banach space and $J : X \rightarrow \mathbb{R}$ a C^1 -functional. We say that J satisfies the Palais-Smale condition, denoted $(P.S)$, if any sequence (u_n) in X such that

$$(J(u_n)) \text{ is bounded, and } J'(u_n) \rightarrow 0, \text{ as } n \rightarrow +\infty \quad (1.3)$$

admits a convergent subsequence.

Any sequence satisfying (1.3) is called a Palais-Smale sequence.

Theorem 1.9. ([24], [29]) Let X be a reflexive Banach space, and J a functional defined on X such that

1. J is $(w.l.s.c.)$,
2. The minimizing sequence of J is bounded on X ,

Then J achieves its infimum on X .

Theorem 1.10. ([27], [28]) Let X be a Banach space and $J : X \rightarrow \mathbb{R}$ a C^1 -functional that is bounded from below and satisfies $(P.S)$. Then, there exists a $u_0 \in X$ such that

$$J(u_0) = \inf_{v \in X} J(v), \text{ and } J'(u_0) = 0.$$

1.2.1 Mountain Pass Theorem

Theorem 1.11. (Mountain Pass Theorem)([24], [30]) Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$ with $J(0) = 0$. Suppose $J(u)$ satisfies $(P.S)$ condition and

- (a) there are $\rho, \alpha > 0$ such that $J(u) \geq \alpha$ when $\|u\|_X = \rho$,
- (b) there is a $J \in X$, $\|e\|_X > \rho$ such that $J(e) < 0$.

Define

$$\Gamma = \{\gamma \in C([0, 1], X) \mid \gamma(0) = 0, \gamma(1) = e\}. \quad (1.4)$$

Then

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J(\gamma(t)) \geq \alpha \quad (1.5)$$

is a critical value of $J(u)$.

1.3 Fourier series

1.3.1 Fourier series of 2π – periodic functions

let $f : \mathbb{R} \longrightarrow \mathbb{R}$ a periodic function of period $T = 2\pi$. We suppose that $\int_{\alpha}^{\alpha+2\pi} |f(t)|dt$ converges for everything $\alpha \in \mathbb{R}$

Definition 1.20. We call the Fourier series associated with f , the trigonometric series denoted σf or

$$\sigma f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

with

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx \text{ and } b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx.$$

Remark 1.3. If the function f is even $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx$ and $b_n = 0$, $n \in \mathbb{N}$.

If the function f is odd $a_n = 0$ and $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$, $n \in \mathbb{N}$.

Theorem 1.12. let $f : \mathbb{R} \longrightarrow \mathbb{R}$ a periodic function of period $T = 2\pi$ satisfying the following conditions (called Dirichlet conditions):

- At any point x_0 , the limits of f to the right and left of x_0 exist and the discontinuities of f are finitely numbered in any finite interval.
- f admits at every point a right and a left derivative.

Then the Fourier series associated with f is convergent and we have:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] = \begin{cases} f(x) & : \text{If } f \text{ is continuous at } x \\ \frac{f(x_0^+) + f(x_0^-)}{2} & : \text{If } f \text{ is discontinuous at } x. \end{cases}$$

Moreover, the convergence is uniform on any interval where the function f is continuous.

1.3.2 Fourier series of an arbitrary period function

Let f be a periodic function with period $T = 2l$. To develop it in series of Fourier on l'interval $[-l, l]$ let's change variable $x = \frac{lt}{\pi}$. Function $g(t) = f(\frac{lt}{\pi})$ will be a 2π – periodic function of t , because

$$g(t + 2\pi) = f\left(\frac{l(t + 2\pi)}{\pi}\right) = f\left(\frac{lt}{\pi} + 2l\right) = f\left(\frac{lt}{\pi}\right) = g(t) \quad (1.6)$$

Then, we can develop it in Fourier series on the interval $[-\pi; \pi]$. Returning to the variable x , by asking $t = \frac{\pi x}{l}$, we obtain

$$\sigma f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right],$$

with

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx, \quad \text{et} \quad b_n = \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx, \quad n \in \mathbb{N}.$$

Examples 1.2. Develop in Fourier series the 2π -periodic function f defined by $f(x) = |x|$, on the interval $[-1; 1]$. As this function is pair, we have $b_n = 0$.

$$a_0 = \frac{2}{\pi} \int_0^1 x dx = 1, \quad a_n = \frac{2}{\pi} \int_0^1 x \cos(n\pi x) dx \implies \begin{cases} a_{2n} = 0, \\ a_{2n+1} = \frac{-4}{\pi^2(2n+1)^2} \end{cases}$$

So,

$$\sigma f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos((2n+1)\pi x)}{(2n+1)^2} = f(x)$$

Theorem 1.13. (Parseval equality) Let f be a function developable in Fourier and period series $T = \frac{2\pi}{w} > 0$, then we have

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{w}{\pi} \int_0^{\frac{2\pi}{w}} (f(x))^2 dx.$$

Lemma 1.4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ periodical of period $T > 0$ and integrable in the interval $[0, T]$.

So:

$$\forall \alpha \in \mathbb{R} : \int_{\alpha}^{\alpha+T} f(t) dt = \int_0^T f(t) dt.$$

STUDY THE EXISTENCE OF A SOLUTION TO AN IMPULSIVE SECOND-ORDER A MIXED BOUNDARY VALUE PROBLEM

In this chapter, we discuss the existence of a solution for an impulsive second-order equation. To obtain our results, we use the variational method. In particular, we use a minimization theorem and the mountain pass lemma. We used as an example the article [1] which was published by authors Jun Wang and Weigao Ge in 2009.

2.1 Introduction

We study the following impulsive problem :

$$\begin{cases} -u''(t) = \lambda u(t) + f(t, u(t)), t \neq t_i, t \in [0, T], \\ -\Delta u'(t_i) = I_i(u(t_i)), i = 1, 2, \dots, l, \\ u'(0) = 0, u(T) = 0, \end{cases} \quad (2.1)$$

where $\lambda \in \mathbb{R}$, $0 = t_0 < t_1 < \dots < t_l < t_{l+1} = T$, $\Delta u'(t_i) = u'(t_i^+) - u'(t_i^-)$, (where $u'(t_i^+)$ (resp. $u'(t_i^-)$) denotes the right limit (resp. left limit) of $u'(t)$ at $t = t_i$, $u'(t_i^-) = u'(t_i)$, $I_i \in C(\mathbb{R}, \mathbb{R})$, $i = 1, 2, \dots, l$, $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$.

2.1.1 The basic hypothesis

(H1) $f(x, t) = o(|x|)$, $I_i(x) = o(|x|)$ as $|x| \rightarrow 0$, $i = 1, 2, \dots, l$

(H2) There exist constants $\mu > 2$ and $r \geq 0$ such that for $|\epsilon| \geq r$,

$$0 < \mu F(t, \xi) \leq \xi f(t, \xi), \quad 0 < \mu \int_0^\xi I_i(s) ds \leq \xi I_i(\xi), \quad i = 1, 2, \dots, l$$

(H3) $f(t, u), I_i(u)$ are odd in u .

(H4) There exist $x_1 > 0, r > \lambda_k, \lambda_k$ is the k -th eigenvalue of (2.2) such that

$$rx_1 + f(t, x_1) \leq 0, t \in [0, T], I_i(x_1) \leq 0, i = 1, 2, \dots, l$$

Remark 2.1. By (H2), there exist $b_1, b_2, c_i, d_i > 0, i = 1, 2, \dots, l$ such that

$$F(t, \xi) \geq b_1|\xi|^\mu - b_2, \int_0^\xi I_i(s)ds \geq c_i|\xi|^\mu - d_i$$

for all $t \in [0, T], \xi \in \mathbb{R}$.

2.1.2 The spaces provided for our problem

We now present the Hilbert space $H_0^1(0, T)$ which is suitable for the study of our problem. Let the space H be such that:

$$H^1([0, T]) = \{u \in L^2([0, T]) : u' \in L^2([0, T])\},$$

$$C[0, T] = \{u : [0, T] \rightarrow \mathbb{R}, \text{ telle que } u \text{ continue}\},$$

muni par sa norme usuelle

$$\|u\|_\infty = \sup_{x \in [0, T]} |u(x)|.$$

Remark 2.2. we say that the space H is space H_0^1

Define the space H by:

$$H = \{u \in C([0, T]) : u'(\cdot) \in L^2([0, T]), u(T) = 0\},$$

with the inner product

$$(u, v) = \int_0^T u'(t)v'(t)dt,$$

inducing the norm

$$\|u\| = \left(\int_0^T |u'(t)|^2 dt \right)^{\frac{1}{2}}.$$

2.1.3 The eigenvalues of the linear boundary value problem

which is defined as follows.

$$\begin{cases} -u''(t) = \lambda u(t), t \in [0, T], \\ u'(0) = 0, u(T) = 0. \end{cases} \quad (2.2)$$

Lemma 2.1. *As is well known, (2.2) possesses a sequence of eigenvalues (λ_i) , where $\lambda_i = \left(\frac{(2i-1)\pi}{2T}\right)^2$ and $\lambda_1 = \frac{\pi^2}{(2T)^2}$ is the first nonzero eigenvalue of the problem (2.2) with*

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_j < \dots$$

Proof.

$$-r^2 = \lambda \implies r = \pm i\sqrt{\lambda}$$

The solution to this problem is as follows:

$$u(t) = \alpha \cos(\sqrt{\lambda}t) + \beta \sin(\sqrt{\lambda}t).$$

Conditions are applied to the limit One finds

$$u'(t) = -\alpha\sqrt{\lambda}\sin(\sqrt{\lambda}t) + \beta\sqrt{\lambda}\cos(\sqrt{\lambda}t)$$

$$u'(0) = 0 \implies \beta = 0$$

So,

$$u(t) = \alpha \cos(\sqrt{\lambda}t)$$

we have,

$$u(T) = 0 \implies \alpha \cos(\sqrt{\lambda}T) = 0.$$

$$\cos(\sqrt{\lambda}T) = 0 \implies \lambda_i = \left(\frac{2i\pi - \pi}{2T}\right)^2$$

□

Remark 2.3. The corresponding eigenfunctions are normalized so that :

$$||\varphi_j|| = 1 = \lambda_j \int_0^T |\varphi_j(t)|^2 dt,$$

her

$$\varphi_j(t) = \sqrt{\frac{2}{T\lambda_j}} \cos(\sqrt{\lambda_j}t), \quad j = 1, 2, \dots \quad (2.3)$$

2.2 Some necessary lemmas and theorems

2.2.1 Some necessary lemmas and theorem

Lemma 2.2. $H_0^1(0, T)$ embeds continuously in $L^2(0, T)$

$$\left(\int_0^T u^2(t) dt \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\lambda_1}} \left(\int_0^T |u'(t)|^2 dt \right)^{\frac{1}{2}} \quad (2.4)$$

Proof.

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_j < \dots$$

In fact, let,

$$u_1(t) = \begin{cases} u(t - 4kT), & t \in [4kT, (4k + 1)T], \\ u(-t + 4kT), & t \in [(4k - 1)T, 4kT], \\ -u(t - 4kT - 2T), & t \in [(4k + 1)T, (4k + 2)T], \\ -u(t - 4kT + 2T), & t \in [(4k - 2)T, (4k - 1)T]. \end{cases}$$

Then u_1 is a $4T$ -periodic function on $(-\infty, +\infty)$. By the expression of Fourier expansion, we have

$$\begin{aligned} \int_0^T u^2(t) dt &= \frac{1}{4} \int_{-2T}^{2T} |u_1(t)|^2 dt \\ &= \frac{1}{4} \int_{-2T}^{2T} \left(\sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi t}{2T}\right) \right)^2 dt \\ &= \frac{1}{4} \sum_{k=1}^{\infty} a_k^2 \int_{-2T}^{2T} \cos^2\left(\frac{k\pi t}{2T}\right) dt \\ &= \frac{1}{4} \sum_{k=1}^{\infty} a_k^2 \int_{-2T}^{2T} \frac{1}{2} (1 + \cos\left(\frac{k\pi t}{T}\right)) dt = \frac{1}{4} \sum_{k=1}^{\infty} a_k^2 2T. \end{aligned}$$

Besides, by the above equality, and by Parseval equality, we obtain

$$\begin{aligned}
\int_0^T |u'(t)|^2 dt &= \frac{1}{4} \int_{-2T}^{2T} |u'_1(t)|^2 dt \\
&= \frac{1}{4} \int_{-2T}^{2T} \left(\sum_{k=1}^{\infty} a_k \frac{k\pi}{2T} \sin \frac{k\pi t}{2T} \right)^2 dt \\
&= \frac{1}{4} \int_{-2T}^{2T} \sum_{k=1}^{\infty} a_k^2 \frac{k^2 \pi^2}{4T^2} \sin^2 \frac{k\pi t}{2T} dt \\
&= \frac{1}{4} \int_{-2T}^{2T} \sum_{k=1}^{\infty} a_k^2 \frac{k^2 \pi^2}{4T^2} dt \\
&= \frac{1}{4} \sum_{k=1}^{\infty} a_k^2 \frac{k^2 \pi^2}{4T^2} \int_{-2T}^{2T} dt \\
&= \frac{1}{4} \sum_{k=1}^{\infty} a_k^2 \frac{k^2 \pi^2}{4T^2} \cdot 2T \geq \lambda_1 \int_0^T u^2(t) dt.
\end{aligned}$$

□

Lemma 2.3. $H_0^1(0, T)$ embeds continuously in $C(0, T)$

If $u \in H$, then,

$$\|u\|_{\infty} \leq T^{\frac{1}{2}} \|u\|,$$

where $\|u\|_{\infty} = \max_{t \in [0, T]} |u(t)|$

Proof. By Holder inequality, for $u \in H$,

$$|u(t)| = |u(T) - \int_t^T u'(s) ds| \leq T^{\frac{1}{2}} \left(\int_0^T |u'(s)|^2 ds \right)^{\frac{1}{2}} = T^{\frac{1}{2}} \|u\|.$$

Therefore, we have,

$$\|u\|_{\infty} \leq T^{\frac{1}{2}} \|u\|.$$

□

Lemma 2.4. If $\lambda < \lambda_1$, then $\left(\int_0^T (u'(t))^2 - \lambda u^2(t) dt \right)^{\frac{1}{2}}$ can be taken as a norm on H . then there exist $\theta_1, \theta_2 > 0$ satisfying

$$\theta_1 \|u\|^2 \leq \int_0^T (u'(t))^2 - \lambda u^2(t) dt \leq \theta_2 \|u\|^2$$

for $u \in H$.

Proof. If $0 < \lambda < \lambda_1$, we have by (2.4)

$$\left(1 - \frac{\lambda}{\lambda_1}\right) \int_0^T (u'(t))^2 dt = \int_0^T (u'(t))^2 dt - \frac{\lambda}{\lambda_1} \int_0^T (u'(t))^2 dt \leq \int_0^T (u'(t))^2 dt - \lambda \int_0^T u^2(t) dt \leq \int_0^T (u'(t))^2 dt \quad (2.5)$$

So, $\theta_1 = \left(1 - \frac{\lambda}{\lambda_1}\right)$, $\theta_2 = 1$

If $\lambda < 0$, we have by (2.4)

$$\begin{aligned} \int_0^T (u'(t))^2 dt &\leq \int_0^T (u'(t))^2 dt - \lambda \int_0^T u^2(t) dt \\ &< \int_0^T (u'(t))^2 dt - \frac{\lambda}{\lambda_1} \int_0^T (u'(t))^2 dt \\ &= \left(1 - \frac{\lambda}{\lambda_1}\right) \int_0^T (u'(t))^2 dt \end{aligned} \quad (2.6)$$

so, $\theta_1 = \theta_2 = \left(1 - \frac{\lambda}{\lambda_1}\right)$ □

Remark 2.4. By Lemma 2.4, if $\lambda < \lambda_1$, then there exist $\theta_1, \theta_2 > 0$ satisfying

$$\theta_1 \|u\|^2 \leq \int_0^T (u'(t))^2 dt - \lambda \int_0^T u^2(t) dt \leq \theta_2 \|u\|^2$$

for $u \in H$.

Theorem 2.1. [14] (*Generalizations of the Mountain Pass Theorem.*) Let E be a real Banach space with $E = V \oplus X$, where V is finite dimensional. Suppose $J \in C^1(E, \mathbb{R})$, satisfies (PS), and

(C3) There are constant $\rho, \alpha > 0$ such that $J \setminus \partial B_\rho \cap X \geq \alpha$, and

(C4) There is an $e \in \partial B_1 \cap X$ and $R > \rho$ such that if $Q \equiv (\bar{B}_R \cap V) \oplus \{re : 0 < r < R\}$, then $J \setminus \partial Q \leq 0$.

Then I possesses a critical value $c \geq \alpha$ which can be characterized as

$$c = \inf_{h \in \Gamma} \max_{u \in Q} J(h(u)),$$

where

$$\Gamma = \{h \in C(\bar{Q}, E) : h = id \text{ on } \partial Q\}.$$

2.3 Variational structure

2.3.1 The Euler-Lagrange functional

For everything $u \in H_0^1(0, T)$ we define the functional J by:

$$J(u) = \frac{1}{2} \int_0^T (u'(t))^2 - \lambda u^2(t) dt - \int_0^T F(t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds.$$

such that : $F(t, u) = \int_0^u f(t, s) ds$.

Remark 2.5. We show that, for all $u, v \in H$ and for everything $x \in [0, T]$

$$- \int_0^T u''(t)v(t) dt = \int_0^T u'(t)v'(t) dt - \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds$$

$u, v \in H(0, T)$. so ,we have :

$$- \int_0^T u''(t)v(t) dt = - \int_0^{t_1^-} u''(t)v(t) dt - \int_{t_1^+}^{t_2^-} u''(t)v(t) dt - \dots - \int_{t_{l-1}^+}^{t_l^-} u''(t)v(t) dt - \int_{t_l^+}^T u''(t)v(t) dt$$

By integrating by parts,we have

$$- \int_0^{t_1^-} u''(t)v(t) dt = [u'(t)v(t)]_0^{t_1^-} + \int_0^{t_1^-} u'(t)v'(t) dt = -u'(t_1^-)v(t_1^-) + u'(0)v(0) + \int_0^{t_1^-} u'(t)v'(t) dt$$

using the boundary conditions in 2.1, we find

$$- \int_0^{t_1^-} u''(t)v(t) dt = -u'(t_1^-)v(t_1^-) + \int_0^{t_1^-} u'(t)v'(t) dt$$

and ,

$$- \int_{t_1^+}^{t_2^-} u''(t)v(t) dt = -[u'(t)v(t)]_{t_1^+}^{t_2^-} + \int_{t_1^+}^{t_2^-} u'(t)v'(t) dt = -u'(t_2^-)v(t_2^-) + u'(t_1^+)v(t_1^+) + \int_{t_1^+}^{t_2^-} u'(t)v'(t) dt$$

And so on,

$$- \int_{t_{l-1}^+}^{t_l^-} u''(t)v(t) dt = -[u'(t)v(t)]_{t_{l-1}^+}^{t_l^-} + \int_{t_{l-1}^+}^{t_l^-} u'(t)v'(t) dt = -u'(t_l^-)v(t_l^-) + u'(t_{l-1}^+)v(t_{l-1}^+) + \int_{t_{l-1}^+}^{t_l^-} u'(t)v'(t) dt$$

and,

$$- \int_{t_l^+}^T u''(t)v(t) dt = -[u'(t)v(t)]_{t_l^+}^T + \int_{t_l^+}^T u'(t)v'(t) dt = -u'(T)v(T) + u'(t_l^+)v(t_l^+) + \int_{t_l^+}^T u'(t)v'(t) dt$$

And like ,

$$\Delta u'(t_1)v(t_1) = (u'(t_1^+) - u'(t_1^-))v(t_1) \dots \Delta u'(t_l)v(t_l) = (u'(t_l^+) - u'(t_l^-))v(t_l).$$

So by definition of impulsive and the boundary conditions in (2.1), we have

$$\begin{aligned}\Delta u'(t_1)v(t_1) + \Delta u'(t_2)v(t_2) + \dots + \Delta u'(t_l)v(t_l) &= \sum_{i=1}^l \Delta u'(t_i)v(t_i) \\ &= -\sum_{i=1}^l I_i(u(t_i))v(t_i),\end{aligned}$$

So, by substitution we have:

$$\int_0^{t_1^-} u'(t)v'(t) dt + \int_{t_1^+}^{t_2^-} u'(t)v'(t) dt + \int_{t_{l-1}^+}^{t_l^-} u'(t)v'(t) dt + \int_{t_l^+}^T u'(t)v'(t) dt = \int_0^T u'(t)v'(t) dt,$$

so, it follows that

$$-\int_0^T u''(t)v(t) dt = \int_0^T u'(t)v'(t) dt - \sum_{i=1}^l I_i(u(t_i))v(t_i)$$

Proposition 2.1. *The functional J is continuously differentiable and the Fréchet derivative of J is written Under the form*

$$\langle J'(u), v \rangle = \int_0^T u'(t)v'(t) dt - \lambda \int_0^T u(t)v(t) dt - \int_0^T f(t, u(t))v(t) dt - \sum_{i=1}^l I_i(u(t_i))v(t_i).$$

for all $v \in H^1([0, t])$

Proof. Step 1: J is Gâteaux-differentiable.

we have $v \in H^1([0, t])$ and $0 < t < T$

$$\begin{aligned}J(u + \mu v) - J(u) &= \frac{1}{2} \int_0^T ((u + \mu v)')^2 dt - \lambda \int_0^T (u + \mu v)^2 dt - \int_0^T F(t, u + \mu v) dt \\ &\quad - \sum_{i=1}^l \int_0^{u+\mu v} I_i(s) ds - \frac{1}{2} \int_0^T (u'(t))^2 dt + \lambda \int_0^T u^2(t) dt + \int_0^T F(t, u(t)) dt \\ &\quad + \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds. \\ &= \frac{1}{2} \int_0^T u'^2(t) dt + \frac{1}{2} \mu^2 \int_0^T v'^2(t) dt + \mu \int_0^T v'(t)u'(t) dt - \frac{\lambda}{2} \int_0^T u^2(t) dt \\ &\quad + \mu \int_0^T u(t)v(t) dt + \mu^2 \int_0^T v^2(t) dt - \frac{1}{2} \int_0^T u'^2(t) dt + \frac{\lambda}{2} \int_0^T F(u) dt \\ &\quad + \sum_{i=1}^l \int_0^{u(t)} I_i(s) ds - \int_0^T F(u + \mu v) dt - \sum_{i=1}^l \int_0^{u+\mu v} I_i(s) ds \\ &= \mu \int_0^T u'(t)v'(t) dt - \lambda \mu \int_0^T u(t)v(t) dt + \frac{\mu^2}{2} \int_0^T v'^2(t) dt - \lambda \mu \int_0^T v^2(t) dt \\ &\quad + \int_0^T F(u) - F(u + \mu v) dt + \sum_{i=1}^l \left(\int_0^u I_i(s) ds - \int_0^{u+\mu v} I_i(s) ds \right)\end{aligned}$$

using the increment finite theorem, we have

$$F(u + \mu v) - F(u) = \mu \int_0^T f(u + \mu \theta v) v \, dt,$$

and

$$\int_0^{u+\mu v} I_i(s) \, ds - \int_0^u I_i(s) \, ds = \mu v(t_i) I_i(u(t_i) + \theta v(t_i)),$$

where $0 < \theta < T$ and then

$$\begin{aligned} \frac{J(u + \mu v) - J(u)}{\mu} &= \int_0^T u'(t) v'(t) dt - \lambda \int_0^T u(t) v(t) dt \\ &+ \frac{\mu}{2} \int_0^T v'^2(t) - \lambda v^2(t) dt + \int_0^T f(u + \mu \theta v) v \, dt \\ &+ \sum_{i=1}^l v(t_i) I_i(u(t_i) + \theta v(t_i)) \end{aligned}$$

Let $\mu \rightarrow 0$, so

$$\langle J'(u), v \rangle = \int_0^T u'(t) v'(t) dt - \lambda \int_0^T u(t) v(t) dt - \int_0^T f(t, u(t)) v(t) dt - \sum_{i=1}^l I_i(u(t_i)) v(t_i).$$

Step 2: J' is continuous.

We show that J' is continuous, then let $(u_n) \subset H_0^1(0, T)$ with $u_n \rightarrow u$ When $n \rightarrow +\infty$, SO by the dominated convergence theorem of lebeuge we obtain:

$$\lim_{n \rightarrow +\infty} \int_0^T u'_n(t) v'(t) dt = \int_0^T u'(t) v'(t) dt,$$

and

$$\begin{aligned} \lim_{n \rightarrow +\infty} \lambda \int_0^T u_n(t) v(t) dt &= \lambda \int_0^T u(t) v(t) dt, \\ \lim_{n \rightarrow +\infty} \int_0^T f(t, u_n(t)) v(t) dt &= \int_0^T f(t, u(t)) v(t) dt, \end{aligned}$$

therefore, we have for everything $v \in H_0^1(0, T)$:

$$\begin{aligned} \langle J'(u_n) - J'(u), v \rangle &= \int_0^T u'_n(t) v'(t) dt - \lambda \int_0^T u_n(t) v(t) dt - \int_0^T f(t, u_n(t)) v(t) dt - \sum_{i=1}^l I_i(u_n) v \\ &- \int_0^T u'(t) v'(t) dt + \lambda \int_0^T u(t) v(t) dt + \int_0^T f(t, u(t)) v(t) dt - \sum_{i=1}^l I_i(u) v \\ &= \int_0^T (u'_n - u') v' dt + \lambda \int_0^T (u_n - u) v dt + \int_0^T f(t, u_n - u) v dt - \sum_{i=1}^l I_i(u_n - u) v \end{aligned}$$

and according to the continuity of f passes to the limit $\langle J'(u_n) - J'(u), v \rangle$, When $n \rightarrow +\infty$, we obtain that $J'(u_n) \rightarrow J'(u)$, quand $n \rightarrow +\infty$. So J' is continuous. $\square$

Definition 2.1. We say that $u \in H_0^1(0, T)$ is a weak solution of the problem (2.1) studied, if for all $v \in H_0^1(0, T)$, we have

$$\langle J'(u), v \rangle = \int_0^T u'(t)v'(t) - \lambda u(t)v(t)dt - \int_0^T f(t, u(t))v(t)dt - \sum_{i=1}^l I_i(u(t_i))v(t_i) = 0.$$

2.3.2 Weak and classical solution

Lemma 2.5. If $u \in H$ is a weak solution of (2.1), then u is a classical solution of (2.1).

Proof. By Definition 2.1, if u is a weak solution of (2.1), then $\langle J'(u), v \rangle = 0$ holds for all $v \in H$, i.e.

$$\int_0^T u'(t)v'(t) - \lambda u(t)v(t)dt - \int_0^T f(t, u(t))v(t)dt - \sum_{i=1}^l I_i(u(t_i))v(t_i) = 0.$$

By integrating by parts, we have

$$\int_0^T u'(t)v'(t)dt = [u'(t)v(t)]_0^T - \int_0^T u''(t)v(t)dt = u'(T)v(T) - \int_0^T u''(t)v(t)dt$$

so

$$\begin{aligned} & \int_0^T u'(t)v'(t) - \lambda u(t)v(t)dt - \int_0^T f(t, u(t))v(t)dt - \sum_{i=1}^l I_i(u(t_i))v(t_i) \\ &= \sum_{i=0}^l u'(t)v(t)|_{t=t_0^+}^{t_{i+1}} + \int_0^T [-u''(t) - \lambda u(t) - f(t, u(t))]v(t)dt - \sum_{i=1}^l I_i(u(t_i))v(t_i) \\ &= \int_0^T [-u''(t) - \lambda u(t) - f(t, u(t))]v(t)dt - \sum_{i=1}^l [\Delta u'(t_i) + I_i(u(t_i))]v(t_i) - u'(0)v(0) + u'(T)v(T) \\ &= \int_0^T [-u''(t) - \lambda u(t) - f(t, u(t))]v(t)dt - \sum_{i=1}^l [\Delta u'(t_i) + I_i(u(t_i))]v(t_i) - u'(0)v(0). \end{aligned}$$

Thus

$$\int_0^T [-u''(t) - \lambda u(t) - f(t, u(t))]v(t)dt - \sum_{i=1}^l [\Delta u'(t_i) + I_i(u(t_i))]v(t_i) - u'(0)v(0) = 0 \quad (2.7)$$

holds for all $v \in Y$. Without loss of generality, we assume that $v \in C_0^\infty(t_i, t_{i+1})$, $v(t) \equiv 0$, $t \in [0, t_i] \cup [t_{i+1}, T]$, then substituting v into 2.7 we get

$$-u''(t) - \lambda u(t) - f(t, u(t)) = 0, \quad t \in (t_i, t_{i+1})$$

Thus u satisfies the equation in (2.1). So 2.7 becomes

$$-\sum_{i=1}^l |\Delta u'(t_i) + I_i(u(t_i))| v(t_i) - u'(0)v(0) = 0 \quad (2.8)$$

Now we will show that u satisfies impulsive condition in (2.1). If not, without loss of generality, we assume that there exists $i \in 1, 2, \dots, l$ such that

$$\Delta u'(t_i) + I_i(u(t_i)) \neq 0$$

Let $v(t) = \prod_{j=0, j \neq i}^{l+1} (t - t_j)$,

$$\prod_{j=0, j \neq i}^{l+1} (t - t_j) = (t - t_1)(t - t_2) \dots (t - t_l) \dots (t - T)$$

so,

$$\begin{aligned} & -\sum_{i=1}^l [\Delta u'(t_i) + I_i(u(t_i))] v(t_i) - u'(0)v(0) \\ &= -\sum_{i=1}^l [\Delta u'(t_i) + I_i(u(t_i))] \prod_{j=0, j \neq i}^{l+1} (t - t_j) - u'(0) \prod_{j=0, j \neq i}^{l+1} (t_0 - t_j) \end{aligned}$$

by conditions impulsive we have,

$$-\Delta u'(t_1) - I_i(u(t_1)) = I_i(u(t_1)) - I_i(u(t_1)), \quad \forall j \neq i = 0,$$

and

$$-\Delta u'(t_2) - I_i(u(t_2)) = I_i(u(t_2)) - I_i(u(t_2)), \quad \forall j \neq i = 0.$$

And so on,

$$-\Delta u'(t_l) - I_i(u(t_l)) = I_i(u(t_l)) - I_i(u(t_l)), \quad \forall j \neq i = 0.$$

then,

$$\begin{aligned} & -\sum_{i=1}^l \left(\Delta u'(t_i) + I_i(u(t_i)) \right) v(t_i) + u'(0)v(0) \\ &= -\left(\Delta u'(t_i) + I_i(u(t_i)) \right) v(t_i) \neq 0. \end{aligned}$$

which contradicts to (2.8). So u satisfies impulsive condition in (2.1), which yields that u satisfies boundary condition. Therefore, u is a solution of (2.1). $\square$

2.4 Nontrivial weak solution

Theorem 2.2. *Suppose that (H1) (H2) hold. Then for all $\lambda \in \mathbb{R}$, problem (2.1) has at least one non-trivial solution.*

Proof. We will finish the proof by two cases (i) $\lambda < \lambda_1$, (ii) $\lambda \geq \lambda_1$.

(i) $\lambda < \lambda_1$.

Step 1: J satisfies the condition (P.S)

Let (u_m) sequence in H such that $J(u_m)$ is bounded and $\lim_{m \rightarrow \infty} J'(u_m) \rightarrow 0$ Let us show that u_m is bounded by (H2) and definitions of J, J' , Lemma 2.4. Choose $\beta = \frac{1}{\mu}$ we have

$$\begin{aligned}
 M + \beta \|u_m\| &\geq J(u_m) - \frac{1}{\mu} \langle J'(u_m), u_m \rangle \\
 &= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_0^T (u'_m(t))^2 - \lambda u_m^2(t) dt - \int_0^T \left[F(t, u_m(t)) - \frac{1}{\mu} f(t, u_m(t)) u_m(t)\right] dt \\
 &\quad - \left[\sum_{i=1}^l \int_0^{u_m(t_i)} I_i(s) ds - \frac{1}{\mu} \sum_{i=1}^l I_i(u_m(t_i)) u_m(t_i) \right] \\
 &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \theta_1 \|u_m\|^2 + c
 \end{aligned}$$

So (u_m) is bounded in H .

The fact (u_m) is bounded in H means that $u_m \rightharpoonup u$.

Following we will show (u_m) strongly converges to u in H . By the definition of E' , Lemma 2.7, Remark 2.4, one has

$$\begin{aligned}
 0 &\leftarrow \langle J'(u_m) - J'(u), u_m - u \rangle \\
 &= \int_0^T (u'_m(t) - u'(t))^2 - \lambda (u'_m(t) - u'(t))^2 dt \\
 &\quad - \int_0^T [f(t, u_m(t)) - f(t, u(t))] [u_m(t) - u(t)] dt \\
 &\quad - \sum_{i=1}^l [I_i(u_m(t_i)) - I_i(u(t_i))] [u_m(t_i) - u(t_i)] \\
 &\leq \theta_2 \|u_m - u\|^2 - \int_0^T [f(t, u_m(t)) - f(t, u(t))] [u_m(t) - u(t)] dt \\
 &\quad - \sum_{i=1}^l [I_i(u_m(t_i)) - I_i(u(t_i))] [u_m(t_i) - u(t_i)].
 \end{aligned} \tag{2.9}$$

Similar to the proof of Proposition 1.2 in [10], implies (u_m) uniformly converges to u in

$C([0, T])$. So

$$\begin{cases} \int_0^T [f(t, u_m(t)) - f(t, u(t))] [u_m(t) - u(t)] dt \rightarrow 0 \\ \sum_{i=1}^l [I_i(u_m(t_i)) - I_i(u(t_i))] [u_m(t_i) - u(t_i)] dt \rightarrow 0 \end{cases} \quad (2.10)$$

as $n \rightarrow \infty$, $\|u_m - u\| \rightarrow 0$. We say that (u_m) strongly converges to u in H .

Step 2: J satisfies the first geometric condition.

for all $u \in H$

$$J(u) = \frac{1}{2} \int_0^T (u'(t))^2 - \lambda u^2(t) dt - \int_0^T F(t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds$$

using hypothesis (H1), we have

$$\begin{aligned} |F(t, u)| &= \int_0^u |f(t, s)| ds \\ &\leq \int_0^u \varepsilon |s| ds \\ &\leq \frac{1}{2} \varepsilon |u|^2 \\ \int_0^T |F(t, u)| dt &\leq \int_0^T \frac{1}{2} \varepsilon |u|^2 dt \\ &\leq T \frac{1}{2} \varepsilon |u|^2 \end{aligned}$$

and we have

$$\begin{aligned} \int_0^{u(t_i)} |I_i(u)| dt &= \int_0^{u(t_i)} |s| ds \\ &\leq \int_0^u \varepsilon |s| ds \\ &\leq \frac{1}{2} \varepsilon |u|^2 \\ \sum_{i=1}^l \int_0^{u(t_i)} |I_i(u)| dt &\leq \sum_{i=1}^l \left(\frac{1}{2} \varepsilon |u|^2 \right) \end{aligned}$$

for all $t \in [0, T]$. Consequently, by Lemma 2.6, for $\|u\| \leq \frac{\delta}{\sqrt{T}}$

$$\int_0^T F(t, u(t)) dt + \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds = \frac{1}{2} \varepsilon |u|^2 (T + 1) \leq \frac{1}{2} \varepsilon T (T + 1) \|u\|^2 \quad (2.11)$$

Since ε is arbitrary, 3.8 shows

$$\int_0^T F(t, u(t)) dt + \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds = o(\|u\|^2)$$

as $u \rightarrow 0$.

By Lemma 2.4 we have,

$$J(u) = \frac{1}{2} \int_0^T (u'(t))^2 - \lambda u^2(t) dt - \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds \geq \frac{\theta_1}{2} \|u\|^2 + o(\|u\|)^2 \quad (2.12)$$

as $u \rightarrow 0$. So (C1) holds.

Step 3: J satisfies the second geometric condition.

we choose $e(t) = \varphi_1(t) \in H, \gamma \in \mathbb{R}$.

$$J(\gamma e) = \frac{\gamma^2}{2} \int_0^T [(e'(t))^2 - \lambda e^2(t)] dt - \int_0^T F(t, \gamma e(t)) dt - \sum_{i=1}^l \int_0^{\gamma e(t_i)} I_i(s) ds$$

Using hypothesis (H2), we have

(H2) There exist constants $\mu > 2$ and $r \geq 0$ such that for $|\epsilon| \geq r$,

$$0 < \mu F(t, \gamma e) \leq \gamma e f(t, \gamma e), \quad 0 < \mu \int_0^{\gamma e} I_i(s) ds \leq \gamma e I_i(\gamma e), \quad i = 1, 2, \dots, l$$

The condition (H2) implies, there exist $b_1, b_2, c_i, d_i > 0, i = 1, 2, \dots, l$ such that

$$\begin{aligned} F(t, \gamma e) &\geq b_1 |\gamma e|^\mu - b_2, \\ \int_0^T F(t, \gamma e) dt &\geq \int_0^T (b_1 |\gamma|^\mu |e(t)|^\mu - b_2) dt \end{aligned}$$

and

$$\begin{aligned} \int_0^{\gamma e} I_i(s) ds &\geq c_i |\gamma e|^\mu - d_i \\ \sum_{i=1}^l \int_0^{\gamma e} I_i(s) ds &\geq \sum_{i=1}^l (c_i |\gamma|^\mu - |e(t_i)|^\mu - d_i) \end{aligned}$$

So

$$\begin{aligned} J(\gamma e) &= \frac{\gamma^2}{2} \int_0^T [(e'(t))^2 - \lambda e^2(t)] dt - \int_0^T F(t, \gamma e(t)) dt - \sum_{i=1}^l \int_0^{\gamma e(t_i)} I_i(s) ds \\ &\leq \frac{\theta_2 \gamma^2}{2} \|e\|^2 - \int_0^T (b_1 |\gamma|^\mu |e(t)|^\mu - b_2) dt - \sum_{i=1}^l (c_i |\gamma|^\mu - |e(t_i)|^\mu - d_i) \end{aligned} \quad (2.13)$$

By computation,

$$\|e\|^2 = \|\varphi_1\|^2 = 1 \quad (2.14)$$

and by Holder inequality, we have

$$\int_0^T |e(t)|^\mu dt \geq \left[\int_0^T e^2(t) dt T^{\frac{2-\mu}{\mu}} \right]^{\frac{\mu}{2}} = \left(\frac{1}{\lambda_1} \right)^{\frac{\mu}{2}} T^{\frac{2-\mu}{2}} = \left(\frac{2T}{\pi} \right)^{\mu} T^{\frac{2-\mu}{2}} \quad (2.15)$$

Substituting (2.14) and (2.15) into (2.13), we have,

$$J(\gamma e) \leq \frac{\theta_2 \gamma^2}{2} - b_1 |\gamma|^\mu \left(\frac{2T}{\pi} \right)^\mu T^{\frac{2-\mu}{2}} + b_2 T + \sum_{i=1}^l d_i \rightarrow -\infty$$

as $|\gamma| \rightarrow -\infty$. Hence (C2) holds.

(ii) $\lambda \geq \lambda_1$. $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k > 0$. We will show that J satisfies the hypotheses of Theorem 2.1. Clearly $J \in C^1(Y, R)$. Set $V = \text{span} \{\varphi_1, \varphi_2, \dots, \varphi_k\}$ and $X = V^\perp$, where φ_j 's are the eigenfunctions defined in (2.3). For $u \in X, u = \sum_{i=k+1}^\infty a_i \varphi_i$

Step 1: J satisfies the condition (P.S)

Remark 2.6. $L^\mu(0, T)$ embeds continuously in $L^2(0, T)$ By Holder inequality,

$$\|u\|_{L^\mu}^\mu \geq \left(\|u\|_{L^2}^2 T^{\frac{2-\mu}{2}} \right)^{\frac{\mu}{2}} = \|u\|_{L^2}^\mu T^{\frac{2-\mu}{2}}. \quad (2.16)$$

Let (u_m) sequence in H such that $J(u_m)$ is bounded and $\lim_{m \rightarrow \infty} J'(u_m) \rightarrow 0$. Let us show that u_m is bounded by (H2) and definitions of J, J' . Choose $\beta \in \left(\frac{1}{\mu}, \frac{1}{2} \right)$ we have

$$\begin{aligned} M + \beta \|u_m\| &\geq J(u_m) - \beta \langle J'(u_m), u_m \rangle \\ &= \left(\frac{1}{2} - \beta \right) \int_0^T (u_m'(t))^2 - \lambda u_m^2(t) dt \\ &\quad - \int_0^T [F(t, u_m(t)) - \beta f(t, u_m(t)) u_m(t)] dt \\ &\quad - \left[\sum_{i=1}^l \int_0^{u_m(t_i)} I_i(s) ds - \beta \sum_{i=1}^l I_i(u_m(t_i)) u_m(t_i) \right] \\ &\geq \left(\frac{1}{2} - \beta \right) \|u_m\|^2 - \left(\frac{1}{2} - \beta \right) \lambda \|u_m\|_{L^2}^2 + (\beta\mu - 1) \int_0^T F(t, u_m(t)) dt \\ &\quad + (\beta\mu - 1) \sum_{i=1}^l \int_0^{u_m(t_i)} I_i(s) ds \\ &\geq \left(\frac{1}{2} - \beta \right) \|u_m\|^2 - \left(\frac{1}{2} - \beta \right) \lambda \|u_m\|_{L^2}^2 \\ &\quad + (\beta\mu - 1) \int_0^T [b_1 |u_m(t)|^\mu - b_2] dt + (\beta\mu - 1) \sum_{i=1}^l [c_i |u_m(t_i)|^\mu - d_i] \\ &\geq \left(\frac{1}{2} - \beta \right) \|u_m\|^2 - \left(\frac{1}{2} - \beta \right) \lambda \|u_m\|_{L^2}^2 \\ &\quad + (\beta\mu - 1) [b_1 T^{\frac{2-\mu}{2}} \|u\|_{L^2}^\mu - b_2 T] + (\beta\mu - 1) \sum_{i=1}^l [c_i \|u\|_{L^2}^\mu - d_i] \\ &\geq \left(\frac{1}{2} - \beta \right) \|u_m\|^2 + M \|u_m\|_{L^2}^\mu \end{aligned} \quad (2.17)$$

for $M > 0$. Therefore, (u_m) is bounded in H and $L^2([0, T])$. The fact (u_m) is bounded in H means that $u_m \rightharpoonup u$. Following we will show (u_m) strongly converges to u . By the definition of J' , one has

$$\begin{aligned}
0 &\longleftarrow \langle J'(u_m) - J'(u), u_m - u \rangle \\
&= \int_0^T u'_m(t) - u'(t))^2 - \lambda(u_m(t) - u(t))^2 dt \\
&\quad - \int_0^T [f(t, u_m(t)) - f(t, u(t))] [u_m(t) - u(t)] dt \\
&\quad - \sum_{i=1}^l [I_i(u_m(t_i)) - I_i(u(t_i))] [u_m(t_i) - u(t_i)] \\
&= \|u_m - u\|^2 - \lambda \int_0^T (u_m(t) - u(t))^2 dt \\
&\quad - \int_0^T [f(t, u_m(t)) - f(t, u(t))] [u_m(t) - u(t)] dt \\
&\quad - \sum_{i=1}^l [I_i(u_m(t_i)) - I_i(u(t_i))] [u_m(t_i) - u(t_i)]
\end{aligned} \tag{2.18}$$

Similar to the proof of Proposition 1.2 in [10], $u_m \rightharpoonup u$ implies (u_m) uniformly converges to u in $C([0, T])$. So

$$\begin{cases} \int_0^T (u_m(t) - u(t))^2 dt \longrightarrow 0 \\ \int_0^T [f(t, u_m(t)) - f(t, u(t))] [u_m(t) - u(t)] dt \longrightarrow 0 \\ \sum_{i=1}^l [I_i(u_m(t_i)) - I_i(u(t_i))] [u_m(t_i) - u(t_i)] \longrightarrow 0 \end{cases} \tag{2.19}$$

as $n \longrightarrow \infty$. $\|u_m - u\| \longrightarrow 0$. We say that (u_m) strongly converges to u in H .

Step 2: J satisfies the first hypotheses of Theorem 2.1.

By $(H1)$, we have for $u \in X$

$$\int_0^T F(t, u(t)) dt + \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds = o(\|u\|^2)$$

For $u \in X$ and $u = \sum_{i=k+1}^{\infty} a_i \varphi_i$

$$\begin{aligned}
\int_0^T (u'(t))^2 - \lambda u^2(t) dt &= \int_0^T \left(\sum_{i=k+1}^{\infty} a_i \varphi'_i(t) \right)^2 - \lambda \left(\sum_{i=k+1}^{\infty} a_i \varphi_i(t) \right)^2 dt \\
&= \sum_{i=k+1}^{\infty} a_i^2 \int_0^T (\varphi'_i(t))^2 dt - \lambda \sum_{i=k+1}^{\infty} a_i^2 \int_0^T \varphi_i^2(t) dt \\
&= \sum_{i=k+1}^{\infty} a_i^2 \left(1 - \frac{\lambda}{\lambda_1} \right) \int_0^T (\varphi'_i(t))^2 dt \\
&= \sum_{i=k+1}^{\infty} a_i^2 \left(1 - \frac{\lambda}{\lambda_i} \right) \|\varphi_i\|^2 \geq \left(1 - \frac{\lambda}{\lambda_{k+1}} \right) \|u\|^2
\end{aligned} \tag{2.20}$$

so,

$$J(u) \geq \left(1 - \frac{\lambda}{\lambda_{k+1}} \right) \|u\|^2 + o(\|u\|^2),$$

as $u \rightarrow 0$. So J satisfies (C3) in Theorem 2.1

Step 3: J satisfies the second hypotheses of Theorem 2.1.

For $u = \sigma + re$, where $\sigma \in V, e = \varphi_{k+1}$, we have

$$\begin{aligned}
J(u) &= \frac{1}{2} \int_0^T (u'(t))^2 - \lambda u^2(t) dt \\
&\quad - \int_0^T F(t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds \\
&= \frac{1}{2} \int_0^T (\sigma'(t))^2 - \lambda \sigma^2(t) dt + \frac{r^2}{2} \int_0^T (e'(t))^2 - \lambda e^2(t) dt \\
&\quad - \int_0^T F(t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds
\end{aligned} \tag{2.21}$$

Since $\sigma \in V$, we assume $\sigma = \sum_{i=1}^k a_i \varphi_i$, where $\varphi_i (i = 1, 2, \dots, k)$ is defined in 2.3. Thus

$$\int_0^T (\sigma'(t))^2 - \lambda \sigma^2(t) dt \leq \left(1 - \frac{\lambda}{\lambda_k}\right) \sum_{i=1}^k a_i^2 = \left(1 - \frac{\lambda}{\lambda_k}\right) \|\sigma\|^2 \leq 0. \tag{2.22}$$

$$r^2 \int_0^T (e'(t))^2 - \lambda e^2(t) dt = r^2 \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \leq r^2. \tag{2.23}$$

Substituting (2.22) (2.23) into (2.21), noticing Remark 2.1, we have

$$J(u) \leq \frac{r^2}{2} - \int_0^T (b_1 |u(t)|^\mu - b_2) dt - \sum_{i=1}^l (c_i |u(t_i)|^\mu - d_i) \tag{2.24}$$

For any finite dimensional subspace $V_1 \subset Y$, the norms $\|\cdot\|$ and $\|\cdot\|_{V_1}$ are equivalent. So there exists $\bar{c} > 0$ satisfying

$$\|u\| \leq \bar{c} \|u\|_{L^\mu} \tag{2.25}$$

for $u \in V_1$. Thus

$$\int_0^T b_1 |u(t)|^\mu dt \geq b_1 \bar{c}^{-\mu} \|u\|^\mu = b_1 \bar{c}^{-\mu} (\|\sigma\|^2 + \|re\|^2)^{\frac{\mu}{2}} \geq b_1 \bar{c}^{-\mu} (\|\sigma\|^\mu + r^\mu). \tag{2.26}$$

By (2.24) (2.25) (2.26), one has

$$J(u) \leq \frac{r^2}{2} - b_1 \bar{c}^{-\mu} (\|\sigma\|^\mu + r^\mu) + b_2 T + \sum_{i=1}^l d_i \tag{2.27}$$

Let

$$g_1(r) = \frac{r^2}{2} - b_1 \bar{c}^{-\mu} r^\mu + b_2 T + \sum_{i=1}^l d_i, g_2(r) = -b_1 \bar{c}^{-\mu} r^\mu.$$

Then $\lim_{r \rightarrow +\infty} g_1(r) = \lim_{r \rightarrow -\infty} g_2(r) = -\infty$, and $g_1(r), g_2(r)$ are bounded from above.

Thus there exists $R_1 > 0$ such that $J(u) \leq 0$ for all $u \in \partial Q$, where $Q = (\bar{B}_{R_1} \cap V) \oplus \{re : 0 < r < R_1\}$. So J satisfies (C4) in Theorem 2.1 $\square$

Example 2.1. Let $T > 0, t_i \in (0, T), a, b_i \in C([0, T], \mathbb{R}^+), i = 1, 2, \dots, l$. Consider mixed boundary value problem with impulse

$$\begin{cases} -u''(t) = \lambda u(t) + a(t)u^3(t), t \neq t_i, t \in [0, T] \\ -\Delta u'(t_i) = b_i(t)u^3(t_i), i = 1, 2, \dots, l, \\ u'(0) = 0, u(T) = 0. \end{cases} \quad (2.28)$$

Compared with (2.1), $f(t, u) = a(t)u^3, I_i(u) = b_i(t)u^3$.

The conditions (H1) and (H2) are satisfied. Applying Theorem 3.3, problem 3.10 has at least one nontrivial solution for $\lambda \in \mathbb{R}$.

EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO AN IMPULSIVE SECOND-ORDER MIXED BOUNDARY VALUE PROBLEM

In this chapter, we discuss the existence and multiplicity of solutions for the same impulsive second-order equation (2.1) that was published in the article [1].

3.1 Introduction

Consider the following impulsive problem which is studied in the previous chapter:

$$\begin{cases} -u''(t) = \lambda u(t) + f(t, u(t)), & t \neq t_i, \ t \in [0, T], \\ -\Delta u'(t_i) = I_i(u(t_i)), & i = 1, 2, \dots, l, \\ u'(0) = 0, \ u(T) = 0, \end{cases} \quad (3.1)$$

where $\lambda \in R$, $0 = t_0 < t_1 < \dots < t_l < t_{l+1} = T$, $\Delta u'(t_i) = u'(t_i^+) - u'(t_i^-)$, (where $u'(t_i^+)$ (resp. $u'(t_i^-)$)) denotes the right limit (resp. left limit) of $u'(t)$ at $t = t_i$, $u'(t_i^-) = u'(t_i)$, $I_i \in C(R, R)$, $i = 1, 2, \dots, l$, $f \in C([0, T] \times R, R)$.

In this chapter, we assume that the following assumptions are satisfied:

(H1) $f(x, t) = o(|x|)$, $I_i(x) = o(|x|)$ as $|x| \rightarrow 0$, $i = 1, 2, \dots, l$

(H2) There exist constants $\mu > 2$ and $r \geq 0$ such that for $|\epsilon| \geq r$,

$$0 < \mu F(t, \xi) \leq \xi f(t, \xi), \quad 0 < \mu \int_0^\xi I_i(s) ds \leq \xi I_i(\xi), \quad i = 1, 2, \dots, l$$

(H3) $f(t, u)$, $I_i(u)$ are odd in u .

(H4) There exist $x_1 > 0$, $r > \lambda_k$, λ_k is the k -th eigenvalue of 2.4 such that

$$rx_1 + f(t, x_1) \leq 0, \ t \in [0, T], \ I_i(x_1) \leq 0, \ i = 1, 2, \dots, l$$

3.2 Some necessary theorems and lemmas

Theorem 3.1. (*Clark's Theorem.*) Let J be a real Banach space, $I \in C^1(E, \mathbb{R})$ with I even, bounded from below, and satisfying (PS). Suppose $I(0) = 0$, there is a set $K \subset E$ such that K is homeomorphic to S^{j-1} by an odd map, and $\sup_K I < 0$. Then I possesses at least j distinct pairs of critical points.

Theorem 3.2. Let E be an infinite dimensional Banach space and let $J \in C^1(E, \mathbb{R})$ be even, satisfy (PS), and $I(0) = 0$. If $E = V \oplus X$, where V is finite dimensional, and I satisfies

(C5) there are constant $\rho, \alpha > 0$ such that $I_{\partial B_\rho \cap X} \geq \alpha$, and

(C6) for each finite dimensional subspace $\tilde{E} \subset E$, there is an $R = R(\tilde{E})$ such that $I \leq 0$ on $\tilde{E} \setminus B_{R(\tilde{E})}$,

then J possesses an unbounded sequence of critical values.

3.3 Existence results and multiplicity

3.3.1 Existence and multiplicity of distinct solutions

Theorem 3.3. For $k \in \mathbb{Z}$. Assume that (H1) (H3) (H4) hold. Then for $\lambda \in (\lambda_k, r]$, problem 3.1 has at least k distinct pairs of solutions

Proof. We consider the problem:

$$\begin{cases} -u''(t) = q(\lambda, t, u(t)), & t \in [0, T], \quad t \neq t_i, \\ -\Delta u'(t_i) = J_i(u(t_i)), & i = 1, 2, \dots, l, \\ u'(0) = 0, \quad u(T) = 0. \end{cases} \quad (3.2)$$

such that,

$$q(\lambda, t, x) = \begin{cases} \lambda x + f(t, x), & x \in [-x_1, x_1] \\ \lambda x_1 + f(t, x_1), & x \in [x_1, \infty), \\ -\lambda x_1 + f(t, -x_1), & x \in (-\infty, -x_1], \end{cases} \quad J_i(x) = \begin{cases} I_i(x), & x \in [-x_1, x_1], \\ I_i(x_1), & x \in [x_1, \infty), \\ I_i(-x_1), & x \in (-\infty, -x_1]. \end{cases}$$

Remark 3.1. We claim any solution of 3.2 is a solution of 3.1, that is, any solution u of 3.2 satisfies $u(t) \in [-x_1, x_1], t \in [0, T]$.

Proof. Using hypothesis $H4$ and $H3$ we have

let $\beta_1 = \{t \in (a_1, b_1) \subseteq [0, T] : u(t) > x_1\}$. By the definitions of $q(\lambda, t, x)$ and $J_i(x)$, 3.2 is reduced to

$$\begin{cases} -u''(t) = q(\lambda, t, x_1) = \lambda x_1 + f(t, x_1) \leq r x_1 + f(t, x_1) \leq 0, & t \in (a_1, b_1), \quad t \neq t_i, \\ -\Delta u'(t_i) = J_i(u(t_i)) = I_i(x_1) \leq 0, & i = 1, 2, \dots, l, \\ u(a_1) = u(b_1) = x_1. \end{cases} \quad (3.3)$$

The solution $u(t)$ of 3.3 satisfies $u(t) \leq x_1, t \in (a_1, b_1)$. So $\beta_1 = \emptyset$ and $u(t) \leq x_1$.

Let $\beta_2 = \{t \in (a_2, b_2) \subseteq [0, T] : u(t) < -x_1\}$. By the definitions of $q(\lambda, t, x)$ and $J_i(x)$, 3.2 is reduced to

$$\begin{cases} -u''(t) = q(\lambda, t, -x_1) = -\lambda x_1 + f(t, -x_1) \geq -r x_1 - f(t, x_1) \geq 0, & t \in (a_2, b_2), \quad t \neq t_i, \\ -\Delta u'(t_i) = J_i(u(t_i)) = -I_i(x_1) \geq 0, & i = 1, 2, \dots, l, \\ u(a_2) = u(b_2) = -x_1. \end{cases} \quad (3.4)$$

The solution $u(t)$ of 3.4 satisfies $u(t) \geq -x_1, t \in (a_2, b_2)$. So $\beta_2 = \emptyset$ and $u(t) \geq -x_1$.

Therefore, any solution of 3.2 is a solution of 3.1. Hence to prove Theorem 3.9, it suffices to produce at least k distinct pairs of critical points of

$$J(u) = \frac{1}{2} \int_0^T |u'(t)|^2 dt - \int_0^T Q(\lambda, t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} J_i(s) ds,$$

where $Q(\lambda, t, x) = \int_0^x q(\lambda, t, s) ds$ □

Step 1: J satisfies the condition $(p.s)$ (see 3.3)

Step 2: K is homeomorphic to S^{j-1}

Now set $K = \left\{ \sum_{i=1}^k a_i \varphi_i : \sum_{i=1}^k a_i^2 = a^2 \right\}$, where φ_i is defined in 2.3.

K is homeomorphic to S^{k-1} by an odd map for any $a > 0$.

$$\langle k, k \rangle = \int_0^T (\sum_{i=1}^k a_i \varphi_i)^2 dt = \sum_{i=1}^k a_i^2 \int_0^T \varphi_i^2 dt = \sum_{i=1}^k a_i^2 = a^2 \in K$$

Step 3: We claim $J(u) < 0, \forall u \in K$ if a is sufficiently small.

For any $u \in K, u = \sum_{i=1}^k a_i \varphi_i$. By (H1)

$$\begin{aligned} J(u) &= \frac{1}{2} \int_0^T |u'(t)|^2 dt - \int_0^T Q(\lambda, t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} J_i(s) ds \\ &= \frac{1}{2} \int_0^T \left[\left(\sum_{i=1}^k a_i \varphi_i(t) \right)^2 \right] dt - \int_0^T Q(\lambda, t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} J_i(s) ds, \\ &= \frac{1}{2} \sum_{i=1}^k a_i^2 \left(1 - \frac{\lambda}{\lambda_i} \right) - \int_0^T F(t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} J_i(s) ds \\ &\leq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_k} \right) a^2 + o(a^2) + o(a^2) \end{aligned}$$

for small $a > 0$. Since $\lambda \in (\lambda_k, r]$, $J(u) < 0$.

Step 4: We show that J is even.

We have $J \in C^1(H, \mathbb{R})$ is even since $f(t, u)$ and $I_i(u)$ are odd with respect to u , and $J(0) = 0$.

Let $t \in [0, T]$ $u \in \mathbb{R}$, we pose $s = -v$, then we have $ds = -dv$ and,

$$\begin{aligned} Q(t, -u) &= \int_0^{-u} q(t, s) ds \\ &= - \int_0^{-u} q(t, -v) dv \\ &= \int_0^u q(t, v) dv = Q(t, u), \quad (\text{because } q \text{ is odd}). \end{aligned}$$

Also, we have $L(u) = \int_0^u I_j(s) ds$, we pose $s = -v$, then we have $ds = -dv$ and,

$$\begin{aligned} L(-u) &= \int_0^{-u} J_i(s) ds \\ &= - \int_0^{-u} J_i(-v) dv \\ &= \int_0^u J_i(v) dv = L(u), \quad (\text{because } I \text{ is odd}). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} J(-u) &= \frac{1}{2} \| -u \|^2 - \int_0^T Q(t, -u(t)) dt - \sum_{i=1}^l \int_0^{-u(t_i)} J_i(t) dt \\ &= \frac{1}{2} \| u \|^2 - \int_0^T Q(t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} J_i(t) dt \\ &= J(u). \end{aligned}$$

That is, J is an even functional. □

Therefore, according to theorem 3.1 J has at least k distinct pairs of solutions, that is to say that the system (3.1) admits at least k distinct pairs of solutions.

Example 3.1. let $T = \frac{\pi}{2}$, $t_i \in (0, \frac{\pi}{2})$, $i = 1, 2, \dots, l$. consider mixed boundary value problem with impulse

$$\begin{cases} -u''(t) = \lambda u(t) - (t+1)u^3(t), t \neq t_i, t \in [0, T] \\ -\Delta u'(t_i) = -10u^3(t_i), i = 1, 2, \dots, l, \\ u'(0) = 0, u(T) = 0. \end{cases} \quad (3.5)$$

Compared with 3.1, $f(t, u) = -(t+1)u^3$, $I_i(u) = -10u^3$. Clearly (H1) (H3) are satisfied. Let $x_1 = 10$, then (H4) is satisfied with $r = 100$. Applying Theorem 3.4, for $\lambda \in (\lambda_k, 100] = ((2k-1)^2, 100]$, $k = 1, 2, 3, 4, 5$ problem 3.5 has at least k distinct pairs of solutions

3.3.2 Existence of an infinity of solutions

Theorem 3.4. *Suppose that (H1) (H2) (H3) hold. Then for $\lambda < \lambda_1$, problem 3.1 possesses infinitely many solutions.*

Proof. Step 1: J satisfies the condition (p.s)

Let (u_m) sequence in H such that $J(u_m)$ is bounded and $\lim_{m \rightarrow \infty} J'(u_m) \rightarrow 0$ Let us show that u_m is bounded by definitions of J, J' , Lemma 2.4 . Choose $\beta = \frac{1}{\mu}$ we have

$$\begin{aligned}
M + \beta \|u_m\| &\geq J(u_m) - \frac{1}{\mu} \langle J'(u_m), u_m \rangle \\
&= \frac{1}{2} \int_0^T (u'_m(t))^2 - \lambda u_m^2(t) dt - \int_0^T F(t, u_m(t)) dt - \sum_{i=1}^l \int_0^{u_m(t_i)} I_i(s) ds \\
&\quad - \frac{1}{\mu} \int_0^T (u'_m(t))^2 - \lambda u_m^2(t) dt + \frac{1}{\mu} \int_0^T f(t, u_m(t)) u_m(t) dt + \frac{1}{\mu} \sum_{i=1}^l I_i(u_m(t_i)) u_m(t_i) \\
&= \left(\frac{1}{2} - \frac{1}{\mu} \right) \int_0^T (u'_m(t))^2 - \lambda u_m^2(t) dt - \int_0^T \left[F(t, u_m(t)) - \frac{1}{\mu} f(t, u_m(t)) u_m(t) \right] dt \\
&\quad - \sum_{i=1}^l \left[\int_0^{u_m(t_i)} I_i(s) ds - \frac{1}{\mu} I_i(u_m(t_i)) u_m(t_i) \right]
\end{aligned}$$

Using hypothesis H2 we have

$$\begin{aligned}
J(u_m) - \frac{1}{\mu} \langle J'(u_m), u_m \rangle &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \theta_1 \|u_m\|^2 - \int_0^T \max_{u_n(x) \in [-r, r]} \left| \frac{1}{\mu} f(t, u_m(t)) u_m(t) - F(t, u_m(t)) \right| dt \\
&\quad - \sum_{i=1}^l \max_{u_n(x) \in [-r, r]} \left| \frac{1}{\mu} I_i(u_m(t_i)) u_m(t_i) - \int_0^{u_m(t_i)} I_i(s) ds \right| \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \theta_1 \|u_m\|^2 - c
\end{aligned}$$

such that

$$\begin{aligned}
c &= \int_0^T \max_{u_n(x) \in [-r, r]} \left| \frac{1}{\mu} f(t, u_m(t)) u_m(t) - F(t, u_m(t)) \right| dt \\
&\quad - \sum_{i=1}^l \max_{u_n(x) \in [-r, r]} \left| \frac{1}{\mu} I_i(u_m(t_i)) u_m(t_i) - \int_0^{u_m(t_i)} I_i(s) ds \right|
\end{aligned}$$

So (u_m) is bounded in H .

The fact (u_m) is bounded in H means that $u_m \rightarrow u$.

Following we will show (u_m) strongly converges to u in H . By the definition of J' , Lemma 2.7, Remark 2.4, one has

$$\begin{aligned}
0 &\leftarrow \langle J'(u_m) - J'(u), u_m - u \rangle \\
&= \int_0^T (u'_m(t) - u'(t))^2 - \lambda(u'_m(t) - u'(t))^2 dt \\
&\quad - \int_0^T [f(t, u_m(t)) - f(t, u(t))][u_m(t) - u(t)] dt \\
&\quad - \sum_{i=1}^l [I_i(u_m(t_i)) - I_i(u(t_i))][u_m(t_i) - u(t_i)] \\
&\leq \theta_2 \|u_m - u\|^2 - \int_0^T [f(t, u_m(t)) - f(t, u(t))][u_m(t) - u(t)] dt \\
&\quad - \sum_{i=1}^l [I_i(u_m(t_i)) - I_i(u(t_i))][u_m(t_i) - u(t_i)].
\end{aligned} \tag{3.6}$$

Similar to the proof of Proposition 1.2 in [10], implies (u_m) uniformly converges to u in $C([0, T])$. the injection of H_0^1 into $C(0, T)$ with its natural norm $\|\cdot\|_\infty$, is continuous. Since $u_m \rightharpoonup u$ in H_0^1 it follows that $u_m \rightharpoonup u$ in $C(0, T)$. By the Banach-Steinhaus theorem, (u_m) is bounded in H_0^1 and, hence, in $C(0, T)$. Moreover, the sequence (u_m) is equi-uniformly continuous since, for $0 \leq s \leq t \leq T$, we have

$$\begin{aligned}
|u_m(t) - u_m(s)| &\leq \int_s^t u'_k(\tau) d\tau \leq (t-s)^{\frac{1}{q}} \left(\int_s^t |u'_m(\tau)|^p d\tau \right)^{\frac{1}{p}} \\
&\leq (t-s)^{\frac{1}{q}} \|u_m\|_{H_0^1} \leq c(t-s)^{\frac{1}{q}}
\end{aligned}$$

So

$$\begin{cases} \int_0^T [f(t, u_m(t)) - f(t, u(t))][u_m(t) - u(t)] dt \rightarrow 0 \\ \sum_{i=1}^l [I_i(u_m(t_i)) - I_i(u(t_i))][u_m(t_i) - u(t_i)] dt \rightarrow 0 \end{cases} \tag{3.7}$$

as $n \rightarrow \infty$, $\|u_m - u\| \rightarrow 0$. we say that (u_m) strongly converges to u in H .

Step 2: J satisfies the first hypotheses of Theorem 3.2.

for all $u \in H$

$$J(u) = \frac{1}{2} \int_0^T (u'(t))^2 - \lambda u^2(t) dt - \int_0^T F(t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds$$

using hypothesis (H1), we have

$$\begin{aligned}
|F(t, u)| &= \int_0^u |f(t, s)| ds \\
&\leq \int_0^u \varepsilon |s| ds \\
&\leq \frac{1}{2} \varepsilon |u|^2 \\
\int_0^T |F(t, u)| dt &\leq \int_0^T \frac{1}{2} \varepsilon |u|^2 dt \\
&\leq T \frac{1}{2} \varepsilon |u|^2
\end{aligned}$$

and we have

$$\begin{aligned}
\int_0^{u(t_i)} |I_i(u)| dt &= \int_0^{u(t_i)} |s| ds \\
&\leq \int_0^u \varepsilon |s| ds \\
&\leq \frac{1}{2} \varepsilon |u|^2 \\
\sum_{i=1}^l \int_0^{u(t_i)} |I_i(u)| dt &\leq \sum_{i=1}^l \left(\frac{1}{2} \varepsilon |u|^2 \right)
\end{aligned}$$

for all $t \in [0, T]$. Consequently, by Lemma 2.6, for $\|u\| \leq \frac{\delta}{\sqrt{T}}$

$$\int_0^T F(t, u(t)) dt + \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds = \frac{1}{2} \varepsilon |u|^2 (T+1) \leq \frac{1}{2} \varepsilon T (T+1) \|u\|^2 \quad (3.8)$$

Since ε is arbitrary, 3.8 shows

$$\int_0^T F(t, u(t)) dt + \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds = o(\|u\|)^2$$

as $u \rightarrow 0$.

By Lemma 2.4 we have ,

$$J(u) = \frac{1}{2} \int_0^T (u'(t))^2 - \lambda u^2(t) dt - \int_0^T F(t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds \geq \frac{\theta_1}{2} \|u\|^2 + o(\|u\|)^2 \quad (3.9)$$

as $u \rightarrow 0$. So (C5) holds.

Step 3: J satisfies the second hypotheses of Theorem 3.2 .

set $\tilde{E} = \text{span} \{\varphi_1, \varphi_2, \dots, \varphi_k\}$. For any $u \in \tilde{E}$, then $u = \sum_{i=1}^k a_i \varphi_i$ for all $u \in H$

$$J(u) = \frac{1}{2} \int_0^T (u'(t))^2 - \lambda u^2(t) dt - \int_0^T F(t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds$$

using hypothesis (H2), there exist $b_1, b_2, c_i, d_i > 0, i = 1, 2, \dots, l$ such that

$$|F(t, u)| \geq b_1|u|^\mu - b_2, \quad \int_0^u I_i(s)ds \geq c_i|u|^\mu - d_i, \quad t \in [0, T]$$

By Lemma 2.4, and remark 2.4,

$$\begin{aligned} J(u) &= \frac{1}{2} \int_0^T (u'(t))^2 - \lambda u^2(t) dt - \int_0^T F(t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} I_i(s) ds \\ &\leq \frac{1}{2} \theta_2 \|u\|^2 - \int_0^T (b_1|u(t)|^\mu - b_2) dt - \sum_{i=1}^l (c_i|u(t_i)|^\mu - d_i) \\ &\leq \frac{1}{2} \theta_2 \|u\|^2 - b_1 \int_0^T |u(t)|^\mu dt + b_2 T - \sum_{i=1}^l (c_i|u(t_i)|^\mu - d_i) \\ &\leq \frac{1}{2} \theta_2 \|u\|^2 - b_1 \|u(t)\|_{L^\mu}^\mu + b_2 T - \sum_{i=1}^l (c_i|u(t_i)|^\mu - d_i) \end{aligned}$$

By remark 2.16 $L^\mu(0, T)$ embeds continuously in $L^2(0, T)$

$$\begin{aligned} J(u) &\leq \frac{1}{2} \theta_2 \|u\|^2 - b_1 T^{\frac{2-\mu}{2}} \|u\|_{L^2}^\mu - \sum_{i=1}^l (c_i|u(t_i)|^\mu - d_i) \\ &\leq \frac{1}{2} \theta_2 \|u\|^2 - k_5 \|u\|_{L^2}^\mu + k_6 \end{aligned}$$

for $k_5, k_6 > 0$. So $J(u) \rightarrow -\infty$ as $\|u\| \rightarrow +\infty$. That is, there exists $R > 0$ such that $J(u) < 0$ for $u \in \tilde{E} \setminus B_{R(\tilde{E})}$

Step 4: We show that J is even.

We have $J \in C^1(H, \mathbb{R})$ is even since $f(t, u)$ and $I_i(u)$ are odd with respect to u , and $J(0) = 0$.

Let $t \in [0, T]$ $u \in \mathbb{R}$, we pose $s = -v$, then we have $ds = -dv$ and,

$$\begin{aligned} F(t, -u) &= \int_0^{-u} f(t, s) ds \\ &= - \int_0^{-u} f(t, -v) dv \\ &= \int_0^u f(t, v) dv = F(t, u), \quad (\text{because } f \text{ is odd}). \end{aligned}$$

Also, we have $L(u) = \int_0^u I_i(s) ds$, we pose $s = -v$, then we have $ds = -dv$ and,

$$\begin{aligned} L(-u) &= \int_0^{-u} I_i(s) ds \\ &= - \int_0^{-u} I_i(-v) dv \\ &= \int_0^u I_i(v) dv = L(u), \quad (\text{because } I \text{ is odd}). \end{aligned}$$

Furthermore, we have

$$\begin{aligned}
J(-u) &= \frac{1}{2} \| -u \|^2 - \int_0^T F(t, -u(t)) dt - \sum_{i=1}^l \int_0^{-u(t_i)} I_i(t) dt \\
&= \frac{1}{2} \| u \|^2 - \int_0^T F(t, u(t)) dt - \sum_{i=1}^l \int_0^{u(t_i)} I_i(t) dt \\
&= J(u).
\end{aligned}$$

That is, J is an even functional. Therefore, according to theorem 3.2 J has an infinity of critical points, that is to say that the system (3.1) admits an infinity of solutions. $\square$

Remark 3.2. Study of the existence and multiplicity of solutions using the mountain pass lemma for some boundary problems, whether on a bounded domain or not, is in diffusion among many researchers, where let us suppose that the member non-linear is verified the Ambrosetti-Rabinowitz condition in order to prove the Palais-Smale condition (PS), and for more information you can look at the references see the references, [24], [28] [40] and [41]

Example 3.2. Let $T > 0, t_i \in (0, T), a, b_i \in C([0, T], R^+), i = 1, 2, \dots, l$. Consider mixed boundary value problem with impulse

$$\begin{cases} -u''(t) = \lambda u(t) + a(t)u^3(t), t \neq t_i, t \in [0, T] \\ -\Delta u'(t_i) = b_i(t)u^3(t_i), i = 1, 2, \dots, l, \\ u'(0) = 0, u(T) = 0. \end{cases} \quad (3.10)$$

Compared with 3.1, $f(t, u) = a(t)u^3, I_i(u) = b_i(t)u^3$.

The condition (H3) is satisfied. Applying Theorem 3.4, problem 3.10 has infinitely many solutions for $\lambda < \lambda_1 = \frac{\pi^2}{(2T)^2}$

Conclusion

IN this work, we studied some theories of critical points, and we studied a impulsive second order limit problem posed on the interval bounded $[0, T]$ by variational methods, minimization theorem, Mountain Pass Theorem and saddle point theorem .

In the first chapter, we gave some basic tools, which are necessary for this work, such as the theory of critical points, the differentiability of an operator in space of Banach, and Mountain Pass Theorem, and also we give some minimization theorems. We also introduced known compactness conditions under the name Palais-Smale .

In the second chapter, we studied the existence of solutions for the following impulsive second order boundary problem,

$$\begin{cases} -u''(t) = \lambda u(t) + f(t, u(t)), & t \neq t_i, t \in [0, T], \\ -\Delta u'(t_i) = I_i(u(t_i)), & i = 1, 2, \dots, l, \\ u'(0) = 0, u(T) = 0, \end{cases}$$

where $f \in C([0, T] \times \mathbb{R}, \mathbb{R}), I_i \in C(\mathbb{R}, \mathbb{R})$.

Our approach is variational, we used minimization theorems. The solutions are obtained as minima or critical points of a certain functional .

Finally, we studied the existence and multiplicity of solutions for the impulsive boundary problem in the second chapter.

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Résumé

Dans ce mémoire, nous avons étudié un problème aux limites de seconde ordre impulsif sur le domaine borné $[0, T]$, ainsi que la théorie des points critiques, lemme du col de la montagne et théorème de points selle. Notre objectif dans cette étude était d'appliquer la théorie des points critiques, lemme du col et théorème du points selle pour vérifier l'existence et la multiplicité des solutions au problème aux limites du second ordre impulsif suivant :

$$\begin{cases} -u''(t) = \lambda u(t) + f(t, u(t)), t \neq t_i, t \in [0, T], \\ -\Delta u'(t_i) = I_i(u(t_i)), i = 1, 2, \dots, l, \\ u'(0) = 0, \quad u(T) = 0. \end{cases}$$

Mots clés : Point critique. Méthode variationnelle. Existence de solution. Seconde ordre. Impulsif. Lemme de Col. Théorème de point selle. Condition de Palais-Smale. Dépendance de dérivée. Fonctionnelle d'énergie.

Abstract

In this memoiry, we have studied an impulsive second-order boundary value problem on the bounded domain $[0, T]$, as well as the theory of critical points, the mountain pass lemma and the saddle point theorem. Our goal in this study was to apply the critical point theory, neck lemma and saddle point theorem to verify the existence and multiplicity of solutions to the following impulsive second-order boundary value problem :

$$\begin{cases} -u''(t) = \lambda u(t) + f(t, u(t)), t \neq t_i, t \in [0, T], \\ -\Delta u'(t_i) = I_i(u(t_i)), i = 1, 2, \dots, l, \\ u'(0) = 0, \quad u(T) = 0. \end{cases}$$

Key words : Critical point. Variational method. Existence of solutions. Second order. Impulsive differential equation. Mountain pass lemma. Saddle point theorem. Palais-Smale Condition. Derivative dependence. Energy functional.

ملخص

في هذه الذاكرة ، درسنا مسألة قيمة حد من الدرجة الثانية اندفاعية على المجال المحدود $[0, T]$ ، بالإضافة إلى نظرية النقاط الحرجة ، توطئة ممر الجبل ، ونظرية نقطة السرج . كان هدفنا في هذه الدراسة هو تطبيق نظرية النقاط الحرجة ، و توطئة ممر الجبل ، ونظرية نقطة السرج للتحقق من وجود وتعدد الحلول لمسألة قيمة حدية من الدرجة الثانية اندفاعية الآتية:

$$\begin{cases} -u''(t) = \lambda u(t) + f(t, u(t)), t \neq t_i, t \in [0, T], \\ -\Delta u'(t_i) = I_i(u(t_i)), i = 1, 2, \dots, l, \\ u'(0) = 0, \quad u(T) = 0. \end{cases}$$

كلمات مفتاحية: نقطة حرجة. طريقة التغيرات. وجود الحلول. الدرجة الثانية. المعادلة التفاضلية المندفعة. توطئة ممر الجبل. نظرية نقطة السرج. شرط بالي-سمال. اعتماد المشتق. دالة الطاقة.