جامعة محمد بوضياف - المسيلة
Université Mohamed Boudiaf - M'sila

DEMOCRATIC AND POPULAR REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf University of M'sila
Faculty of Mathematics and Informatics
Department of Mathematics

Master dissertation

Field : Mathematics and Informatics

Branch : Mathematics

Speciality : Algebra and Discrete Mathematics

Theme

Aggregation operators on bounded lattices

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University year 2020/2021

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28/06/2021

Acknowledgements

I cannot begin and finish my work without thanking the greatest and the most powerful "Allah" for blessing me to complete this memoir.

I am highly indebted to my supervisors **Prof. dr. Lemnaour Zedam**, who taught and supervised me during these months of research.

I sincerely thank the president of the jury committce **Soheyb Milles**.

I sincerely thank the examiner of the jury committce **Abdelaziz Amroune**.

I would like to express my gratitude to all my colleague in M'sila university who have helped me during my study

I would like to thank our family, especialy my parents and my husband, for their encouragement, patience, and assistance over the years. We are forever indebted to our parents, who have alwayse kept me in the prayers.

Contents

Introduction

Aggregation operators on the unit interval $[0, 1]$ has been widely studied in the fuzzy set theory and some other relatsd topies. The importance of aggregation operators is made apparent by their wide use, not only in pure mathematics (e.g. in the theory of functional equations, measure and integration theory), but also in several applied fields such as operations research, computer and information sciences, economics and social sciences, as well as in other experimental areas of physics and natural sciences[1],[2].

The aim of this dissertation is to study aggregation operators on bounded lattices, including the introduction of some new construction methods for aggregation operators acting on bounded lattices.

The memoire is organized as follows. The first chapter is devoted to preliminaries on orderd sets and lattice In the second chapter, we study the concept of aggregation operators and some main properties on bounded lattices. we give some general methods for construction of aggregation operators on bounded lattices and some interesting types of aggregation operators, like, t-normes and t-conormes.

Chapter 1

Preliminaries on ordered sets and lattice

In this chapter, recall the some basic order-theoretic concepts as binary relation, ordered sets and lattices.

1.1 Binary relations

In this part we define the binary relation and some relations.

Cartisian prouduit

Definition 1.1.

Let E and F be two nonempty sets. We call the cartesian product $E \times F$ any pair(x,y) such that $x \in E$ and $y \in F$.

$$E \times F = \{\ (\text{x,y})\ /\ x \in E,\ y \in F\}$$

Remark 1.1.

The pair (x, y) is an ordered pair i.e., the important order (x,y) $\neq$ (y,x).

Example 1.1.

Let E={2,4,6} and F={x,y,z} tow sets, then their cartisian prouduit is :

$E \times F$ = {(2,x),(2,y),(2,z),(4,x),(4,y),(4,z),(6,x),(6,y),(6,z)}.

1.1.1 Binary relation

Definition 1.2.

Let E and F be the nonempty set. And let $\Re$ be part of the cartisian product, then the binary relation is a part $\Re$ such that any pair $(x, y) \in \Re$ we say that x $\Re$ y or $\Re$ (x,y).

Example 1.2.

$E \times F$ = {(2,x),(2,y),(2,z),(4,x),(4,y),(4,z),(6,x),(6,y),(6,z)}.

$\Re_1$={(2,x),(2,y)}.

$\Re_2$={(4,x),(6,z)}.

$\Re_3=\{\varnothing\}$, as $\varnothing \subseteq E \times F$

1.1.2 Basic properties

Let $\Re$ be a binary relation on a set E we say that $\Re$ is :

- Reflexive if : $\forall\ x \in E$, x $\Re$ x, i.e., $\Delta_E \subseteq \Re$.

  Example 1.3.

  Let E= {1,2,3} and $\Re$= {(1,1),(2,2),(3,3)}, so $\Re$ if a reflexive relation on E.

- irreflexive if : $\forall\ x \in E$, (x, x) $\notin \Re$, i.e., $\Delta_E \cap \Re = \varnothing$.

  Example 1.4.

  On the set of lines in space, " (D) is orthogonal to (D') " if irreflexive, since a line cannot be orthogonal to itself.

- symmetric if : $\forall\ x, y \in E$, x $\Re$ y $\Longrightarrow$ y $\Re$ x, i.e., $\Re = \Re^{-1}$

  Example 1.5.

  In $\mathbb{R}$ addition,multiplication et equality are symmetric.

- Antisymmetric if : $\forall\ x, y \in E$, x $\Re$ y et y $\Re$ x $\Longrightarrow$ x = y, i.e., $\Re \cap \Re^{-1} = \Delta_E$.

  Example 1.6.

  The usual relation $\geq$ is antisymmetric on $\Re$ because if x $\Re$ y and y $\Re$ x, so x = y.

- Transitive : $\forall\ x, y, z \in E$, x $\Re$ y and y $\Re$ z $\Longrightarrow$ x $\Re$ z, i.e., $\Re^2 = \Re \circ \Re \subseteq \Re$.

  Example 1.7.

  The inclusion relation on the sets is transitive, because if A $\subseteq$ B and B $\subseteq$ C, so A $\subseteq$ C.

1.2 Ordered sets

This section is devoted to the definition of the order relation and some of its types

1.2.1 Order relation

Definition 1.3.

A binary relation $\Re$ on a set e is called an order relation if the following conditions are satisfied :

1. Reflexive $\forall\ x \in E$, x $\Re$ x.

2. Antisymmetric $\forall\ x, y \in E$, x $\Re$ y et y $\Re$ x $\Longrightarrow$ x $=$ y.

3. Transitive $\forall\ x, y, z \in E$, x $\Re$ y and y $\Re$ z $\Longrightarrow$ x $\Re$ z.

1.2.2 Partially ordered and totally ordered sets

Definition 1.4 (partially ordered set).

Let $\leq_p$ be the partial order relation on the set E (i.e., is reflexive, antisymmetric, transitive). In this case the pair (E,$\leq_p$) is said to be ordered set (or else partially ordered set, simply poset).

Definition 1.5(Totally ordered set).

Let E be a set and let $\leq_p$ be an order relation on E. The order is total if any two elements of E are always comparable :

$\forall\ x, y, \in$ E $\times$E, x $\leq$ y or y $\leq$ x.

In this case the pair (E,$\leq$) is said to be strictly ordered set.

Example 1.8.

1. The divisibility relation (a $\Re$ b $\Longrightarrow a \setminus b$) is a partial order on $\mathbb{N}^*$ so the pair ($\mathbb{N}^*$, $a \setminus b$) is a partially ordered set.

2. For any set E,($P(E)$, $\subseteq$) is a partially ordered set with $P(E)$ is the set of partial of E.

1.2.3 Remarkable elements in an ordered set

Let (E,$\leq$) be an ordered set and $A \subseteq E$.

1. We say that $x \in E$ is a upper bound of A if $a \leq x$, $\forall\, a \in A$.

2. We say that $x \in E$ is a lower bound of A if $x \leq a$, $\forall\, x \in A$.

3. We say that $M \in A$ is a maximum of A if $a \leq M$, $\forall\, a \in A$.

4. We say that $m \in A$ is a minimum of A if $m \leq a$, $\forall\, a \in A$.

5. We say that $x \in A$ is a maximal element of A if $\exists\, a \in A$ such that x $\leq$ a $\Longrightarrow$ a = x.

6. We say that $x \in A$ is a minimal element of A if $\exists\, a \in A$ such that a $\leq$ x $\Longrightarrow$ a = x.

Notation 1.1.

- If A has a maximum (or minimum) we notice Max (A) (or Min(A)).

- If A has a least upper bound (or greatest lower bound) we notice Sup(A) (or Inf(A)).

Remark 1.2. The min and the max elements if they exist they are unique.

Example 1.9.

1. The set of integers natural provided with its order natural, it has 0 as minimum but does not has a maximum.

2. Let X= [0, 1].
   Sup X =1.
   Inf X =0, so Max =1 and Min=0.

3. Let A be a subset of $\mathbb{N}^*$ provided the order relation of the divisibility, so inf (A) is the pgcd of the elements of A while sup (A) is the ppcm of these elements.

1.2.4 Chains and Antichains

Definition 1.6.[3]

Partial ordered sets provide a common frame for many combinatorial configurations. Formally, a partially ordered set (or poset, for short) is a set P together with a binary relation < between its elements which is transitive and antysymmetric: if x<y and y<z then x<z, but x<y and y<x cannot both hold. We write x $\leq$ y if x<y or x = y. Elements x and y are comparable if either x $\leq$ y or y $\leq$ x (or both) hold. A chain in a poset P is a subset C $\subseteq$P such that any

two of its points are comparable. Dually, an antichain is a subset $A \subseteq P$ such that no two of its points are comparable. Observe that $|C \cap A| \leq 1$, i.e., every chain C and every antichain A can have at most one element in common (for two points in their intersection would be both comparable and incomparable). Here are some frequently encountered examples of posets: a family of sets is partially ordered by set inclusion; a set of positive integers is partially ordered by division; a set of vectors in Rn is partially ordered by (a1,...,an) < (b1,...,bn) iff ai $\leq$ bi for all i, and ai < bi for at least one i. Small posets may be visualized by drawings, known as Hasse diagrams: x is lower in the plane than y whenever x<y and there is no other point $z \in P$ for which both x<z and z<y.

Example 1.10

The ordered set $(\mathbb{N}, \leq)$ is a Chain.

Definition 1.7

Let C be a chain of an ordered set X . C is said to be maximal if it is not contained (strictly) in any chain of X.

Porposition 1.1.

the product of two chains is not necessarily a chain.

Example 1.11

Let the chain of 5 elements C_5={0,1,2,3,4}

$C_5 \times C_5 = \{$(x,y) $x \in C_5$, $y \in C_5\}$.

(a,b) $\leq$ (c,d) $\Longleftrightarrow$ a $\leq$ b and b $\leq$ d.

(0,4) $\leq$ (4,1) are not comparable.

1.3 Lattices

Many important propertier of an ordered set E are expressed in terms of the existnce of certain upper bounds or lower bounds of subsets of E. One of the most important classes of ordered sets defined in this way is the lattice structure.

Definition 1.7.

A lattice is consists of a partially ordered set in which every two elements have a unique supremum (also called a least upper bound or join) and a unique infimum (also called a greatest lower bound or meet).

Definition 1.8.

Let (X, $\leq$) be an ordered set.

- If x $\vee$ y, $x \wedge y$ exist for all $x, y \in X$, then (X,$\leq$) is called a lattice.

- If $\vee S$,$\wedge S$ exist for all S $\subseteq$ X, then (X,$\leq$) is called a complete lattice.

Example 1.12.

1. ($\mathbb{N}^*$,|) is a lattice with x$\vee$y $=$ ppcm (x , y) and $x \wedge y$ $=$ pgcd(x , y) for all $x, y \in \mathbb{N}^*$

2. Every chain is a lattice such that x $\vee$ y $=$ max(x , y) and $x \wedge y$ $=$ min(x , y).

Definition 1.9.

Let L be a lattice and $A \subseteq L$. A is called sublattice of L if for all $x, y \in A$, $x \wedge y \in A$ and x $\vee$ y $\in A$.

1.3.1 Lattice as ordered set

Let (E, $\leq$) be an ordered set.

Definition 1.9. (**join.semi-lattice**).

E is said to be **reticulated join**, or is called a **join.semi-lattice**, if any pair {x, y} of elements of E has an upper bound in E, we will note: $sup\{x, y\}$ $=$ x $\vee$ y, (which can be read: x or y).

Definition 1.10. (**meet.semi-lattice**).

E is said to be **reticulated meet**, or is called a **meet.semi-lattice**, if any pair {x, y} of elements of E has an lower bound in E, we will note: $inf\{x, y\}$ $=$ x $\wedge$ y, (which can be read: x and y).

Definition 1.11 (**Lattice**)

E is said to be **reticle**, or else is called a lattice, if it is both **reticulated join** and **reticulated meet**, that is to say, it is both **join.semi-lattice** and **meet.semi-lattice**.

Example 1.13

For any set E, ($P(E)$, $\subseteq$) is a lattice with A $\vee$ B $=$ A $\cup$B and A $\wedge$B $=$ A $\cap$ B
for all A, $B \in P(E)$.

Proposition 1.2.

In an join.semi-lattice (E, $\leq$) the internal composition law $\vee$ is :

- **idempotente** (that is to say : x $\vee$ x = x).
- **commutative** (that is to say : x $\vee$ y = y $\vee$ x).
- **associative** (that is to say : x $\vee$ (x $\vee$ y) = (x $\vee$ y) $\vee$ y).

1.3.2 lattice as algebraic structure

. Let E be a set with an internal composition law that we denote by $\vee$, we can ask if it is possible to define on E a relation of order $\leq$ which makes it a **join.semi-lattice**, such that $sup_E\{x,y\}$= x $\vee$ y. The previous study shows us the conditions obligatory :

- The law $\vee$ must be idempotent, commutative and associative.
- If x $\leq$ y, we will have $sup_E\{x,y\}$ = y, or x $\vee$ y = y and reciprocally.

Proposition 1.3.

Let E be a set provided with an internal composition law $\vee$, which is **idempotent**, **commutative** and **associative**, then there exists a unique relation of order $\leq$ on E such that E is a **join.semi-lattice** and whatever x and y are: $sup_E\{x,y\}$= x $\vee$ y. (analogous proposal for a **meet.semi-lattice**).

Remark 1.2.

If (E, $\leq$) is a lattice, it is provided with two internal composition laws $\vee$and $\wedge$, each of them being idempotent, commutative and associative.

Theorem 1.1 [4]

Let E be a set provided with two internal laws of composition, $\vee$ and $\wedge$, such that: these laws are idempotent, commutative and associative, and verify the absorption laws, that is to say for all x and y: x $\wedge$ (x $\vee$y) = x = x $\vee$ (x $\wedge$ y).

Then we can define on E a single relation of order $\leq$ such that (E, $\leq$) is a lattice, with $inf_E\{x,y\}$ = x $\wedge$ y and $sup_E\{x,y\}$ = x $\vee$ y. This order relation is defined by x $\vee$ y = y or x $\wedge$ y = x which are equivalent relations.

Proof .

From the above, it suffices to show that the following two relations are equivalent :

x $\leq_1$ y defined by x $\vee$ y = y

x $\leq_2$ y defined by x $\wedge$ y = x

Suppose x $\leq_1$ y, so x $\vee$ y = y hence x $\wedge$y = x$\wedge$ (x $\vee$ y) = x (absorption law), therefore x $\leq_2$ y.

Suppose x $\leq_2$ y, so x $\wedge$ y = y hence x $\vee$ y = (x $\wedge$ y)$\vee$ y = y (absorption law), therefore x $\leq_1$ y.

1.3.3 Sub-lattice and morphisms

. **Definition 1.12.** (sub-join.semi-lattice) A non-empty part A of a join.semi-lattice E, is called a sub-join.semi-lattice if we have one of the two equivalent conditions:

- $sup_A\{x, y\}$ exists and equal to x $\vee$ y, whatever x, y in A.
- $x \vee y \in A$ whatever x, y in A.

A is then a join.semi-lattice for the induced structure (analogous definition for a meet.semi-lattice).

Example 1.14

In ($\mathbb{N}^*$,|), $A = \{1,\ 3,\ 4,\ 24\}$ is not a sub-join.semi-lattice because $3 \vee 4 = 12 \notin A$. But A is always a join.semi-lattice bound, because all pairs have an upper bound in A, in particular $sup_A\{3, 4\} = 24$.

Definition 1.13. (Sub-lattice).

A nonempty part A of a lattice (E, $\leq$) is said to be a sub-lattice if it is both a sub-join.semi-lattice and a sub-meet.semi-lattice, this is equivalent to saying whatever x and y elements of A: x $\vee$ y $\in$A and x $\wedge$ y $\in$ A. A is then a lattice for the induced structure, with: $sup_A\{x, y\}$ = x $\vee$ y and $inf_A\{x, y\}$ = x $\wedge$ y.

Example 1.15

We consider here the lattice ($\mathbb{N}^*$, |) and let n be a fixed non-zero positive integer, we denote by $D(n)$ the set of positive divisors of n, we then have $D(n)$ is a sub-lattice of $\mathbb{N}^*$ Indeed: if x | n and y | n, then *pgcd* (x, y) | n and *ppcm* (x, y) | n, so x$\vee$y = *ppcm* (x, y) $\in D(n)$ and
x $\wedge$ y = *pgcd* (x, y) $\in D(n)$.

Proposition 1.4.

A $\vee$-morphism (resp. A $\wedge$-morphism) is an ingrowing application.

Proof .

For a $\vee$-morphism if x $\leq$ y $\Longleftrightarrow$ x $\vee$ y = y $\Longrightarrow$ f (x $\vee$ y) = f (x) $\vee$ f (y) = f (y)$\Longrightarrow$ f (x) $\leq$ f (y).

The notion of half-lattice morphism is therefore more precise than that of order morphisms.

Chapter 2

Basic aggregation operators on bounded lattices

Aggregation is the process of combining several elements(values, members,...)into a single element. Mathematical functions which provide a mechanism for doing so are called aggregation functions operators. Aggregation operators are functions with special properties we shall discuss in this section.

2.1 Definitions and basic properties

2.1.1 Definitions and examples

Definition 2.1. (Aggregation operators)

An aggregation operators is an application $n > 1$ arguments that maps the (n-dimensional) cube onto an interval $\mathbb{I} = [a, b]$, $f : \mathbb{I}^n \longrightarrow \mathbb{I}$, with the properties

- $f(a, a, ..., a) = a$ and $f(b, b, ..., b) = b$.
- $\mathbf{x} \leq \mathbf{y}$ implies $f(\mathbf{x}) \leq f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$.

Remark 2.1.

1. Let $\vee$, $\wedge$: $\mathbb{I}^2 \longrightarrow \mathbb{I}$ be two binary idempotent aggregation functions defined as $\vee(\mathrm{x, y}) = \max(\mathrm{x, y})$ and $\wedge(\mathrm{x, y}) = \min(\mathrm{x, y})$. So, when f is an idempotent aggregation function, then $\wedge(\mathrm{x, y}) \leq f(x, y) \leq \vee(\mathrm{x, y})$ for all x, $y \in \mathbb{I}$.

2. For all $X, Y \in \mathbb{I}^n$., we have

   (1) $f(X \vee Y) \geq f(X) \vee f(Y)$, where $X \vee Y = (x_1 \vee y_1,...,x_n \vee y_n)$,

   (2) $A(X \wedge Y) \leq A(X) \wedge A(Y)$, where $X \wedge Y = (x_1 \wedge y_1,..., x_n \wedge y_n)$,

   (3) $(\text{A(X.Y)})^2 \leq A(X).A(Y)$, where $X.Y = (x_1.y_1,..., x_n.y_n)$.

Definition 2.2. [5]

An aggregation A_1 dominates another aggregation A_2 if and only if the following inequality holds

$$A_1\ (A_2\ (\text{x, y}), A_2\ (\text{u, v})) \geq A_2\ (A_1\ (\text{x,u}), A_1\ (\text{y, v})), \text{ for all} x, y, u, v \in U.$$

Definition 2.3. [5]

An aggregation A_1 bidominates another aggregation A_2 if and only if the following equality holds

$$A_1\ (A_2\ (\text{x, y}), A_2\ (\text{u, v})) = A_2\ (A_1\ (\text{x,u}), A_1\ (\text{y, v})), \text{ for all} x, y, u, v \in U.$$

Definition 2.4. [6]

Let A be an aggregation, we said that the t-norms T satisfies the distributive property if and only if for all x, y_1,..., $y_n \in X$,

$$A(T\ (\text{x}, y_1),...,T\ (\text{x}, y_n)) = T\ (\text{x}, A(y_1,..., y_n)).$$

Example 2.1.

Arithmetic mean $A_n(x) = \dfrac{1}{n}(x_1 + x_2 + ... + x_n)$.

Geometric mean $An(x) = \sqrt[n]{x_1 x_2 ... x_n}$.

Minimum min(x) = $\min\{x_1, ..., x_n\}$.

Maximum max(x) = $\max x_1, ..., x_n$.

2.1.2 Main Classes

There are various semantics of aggregation, and the main classes are determined according to these semantics. In some cases we require that high and low inputs average each other, in other cases aggregation functions model logical connectives (disjunction and conjunction), so that the inputs reinforce each other, and sometimes the behavior of aggregation functions depends on the inputs. The four main classes of aggregation functions are [7], [8], [9]

- Averaging,
- Conjunctive,
- Disjunctive,
- Mixed.

Definition 2.5. (Averaging aggregation)

An aggregation function f has averaging behavior (or is averaging) if for every $x \in I_n$ it is bounded by

$$\min(\mathrm{x}) \leq \mathrm{f}\ (\mathrm{x}) \leq \max(\mathrm{x}).$$

Definition 2.6. (Conjunctive aggregation)

An aggregation operator f has conjunctive behavior (or is conjunctive) if for every $\mathbf{x}$ it is bounded by

$$f(x) \leq \min\ (\mathrm{x}) = \min\ (x_1, x_2, ..., x_n).$$

Definition 2.7. (Disjunctive aggregation)

An aggregation function f has disjunctive behavior (or is disjunctive) if for every x it is bounded by

$$\mathrm{f}\ (\mathrm{x}) \geq \max(\mathrm{x}) = \max(x_1, x_2, ..., x_n).$$

Definition 2.8. (Mixed aggregation)

An aggregation function f is mixed if it does not belong to any of the above classes, i.e., it exhibits different types of behavior on different parts of the domain.

2.1.3 Main Properties

The main properties are these:

1. An aggregation A is called **idempotent** if for every input $x = (t, t, ..., t)$, $t \in \mathbb{I}$ the output is $A(t, t..., t) = t$.

2. An aggregation operator A is **strictly monotone increasing** if

$$\mathbf{x} \leq \mathbf{y} \text{ but } \mathbf{x} \neq \mathbf{y} \text{ implies } f(\mathbf{x}) < f(\mathbf{y}) \text{ for every } \mathbf{x}, \mathbf{y} \in \mathbb{I}^n.$$

3. An aggregation operator A is called **symmetric**, if its value does not depend on the permutation of the arguments, i.e., $A(x_1, x_2, ..., x_n) = A(x_{P(1)}, x_{P(2)}, ..., x_{P(n)})$, for every x and every permutation $P = (P(1), P(2), ..., P(n)) of (1, 2..., n)$.

4. An aggregation operator A has a **neutral element** $e \in \mathbb{I}$, if for every $t \in \mathbb{I}$in any position it holds

$$f(e, ..., e, t, e, ..., e) = t.$$

5. An aggregation oprator A has an **absorbing element** $a \in \mathbb{I}$if

$$A(x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_n) = a,$$

for every x such that $x_i = a$ with a in any position.

Note 2.1 [10]

An absorbing element, if it exists, is unique. It can be any number from $\mathbb{I}$.

6. An element $a \in]0, 1[$ is a **zero divisor** of an aggregation operator A, if for all $i \in 1, ..., n$ there exists some $\mathbf{x} \in]0, 1]^n$ such that its i-th component is $x_i = a$, and it holds $A(\mathbf{x}) = 0$, i.e., the equality

$$A(x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_n) = 0,$$

can hold for some $x > 0$ with a at any position

7. An element $a \in]0, 1[$ is a **one divisor** of an aggregation operator A, if for all $i \in 1, ..., n$ there exists some $\mathbf{x} \in [0, 1[^n$ such that its i-th component is $x_i = a$, and it holds $A(\mathbf{x}) = 1$, i.e., the equality

$$A(x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_n) = 1,$$

can hold for some $x < 1$ with a at any position

8. A two-argument operator A is **associative** if $A(A(x_1, x_2), x_3) = A(x_1, A(x_2, x_3))$ holds for all x_1, x_2, x_3 in its domain. Consequently, the n-ary aggregation operator can be constructed inaunique way by iteratively applying A_2as

$$A_n(x_1, ..., x_n) = A_2(A_2(...A_2(x_1, x_2), x_3), ..., x_n).$$

9. . An aggregation operator $A : \mathbb{I}^n \longrightarrow \mathbb{I}$ is **homogeneous** of order 1 if for all λ and for all $(x_1, ..., x_n) \in \mathbb{I}^n$ it is

$$A(\lambda + x_1, ..., \lambda + x_n) = \lambda + A(x_1, ..., x_n).$$

10. An aggregation A is said to be **(left-) rigth-continuous** for the first component, if for any(non-decreasing) non-increasing sequence $(x_n)_{n\in\mathbb{N}}$.

2.2 Particuler aggregation operators on bounded lattices

2.2.1 Definitions properties

Definition 2.9.

A bounded lattice is a lattice ($\mathbf{L}$,$\leq$) which has a top element 1 and a bottom element 0. For all $\mathbf{x}, \mathbf{y} \in L$, we write $\mathbf{x} < \mathbf{y}$ if $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. If x is incomparable with y, then we write $\mathbf{x} \parallel \mathbf{y}$.

Definition 2.10.

Let (L,$\leq$,0,1) be a bounded lattice and $n \in \mathbb{N}$ be fixed. A mapping $A : L^n \longrightarrow L$ is called an n-ary aggregation operator on L if it is increasing, $A(x) \leq A(y)$ whenever $x \leq y$ (i.e., $x_1 \leq y_1, ..., x_n \leq y_n$) and it satisfies boundary conditions A(0,. . . ,0)= 0, A(1,. . . ,1)= 1.

Properties 2.1. [11]

- An aggregation operator A is called idempotent if $A(x, \ldots, x) = x$ for all $x \in L$.
- An aggregation operator A is called 0-positive (resp. 1-positive) if $A(x_1, \ldots, x_n) = 0$ (resp. $A(x_1, \ldots, x_n) = 1$) if and only if $x_i = 0(resp.x_i = 1)$ for all $i \in \{1, 2, \ldots, n\}$.
- The greatest aggregation operator on L is $A_\top$ which takes value 0 if $x = [0]^n$ and 1 otherwise, while the smallest aggregation operator on L is $A_\perp$ which takes value 1 if $x = [1]^n$ and 0 otherwise.

- The greatest idempotent aggregation operator on L is $\mathbf{A}_\vee$ defined by $\mathbf{A}_\vee(\mathbf{x}) = \vee_{i=1}^{n}\mathbf{x}_i$ for all $\mathbf{x} \in L^n$, and the smallest idempotent aggregation operator on L is$\mathbf{A}_\wedge$ defined by $\mathbf{A}_\wedge(x) = \wedge_{i=1}^{n}\mathbf{x}_i$ for all $\mathbf{x} \in L^n$

- $\mathbf{A}_\wedge$ is 1-positive and $\mathbf{A}_\vee$ is 0-positive. We call $\mathbf{A}_\wedge$ is the $\wedge$-aggregation operator on L and $\mathbf{A}_\vee$ is the $\vee$-aggregation operator on L.

2.2.2 The lattice structure of some sets of aggregation operators

Definition 2.11.

Denote by $A_n(L)$ the set of all n-ary aggregation functions. Consider $A_n(L)$ with the following order: For $A, B \in A_n(L)$,

$$A \leq B \text{ whenever } A(x) \leq B(x), \text{ forall } x \in L^n.$$

The smallest and the greatest aggregation operator in $A_n(L)$ are, respectively, defined by

$$A_\perp(\mathbf{x}) = \begin{cases} 1 & , if\ x_i = 1\ for\ all\ i \in [n] \\ 0 & , otherwise \end{cases}$$

$$A_\top(\mathbf{x}) = \begin{cases} 0 & , if\ x_i = 0\ for\ all\ i \in [n] \\ 1 & , otherwise \end{cases}$$

Thus we have $A_\perp \leq A \leq A_\top$ for any aggregation operator $A : L^n \longrightarrow L$.

Remark 2.2.

Each aggregation operator $A \in An_n(L)$ contains the set{0,1} as a subset of its range. The extremal aggregation operator $A_\perp$ and $A_\top$ are particular examples of aggregation operator on L with the minimal range{0,1}. For any aggregation operator $A \in An(L)$, denote $D_A = \{\mathbf{x} \in L_n | A(x) = 0\}$ and $U_A = \{\mathbf{x} \in L^n | A(\mathbf{x}) = 1\}$. Then D_A is a downset on L^n, i.e., a proper subset of Ln such that if $x \in D_A$ and $\mathbf{y} \leq \mathbf{x}$, then also $\mathbf{y} \in D_A$

Properties 2.2.

Fix an element $u \in L$ and a downset D on L^n. Then the mapping $A_{u,D}$: $L^n \longrightarrow L$ given by

$$A_{u,D} = \begin{cases} 0 & , if\ x \in D \\ 1 & , if\ x = (1, ..., 1) \\ u & ,_i f:\ otherwise \end{cases}$$

is an aggregation operator L.

proof.

Proof is again trivial and we do not go in details. Observe that in extreme cases, i.e., when $u \in \{0, 1\}$, we obtain, in fact, two valued aggregation operator. Indeed,$A_{0,D} = A_{\perp}$ is the smallest aggregation operator on L, and $A_{1,D} = A_D$.

The introduced aggregation operator $A_{u,D}$ play an important role in the structure of the set $A_n(L)$. Following, any aggregation operator $A \in A_n(L)$ can be obtained by means of composition of supremum and binary infimum on L, and aggregation operator A_{u,D_y}, where for $y \in L_n\{(1, ..., 1)\}$, the downset D_y on L_n is given by D_y =$L_n\{x \in L_n \backslash x \geq y\}$. Observe that the aggregation operator A_u, D_y is the smallest aggregation operator on L attaining the value u in the point y. As an important property of the lattice $A_n(L)$ we show that any of lattices $L, L^2, ..., L^n$ can be ambedded into $A_n(L)$.

Properties 2.3.

Let (L,≤,0,1) be a complete lattice. Then $(A_n(L), \leq, A_{\perp}, A_{\top})$is a complete lattice whose bounded $A_{\perp}$ and $A_{\top}$ any given in Definition 2.7

Proof. Let $\{A_{\tau} \backslash \tau \in T\}$ be a non-empty subset of $A_n(L)$. Obviously,

$$\bigvee_{\tau \in T} A_{\tau}(0,...,0) = \bigvee_{\tau \in T} 0 = 0$$
$$\bigvee_{\tau \in T} A_{\tau}(1,...,1) = \bigvee_{\tau \in T} 1 = 1$$

and hence $\bigvee_{\tau \in T}$ satisfies the condition (i) of Definition 2.6; for $x, y \in L_n$ such that $x \leq y$ it holds $A_{\tau}(x) \leq A_{\tau}(y)$ for all $\tau \in T$. Thus we have that

$$\bigvee_{\tau \in T} A_{\tau}(\mathbf{x}) \leq \bigvee_{\tau \in T} A_{\tau}\ (\mathbf{y})$$

That is $\bigvee_{\tau \in T} A_{\tau}$ satisfies the second condition of Definition 2.6 and hance $\bigvee_{\tau \in T} A_{\tau}$ is an aggregation operator. Analogously one can prove that $\bigwedge_{\tau \in T} A_{\tau}$ is an aggregation operator Observe that the set of all aggregation operator form $A_n(L)$ with minimal range {0,1} is a proper sublattice of the lattice $A_n(L)$ (supposing L contains more that two elements) with $A_\perp$ and $A_\top$ being its bottom and top elements, and satisfying $A_{D_1} \vee A_{D_2} = A_{D_1 \cap D_2}$ and $A_{D_1} \wedge A_{D_2} = A_{D_1 \cup D_2}$

Properties 2.4.

Fix a downset D on L^n differing form the maximal downset $D_{A_\perp}$.

Let a mapping $\theta : L \longrightarrow A_n(L)$ be given by $\theta(u) = A_{u,D}$. Then θ is a lattice monomorphism.

Proof.

Cleary, θ is an injective mapping (this will fail once $D_{A_\perp}$ Will be considered) and $\theta(u \vee v) = A_{u \vee v},\ D = A_{u,D} \vee A_{u,D}$. Similarly, θ preserves the meet operation. Observe that considering the smallest downset $D_{A_\top}$, also the bounds of lattice are preserved, i.e.,then $\theta(1) = A_\top$ and $\theta(0) = A_\perp$. We have just shown that L can be concluded as a sublattice of $A_n(L)$. The next proposition shows a similar result for L^m, m = 2,...,n.

Definition 2.12.

Let 2 $\leq$ m $\leq$ n. for any fixed m-tuple $(a_1, a_2, ..., a_m) \in L^m$, we denote by $\theta(a_1, a_2, ..., a_m)$ a mapping $A(a_1, a_2, ..., a_m) : L_n \longrightarrow L$ given by

$$A(a_1, a_2, ..., a_m)(\mathbf{x}) = \begin{cases} 0 & , if\ \mathbf{x} = (0, ..., 0) \\ a_i & , if\ \exists i \in \{1, 2, ..., m\},\ such\ that\ x_j = 0 \text{ for } j \neq i\ and\ x_j \neq 0 \\ 1 & ,\ otherwise \end{cases}$$

Properties 2.5. [12]

Let 2 $\leq$ m $\leq$ n. Then the mapping $\theta : L^m \longrightarrow A_n(L)$ given in Definition 2.8 is a lattice monomorphism.

Proof.

First, we must show that $A_{(a_1, a_2, ..., a_m)}$ is aggregation operator. It is trivial that

$$A_{(a_1, a_2, ..., a_m)}(0,...,0) = 0$$

and

$$A_{(a_1, a_2, ..., a_m)}(1,...,1) = 1$$

Choose $\mathbf{x}, \mathbf{y} \in L^n$ such that $\mathbf{x} \leq \mathbf{y}$.

- If $A_{(a_1, a_2, ..., a_m)}(\mathbf{x}) = 0$, then obviously $A_{(a_1, a_2, ..., a_m)}(\mathbf{x}) \leq A_{(a_1, a_2, ..., a_m)}(\mathbf{y})$.

- Let $A_{(a_1,a_2,\ldots,a_m)}(\mathbf{x}) = a_i > 0$ for some $i \in [m]$. Then we have that $x_i \neq 0$ and, thus also $y_i \neq 0$. Consequently, $A_{(a_1,a_2,\ldots,a_m)}(j) \in \{a_i, 1\}$ and hence $A_{(a_1,a_2,\ldots,a_m)}(\mathbf{x}) \leq A_{(a_1,a_2,\ldots,a_m)}(\mathbf{y})$

- If $A_{(a_1,a_2,\ldots,a_m)}(\mathbf{x}) = 1$, then either there exist $i, j \in [m]$ such that $x_i, x_j \neq 0$,or there is $k \in \{m+1, ..., n\}$ such that $x_k \neq 0$. Thus either for $i, j \in [m]$, it holds that$y_i, y_j \neq 0$ or $y_k \neq 0$ for $k \in \{m+1, ..., n\}$. In both cases.

$$A_{(a_1,a_2,\ldots,a_m)}(\mathbf{x}) = 1 = A_{(a_1,a_2,\ldots,a_m)}(\mathbf{y})$$

Now we must show that θ is a lattice monomorphism. we suppose that $\theta(a_1, ..., a_m) = \theta(b_1, ..., b_m)$ for some $(a_1, a_2, ..., a_m), (b1, b2, ..., bm) \in L^m$. Then have that

$$\theta(a_1, ..., a_m)(1,0,...,0) = \text{a1} = \text{b1} = \theta(b_1, ..., b_m)(1,0,...,0)$$

Similarly we can see that a2 = b2...am = bm. diffcult or not diffcult

$$\theta(\ (a_1, ..., a_m) \vee (b_1, ..., b_m)\ ) = \theta\ (a_1, ..., a_m) \vee \theta\ b_1, ..., b_m)$$

and

$$\theta(\ (a_1, ..., a_m) \wedge (b_1, ..., b_m)\ ) = \theta\ (a_1, ..., a_m) \wedge \theta\ b_1, ..., b_m)$$

properties of the considered lattice L are herited by $A_n(L)$, but not all of them.

Remark 2.3.

(**i**) If is a distributive (complete)lattice, then it is obvious that $A_n(L)$ is a distributive (complete) lattice.

(**ii**) If is a complete lattice and $a \in L$ is a compact element of L, then $A_{a,D_{A_T}}$ is a compact element of $A_n(L)$. To see this fact, choose a family $\{A_\tau \mid \tau \in T\} \subset A_n(L)$ such that $A_{a,D_{A_T}} \leq \bigvee_{\tau \in T} A_\tau$. For $\mathbf{x} \notin \{\mathbf{0}, \mathbf{1}\}$, $A_{a,D_{A_T}}(\mathbf{x}) \leq \bigvee_{\tau \in T} A_\tau\ (\mathbf{x})$, a $\leq \bigvee_{\tau \in T} A_\tau\ (\mathbf{x})$. Since a is an atom there exist element $\tau_1, ..., \tau_n \in T$ such that

$$A_{a,D_{A_T}} = \text{a} \leq A_{\tau_1}(\mathbf{x}) \vee A_{\tau_2}(\mathbf{x}) \vee ... \vee A_{\tau_n}(\mathbf{x})$$

For x =(0,...,0) or x =(1,...,1) we already have that

$$A_{a,D_{A_T}}(0,...0,) \leq A_{\tau_1}(0,...,0) \vee A_{\tau_2}(0,...,0) \vee A_{\tau_n}(0,...,0)$$
$$A_{a,D_{A_T}}(1,...1,) \leq A_{\tau_1}(1,...,1) \vee A_{\tau_2}(1,...,1) \vee A_{\tau_n}(1,...,1)$$

Thuse $A_{a,D_{A_T}} \leq A_{\tau_1} \vee A_{\tau_2} \vee \ . \ . \ . \ \vee A_{\tau_n}$ and $A_{a,D_{A_T}}$ is a compact element of $A_n(L)$.

(iii) Let be a bounded lattice and let $a \in L$ be an atom of L. Then The n-ary aggregation operator $A_{a,D_{A_T}}$ is not an atom of $A_n(L)$, To see this fact, it is enough to observe that for any two downset D_1 and D_2 such that D_1 is a proper subset of D_2, and for any $u \in L\backslash\{0,1\}$ it holds $A_{u,D_1} > A_{u,D_2}$. Now, the result follows form the fat that the considered lattice L has at least three elements, and hence the minimal downset on L^n is a proper subset of the maximal downset on L^n. A similar conclusion holds for coatoms. Namely, for any lattice L having a coatom c, the aggregation operator $A_{c,D_{A_T}}$.

Example 2.2

Let L be a distributive lattice. For any $(a_1, ...a_n) \in L^n$, consider

$b_1 = a_1 \vee ... \vee a_n$.

$b_2 = [(a_1 \wedge a_2) \vee ... \vee (a_1 \wedge a_n)] \vee [(a_2 \wedge a_3) \vee ... \vee (a_2 \wedge a_n)] \vee ... \vee [(a_{n-1} \wedge a_n)]$.

... ...

$b_s = \vee\{a_{j_1} \wedge ... \wedge a_{j_s} | \{j_1, ..., j_s\} \subseteq \{1, 2, ..., n\}\}$, ...

$b_n = a_1 \wedge ... \wedge a_n$

Then the operator $A_k : L^n \longrightarrow L$, $A_K(a_1, ..., a_n) = b_k$ is aggregation operator for each $k \in \{1, 2, ..., n\}$.

Observe that if L is a chain, then A_K is the k-th order statistics.

2.2.3 General methods for construction of aggregation operators on bounded lattices

In this section, we introduce some general construction methods for aggregation operator on bounded lattices. Note that these methods are well known when $L = [a, b]$ (mostly $L = [0, 1]$) is a real interval, see Beliakov, Pradera, and Calvo (2007), and we show their validity in the framework of general bounded lattices.

Properties 2.6.

Let L be a bounded lattice, $K, n \in \mathbb{N}$ and let $A_1, ..., A_k : L^n \longrightarrow L$ $B : L^k \longrightarrow L$ be aggregation operator on L . Then thecomposite operator

$$C_{B,A_1,...,A_K(\mathbf{x})} = B(A_1(\mathbf{x}), ..., A_K(\mathbf{x}))$$

is an n-ary aggregation operator on L.

Proof.

The non-decreasing monotonicity of $C_{B,A_1,...,A_K}$ follows from the non-decreasing monotonicity of $B, A_1, ..., A_k$. We have that

$$C_{B,A_1,...,A_K(\mathbf{0})} = B(A_1(\mathbf{0}), ..., A_K(\mathbf{0}))$$

and

$$C_{B,A_1,...,A_K(\mathbf{1})} = B(A_1(\mathbf{0}), ..., A_K(\mathbf{1}))$$

proving the result.

Definition 2.13.

Let L be a bounded lattice. A mapping $f : L \longrightarrow L$ is called a distortion if and only if f is non-decreasing, $f(0) = 0$ and $f(1) = 1$

Denote by

$\mathcal{F}\ (L)$ the set of all distortions on L. Formally, distortions on L can be seen as unary aggregation operator on L, i.e., $\mathcal{F}\ (L) = A_1(L)$.

Corollary 2.1.

Let L be a bounded lattice and $A : L^n \longrightarrow L$be an aggregation operator and $f_1, ..., f_n, g \in F(L)$. Then the function $B : L^n \longrightarrow L$ defined by $B(x) = g(A(f_1(x_1), ..., f_k(x_n)))$ is an aggregation operator.

Proof.

The result can be obtained applying consecutively Proposition 2.6. In the first step, we consider composite aggregation operator $f_i(P_i(x))$, where $P_i(x) = x_i$ is the ith projection. Next, we consider the composite operator $C(x) = A(f_1(P_1(\mathbf{x}), ..., f_n(P_n(\mathbf{x}))) = A(f_1(x_1), ..., f_n(x_n))$, and finally B$(\mathbf{x})$ = g$(C(\mathbf{x}))$.

The ordinal sum of triangular norms on unit interval has been used to construct other triangular norms. But, in general, ordinal sum construction of triangular norms and triangular conorms may not work on an arbitrary bounded lattice.

In this study, we focus on ordinal sums of triangular norms and triangular conorms on an arbitrary bounded lattice. Moreover, we show a complete generalization of our ordinal sums of triangular norms and triangular conorms on an arbitrary bounded lattice. Also some illustrative examples are added for clarity.

Corollary 2.2.

An function $A : L^n \longrightarrow L$given by

$$A(\mathbf{x}) = B(\mathbf{x}_I)$$

is an aggregation operator, where, for $\mathbf{x} \in L^n$, $x_I = (x_{i1}, ..., x_{ik})$, k = |I|, $i_1 < ... < i_k$, and $i_1, ..., i_k = I$.

Proof.

The proof is similar to that of Corollary 2.1 and therefore omitted.

Note that we can combine the introduced constructions to build rather complex aggregation operator (if necessary). So, for example, we can introduce a step-wise aggregation based on Proposition 2.6 and Corollaries 2.1, 2.2. We sketch this idea using two steps only. In the first step, consider a system $(I_1, ..., I_k)$ of non-empty subsets of [n], and a system of aggregation operator $(A_1, ..., A_k)$, $A_i : L^{|I_i|} \longrightarrow L$. I_i can be seen as the i-th group of criteria chosen from the global criteria set [n]. Evidently, for any $\mathbf{x} \in L^n$, the k-tuple $(A_1(x_{I_1}), ..., A_k(x_{I_k}))$ belongs to L^k. Now, considering an aggregation operator $B : L^k \longrightarrow L$, our composite (and rather complex) aggregation operator $C : L^n \longrightarrow L$ is obtained by aggregating of partial outputs $(A_1(\mathbf{x}_{I_1}), ..., A_k(\mathbf{x}_{I_k}))$ by means of B, i.e., C($\mathbf{x}$)= $B(A_1(\mathbf{x}_{I_1}), ..., A_k(\mathbf{x}_{I_k}))$.

2.3 Triangular norms and trianglar conorms on bounded lattices

The ordinal sum of triangular norms on unit interval has been used to construct other triangular norms. But, in general, ordinal sum construction of triangular norms and triangular conorms may not work on an arbitrary bounded lattice. In this study, we focus on ordinal sums of triangular norms and triangular conorms on an arbitrary bounded lattice. Moreover, we show a complete generalization of our ordinal sums of triangular norms and triangular conorms on an arbitrary bounded lattice. And some illustrative examples are added for clarity

2.3.1 Definitions

Definition 2.14.

A bounded lattice $(L, \leq)$ is a lattice which has the top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$, for all $x \in L$.

Definition 2.15.

Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, if a and b are incomparable, in this case, we use the notation $a \parallel b$. We denote the set of elements which are incomparable with a by I_a. So $I_a = \{x \in L \mid x \parallel a\}$.

Definition 2.16.

Let $(L, \leq, 0, 1)$ be a bounded lattice. Operation $T : L^2 \longrightarrow L$ is called a triangular norm (t-norm) if it is :

- $T(x, y) = T(y, x)$. (commutative)
- $T(x, T(y, z)) = T(T(x, y), z)$. (associative)
- Increasing with respect to both variables and it satisfies $T(x, 1) = x$ for all $x \in L$.

Definition 2.17.

Let $(L, \leq, 0, 1)$ be a bounded lattice. Operation $S : L^2 \longrightarrow L$ is called a triangular conorm (t-conorm) if it is :

- $T(x, y) = T(y, x)$. (commutative)
- $T(x, T(y, z)) = T(T(x, y), z)$. (associative)
- Increasing with respect to both variables and it satisfies $S(x, 0) = x$ for all $x \in L$.

2.3.2 Construction of t-norms and t-conorms on bounded lattices

Consider a bounded lattice $(L, \leq, 0, 1)$, an element $a \in L \backslash \{0, 1\}$, a t-norm $V : [a, 1]^2 \longrightarrow [a, 1]$ and a t-conorm $W : [0, a]^2 \longrightarrow [0, a]$. An ordinal sum extension T of V to L and S of W to L is given by

$$T(x, y) = \begin{cases} V(x, y) & if (x, y) \in [a, 1]^2, \\ x \wedge y & otherwise. \end{cases}$$

and

$$S(x,y)=\begin{cases} W(x,y) & if(x,y)\in[0,a]^2,\\ x\vee y & otherwise.\end{cases}$$

However, the above-defined mapping T need not be a t-norm, in general. Similarly, S need not be a t-conorm, in general

Example 2.3

Given the lattice $L=\{0,t,s,a,k,1\}$ with order given in Figure 1 and consider the t-norm $V:[a,1]^2\longrightarrow[a,1]$, $V(x,y)=\begin{cases} x\wedge y & 1\in\{x,y\},\\ a & otherwise.\end{cases}$ for all $x,y\in[a,1]$. Then the operation T is constructed as Table 1 by using the formula (1), but T is not a t-norm on L.

T	0	t	s	a	k	1
0	0	0	0	0	0	0
t	0	t	0	t	t	t
s	0	0	s	0	s	s
a	0	t	0	a	a	a
k	0	t	s	a	a	k
1	0	t	s	a	k	1

Table 1: The operation T on L

If we take elements $s,k\in L$, then $s\le k$. But we have that $T(s,k)=s\|a=T(k,k)$. Hence, the operation T does not satisfy monotonicity. Moreover, $T(T(k,k),s)=T(a,s)=0$ and $T(k,T(k,s))=T(k,s)=s$ for elements $s,k\in L$. Hence, the operation T does not satisfy associativity. So, we obtain that T is not a t-norm on L.

Example 2.3

Given the lattice $L=\{0,t,s,a,k,1\}$ with order given in Figure 2 and consider the t-conorm $W:[0,a]^2\longrightarrow[0,a]$, $W(x,y)=\begin{cases} x\vee y & 0\in\{x,y\},\\ a & otherwise.\end{cases}$ for all $x,y\in[0,a]$. Then the operation S is constructed as Table 2 by using the formula (2), but S is not a t-conorm on L.

S	0	k	a	s	t	1
0	0	k	a	s	t	1
k	k	a	a	s	t	1
a	a	a	a	1	t	1
s	s	s	1	s	1	1
t	t	t	t	1	1	1
1	1	1	1	1	1	1

Table 2: The operation S on L

If we take elements $k, s \in L$, then $k \leq s$. But we have that $S(k,k) = a \| s = S(s,k)$. Hence, the operation S does not satisfy monotonicity. Moreover, $S(S(k,k),s) = S(a,s) = 1$ and $S(k, S(k,s)) = S(k,s) = s$ for elements $K, s \in L$. Hence, the operation S does not satisfy associativity. So, we obtain that S is not a t-conorm on L.

Theorem 2.1 [13]

Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \backslash \{0,1\}$. If V is a t-norm on $[a,1]$ and W is a t-conorm on $[0,a]$, then the functions T :$L^2 \longrightarrow L$ and $S : L^2 \longrightarrow L$are, respectively, a t-norm and a t-conorm on L, where

$$T(x,y) = \begin{cases} V(x,y) & if (x,y) \in [a,1]^2, \\ x \wedge y & if\ 1 \in \{x,y\}, \\ 0 & otherwise. \end{cases}$$

$$S(x,y) = \begin{cases} W(x,y) & if (x,y) \in [0,a]^2, \\ x \vee y & if\ 0 \in \{x,y\}, \\ 1 & otherwise. \end{cases}$$

Proof.

We have $T(x,1) = x \wedge 1 = x$for all $x \in L$.. So, the fact that $1 \in L$is a neutral element of T. It is easy to see commutativity of T. i) Monotonicity: We prove that if$x \leq y$then for all $z \in L, T(x,z) \leq T(y,z)$.. The proof is split into all possible cases. If $z = 1$, then we have that$T(x,z) = T(x,1) =x \leq y= T(y,1) = T(y,z)$.

1. Let $x \leq a$.

1.1. $y \leq a$ and $z \in L \setminus \{1\}$

$$T(x,z) = 0 = T(y,z)$$

1.2. $1 > y > a$,

1.2.1. $z \leq a$ *or* $z \parallel a$,

$$T(x,z) = 0 = T(y,z)$$

1.2.2. $1 > z > a$,

$$T(x,z) = 0 \leq V\ (y,z) = T(y,z)$$

1.3. $y \| a$ and $z \in L \setminus \{1\}$

$$T(x,z) = 0 = T(y,z)$$

1.4. $y = 1$ and $z \in L \setminus \{1\}$

$$T(x,z) = 0 \leq z = T(y,z)$$

2. Let $1 > x > a$.

2.1. $1 > y > a$,

2.1.1. $z \leq a$ *or* $z \parallel a$,

$$T(x,z) = 0 = T(y,z)$$

2.1.2. $1 > z > a$,

$$T(x,z) = V\ (x,z) \leq V\ (y,z) = T(y,z)$$

2.2. $y = 1$,

2.2.1. $z \leq a$ *or* $z \parallel a$,

$$T(x,z) = 0 \leq z = T(y,z)$$

2.2.2. $1 > z > a$,

$$T(x,z) = V\ (x,z) \leq V\ (1,z) = z = T(y,z)$$

3. Let $x \parallel a$.

3.1. $1 > y > a$, and $z \in L \setminus \{1\}$,

$$T(x,z) = 0 \leq z = T(y,z)$$

3.2. $y \parallel a$ and $z \in L \setminus \{1\}$,

$$T(x,z) = 0 = T(y,z)$$

3.3. $y = 1$ and $z \in L \setminus \{1\}$,

$$T(x,z) = 0 \le z = T(y,z)$$

4. Let $x = 1$. Then, since $y = 1$, $T(x,z) = z = T(y,z)$.

ii) Associativity: We demonstrate that $T(x,T(y,z)) = T(T(x,y),z)$ for all $x,y,z \in L$. Again the proof is split into all possible cases considering the relationships of the elements x,y,z and a.

1. Let $x \le a$,

1.1. $y \le a$,

1.1.1. $z \in L \setminus \{1\}$,

$$\begin{aligned} T(x,T(y,z)) &= T(x,0) = 0 \\ &= T(x,0) \\ &= T(T(x,y),z) \end{aligned}$$

1.1.2. z = 1,

$$\begin{aligned} T(x,T(y,z)) &= T(x, y \wedge z) = T(x,y) = 0 \\ &= 0 \wedge z \\ &= T(x,0) \\ &= T(T(x,y),z) \end{aligned}$$

1.2. $1 > y > a$,

1.2.1. $z \le a$ or $z \parallel a$,

$$\begin{aligned} T(x,T(y,z)) &= T(x,0) = 0 \\ &= T(0,z) \\ &= T(T(x,y),z) \end{aligned}$$

1.2.2. $1 > z > a$,

$$
\begin{aligned}
T(x,T(y,z)) &= T(x,V\ (y,z)) = 0 \\
&= T(0,z) \\
&= T(T(x,y),z)
\end{aligned}
$$

1.2.3. z $=1$,

$$
\begin{aligned}
T(x,T(y,z)) &= T(x,y\wedge z) = T(x,y) = 0 \\
&= 0\wedge z \\
&= T(0,z) \\
&= T(T(x,y),z)
\end{aligned}
$$

1.4. y $=1$ and $z\in L\setminus\{1\}$,

$$
\begin{aligned}
T(x,T(y,z)) &= T(x,y\wedge z) \\
&= T(x,z) \\
&= T(x\wedge y,z) \\
&= T(T(x,y),z)
\end{aligned}
$$

2. Let $1>x>a$,

2.1. $y\leq a$,

2.1.1.$z\in L\setminus\{1\}$,

$$
\begin{aligned}
T(x,T(y,z)) &= T(x,0) = 0 \\
&= T(0,z) \\
&= T(T(x,y),z)
\end{aligned}
$$

2.1.2. z $=1$,

$$
\begin{aligned}
T(x,T(y,z)) &= T(x,y\wedge z) = T(x,y) \\
&= 0\wedge z \\
&= T(0,z) \\
&= T(T(x,y),z)
\end{aligned}
$$

2.2. $1>y>a$,

2.2.1. $z\leq a$ or $z\parallel a$,

$$\begin{aligned}T(x,T(y,z)) &= T(x,0) = 0\\ &= T(V\ (x,y),z)\\ &= T(T(x,y),z)\end{aligned}$$

2.2.2. $1 > z > a$,

$$\begin{aligned}T(x,T(y,z)) = T(x,V\ (y,z)) &= V\ (x,V\ (y,z))\\ &= V(V\ (x,y),z)\\ &= T(V(x,y),z)\\ &= T(T(x,y),z)\end{aligned}$$

2.2.3. z $= 1$,

$$\begin{aligned}T(x,T(y,z)) = T(x,y) &= V\ (x,y)\\ &= V(x,y) \wedge z\\ &= T(V(x,y),z)\\ &= T(T(x,y),z)\end{aligned}$$

2.3. $y \parallel a$,

2.3.1. $z \in L \setminus \{1\}$,

$$\begin{aligned}T(x,T(y,z)) &= T(x,0) = 0\\ &= T(0,z)\\ &= T(T(x,y),z)\end{aligned}$$

2.3.2. $z = 1$,

$$\begin{aligned}T(x,T(y,z)) = T(x,y\wedge z) = T(x,y) &= 0\\ &= 0 \wedge z\\ &= T(0,z)\\ &= T(T(x,y),z)\end{aligned}$$

2.4. $y = 1$,

2.4.1. $z \leq a$ or $z \parallel a$,

$$\begin{aligned}T(x,T(y,z)) &= T(x,y\wedge z)\\ &= T(x,z)\end{aligned}$$

$$
\begin{aligned}
&= T(x \wedge y, z) \\
&= T(T(x, y), z)
\end{aligned}
$$

2.4.2. $1 > z > a$,

$$
\begin{aligned}
T(x, T(y, z)) &= T(x, y \wedge z) \\
&= T(x, z) \\
&= T(x \wedge y, z) \\
&= T(T(x, y), z)
\end{aligned}
$$

2.4.3. $z = 1$,

$$
\begin{aligned}
T(x, T(y, z)) &= T(x, 1) = x \wedge 1 = x \\
&= x \wedge z \\
&= T(x, z) \\
&= T(x \wedge y, z) \\
&= T(T(x, y), z)
\end{aligned}
$$

3. Let $x \parallel a$,

3.1. $y \leq a$,

3.1.1. $z \in L \setminus \{1\}$,

$$
\begin{aligned}
T(x, T(y, z)) &= T(x, 0) = 0 \\
&= T(0, z) \\
&= T(T(x, y), z)
\end{aligned}
$$

3.1.2. $z = 1$,

$$
\begin{aligned}
T(x, T(y, z)) &= T(x, y \wedge z) = T(x, y) = 0 \\
&= 0 \wedge z \\
&= T(0, z) \\
&= T(T(x, y), z)
\end{aligned}
$$

3.2. $1 > y > a$,

3.2.1. $z \leq a$ or $z \parallel a$,

$$
\begin{aligned}
T(x,T(y,z)) &= T(x,0)=0\\
&= T(0,z)\\
&= T(T(x,y),z)
\end{aligned}
$$

3.2.2. $1>z>a$,

$$
\begin{aligned}
T(x,T(y,z)) &= T(x,V(y,z))=0\\
&= T(0,z)\\
&= T(T(x,y),z)
\end{aligned}
$$

3.2.3. $z=1$,

$$
\begin{aligned}
T(x,T(y,z)) &= T(x,y\wedge z)=T(x,y)=0\\
&= 0\wedge z\\
&= T(0,z)\\
&= T(T(x,y),z)
\end{aligned}
$$

3.3. $y \parallel a$,

3.3.1. $z\in L\setminus\{1\}$,

$$
\begin{aligned}
T(x,T(y,z)) &= T(x,0)=0\\
&= T(0,z)\\
&= T(T(x,y),z)
\end{aligned}
$$

3.3.2. $z=1$,

$$
\begin{aligned}
T(x,T(y,z)) &= T(x,y\wedge z)=T(x,y)=0\\
&= 0\wedge z\\
&= T(0,z)\\
&= T(T(x,y),z)
\end{aligned}
$$

3.4. $y=1$,

3.4.1. $z\le 1$,

$$
\begin{aligned}
T(x,T(y,z)) &= T(x,y\wedge z)\\
&= T(x,z)\\
&= T(x\wedge y,z)\\
&= T(T(x,y),z)
\end{aligned}
$$

3.4.2. $1 > z > a$,

$$\begin{aligned} T(x,T(y,z)) &= T(x, y\wedge z) \\ &= T(x,z) \\ &= T(x\wedge y, z) \\ &= T(T(x,y),z) \end{aligned}$$

3.4.2. $z \parallel a$,

$$\begin{aligned} T(x,T(y,z)) &= T(x, y\wedge z) \\ &= T(x,z) \\ &= T(x\wedge y, z) \\ &= T(T(x,y),z) \end{aligned}$$

3.4.4. $z = 1$,

$$\begin{aligned} T(x,T(y,z)) &= T(x,1) = x\wedge 1 = x \\ &= x\wedge z \\ &= T(x,z) \\ &= T(x\wedge y, z) \\ &= T(T(x,y),z) \end{aligned}$$

4. Let $x = 1$.

4.1. $y \leq a$

4.1.1. $z \in L \setminus \{1\}$,

$$\begin{aligned} T(x,T(y,z)) &= T(x,0) = x\wedge 0 = 0 \\ &= T(y,z) \\ &= T(x\wedge y, z) \\ &= T(T(x,y),z) \end{aligned}$$

4.1.2. $z = 1$,

$$\begin{aligned} T(x,T(y,z)) &= T(x,y\wedge z) = T(x,y) = x\wedge y = y \\ &= y\wedge z \\ &= T(y,z) \\ &= T(x\wedge y, z) \\ &= T(T(x,y),z) \end{aligned}$$

4.2. $1 > y > a$,

4.2.1. $z \leq a$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, 0) = x \wedge 0 = 0 \\ &= T(y, z) \\ &= T(x \wedge y, z) \\ &= T(T(x, y), z) \end{aligned}$$

4.2.2. $1 > z > a$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, V(y, z)) = x \wedge V(y, z) = V(y, z) \\ &= T(y, z) \\ &= T(x \wedge y, z) \\ &= T(T(x, y), z) \end{aligned}$$

4.2.3. $z \parallel a$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, 0) = x \wedge 0 = 0 \\ &= T(y, z) \\ &= T(x \wedge y, z) \\ &= T(T(x, y), z) \end{aligned}$$

4.2.4. $z = 1$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, y \wedge z) = T(x, y) = x \wedge y = y \\ &= y \wedge z \\ &= T(y, z) \\ &= T(x \wedge y, z) \\ &= T(T(x, y), z) \end{aligned}$$

4.3 $y \parallel a$,

4.3.1. $z \in L \setminus \{1\}$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, 0) = x \wedge 0 = 0 \\ &= T(y, z) \\ &= T(x \wedge y, z) \\ &= T(T(x, y), z) \end{aligned}$$

4.3.2. $z = 1$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, y \wedge z) = T(x, y) = x \wedge y = y \\ &= y \wedge z \\ &= T(y, z) \\ &= T(x \wedge y, z) \\ &= T(T(x, y), z) \end{aligned}$$

4.4 $y = 1$ and $z \in L$

$$\begin{aligned} T(x, T(y, z)) &= T(x, y \wedge z) = T(x, z) = x \wedge z = z \\ &= 1 \wedge z \\ &= T(1, z) \\ &= T(T(x, y), z) \end{aligned}$$

So, we have the fact that T is a t-norm on L. Similarly, it can be shown the fact that S is a t-conorm L

Corollary 2.3.

Let $a \in L \setminus \{0, 1\}$. If we put $V(x, y) = \begin{cases} x \wedge y & 1 \in \{x, y\}, \\ a & otherwise. \end{cases}$ on $[a, 1]$ in the formula (3) in Theorem 1, the following t-norm is the smallest t-norm on L that extends V.

$$T(x, y) = \begin{cases} a & if (x, y) \in [a, 1]^2, \\ x \wedge y & if\ 1 \in \{x, y\}, \\ 0 & otherwise. \end{cases}$$

Corollary 2.4.

Let $a \in L \setminus \{0, 1\}$. If we put $W(x, y) = \begin{cases} x \vee y & 0 \in \{x, y\}, \\ a & otherwise. \end{cases}$ on $[0, a]$ in the formula (4) in Theorem 1, the following t-conorm is the greatest t-conorm on L that extends W.

$$S(x, y) = \begin{cases} a & if (x, y) \in [0, a]^2, \\ x \vee y & if\ 1 \in \{x, y\}, \\ 1 & otherwise. \end{cases}$$

Example 2.4

Given a bounded lattice $L = \{0, t, k, a, s, m, 1\}$ with order given in Figure 3.

(i) Consider the t-norm $V : [a,1]^2 \to [a,1], V(x,y) = x \wedge y$. By using Theorem 1, the corresponding t-norm $T : L^2 \to L$ is given as Table 3

T	0	t	a	s	k	m	1
0	0	0	0	0	0	0	0
t	0	0	0	0	0	0	t
a	0	0	a	0	0	a	a
s	0	0	0	0	0	0	s
k	0	0	0	0	0	0	k
m	0	0	a	0	0	m	m
1	0	t	a	s	k	m	1

Table 3: The t-norm T on L

(ii) Consider the t-norm $V : [a,1]^2 \to [a,1]$, $V(x,y) = \begin{cases} x \wedge y & 1 \in \{x,y\}, \\ a & otherwise. \end{cases}$. By using Theorem 1, the corresponding t-norm $T : L^2 \to L$ is given as Table 4.

T	0	t	s	k	a	m	1
0	0	0	0	0	0	0	0
t	0	0	0	0	0	0	t
s	0	0	0	0	0	0	s
k	0	0	0	0	0	0	k
a	0	0	0	0	a	a	a
m	0	0	0	0	a	a	m
1	0	t	s	k	a	m	1

Table 4: The t-norm T on L

Example 2.5

Given a bounded lattice$L = \{0, t, a, s, k, m, 1\}$ with order given in Figure 4.

(i) Consider the t-conorm $W : [0,a]^2 \to [0,a], W(x,y) = x \vee y$. By using Theorem 1, the corresponding t-conorm $S : L^2 \to L$ is given as Table 5.

S	0	t	a	k	s	m	1
0	0	t	a	k	s	m	1
t	t	t	a	1	1	1	1
a	a	a	a	1	1	1	1
k	k	1	1	1	1	1	1
s	s	1	1	1	1	1	1
m	m	1	1	1	1	1	1
1	1	1	1	1	1	1	1

Table 5: The t-conorm T on L

(ii) Consider the t-conorm $W : [0,a]^2 \to [0,a]$, $W(x,y) = \begin{cases} x \vee y & 0 \in \{x,y\}, \\ a & otherwise. \end{cases}$ By using Theorem 1, the corresponding t-conorm $S : L^2 \to L$ is given as Table 6.

S	0	t	a	k	s	m	1
0	0	t	a	k	s	m	1
t	t	a	a	1	1	1	1
a	a	a	a	1	1	1	1
k	1k	1	1	1	1	1	1
s	1	1	1	1	1	1	1
m	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1

Table 6: The t-conorm T on L

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الخلاصة

في هذه المذكرة، نوضح أن المجموعة An (L) لجميع التجميعات n-ary الوظائف على شبكة كاملة L هي شبكة كاملة ونحن ندرسبعض خصائص هذه الشبكة. إذا كانت L عبارة عن شبكة محدودة و n هي a عدد طبيعي مثل n ≠ 0 نحصل على Lm لـــ m ∈ [n] يمكن تضمينها في الشبكة An (L). نحن نولد التجميع وظائف من وظائف رتيبة. نقدم مفهوم المنتج الداخلي لوظائف التجميع. نعطي بعض الأمثلة على وظائف التجميع على المشابك المحدودة.

Résume

Dans ce mémoire, nous montrons que l'ensemble An(L) de toutes les agrégations n-aires fonctions sur un réseau complet L est un réseau complet et nous étudions quelques propriétés de ce réseau. Si L est un réseau borné et n est un entier naturel tel que n ≠ 0, on obtient que Lm pour m ∈ [n] peut être noyé dans le treillis An(L). Nous générons l'agrégation fonctions à partir de fonctions monotones. Nous introduisons le concept de produit interne des fonctions d'agrégation. Nous donnons quelques exemples de fonctions d'agrégation sur les réseaux bornés.

Abstract

In this memoire, we show that the set $A_n(L)$ of all *n*-ary aggregation functions on a complete lattice *L* is a complete lattice and we study some properties of this lattice. If *L* is a bounded lattice and *n* is a natural number such that $n \neq 0$, we obtain that L^m for $m \in [n]$ can be embedded into the lattice $A_n(L)$. We generate aggregation functions from monotone functions. We introduce the concept of internal product of aggregation functions.We give some examples of aggregation functions on bounded lattices.