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OPERATOR IDEALS AND s-NUMBERS

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OPERATOR IDEALS AND s-NUMBERS

Thanks

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Contents

Notations

$\mathbb{K}$	The field of real or complex numbers.
$\mathbb{R}$	The reel space.
X^*	The topological dual of X.
B_X	The closed unit ball of X.
$\overset{\circ}{B}_X$	The open unit ball of X.
$\mathcal{L}(X,Y)$	The set of all bonded linear operators.
$\mathcal{L}_f$	The set of all finite rank linear operators.
$\sigma(X,X^*)$	The weak topology is called w-convergence.
T^*	The adjoint linear operator of T.
$\mathcal{I}$	The deal of all linear operator.
$\mathcal{K}$	The set of all compact linear operators.
J_M^X	The canonical injection from the subspace M of X into X.
$\mathcal{K}$	The closed ideal of compact operators.
ℓ_p	The Banach spaces of p-summable scalar sequences ($1 \leq p < \infty$).
ℓ_∞	The Banach space of bounded scalar sequences.
c_0	The subspace of ℓ_∞ consisting of the scalar sequences which converges to 0.

0.1 Introduction

The operator ideal theory has a special importance in functional analysis. One of the most important methods to construct operator ideals is via s-numbers. The definition of s-numbers goes back to Schmidt [21], who used this concept in the theory of non-selfadjoint integral equations. In Banach spaces there are many diferent possibilities of defining some equivalents of s-numbers, namely Kolmogorov numbers, Gelfand numbers, approximation numbers, and several others (see [15], [3] and the references therein). The theories of s-numbers and operator ideals, which are both closely related to geometry and local theory of Banach spaces, and also to probability on Banach spaces, were already developed in the 1970s and 1980s, with main contributions due to Pietsch. During the last 15 years these by now almost classical abstract functional-analytic concepts appeared quite naturally in several, more applied branches of mathematics [11]. In particluar (see [11]), they have found important applications in areas such as:

• Compressed Sensing and Image Processing (Gelfand numbers, Johnson-Lindenstrauss lemma)

• Numerical Analysis and Information-based Complexity (approximation and entropy numbers, 2-summing operators, Banach spaces of type)

• Function Spaces (various s-numbers, operator ideal techniques)

• Approximation Theory (abstract approximation spaces)

• Small Deviations of Gaussian Processes (entropy numbers)

• Statistical Learning Theory (covering numbers).

The main aims of this memory were to present an axiomatic theory of s-numbers and we discuss related operator ideals obtained in this way. In order to achieve these goals the memory was organized as follows.

In the first chapter (preliminaries), we recall some basic notions, properties, and terminologies needed in our memory. For example metric injection and metric surjection, finite rank operators, compact operators,...etc.

In the second chapter, we present an axiomatic theory of s-numbers (see [15], [19] and the references therein) and some special properties of s-numbers such as injectivity, surjectivity, additivity ...etc. Then we treat several important example of s-numbers starting from the approximation numbers, then Galfand numbers and Weyl numbers, after them Kolmogorov numbers and Chang numbers, finally Hilbert numbers. Next we study the dual s-numbers. Furthermore, we investigate the relationship between different s-numbers (see [15], [18]).

In the last chapter, we deal with operator ideals related to s-numbers, we recall the main definitions and properties of the theory of operator ideals that we will use in this chapter. We study the operator ideals generated by an additive s-function and we recall the definition and the basic properties of entropy numbers after this we study the quasi-normed operator ideals related to outer entropy numbers and the quasi-normed operator ideals generated by the approximation numbers, as application, we characterize the ”compactness” of an operator. Finally we investigate the relation between them.

CHAPTER 1

PRELIMINARIES

In this chapter we present a collection of some definitions, properties and basic formulas that will benefit us during this work (see [1], [19] and [17]). For example operator on Banach spaces, isomorphisms, injections, surjections, and projections, finite rank operators, compact linear operators, the metric extension property and the metric lifting property.

We will write $\mathbb{K}$ for the real numbers field $\mathbb{R}$ or the complex numbers field $\mathbb{C}$. The set of all natural numbers $\{0, 1, \ldots\}$ is denoted by $\mathbb{N}$. Along this work the letters X and Y denotes Banach spaces with the norm $\|.\|$. The open unit ball of X is denoted by $\overset{\circ}{B}_X$ that is the set $\{x \in X : \|x\| < 1\}$ and the closed unit ball of X is denoted by B_X that is the set $\{x \in X : \|x\| \leq 1\}$. The set of all *functionals* of a normed space X (that is the continuous linear mapping from X into the scalars) is a Banach space denoted by X^* and called the *topological dual* of X. For $x \in X$ we shall write $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$ for the action of the functional x^* on x. The norm of $x^* \in X^*$ is

$$\|x^*\| = \sup\{|\langle x, x^*\rangle| : x \in B_X\}.$$

1.1 Operators on Banach spaces

We denote by $\mathcal{L}(X, Y)$ the Banach space of all continuous linear operators between X and Y with the norm

$$\|T\| = \sup_{x \in B_X} \|T(x)\|.$$

We write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$. If $T \in \mathcal{L}(X, Y)$, the continuous linear operator $T^* : Y^* \longrightarrow X^*$ defined as

$$T^*(y^*)(x) = y^*(T(x)),$$

for every $y^* \in Y^*$ and $x \in X$ is called the *adjoint operator of* T with $\|T\| = \|T^*\|$.

Definition 1.1.1. For every operator $T \in \mathcal{L}(X, Y)$ we define **null space**

$$\ker(T) = \{x \in X : T(x) = 0\}.$$

And the **range** (image of T)

$$range(T) = \{T(x) : x \in X\}.$$

Both of these subsets are linear, and $range(T)$ is always closed.

Definition 1.1.2. i) (Identity operator) The identity operator $Id_X : X \longrightarrow X$ is defined by $Id_X(x) = x$ for all $x \in X$, we simply use the symbol I. In case X is a non-empty normed linear space, the operator Id_X is bounded and the norm $\|Id_X\| = 1$.

ii) The zero operator $0 : X \to Y$ where X and Y are normed linear spaces, is bounded and the norm $\|0\| = 0$.

Definition 1.1.3. i) An operator $T \in \mathcal{L}(X,Y)$ is invertible if there exists $S \in \mathcal{L}(Y,X)$ such that $ST = Id_X$ and $TS = Id_Y$. If so, then the inverse operator S, usually denoted by T^{-1}, is uniquely determined.
ii) An operator $T \in \mathcal{L}(X,Y)$ is invertible if and only if it is one-to-one (injective) and onto (surjective). i.e., $\ker(T) = \{0\}$ and $range(T) = Y$.

Definition 1.1.4. (The quotient space)

Let N be any subspace of a Banach space E. Then the quotient space $E|N$ consists of all equivalence classes $x+N$. It turns out that $E|N$ becomes a Banach space with respect to the norm

$$\|x + N\| = \inf \{\|x + y\| : y \in N\}.$$

Proposition 1.1.5. *Let N be a linear subset of a linear space E such that the quotient space $E|N$ is finite dimensional. Then, the codimension of N in E is defined by the formula*

$$codim(N) = \dim(E|N).$$

1.2 Isomorphisms, injections, surjections, and projections

1) Isomorphisms

$U \in \mathcal{L}(X,Y)$ is said to be an *isomorphism* if U admits an inverse $U^{-1} \in \mathcal{L}(Y,X)$, or

$$a\|x\| \le \|U(x)\| \le b\|x\| \quad \text{for all } x \in X$$

where a and b are positive constants. Banach spaces X and Y are called isomorphic if there exists an isomorphism $U \in \mathcal{L}(X,Y)$. Then we write $X \cong Y$.
If $\|U\| = \|U^{-1}\| = 1$, then U is said to be a metric isomorphism.

One refers to a linear map U from X onto Y as an isometry if it preserves the norm:

$$\|U(X)\| = \|x\| \quad \text{for all } x \in X.$$

2) Injections

An continuous linear operator $J : X \to Y$ is said to be an *injection* (is an isomorphism) if there exists a constant$C > 0$ such that $\|J(x)\| \geq C\|x\|$ for all $x \in X$.

In this case that $\|J(x)\| = \|x\|$, we use the term *metric injection* (isometric isomorphism).

For every closed subspace M of X, the canonical embedding J_M^X from M into X has this property. If there exists an injection from X into Y, then we say that Y contains an isomorphic copy of X.

A linear operator T is an *embedding* of X into Y if T is an isomorphism onto its image $T(X)$. In this case we say that X *embeds* in Y. If $T : X \longrightarrow Y$ is an embedding such that $\|T(x)\| = \|x\|$ for all $x \in X$, then T is said to be an *isometric embedding*.

3) Surjections

An operator $Q \in \mathcal{L}(X, Y)$ is called a *surjection* if $q(Q) > 0$ such that $q(Q) = \sup\{C \geq 0 : Q(B_X) \supseteq CB_Y\}$. In the case that the open unit ball of X is mapped onto the open unite ball of Y, we use the term *metric surjection*.

For every closed subspace N of Y, the quotient map Q_N^Y from Y onto $Y|N$ has this property. If there exists a *surjection* from X onto Y, then we say that Y is a continuous image of X.

3) Projections

Definition 1.2.1. The map $P \in \mathcal{L}(X)$ is an *projection* if $P^2 = P$.

Definition 1.2.2. A Banach space E is the direct sum of the subspaces M and N if $E = M + N$ and $M \cap N = \{0\}$. In this case, we write $E = M \oplus N$.

The above conditions mean that every element $x \in E$ admits a unique decomposition $x = u + v$ with $u \in M$ and $v \in N$. Then $P \in \mathcal{L}(E)$ defined by $P(x) = u$ is called the *projection of* E *onto* M *along* N.

Definition 1.2.3. Let M be any subspace of a Hilbert space H. Then

$$M^{\perp} = \{x \in H : (x, y) = 0, \quad \text{for all } y \in M\}$$

is called the orthogonal complement of M. It turns out that $H = M \oplus M^{\perp}$. The projection P from H onto M along $M^{\perp}$ is said to be the *orthogonal projection* from H onto M. Note that $\|P\| = 1$ whenever $M \neq \{0\}$.

Proposition 1.2.4. *The concepts of an* injection *and* surjection *are dual to each other. This means that:*

- J *is an* injection *if and only if* J^* *is a* surjection.
- Q *is a* surjection *if and only if* Q^* *is an* injection.

1.3 Finite rank operators

Definition 1.3.1. A linear operator $T \in \mathcal{L}(X, Y)$ is said to be *finite rank* if $T(X)$ is finite dimensional (and that dimension is called the rank of T). The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(X, Y)$.
- Note that

$$rank(T) = codim(\ker(T)) = \dim(range(T)).$$

Example 1.3.2.

- The zero operator is a *finite rank* operator with zero rank.
- The linear forms $\varphi : X \to \mathbb{K}$ are finite rank.

- If $\dim X = n$ then $T \in \mathcal{L}_f(X, Y)$.

Remark 1.3.3. ([1]) We can find an example of an unbounded operator on a Banach space whose rank is finite. In fact, the problem then reduces to finding an unbounded functional (which has finite rank, of course); such functionals exist iff X is infinite-dimensional. To construct an unbounded functional, pick an infinite independent set in X, say $\{e_1, e_2, e_3, \ldots\}$ with $\|e_n\| = 1, \forall n \in \mathbb{N}$. Define $T(e_n) = n, \forall n \in \mathbb{N}$. This defines T on $Span\{e_1, e_2, e_3, \ldots\}$. Now complete the set $\{e_1, e_2, e_3, \ldots\}$ to a basis and define T to be 0 on other basis elements. We have thus defined a linear functional T on X; it is easy to check that it is not bounded. Namely, since $\|e_n\| = 1$ and $T(e_n) = n$, we have $\|T\| \geq n$, and this holds for every $n \in \mathbb{N}$.

Proposition 1.3.4. *Here are two immediate properties of finite rank operators.*
1) Sums and scalar products of finite rank operators are finite rank.
2) If in a scheme of operators $X \xrightarrow{T} Y \xrightarrow{S} E$ between Banach spaces either T or S is finite rank, then $S \circ T$ is finite rank.

Proof. 1) i) Clearly the sum of two operators of finite rank has finite rank, since the range is contained in the sum of the ranges (but is often smaller):

$$(T + S)(x) \in range(T) + range(S).$$

ii) Since the range of a constant multiple of T is contained in the range of T it follows that the finite rank operators form a linear subspace of $\mathcal{L}_f(X, Y)$.
2) Indeed, the range of ST is the range of S restricted to the range of T and this is certainly finite dimensional since it is spanned by the image of a basis of range(T). Similarly $ST \in \mathcal{L}_f$ since the range of ST is contained in the range of T. □

Proposition 1.3.5. *The space $\mathcal{L}_f(X, Y)$ is generated by the mappings of the special form*

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle\, y$$

i.e., if $T \in \mathcal{L}_f(X,Y)$ we have

$$T = \sum_{i=1}^{n} x_i^* \otimes y_i = \sum_{i=1}^{n} x_i^*(.)y_i, \tag{1.1}$$

where $(x_i^*)_{i=1}^n \subset X^*$ and $(y_i)_{i=1}^n \subset Y$.

Proof. First, since $range(T)$ is finite dimensional, we can choose a basis $v_1, \ldots, v_n$, for it. Then, for any element $x \in X$,

$$T(x) = \sum_{j=1}^{n} \alpha_j v_j. \tag{1.2}$$

The constants α_j are determined, since the v_j are a basis, and so define linear functionals $x_j^* : x \to \alpha_j$. These are continuous. In fact, there are unit vectors $v_1^*, \ldots, v_n^* \in range(T)^*$, such that

$$\langle v_j, v_i^* \rangle = \delta_{ij}, \quad 1 \leq i, j \leq n.$$

and then, pairing (1.2) with v_i^* we see that

$$\langle T(x), v_i^* \rangle = \langle x, T^*(v_i^*) \rangle = \sum_{j=1}^{n} \alpha_j \langle v_j, v_i^* \rangle = \alpha_i.$$

By Hahn-Banach Theorem, there exists $y_j^* \in Y^*$, such that y_j^* is v_i^* on $range(T)$ (i.e., $x_j^* = T^*(y_j^*) = T^*(v_i^*) \in X^*$ such that

$$T(x) = \sum_{j=1}^{n} T^*(y_j^*)(x) v_j = \sum_{j=1}^{n} x_j^*(x) v_j. \tag{1.3}$$

That is,$T = \sum_{j=1}^{n} x_j^* \otimes v_j$. Conversely, if T can be written in the form (1.1), then it is of finite rank, since its range is contained in the span of the y_i. □

Proposition 1.3.6. *If $T : X \to Y$ is an operator of rank n between Banach spaces with a representation (1.1), then its norm adjoint $T^* : Y^* \to X^*$ is also an operator of rank n and $T^* = \sum_{i=1}^{n} y_i \otimes x_i^*$.*

Proof. From (1.1) it follows that T^* is also of finite rank since

$$\langle T^*(y^*), x\rangle = \langle y^*, T(x)\rangle = \sum_{j=1}^{n} x_j^*(x)y^*(y_j) = \sum_{j=1}^{n} k_Y(y_j)(y^*)x_j^*(x),$$

where, $k_Y : Y \to Y^{**}$ is the canonical embedding defined by $k_Y(y)(y^*) = y^*(y)$. Consequently,

$$T^* = \sum_{j=1}^{n} k_Y(y_j) \otimes x_j^*.$$

This completes the proof of the claim. □

1.4 Compact linear operators

The concept of *compact* operators goes back to F. Riesz's work Circa 1918 (see [20]).

Definition 1.4.1. A continuous linear operator T is *compact* if the closed unit ball B_X is mapped into a subset $T(B_X)$ which is *relatively compact* in the norm topology.

An equivalent formulation is that T is *compact* if and only if every bounded sequence $(x_i)_{i=1}^{\infty}$ in X has a subsequence $(x_{i_k})_{k=1}^{\infty}$ such that $(Tx_{i_k})_{k=1}^{\infty}$ converg in Y. Equivalently (see[6, p.6] and [12]), T is *compact* if and only if for every $\varepsilon > 0$, there exists elements $y_1, y_2, \dots, y_n \in Y$ such that

$$T(B_X) \subseteq \bigcup_{k=1}^{n} \{y_k + \varepsilon B_Y\},$$

where by B_X and B_Y we mean the closed unit balls of X and Y, respectively. Every *compact* linear operator is bounded, hence continuous, but clearly not every bounded linear map is *compact* since one can take the identity operator on an infinite dimensional space X.

The class of all compact operator from a Banach space X into another Banach space Y will be denoted by $\mathcal{K}(X, Y)$.

Example 1.4.2. Every *finite rank* operator between Banach spaces is *compact*.

Proposition 1.4.3. *$\mathcal{K}(X,Y)$ is closed subspace of $\mathcal{L}(X,Y)$.*

Proposition 1.4.4.

- *Sums and scalar multiples of compact operators are compact operators.*
- *The composition of operators at least one of which is compact is a compact operators.*
- *An operator is compact if and only if its adjoint is compact.*

1.5 The metric extension property and the metric lifting property

1) The metric extension property: A Banach space Y has *the extension property* if every operator T_0 defined on a closed subspace M of an arbitrary Banach space X into Y admits an extension T from X into Y such that the following diagram commutes

$$\begin{array}{ccc} X & \overset{T}{\searrow} & \\ J_M^X \uparrow & & \\ M & \underset{T_0}{\longrightarrow} & Y \end{array}$$

Here J_M^X denotes the canonical injection from M into X. We use the term *metric extension property* if the operator T can be chosen such that $\|T\| = \|T_0\|$.

1) The metric lifting property: A Banach space X has *the lifting property* if every operator T_0 mapping X into a quotient space $Y|N$ of an arbitrary Banach space Y, admits a lifting T.

Here Q denotes the canonical surjection from Y onto N. We use the term *metric lifting property* if for $\varepsilon > 0$, the operator T from X into Y can be chosen such that $\|T\| \leq (1+\varepsilon)\|T_0\|$.

CHAPTER 2

s-NUMBER FUNCTIONS

The theory of s-numbers of linear bounded operators among Banach spaces was introduced and studied by Pietsch [19, 18]. It plays a fundamental role in the theory of operators and the local theory of Banach spaces and it is a powerful tool in the study of eigenvalue distribution of Riesz operators in Banach spaces (see, e.g., [14, 15]).

2.1 Definition and general properties

In the theory of s-numbers, one associates with every operator T various kinds of scalar sequences $s_1(T) \geq s_2(T) \geq ... \geq 0$. The main purpose is to classify operators by the behavior of $s_n(T)$ as $n \to \infty$. The operator ideals obtained in this way will be discussed in Chapter.3.

Definition 2.1.1. *A map $s = (s_n) : \mathcal{L} \to [0, \infty]$, assigning to every operator $T \in \mathcal{L}$ a non-negative scalar sequence $(s_n(T))_{n\in\mathbb{N}}$ is called an s-number function if the following conditions are satisfied:*

(SN_1) Monotonicity: $\|T\| = s_1(T) \geq s_2(T) \geq \ldots \geq 0$ *for* $T \in \mathcal{L}(X, Y)$.

(SN_2) Additivity: $s_{m+n-1}(S+T) \leq s_m(S) + s_n(T)$ *for* $S, T \in \mathcal{L}(X, Y)$.

(SN_3) Ideal property: $s_n(STR) \leq \|S\|\, s_n(T)\, \|R\|$ *for* $R \in \mathcal{L}(E, X)$, $T \in \mathcal{L}(X, Y)$ *and* $S \in \mathcal{L}(Y, F)$.

(SN_4) Rank property: if $rank(T) < n$, *then* $s_n(T) = 0$.

(SN_5) Norming property: $s_n(I : \ell_2^n \to \ell_2^n) = 1$ *for* $n \in \mathbb{N}$ *, where* I *denote the identity operator on the* n-dimensional *Hilbert space* ℓ_2^n.

The number $s_n(T)$ is said to be the *n-th* s*-number* of the operator T.

Lemma 2.1.2. *The converse of (SN_4) is true i.e., if* $s_n(T) = 0$, *then* $rank(T) < n$

Proof. Assume that $rank(T) \geq n$, then we can find $S \in \mathcal{L}(\ell_2^n, X)$ and $B \in \mathcal{L}(Y, \ell_2^n)$ with $BTS = I$ is the identity operator of ℓ_2^n, we have

$$1 = s_n(I) \overset{(SN_5)}{=} s_n(BTS) \overset{(SN_3)}{\leq} \|B\| s_n(T) \|S\|.$$

Hence $s_n(T) > 0$. □

Theorem 2.1.3. *(The* s*-numbers are continuous functions since*

$$|s_n(T) - s_n(S)| \leq \|S - T\| \quad \textit{for } S, T \in \mathcal{L}(X, Y) \textit{ and } n = 1, 2, \ldots$$

Proof. Taking $m = 1$ by (SN_2)

$$\begin{aligned} s_{n+0}(T) = s_{n+1-1}(T - S + S) &\leq s_n(S) + s_1(T - S) \\ &\leq s_n(S) + \|T - S\|. \end{aligned}$$

Hence

$$s_n(T) - s_n(S) \leq \|T - S\|.$$

Interchanging the roles of T and S, we obtain

$$s_n(S) - s_n(T) \leq \|S - T\|.$$

This proof the desired inequality. □

Definition 2.1.4.

(A) An s-number function s is called injective if, given any metric injection $J \in \mathcal{L}(Y, F)$, that is, $\|Jy\| = \|y\| \quad$ for $y \in Y$, we have $s_n(T) = s_n(JT)$ for all $T \in \mathcal{L}(X, Y)$ and all Banach spaces X.

(B) An s-number function s is called surjective if, given any metric surjection $Q \in \mathcal{L}(E, X)$, that is, $Q(\overset{\circ}{B}_E) = \overset{\circ}{B}_X$, we have $s_n(T) = s_n(TQ)$ for all $T \in \mathcal{L}(X, Y)$ and all Banach spaces Y.

(C) An s-number function s is called multiplicative if

$$s_{m+n-1}(ST) \leq s_m(S)s_n(T) \quad \text{for all } T \in \mathcal{L}(X, Y) \text{ and } S \in \mathcal{L}(Y, F)$$

.

2.2 Examples of s-number functions

We will give examples of s-number functions. In 1963, Pietsch [16] firstly introduced the approximation numbers of bounded linear operator in Banach spaces. Subsequently, different s-numbers, namely Kolmogorov numbers, Gelfand numbers, etc.

1) Approximation numbers

For every operator $T \in \mathcal{L}(X, Y)$ the n-th approximation number ($n \in \mathbb{N}$) is defined by

$$a_n(T) = \inf \{\|T - S\| : S \in \mathcal{L}(X, Y), rank(S) < n\}.$$

Theorem 2.2.1. *A map*

$$app : T \to (a_n(T))$$

is an s-number function.

Proof. *(SN_1) Monotonicity: Let $T \in \mathcal{L}(X,Y)$*, then

$$\begin{aligned} a_1(T) &= \inf \{\|T - S\| : S \in \mathcal{L}(X,Y), rank(S) < 1\} \\ &= \|T\| . \end{aligned}$$

On the other hand,

$$\begin{aligned} \{\|T - S\| : S \in \mathcal{L}(X,Y), rank(S) < 1\} &\subset \{\|T - S\| : S \in \mathcal{L}(X,Y), rank(S) < 2\} \\ &\vdots \\ &\subset \{\|T - S\| : S \in \mathcal{L}(X,Y), rank(S) < n\} . \end{aligned}$$

Hence

$$\begin{aligned} \inf \{\|T - S\| : S \in \mathcal{L}(X,Y), rank(S) < 1\} &\geq \inf \{\|T - S\| : S \in \mathcal{L}(X,Y), rank(S) < 2\} \\ &\vdots \\ &\geq \inf \{\|T - S\| : S \in \mathcal{L}(X,Y), rank(S) < n\} \\ &\geq 0. \end{aligned}$$

(SN_2) Additivity: Let $T, S \in \mathcal{L}(X,Y)$: Given $\varepsilon > 0$, we choose $u, v \in \mathcal{L}(X,Y)$ such that $rank(u) < n$ and $rank(v) < m$ and $\|T - u\| \leq (1+\varepsilon)a_n(T)$ and $\|S - v\| \leq (1+\varepsilon)a_m(S)$. Then $rank(T+S) < n + m - 1$ and

$$\begin{aligned} a_{n+m-1}(T+S) &\leq \|T + S - (u+v)\| \\ &\leq \|T - u + S - v\| \\ &\leq \|T - u\| + \|S - v\| \\ &\leq (1+\varepsilon)\left[a_n(T) + a_m(S)\right] . \end{aligned}$$

(SN_3) Ideal property: Let $R \in \mathcal{L}(E,X)$ and $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{L}(Y,F)$. We have

$$a_n(STR) = a_{n+1-1}(STR) \leq a_n(ST)a_1(R) = a_n(ST)\,\|R\|$$

and we have

$$a_n(ST) = a_{n+1-1}(ST) \leq a_n(T)a_1(S) = a_n(T)\,\|S\| .$$

Hence

$$a_n(STR) \leq \|S\|\, a_n(T)\, \|R\| .$$

(SN_4)Rank property: Let $T \in \mathcal{L}_f(X,Y)$ such that $rank(T) < n$, then

$$a_n(T) \leq \|T - S\| \text{ for every } S \in \mathcal{L}(X,Y) \text{ and } rank(S) < n.$$

Therefore, $a_n(T) \leq \|T - T\| = 0$. *Hence* $a_n(T) = 0$.

(SN_5) *Norming property: suppose that*

$a_n(I : \ell_2^n \to \ell_2^n) < 1$, *then there exists* $S \in \mathcal{L}(\ell_2^n, \ell_2^n)$ *with* $rank(S) < n$ *and* $\|I-S\| < 1$.

Hence $S = I-(I-S)$ is invertible. Therefore we obtain $rank(S) = \dim(range(S)) \geq n$. Contradiction. □

Corollary 2.2.2. *Let* $T \in \mathcal{L}(X, Y)$

$a_n(T) = 0$ *if and only if* $rank(T) < n$.

Proposition 2.2.3. *Let* X, Y, F *be Banach spaces and* $m, n \in \mathbb{N}$. *Then*

(a) $a_{m+n}(S + T) \leq a_m(S) + a_n(T)$, $\forall T, S \in \mathcal{L}(X, Y)$.

(b) $|a_m(S) - a_m(T)| \leq \|S - T\|$, $\forall T, S \in \mathcal{L}(X, Y)$.

(c) $a_{m+n}(S \circ T) \leq a_m(S)a_n(T)$, $\forall S \in \mathcal{L}(Y, F), \forall T \in \mathcal{L}(X, Y)$.

(d) $a_n(\lambda T) = \lambda a_n(T)$, $\forall n \in \mathbb{N}$ *and all* $\lambda \in \mathbb{K}$.

Proof. (a) *For* $u, v \in \mathcal{L}(X, Y)$ *with* $rank(u) < m$ *and* $rank(v) < n$. *We have* $rank(u + v) < n + m$ and

$$\begin{aligned} a_{m+n}(S+T) &\leq \|S + T - (u + v)\| \\ &\leq \|S - u + T - v\| \\ &\leq \|S - u\| + \|T - v\| . \end{aligned}$$

This implies $a_{m+n}(S + T) \leq a_m(S) + a_n(T)$.

(b) *From (a) we get*

$$a_m(S) = a_{m+0}(T + S - T) \leq a_m(T) + a_0(S - T).$$

Hence

$$a_m(S) - a_m(T) \leq \|S - T\| .$$

In the same way, $a_m(T) - a_m(S) \leq \|S - T\|$.

(c) *Let* $T \in \mathcal{L}(X, Y)$ *and* $S \in \mathcal{L}(Y, F)$. *For* $u \in \mathcal{L}(Y, F), v \in \mathcal{L}(X, Y)$ *with* $rank(u) < n$ *and* $rank(v) < m$ *the operator* $w = u \circ T + (S - u) \circ v$ *is in* $\mathcal{L}(X, F)$ *and* $rank(w) < m + n$.

Hence

$$\begin{aligned} a_{m+n}(S \circ T) &\le \|S \circ T - w\| \\ &= \|S \circ T - u \circ T - (S-u) \circ v\| \\ &= \|(S-u) \circ T - (S-u) \circ v\| \\ &= \|(S-u) \circ (T-v)\| \le \|S-u\| \, \|T-v\| \, . \end{aligned}$$

This implies $a_{m+n}(S \circ T) \le a_m(S) a_n(T)$.

(d) Let $T \in \mathcal{L}(X,Y)$. For $S \in \mathcal{L}(X,Y)$ with $rank(S) < n$. We have $rank(\frac{S}{\lambda}) < n$ and

$$\begin{aligned} a_n(\lambda T) &\le \|\lambda T - S\| \\ &\le \left\|\lambda (T - \tfrac{S}{\lambda})\right\| \\ &= |\lambda| \left\|T - \tfrac{S}{\lambda}\right\| . \end{aligned}$$

Therefore $a_n(\lambda T) \le |\lambda| \, a_n(T)$. □

Theorem 2.2.4. *The n-th approximation numbers are the largest s-numbers.*

Proof. Let $T \in \mathcal{L}(X,Y)$. Then for each s-number function s and $L \in \mathcal{L}(X,Y)$ with $rank(L) < n$ we have

$$s_n(T) = s_{n+1-1}(T - L + L) \le s_n(L) + s_1(T - L) = \|T - L\| \, .$$

Hence,

$$s_n(T) \le a_n(T) \quad \text{for all } T \in \mathcal{L}(X,Y).$$

□

2) Gelfand and Wely numbers "Injective s-numbers"

Definition 2.2.5. (**Gelfand numbers**) The n-th Gelfand number of $T \in \mathcal{L}(X,Y)$ is defined by

$$c_n(T) = \inf \left\{ \left\| T J_M^X \right\| : M \subset X, Codim(M) < n \right\},$$

with J_M^X denotes the canonical injection from the subspace M of X into the Banach space X. Hence $T J_M^X$ is the restriction of T to M.

Theorem 2.2.6. *The map*

$$gel : T \to (c_n(T))$$

is an injective s-number function.

Proof. i) (SN_1) Monotonicity: Let $T \in \mathcal{L}(X,Y)$, we have

$$c_1(T) = \inf\left\{\left\|TJ_M^X\right\| : M \subset X, Codim(M) < 1\right\} = \|T\|.$$

On other hand

$$\begin{aligned}\left\{\left\|TJ_M^X\right\| : M \subset X, Codim(M) < 1\right\} &\subset \left\{\left\|TJ_M^X\right\| : M \subset X, Codim(M) < 2\right\}\\ &\vdots\\ &\subset \left\{\left\|TJ_M^X\right\| : M \subset X, Codim(M) < n\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}\inf\left\{\left\|TJ_M^X\right\| : M \subset X, Codim(M) < 1\right\} &\geq \inf\left\{\left\|TJ_M^X\right\| : M \subset X, Codim(M) < 2\right\}\\ &\geq \ldots\\ &\geq \inf\left\{\left\|TJ_M^X\right\| : M \subset X, Codim(M) < n\right\}\\ &\geq 0.\end{aligned}$$

Therefore,

$$\|T\| = c_1(T) \geq c_2(T) \geq \ldots \geq c_n(T) \geq 0.$$

(SN_2) Additivity: Given $\varepsilon > 0$, and $T, S \in \mathcal{L}(X,Y)$ and M_1, M_2 subspaces of X such that $codim(M_1) < n$, $codim(M_2) < m$ and $\left\|TJ_{M_1}^X\right\| \leq (1+\varepsilon)c_n(T)$, $\left\|SJ_{M_2}^X\right\| \leq (1+\varepsilon)c_m(S)$, put $M = M_1 \oplus M_2$ then $codim(M) \leq n+m+1$ and

$$\begin{aligned}c_{n+m-1}(T+S) &\leq \left\|(T+S)J_M^X\right\|\\ &\leq \left\|TJ_{M_1}^X\right\| + \left\|SJ_{M_2}^X\right\|\\ &\leq (1+\varepsilon)(c_n(T) + c_m(S)).\end{aligned}$$

(SN_3) Ideal property: Let $R \in \mathcal{L}(E, X)$, $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, F)$. We have

$$\begin{aligned} c_n(STR) &= c_{n+1-1}(STR) \\ &\leq c_n(ST)c_1(R) \\ &= c_{n+1-1}(ST)\,\|R\| \\ &\leq \|S\|\, c_n(T)\,\|R\|\,. \end{aligned}$$

(SN_4) Rank property: Let $T \in \mathcal{L}_f(X, Y)$ such that $rank(T) < n$, then

$$c_n(T) \leq \left\|TJ_M^X\right\|.$$

If we take $M = \{0\}$ then, $c_n(T) \leq \|0\|$. Hence $c_n(T) = 0$.

(SN_5) Norming property: Let M be a subspace of ℓ_2^n with $codim(M) < n$. Then $M \neq \{0\}$ and

$$\left\|Id_{\ell_2^n} J_M^{\ell_2^n}\right\| = \left\|J_M^{\ell_2^n}\right\| = 1.$$

Therefore $c_n(Id_{\ell_2^n}) = 1$.

ii) Let M is any subspace of X with $codim(M) < n$, then

$$c_n(T) \leq \left\|TJ_M^X\right\| = \left\|JTJ_M^X\right\|.$$

Therefore $c_n(T) \leq c_n(JT)$. □

Proposition 2.2.7. *Let H is a Hilbert space. Then*

$$c_n(T) = a_n(T) \quad \textit{for all } T \in \mathcal{L}(H, Y).$$

Proof. Since $a_n(T)$ are the largest s-numbers, then $c_n(T) \leq a_n(T)$ for all $T \in \mathcal{L}(H, Y)$. We show $a_n(T) \leq c_n(T)$. Fix any natural number n, and consider an arbitrary subspace M of H with $codim(M) < n$. Let $P \in \mathcal{L}(H)$ denote the orthogonal projection from H onto M, and define $L = T - TP$. Since $M \subseteq \ker(L)$, it follows that

$$rank(L) = codim(\ker(L)) \leq codim(M) < n.$$

Therefore

$$a_n(T) \leq \|T - L\| = \|T - (T - TP)\| = \|TP\| = \left\|TJ_M^H\right\|.$$

This implies that $a_n(T) \leq c_n(T)$. □

Theorem 2.2.8. *If Y has the extension property, then*

$$c_n(T) = a_n(T) \quad \text{for all } T \in \mathcal{L}(X,Y).$$

Proof. Since $a_n(T)$ are the largest s-numbers, it is enough to show $a_n(T) \leq c_n(T)$ for all $T \in \mathcal{L}(X,Y)$. Fix any natural number n, we choose a subspace M of X such that $codim(M) < n$, then there exists an extension $T_0 \in \mathcal{L}(X,Y)$ of TJ_M^X with $\|T_0\| = \left\|TJ_M^X\right\|$. We set $L = T - T_0$. Since $M \subseteq \ker(L)$, it follows that

$$rank(L) = codim(\ker(L)) \leq codim(M) < n.$$

Therefore

$$a_n(T) \leq \|T - L\| = \|T_0\| = \left\|TJ_M^X\right\|.$$

This implies that $a_n(T) \leq c_n(T)$. □

Proposition 2.2.9. *([2, p.470]) The Gelfand number can be characterized by the approximation numbers as follows*

$$c_n(T) = a_n(J_\infty T) \quad \text{for all } T \in \mathcal{L}(X,Y),$$

where $J_\infty : Y \to \ell_\infty(B_{Y^*})$ is the metric injection defined by

$$J_\infty(y) = (\langle y, a\rangle)_{a \in B_{Y^*}},$$

with the space $\ell_\infty(B_{Y^*})$ of bounded sequences.

Proof. By the injectivity of the Gelfand numbers and Theorem 2.2.8, we have

$$c_n(T) = c_n(J_\infty T) = a_n(J_\infty T).$$

(because $\ell_\infty(B_{Y^*})$ has the extension property). □

Theorem 2.2.10. *The Gelfand numbers are the largest injective s-numbers.*

Proof. Let $T \in \mathcal{L}(X,Y)$ and let s be any injective s-function. Then

$$s_n(T) = s_n(J_\infty T) \leq a_n(J_\infty T) = c_n(T).$$

□

Definition 2.2.11. (**Weyl numbers**) The n-th Weyl number of $T \in \mathcal{L}(X,Y)$ is defined by

$$x_n(T) = \sup\left\{a_n(TS) : S \in \mathcal{L}(\ell_2, X), \|S\| \leq 1\right\}.$$

Theorem 2.2.12. *([15, Theorem 2.4.14]) The map*

$$x : T \to (x_n(T)),$$

is an injective s-number function.

Proposition 2.2.13. *([15, Proposition 2.4.20]) Let H is a Hilbert space. Then*

$$x_n(T) = a_n(T) \quad \textit{for all } T \in \mathcal{L}(H,Y).$$

3) Kolmogorov and chang numbers "Surjective s-numbers".

Definition 2.2.14. (**Kolmogorov numbers**) The n-th Kolmogorov number of $T \in \mathcal{L}(X,Y)$ is defined by

$$d_n(T) = \inf\left\{\left\|Q_N^Y T\right\| : N \subset Y, \dim(N) < n\right\},$$

with Q_N^Y denotes the canonical surjection from the Banach space Y onto the quotient space $Y|N$.

Theorem 2.2.15. *The map*

$$Kol : T \to (d_n(T))$$

is a surjective s-number function.

Proof. i) (SN_1) Monotonicity: Let $T \in \mathcal{L}(X,Y)$

$$d_1(T) = \inf\left\{\left\|Q_N^Y T\right\| : N \subset Y, \dim(N) < 1\right\} = \|T\|.$$

and, we have

$$\begin{aligned}\left\{\left\|Q_N^Y T\right\| : N \subset Y, \dim(N) < 1\right\} &\subset \left\{\left\|Q_N^Y T\right\| : N \subset Y, \dim(N) < 2\right\}\\ &\vdots\\ &\subset \left\{\left\|Q_N^Y T\right\| : N \subset Y, \dim(N) < n\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}\inf\left\{\left\|Q_N^Y T\right\| : N \subset Y, \dim(N) < 1\right\} &\geq \inf\left\{\left\|Q_N^Y T\right\| : N \subset Y, \dim(N) < 2\right\}\\ &\vdots\\ &\geq \inf\left\{\left\|Q_N^Y T\right\| : N \subset Y, \dim(N) < n\right\}\\ &\geq 0.\end{aligned}$$

Therefore,

$$\|T\| = d_1(T) \geq d_2(T) \geq \ldots \geq d_n(T) \geq 0.$$

(SN_2) Additivity: Given $\varepsilon > 0$, let $T, S \in \mathcal{L}(X,Y)$ and $N_1, N_2 \subset Y$ such that $\dim(N_1) < n_1$, $\dim(N_2) < n_2$ and $\left\|Q_{N_1}^Y T\right\| \leq (1+\varepsilon)d_{n_1}(T)$, $\left\|Q_{N_2}^Y S\right\| \leq (1+\varepsilon)d_{n_2}(S)$, put $N = N_1 \oplus N_2$. Then $\dim(N) < n_1 + n_2 + 1$

and

$$\begin{aligned}d_{n_1+n_2-1}(T+S) &\leq \left\|Q_N^Y (T+S)\right\|\\ &\leq \left\|Q_{N_1}^Y T\right\| + \left\|Q_{N_2}^Y S\right\|\\ &\leq (1+\varepsilon)(d_{n_1}(T) + d_{n_2}(S)).\end{aligned}$$

(SN_3) Ideal property: Let $R \in \mathcal{L}(E,X)$, $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,F)$. We have

$$\begin{aligned}d_n(STR) &= d_{n+1-1}(STR)\\ &\leq d_n(ST) d_1(R)\\ &= d_{n+1-1}(ST)\|R\|\\ &\leq \|S\| d_n(T) \|R\|.\end{aligned}$$

(SN_4) Rank property: Let $T \in \mathcal{L}_f(X,Y)$ such that $rank(T) < n$. Then

$$d_n(T) \leq \left\| Q_N^Y T \right\| \quad \textit{such that } \dim(N) < n.$$

If we take $N = \{0\}$ then, $d_n(T) \leq \|0\|$. Hence $d_n(T) = 0$.

(SN_5) Norming property: Let N be a subspace of ℓ_2^n with $\dim(N) < n$. Then $N \neq \ell_2^n$

Hence

$$\left\| Q_N^{\ell_2^n} Id_{\ell_2^n} \right\| = \left\| Q_N^{\ell_2^n} \right\| = 1.$$

Therefore $d_n(Id_{\ell_2^n}) = 1$.

ii) Let N is any subspace of Y with $dimN < n$. For any metric surjection $Q \in \mathcal{L}(E,X)$, we have

$$Q_N^Y T(B_X^\circ) = Q_N^Y TQ(B_E^\circ).$$

Then,

$$d_n(T) \leq \left\| Q_N^Y T \right\| \leq \left\| Q_N^Y TQ \right\|.$$

Therefore $d_n(T) \leq d_n(TQ)$. □

Proposition 2.2.16. *Let H is be a Hilbert space and $T \in \mathcal{L}(X,H)$. Then*

$$d_n(T) = a_n(T) = \inf \left\{ \|T - TP\| : rank(P) < n \right\},$$

with $P \in \mathcal{L}(H)$ is the orthogonal projection.

Proof. Let $T \in \mathcal{L}(X,H)$, $d_n(T) \leq a_n(T)$ is trivial. We show $a_n(T) \leq d_n(T)$. Given $\varepsilon > 0$, fix any natural number n. We choose $L \in \mathcal{L}(X,H)$ such that $rank(L) < n$ and $\|TQ_N - L\| \leq (1+\varepsilon)d_n(T)$. Let $P \in \mathcal{L}(H)$ be the orthogonal projection with $range(P) = range(L)$. Then

$$rank(P) = \dim(range(P)) = \dim(range(L)) = rank(L) < n.$$

Moreover,

$$a_n(T) \leq \|T - TP\| = \|(Id_H - P)TQ_N\| = \|(Id_H - P)(TQ_N - L)\| \leq \|TQ_N - L\| \leq (1+\varepsilon)d_n(T).$$

□

Theorem 2.2.17. *If X has the lifting property. Then*

$$d_n(T) = a_n(T) \quad \textit{for all } T \in \mathcal{L}(X,Y).$$

Proof. Since $a_n(T)$ are the largest s-numbers, it is enough to show $a_n(T) \leq d_n(T)$ for all $T \in \mathcal{L}(X,Y)$. Given $\varepsilon > 0$ and fix any natural number n, we choose a subspace $N \subset Y$ such that $\dim(N) < n$ and $\left\|Q_N^Y\right\| \leq d_n(T) + \varepsilon$. Then there exists a lifting $T_0 \in \mathcal{L}(X,Y)$ of $Q_N^Y T$ with $\|T_0\| \leq (1+\varepsilon)\left\|Q_N^Y T\right\|$. We set $L = T - T_0$. Since $L(x) \in N$ for all $x \in X$, we have

$$rank(L) = \dim(range(L)) \leq \dim(N) < n.$$

Therefore

$$\begin{aligned} a_n(T) &\leq \|T - L\| \\ &= \|T_0\| \\ &\leq (1+\varepsilon)\left\|Q_N^Y T\right\| \\ &\leq (1+\varepsilon)\left[d_n(T) + \varepsilon\right]. \end{aligned}$$

This implies that $a_n(T) \leq d_n(T)$. □

Proposition 2.2.18. *([4, p.334]) The kolmogorov numbers can be characterized by the approximation numbers as follows*

$$d_n(T) = a_n(TQ_1) \quad \textit{for all } T \in \mathcal{L}(X,Y),$$

where $Q_1 : \ell_1(B_X) \to X$ is the metric surjection from the space of summable sequences $\ell_1(B_X)$ onto X, defined by

$$Q_1((\xi_x)) = \sum_{x \in B_X} \xi_x x.$$

Proof. By the surjectivity of the kolmogorov numbers and Theorem 2.2.17, we have

$$d_n(T) = d_n(TQ_1) = a_n(TQ_1).$$

□

Theorem 2.2.19. *([18, Theorem 11.6.5]) The map*

$$Kol : T \to (d_n(T))$$

is the largest surjective s-function.

Proof. For each surjective s-number s. We have

$$s_n(T) = s_n(TQ_1) \leq a_n(TQ_1) = d_n(T), \quad \text{for all } T \in \mathcal{L}(X, Y).$$

□

Definition 2.2.20. **(Chang numbers)** The n-th Chang number of $T \in \mathcal{L}(X, Y)$ is defined by

$$y_n(T) = \sup \{a_n(RT) : R \in \mathcal{L}(Y, \ell_2), \|R\| \leq 1\}.$$

Theorem 2.2.21. *([15, Theorem 2.5.9]) The map*

$$y : T \to (y_n(T)),$$

is a surjective s-number function.

4) Hilbert numbers

Definition 2.2.22. The n-th Hilbert number of $T \in \mathcal{L}(X, Y)$ with X, Y be Banach spaces is defined by

$$h_n(T) = \sup \{a_n(BTA) : \|A : \ell_2 \to X\| \leq 1, \|B : Y \to \ell_2\| \leq 1\}.$$

Theorem 2.2.23. *([15, Theorem 2.6.2]) The map*

$$h : T \to (h_n(T)),$$

is the smallest s-number function.

2.3 Dual s-numbers

Definition 2.3.1. For each s-number function $s = (s_n)$, a dual s-number function $s^D = (s_n^D)$ is defined by

$$s_n^D(T) = s_n(T^*) \quad \text{for all } T \in \mathcal{L}(X,Y).$$

Definition 2.3.2. An s-number function s is called *symmetric* if

$$s_n(T) \geq s_n(T^*) \quad \text{for all } T \in \mathcal{L}(X,Y).$$

If $s_n(T) = s_n(T^*)$, then the s-number function is said to be *completely symmetric*.

Proposition 2.3.3. *([18, Proposition 11.7.3]) The approximation numbers are* symmetric, *i.e.,*

$$a_n(T^*) \leq a_n(T) \quad \textit{for } T \in \mathcal{L}(X,Y).$$

Proof. Let $T \in \mathcal{L}(X,Y)$ and $\varepsilon > 0$. We choose $L \in \mathcal{L}(X,Y)$ with $rank(L) < n$ such that $\|T - L\| \leq (1+\varepsilon)a_n(T)$. An application of Proposition 1.3.6 ensures that $rank(L^*) < n$ and $\|T^* - L^*\| \leq (1+\varepsilon)a_n(T)$.

We conclude that $a_n(T^*) \leq (1+\varepsilon)a_n(T)$. This proves that $a_n(T^*) \leq a_n(T)$. □

Proposition 2.3.4. *([10]) Let Y be a Banach space such that there exists a linear projection P from Y^{**} onto $k_Y(Y)$. Then*

$$a_n(T^*) = a_n(T), \quad \textit{for every } T \in \mathcal{L}(X,Y) \textit{ and } n \in \mathbb{N}\,.$$

Proof. Fix $\varepsilon > 0$. Then there exists a linear operator $S : X^{**} \to Y^{**}$ with $rank(S) < n$, such that $\|T^{**} - S\| < a_n(T^{**}) + \varepsilon$. Let $A = PSk_X$, where $k_X : X \to X^{**}$ is given by $k_X(x)(\xi) = \xi(x) \quad$ for all $x \in X$ and $\xi \in \mathcal{L}(X)$. Then $A : X \to Y^{**}$ with $rank(A) < n$. Our hypothesis $\|P\| = 1$ in combination with $\|k_X\| = 1$ yields

$$\|T - P\| = \|PT^{**}k_X - PSk_X\| \leq \|T^{**} - S\| < a_n(T^{**}) + \varepsilon.$$

Since ε is arbitrary, we conclude that

$$a_n(T) \leq a_n(T^{**}).$$

By the Proposition 2.3.3, we have

$$a_n(T^*) \leq a_n(T) \leq a_n(T^{**}) \leq a_n(T^*).$$

This completes of the proof . □

Definition 2.3.5. ([10, Definition 2.1]) The Kuratowski measure of non-compactness of $T \in \mathcal{L}(X,Y)$ are defined by

$\gamma(T) = \inf\{\varepsilon > 0 : T(B_X)$ may be covered by finitely many sets of diameter $\leq \varepsilon$ $\}$.

Lemma 2.3.6. *([10]) Let X, Y be Banach spaces and let $T \in \mathcal{L}(X,Y)$. Then*

T is compact if, and only if, $\gamma(T) = 0$.

Proposition 2.3.7. *([9, Proposition 3.2]) If $T \in \mathcal{L}(X,Y)$, then for all $n \in \mathbb{N}$,*

$$a_n(T) \leq a_n(T^{**}) + 2\gamma(T),$$

Proof. Let $\varepsilon > 0$ and $\lambda > \gamma(T)$. Then there are $S \in \mathcal{L}(X^{**}, Y^{**}), rank(S) < n$ such that

$$\|T^{**} - S\| \leq a_n(T^{**}) + \varepsilon,$$

Let $y_1, \ldots, y_k \in Y$ with $T(B_X) \subset \{y_1, \ldots, y_k\} + \lambda B_Y$. Let M be the linear span of $Im(S) \bigcup \{k_Y(y_i) : 1 \leq i \leq k\}$. By the principle of local reflexivity, there exists $R : M \to Y$ such that $\|R\| \leq 1 + \varepsilon$ and $Rk_Y(y_i) = y_i \quad (1 \leq i \leq k)$. Define $A = RSk_X \in \mathcal{L}(X,Y)$. Then $rank(A) < n$. For every $x \in B_X$, we choose y_i with $\|T(x) - y_i\| \leq \lambda$. Consequently

$$\begin{aligned}
\|T(x) - A(x)\| &\leq \|T(x) - y_i\| + \|y_i - A(x)\| \\
&\leq \lambda + (1+\varepsilon)\left(\|k_Y(y_i) + k_Y T(x)\| + \|T^{**}k_X(x) - Sk_X(x)\|\right) \\
&\leq \lambda + (1+\varepsilon)\left(\lambda + a_n(T^{**}) + \varepsilon\right).
\end{aligned}$$

□

Corollary 2.3.8. *$a_n(T^*) = a_n(T)$ for every compact operator $T \in \mathcal{L}(X,Y)$ and $n \in \mathbb{N}$.*

Proof. Since T^* is a linear operator among Banach spaces,

$$a_n(T^{**}) \leq a_n(T^*).$$

By the Lemma 2.3.6 If T is compact operator, $\gamma(T) = 0$ and by Proposition 2.3.3 and Proposition 2.3.7 we have

$$a_n(T) \leq a_n(T^{**}) \leq a_n(T^*) \leq a_n(T).$$

□

Proposition 2.3.9. *([18, Proposition 11.7.6]) Let $T \in \mathcal{L}(X,Y)$, then*

$$c_n(T) = d_n(T^*).$$

Proof. Let $T \in \mathcal{L}(X,Y)$. By duality there is one-to-one correspondence between subspace M of X with $codim(M) < n$ and subspace N of X^* with $\dim(N) < n$,

$$\begin{aligned} M \longrightarrow N &= \{a \in X^* : \langle x, a\rangle = 0, \text{for all } x \in M\}. \\ N \longrightarrow M &= \{x \in X : \langle x, a\rangle = 0, \text{for all } a \in N\}. \end{aligned}$$

Therefore

$$\left\|TJ_M^X\right\| = \left\|Q_N^{X^*}T^*\right\|.$$

□

Theorem 2.3.10. *Let $T \in \mathcal{L}(X,Y)$ and T is a compact operator, then*

$$c_n(T^*) = d_n(T).$$

Proof. Let $T \in \mathcal{L}(X,Y)$

- Since Q_1^* is a metric injection, the injectivity of the Gelfand numbers implies $c_n(T^*) = c_n(Q_1^*T^*) \leq a_n(Q_1^*T^*) \leq a_n(TQ_1) = d_n(T)$.

- The inverse inequality: we have X^* has the lifting property and by the Corollary 2.3.8.

  Hence

  $d_n(T) = d_n(TQ_1) = a_n(TQ_1) \overset{TQ_1 compact}{=} a_n((TQ_1)^*) = a_n(Q_1^* T^*) = c_n(Q_1^* T^*) \leq c_n(T^*)$.

□

Corollary 2.3.11. *Gelfand and Kolmogorov numbers are dual to each other.*

Proposition 2.3.12. *[15, proposition 2.5.12]*

$x_n(T) = y_n(T^*)$ *and* $y_n(T) = x_n(T^*)$ *for all* $T \in \mathcal{L}(X, Y)$.

i.e., Weyl and Chang numbers are dual to each other.

2.4 Relation between some s-numbers

In this section the approximation numbers, the Gelfand numbers, Kolmogorov numbers, Weyl numbers, Chang numbers and Hilbert numbers are compared with each other. We can consult ([15, 18]) for this.

Theorem 2.4.1. *Let* $T \in \mathcal{L}(X, Y)$, *then*

$h_n(T) \leq x_n(T) \leq c_n(T) \leq a_n(T)$ *and* $h_n(T) \leq y_n(T) \leq d_n(T) \leq a_n(T)$.

Proof. Let $T \in \mathcal{L}(X, Y)$

- $a_n(TA) = c_n(TA) \leq c_n(T) \|A\|$ *for* $A \in \mathcal{L}(\ell_2, X)$, ℓ_2 Hilbert.

  This implies $x_n(T) \leq c_n(T)$.

- $a_n(BT) = d_n(BT) \leq \|B\| d_n(T)$ *for* $B \in \mathcal{L}(Y, \ell_2)$.

  This implies $y_n(T) \leq d_n(T)$.

□

Theorem 2.4.2. *Let $T \in \mathcal{L}(X, Y)$, then*

$a_n(T) \leq 2n^{1/2} c_n(T)$ *and* $a_n(T) \leq 2n^{1/2} d_n(T)$.

Lemma 2.4.3. *(D. J. H. Garling / Y. Gordon 1971) [15, Lemma 1.7.17]. Let M be any n-codimensional subspace of X. Then, given $\varepsilon > 0$, there exists a projection $P \in \mathcal{L}(X)$ such that $M(P) = M$ and $\|P\| \leq (1+\varepsilon)n^{1/2}$.*

Proof of the **Theorem 2.4.2.** Fix any natural number n. Given $\varepsilon > 0$, we choose a subspace M of X such that

$\left\|TJ_M^X\right\| \leq (1+\varepsilon)c_n(T)$ and $codim(M) < n$.

By Lemma 2.4.3, there exists a projection $P \in \mathcal{L}(X)$ such that $M = M(P)$ and $\|P\| \leq n^{1/2}(1+\varepsilon)$. Define $L = TP$. Then $rank(L) \leq rank(P) = codim(M) < n$.

Therefore

$$\begin{aligned} a_n(T) &\leq \|T - L\| \\ &= \|T - TP\| \\ &= \|T(I-P)\| \\ &\leq \left\|TJ_M^X\right\| \|I - P\| \\ &\leq (1+\varepsilon)^2 c_n(T)(1 - n^{1/2}). \end{aligned}$$

Letting $\varepsilon \to 0$ yields $a_n(T) \leq 2n^{1/2} c_n(T)$.

The remaining estimate can be proved by the same technique.

CHAPTER 3

THE OPERATOR IDEALS RELATED TO s-NUMBERS

In this chapter, we deal with operator ideals related to s-numbers, we study the operator ideals generated by an additive s-numbers and we recall the definition and the basic properties of entropy numbers after this we study the quasi-normed operator ideals related to outer entropy numbers and the quasi-normed operator ideals generated by the approximation numbers, as application, we characterize the "compactness" of an operator. Finally we investigate the relation between them (see for instance [18, 19, 3] and the references therein).

Let X be a Banach space over $\mathbb{K}$ and $1 \leq p \leq \infty$. The classical Banach sequence spaces ℓ_p, ℓ_∞ and c_0 are defined by

$$\begin{aligned}
\ell_p &= \left\{ (x_n)_{n=1}^{\infty} \subset \mathbb{K} : \|(x_n)_{n=1}^{\infty}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\}, \quad 1 \leq p \leq \infty, \\
\ell_\infty &= \left\{ (x_n)_{n=1}^{\infty} \subset \mathbb{K} : \|(x_n)_{n=1}^{\infty}\|_\infty = \sup_{n \in \mathbb{N}} |x_n| < \infty \right\}, \quad p = \infty, \\
c_0 &= \left\{ (x_n)_{n=1}^{\infty} \subset \mathbb{K} : \lim_{n \to +\infty} x_n = 0 \right\}.
\end{aligned}$$

3.1 Banach operator Ideals

We present the definition and some concepts about the operator ideals, the reader can see ([18, 7]) for more details.

Definition 3.1.1. *An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces X* and *Y its components $\mathcal{I}(X,Y) := \mathcal{L}(X,Y) \cap \mathcal{I}$ satisfy:*

(i) $\mathcal{I}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$ *which contains the finite rank operators.*

(ii) The ideal property: if $v \in \mathcal{L}(E,X)$, $T \in \mathcal{I}(X,Y)$ and $w \in \mathcal{L}(Y,F)$, then the composition $w \circ T \circ v$ is in $\mathcal{I}(E,F)$.

If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies:

(i') $(\mathcal{I}(X,Y), \|.\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces X and Y,

(ii') $\|Id_{\mathbb{K}}\|_{\mathcal{I}} = 1$,

(iii') if $v \in \mathcal{L}(E,X)$, $T \in \mathcal{I}(X,Y)$ and $w \in \mathcal{L}(Y,F)$,

$$\|w \circ T \circ v\|_{\mathcal{I}} \leq \|w\| \, \|T\|_{\mathcal{I}} \, \|v\| \,,$$

then $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal.

The operator ideal $\mathcal{I}$ is said to be *closed* if each $\mathcal{I}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$ for the sup norm.

Proposition 3.1.2. *Let $(\mathcal{I}, \|.\|_{\mathcal{I}})$ be a Banach ideal. Then*

$$\|T\| \leq \|T\|_{\mathcal{I}} \quad \textit{for all } T \in \mathcal{I}(X,Y)\,.$$

Proof. For all $x \in X$, we define the operator

$$\begin{array}{rcl} R : \mathbb{K} & \longrightarrow & X \\ \lambda & \longrightarrow & \lambda x. \end{array}$$

R is linear and continuous, with $\|R\| = \|x\|$. Let $y^* \in Y^*$ and $T \in \mathcal{I}(X,Y)$ we have

$$y^* \circ T \circ R = y^* \circ T(x).Id_{\mathbb{K}}.$$

Then,

$$\begin{aligned}\|y^* \circ T\| &= \sup_{x\in B_X} |y^* \circ T(x)| \\ &= \sup_{x\in B_X} |y^* \circ T(x)| \, \|Id_{\mathbb{K}}\|_{\mathcal{I}} \\ &= \sup_{x\in B_X} \|y^* \circ T(x).Id_{\mathbb{K}}\|_{\mathcal{I}} \\ &= \sup_{x\in B_X} \|y^* \circ T \circ R\|_{\mathcal{I}} \\ &\leq \sup_{x\in B_X} \|y^*\| \, \|T\|_{\mathcal{I}} \, \|R\|\end{aligned}$$

Hence

$$\|T\| = \|T^*\| = \sup_{y^*\in B_{Y^*}} \|y^* \circ T\| \leq \sup_{y^*\in B_{Y^*}} \sup_{x\in B_X} \|y^*\| \, \|T\|_{\mathcal{I}} \, \|x\| = \|T\|_{\mathcal{I}} \, .$$

□

Definition 3.1.3. (**Injective operator ideal**)

A normed operator ideal $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is said to be *injective* if for every metric injection $i : Y \hookrightarrow F$ and every $T \in \mathcal{L}(X, Y)$ it follows from $i \circ T \in \mathcal{I}(X, F)$ that $T \in \mathcal{I}(X, Y)$. Moreover

$$\|i \circ T\|_{\mathcal{I}} = \|T\|_{\mathcal{I}} \, .$$

Definition 3.1.4. (**Surjective operator ideal**)

An Banach operator ideal $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is *surjective* if

$$\|v \circ L\|_{\mathcal{I}} = \|v\|_{\mathcal{I}} \, ,$$

whenever X, Y, F are Banach spaces, $L \in \mathcal{L}(F, X)$ is isometric and $v \in \mathcal{I}(X, Y)$.

Example 3.1.5. The ideal $\mathcal{L}_f$ of finite rank linear operators is the smallest operator ideal and $\mathcal{L}$ the largest one.

Definition 3.1.6. (**Dual ideal**) [7]. *The dual of an operator ideal $\mathcal{I}$ is defined as follows: For Banach spaces X and Y,*

$$\mathcal{I}^{dual}(X, Y) = \{T \in \mathcal{L}(X, Y) : T^* \in \mathcal{I}(Y^*, X^*)\} \, .$$

Where $T^* : Y^* \longrightarrow X^*$ is the adjoint of T.

In this case we define

$$\|T\|_{\mathcal{I}^{dual}} = \|T^*\|_{\mathcal{I}}.$$

And $\left(\mathcal{I}^{dual}, \|.\|_{\mathcal{I}^{dual}}\right)$ is a Banach operator ideal is called the dual ideal of $(\mathcal{I}, \|.\|_{\mathcal{I}})$.

Definition 3.1.7. (Symmetric operator ideal)

An operator ideal $\mathcal{I}$ is called *symmetric* if $\mathcal{I} \subset \mathcal{I}^{dual}$ and is called *completely symmetric* if $\mathcal{I} = \mathcal{I}^{dual}$.

3.2 Operator ideals related to s-number functions

Definition 3.2.1. Let s be an s-number function. Then, we define

$$\mathcal{L}_p^{(s)} = \left\{ T \in \mathcal{L} : \sum_{n=1}^{\infty} (s_n(T))^p < \infty \right\} \quad \textit{for } 0 < p < \infty ,$$

and

$$\mathcal{L}_\infty^{(s)} = \left\{ T \in \mathcal{L} : \lim_n s_n(T) = 0 \right\}.$$

Theorem 3.2.2. *Let s be an additive s-number function then,*

$\mathcal{L}_p^{(s)}$ *is an operator ideal,* $\quad 0 < p \leq \infty$.

Proof. A) The case $0 < p < \infty$.

(i) is clear.

(ii) Ideal property: Let E and F be a Banach spaces and let $u \in \mathcal{L}(Y,F), v \in \mathcal{L}_p^{(s)}(X,Y)$ and $w \in \mathcal{L}(E,X)$. Then $uvw \in \mathcal{L}(E,F)$ and

$$\begin{array}{lcl} s_n(uvw) & \leq & \|u\| \, s_n(v) \, \|w\| \\ (s_n(uvw))^p & \leq & \|u\|^p \, s_n(v)^p \, \|w\|^p \\ \displaystyle\sum_{n=1}^{\infty} (s_n(uvw))^p & \leq & \|u\|^p \, \|w\|^p \displaystyle\sum_{n=1}^{\infty} (s_n(v))^p \\ & \leq & \infty. \end{array}$$

B) The case $p = \infty$.

(i) is trivial.

(ii) Ideal property: Let E, F be a Banach spaces and let $u \in \mathcal{L}(Y,F), v \in \mathcal{L}_p^{(s)}(X,Y)$ and $w \in \mathcal{L}(E,X)$. Then $uvw \in \mathcal{L}(E,F)$ and

$$\begin{aligned}\lim_n s_n(uvw) &\leq \lim_n \|u\|\, s_n(v)\, \|w\| \\ &= \|u\|\, \|w\| \lim_n s_n(v).\end{aligned}$$

Hence $\lim_n s_n(uvw) = 0$.

(iii)

$$\begin{aligned}\lim_n(s_n(T_1+T_2)) &= \lim_n(s_{2n-1}(T_1+T_2)) \\ &\leq \lim_n(s_n(T_1)+s_n(T_2)) = 0.\end{aligned}$$

□

Proposition 3.2.3. *[18, p.190]*

(1) For every (completely) symmetric additive s-function the operator ideal $\mathcal{L}_p^{(s)}$ is (completely) symmetric.

(2) For every injective additive s-function the operator ideal $\mathcal{L}_p^{(s)}$ is injective.

(3) For every surjective additive s-function the operator ideal $\mathcal{L}_p^{(s)}$ is surjective.

Theorem 3.2.4. *The operator ideal $\mathcal{L}_p^{(s)}$ is* closed.

Proof. Let $T \in \mathcal{L}(X,Y)$.

We suppose that, for every $\varepsilon > 0$, there is $T_0 \in \mathcal{L}_p^{(s)}(X,Y)$, with $\|T - T_0\| \leq \varepsilon$. We now choose a natural number n_0 such that

$$s_n(T_0) \leq \varepsilon \quad \text{for } n \geq n_0.$$

Consequently,

$$\begin{aligned}s_n(T) &= s_{n-1+1}(T) \\ &= s_{n+1-1}(T - T_0 + T_0) \\ &\leq s_n(T_0) + \|T - T_0\| \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \qquad \text{for } n \geq n_0.\end{aligned}$$

This proves that $T \in \mathcal{L}_p^{(s)}(X,Y)$. □

Theorem 3.2.5. *Let $\mathcal{K}$ be the class of compact operators. Then,*

$$\mathcal{L}_\infty^{gel} = \mathcal{L}_\infty^{Kol} = \mathcal{K}.$$

Proof. Let $T \in \mathcal{K}(X,Y)$. If $\varepsilon > 0$, We choose $y_1, \ldots, y_m \in Y$ such that

$$T(B_X) \subset \bigcup_1^m \{y_i + \varepsilon B_Y\}.$$

Let N be a finite dimensional subspace of Y with $y_1, \ldots, y_m \in N$. Then $\left\|Q_N^Y T\right\| \leq \varepsilon$. Consequently,

$$d_n(T) \leq \varepsilon \quad \text{for all } n \geq n_0 = \dim(N)\ .$$

This proves $\mathcal{K} \subset \mathcal{L}_\infty^{Kol}$.

Now the inverse statement, let $\mathcal{L}_\infty^{Kol}(X,Y)$. If $\varepsilon > 0$, we choose a natural numbers n with $d_n(T) < \varepsilon$. Hence there is a subspace N of Y such that $\left\|Q_N^Y T\right\| < \varepsilon$ and $\dim(N) < n$. Since B_N is compact, we find $y_1, \ldots, y_m \in Y$ such that

$$(\|T\| + \varepsilon)B_N \subset \bigcup_1^m \{y_i + \varepsilon B_Y\}.$$

Let $x \in B_X$. Then $\left\|Q_N^Y T(x)\right\| < \varepsilon$, therefore

$$\|T(x) - y\| < \varepsilon \quad \text{for some } y \in N.$$

Since $\|y\| \leq \|T\| + \varepsilon$, we have

$$y \in \bigcup_1^m \{y_i + \varepsilon B_Y\}.$$

Consequently,

$$T(x) \in \bigcup_1^m \{y_i + 2\varepsilon B_Y\} \quad \text{for all } x \in B_X.$$

This proves $\mathcal{L}_\infty^{Kol} \subset \mathcal{K}$.

Finally, $\mathcal{L}_\infty^{gel} = \mathcal{L}_\infty^{Kol}$ follows from the Proposition 2.3.9 and Schauder's theorem.

□

3.3 Quasi-normed operator ideals related to approximation numbers

Definition 3.3.1. Let $a_n(T)$ the n-th approximation number and for every operator $T \in \mathcal{L}(X,Y)$ and $n \in \mathbb{N}$ we define

$$\mathcal{L}_p^{(a)} = \left\{ T \in \mathcal{L} : \sum_{n=1}^{\infty} a_n(T)^p < \infty \right\} \quad \text{for } 0 < p < \infty ,$$

and we put

$$\|T\|_{\mathcal{L}_p^{(a)}} = \left(\sum_{n=1}^{\infty} a_n(T)^p \right)^{1/p} .$$

$$\mathcal{L}_\infty^{(a)} = \{ T \in \mathcal{L} : (a_n(T))_n \in c_0 \} ,$$

and we put

$$\|T\|_{\mathcal{L}_\infty^{(a)}} = \|(a_n(T))_n\|_\infty .$$

In that case, $T \in \mathcal{L}(X,Y)$ is called p-approximable. We denote the set of all p-approximable operator in $\mathcal{L}(X,Y)$ by $\mathcal{L}_p^{(a)}(X,Y)$ (see [3, 13]).

Theorem 3.3.2. $\left(\mathcal{L}_p^{(a)}, \|.\|_{\mathcal{L}_p^{(a)}}\right)$ *is a quasi normed ideal for which every component* $\mathcal{L}_p^{(a)}(X,Y)$ *becomes a complete metric linear space.*

Proof. (i') We now that $(a+b)^p \leq \rho_p(a^p + b^p)$ *for* $a, b \geq 0$*, with* $\rho_p = \max(2^{p-1}, 1)$*. Let* $T_1, T_2 \in \mathcal{L}_p^{(a)}(X,Y)$ *and for every* $k \in \mathbb{N}$ *we have*

$$\begin{aligned}
\sum_{n=1}^{k} a_n(T_1+T_2)^p &\leq \sum_{n=1}^{k} a_{2n+1}(T_1+T_2)^p + \sum_{n=1}^{k} a_{2n}(T_1+T_2)^p \\
&\leq \sum_{n=1}^{k} a_{2n}(T_1+T_2)^p + \sum_{n=1}^{k} a_{2n}(T_1+T_2)^p \\
&\leq 2 \sum_{n=1}^{k} a_{2n}(T_1+T_2)^p \\
&\leq 2 \sum_{n=1}^{k} (a_n(T_1) + a_n(T_2))^p \\
&\leq 2\rho_p \left[\sum_{n=1}^{k} a_n(T_1)^p + a_n(T_2)^p \right] .
\end{aligned}$$

Hence, $T_1 + T_2 \in \mathcal{L}_p^{(a)}(X, Y)$,

and

$$\|T_1 + T_2\|_{\mathcal{L}_p^{(a)}} \leq 2^{1/p} \rho_p^{1/p} \rho_{1/p} \left[\|T_1\|_{\mathcal{L}_p^{(a)}} + \|T_2\|_{\mathcal{L}_p^{(a)}} \right].$$

(i) is trivial,

(ii and iii'): Let $T \in \mathcal{L}_p^{(a)}(X, Y), S \in \mathcal{L}(Y, F)$ *and* $R \in \mathcal{L}(E, X)$ *be given, for* $n \in \mathbb{N}$, $a_n(STR) = a_{o+n+o}(STR) \leq \|S\| \, a_n(T) \, \|R\|$, *by Proposition 2.2.3 (c), so that* $\sum_{n=1}^{\infty} a_n(STR)^p \leq \|S\|^p \|R\|^p \sum_{n=1}^{\infty} a_n(T)^p$. □

3.4 Quasi-normed operator ideals related to outer entropy numbers

In the following we introduce the entropy numbers of operator in Banach spaces. The concept is more suitable for generating operator ideals. First we state a definition and some basic properties of entropy numbers. Then we deal with the quasi-normed operators ideals related to it (see [3]). Throughout this section we take X, Y and E, F are reals Banach spaces.

Definition 3.4.1. ([10, Definition 2.1]) The Hausdorff measure of non-compactness of $T \in \mathcal{L}(X, Y)$ denote by $\widetilde{\gamma}(T)$, are defined by

$$\widetilde{\gamma}(T) = \inf\{\varepsilon > 0 : T(B_X) \text{ can be covered by finitely many balls of radius} \leq \varepsilon \}.$$

Lemma 3.4.2. *Let* X, Y *be Banach spaces and let* $T \in \mathcal{L}(X, Y)$*. Then*

T *is compact if, and only if,* $\widetilde{\gamma}(T) = 0$.

Definition 3.4.3. ([8, Definition 1.1]) For every operator $T \in \mathcal{L}(X, Y)$ the n-th outer entropy number $e_n(T)$ is defined by

$$e_n(T) = \inf \left\{ \sigma \geq 0 : \exists y_1, \ldots, y_q \in Y, q \leq 2^{n-1} \quad \text{with } T(B_X) \subseteq \bigcup_1^q \{y_i + \sigma B_Y\} \right\}, \quad (n \in \mathbb{N}).$$

Since the $e_n(T)$ are monotonic decreasing as n increases (see the proposition below), their limit exists. Clearly

$$\lim_{n\to\infty} e_n(T) = \widetilde{\gamma}(T).$$

Proposition 3.4.4. *If $T \in \mathcal{L}(X,Y)$, then*

$$\|T\| = e_1(T) \geq e_2(T) \geq \cdots \geq e_n(T) \geq e_{n+1}(T) \geq 0, \quad (n \in \mathbb{N}).$$

Proof. We check $\|T\| = e_1(T)$. It follows from

$$\|T\| = \inf\left\{\sigma \geq 0 : T(B_X) \subseteq \{y_1 + \sigma B_Y\}\right\}.$$

That $e_1(T) \leq \|T\|$. We now assume that $T(B_X) \subseteq y_1 + \sigma B_Y$ for some $y_0 \in Y$. If $x \in X$, then there are $y_+, y_- \in B_Y$ with $+T(x) = y_0 + \sigma y_+$ and $-T(x) = y_0 + \sigma y_-$. Hence $2\|T(x)\| = \sigma\|y_+ - y_-\| \leq 2\sigma$.

So we have $\|T\| = \sup\limits_{x\in B_X} \|T(x)\| \leq \sigma$ and therefore

$$\|T\| \leq e_1(T).$$

This shows that $\|T\| = e_1(T)$; and the rest is obvious. □

Proposition 3.4.5. *The outer entropy numbers are additive i.e.,*

If $T_1, T_2 \in \mathcal{L}(X,Y)$ then

$$e_{n_1+n_2-1}(T_1 + T_2) \leq e_{n_1}(T_1) + e_{n_2}(T_2), \quad (n_1, n_2 \in \mathbb{N}).$$

Proof. Let $T_1, T_2 \in \mathcal{L}(X,Y)$. If $\sigma_k > e_{n_k}(T_k)$, for $k = 1, 2$. Then there are $y_1^{(k)}, \ldots, y_{q_k}^{(k)} \in Y$ such that

$$T_k(B_X) \subseteq \bigcup_{1}^{q_k} \{y_i^{(k)} + \sigma_k B_Y\}, \quad \text{with } q_k \leq 2^{n_k-1} \text{ for } k = 1, 2.$$

Hence, given $x \in B_X$, and we can find i_k and $y_k \in B_Y$ with

$$T_k(x) = y_{i_k}^{(k)} + \sigma_k y_k \quad \text{for } k = 1, 2.$$

This implies

$$(T_1 + T_2)(x) \in y_{i_1}^{(1)} + y_{i_2}^{(2)} + (\sigma_1 + \sigma_2) B_Y.$$

Therefore

$$(T_1 + T_2)(B_X) \subseteq \bigcup_{i_1=1}^{q_1} \bigcup_{i_2=1}^{q_2} \{y_{i_1}^{(1)} + y_{i_2}^{(2)} + (\sigma_1 + \sigma_2) B_Y\}.$$

Since $q_1 q_2 \leq 2^{n_1-1} 2^{n_2-1} = 2^{(n_1+n_2-1)-1}$, we get

$$e_{n_1+n_2-1}(T_1 + T_2) \leq \sigma_1 + \sigma_2.$$

This shows the desired inequality. □

Theorem 3.4.6. *The outer entropy numbers are multiplicative. i.e.,*

If $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, F)$, then

$$e_{n+m-1}(ST) \leq e_n(S) e_m(T), \quad (n, m \in \mathbb{N}).$$

Proof. The proof of this is similar to that of the Proposition 3.4.5. □

Definition 3.4.7. Let e be an outer entropy numbers. Then, we define (see [3])

$$\mathcal{L}_p^{(e)} = \{T \in \mathcal{L} : (e_n(T))_n \in \ell_p\}. \quad \text{for } 0 < p < \infty, n \in \mathbb{N},$$

and we put

$$\|T\|_{\mathcal{L}_p^{(e)}} = \left(\sum_{n=1}^{\infty} e_n(T)^p\right)^{1/p}.$$

Let e be an outer entropy numbers and $\mathcal{K}$ the closed ideal of compact operators.

By Lemma 3.4.2, we have

$$\mathcal{K} = \{T \in \mathcal{L} : (e_n(T))_n \in c_0\} \quad n \in \mathbb{N}.$$

Theorem 3.4.8. *$\mathcal{L}_p^{(e)}$ is an operator ideal if every component $\mathcal{L}_p^{(e)}(X,Y)$ becomes a complete metric linear space with respect to the quasi-norm $\|.\|_{\mathcal{L}_p^{(e)}}$.*

The proof of this theorem is through the following two theorems.

Theorem 3.4.9. *If $T_1, T_2 \in \mathcal{L}_p^{(e)}(X,Y)$, then $T_1 + T_2 \in \mathcal{L}_p^{(e)}(X,Y)$ and*

$$\|T_1 + T_2\|_{\mathcal{L}_p^{(e)}} \le c\left(\|T_1\|_{\mathcal{L}_p^{(e)}} + \|T_2\|_{\mathcal{L}_p^{(e)}}\right),$$

where

$$c = 2^{1/p}\max(2^{1/p-1}, 1).$$

Proof. Let $T_1, T_2 \in \mathcal{L}_p^{(e)}(X,Y)$, we now that $(a+b)^p \le \rho_p(a^p + b^p)$ with $\rho_p = \max(2^{p-1}, 1)$ and by Proposition 3.4.5 we have

$$\begin{aligned}
\|T_1 + T_2\|_{\mathcal{L}_p^{(e)}} &= \left(\sum_{n=1}^{\infty} e_n(T_1+T_2)^p\right)^{1/p} \\
&\le \left(2\sum_{n=1}^{\infty} e_{2n-1}(T_1+T_2)^p\right)^{1/p} \\
&\le 2^{1/p}\left(\sum_{n=1}^{\infty} e_{2n-1}(T_1+T_2)^p\right)^{1/p} \\
&\le 2^{1/p}\left(\sum_{n=1}^{\infty} [e_n(T_1)+e_n(T_2)]^p\right)^{1/p} \\
&\le 2^{1/p}\max(2^{p-1},1)^{1/p}\left(\sum_{n=1}^{\infty} e_n(T_1)^p + \sum_{n=1}^{\infty} e_n(T_2)^p\right)^{1/p} \\
&\le 2^{1/p}\max(2^{p-1},1)^{1/p}\max(2^{\frac{1}{p}-1},1)\left[\|T_1\|_{\mathcal{L}_p^{(e)}} + \|T_2\|_{\mathcal{L}_p^{(e)}}\right].
\end{aligned}$$

□

Theorem 3.4.10. *If $S \in \mathcal{L}(E,X), T \in \mathcal{L}_p^{(e)}(X,Y)$ and $R \in \mathcal{L}(Y,F)$, then $RTS \in \mathcal{L}_p^{(e)}(E,F)$ and*

$$\|RTS\|_{\mathcal{L}_p^{(e)}} \le \|R\|\,\|T\|_{\mathcal{L}_p^{(e)}}\,\|S\|.$$

Proof. Let $S \in \mathcal{L}(E,X), T \in \mathcal{L}_p^{(e)}(X,Y)$ and $R \in \mathcal{L}(Y,F)$,then $RTS \in \mathcal{L}(E,F)$ and

$$\|RTS\|_{\mathcal{L}_p^{(e)}} = \left(\sum_{n=1}^{\infty} e_n(RTS)^p\right)^{1/p}.$$

By (SN_3)

$$\|RTS\|_{\mathcal{L}_p^{(e)}} \leq \|R\| \left(\sum_{n=1}^{\infty} e_n(T)^p\right)^{1/p} \|S\| = \|R\| \|T\|_{\mathcal{L}_p^{(e)}} \|S\|.$$

Therefore $RTS \in \mathcal{L}_p^{(e)}(E, F)$. □

Proposition 3.4.11. *If $0 < p_1 < p_2 < \infty$, then $\mathcal{L}_{p_1}^{(e)} \subset \mathcal{L}_{p_2}^{(e)}$ and the embedding map is continuous.*

Theorem 3.4.12. *If $0 < p, q < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then $T \in \mathcal{L}_q^{(e)}(X, Y)$ and $S \in \mathcal{L}_p^{(e)}(Y, F)$ imply that $ST \in \mathcal{L}_r^{(e)}(X, F)$ and*

$$\|ST\|_{\mathcal{L}_r^{(e)}} \leq 2^{1/r} \|S\|_{\mathcal{L}_p^{(e)}} \|T\|_{\mathcal{L}_q^{(e)}}.$$

Proof. By Theorem 3.4.6 we have

$$\begin{aligned}
\|ST\|_{\mathcal{L}_r^{(e)}} &= \left(\sum_{n=1}^{\infty} e_n(ST)^r\right)^{1/r} \\
&\leq \left(2\sum_{n=1}^{\infty} e_{2n-1}(ST)^r\right)^{1/r} \\
&\leq 2^{1/r} \left(\sum_{n=1}^{\infty} e_n(S)^r e_n(T)^r\right)^{1/r} \\
&\leq 2^{1/r} \left(\sum_{n=1}^{\infty} e_n(S)^p\right)^{1/p} \left(\sum_{n=1}^{\infty} e_n(T)^q\right)^{1/q} \\
&\leq 2^{1/r} \|S\|_{\mathcal{L}_p^{(e)}} \|T\|_{\mathcal{L}_q^{(e)}}.
\end{aligned}$$

□

Theorem 3.4.13. *[3]*

(i) If $0 < p < \infty$, then $\mathcal{L}_p^{(a)} \subseteq \mathcal{L}_p^{(e)}$.

(ii) If $0 < p < 2$ and $\frac{1}{q} = \frac{1}{p} - 2$, then $\mathcal{L}_p^{(e)} \subseteq \mathcal{L}_p^{(a)}$.

3.5 Entropy numbers of operators in Hilbert spaces

Theorem 3.5.1. *Let $T \in \mathcal{L}(\ell_2, \ell_2)$ such that $T(\xi_n) = (\sigma_n \xi_n)$ and $(\sigma_n) \in c_0$.*

If $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$, then

$$\sigma_n \leq 2e_n(T).$$

Proof. If $\sigma_n = 0$, then the assertion is trivial. We assume that $\sigma_1 \geq \sigma_2 \geq \cdots > 0$. Put

$$J_n(\xi_1, \ldots, \xi_n) = (\xi_1, \ldots, \xi_n, 0, \ldots)$$

and

$$Q_n(\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots) = (\xi_1, \ldots, \xi_n).$$

Then $T_n = Q_n T J_n$ is invertible. If I denotes the identity map of ℓ_2^n, it follows from $e_n(I) \geq 1/2$ and Theorem 3.4.6 that

$$1/2 \leq e_n(I) = e_n(T_n T_n^{-1}) \leq e_n(T_n) \left\| T_n^{-1} \right\| \leq \|Q_n\| \, e_n(T) \, \|J_n\| \, \sigma_n^{-1} \leq e_n(T)\sigma_n^{-1}.$$

□

Theorem 3.5.2. *([3, Theorem 5]) Let $T \in \mathcal{L}(\ell_2, \ell_2)$ such that $T(\xi_n) = (\sigma_n \xi_n)$ and $(\sigma_n) \in c_0$. Then*

$$\left(\sum_{n=1}^{\infty} e_n(S)^p \right)^{1/p} \leq c_p \left(\sum_{n=1}^{\infty} |\sigma_n|^p \right)^{1/p} \quad \text{for } 1 < p < \infty,$$

where c_p is some positive constant.

Corollary 3.5.3. *For any Hilbert space H the operator ideal $\mathcal{L}_p^{(e)}(H, H)$ coincides with the operator ideal $\mathcal{L}_p^{(a)}(H, H)$.*

and in particular, $\mathcal{L}_2^{(e)}(H, H)$ is the ideal of so-called Hilbert-schmidt operator.

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ملخص

في نظرية الاعداد s، نرفق لكل مؤثر T معرف بين فضاءات بناخ X و Y العديد من المتتاليات العددية $0 \geq ... \geq s_2(T) \geq s_1(T) = \|T\|$. في هذه المذكرة نقدم عرض مفصل حول نظرية الاعداد s ،ونناقش المثاليات المرتبطة بهذه الاعداد والتي تم الحصول عليها بهذه الطريقة.

الكلمات المفتاحية: المؤثر ذو الرتبة المنتهية، الأعداد s، مثالي المؤثرات، أعداد الانتروبي.

Abstract

In the theory of s-numbers, on associates with every bounded linear operator T from a Banach space X into a Banach space Y various kinds of scalar sequences $\|T\| = s_1(T) \geq s_2(T) \geq ... \geq 0$. In this memory we present an axiomatic theory of s-numbers and we discuss related operator ideals obtained in this way.

Key-words : Finite rank operator, s-number, Operator ideal, Entropy number.

Résumé

Dans la théorie des s-nombres, on associe à tout opérateur linéaire continu T d'un espace Banach X dans un espace Banach Y différents types de suites scalaires $\|T\| = s_1(T) \geq s_2(T) \geq ... \geq 0$. Dans ce mémoire, nous présentons une théorie axiomatique des s-nombres et nous discutons sur les idéaux d'opérateurs liées obtenus de cette manière.

Mots-Clés : Opérateur de rang fini, s-nombres, Ideal d'opérateur, Entropie nombres.