PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESERACH

Mohamed Boudiaf University – M'sila-

Faculty of Mathematics and Informatics

Departement of Mathematics

Master of Mathematics

Specialty : Mathematics

Option : Numerical and Mathematical Analysis

Entitled

On finite rank approximation of integral operators

Persented by: **Fatma Zohra Bouras**

Publicly presented on **19/06/2023 in front of the jury :**

Noui DJAIDJA	**M.C.B.**	University of M'sila	**Chairperson**
Toufik HERAIZ	**M.C.B.**	University of M'sila	**Supervisor**
S. Amel BENMERROUCH	**M.A.A .**	University of M'sila	**Examiner**
Khalifa CHADI	**M.A.A.**	University of M'sila	**Examiner**

University years**: 2022/2023**

Acknowlegments

First of all, I thank my God who gave me the strength to finish this these and for evry thing.

I would like to thank formost my supervisor, professor HERAIZ Toufik and I am grateful for his patience, time and advice.

Of course, I am grateful to my familly for their love and supporting me, whithout them this work would never come into existence. Finally, I will never forget to thank my colleagues for the exchange of scientific ideas that we did from time in order to enrich our knowledge in the field of our specialty.

Notations

$B(X,Y)$	The algebra of all bounded linear operators from X into Y
$N(T)$	The null space of T
$R(T)$	The range space of T,
$\mathcal{K}(X)$	The ideal of all compact operators on X
$\mathcal{C}(X)$	The set of all closed densly defined
$\sigma(T)$	The spectrum of T,
$\rho(T)$	The resolvent set of T
$\alpha(T)$	The nullity of T is defined as the dimension of $N(T)$
$\beta(T)$	The deficiency of T is defined as the codimension of $R(T)$
I	Operator of identity
$\Phi(X,Y)$	The set of fredholm operators
$\Phi_+(X,Y)$	The set of upper semi-Fredholm operators
$\Phi_-(X,Y)$	The set of lower semi-Fredholm operators
$D(T)$	The demain of T
$G(T)$	The graph of T
$L(X,Y)$	The algebra of all bounded linear operators from X into Y
$\sigma_p(T)$	The points spectrum of T
$\sigma_r(T)$	The residuel spectrum of T
$\sigma_c(T)$	The continous spectrum of T
$\Phi_{\pm}(X,Y)$	The set of semi-fredholm operator
$T_n \xrightarrow{n} T$	The norm convergence of T_n to T
$T_n \xrightarrow{p} T$	The pointwise convergence of T_n to T
$T_n \xrightarrow{c} T$	The compact convergence of T_n to T
$T_n \xrightarrow{cc} T$	The collectively compact convergence of T_n to T
$T_n \xrightarrow{\nu} T$	The ν-convergence of T_n to T

Table of contents

Introduction

For a bounded operator T on a complex Banach space X. In order to obtain an approximate solution to the the eigenvalue problem $T\varphi = \lambda\varphi$, a well known process is to construct a sequences (T_n) of bounded operators on X which converge to T in an appropriate manner, and have given error estimates for the solutions of the approximate eigenvalue problem $T_n\varphi_n = \lambda_n\varphi_n$. The approximation of T by a sequences of finite rank operators (T_n) allows to reduce the eigenvalue problem $T_n\varphi_n = \lambda_n\varphi_n$ to a matrix eigenvalue problem $M_n u_n = \lambda_n u_n$ Typically, the size of the matrix M_n increases with n. Several authors have studied this problem and other related topics using various types of convergence in $B(X)$. There are several notions of convergence of a sequence of operators which yield spectral results: the norm convergence, collectively compact convergence ([17]), compact convergence and regular convergence ([5]), stable and strongly stable convergence ([8]), resolvent operator convergence ([15]), convergence of $(T_n)_n$ to T in the sense that the spectral radius $r(T_n - T)$ of $T_n - T$ tends to zero and $\|(T_n - T)T_n\|$ tends to zero ([3]). Moreover, in 1994 M. Ahues, and A. Largillier introduced the ν-convergence in ([1]). In order to study spectral continuity properties, this type of convergence is useful ([7], [9], [4]).. This work is devoted to study of the finit rank approximation of integral operators by using methods of approximation wich provided by M,Ahus and A,Largilier in 1994. This work is composed of three chapter

The first chapter, we recal the basic properities of the bounded linear operator in a Banach space which is an impaortant part of functional analysis, we beging by the basic notaions necessary to study linear operators for norme spaces and we present the spectrum of bounded linear operators and the patrs af the spectrum. In the last part we present the compact operator.

The second chapter, we study the convergence, it consists of two section. In first section, introdusing the notion of the spectral approximation of the operators and the existence theorem, we defined some mode of convergence in order of as follows (the norm convergence $T_n \xrightarrow{n} T$, pointwise convergence $T_n \xrightarrow{p} T$, collectivly compact convergence $T_n \xrightarrow{cc} T$, and the ν convergence $T_n \xrightarrow{\nu} T$). In the last section, we will be paid to the study of .the spectrul opproximation.and we study there the spectrum continuity (the upper semicontinuity and the lower semicontinuity).

The third chapter, we study the integral operators in the first section we define the integral operator and fredholm operator. In the second section we study the finit rank approximation it consists tow subsections in the first we show the approximation based on projections and gives some ways of constructing a sequense (π_n) and in the second subsection we show the approximation of integral operator we used the degenerate kernel approximation an the approximation based on numerical integration.

Chaptre 1

Basic Properties

1.1 Bounded linear operator

1.1.1 Linear operator

Definition 1.1.1 *Let X and Y be any two normed spaces over a field K, T is function from X to Y, then we say T is linear operator if :*

1 $T(x+y) = T(x) + T(y)$;for all $x, y \in X$,

2 $T(\alpha x) = \alpha T(x)$; for each $\alpha \in K$, $x \in X$.

Example 1.1.1 *Let $T : X \longrightarrow Y$ such that $X = C([0,1])$ and $Y = C([0,1])$ be defined by $T(f)(s) = \int_0^1 k(s,t)f(t)dt$; where $f \in C([0,1]), k(s,t)$ is a real valued continous function over $[0,1] \times [0,1]$, that is $F(s) = T(f)(s)$, where $F \in c[0,1]$ $s, t \in [0,1]$, cleary T is a linear operator.*

Definition 1.1.2 *X is called the demain of T denoted by $D(T)$, the image of X under T is called the range of T denoted by ($R(T)$ or $\operatorname{Im}(T)$) is the set of all $y \in Y$ such that $Tx = y$ for some $x \in X$.*

in other word

$$R(T) = \{y \in Y, y = Tx\}.$$

The null space (or kernel) of linear operator T denoted by ($N(T)$ or $ker(T)$) is the set of all $x \in X$ such that $T(x) = 0$.

$$N(T) = \{x \in X, T(x) = 0\}.$$

The graph of T is the set $G(T)$ such that

$$G(T) = \{(x, Tx)/x \in D(T)\} \subset X \times Y.$$

$X \times Y$ is the norm $\|(x, y)\|^2_{X\times Y} = \|x\|^2_X + \|y\|^2_X$ wich makes $X \times Y$ is a Banach space

Definition 1.1.3 *A linear operator $T : X \longrightarrow Y$ is said to be continous at $x_0 \in X$ if there exists a sequence $\{x_n\} \in X$ such that $\|x_n - x_0\| \longrightarrow 0$ as $n \longrightarrow \infty$. Implies* $\|T(x_n) - T(x_0)\| \longrightarrow 0$.

As $n \longrightarrow \infty$. Equivalently, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(x_0)\| < \varepsilon$ when ever $\|x - x_0\| < \delta$.

Theorem 1.1.1 *Let $T : X \longrightarrow Y$ be a linear operator, if T is continous at point of X, then T is continous at every other point of X.*

Proof. Suppose T is continous at $x_0 \in X$, so given $\varepsilon > 0$, there is a positive $\delta > 0$ such that $\|T(x) - T(x_0)\| < \varepsilon$, whenever $\|x - x_0\| < \delta$, soppose x_1 another point of X. Then if $\|x - x_1\| < \delta$, we write $\|x - x_1\| = \|(x - x_1 + x_0) - x_0\|$.

Thus $\|x - x_1\| < \delta$ i.e, $\|(x - x_1 + x_0) - x_0\| < \delta$ implies by virtue of contunuity of T,

$$\|T(x - x_1 + x_0) - T(x_0)\| < \varepsilon$$

or $\|T(x) - T(x_1) + T(x_0) - T(x_0)\| < \varepsilon$

or $\|T(x) - T(x_1)\| < \varepsilon$.

So T is continous at $x = x_1$. ■

1.1.2 Bounded linear operator

Definition 1.1.4 *Let X, Y be normed vector spaces and $T : X \longrightarrow Y$ a linear operator we say that T is bounded if there exists a number $C > 0$ such that*

$$\|T(x)\| \leq C \|x\| \text{ for each } x \in X$$

the lower bound of all $C > 0$ such that $\|T(x)\| \leq C\|x\|$ for each $x \in X$ is called norm of the operator T and is denoted by $\|T\|$

Theorem 1.1.2 [2]*If T is bounded define*

$$\begin{aligned} \|T\| &= \sup_{x\neq 0}\tfrac{\|Tx\|}{\|x\|} = \sup_{x=1}\|Tx\| \\ &= \sup\{\|Tx\|\,;\|x\| \leq 1\} \end{aligned}$$

If $\|Tx\| \leq C$, for every $x \in X$ with $\|x\| = 1$, then $\|T\| \leq C$.

. We denote the set of bounded linear operators from X into Y by $B(X, Y)$ and by $B(X)$ if $X = Y$.

Proposition 1.1.1 *T a linear operator is bounded if only if it is continous at one point $x_0 \in X$, and hence continous at every point.*

Proof. If T is not bounded then for each n we could find an element $x_n \in X$ such that

$$\|Tx_n\| > n\|x_n\|.$$

Set

$$y_n = \frac{x_n}{n\|x_n\|} + x_{0.}$$

Then $y_n \longrightarrow x_0$. Since T is continous at x_0, we must have $Ty_n \longrightarrow Tx_0$. But

$$Ty_n = \frac{Tx_n}{n\|x_n\|} + Tx_{0.}$$

Hence

$$\frac{Tx_n}{n\|x_n\|} \longrightarrow 0.$$

But

$$\frac{\|Tx_n\|}{n\|x_n\|} > 1.$$

Providing a contradiction. ■

Theorem 1.1.3 *Let X and Y be two Banach spaces and $\{T_n\}$ a sequence of linear and continous operators defined on X to Y, if the sequence T_n converges to an operator T, then this operator is linear and continous moreover, we have*

$$\|T\| \leq \liminf \|T_n\| .$$

Proof. Let T be a linear operator then

$$\begin{aligned} T(\alpha x + \beta y) &= \lim_{n\to\infty} T_n(\alpha x + \beta y) \\ &= \lim_{n\to\infty} (\alpha T_n(x) + \beta T_n(y)) \\ &= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y) \\ &= \alpha T(x) + \beta T(y). \end{aligned}$$

The sequence of perators $\{T_n\}$ be continous then, we have

$$\|T_n(x)\| \leq\| C \|x\| , \qquad \forall n \in \mathbb{N} \text{ and } \forall x \in X.$$

We obtain

$$\lim_{n\to\infty} \|T_n(x)\| \quad = \quad \|T(x)\| \leq C \|x\| , \text{ for every } x \in X.$$

Hence the cotinuity of the operator$T$.

For the evaluation of the norm $\|T\|$ of the operator T, we write

$$\| T_n(x) \| \leq \| T_n \| \| x \|,$$

for the limit of two members, we obtain

$$\begin{aligned} T(x) &= \lim_{n\to\infty} \| T_n(x) \| \\ &\leq \lim_{n\to\infty} \inf \| T_n \| \| x \| . \end{aligned}$$

Hence

$$\| T \| \leq \lim_{n\to\infty} \inf \| T_n \| .$$

■

Corollary 1.1.1 *Let X, Y be normed vector spaces and $T : X \longrightarrow Y$ linear then the following are equivalent:*

1. T is bounded.
2. T is continous at $x_0 \in X$.
3. T is uniformaly continous.

Theorem 1.1.4 *Let X be a normed linear spaces and Y be a Banach space, then $B(X, Y)$ is a Banach space.*

Proof. Let (T_n) be Cauchy sequence of bounded linear operator from X to Y then for $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $\|T_n - T_m\| < \varepsilon$, for every $n, m \geq N$

Let $x \in X$ be arbitrary then

$$\begin{aligned} \| T_n(x) - T_m(x) \| &= \| (T_n - T_m)(x) \|_Y \\ &\leq \| T_n - T_m \| \| x \| \\ &\leq \varepsilon \| x \| . \text{ for every } n, m \in \mathbb{N} \end{aligned}$$

So that for each $x \in X$, $(T_n(x))$ is a Cauchy sequence in Y.

Since Y is a Banach space $(T_n(x))$ converge in Y for each $x \in X$

$$\begin{aligned} T(x) &= \lim_{n\to\infty} T_n(x) \qquad \forall x \in X \\ T(\alpha x + \beta y) &= \lim_{n\to\infty} (T_n(\alpha x) + T_n(\beta y)) \\ &= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y) \\ &= \alpha T(x) + \beta T(y) \end{aligned}$$

T is linear

Since T_n is a Cauchy sequence , is a bounded hence there exists some $M > 0$ with $M = \sup\{\|T_n\|, n \in \mathbb{N}\}$

so that

$$\begin{aligned} \| T_n(x) \|_Y &\leq \| T_n \| \| x \|_Y \leq M \| x \|_X \text{ for every } n \in \mathbb{N} \text{ and for each } x \in X \\ \| T(x) \| &= \| \lim_{n\to\infty} T_n(x) \| \\ &= \| T_n(x) \|_Y \leq M \| x \|_X \text{ for every } x \in X \end{aligned}$$

Hence T is a bounded linear operator from X to Y . ■

Theorem 1.1.5 ***(Banach_ Steinhaus1)***

Let $\{T_n(x)\}$ be a sequence of operators defined on a Banach space X in a normed space Y, if the sequence $T_n(x)$ is bounded for every $x \in X$, then the norm of these operators $\|T_n\|$ are also bounded. In other words,

$$\forall x \in X, \ \sup \| T_n(x) \| < \infty \Rightarrow \sup \| T_n \| < \infty$$

Theorem 1.1.6 ***(Banach_ Steinhaus 2)***

Let $\{T_n\}$ a sequence of continous linears operators, defined on a Banach space X in a Banach space Y, the sequence $\{T_n\}$ converges to a continous linear operator, iff

1. The normes $\| T_n \|$ of the operators T_n are bounded.

2. The sequence $\{T_n(x)\}$ is Cauchy for any element of the set G dense in X.

Adjoints linears operators in the normed space

Definition 1.1.5 *Let T a bounded linear operator defined for a normed space X in a normed space Y so, for all $\varphi \in X$ and $\psi \in Y$, we define $U \in Y^* = B(Y, \mathbb{k})$ and $V \in X^* = B(X, \mathbb{k})$ with $\mathbb{k} = (\mathbb{R}$ or $\mathbb{C})$*

$$\begin{aligned} & F \longrightarrow \mathbb{k} \\ U: \ & \psi \longmapsto U(\psi) \end{aligned}$$

et

$$\begin{aligned} & E \longrightarrow \mathbb{k} \\ V: \ & \varphi \longmapsto V(\varphi) \end{aligned}$$

The operator noted by T^* defined for Y^* on X^* is called adjoint operator of T if, for all $U \in Y^*$ and $V \in X^*$

$$\begin{aligned} & Y^* \longrightarrow X^* \\ T^*: \ & U \longmapsto T^*(U) = U(T(\varphi)) = V(\varphi) \end{aligned}$$

and

$$\begin{aligned} & X \longrightarrow Y \qquad \longrightarrow \mathbb{k} \\ T^* = U \circ T: \ & \varphi \longmapsto T(\varphi) \ \ \longmapsto U(T(\varphi)). \end{aligned}$$

1.1.3 Spectrum of a bounded linear operator

Definition 1.1.6 [18]*Let T be a bounded linear operator acting on a Banach space X over the complex scalar field $\mathbb{C}$, and I be the identity operator on X, the spectrum of T is the set of all $\lambda \in \mathbb{C}$ for with the operator $T - \lambda I$ does not have an inverse that is a bounded linear operator*

$$\sigma(T) = \{\lambda \in \mathbb{C},\ (T - \lambda I) \text{ is not invertible}\}$$

The spectrum of operator T is often denotted $\sigma(T)$, and its complement, the resolvent set, is denotted $\rho(T)$

$$\rho(T) = \mathbb{C}/\sigma(T)$$

Parts of the spectrum [10]

The spectrum $\sigma(T)$ is the disjoint union of the point spectrum $\sigma_p(T)$, (set of eigenvalues), continous spectrum $\sigma_c(T)$, and residual spectrum $\sigma_r(T)$:

The points spectrum $\sigma_p(T)$ That gathers all $\lambda \in \mathbb{C}$ such that the operators $T - \lambda I$ is not injective. Such a complex number λ is called an eigenvalue of T and the dimension of the kernel $ker(T - \lambda I)$ is the geometric multiplicity associated with this eigenvalue. An element of $ker(T - \lambda I)$ is called eigenvector or often, an eigenfunction in the case the Banach space is a function space.

$$\begin{aligned} \sigma_p(T) &= \{\lambda \in \mathbb{C}/(T - \lambda I) \text{is not injective}\} \\ &= \{\lambda \in \mathbb{C}/N(T - \lambda I) \neq \{0\}\} \end{aligned}$$

$\lambda \in \sigma_p(T)$ eigenvalue and $x \neq 0$.e.i $Tx = \lambda x$ eigenvectors.

The residuel spectrum $\sigma_r(T)$ That gathers all $\lambda \in \mathbb{C}$ such that the operator $T - \lambda I$ is injective but does not have a dense image.

$$\begin{aligned}\sigma_r(T) &= \{\lambda \in \mathbb{C}/(T-\lambda I) \text{ is injective and } \overline{Ran(T-\lambda I)} \neq X\} \\ &= \{\lambda \in \mathbb{C}; N(T-\lambda I)=\{0\}, \overline{Ran(T-\lambda I)} \neq X\}\end{aligned}$$

The continous specrtum $\sigma_c(T)$ That gathers all $\lambda \in \mathbb{C}$ such that the operator $T-\lambda I$ is injective, has a dense image, but its inverse $(T-\lambda I)^{-1}$ is not bounded.

$$\begin{aligned}\sigma_c(T) &= \{\lambda \in \mathbb{C}/(T-\lambda I) \text{ is injective}, \overline{Ran(T-\lambda I)} = X, \text{ but } Ran(T-\lambda I) \neq X\} \\ &= \{\lambda \in \mathbb{C}, N(T-\lambda I)=\{0\}, \overline{Ran(T-\lambda I)} = X\}\end{aligned}$$

1.1.4 Compact operator

Definition 1.1.7 [6] ***(Compact linear operator)*** *Let X and Y be normed space. A linear operator $T : X \longrightarrow Y$ is compact linear operator if for any bounded sequence $(x_n)_{n\geq 1}$ in X, the sequence $(Tx_n)_{n\geq 1}$ in Y has a convergent subsequence.*

We note that every compact operator T is bounded. Indeed, if $\|T\| = \infty$, then there exists a sequence $(x_n)_{n\geq 1}$ such that $\|x_n\| \leq 1$ and $\| Tx_n \| \rightarrow \infty$. Then $(Tx_n)_{n\geq 1}$ cannot have a convergent subsequence. Hence, $\|T\| < \infty$.

Theorem 1.1.7 [18]*Let X be a normed space and Y be a Banach space. Let $(T_n)_{n\geq 1}$ be a sequence of compact linear operator from X into Y. If $T_n \longrightarrow T$ (that is, $\|T_n - T\| \longrightarrow 0$), then the limit operator T is compact.*

Proof. Since T_1 is acompact operator, hense Cauchy we know that the sequence $(T_1(x_n))$ has a convergent subsequence $(T_1(x_{1,m}))$, where $(x_{1,m})$ is a subsequence of (x_n). $(x_{1,m})$ is bounded, so we can repeat the argument with T_2 to produce a subsequence $(x_{2,m})$ of $(x_{1,m})$ with the property that $(T_2(x_{2,m}))$ converges. We continue in the same way, and then define a sequence $(y_m) = (x_{m,m})$. Notice that (y_m) is a subsequence of (x_n), so it is bounded, say by $\| y_n \| \leq c$, and it has the property that for every fixed n, the sequence $(T_n(y_m))$ is convergent, and hence Cauchy.

We claim that $(T(y_m))$ is a Cauchy sequence in Y, let $\varepsilon > 0$. Since $\|T_n - T\| \longrightarrow 0$, thare is some $p \in \mathbb{N}$ such that $\|T_p - T\| < \frac{\varepsilon}{3c}$. Also, sinse $(T_p(y_m))$ is Cauchy, there is some $N > 0$ such that $\|T_p(y_j) - T_p(y_k)\| < \frac{\varepsilon}{3}$ whenever $j, k > N$. There for $j, k > N$, we have

$$\begin{aligned}
\| T(y_j) - T(y_k) \| &\leq \| T(y_j) - T_p(y_j) + T_p(y_j) - T_p(y_k) + T_p(y_k) - T(y_k) \| \\
&\leq \| T(y_j) - T_p(y_j) \| + \| T_p(y_j) - T_p(y_k) \| + \| T_p(y_k) - T(y_k) \| \\
&< \| T - T_p \| \| y_j \| + \tfrac{\varepsilon}{3} + \| T - T_p \| \| y_k \| \\
&< \tfrac{\varepsilon}{3c} c + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3c} c = \varepsilon.
\end{aligned}$$

Which proves that $(T(y_m))$ is Cauchy. Since Y is a Banach space, it is by definition complete, so $(T(y_m))$ converges. We have thus produced, for an arbitrary bounded sequence $(x_n) \subset X$, a convergent subsequence of its image under T. therefore,T is compact. ■

Lemma 1.1.1 *Let X be a normed linear space, and let T and S be a boounded linear operator on X. if T is compact, then so are ST and TS.*

Theorem 1.1.8 *Let X and Y be tow normed linear spaces; suppose $T : X \longrightarrow Y$, is a linear operator, then the following statements:*

1. T is compact.

2. The image of the open unit ball under T is relatively compact in Y.

3. For any bounded sequence $\{x_n\}$ in X, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ that converges in Y.

Chaptre 2

Spectrum continuity

2.1 Modes of convergence

Since exact computations of the spectrum are almost always impossible, it is relevant to know the spectrum or any of its parts in approximate way. Several authors have studied this problem and other related topics using various types of convergence in $B(X)$. In addition to the norm convergence; There are several notions of convergence of a sequence of operators which yield spectral results:

Definition 2.1.1 [1]*Let X, Y two normed spaces, T_n is a sequence of linear operator from X into Y. And $T : X \longrightarrow Y$ is a bounded operator*

1. We said that T_n converges in norm to T, denoted by $T_n \xrightarrow{n} T$ if

$$\| T_n - T \| \longrightarrow 0$$

2. T_n is said be convergent to T by pointwise convergence, written $T_n \xrightarrow{p} T$ if

$$\| T_n x - Tx \| \longrightarrow 0 \text{ for every } x \in X$$

3. T_n is said be convergent to T by compact convergence, written $T_n \xrightarrow{c} T$ if

The following conditions is satisfed

a) $T_n \xrightarrow{p} T$.

b) for any sequence $\{x_n\}$ in Y, the sequence $\{(T - T_n)x_n\}_N$ is a relatively compact in X.

4. T_n is said be convergent to T by collectively compact convergence, denotted by $T_n \xrightarrow{cc} T$ if

$T_n \xrightarrow{p} T$, and for some positive integer n_0,

$$\underset{n \geq n_0}{\cup} \{(T_n - T)x, x \in X, \| x \| \leq 1\},$$

is a relatively compact subset of X, if T is compact, then the letter condition is equivalent to the condition that for some positive integer n_0, the set :

$$\underset{n \geq n_0}{\cup} \{T_n x; x \in X, \| x \| \leq 1\},$$

is a relatively compact subset of X.

Proposition 2.1.1 [1]*If $T_n \xrightarrow{n} T$ or $T_n \xrightarrow{cc} T$, then clearly $T_n \xrightarrow{p} T$ But the converse is not true.*

The ν _Convergence

Definition 2.1.2 *Let X, Y two normed spaces, T_n is a sequence of linear operator from X into Y. And $T : X \longrightarrow Y$ is a bounded operator. T_n is said to be convergent to T by the $\nu-$ convergence, denoted by $T_n \xrightarrow{\nu} T$ if*

$(\| T_n \|)$ is bounded, $\| (T_n - T)T \| \longrightarrow 0$, and $\|(T_n - T)T_n\| \longrightarrow 0$.

Lemma 2.1.1 [1] *We have*

a) If $T_n \xrightarrow{n} T$, then $T_n \xrightarrow{\nu} T$, conservely if $0 \notin \sigma(T)$ and $T_n \xrightarrow{\nu} T$ then $T_n \xrightarrow{n} T$.

b) Let $T_n \xrightarrow{\nu} T$ and $U_n \xrightarrow{n} U$, then $T_n + U_n \xrightarrow{\nu} T + U$ iff $(T_n - T)U \xrightarrow{n} 0$. In particular

i) If $T_n \xrightarrow{\nu} T$ and $U_n \xrightarrow{n} 0$, then $T_n + U_n \xrightarrow{\nu} T$.

ii) If $T_n \xrightarrow{\nu} 0$, $U_n \xrightarrow{n} U$ and $T_nU \xrightarrow{n} 0$, then $T_n + U_n \xrightarrow{\nu} U$.

c) If $T_n \xrightarrow{cc} T$ and T is a compact operator, then $T_n \xrightarrow{\nu} T$.

Proof.

a) Let $T_n \xrightarrow{n} T$. Since $\|T_n\| \leq \|T_n - T\| + \| T \|$, $\|(T_n - T)T\| \leq \|T_n - T\| \|T\|$ and $\|(T_n - T)T_n\| \leq \|T_n - T\| \|T_n\|$, we see that $T_n \xrightarrow{\nu} T$.

Conservely, let $0 \notin \sigma(T)$ and $T_n \xrightarrow{\nu} T$. Then T is invertible and $\|T_n - T\| = \|(T_n - T)TT^{-1}\| \leq \|(T_n - T)T\| \|T^{-1}\|$, so that $T_n \xrightarrow{n} T$.

b) Since $\| T_n + U_n \| \leq \| T_n \| + \| U_n \|$, we see that the sequence $(\|T_n + U_n\|)$ is bounded. Assume that $(T_n - T)U \xrightarrow{n} 0$. As

$$\| (T_n + U_n - T - U)(T + U) \| \leq \| (T_n - T)T \| + \|(T_n - T)U\| + \|U_n - U\| \|T + U\|,$$
$$\| (T_n + U_n - T - U)(T_n + U_n) \| \leq \| (T_n - T)T_n \| + \|(T_n - T)U_n\| + \|U_n - U\| (\| T_n \| + \| U_n \|).$$

Where $\| (T_n - T)U_n \| \leq \| T_n - T \| \| U_n - U \| + \|(T_n - T)U\|$, we see that $T_n + U_n \xrightarrow{\nu} T + U$. Conservely, assume that $T_n + U_n \xrightarrow{\nu} T + U$. Since

$$(T_n - T)U = (T_n + U_n - T - U)(T + U) - (T_n - T)T - (U_n - U)(T + U)$$

We obtain $(T_n - T)U \xrightarrow{n} 0$. The particular cases (i) and (ii) follow easily.

c) Let $T_n \xrightarrow{cc} T$. By the Banach_ Steinhaus theorem, the sequence $(\|T_n\|)$ is bounded and the pointwise convergence of (T_n) to T is uniform on the relatively compact sets $\{Tx : x \in X, \| x \| \leq 1\}$ and $\bigcup_{n \geq n_0} \{T_nx : x \in X, \| x \| \leq 1\}$, hence $\|(T_n - T)T\| \longrightarrow 0$ and $\|(T_n - T)T_n\| \longrightarrow 0$. Thus $T_n \xrightarrow{\nu} T$.

■

2.2 Spectral approximation

2.2.1 Spectrum continuity

Property U

This property is known by **The upper semicontinuity of the specrtum** holds if, whenever $T_n \longrightarrow T$, $\lambda_n \in \sigma(T_n)$ and $(\lambda_n) \longrightarrow \lambda$, we have $\lambda \in \sigma(T)$.

Proposition 2.2.1 [1]*If* $T_n \xrightarrow{p} T$, *then we may have* $\lambda_n \in \sigma(T_n)$ *with* $\lambda_n \longrightarrow \lambda$, *but* $\lambda \notin \sigma(T)$

Example 2.2.1 *Property U does not under pointwise convergence :*

Consider $x = \sum_{k=1}^{\infty} x(k)e_k \in X$, $X = l^2$, let $Tx = x(1)e_1$ and for each $n \geq 2$

$$T_n x = x(1)e_1 - x(n)e_n.$$

Since $\| T_n x - Tx \|_2 = | x(n) | \to 0$, for every $x \in X$, we see that $T_n \longrightarrow T$. Now

$$\sigma(T) = \{0, 1\} \text{ and } \sigma(T_n) = \{-1, 0, 1\}.$$

Since $\lambda = -1 \in \sigma(T_n) \quad \forall n$, but $-1 \notin \sigma(T)$, the property U does not hold. Also, if for $x \in X$, we let

$$T_n x = x(1)e_1 + x(n)e_n.$$

Then $T_n \xrightarrow{p} T$, 1 is an eigenvalue of T_n of algebriac multiplicity 2, but 1 is an eigenvalue of T of algebriac multiplicity 1.

On the other hand, if we let for $x \in X$,

$$\tilde{T}x = x(1)e_1 + x(2)e_2 \text{ and } \tilde{T}_n x = x(1)e_1 + \frac{n-1}{n}x(2)e_2$$

Then again $\| \tilde{T}_n x - \tilde{T}x \|_2 = | x(2) | /n \to 0$ for every $x \in X$, so that $\tilde{T}_n \xrightarrow{p} \tilde{T}$, 1 is an eigenvalue of each $\tilde{T}_n$ of algebriac multiplicity 1, while 1 is eigenvalue of $\tilde{T}$ of algebriac multiplicity 2.

Corollary 2.2.1 [1]*Property U holds under* $\nu - convergence$, *that is, if* $T_n \xrightarrow{\nu} T$, $\lambda_n \in \sigma(T_n)$ *and* $\lambda_n \longrightarrow \lambda$, *then* $\lambda \in \sigma(T)$.

Proof. Suppose suppose that $\lambda \in \rho(T)$. Since the set $\rho(T)$ is open in $\mathbb{C}$, there is some $r > 0$ such that $E = \{z \in \mathbb{C} :| z - \lambda |\leq r\} \subset \rho(T)$. $E \subset \rho(T_n)$ for all n. Since $\lambda_n \longrightarrow \lambda$, we see that $\lambda_n \in E \subset \rho(T)$ for every $n \in \mathbb{N}$, which is contradictory to the hypothesis $\lambda_n \in \sigma(T_n)$ for every $n \in \mathbb{N}$, hence λ must belong to $\sigma(T)$.

We note that the previeus corollary does not hold if, from the definition of the $\nu - convergence$, one of the two conditions $\|(T_n - T)T\| \longrightarrow 0, \|(T_n - T)T_n\| \longrightarrow 0$ is omitted or if these two conditions are replaced by the condition $\|(T_n - T)^2\| \longrightarrow 0$. ■

Property L

This property is known by **The lower semicontinuity of the spectrum** holds if, whenever $T_n \longrightarrow T$ and $\lambda \in \sigma(T)$, there exists $\lambda_n \in \sigma(T_n)$ for each n such that $\lambda_n \longrightarrow \lambda$.

Proposition 2.2.2 [1]*Some spectral values of T may not be approximable by spectral values of T_n, even when $T_n \xrightarrow{n} T$.*

Example 2.2.2 [1]*Property L does not hold under norm convergence:*

Let

$$(Tx)(k) = \begin{cases} x(k+1) & \text{if } k \neq -1 \\ 0 & \text{if } k = -1 \end{cases}$$

For $X = l^2(\mathbb{Z})$, for $x \in X$

And for every n,

$$(T_n x)(k) = \begin{cases} x(k+1) & \text{if } k \neq -1 \\ \frac{x(0)}{n} & \text{if } k = -1 \end{cases}$$

Since $\| T_n x - Tx \|=| x(0) | /n$ for all $x \in X$, we see that $\| T_n - T \|= 1/n \longrightarrow 0$.

Suppose that $\sigma(T) = \{\lambda \in \mathbb{C} :| \lambda |< 1\}$. if $| \lambda |< 1$, consider $x_\lambda \in X$ defined by $x_\lambda(k) = 0$ $\forall k < -1$, $x_\lambda(0) = 1$ and $x_\lambda(k) = \lambda^k$ for all $k \geq 1$, and note that $Tx_\lambda = \lambda x_\lambda$ with $x_\lambda \neq 0$. Thus every λ satisfing $| \lambda |< 1$ is an eigenvalue and hence a spectral value of T. Since $\sigma(T)$ is closed, it follows that $\{\lambda \in \mathbb{C} :| \lambda |\leq 1\} \subset \sigma(T)$, but $\sigma_r(T) \leq T = 1$, we see that $\sigma(T) \subset \{\lambda \in \mathbb{C} :| \lambda |\leq 1\}$,

On the other hand, we claim that $\sigma(T_n) = \{\lambda \in \mathbb{C} :| \lambda |= 1\}$ for each n, if $| \lambda |= 1$, consider $y \in X$ defined by $y(k) = 0$ for all $k \neq -1$ and $y(-1) = 1$, and note that there is no $x \in X$ with $T_n x - \lambda x = y$. Thus every λ satisfying $\mid \lambda \mid = 1$ is a spectral value of T.

$$\{\lambda \in \mathbb{C} :\mid \lambda \mid = 1\} \subset \sigma(T_n) \subset \{\lambda \in \mathbb{C} :\mid \lambda \mid \leq 1\}.$$

It can be seen that T_n is bijective and for $x \in X$,

$$(T_n^{-1}x)(k) \quad = \begin{cases} x(k-1) & \text{if } k \neq 0 \\ nx(-1) & \text{if } k = 0 \end{cases}$$

So that $\parallel T_n^{-1} \parallel = n$. similarly, for $j = 2, 3, ...$and $x \in X$,

$$(T_n^{-j}x)(k) \quad = \begin{cases} x(k-j) & \text{if } k \neq 0, 1, ..., j-1, \\ nx(k-j) & \text{if } k = 0, 1, ..., j-1, \end{cases}$$

So that $\parallel T_n^{-j} \parallel = n$, for every j. Hence $\sigma_r(T_n^{-1}) = \lim_{j \to \infty} \parallel T_n^{-j} \parallel^{1/j} = \lim_{j \to \infty} n^{1/j} = 1$.

Thus $\sigma(T_n^{-1}) \subset \{\lambda \in \mathbb{C} :\mid \lambda \mid \leq 1\}$, $\sigma(T_n) \subset \{\lambda \in \mathbb{C} :\mid \lambda \mid \geq 1\}$, and it is known that if $\mid \lambda \mid < 1$, then $\lambda \in \sigma(T)$, but there is no λ_n in $\sigma(T_n)$ such that $\lambda_n \longrightarrow \lambda$.

Proposition 2.2.3 [1]*Let Λ be a spectral set for T, $C \in C(T, \Lambda)$ and $T_n \xrightarrow{\nu} T$.*

(a) There is a positive integer n_0 such that for each $n \geq n_0$, C lies in $\rho(T_n)$. And if

$$\Lambda_n = \sigma(T_n) \cap int(C) \;\; \text{and } P_n = -\frac{1}{2\pi i} \int_C R_n(z) dz,$$

Then Λ_n and P_n do not depond on $C \in C(T, \Lambda)$ for each n.

(b) Let E be a nonempty closed subset of $\rho(T)$. Let $E = C$, then for each $n \geq n_0$:

$$\parallel P \parallel \leq \frac{l(C)}{2\pi} \alpha_1(C), \quad \parallel P_n \parallel \leq \frac{l(C)}{2\pi} \alpha_2(C),$$

And

$$\begin{aligned} \parallel P_n - P \parallel &\leq \tfrac{l(C)}{2\pi} \alpha_1(C) \alpha_2(C) \parallel T_n - T \parallel \\ \parallel (P_n - P)P \parallel &\leq \tfrac{l(C)}{2\pi} \alpha_1(C) \alpha_2(C) \parallel (T_n - T)P \parallel \\ \parallel (P_n - P)P_n \parallel &\leq \tfrac{l(C)}{2\pi} \alpha_1(C) \alpha_2(C) \parallel (T_n - T)P_n \parallel \end{aligned}$$

And if $0 \in ext(C)$, then

$$\begin{aligned} \parallel (T_n - T)P \parallel &\leq \tfrac{l(C)}{2\pi} \tfrac{\alpha_1(C)}{\delta(C)} \parallel (T_n - T)T \parallel, \\ \parallel (T_n - T)P_n \parallel &\leq \tfrac{l(C)}{2\pi} \tfrac{\alpha_2(C)}{\delta(C)} \parallel (T_n - T)T_n \parallel, \end{aligned}$$

Corollary 2.2.2 [1]*Let Λ be a spectral set for T,*

(a) If $T_n \xrightarrow{n} T$, then $P_n \xrightarrow{n} P$,

(b) If $0 \notin \Lambda$ and $T_n \xrightarrow{\nu} T$, then $\|(T_n - T)P\| \longrightarrow 0, \|(T_n - T)P_n\| \longrightarrow 0$, and $P_n \xrightarrow{\nu} P$

Remark 2.2.1 *the upper semicontinuity of the spectrum and the lower semicontinuity of the spectrum, taken together, give the continuity of the spectrum in the sense of kuratowski.*

Chaptre 3

Integral operator

3.1 Integral operator

Definition 3.1.1 *The integral operator is any linear operator T defined on a normed space X in a normed space Y given by*

$$T\varphi(x) = \int_{G_2} k(x,y)\varphi(y)dy \qquad x \in G_1.$$

Where $k(x,y)$ is a measurable function defined on a measure set $G_1 \times G_2$ and $\varphi(y)$ is a measurable function defined on G_2.

Remark 3.1.1 *The measurable function $k(x,y)$ is said kernel of the integral operator T.*

3.1.1 Norms of the integral operators

Let T be an integral operator defined on $L_p(G_1)$, then for all p and q conjugate ($\frac{1}{p} + \frac{1}{q} = 1$), with ($1 \leq p, q \leq \infty$), the norm of the operator T is given by

$$\| T \|_p = \begin{cases} (\int_{G_1} \mid k(x,y) \mid^q dy)^{\frac{p}{q}} dx)^{\frac{1}{p}}, & \text{for } 1 < p < \infty \\ \int_{G_1} es \sup\limits_{y} \mid\mid k(x,y) \mid dx, & \text{for } p = 1 \\ es \sup\limits_{x} \int_{G_2} \mid k(x,y) \mid dy, & \text{for } p = \infty \end{cases}$$

Remark 3.1.2 *The norm of the integral operator T for $p = q = 2$ given by*

$$\parallel T \parallel_2 = \left(\int_{G_1}\int_{G_2} \mid k(x,y) \mid^2 dxdy\right)^{\frac{1}{2}} < \infty$$

Theorem 3.1.1 *The integral operator T defined from X*

$$T\varphi(x) = \int_X k(x,y)\varphi(y)dy, \quad x \in X,$$

with continous kernel $k(x,y)$ is a compact operator.

3.1.2 Fredholm operator

Fredholm integral operator

Definition 3.1.2 [18]*let X, Y be Banach space. An operator $T \in B(X,Y)$ is said to be a Fredholm operator from X to Y if*

(1) $\alpha(T) = \dim N(T)$ is finite,

(2) $R(T)$ is closed in Y,

(3) $\beta(T) =$co $\dim R(T)$ is finite.

The set of Fredholm operators from X to Y is denoted by $\Phi(X,Y)$.

The index of a Fredholm integral operator is defined as

$$i(T) = \alpha(T) - \beta(T).$$

Theorem 3.1.2 *If T be a cmpact operator, then $Id - T$ is of Fredholm, and $i(Id - T) = 0$.*

Theorem 3.1.3 *If $A, B \in L(X)$ are two Fredholm operator, then BA is Fredhom operator*

$$i(BA) = i(A) - i(B).$$

Proposition 3.1.1 [11]*The set of* ***upper semi-Fredholm*** *operators from X into Y denoted by $\Phi_+(X,Y)$, defined by*

$$\Phi_+(X,Y) = \{T \in C(X,Y), \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } Y\}.$$

The set of **lower semi-Fredholm** operators from X into Y denoted by $\Phi_-(X,Y)$, defined by

$$\Phi_-(X,Y) = \{T \in C(X,Y), \beta(T) < \infty \text{ and } R(T) \text{ is closed in } Y\}.$$

The set of **semi-Fredholm** operators from X into Y denoted by $\Phi_\pm(X,Y)$, defined by

$$\Phi_\pm(X,Y) = \Phi_+(X,Y) \cup \Phi_-(X,Y).$$

The set of Fredholm operators from X into Y defined by

$$\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y).$$

3.2 Finit rank approximation

3.2.1 Approximation based on projections

Definition 3.2.1 *Let π_n be a sequence of bounded projections defined on X, that is, each π_n is in $B(X)$ and $\pi_n^2 = \pi_n$.and $T \in B(X)$ define*

$$T_n^p = \pi_n T, \qquad T_n^S = T\pi_n \text{ and } T_n^G = \pi_n T \pi_n.$$

Such that, the bounded operators T_n^p is **the projection approximation** of T, T_n^S is **the Sloan approximation** of T, and T_n^G is **the Galerkin approximation** of T, all three are finite rank operators if the rank of the projection π_n is finite.

Theorem 3.2.1 *Let $T \in B(X)$ and $\pi_n \xrightarrow{p} I$, such that I be the identity operator, then*

(1) $T_n^p \xrightarrow{P} T, T_n^S \xrightarrow{P} T$ and $T_n^G \xrightarrow{P} T$.

(2) If T is a compact operator, then $T_n^p \xrightarrow{n} T$, $T_n^S \xrightarrow{\nu} T$ and $T_n^G \xrightarrow{\nu} T$.

(3) If T is a compact operator and $\pi_n^* \xrightarrow{p} I^*$, then $T_n^S \xrightarrow{n} T$ and $T_n^G \xrightarrow{n} T$.

Proof. We have the sequence $(\| \pi_n \|)$ is bounded.

(1) Since $\pi_n \xrightarrow{p} I$, and $T_n^p x = \pi_n T x \longrightarrow Tx$ for every $x \in X$, that is $T_n^p \xrightarrow{P} T$. Also, since T is continous and

$$T_n^S - T = T(\pi_n - I), \quad T_n^G - T = (T_n^p - T)\pi_n + T_n^S - T,$$

And we see that $T_n^s \xrightarrow{P} T$ and $T_n^G \xrightarrow{P} T$.

(2) Let the operator T be compact. The set $E = \{Tx : x \in X, \| x \| \leq 1\}$ is relatively compact in X (Banach space). The convergence $\pi_n \xrightarrow{p} I$ is uniform on the set E, that is,

$$\| T_n^P - T \| = \| \pi_n T - T \| = \sup\{\| (\pi_n - I)y \| : y \in E\} \longrightarrow 0.$$

So, $T_n^p \xrightarrow{n}$T.$Since \| T_n^S \| \leq \| \pi_n \| \| T \|$ and $\| T_n^G \| \leq \| \pi_n \|^2 \| T \|$ for each n, the sequence (T_n^S) and (T_n^G) are bounded in $B(X)$. And we note that

$$(T_n^S - T)T = T(T_n^p - T), \quad (T_n^G - T)T = (T_n^p - T)T_n^p + (T_n^S - T)T, \quad (T_n^S - T)T_n^S = (T_n^S - T)T\pi_n, \quad (T_n^G - T)T$$

Hence $T_n^S \xrightarrow{\nu} T$ and $T_n^G \xrightarrow{\nu} T$.

(3) Let the operator T be compact and $\pi_n^* \xrightarrow{p} I^*$ in addition to $\pi_n \xrightarrow{p} I$. Since T^* is a compact operator on X^*, and

$$\| T_n^S - T \| = \| (T_n^S - T)^* \| = \| \pi_n^* T^* - T^* \| \longrightarrow 0$$

By (1) we have

$$\begin{aligned} \| T_n^G - T \| &= \| \pi_n (T_n^S - T) + T_n^p - T \| \\ &\leq \alpha \| T_n^S - T \| + \| T_n^P - T \| \longrightarrow 0. \end{aligned}$$

So $T_n^S \xrightarrow{n} T$ and $T_n^G \xrightarrow{n} T$.

■

Remark 3.2.1 [1]*If T is compact operator (T_n^S) and (T_n^G) in fact converge to T in a collectively compact manner.*

Ways of constructing a sequence (π_n)

We now gives deffirent ways of constructing a sequence (π_n) of bounded projections on X such that $\pi_n \xrightarrow{p} I$ and the rank of π_n is finite.

Truncation of a Schauder Expansion

Definition 3.2.2 ***(Schauder basis)***

The Banach space X has a Schauder basis, that is, there are $e_1, e_2, \ldots$ in X such that $\|e_j\| = 1$ for each integer $j > 0$ and $\forall x \in X$, there are unique scalars $c_1(x), c_2(x), \ldots$ for which

$$x = \sum_{j=1}^{\infty} c_j(x) e_j = \lim_{n=\infty} \sum_{j=1}^{n} c_j(x) e_j.$$

Proposition 3.2.1 *Each linear functional $x \longrightarrow c_j(x), x \in X,$ is continous on X.*

For each integer $n > 0$, define

$$\pi_n x = \sum_{j=1}^{n} c_j(x) e_j, \quad x \in X.$$

Cleary, $\pi_n \in B(X)$, $\pi_n^2 = \pi_n$, $\pi_n \xrightarrow{p} I$ and rank $\pi_n = n$.

If an operator $T \in B(X)$ is represented by the infinite matrix $(t_{i,j})$ with respect to a scauder basis $e_1, e_2, \ldots$ that is

$$Te_{j=} \sum_{i=1}^{\infty} t_{i,j} e_i, \quad j = 1, 2, \ldots$$

then

$$T_n^p e_j = \pi_n T e_j \quad = \sum_{i=1}^{n} t_{i,j} e_i, \quad j = 1, 2, \ldots$$

$$T_n^S e_j = T \pi_n e_j \quad = \begin{cases} \sum_{i=1}^{\infty} t_{i,j} e_i & \text{if } j = 1, \ldots, n, \\ 0 & \text{if } j > n, \end{cases}$$

$$T_n^G e_j = \pi_n T \pi_n e_j \quad = \begin{cases} \sum_{i=1}^{n} t_{i,j} e_i\ , & \text{if } j = 1, \ldots, n, \\ 0 & \text{if } j > n, \end{cases}$$

Hence the matrix representing **the projection approximation** T_n^p of T is obtained by truncating each column of the matrix $(t_{i,j})$ at the n^{th} entry and putting all zeros thereafter. The matrix represinting **the Sloan approximation** T_n^S of T is obtained by replacing every column after the n^{th} column of the matrix $(t_{i,j})$ by a column of all zeros. While the matrix representing **the Galerkin approximation** T_n^G of T is obtained by carrying out both these operations, so that the entries in the top left $n \times n$ corner of this matrix are the same as the corresponding entries of the matrix $(t_{i,j})$ and all other entries are equal to zero.

Special case

Let X be a separable Hilbert space and let $e_1, e_2, ...$ form an orthonormal basis for X. If $\langle ., . \rangle$ denotes the inner product in X, then,

$$x = \sum_{j=1}^{n} \langle x, e_j \rangle e_j, \quad x \in X.$$

So for each integer $n > 0$,

$$\pi_n x = \sum_{j=1}^{n} \langle x, e_j \rangle e_j, \quad x \in X.$$

Since $\langle \pi_n x, y \rangle = \sum_{j=1}^{n} \langle x, e_j \rangle \langle e_j, y \rangle = \langle x, \pi_n x \rangle$ for every $x, y \in X$, we see that $\pi_n^* = \pi_n$, that is, the projection π_n is orthogonal. If T is a compact operator on X, then we obtain not only $T_n^p \xrightarrow{n} \mathrm{T} butalso \mathrm{T}_n^S \xrightarrow{n} \mathrm{T} and\ \mathrm{T}_n^G \xrightarrow{n} \mathrm{T}$.

Example 3.2.1 *Some classical Schauder bases and orthonormal bases:*

(1) Standard Schauder basis for l^p, $1 \leq p < \infty$,

$\forall j > 0, j \in \mathbb{Z}$, let

$$e_k = (0, ..., 0, 1, 0, 0, ...),$$

Where only the k^{th} entry is 1. then (e_k) forms a Schauder basis for l^p if $p = 2$, this is orthonormal basis.

(2) Schauder basis of saw-tooth functions for $C^0([0, 1])$

for $t \in [0, 1]$, and $e_0(t) = t, e_1(t) = 1 - t$,

$$e_2(t) = \begin{cases} 2e_0(t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 2e_1(t) & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

for each integer $m > 0$ and $j = 1, ..., 2^m$, let

$$e_{2^m} + j(t) = e_2(2^m t - j + 1),$$

where we let $e_2(t) = 0$ for $t \notin [0, 1]$. The sequence of saw-tooth functions forms a schauder basis for $C^0([0, 1])$.

(3) Orthonormal basis of haar functions for $L^2([0,1])$.

t$\in [a,b]$, let $e_{0,0}(t) = 1$,

$$e_{1,0}(t) = \begin{cases} 1 & \text{if } 0 \leq t < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} < t < 1, \\ 0 & \text{if } \ t = \frac{1}{2} \end{cases}$$

For each integer $m > 0$ and $j = 1, ..., 2^m$, let

$$e_{m,j}(t) = \begin{cases} 2^{m/2} & \text{if } \frac{j-1}{2^m} \leq t < \frac{2j-1}{2^{m+1}}, \\ -2^{m/2} & \text{if } \frac{2j-1}{2^{m+1}} < t < \frac{j}{2^m}, \\ 0 & \text{otherwise.} \end{cases}$$

The sequence of these haar functions forms an orthonormal basis for the space $L^2([0,1])$.

Interpolatory Projections

Also known as a collocation approximation .

Definition 3.2.3 *Let* $X = C^0([a,b])$ *with the norm* $\| \, . \, \|_\infty$*. For each integer* $n > 0$ *and* $r(n)$*, consider the nodes* $t_{n,1}, ..., t_{n,r(n)}$ *in* $[a,b]$*:*

Where $t_{n,0} = a$ and $t_{n,r(n)+1} = b$

$$a \leq t_{n,1} < t_{n,2} < ... < t_{n,r(n)-1} < t_{n,r(n)} \leq b$$

Consider functions $e_{n,1}, ..., e_{n,r(n)}$ in X such that

$$e_{n,j}(t_{n,k}) = \delta_{j,k}, \quad j,k = 1, ..., r(n).$$

We define

$$(\pi_n x)(t) = \sum_{j=1}^{r(n)} x(t_{n,j}) e_{n,j}(t), \quad x \in X, t \in [a,b].$$

Then $\pi_n : X \longrightarrow X$, and we see that $\pi_n^2 = \pi_n$. Also

$$\mathcal{R}(\pi_n) = span\{e_{n,1}, ..., e_{n,r(n)}\}$$

The set $\{e_{n,1},...,e_{n,r(n)}\}$ is lineary independent. Hence $rank(\pi_n)=r(n)$. Since for every $x\in X$

$$(\pi_n x)(t_{n,j})=x(t_{n,j}),\qquad j=1,...,r(n),$$

$\pi_n x$ interpolates x at $t_{n,1}1,...,t_{n,r(n)}$, we say that π_n is an interpolatory projection.

We shows that $\|\pi_n\|=\|\sum_{j=1}^{r(n)}|e_{n,j}|\|_\infty$. foe every $x\in X$ and $t\in[a,b]$, we have

$$\begin{aligned}|(\pi_n x)(t)| &\leq \|\sum_{j=1}^{r(n)}|x(t_{n,j})||e_{n,j}(t)|\| \\ &\leq \|x\|_\infty \sum_{j=1}^{r(n)}|e_{n,j}(t)| \\ &\leq \|x\|_\infty \|\sum_{j=1}^{r(n)}|e_{n,j}|\|_\infty .\end{aligned}$$

Hence $\|\pi_n\|\leq\|\sum_{j=1}^{r(n)}|e_{n,j}|\|_\infty$. On the other hand, choose $t_0\in[a,b]$ such that

$$(\sum_{j=1}^{r(n)}|e_{n,j}|)(t_0)=\|\sum_{j=1}^{r(n)}|e_{n,j}|\|_\infty$$

and define $x_0\in X$ as follows:

$$x_0(t_{n,j})=sgn\ \ e_{n,j}(t_0),\qquad j=1,...,r(n),$$

$x_0(a)=x(t_{n,1}),x_0(b)=x(t_{n,r(n)})$ and x_0 is a polynomial of degree ≤ 1 on each of the subintervals $[a,t_{n,1}],[t_{n,1},t_{n,2}],...,[t_{r(n)-1},t_{r(n)}],[t_{r(n)},b]$. then $\|x_0\|_\infty=1$ and

$$(\pi_n x_0)(t_0)=\sum_{j=1}^{r(n)}x_0(t_{n,j})e_{n,j}(t_0)=\sum_{j=1}^{r(n)}|e_{n,j}(t_0)|=\|\sum_{j=1}^{r(n)}|e_{n,j}|\|_\infty .$$

So, $\|\pi_n\|\geq\|\sum_{j=1}^{r(n)}|e_{n,j}|\|_\infty$.

Example 3.2.2 *We often have $r(n)=n$*

(a) Piecewise Linear Interpolation: For each integer $n>0$, define $e_{n,j}$ in $C^0([a,b])$

$$e_{n,j}(t_{n,k})=\delta_{j,k},\quad j,k=1,...,n,$$

$e_{n,1}(a) = 1, e_{n,n}(b) = 1, e_{n,j}(a) = 0$, for $j = 2, ..., n$, $e_{n,j}(b) = 0$ for $j = 1, ..., n-1$ and $e_{n,j}$ is a polynomial of degree ≤ 1 on each of the subintervals $[a, t_{n,1}], [t_{n,1}, t_{n,2}], ..., [t_{n,n-1}, t_{n,n}]$ and $[t_{n,n}, b]$.

Note that $e_{n,j}(t) \geq 0$ for all integer $n > 0$, $j = 1, ..., n$ and $t \in [a, b]$. If $t \in [t_{n,j-1}, t_{n,j}]$ for some $j = 1, ..., n+1$, then $e_{n,j-1}(t) + e_{n,j}(t) = 1$, and $e_{n,k}(t) = 0$ for all $k \neq j-1$,

$$e_{n,1}(t) + e_{n,n}(t) = 1 \text{ for all } t \in [a, b].$$

Let $t_{n,0} = a$ and $t_{n,n+1} = b$.

We show that if

$$h_n = \max\{t_{n,j} - t_{n,j-1} : j = 1, ..., n+1\} \longrightarrow 0, \text{ then } \pi_n \xrightarrow{p} I.$$

Fix $x \in C^0([a, b])$ and let $\epsilon > 0$. $\exists \delta > 0$ such that $|x(s) - x(t)| < \varepsilon$, whenever $s, t \in [a, b]$ and $|s - t| < \delta$, let n_0 such that $h_n < \delta, \forall n \geq n_0$. If $t \in [t_{n,j-1}, t_{n,j}]$, then $|t_{n,j} - t| \leq h_n < \delta$ and $|x(t_{n,j}) - x(t)| < \varepsilon$.

Thus

$$\begin{aligned} |\ (\pi_n x)(t) - x(t)\ | &= \Big|\ \sum_{i=1}^{n} [x(t_{n,i}) - x(t)] e_{n,i}(t)\ \Big| \\ &\leq |\ x(t_{n,j-1}) - x(t)\ |\ e_{n,j-1}(t) + |\ x(t_{n,j}) - x(t)\ |\ e_{n,j}(t) \\ &< \epsilon [e_{n,j-1}(t) + e_{n,j}(t)] = \varepsilon. \end{aligned}$$

So, $\| \pi_n x - x \|_\infty \longrightarrow 0$ for every $x \in C^0([a, b])$, that is $\pi_n \xrightarrow{p} I$.

If $x \in C^1([a, b])$ and $e \in [t_{n,j-1}, t_{n,j}]$ for $j = 1, ..., n+1$, then $|x(t_{n,j-1}) - x(t)| = |x'(s_{n,j})|\,(t - t_{n,j-1})$ and $|x(t_{n,j}) - x(t)| = |x'(u_{n,j})|\,(t_{n,j} - t)$ for $s_{n,j} \in [t_{n,j-1}, t]$ and $u_{n,j} \in [t, t_{n,j}]$, so

$$\begin{aligned} |\ (\pi_n x)(t) - x(t)\ | &\leq |\ x'(s_{n,j})\ |\ (t - t_{n,j-1}) e_{n,j-1}(t) + |\ x'(u_{n,j})\ |\ (t_{n,j} - t) e_{n,j}(t) \\ &\leq \|\ x'\ \|_\infty\ [(t - t_{n,j-1}) + (t_{n,j} - t)] \\ &\leq \|\ x'\ \|_\infty\ h_n. \end{aligned}$$

So,

$$\|\ \pi_n x - x\ \|_\infty \leq \|\ x'\ \|_\infty\ h_n \text{ for every } x \in C^1([a, b]).$$

(b) Cubic Spline Interpolation: For each integer $n > 0$ and $a = t_{n,1} < ...t_{n,n} = b$ in $[a, b]$, X_n denote the subspace of $X = C^0([a, b])$ consisting of all $x \in C^2([a, b])$ such that x is a polynomial of degree ≤ 3 on each of the $n-1$ subintervals $[t_{n,1}, t_{n,2}], ..., [t_{n,n-1}, t_{n,n}]$. Then dim $X_n = n+2$. An element of X_n is known as a **Cubic Spline function with knots** at $t_{n,1}, ..., t_{n,n}$. for each $j = 1, ..., n,$ there is a unique function $e_{n,j}$ in X_n such that

$$e_{n,j}(t_{n,k}) = \delta_{j,k}, \quad k = 1, ..., n,$$

The third derivative of $e_{n,j}$ exists at $t_{n,2}$ and $t_{n,n-1}$. This means that for each $e_{n,j}$, the two points $t_{n,2}$ and $t_{n,n-1}$ are not really knots. For $x \in X$, let $\pi_n x = \sum_{j=1}^{n} x(t_{n,j})e_j$, $x \in X$, let h_n and $\tilde{h}_n$ denote the maximum and the minimum of $\{t_{n,j} - t_{n,j-1} : j = 2, ..., n\}$. If there is a constant α such that $(h_n/\tilde{h}_n) \leq \alpha \;\; \forall n$, and $h_n \longrightarrow 0$, then it is known that $\pi_n \stackrel{p}{\longrightarrow} I$. Further it can be shown that

$$\| \pi_n x - x \|_\infty \leq \alpha \| x^{(4)} \|_\infty (h_n)^4 \qquad \forall x \in C^4([a,b]).$$

Where α is a constant.

Remark 3.2.2 *It is possible to choose a basis for the space X_n of cubic splines such that each functions in the basis is nonzero on at most four of the subintervals $[a, t_{n,1}], ..., [t_{n,n-1}, b]$. Such functions are known as* ***B-splines*** *and have proved to be well suited for numerical computations.*

Orthogonal Projections on Subspaces of Piecewise Constant Functions

Let $X = L^2([a,b])$. For each positive integer n, consider

$$a = t_{n,0} < t_{n,1} < ... < t_{n,n-1} < t_{n,n} = b$$

and $X_n = \{x \in X : x \text{ is constant on } [t_{n,j-1}, t_{n,j}], j = 1, ..., n\}$ of X. Let

$$\begin{aligned} (\pi_n x)(t) &= \frac{1}{t_{n,j} - t_{n,j-1}} \int_{t_{n,j-1}}^{t_{n,j}} x(s)ds, t \in [t_{n,j-1}, t_{n,j}[, j = 1, .., n, \\ (\pi_n x)(b) &= (\pi_n x)(t_{n,n-1}). \end{aligned}$$

We have $\pi_n : X \longrightarrow X$ is linear,$\mathcal{R}(\pi_n) = X_n$ and if $x \in X_n$, then $\pi_n x = x$. Hence $\pi_n^2 = \pi_n$ and $rank\ \pi_n = \dim X_n = n$. let $x \in X$ for $j = 1, ..., n$, then

$$\begin{aligned} \int_{t_{n,j-1}}^{t_{n,j}} \mid (\pi_n x)(t) \mid^2 dt &= \int_{t_{n,j-1}}^{t_{n,j}} \mid \tfrac{1}{t_{n,j}-t_{n,j-1}} \int_{t_{n,j-1}}^{t_{n,j}} x(s)ds \mid^2 dt \\ &\leq \int_{t_{n,j-1}}^{t_{n,j}} [\tfrac{1}{t_{n,j}-t_{n,j-1}} \int_{t_{n,j-1}}^{t_{n,j}} \mid x(s) \mid^2 ds]dt \\ &= \int_{t_{n,j-1}}^{t_{n,j}} [\mid x(s) \mid^2 (\int_{t_{n,j-1}}^{t_{n,j}} \tfrac{1}{t_{n,j}-t_{n,j-1}} dt)]ds \\ &= \int_{t_{n,j-1}}^{t_{n,j}} \mid x(s) \mid^2 ds, \end{aligned}$$

hence

$$\| \pi_n x \|_2^2 = \sum_{j=1}^{n} \int_{t_{n,j-1}}^{t_{n,j}} \mid (\pi_n x)(t) \mid^2 dt \leq \sum_{j=1}^{n} \int_{t_{n,j-1}}^{t_{n,j}} \mid x(s) \mid^2 ds = \| x \|_2^2 .$$

So $\| \pi_n \|_2 \leq 1$.

Corollary 3.2.1 *π_n is an orthogonal projection defined on the Hilbert space $L^2([a, b])$ and the range X_n of π_n is contained in the set of all piecewise constant functions.*

We shown that if $h_n = \max\{t_{n,j} - t_{n,j-1} : j = 1, ..., n\} \longrightarrow 0$, then $\| \pi_n x - x \|_2 \longrightarrow 0$ $\forall x \in L^2([a, b])$. Since $(\| \pi_n \|_2)$ is bounded and $C^0([a, b])$ is a dence subset of $L^2([a, b])$, we will prove that

$$\| \pi_n x - x \|_2 \longrightarrow 0 \ \text{ for every } x \in C^0([a, b]).$$

Let $\epsilon > 0$. x is uniformly continous on $[a, b]$, there is $\delta > 0$ such that $|x(s) - x(t)| < \epsilon$ whenever $s, t \in [a, b]$ and $|s - t| < \delta$. Sinse $h_n \longrightarrow 0$, let n_0 such that $h_n < \delta$ for all $n \geq n_0$.

Then for $n \geq n_0$ and $j = 1, ..., n$, we have

$$\begin{aligned} \int_{t_{n,j-1}}^{t_{n,j}} \mid (\pi_n x)(t) - x(t) \mid^2 dt &= \int_{t_{n,j-1}}^{t_{n,j}} \mid \tfrac{1}{t_{n,j}-t_{n,j-1}} \int_{t_{n,j-1}}^{t_{n,j}} [x(t) - x(s)]ds \mid^2 dt \\ &\leq \int_{t_{n,j-1}}^{t_{n,j}} [\tfrac{1}{t_{n,j}-t_{n,j-1}} \int_{t_{n,j-1}}^{t_{n,j}} \mid x(t) - x(s) \mid^2 ds]dt \\ &\leq \int_{t_{n,j-1}}^{t_{n,j}} \epsilon^2 ds \\ &= \epsilon^2 (t_{n,j} - t_{n,j-1}). \end{aligned}$$

So

$$\begin{aligned} \| \pi_n x - x \|_2^2 &= \sum_{j=1}^{n} \int_{t_{n,j-1}}^{t_{n,j}} \mid (\pi_n x)(t) - x(t) \mid^2 dt \\ &= \sum_{j=1}^{n} \epsilon^2 (t_{n,j} - t_{n,j-1}) \\ &= \epsilon^2 (b - a), \end{aligned}$$

for all $n \geq n_0$.

Thus $\| \pi_n x - x \|_2 \longrightarrow 0$ for every $x \in C^0([a, b])$. And $\pi_n \stackrel{p}{\longrightarrow} I$.

Finite Element Approximation

Definition 3.2.4 ***(Generalized eigenvalue problem)***

The problem of finding a nonzero element φ of X and a scalar λ such that $A\varphi = \lambda B\varphi$, when A and B are operators on a banach space X.

A 'weak formulation ' of the generalized eigenvalue problem can be given as follows

Let X be a Hilbert space with an inner product $\langle .,. \rangle$. a sesquilinear functional $a(.,.)$ on $X \times X$ is a complex-valued function on $X \times X$ which is linear in the first variable and conjugate-linear in the second variable. We said it is bounded if $\mid a(x,y) \mid \leq \alpha \parallel x \parallel \parallel y \parallel$ $\forall \alpha > 0$ and $\forall x, y \in X$. Consider bounded sesquilinear functionals $a(.,.)$ and $b(.,.)$ on $X \times X$, where $a(.,.)$ is strongly coercive,

$$\Re a(x,x) \geq \alpha \parallel x \parallel^2, \text{ for every } \alpha > 0 \text{ and } x \in X.$$

Consider the problem of a **weakly posed generalized eigenvalue problem**

$$a(\varphi, y) = \lambda b(\varphi, y), \text{ for every } y \in X$$

when $A \in B(X)$ and $B \in B(X)$ such that

$$a(x,y) = \langle Ax, y \rangle \text{ and } b(x,y) = \langle Bx, y \rangle, \text{ for every } x, y \in X$$

The strong coercivity of the sesquilinear functional $a(.,.)$ shows that the linear operator A is bounded below and its adjoint A^* is injective. And the bounded operator A is invertible in $B(X)$. $T = A^{-1}B$.

By constructing, for each integer $n > 0$, a finite dimentional subspace X_n of X and by requiring to find a nonzero element φ_n of X_n and a scalar λ, we obtained a finite element approximation of the weakly posed generalized eigenvalue problem such that

$$a(\varphi_n, y) = \lambda_n b(\varphi_n, y) \text{ for every } y \in X_n.$$

Let the orthogonal projection π_n defined on X with $\mathcal{R}(\pi_n) = X_n$. Then the preceding equation can be written as

$$\pi_n A \varphi_n = \lambda_n \pi_n B \varphi_n, \quad 0 \neq \varphi_n \in X_n.$$

Then $\forall x \in X_n$, we have

$$\begin{aligned}\alpha \parallel x \parallel^2 &\leq \Re a(x,x) \leq \mid a(x,x) \mid = \mid \langle Ax, y \rangle \mid \\ &= \mid \langle Ax, \pi_n x \rangle \mid = \mid \langle \pi_n Ax, x \rangle \mid \\ &\leq \parallel \pi_n Ax \parallel \parallel x \parallel .\end{aligned}$$

Define $A_n = \pi_n A_{|X_n,,X_n}$. Then $A_n \in B(X_n)$ is injective.

The linear space X_n is finite dimentional, A_n is invertible and $\|A_n^{-1}\| \leq 1/\alpha$. Then the ealier equation can be written as

$$\varphi_n = \lambda_n A_n^{-1} \pi_n B \varphi_n = \lambda_n A_n^{-1} \pi_n AT \varphi_n, \quad 0 \neq \varphi_n \in X_n.$$

Define $\breve{\pi}_n x = A_n^{-1} \pi_n Ax$ for $x \in X$, then $\breve{\pi}_n \in B(X)$, $\mathcal{R}(\breve{\pi}_n) = X_n$ and

$$\breve{\pi}_n^2 = A_n^{-1}(\pi_n A_{|X_n} A_n^{-1}) \pi_n A = A_n^{-1} \pi_n A = \breve{\pi}_n.$$

So $\breve{\pi}_n$ is a bounded finite rank projection; and if we let

$$\breve{T}_n^p = \breve{\pi}_n T,$$

and

$$\breve{T}_n^p \varphi_n = \frac{1}{\lambda_n} \varphi_n, \quad 0 \neq \varphi_n \in X_n, \lambda_n \neq 0.$$

In most operations, the operator$T$ is compact and the finite dimentional subspaces $X_1, X_2, ...$ of X are so chosen that $\pi_n \xrightarrow{p} I$. Since $\breve{\pi}_n = \breve{\pi}_n - \pi_n + \pi_n = A_n^{-1} \pi_n A(I - \pi_n) + \pi_n$, we see that $\breve{\pi}_n \xrightarrow{p} I$ and so $\breve{T}_n^p \xrightarrow{n} T$.

3.2.2 Approximation of Integral Operators

Let X is a suitable function space and T is a fredholm integral operator on X.

We consider $X = L^2([a,b])$ with the 2-norm and $k(.,.) \in L^2([a,b] \times [a,b])$, or $X = C^0([a,b])$ with the sup norm and a function $k(.,.) \in C^0([a,b] \times [a,b])$. For $x \in X$, let

$$(Tx)(s) = \int_a^b k(s,t) x(t) dt, \quad s \in [a,b].$$

T is a fredholm integral operator with kernel $k(.,.)$. It is easy to see that $T \in B(x)$.

If $X = L^2([a,b])$, then

$$\| T \| = \left(\int\int_{[a,b]\times[a,b]} | k(s,t) |^2 \, dm(s,t)\right)^{1/2} = \| k(.,.) \|_2, \tag{3.2.1}$$

if $X = C^0([a,b])$, then

$$\| T \| \leq (b-a) \sup\{| k(s,t) | : s,t \in [a,b]\} = (b-a) \| k(.,.) \|_\infty . \tag{3.2.2}$$

Degenerate kernel approximation

Definition 3.2.5 *A kernel $\tilde{k}(.,.)$ is degenerate if there are $x_1, ..., x_r$ and $y_1, ..., y_r$ in X such that*

$$\tilde{k}(s,t) = \sum_{j=1}^{r} x_j(s) y_j(t), \quad s,t \in [a,b].$$

If $\tilde{T}$ is a fredholm operator with a degenerate kernel $\tilde{k}(.,.)$ then

$$(\tilde{T}x)(s) = \sum_{j=1}^{r} x_j(s) \int_a^b y_j(t)x(t)dt, \quad s \in [a,b], x \in X$$

so that $R(T) \subset span\{x_1, ..., x_r\}$.

Then $\tilde{T}$ is a finite rank operator .

Theorem 3.2.2 *Let T be a fredholm integral operator with kernel $k(.,.)$ on X, such that $X = L^2([a,b])$ or $X = C^0([a,b])$. Let $k_n(.,.)$ be a degenate kernal such that $\| k_n(.,.) - k(.,.) \| \longrightarrow 0$ for each integer $n > 0$. Consider the degenerate kernel approximation of T given by:*

$$(T_n^D x)(s) = \int_a^b k_n(s,t)x(t)dt, \quad x \in X, s \in [a,b].$$

(T_n^D) is a sequence of bounded finite rank operator on X and $T_n^D \xrightarrow{n} T$.

Proof. If $X = L^2([a,b])$

For (3.2.1) we have

$$\| T_n^D - T \|_2 \leq \| k_n(.,.) - k(.,.) \|_2 \longrightarrow 0$$

such that $T_n^D - T$ is a fredholm integral operator with kernel $k_n(.,.) - k(.,.)$ on X.

If $X = C^0([a,b])$,

For (3.2.2) we have

$$\| T_n^D - T \|_\infty \leq (b-a) \| k_n(.,.) - k(.,.) \|_\infty \longrightarrow 0.$$

Hence the result follows. ■

Different methods for constructing a sequence of degenerate kernels which converges to given kernel in the norm

(1) Piecewise linear interpolation in the second variable: Let $k(.,.) \in C^0([a,b] \times [a,b])$, and $s \in [a,b]$, consider $k_s(t) = k(s,t), t \in [a,b]$. $\forall n \in \mathbb{Z}, n > 0$, let $a = t_{n,0} \leq t_{n,1} < ... < t_{n,n} \leq t_{n,n+1} = b$ and π_n is the piecewise linear interpolatory projection, and $e_{n,j}, j = 1, ..., n$, are the corresponding haat functions. Define

$$k_n(s,t) = (\pi_n k_s)(t) = \sum_{j=1}^{n} k(s, t_{n,j}) e_{n,j}(t), \quad s,t \in [a,b],$$

Thus $k_n(.,.)$ is a degenerate kernel obtained by interpolating the kernel $k(.,.)$ in the second variable. We shows that $\|k_n(.,.) - k(.,.)\|_\infty \longrightarrow 0$ if $h_n = \max\{t_{n,j} - t_{n,j-1} : j = 1, ..., n+1\} \longrightarrow 0$.

Let $\epsilon > 0$. There exists $\delta > 0$ such that $|k(s,t) - k(s,u)| < \epsilon$ whenever $s \in [a,b]$ and $|t-u| < \delta$. Since $h_n \longrightarrow 0$, choose n_0 such that $h_n < \delta$, $\forall n \geq n_0$. if $n \geq n_0$, we have

$$| k_n(s,t) - k(s,t) | = | (\pi_n k_s)(t) - k_s(t) | < \epsilon, \text{ for every } s,t \in [a,b].$$

So $\| k_n(.,.) - k(.,.) \|_\infty \longrightarrow 0$.

Remark 3.2.3 *If we interpolate the kernel $k(.,.)$ in both the variables, we obtain*

$$\tilde{k}_n(s,t) = \sum_{i,j=1}^{n} k(t_{n,i}, t_{n,j}) e_{n,i}(s) e_{n,j}(t), \qquad s,t \in [a,b],$$

As before, it can be seen that $\left\| \tilde{k}_n(.,.) - k(.,.) \right\|_\infty \longrightarrow 0$ if $h_n \longrightarrow 0$.

(2) Bernstein polynomials in two variables: Let $k(.,.) \in C^0([a,b] \times [a,b])$ .for each integer $n > 0$, consider the n^{th} **bernstein polynomials in two variables** given by

$$k_n(s,t) = \sum_{i,j=0}^{n} k(\frac{i}{n},\frac{j}{n}) \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} s^i(1-s)^{n-i}t^j(1-t)^{n-j}, \quad s,t \in [a,b].$$

Then $k_n(.,.)$ is a degenerate kernel and $\| k_n(.,.) - k(.,.) \|_\infty \longrightarrow 0$.

(3) Truncation of a Taylor expansion: Let $k(.,.) \in C^0([a,b]\times[a,b])$ and $(s_0,t_0) \in \mathbb{R}^2$ and $c_{i,j} \in \mathbb{C}$ for $i,j = 0,1,....$ The uniformly and absoluty convergent Tylor series expansion given by

$$k(s,t) = \sum_{i,j=0}^{\infty} c_{i,j}(s-s_0)^i(t-t_0)^j, \quad s,t \text{ belongs to } [a,b],$$

for each integer $n > 0$, let

$$k_n(s,t) = \sum_{i,j=0}^{n} c_{i,j}(s-s_0)^i(t-t_0)^j, \quad s,t \text{ belonges to } [a,b].$$

Then $\| k_n(.,.) - k(.,.) \|_\infty \longrightarrow 0$.

Example 3.2.3 $e^{st} = \sum_{j=0}^{\infty} \frac{s^j t^j}{j!}, \quad s,t \in [a,b].$

where $(s_0,t_0) = (0,0), c_{i,j} = 0$ if $i \neq j$ and $c_{j,j} = \frac{1}{j!}$ for $i,j = 0,1,...$

(4) Truncation of a Fourier expansion: Let $k(.,.) \in L^2([a,b]\times[a,b])$. For each integers $i,j > 0$, let

$$k_{i,j}(s,t) = e_i(s)\overline{e_j(t)}, \quad s,t \in [a,b].$$

So $(k_{i,j})$ is an orthonormal basis for $L^2([a,b]\times[a,b])$.

$$k(.,.) = \sum_{i,j=1}^{\infty} c_{i,j}k_{i,j}(.,.),$$

where

$$c_{i,j} = \int_a^b \int_a^b k(s,t)\overline{k_{i,j}(s,t)}dsdt,$$

for each integer $n > 0$,let

$$k_n(s,t) = \sum_{i,j=1}^{n} c_{i,j}k_{i,j}(s,t) = \sum_{i,j=1}^{n} c_{i,j}e_i(s)\overline{e_j(t)}, \quad s,t \in [a,b].$$

Then $\| k_n(.,.) - k(.,.) \|_2 \longrightarrow 0$.

Remark 3.2.4 *If we let* $\pi_n x = \sum_{j=1}^{n} \langle x, e_j \rangle e_j$ *for* $x \in L^2([a,b])$*, then it is easy to see that* $T_n^D = \pi_n T \pi_n = T_n^G$.

Approximation based on numerical integration

Let $X = C^0([a,b])$ with the sup norm and let $Q : X \longrightarrow \mathbb{C}$ is a continous linear functional on X be defined by $Q(x) = \int_a^b x(t)dt, x \in X$. And $\|Q\| = b - a$. let $\tilde{Q} : X \longrightarrow \mathbb{C}$ is a quadrature formula given by

$$\tilde{Q}(x) = \sum_{j=1}^{r} \omega_j x(t_j), \quad x \in X,$$

where $t_1, ..., t_r$ satisfy $a \leq t_1 < ... < t_r \leq b$ and the weights $\omega_1, ..., \omega_r$ are compplex numbers. Then $\tilde{Q}$ is continous on X and in fact we have

$$\| \tilde{Q} \| = \sum_{j=1}^{r} | \omega_j | .$$

And (Q_n) is a convergent sequence of quadrature formula if

$$Q_n(x) \longrightarrow Q(x), \quad \forall x \in X.$$

Theorem 3.2.3 ***(polya's theorem)***

A sequence of quadrature formula given by

$$Q_n(x) = \sum_{j=1}^{r(n)} \omega_{n,j} x(t_{n,j}), \quad x \in X,$$

is covergent iff

a) $Q_n(y) \longrightarrow Q(y)$,$\forall y \in E$ (a subset span dense in $C^0([a,b])$),

b) $\sum_{j=1}^{r(n)} | \omega_{n,j} | \leq \alpha$ for some constant α and for all integer $n > 0$.

Definition 3.2.6 ***(Nyström approximation of*** T ***based on the quadrature formula*** Q_n***)***

Denoted by T_n^N,

we define T_n^N by replacing the functional Q by the quadrature formula Q_n. Thus for each integer $n > 0$ and for every $x \in X$, let

$$\begin{aligned}(T_n^N x)(s) &= Q_n(k_s x) = \sum_{j=1}^{r(n)} \omega_{n,j}(k_s x)(t_{n,j}) \\ &= \sum_{j=1}^{r(n)} \omega_{n,j} k(s, t_{n,j}) x(t_{n,j}), \quad s \in [a,b].\end{aligned}$$

Definition 3.2.7 ***(Fredholm approximation of*** T ***based on the quadrature formula*** Q_n***)***

Denoted by T_n^F,

let (π_n) be a sequence of bounded projections defined on X , for each nonegative integer n, let

$$T_n^F = \pi_n T_n^N.$$

Proposition 3.2.2 *Let* $X = C^0([a,b])$, $T \in B(x)$ *be a fredholm integral operator with a kernal* $k(.,.)$, *and a sequence* $(T_n) \in B(X)$.

1) $\forall x \in X$ and each $s \in [a,b]$, $(T_n x)(s) \longrightarrow (Tx)(s)$ as $n \longrightarrow \infty$.

2) $\forall n$, there are $t_{n,1}, ..., t_{n,r(n)} \in [a,b]$ such that

1. $T_n x = 0$, for all $x \in X$ and $x(t_{n,1}) = ... = x(t_{n,r(n)} = 0$,

2. $\forall s \in [a,b]$, there is $\delta_n(s) > 0$ such that

$$\sum_{j=1}^{r(n)} \int_{|t_{n,j}-t|<\delta_n(s)} k(s,t)dt \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

Then

$$\liminf_{n\to\infty} \| T_n - T \| \geq 2 \| T \| .$$

Proof. Let $\varepsilon > 0$. $\exists x_0 \in X$ and $s_0 \in [a,b]$ such that $\|x_0\|_\infty \leq 1$ and

$$| (Tx_0)(s_0) | > \| T \| - \varepsilon.$$

Since, by condition (1), $(T_n x_0)(s_0) \longrightarrow (Tx_0)(s_0)$, choose n_0 such that for all $n \geq n_0$. We have

$$| (T_n x_0)(s_0) - (Tx_0)(s_0) | < \varepsilon.$$

$\forall n = 1, 2, \ldots$ consider points $t_{n,1}, \ldots, t_{n,r(n)}$ and let $\delta_n = \delta_n(s_0)$.

$$\| x_n \|_\infty \leq 1, x_n(t_{n,j}) = -x_0(t_{n,j}) \quad \text{for } j = 1, \ldots, r(n)\ .$$

Then

$$\begin{aligned} | (Tx_n)(s_0) - (Tx_0)(s_0) | &= | \sum_{j=1}^{r(n)} \int_{I_{n,j}} k(s_0, t)[x_n(t) - x_0(t)]dt | \\ &\leq 2 \sum_{j=1}^{r(n)} \int_{I_{n,j}} | k(s_0, t) | \, dt, \end{aligned}$$

where $I_{n,j} =]t_{n,j} - \delta_n, t_{n,j} + \delta_n[\cap [a, b], j = 1, \ldots, r(n)$.

So

$$(Tx_n)(s_0) \longrightarrow (Tx_0)(s_0).$$

Then for every n, we have $(x_0 + x_n)(t_{n,j}) = 0$, $j = 1, \ldots, r(n)$, so that $T_n(x_0 + x_1) = 0$. In particulier, $(T_n x_n)(s_0) = -(T_n x_0)(s_0)$. Hence

$$| (T_n x_n - Tx_n)(s_0) | = | -(T_n x_0)(s_0) - (Tx_n)(s_0) | \longrightarrow 2 \, | (Tx_0)(s_0) |\ .$$

Since $\| x_n \|_\infty \leq 1$, we see that

$$\| T_n - T \| \geq \| (T_n - T)x_n \|_\infty \geq | (T_n x_n - Tx_n)(s_0) |\ .$$

Thus

$$\liminf_{n \to \infty} \| T_n - T \| \geq \lim_{n \to \infty} | (T_n x_n - Tx_n)(s_0) | = 2 \, | (Tx_0)(s_0) | \geq 2 \, \| T \| - 2\varepsilon.$$

As $\varepsilon > 0$, we have

$$\liminf_{n \to \infty} \| T_n - T \| \geq 2 \, \| T \|\ .$$

■

Theorem 3.2.4 *Let $X = C^0([a, b])$ and T be a fredholm integral operator on X with a continous kernel.*

(i) Let (T_n^N) be a Nyström approximation of T based on a convergent sequence of quadrature formula. Then $T_n^N \xrightarrow{p} T$ and $T_n^N \xrightarrow{\nu} T$.

(ii) Let π_n be a sequence of bounded projections such that $\pi_n \xrightarrow{p} I$. Then $T_n^F \xrightarrow{p} T$ and $T_n^F \xrightarrow{\nu} T$.

Proof.

(i) Let $x \in X$. We see that

$$(T_n^N x)(s) = Q_n(k_s x) \longrightarrow Q(k_s x) = (Tx)(s), \quad \forall s \in [a,b].$$

The subset $S = \{k_s x : s \in [a,b]\}$ of X is uniformly bounded since for all $s \in [a,b]$,

$$\| k_s x \|_\infty \leq \| k_s \|_\infty \| x \|_\infty \leq \| k(.,.) \|_\infty \| x \|_\infty .$$

Also, it is uniformly equicontinous, since the functions x and $k(.,.)$ are uniformly continous.

$\forall t, u \in [a,b]$, we have

$$\begin{aligned} | k_s(t)x(t) - k_s(u)x(u) | &\leq | k_s(t)x(t) - k_s(t)x(u) | + | k_s(t)x(u) - k_s(u)x(u) | \\ &\leq \| k(.,.) \|_\infty | x(t) - x(u) | + \| x \|_\infty \sup_{s\in[a,b]} | k(s,t) - k(s,u) | . \end{aligned}$$

The set S is relatively compact. So $\left\|T_n^N x - Tx\right\|_\infty \longrightarrow 0$. Thus $T_n^N \xrightarrow{p} T$. In particular $(\left\|T_n^N\right\|)$ is bounded.Let $E = \{Tx : x \in X, \|x\|_\infty \leq 1\}$. T is a compact operator then the set E is relatively compact in X. $T_n^N \xrightarrow{p} T$ is uniform on E, that is

$$\| (T_n^N - T)T \| = \sup\{\| (T_n^N - T)y \|_\infty : y \in E\} \longrightarrow 0.$$

The subset

$$\tilde{E} = \bigcup_{n=1}^{\infty} \{T_n^N x : x \in X, \| x \|_\infty \leq 1\}$$

of X is uniformly bounded since for every $x \in X$ with $\|x\|_\infty \leq 1, \left\|T_n^N x\right\| \leq \sup_{n\geq 1} \left\|(T_n^N)\right\| < \infty$. Also, it is uniformly equicontinous since for all n, all $x \in X$ with $\|x\|_\infty \leq 1$, and all $s, u \in [a,b]$, we have

$$\begin{aligned} | (T_n^N x)(s) - (T_n^N x)(u) | &\leq \sum_{j=1}^{r(n)} | \omega_{n,j} || k(s,t_{n,j}) - k(u,t_{n,j}) || x(t_{n,j}) | \\ &\leq (\sup_{n\geq 1} \sum_{j=1}^{r(n)} | \omega_{n,j} |)(\sup_{t\in[a,b]} | k(s,t) - k(u,t) |). \end{aligned}$$

And $\sup_{n\geq 1}\sum_{j=1}^{r(n)} \mid \omega_{n,j} \mid < \infty$. Then the set $\tilde{E}$ is a relatively compact and $T_n^N \xrightarrow{p} T$ is uniform on $\tilde{E}$, so

$$\| (T_n^N - T)T_n^N \| \leq \sup\{\| (T_n^N - T)\tilde{y} \|_\infty : \tilde{y} \in \tilde{E}\} \longrightarrow 0.$$

Thus $T_n^N \xrightarrow{\nu} T$.

(ii) Let π_n be a sequence of bounded projections defined on X such that $\pi_n \xrightarrow{p} I$. And $\|\pi_n\| \leq \alpha$ for some $\alpha > 0$ and all n. Since$T_n^F - T = \pi_n(T_n^N - T) + \pi_n T - T = \pi_n(T_n^N - T) + T_n^P - T$,

we see that $T_n^F \xrightarrow{p} T$. Then $(\|T_n^F\|)$ is bounded.

The sets E and $\tilde{E}$ is relatively compact, we have

$$\| \ (T_n^F - T)T \| = \sup\{\| (T_n^F - T)y \|_\infty : y \in E\} \longrightarrow 0,$$
$$\| \ (T_n^F - T)T_n^F \| = \sup\{\| (T_n^F - T)\pi_n\tilde{y} \|_\infty : \tilde{y} \in \tilde{E}\} \longrightarrow 0.$$

Since $(T_n^F - T)\pi_n \xrightarrow{p} 0$. so $T_n^F \xrightarrow{\nu} T$.

■

Theorem 3.2.5 *Let a fredholm integral operator T on $X = C^0([a,b])$ with a continous kernel $k(.,.)$. Let p be a positive integer and assume that the sequence (Q_n) satisfies*

$$\mid Q_n(x) - \int_a^b x(t)dt \mid \leq \frac{c_p}{n^p} \| x^{(p)} \|_\infty \textit{ for } n = 1,2,...\textit{and } x \in C^p([a,b])$$

where (Q_n) is a convergent sequence of quadrature formula,and c_p is constant, independent of n and x .

(i) $\forall i = 1,...,p$, the i^{th} partial derivatives of $k(.,.)$ with respect to the first variable as well as the second variable exist and be continous on $[a,b]\times[a,b]$. Then

$$\| (T_n^N - T)T \| = O(\frac{1}{n^p}) = \| (T_n^N - T)T_n^N \| .$$

Where T_n^N be the nystrõm approximation of T based on Q_n.

(ii) $\forall i, j = 0, 1, ..., p$, the $(i,j)^{th}$ partial derivative $\frac{\partial^{i+j} k}{\partial t^j \partial s^i}$ of $k(.,.)$ exists and be continous at every $(s,t) \in [a,b] \times [a,b]$. Then

$$\| (T_n^N - T)^q T \| \leq (b-a)(\frac{d_p}{n^p})^q \text{ and } \| (T_n^N - T)^q T_n^N \| \leq \alpha(\frac{d_p}{n^p})^q,$$

where n,$q = 1, 2, ...$ and d_p is a constant independent of n and q, and α is constant, independent of n, q and p.

Proof. Let $x \in X$.

(i) Tx and $T_n^N x$ belong to $C^p([a,b])$, and for $j = 1, ..., p$,

$$\| (Tx)^{(j)} \| \leq \alpha_{j,0}(b-a) \| x \|_\infty, \| (T_n^N x)^{(j)} \| \leq \alpha_{j,0}\alpha \| x \|_\infty,$$

where $a_{j,0}$ is finite by the continuity of the j^{th} partial derivative of $k(.,.)$ with respect to the first variable, and α is finite by polya's theorem. Hence

$$\begin{aligned} \| (T_n^N - T)Tx \|_\infty &\leq \tfrac{c_p}{n^p} \sum_{j=1}^{p} \begin{pmatrix} p \\ j \end{pmatrix} \alpha_{j,p-j} \| (Tx)^{(j)} \|_\infty \\ &\leq (b-a)\tfrac{c_p}{n^p} \sum_{j=1}^{p} \begin{pmatrix} p \\ j \end{pmatrix} \alpha_{j,p-j}\alpha_{j,0} \| x \|_\infty \end{aligned}$$

and

$$\begin{aligned} \| (T_n^N - T)T_n^N x \|_\infty &\leq \tfrac{c_p}{n^p} \sum_{j=1}^{p} \begin{pmatrix} p \\ j \end{pmatrix} \alpha_{j,p-j} \| (T_n^N x)^{(j)} \|_\infty \\ &\leq \alpha\tfrac{c_p}{n^p} \sum_{j=1}^{p} \begin{pmatrix} p \\ j \end{pmatrix} \alpha_{j,p-j}\alpha_{j,0} \| x \|_\infty . \end{aligned}$$

Hencre $\| (T_n^N - T)T \| \leq 1/n^p$ and $\| (T_n^N - T)T_n^N \| \leq 1/n^p$.

(ii) $(T_n - T)^q Tx \in C^p([a,b])$ $\forall q = 1, 2, ...$ and for $q = 1, 2, ...$,$n = 1, 2, ...$ and $i = 0, 1, ..., p$

$$\| [(T_n^N - T)^q Tx]^{(i)} \|_\infty \leq (b-a)\frac{c_p\gamma_p(c_p\tilde{\gamma}_p)^{q-1}}{n^{pq}} \| x \|_\infty,$$

such that

$$\begin{aligned} \gamma_p &= \max_{i=0,1,...,p} \sum_{j=0}^{p} \begin{pmatrix} p \\ j \end{pmatrix} \alpha_{i,p-j}\alpha_{j,0} \\ \tilde{\gamma}_p &= \max_{0,1,...,p} \sum_{j=0}^{p} \begin{pmatrix} p \\ j \end{pmatrix} \alpha_{i,p-j}, \end{aligned}$$

let $q = 1$. We have

$$\| (Tx)^{(j)} \|_\infty \leq \alpha_{j,0}(b-a) \| x \|_\infty \quad \text{for } j = 0, ..., p,$$

and

$$\begin{aligned} \| [(T_n^N - T)Tx]^{(i)} \|_\infty &\leq \tfrac{c_p}{n^p} \sum_{j=1}^{p} \begin{pmatrix} p \\ j \end{pmatrix} \alpha_{j,p-j} \| (Tx)^{(j)} \|_\infty \\ &\leq (b-a)\tfrac{c_p \gamma_p}{n^p} \| x \|_\infty \end{aligned}$$

for $n = 1, 2, ...$ and$i = 0, ..., p$. Thus our claim holds for $q = 1$. Let $y = (T_n^N - T)^q Tx$, for $q \geq 1$

$$\begin{aligned} \| [(T_n^N - T)^{q+1}Tx]^{(i)} \|_\infty &= \| [(T_n^N - T)y]^{(i)} \|_\infty \\ &\leq \tfrac{c_p}{n^p} \sum_{j=1}^{p} \begin{pmatrix} p \\ j \end{pmatrix} \alpha_{j,p-j} \| y^{(j)} \|_\infty \\ &\leq \tfrac{c_p}{n^p} \sum_{j=1}^{p} \begin{pmatrix} p \\ j \end{pmatrix} \alpha_{j,p-j}(b-a)\tfrac{c_p\gamma_p(c_p\tilde{\gamma}_p)^{q-1}}{n^{pq}} \| x \|_\infty \\ &\leq (b-a)\tfrac{c_p\gamma_p(c_p\tilde{\gamma}_p)^{q}}{n^{pq}} \| x \|_\infty \end{aligned}$$

for $n = 1, 2, ...$ and $i = 0, ..., p$. Thus our claim holds for $q + 1$ and the induction is over. For $q = 1, 2, ...$ and $n = 1, 2,$ letting $i = 0$, we have

$$\| (T_n^N - T)^q Tx \|_\infty \leq (b-a)\frac{c_p\gamma_p(c_p\tilde{\gamma}_p)^{q-1}}{n^{pq}} \| x \|_\infty .$$

We let $d_p = c_p \max\{\gamma_p, \tilde{\gamma}_p\}$ for $q = 1, 2, ...$ we obtain

$$\| (T_n^N - T)^q T \|_\infty \leq (b-a)(\frac{d_p}{n^p})^q \text{ for every } n = 1, 2, ...$$

■

Bibliographie

[1] M. Ahues, A. Largillier and B.V. Limaye, *Spectral computations for bounded operators,* Applied Mathematics (Boca Raton), 18. Chapman Hall/CRC, Boca Raton, FL., xviii+382 pp. ISBN: 1-58488-196-8 (2001).

[2] N.I.Akhiezer and I.M.Glazman, Theory of linear operators in Hilbert space, Dover Publications, Inc. NEW YORK, 1993.

[3] R. Alam, *On spectral approximation of linear operators,* J. Math. Anal. App., 226, 229-244 (1998).

[4] A. Ammar, *Some properties of the Wolf and Weyl essential spectra of a sequence of linear operators ν-convergent,* Indag. Math. (N.S.) 27, no. 1, 282-295 (2016).

[5] P.M. Anselone, and R. Ansorge, *Compactness principles in nonlinear aperators apprximation theory,* Num. Func. Anal. Optimiz., 1, 589-618 (1979).

[6] M. Benharrat. Comparison between the diffrent definitions of the essential spectrum and application, Thesis submited of univ of Oran. 2013.

[7] John. B. Conway, Bernard B. Morrel, *Operators that are points of spectral continuity,* Integral Equations Operator Theory 2(2), 174-198 (1979).

[8] F. Chatelin, *Spectral approximation of linear operators,* Academic press, New York (1983).

[9] S. Sánchez-Perales, S. V. Djordjeviè, *Spectral continuity using ν-convergence,* J. Math. Anal. Appl. 433, 405-415 (2016).

[10] F. Heleine. Spectral theory(cour de m1, univ paris diderot). 2014.

[11] T Heraiz, A. Ammar, and A. Jeribi, Essential approximation point and essential defect spectrum of a sequence of linear operators in Banach space, MFAT. Vol. 25, no. 4, pp. 373-380 (2019).

[12] A.Jeribi and N. Moalla, A characterization of some subsets of Schecher's essential spectrum and application to singular transport equation, J. Math. Ana. Appl. 358, 434-444(2009).

[13] A.Jeribi, Spectral theory and application of linear operators and blocks operator Matrices, University of Sfax, Sfax, Tunisia, ISBN 978-3-319-17566-9(ebook)(2016).

[14] T. Kato, Perturbation theory for linear operators, Springer, 2nd Edition, (1980).roceedings of the centre for mathematical analysis, Vol. 13, Austral. Nat. Univ., Canberra (1987.)

[15] B. Limaye, *Spectral perturbation and approximation with Numerical Experiments,* Pat. 53. 209-227(2007).

[16] L. I. Nicolaescu. On the space of Fredholm operatos, An. Stiint. Univ. Ai. I. Cuza asi. Mroceedings of the centre for mathematical analysis, Vol. 13, Austral. Nat. Univ., Canberra (1987.)

[17] Philip. M. Anselone, Collectively Compact Operator Approximation Theory and Application to Integral Equations, Prentice-Hall, ISBN(0131406736), (1971).

[18] M.Shechter, Principal of functional analysis, academic press, New York, 1971.

[19] C.Schmoeger, The spectral mapping theorem for the essential approximate point specrum, clloq. Math. 12(1959).

Abstract

The spectrum of a linear operator is one of the most useful objects in functional analysis. However, exact eigenvalues, and generalized eigenvectors with infinite dimensional ranges can rarely be found. It is thus imperative to approximate such operator by finite rank operators and solve the original eigenvalue problem approximately.

Key words: Bounded linear operator, spectrum continuity, integral operator;

Résumé

Le spectre d'un opérateur linéaire est l'un des objets les plus utiles en analyse fonctionnelle. Cependant, le calcule des valeurs propres exactes et des vecteurs propres généralisées d'un opérateur de rang infini peuvent rarement être trouvés. Il est donc impératif d'approximer cet opérateur par des opérateurs de rang fini et de résoudre le problème aux valeurs propres d'origine de manière approximative.

Mots clés: Opérateur linéaire borné, continuité du spectre, opérateur intégral;

ملخص

يعد الطيف المؤثر الخطي أحد أكثر الأشياء فائدة في التحليل الدالي. ومع ذلك نادرا ما يمكن العثور على القيم الذاتية بدقة في حالة مؤثر بصورة ذات بعد غير منته. لذلك من الضروري تقريب هذا المشغل من خلال متتالية مؤثرات ذات الرتبة المحدودة وحل مشكلة القيمة الذاتية الأصلية بشكل تقريبي.

الكلمات المفتاحية. مؤثر خطي محدود، استمرارية الطيف، مؤثر التكامل.