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LIPSCHITZ p-CONVEX AND q-CONCAVE MAPS

(JAVIER ALEJANDRO CHAVEZ-DOMINGUEZ)

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Contents

Notations

$\mathbb{K}$	The field of real or complex numbers.
$\mathbb{R}$	The reel space.
p'	The conjugate of the number p $(1 \leq p \leq \infty)$, that is $\frac{1}{p} + \frac{1}{p'} = 1$.
X^*	The topological dual of X.
B_X	The unit ball of X.
$\mathcal{L}(X,Y)$	The set of all bonded linear operators.
$\mathcal{M}_0$	The class of complete metric pointed metric spaces.
$\mathrm{Lip}_0(X,Y)$	The set of all Lipschitz operators between X and Y that vanish in 0.
$X^\# = \mathrm{Lip}_0(X,\mathbb{R})$	The Lipschitz dual of the pointed metric space X.
Æ(X) or Æ(X,d)	The Arens Eells space of X.
$\mathcal{M}(X)$	The linear space of all molecules on the metric space X.
T_L	The linearization of the Lipschitz operator T.
$\mathbb{1}_A$	The characteristic function of the set A.
$m_{xx'}$	The molecule defined by $m_{xx'} = \mathbb{1}_{\{x\}} - \mathbb{1}_{\{x'\}}$ for $x, x' \in X$.
T^*	The adjoint linear operator of T.
$T^\#$	The adjoint Lipschitz map of T.
$B_{X^\#}$	The unit ball of $X^\#$.
$x \vee y$	The $\sup\{x,y\}$.
$x \wedge y$	The $\inf\{x,y\}$.
$x \perp y$	x is orthogonal to y.

Introduction

In this work we generalize the notion of p-convex (resp. q-concave) studied in the linear case to the Lipschitz case. We generalize show some results from the linear theory to the non linear theory which is the Lipschitz case, with makes clear why there is a close parallel between the linear and nonlinear situation.

In the first chapter, we recall an important mathematical concepts concerning Banach lattice and Positive operators, definition the Linear p-convex and q-concave maps. We study the Factorization through L_p spaces and the proofs of the basic factorizations for p-convex (resp. q-concave) operators.

In the second chapter, we study some properties concerning Lipschitz space. We give the Arens Eells space, and universal property of Arens Eells space, also the adjoint of Lipschitz mapping.

In the last chapter, we present the definition of Lipschitz p-convex maps, we then prove that a map is Lipschitz p-convex if and only if its canonical linearization is p-convex. We introduce the concept of Lipschitz q-concave maps, and prove that the composition of a Lipschitz p-convex map followed by a Lipschitz p-concave one factors through an L_p space. A nonlinear version of a theorem due to Krivine, factorizations of Lipschitz p-convex and Lipschitz q-concave maps in terms through p-convex and q-concave Banach lattices, respectively. This is a nonlinear generalization of the work of Raynaud and Tradacete [10].

CHAPTER 1

BANACH LATTICE, LINEAR p-CONVEX AND q-CONCAVE MAPS

In this chapter we give a short introduction to Banach lattices and positive operators. We recall also the linear p-convex and q-concave maps. The most results of this chapter can be found in [7], [10], and [15].

1.1 Banach lattice

Here we will define and give examples of Banach lattices. We start by giving the definition of an ordered set. We can consult [15] for this.

Definition 1.1. *A non empty set M with a relation $\leq$ is said to be an ordered set if the following conditions are satisfied.*

1. $x \leq x$ *for every* $x \in M$,
2. $x \leq y$ *and* $y \leq x$ *implies* $x = y$,
3. $x \leq y$ *and* $y \leq z$ *implies* $x \leq z$.

We will need the definition of supremum and infimum of a subset of an ordered set. These are deifned as follows.

Definition 1.2. *Let A be a subset of an ordered set M. An element $x \in M$ (respectively $z \in M$) is called an upper bound (lower bound respectively) of A if $y \leq x$ for all $y \in A$ (respectively $z \leq y$ for all $y \in A$). The smallest upper bound of A is called the supremum of A, denoted by $\sup(A)$. The largest lower bound of A is called the infimum of A, denoted by $\inf(A)$.*

The next definition is that of a vector lattice. It is the underlying algebraic-order structure required to define a Banach lattice.

Definition 1.3. *A real vector space E which is ordered by an order relation $\leq$ is called a vector lattice if any two elements $x, y \in E$ have a supremum $x \vee y = \sup\{x, y\}$ and an infimum $x \wedge y = \inf\{x, y\}$ and the following properties are satisfied*

1. $x \leq y$ *implies* $x + z \leq y + z$ *for all* $x, y, z \in A$,
2. $0 \leq x$ *implies* $0 \leq tx$ *for all* $x \in E$ *and* $t \in \mathbb{R}_+$.

Let E be a vector lattice and $x \in E$. The absolute value of x is defined by $|x| = x \vee -x$. A norm on a vector lattice E is called a lattice norm if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for $x, y \in E$. We are now in a position to define a Banach lattice.

Definition 1.4. *A Banach lattice is a Banach space with a lattice norm.*

Definition 1.5. *Let E be a Banach lattice and $x \in E$. We define the positive part of x by $x^+ = x \vee 0$ and the negative part of x by $x^- = -x \vee 0$.*

Notation : Let E be a vector lattice. We denote by $E_+ := \{x \in E : 0 \leq x\}$ the positive cone of E.

Proposition 1.1. *For all $x, y, z \in E$ and $a \in \mathbb{R}$, the following assertions are satisfied.*

1. $x + y = (x \vee y) + (x \wedge y)$
   $x \vee y = -(-x) \wedge (-y)$
   $(x \vee y) + z = (x + z) \vee (y + z)$
   and $(x \wedge y) + z = (x \wedge z) + (y \wedge z)$.
2. $x = x^+ - x^-$.
3. $|x| = x^+ + x^-, |ax| = |a|\,|x|$, *and* $|x + y| \leq |x| + |y|$.
4. $x^+ \perp x^-$ *and the decomposition of x into the difference of two orthogonal positive elements in unique.*
5. $x \leq y$ *is equivalent to* $x^+ \leq y^+$ *and* $x^- \leq y^-$.
6. $x \perp y$ *is equivalent to* $|x| \vee |y| = |x| + |y|$ *. In this case we have* $|x + y| = |x| + |y|$.
7. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ *and* $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.
8. *For all $x, y, z \in E_+$ we have* $(x + y) \wedge z \leq (x \wedge z) + (y \wedge z)$.
9. $|x + y| = (x \vee y) - (x \wedge y)$, *and* $|x - y| = |(x \vee z) - (y \vee z)| + |(x \wedge z) - (y \wedge z)|$.

Proof. For the proof we can see [7], [13], and [15]. □

Exemple 1.1. *All of the classical (real) Banach spaces, $l_p, c_0, C(K), L_P(\mu)$(for $1 \leq p \leq \infty$) are Banach lattices for their usual norm and the pointwise (almost everywhere in the case of $L_P(\mu)$) order.*

For the following propositions, we can consult [7]. The next proposition is the Hölder's inequality for lattices.

Proposition 1.2. *Let X be a Banach lattice, For every $1 \leq p, p' \leq \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ andevery choice of $\{x_i\}_{i=1}^n$ in X and $\{x_i^*\}_{i=1}^n$ in X^*,*

$$\sum_{i=1}^n x_i^*(x_i) \leq \left(\left(\sum_{i=1}^n |x_i^*|^{p'} \right)^{\frac{1}{p'}} \right) \left(\left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \right),$$

As usual, if $p = \infty$ an expression of the form $\left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$ means $\bigvee_{i=1}^n |x_i|$.

The following results are consequences of the Hahn-Banach theorem.

Proposition 1.3. *Let E be a Banach lattice .Then $0 \leq x$ is equivalent to $\langle x, x^* \rangle \geq 0$ for all $x^* \in E_+^*$.*

Proposition 1.4. *Let E be a Banach lattice. For each $0 \lneq x$ there exists $x^* \in E_+^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.*

1.2 Positive operators

Having defined Banach lattices in the previous section, we can now discuss positive operators on Banach lattices. The following is the definition of a positive operator between Banach lattices.

Definition 1.6 (Positive Operators). *Let E, F be Banach lattices. A linear operator $T : E \longrightarrow F$ is called positive if $T(x)$ in F_+ whenever x in E_+.*

In other words, a linear operator $T : E \longrightarrow F$ between Banach lattices is positive if it maps the positive elements of E into the postive elements of F.

Exemple 1.2. *Consider the Banach lattice l_∞ of all bounded sequences of real numbers. The following are positive operators on l_∞.*

1. *The zero operator $(a_1, a_2, ...) \longmapsto (0, 0, ...)$.*

2. *The identity operator $(a_1, a_2, ...) \longmapsto (a_1, a_2, ...)$.*

3. *The right shift operator* $(a_1, a_2, ...) \longmapsto (0, a_1, a_2, ...)$.

The next classical results are basic properties of positive operators on Banach lattices.

Proposition 1.5. *If E is a Banach lattice and T a positive operator on E then,*

$$\|T\| = \sup\{\|Tx\| : 0 \leq x, \|x\| \leq 1\}.$$

Theorem 1.1. *Let $T : E \longrightarrow F$ be a Banach lattice between Banach lattices. Then the following hold.*

1. *T is order preserving,*
2. *$|Tx| \leq T|x|$,*
3. *$(Tx)^+ \leq Tx^+$ and $(Tx)^- \leq Tx^-$,*
4. *If $S, T \in \mathcal{L}(E, F)$ and $0 \leq S \leq T$, then $\|S\| \leq \|T\|$.*

1.3 Linear p-convex maps

Definition 1.7 (Linear p-convex maps). *Consider $1 \leq p \leq +\infty$. A linear map $T : X \longrightarrow E$ from a Banach space X to a Banach lattice E is called p-convex if there exists a constant $M < \infty$ such that for all $x_1, ..., x_n \in X$*

$$\left\| \left(\sum_{j=1}^{n} |Tx_j|^p \right)^{\frac{1}{p}} \right\|_E \leq M \left(\sum_{j=1}^{n} \|x_j\|_X^p \right)^{\frac{1}{p}} \qquad \text{if } 1 \leq p < +\infty,$$

and

$$\left\| \sup_{1 \leq j \leq n} |Tx_j| \right\|_E \leq M \max_{1 \leq j \leq n} \|x_j\|_X \qquad \text{if } p = +\infty.$$

The smallest such constant M is denoted $M^{(p)}(T)$.

The space E is p-convex if id_E is p-convex and

$$M^{(p)}(id_E) = M^{(p)}(E).$$

1.4 Linear q-concave maps

Definition 1.8 (Linear q-concave maps). *A linear map $S : E \longrightarrow Y$ from a Banach lattice E to a Banach space Y is called q-concave if there exists a constant $M < \infty$ such that for all $x_1, ..., x_n \in E$,*

$$\left(\sum_{j=1}^{n} \|Sx_j\|_Y^q\right)^{\frac{1}{q}} \leq M \left\| \left(\sum_{j=1}^{n} |x_j|^q\right)^{\frac{1}{q}} \right\|_E \qquad 1 \leq q < +\infty,$$

and

$$\max_{1\leq j\leq n} \|Sx_j\|_Y \leq M \left\| \sup_{1\leq j\leq n} |x_j| \right\|_E \qquad \text{if } q = +\infty.$$

The smallest such constant M is denoted $M_{(q)}(S)$.

The space E is q-concave if id_E is q-concave and

$$M_{(q)}(id_E) = M_{(q)}(E).$$

The following results are classical.

Proposition 1.6. *Let E be a Banaeh lattiee, X a Banach space and let $1 \leq p, p' \leq \infty$ be so that $\frac{1}{p} + \frac{1}{p'} = 1$.*

1. *A linear operator $T : X \longrightarrow E$ is p-convex if, and only if, T^* is p'-concave. In this case, $M_{(p')}(T^*) = M^{(p)}(T)$.*

2. *A linear operator $T : E \longrightarrow X$ is p-concave if, and only if, T^* is p'-convex. In this case, $M_{(p')}(T^*) = M^{(p)}(T)$.*

3. *E is p-convex (resp. concave) if, and only if, E^* is p'-concave (resp. convex) and $M_{(p')}(E^*) = M^{(p)}(E)$(resp. $M^{(p')}(E^*) = M_{(p)}(E)$).*

We study next the dependence of p-convexity and p-concavity. For simplicity of notations, we put $M^{(p)}(T) = \infty$ or $M_{(p)}(T) = \infty$ if T is not p-convex (resp. p-concave).

The following simple proposition will enable us to describe some classes of p-convex and q-concave operators.

Proposition 1.7. *Let X and Y be two Banach lattices and let $T : X \longrightarrow Y$ be a positive operator. Then, for every $1 \leq p \leq \infty$ and every choice of $\{x_i\}_{i=1}^n$ in X, we have*

$$\left\| \left(\sum_{i=1}^{n} |Tx_i|^p \right)^{\frac{1}{p}} \right\| \leq \|T\| \left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\| \qquad \textit{if } p < \infty,$$

and

$$\left\| \sup_{1 \leq i \leq n} |Tx_i| \right\| \leq \|T\| \left\| \sup_{1 \leq i \leq n} |x_i| \right\| \qquad \textit{if } p = \infty.$$

1.5 Factorization through L_p

We present now some factorization theorems for p-convex and p-concave operators, due to J. L. Krivine [6], which were inspired by results of H. P. Rosenthal [11] and B. Maurey [8].

Theorem 1.2. *Let X, Y be banach spaces and E a Banach lattice. Suppose that $T : X \longrightarrow E$ is linear p-convex and $S : E \longrightarrow Y$ is linear p-concave. Then the operator ST can be factorized through an $L_P(\mu)$ space. Moreover, we may arrange to have $ST = S_1 T_1$ with $T_1 : X \longrightarrow L_p(\mu)$, $S_1 : L_p(\mu) \longrightarrow Y$, $\|T_1\| \leq M^{(p)}(T)$ and $\|S_1\| \leq M_{(p)}(S)$*

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[dr, "T_1"'] & E \arrow[r, "S"] & Y \\
 & L_p(\mu) \arrow[ur, "S_1"'] &
\end{tikzcd}$$

Corollary 1.1. *Let X be a Banach space and fix $1 \leq p < \infty$*

1. *Every p-convex operator T from X into a p-concave Banach lattice E can be factorized through an $L_p(\mu)$ space in the sense that $T = T_1 T_2$, where T_1 is a positive operator from $L_p(\mu)$ into X with $\|T_1\| \leq M_{(p)}(X)$ and T_2 is an operator from V into $L_{(p)}(\mu)$ with $\|T_2\| \leq M^{(p)}(T)$.*

2. *Every p-concave operator S from a p-convex Banach lattice E into X can be factorized through an $L_p(\mu)$ space in the sense that $S = S_1 S_2$, where S_1 is an operator from $L_p(\mu)$ into X with $\|S_1\| \leq M_{(p)}(S)$ and S_2 a positive operator from X into $L_p(\mu)$ with $\|S_2\| \leq M^{(p)}(X)$.*

1.6 Factorization of Linear q-concave maps

Theorem 1.3. *[10] Let E be a Banach lattice, X a Banach space and $1 \leq q \leq \infty$. A linear operator $T : E \longrightarrow X$ is q-concave if, and only if, there exist a q-concave Banach lattice V, a positive operator $\phi : E \longrightarrow V$ (in fact, a lattice homomorphism with dense image), and another operator $S : V \longrightarrow X$ such that $T = S\phi$.*

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "\phi"'] & & X \\ & V \arrow[ur, "S"'] & \end{tikzcd}$$

Proof. Let us suppose $q < \infty$. The proof for the case $q = \infty$ is trivial because every Banach lattice is ∞-concave. However, the precise construction carried out here for $q < \infty$ has its analogue for $q = \infty$.

For the if part, let $(x_i)_{i=1}^n$ in E. Since V is q-concave and ϕ is positive, by Proposition 1.7 we have

$$\left(\sum_{i=1}^n \|Tx_i\|^q\right)^{\frac{1}{q}} \leq \|S\| \, M_{(q)}(I_V) \, \|\phi\| \left\| \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}} \right\|,$$

which yields that T is q-concave.

Now, for the other implication, given $x \in E$, let us consider

$$\rho(x) = \sup\left\{ \left(\sum_{i=1}^n \|Tx_i\|^q\right)^{\frac{1}{q}} : \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}} \leq |x| \right\}.$$

If $M_{(q)}(T)$ denotes the q-concavity constant of T, then for $(x_i)_{i=1}^n$ in E, we have

$$\left(\sum_{i=1}^n \|Tx_i\|^q\right)^{\frac{1}{q}} \leq M_{(q)}(T) \left\| \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}} \right\|.$$

In particular, for all $x \in E$

$$\|Tx\| \leq \rho(x) \leq M_{(q)}(T) \, \|x\|.$$

Moreover, ρ is a lattice semi-norm on E. Indeed, for any $x \in E$ and $\lambda \geq 0$ it is clear that $\rho(\lambda x) = \lambda \rho(x)$. In order to prove the triangle inequality, let $x, y \in E$ and $z = |x| + |y|$ and denote $I_z \subset E$ the ideal generated by z in E, which is identified with a space $C(K)$ in which z corresponds to the function identically one [13, II.7]. Now, for every $\epsilon > 0$ let $z_1, ..., z_n \in E$ such that $\left(\sum_{i=1}^n |z_i|^q\right)^{\frac{1}{q}} \leq |z|$ and

$$\rho(z) \leq \left(\sum_{i=1}^{n} \|Tz_i\|^q\right)^{\frac{1}{q}} + \epsilon.$$

Since $x, y \in I_z$, they correspond to functions $f, g \in C(K)$ such that $|f(t)| + |g(t)| = 1$ for every $t \in K$. Similarly, z_i corresponds to $h_i \in C(K)$ with $(\sum_{i=1}^{n} |h_i(t)|^q)^{\frac{1}{q}} \leq 1$ for every $t \in K$. Hence we can consider

$$\begin{cases} f_i(t) = & h_i(t) f(t), \\ g_i(t) = & h_i(t) g(t), \end{cases}$$

which belong to $C(K)$ and satisfy $(\sum_{i=1}^{n} |f_i(t)|^q)^{\frac{1}{q}} \leq |f(t)|$ and $(\sum_{i=1}^{n} |g_i(t)|^q)^{\frac{1}{q}} \leq |g(t)|$. This means that we can consider $(x_i)_{i=1}^{n}$ and $(y_i)_{i=1}^{n}$ such that $(\sum_{i=1}^{n} |x_i|^q)^{\frac{1}{q}} \leq |x|$ and $(\sum_{i=1}^{n} |y_i|^q)^{\frac{1}{q}} \leq |y|$ in E, with $x_i + y_i = z_i$. Thus, it follows that

$$\rho(x+y) \leq \rho(x) + \rho(y) + \epsilon,$$

and since this holds for every $\epsilon > 0$, the triangle inequality is proved.

Now, if $|y| \leq |x|$, then for any $(x_i)_{i=1}^{n}$ such that $\left(\sum_{i=1}^{n} |x_i|^q\right)^{\frac{1}{q}} \leq |y|$, it holds that $\left(\sum_{i=1}^{n} |x_i|^q\right)^{\frac{1}{q}} \leq |x|$ hence for any such $\{x_i\}_{i=1}^{n}$, $\left(\sum_{i=1}^{n} \|Tx_i\|^q\right)^{\frac{1}{q}} \leq \rho(x)$. This implies that $\rho(y) \leq \rho(x)$.

Let V denote the Banach lattice obtained by completing $E/\rho^{-1}(0)$ with the norm induced by ρ. Let ϕ denote the quotient map from E to $E/\rho^{-1}(0)$, seen as a map to V. Now, for $x \in E$ let us define $S(\phi(x)) = T(x)$. Since $\|T(x)\| \leq \rho(x)$, S is well defined and extends to a bounded operator $S : V \longrightarrow X$, such that $T = S\phi$.

Now, let $(x_i)_{i=1}^{n}$ in E. For every $\epsilon > 0$ and for every $i = 1, ..., n$ there exist $\left\{y_j^i\right\}_{j=1}^{k_i}$ in E such that $\left(\sum_{j=1}^{k_i} |y_j^i|^q\right)^{\frac{1}{q}} \leq |x_i|$ and

$$\rho(x_i)^q = \sup\left\{\sum_{j=1}^{k} \|Ty_j\|^q : \left(\sum_{j=1}^{k} |y_j|^q\right)^{\frac{1}{q}} \leq |x_i|\right\} \leq \sum_{j=1}^{k_i} \left\|Ty_j^i\right\|^q + \frac{\epsilon^q}{n},$$

for every $i = 1, ..., n$. Therefore, we have

$$\left(\sum_{i=1}^{n} \rho(x_i)^q\right)^{\frac{1}{q}} \leq \rho\left(\left(\sum_{i=1}^{n} |x_i|^q\right)^{\frac{1}{q}}\right) + \epsilon,$$

and since this holds for every $\epsilon > 0$, the normed lattice $E/\rho^{-1}(0)$ is q-concave; hence, the same holds for its completion V. $\square$

Since the lattice V constructed in the proof depends on the operator $T : E \longrightarrow X$ and q, we will denote it by $V_{T,q}$ whenever needed. Similarly we will denote ρ_T for the expression defining the norm of $V_{T,q}$.

Remark 1.1. *Note that $V_{T,q}$ has q-concavity constant one. In particular if E is q-concave and $T = id_E$ is the identity, then $V_{T,q}$ is the usual lattice renorming of E with q-concavity constant one.*

Remark 1.2. *In [4], it was proved that an order weakly compact operator $T : E \longrightarrow Y$ (i.e., $T[-x, x]$ is relatively weakly compact for every $x \in E_+$) always factors through an order continuous Banach lattice F. The Banach lattice F is constructed by means of the expression*

$$\|x\|_F = \sup \{\|Ty\| : |y| \leq |x|\}, \textit{ for } x \in E,$$

which yields a Banach lattice in the usual way. Notice that if $T : E \longrightarrow Y$ is q-concave, which implies being order weakly compact, then $\|x\|_F \leq \rho_T(x)$,hence we can consider a natural map $V_{T,q} \longrightarrow F$ such that we can factor T as follows.

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "\phi"'] & Y \\ V_{T,q} \arrow[r, hook, "i"] & F \arrow[u, "\tilde{T}"'] \end{tikzcd}$$

Moreover,F coincides with $V_{T,\infty}$, so in a sense the previous Theorem is an extension of [4, Theorem.I.2].

1.7 Factorization of Linear p-convex maps

There is an analogous version of Theorem 1.3 for p-convex operators, which could be considered, in a sense, as a predual construction to that given in Theorem 1.3.

Theorem 1.4. *[10] Let E be a Banach lattice, X a Banach space and $1 \leq p \leq \infty$. A linear operator $T : X \longrightarrow E$ is p-convex if, and only if, there exist a p-convex Banach lattice W, a positive operator (an injective interval preserving lattice homomorphism) $\varphi : W \longrightarrow E$ and another operator $R : X \longrightarrow W$ such that $T = \varphi R$.*

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "R"'] & & E \\ & W \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Proof. Let us suppose $p < \infty$. The proof for the case $p = \infty$ is analogous, with the usual changes. As in the proof of Theorem 1.3, Proposition 1.7 yields one implication, For the non-trivial one, let $T : X \longrightarrow E$ be p-convex. Let us consider the following set

$$S = \left\{ y \in E : |y| \leq \left(\sum_{i=1}^{k} |Tx_i|^p \right)^{\frac{1}{p}}, \text{ where } \sum_{i=1}^{k} \|x_i\| \leq 1 \text{ and } k \in \mathbb{N} \right\}.$$

We can consider the Minkowski functional defined by its closure $\bar{S}$ in E

$$\|z\|_W = \inf \left\{ \lambda > 0, z \in \lambda \bar{S} \right\}.$$

Clearly $\bar{S}$ is solid, and since T is p-convex, it is also a bounded set of E. Let us consider the space $W = \{z \in E : \|z\|_W < \infty\}$. We claim that for any $z_1, ..., z_n$ in W, it holds that $\left(\sum_{i=1}^{k} |z_i|^p \right)^{\frac{1}{p}}$ belongs to W and

$$\left\| \left(\sum_{i=1}^{k} |z_i|^p \right)^{\frac{1}{p}} \right\|_W \leq \left(\sum_{i=1}^{n} \|z_i\|_W^p \right)^{\frac{1}{p}}.$$

Indeed, given $z_1, ..., z_n$ in W, for every $\epsilon > 0$ and for every $i = 1, ..., n$ there exist λ_i with $z_i \in \lambda_i \bar{S}$, such that

$$\lambda_i^p \leq \inf \left\{ \mu^p : z_i \in \mu \bar{S} \right\} + \frac{\epsilon^p}{n},$$

for each $i = 1, ..., n$.

This means that for every $i = 1, ..., n$, and for every $\delta > 0$ there exists y_i^δ in E with $\left\| z_i - y_i^\delta \right\|_E \leq \delta$, and

$$\left| y_i^\delta \right| \leq \left(\sum_{j=1}^{m_{i,\delta}} \left| T x_j^{i,\delta} \right|^p \right)^{\frac{1}{p}},$$

where $\left\{ x_j^{i,\delta} \right\}_j^{m_{i,\delta}}$ satisfy

$$\left(\sum_{j}^{m_{i,\delta}} \left\| x_j^{i,\delta} \right\|^p \right)^{\frac{1}{p}} \leq \lambda_i,$$

for each $i = 1, ..., n$, and each $\delta > 0$.

Now, for each $\delta > 0$ let

$$w_\delta = \left(\sum_{i=1}^{n} \left| \lambda_i^\delta \right|^p \right)^{\frac{1}{p}}.$$

Notice that

$$\left\| \left(\sum_{i=1}^{n} |z_i|^p \right)^{\frac{1}{p}} - w_\delta \right\|_E \leq \left\| \left(\sum_{i=1}^{n} |z_i - y_i^\delta|^p \right)^{\frac{1}{p}} \right\|_E \leq \sum_{i=1}^{n} \left\| z_i - y_i^\delta \right\|_E \leq n\delta.$$

Moreover, note that for every $\delta > 0$, w_δ belongs to $\left(\sum_{i=1}^{n} \lambda_i^p \right)^{\frac{1}{p}} S$. Indeed,

$$|w_\delta| = \left(\sum_{i=1}^{n} \left| y_i^\delta \right|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} \sum_{j=1}^{m_{i,\delta}} \left| T x_j^{i,\delta} \right|^p \right)^{\frac{1}{p}},$$

and

$$\left(\sum_{i=1}^{n} \sum_{j=1}^{m_{i,\delta}} \left\| x_j^{i,\delta} \right\|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} \lambda_i^p \right)^{\frac{1}{p}}.$$

Hence, $\left(\sum_{i=1}^{n} z_i^p \right)^{\frac{1}{p}} \in \left(\sum_{i=1}^{n} \lambda_i^p \right)^{\frac{1}{p}} \bar{S}$. Therefore, it follows that

$$\begin{aligned} \left\| \left(\sum_{i=1}^{n} z_i^p \right)^{\frac{1}{p}} \right\|_W &= \inf \left\{ \mu > 0 : \left(\sum_{i=1}^{n} z_i^p \right)^{\frac{1}{p}} \in \mu \bar{S} \right\} \leq \left(\sum_{i=1}^{n} \lambda_i^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{i=1}^{n} \left(\inf \left\{ \mu^p : z_i \in \mu \bar{S} \right\} + \frac{\epsilon^p}{n} \right) \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} \|z_i\|_W^p \right)^{\frac{1}{p}} + \epsilon. \end{aligned}$$

Since this holds for every $\epsilon > 0$, we finally have

$$\left\| \left(\sum_{i=1}^{n} |z_i|^p \right)^{\frac{1}{p}} \right\|_W \leq \left(\sum_{i=1}^{n} \|z_i\|_W^p \right)^{\frac{1}{p}}.$$

It follows that the Minkowski functional $\|.\|_W$ is a norm on W. Indeed since $\bar{S}$ is bounded, $\|x\|_W = 0$ implies $x = 0$. Moreover if $x, y \in W$ are non zero, set

$$u = \frac{|x|}{\|x\|_W}, v = \frac{|y|}{\|y\|_W}, \alpha = \frac{\|x\|_W}{\|x\|_W + \|y\|_W}, \beta = \frac{\|y\|_W}{\|x\|_W + \|y\|_W}.$$

Since $\|u\|_W = \|v\| = 1, \alpha, \beta \geq 0$ and $\alpha + \beta = 1$ we have

$$\begin{aligned} \|x + y\|_W &\leq \| |x| + |y| \|_W \\ &\leq (\|x\|_W + \|y\|_W) \|\alpha u + \beta v\|_W \\ &\leq (\|x\|_W + \|y\|_W) \left\| (\alpha u^p + \beta v^p)^{\frac{1}{p}} \right\|_W \\ &\leq (\|x\|_W + \|y\|_W) \left(\alpha \|u\|_W^p + \beta \|v\|_W^p \right)^{\frac{1}{p}} \\ &\leq \|x\|_W + \|y\|_W . \end{aligned}$$

Therefore, $(W, \|.\|_W)$ is a p-convex normed lattice. We claim that W is complete, and hence a p-convex Banach lattice. Indeed, let $(w_i)_{i=1}^{\infty}$ be a Cauchy sequence in W. Since for every $z \in E$ it holds that

$$\|z\|_E \leq M^{(p)}(T) \|z\|_W.$$

it follows that $(w_i)_{i=1}^{\infty}$ is also a Cauchy sequence in E. Let $w \in E$ be its limit. Notice that since w_i are bounded in W, there exists some $\lambda < \infty$ such that $w_i \in \lambda \bar{S}$ for every $i = 1, 2, ...$ and since $\bar{S}$ is closed in E, we must have $w \in \lambda \bar{S}$. Thus, w belongs to W, and we will show that $(w_i)_{i=1}^{\infty}$converges to w also in W. To this end, let $\epsilon > 0$. Since $(w_i)_{i=1}^{\infty}$ is a Cauchy sequence, there exists N such that $w_i - w_j \in \epsilon \bar{S}$ when $i, j \geq N$. Thus, if $i \geq N$ we can write

$$w - w_i = (w - w_j) + (w_j - w_i),$$

for every $j \in \mathbb{N}$, and letting $j \longrightarrow \infty$ we obtain that $w - w_i \in \epsilon \bar{S}$. This shows that $w_i \longrightarrow w$ in W, and hence W is complete, as claimed.

Clearly, by the definition of W we have

$$\|Tx\|_W \leq \|x\|_X,$$

for every $x \in X$. Moreover, as noticed above it also holds that $\|z\|_E \leq M^{(p)}(T) \|z\|_W$ for each $z \in E$, therefore the formal inclusion $\varphi : W \hookrightarrow E$ is clearly an injective interval preserving lattice homomorphism, and we have the following diagram

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "R"'] & & E \\ & W \arrow[ur, hook, "\varphi"'] & \end{tikzcd}$$

where R is defined by $Rx = Tx$ for $x \in X$. This finishes the proof. $\square$

As with the Banach lattice constructed in Theorem 1.3, we will denote by $W_{T,p}$ the Banach lattice obtained in the proof of Theorem 1.4.

Remark 1.3. *[10] The operator $\varphi : W_{T,p} \longrightarrow E$ constructed in the proof is an injective, interval preserving lattice homomorphism. Moreover, it satisfies that the image of the closed unit ball $\varphi(W_{T,p})$ is a closed set in E. This let us introduce the class $\mathcal{C}$ consisting of operators $T : E \longrightarrow F$ between Banach* *lattices which are injective, interval preserving lattice homomorphisms, such that the image of the closed unit ball $T(B_E)$ is closed in F. The importance of this class will become clear next.*

Remark 1.4. *[10] Note that if $T : X \longrightarrow E$ is p-convex, then it is also p'-convex for every $1 \leq p' \leq p$. Hence, if we consider the factorization spaces $W_{T,p}$ and $W_{T,p'}$, it always holds that*

$$W_{T,p} \hookrightarrow W_{T,p'},$$

with norm smaller than or equal to one.

Indeed, this follows from the following two facts. First, the set

$$S = \left\{ y \in E : |y| \leq \left(\sum_{i=1}^{k} |Tx_i|^p \right)^{\frac{1}{p}}, \text{with} \sum_{i=1}^{k} \|x_i\| \leq 1 \right\},$$

can be equivalently described by

$$S = \left\{ y \in E : |y| \leq \left(\sum_{i=1}^{k} \alpha_i |Tw_i|^p \right)^{\frac{1}{p}}, \text{with} \|w_i\| \leq 1, \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1 \right\}.$$

Furthermore, for $1 \leq p' \leq p$, and $\alpha_i \geq 0$ with $\sum\limits_{i=1}^{k} \alpha_i = 1$ it always holds that

$$\left(\sum_{i=1}^{k} \alpha_i |Tw_i|^{p'} \right)^{\frac{1}{p'}} \leq \left(\sum_{i=1}^{k} \alpha_i |Tw_i|^p \right)^{\frac{1}{p}}.$$

Hence the unit ball of $W_{T,p'}$ is contained in that of $W_{T,p}$.

Remark 1.5. *[10] $W_{T,p}$ has p-convexity constant equal to one. If E is already p-convex and $T : E \longrightarrow E$ is the identity then $W_{T,p}$ is a renorming of E with p-convexity constant one.*

Recall the classical result proved in [6]. Given Banach spaces X, Y and a Banach lattice E, if $T_1 : X \longrightarrow E$ is p-convex and $T_2 : E \longrightarrow Y$ is p-concave, then T_2T_1 factors through $L_p(\mu)$ for certain measure . We remark that the factorization Theorems 1.3 and 1.4 allow us to reduce Krivine s theorem to the following purely lattice theoretical version.

Proposition 1.8. *[10] If W, V are quasi-Banach lattices with W p-convex and V p-concave, then every lattice homomorphism $h : W \longrightarrow V$ factors through some space $L_p(\mu)$, and the factors are lattice homomorphisms.*

The proof of following can be founded in [10].

Proposition 1.9. *Let $T : X \longrightarrow E$ be p-convex and $S : E \longrightarrow Y$ q-concave. For every $\theta \in (\theta, 1)$ we can factor ST through a Banach lattice U_θ which is p_θ-convex and q_θ-concave (with as usual $p_\theta = \dfrac{p}{p(1-\theta)+\theta}$ and $q_\theta = \dfrac{q_\theta}{1-\theta}$).*

Corollary 1.2. *If $T : E \longrightarrow E$ is p-convex and q-concave, then T^2 factors through a p_θ-convex and q_θ-concave Banach lattice. In particular, it factors through a super reflexive Banach lattice.*

The following factorization for operators which are both p-convex and q-concave.

Theorem 1.5 (Raynaud/Tradacete)**.** *Suppose that a linear operator $T : E \longrightarrow F$ between Banach lattices is p-convex and q-concave, with $1 < p \leq \infty$ and $1 \leq q < \infty$. Then for every $\theta \in (0, 1)$, $T = T_2 T_1$ where T_2 is $p_\theta = \dfrac{p}{\theta + (1-\theta)p}$-convex and T_1 is $q_\theta = \dfrac{q}{1-q}$-concave. In fact, there is a factorization*

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "\phi"'] & F \\ E_\theta \arrow[r, "R"'] & F_\theta \arrow[u, "\varphi"'] \end{tikzcd}$$

where ϕ and φ are positive linear maps, E_θ is q_θ-concave, F_θ is p_θ-convex and R is a bounded linear map.

Chapter 2

Lipschtiz spaces

In this chapter we study the Lipschtiz spaces, and we begin with a brief summary of important facts from functional analysis some with proof, some without. A good reference for the basic theory of the spaces of Lipschitz functions we consult the excellent book of Weaver [14].

2.1 Lipschitz function

Definition 2.1. *A map $f : (X, d_X) \longrightarrow (Y, d_Y)$ between two metric spaces is called Lipschitz if there is a positive constant C such that*

$$\forall x, y \in X, d_Y(f(x), f(y)) \leq C d_X(x, y). \tag{2.1}$$

If $C = 1$, the map is called nonexpansive (and contraction if $C < 1$.

For a Lipschitz map f, we define its Lipschitz constant by

$$\mathrm{Lip}(f) = \sup_{x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} = \inf\{C : C \ \textit{verifying} \ 2.1\}.$$

Let $(X, d_X, e_X), (Y, d_Y, e_Y)$ be pointed metric spaces. We say a map $f : (X, d_X, e_X) \longrightarrow (Y, d_Y, e_Y)$ preserves distinguished point if $f(e_X) = f(e_Y)$.

Proposition 2.1. *Let X, Y and Z be metric spaces and let $f : (X, d_X) \longrightarrow (Y, d_Y)$, $g : (Y, d_Y) \longrightarrow (Z, d_Z)$ be Lipschitz maps. Then $g \circ f(x) : (X, d_X) \longrightarrow (Z, d_Z)$ is Lipschitz and* $\mathrm{Lip}(g \circ f) \leq \mathrm{Lip}(g)\mathrm{Lip}(f)$.

Proof. For $x, y \in X$, we have

$$\begin{aligned} d_Z(g(f(x)), g(f(y))) &\leq \mathrm{Lip}(g) d_Y(f(x), f(y)) \\ &\leq \mathrm{Lip}(g)\mathrm{Lip}(f) d_X(x, y). \end{aligned}$$

And this shows the proposition. □

Proposition 2.2. *Let (X, d) be metric space. For Lipschitz functions $f, g : (X, d) \longrightarrow \mathbb{R}$ and scalar $\alpha \in \mathbb{R}$ the Lipschitz constant has the properties.*

1. $\mathrm{Lip}(f + g) \leq \mathrm{Lip}(f) + \mathrm{Lip}(g)$.

2. $\mathrm{Lip}(\alpha f) = \alpha \mathrm{Lip}(f)$.

Theorem 2.1. *Let X_0, Y_0 be metric spaces and let X, Y be their completions. Let$f_0 : X_0 \longrightarrow Y_0$ be Lipschitz. Then f has a unique Lipschitz extension $f : X \longrightarrow Y$such that* $\mathrm{Lip}(f) = \mathrm{Lip}(f_0)$.

Proof. Since Lipschitz functions are continuous and X_0 is dense in X, there is at most one Lipschitz extension. Consider x in $X \backslash X_0$ and put

$$f(x) = \lim f_0(x_n),$$

where x is a cauchy sequence in X_0 such that $x_n \longrightarrow x$. We have $\mathrm{Lip}(f) = \mathrm{Lip}(f_0)$. Indeed

$$\begin{aligned} d_Y(f(x), f(y)) &= d_Y(\lim f_0(x_n), \lim f_0(y_n)), \\ &\leq \mathrm{Lip}(f_0) d_X(x, y). \end{aligned}$$

This implies that $\mathrm{Lip}(f) \leq \mathrm{Lip}(f_0)$. For the converse, consider the following diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

and we have in the first part

$$\begin{aligned} \mathrm{Lip}(i_Y \circ f_0) &= \sup_{x \neq y} \frac{d_Y(i_Y \circ f_0(x), i_Y \circ f_0(y))}{d_X(x, y)}, \\ &= \sup_{x \neq y} \frac{d_Y(f_0(x), f_0(y))}{d_X(x, y)}, \\ &= \mathrm{Lip}(f_0) d_X(x, y). \end{aligned}$$

And in the second part

$$\mathrm{Lip}(i_Y \circ f_0) = \mathrm{Lip}(f \circ i_X) \leq \mathrm{Lip}(f).$$

This implies that $\mathrm{Lip}(f_0) \leq \mathrm{Lip}(f)$ and this completes the proof. □

Proposition 2.3. *Let X, Y be metric spaces and let f and $\{f_n\}_{n \in \mathbb{N}}$ be Lipschitz functions from X to Y. Suppose that $f_n \longrightarrow f$ pointwise. Then*

$$\mathrm{Lip}(f) \leq \sup_n \mathrm{Lip}(f_n).$$

Proof. Let x, y be in X. We have

$$d_Y(f(x), f(y)) = \lim_{n \longrightarrow \infty} d_Y(f_n(x), f_n(y)).$$

So

$$\begin{aligned}\sup_{x\neq y}\frac{d_Y(f(x),f(y))}{d_X(x,y)} &= \sup_{x\neq y}\lim_{n\longrightarrow\infty}\frac{d_Y(f_n(x),f_n(y))}{d_X(x,y)},\\ &\leq \sup_{x\neq y}\sup_{n\longrightarrow\infty}\frac{d_Y(f_n(x),f_n(y))}{d_X(x,y)}.\end{aligned}$$

By permitting the $\sup$, we obtain the result. □

Corollary 2.1. *If $\sum\limits_{n\geq 1} f_n$ converges pointwise then* $\mathrm{Lip}\left(\sum\limits_{n\geq 1} f_n\right)\leq \sum\limits_{n\geq 1}\mathrm{Lip}(f_n)$.

Proof. Let $g_n=\sum\limits_{i=1}^{n} f_i$ and $f=\sum\limits_{n\geq 1} f_n$ pointwise and $\mathrm{Lip}(g_n)=\sum\limits_{i=1}^{n}\mathrm{Lip}(f_i)$. So by Proposition 2.3 we have

$$\begin{aligned}\mathrm{Lip}(f) &\leq \sup\mathrm{Lip}\left(g_n\right),\\ &\leq \sum_{i=1}^{\infty}\mathrm{Lip}(f_i).\end{aligned}$$

And this ends the proof. □

Proposition 2.4. *Let $(X,d_X),(X_i,d_i)(i\in I)$ be metric spaces in $\mathcal{M}_0$. For each i in I, let $f_i: X\longrightarrow X_i$ be a Lipschitz map which preserves distinguished point. Suppose that $\sup\limits_{i\in I}\mathrm{Lip}(f_i)<\infty$. Then, the the product map $f: X\longrightarrow \prod^{\infty} X_i$ satisfies $\mathrm{Lip}(f)=\sup\limits_{i\in I}\mathrm{Lip}(f_i)$.*

Proof. Let x be in X. We prove that $(f_i(x))\in\prod^{\infty} X_i$. We have

$$\begin{aligned}\sup_{i\in I} d_{X_i}(f_i(x),e_i) &= \sup_{i\in I} d_{X_i}(f_i(x),f_i(e)),\\ (d=\sup_{i\in I} d_i) \qquad &\leq \sup_{i\in I}\mathrm{Lip}(f_i)d(x,e),\\ &< \infty.\end{aligned}$$

For x,y in X. We have by definition

$$\frac{d(f(x),f(y))}{d(x,y)}=\sup_{i\in I}\frac{d_i(f_i(x),f_i(y))}{d(x,y)},$$

and hence

$$\begin{aligned}\sup_{x\neq y}\frac{d(f(x),f(y))}{d(x,y)} &= \sup_{x\neq y}\sup_{i\in I}\frac{d_i(f_i(x),f_i(y))}{d(x,y)},\\ &= \sup_{i\in I}\sup_{x\neq y}\frac{d_i(f_i(x),f_i(y))}{d(x,y)},\\ &= \sup_{i\in I}\mathrm{Lip}(f_i).\end{aligned}$$

This implies that $\mathrm{Lip}(f) = \sup\limits_{i\in I} \mathrm{Lip}(f_i)$, and we obtain the result. □

Definition 2.2. *Let $(X, d_X), (Y, d_Y)$ be pointed metric space. We denote by $\mathrm{Lip}_0(X, Y)$ the set of all base point preserving Lipschitz maps from X to Y with the norm*

$$\mathrm{Lip}(f) = \sup_{x\neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.$$

If E is a Banach space, $\mathrm{Lip}_0(X, E)$ is a Banach space under the Lipschitz norm given by

$$\mathrm{Lip}(f) = \sup\left\{\frac{\|f(x) - f(y)\|}{d(x, y)} : x \neq y\right\}.$$

For $E = \mathbb{K}$, we designate $\mathrm{Lip}_0(X, \mathbb{K}) = \mathrm{Lip}_0(X) = X^{\#}$. The banach space $X^{\#}$ is called also Lipschitz dual of X.

We now see the classical theorem of Hahn-Banach.

Theorem 2.2. *Let E be a subset of a metric space (X, d) and let $f : E \longrightarrow \mathbb{R}$ be a Lipschitz function. Then f can be extended to a Lipschitz function $f_0 : X \longrightarrow \mathbb{R}$ with the same Lipschitz constant.*

Proof. Fix z in $X - E$. We must find a value for $f_0(z)$ such that for all x in E

$$|f_0(z) - f(x)| \leq Lip(f)d(x, z) \qquad \forall x \in X,$$

or equivalently

$$f(y) - \mathrm{Lip}(f)d(y, z) \leq f_0(z) \leq f(x) + \mathrm{Lip}(f)d(x, z) \qquad \forall y \in E.$$

Hence

$$\sup_{y\in E}(f(y) - \mathrm{Lip}(f)d(y, z)) \leq f_0(z) \leq \inf_{x\in E}(f(x) + \mathrm{Lip}(f)d(x, z)).$$

It is possible because for all x, y in E, we have

$$f(x) - f(y) \leq \mathrm{Lip}(f)d(x, y) \leq \mathrm{Lip}(f)(d(x, z) + d(y, z)).$$

We put

$$f_0(z) = \inf_{x\in E}(f(x) + \mathrm{Lip}(f)d(x, z)),$$

and Zorn's Lemma ends the proof.

Direct proof see (Metric Embeddings and Lipschitz Extensions, Lecture Notes, Assaf Naor 2015). Define the function $f_0 : X \longrightarrow \mathbb{R}$ by the formula

$$f_0(z) = \inf_{x\in E}(f(x) + \mathrm{Lip}(f)d(x,z)) \qquad z \in X.$$

To see that this function satisfies the results, fix an arbitrary $x_0 \in E$. Then, for any $x \in E$.

$$\begin{aligned} f(x_0) - f(x) &\leq \mathrm{Lip}(f)d(x_0,x), \\ &\leq \mathrm{Lip}(f)(d(x_0,z) + d(z,x)). \end{aligned}$$

This implies (that $f(x) + \mathrm{Lip}(f)d(x,z)$ is bounded below)

$$f(x_0) - Lip(f)d(x_0,z) \leq f(x) + Lip(f)d(x,z).$$

So $f_0(z)$ is well-defined. Also, if $z \in E$, the above shows that $f_0(z) = f(z)$. Finally (by definition of the inf), for $z, y \in X$ and $\epsilon > 0$, choose $x_z \in E$ such that

$$f_0(z) \geq f(x_z) + \mathrm{Lip}(f)d(z,x_z) - \epsilon,$$

$$-f_0(z) \leq -f(x_z) - Lip(f)d(z,x_z) + \epsilon.$$

Then

$$\begin{aligned} f_0(y) - f_0(z) &\leq f(x_z) + \mathrm{Lip}(f)d(y,x_z) - f(x_z) - Lip(f)d(z,x_z) + \epsilon, \\ &\leq \mathrm{Lip}(f)d(y,z) + \epsilon. \end{aligned}$$

Thus, we see that f_0 is indeed $\mathrm{Lip}(f)$-Lipschitz. □

2.2 Molecules and the Arens Eells space

Now we are going to present some concepts about the space of molecules, the reader can see [14] for more details.

Definition 2.3. *Let (X, d, e) be pointed a metric space. A molecule on X is a real valued function m on X with finite support (i.e., the set where m has non-zero values) and satisfies*

$$\sum_{x \in supp(m)} m(x) = 0.$$

Denote by $\mathcal{M}(X)$ the real linear space of molecules on X. Define the basic molecule $m_{xy} = \mathbb{1}_{\{x\}} - \mathbb{1}_{\{y\}}$ (with $x, y \in X$ are called atoms).

Put now

$$\|m\|_{\mathcal{M}(X)} = \inf \left\{ \sum_{j=1}^{l} |\lambda_j| \, d(x_j, x'_j) : m = \sum_{j=1}^{l} \lambda_j m_{x_j x'_j} \right\},$$

where the infimum is taken over all representations of the molecule m. Denote by $\text{Æ}(X)$ the completion of the normed space $(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)})$. This space was first introduced by Arens and Eells [1] in 1956. The terminology Arens-Eells space $\text{Æ}(X)$ is due to Weaver [14]. A different notation was used in [5] by Godefroy and Kalton. It is the Lipschitz-free space denoted by $\mathcal{F}(X)$.

Remark 2.1. *Every molecule m is uniquely expressible in the form*

$$m = \sum_{j=1}^{l} \lambda_j (\mathbb{1}_{\{x_j\}} - \mathbb{1}_{\{e\}}),$$

where the points x_j are all distinct and none equals to e.

This theorem is very important in this chapter, to satisfy the facilitde study Lipschitz dual of the pointed metric space X.

Theorem 2.3. *Let (X, d, e) be a pointed metric space. Then $\text{Æ}^*(X, d)$ is isometrically isomorphic to* $\mathrm{Lip}_0(X)$.

Proof. Define

$$S : \text{Æ}^*(X, d) \longrightarrow \mathrm{Lip}_0(X),$$

by

$$(S\varphi)(x) = \varphi((\mathbb{1}_{\{x\}} - \mathbb{1}_{\{e\}})).$$

Since $\left\| \mathbb{1}_{\{x\}} - \mathbb{1}_{\{x'\}} \right\| = d(x, x')$ for all $x, x' \in X$, we have

$$\begin{aligned} |(S\varphi)(x) - (S\varphi)(x')| &= \left| \varphi((\mathbb{1}_{\{x\}} - \mathbb{1}_{\{e\}})) - \varphi((\mathbb{1}_{\{x'\}} - \mathbb{1}_{\{e\}})) \right|, \\ &= \left| \varphi((\mathbb{1}_{\{x\}} - \mathbb{1}_{\{x'\}})) \right|, \\ &\leq \|\varphi\| \, d(x, x'). \end{aligned}$$

Also $(S\varphi)(e) = \varphi(0)$, so indeed $S\varphi \in \mathrm{Lip}_0(X)$ It follows that S is a nonexpansive linear mapping from Æ(X, d) to $\mathrm{Lip}_0(X)$.

Define now

$$R : \mathrm{Lip}_0(X) \longrightarrow Æ^*(X, d),$$

by

$$(Rf)(m) = \sum_x m(x)f(x),$$

for $f \in \mathrm{Lip}_0(X)$ and m a molecule. if $m = \sum_{j=1}^{l} \lambda_j d(\mathbb{1}_{\{x_j\}} - \mathbb{1}_{\{x'_j\}})$, we have

$$\begin{aligned} |(Rf)(m)| &= \left|\sum_x m(x)f(x)\right|, \\ &\leq \sum_{j=1}^{l} |\lambda_j| \left|f(x_j) - f(x'_j)\right|, \\ &\leq \mathrm{Lip}(f) \sum_{j=1}^{l} |\lambda_j| \, d(\mathbb{1}_{\{x_j\}} - \mathbb{1}_{\{x'_j\}}). \end{aligned}$$

Hence $|(Rf)(m)| \leq \mathrm{Lip}(f) \, \|m\|_{\mathcal{M}(X)}$, which uniquely extends to a continuous linear functional on the completion Æ(X, d) of $\mathcal{M}(X)$, denoted by the same symbol Rf. Thus $Rf \in$ Æ(X, d) and $\|Rf\| \leq Lip(f)$. Straightforward calculations show that R and S are inverses. Indeed, for all $x \in X$

$$\begin{aligned} (S \circ R)(f)(x) &= S(R(f))(x), \\ &= R(f)(\mathbb{1}_{\{x\}} - \mathbb{1}_{\{e\}}), \\ &= f(x). \end{aligned}$$

And for all $m \in M(x)$

$$\begin{aligned}(R \circ S)(\varphi)(m) &= R(S(\varphi))(m), \\ &= \sum_{x} m(x) S(\varphi)(x), \\ &= \sum_{j=1}^{l} \lambda_j (S(\varphi)(x_j) - S(\varphi)(x_j')), \\ &= \sum_{j=1}^{l} \lambda_j \varphi(\mathbb{1}_{\{x\}} - \mathbb{1}_{\{x_j'\}}), \\ &= \varphi(m).\end{aligned}$$

The operators R, S are non expansive and $R \circ S = S \circ R = Id$ so S is isometric ($\|x\| = \|R \circ S(x)\| \leq \|R\| \, \|S(x)\|) \leq \|S(x)\|$) and hence $\mathrm{Lip}_0(X)$ is isometrically isomorphic to $Æ^*(X, d)$. $\square$

Proposition 2.5. *Let (X, d, e) be a pointed metric space*

1. *For any molecule m we have $\|m\|_{Æ(X)} = \sup\{|< m, f >| = \left|\sum_{x \in X} m(x) f(x)\right|, f \in B_{X^{\#}}\}$ and there exists $f \in B_{X^{\#}}$ such that $< m, f >= \|m\|_{Æ(X)}$.*
2. *$\|.\|_{Æ(X)}$ is a norm on $\mathcal{M}(X)$ and $\left\|\mathbb{1}_{\{x\}} - \mathbb{1}_{\{y\}}\right\|_{Æ(X)} = d(x, y)$ for all x, y in X.*
3. *$\|.\|_{Æ(X)}$ is the largest seminorm on $\mathcal{M}(X)$ which satfies forall x, y in X, $\left\|\mathbb{1}_{\{x\}} - \mathbb{1}_{\{y\}}\right\|_{Æ} = d(x, y)$.*

Proof. 1. This follows from the identification of $\mathrm{Lip}_0(X, d)$ with $Æ(X, d)$ and the Hahn-Banach theorem.

2. The inequality $\left\|\mathbb{1}_{\{x\}} - \mathbb{1}_{\{y\}}\right\|_{Æ} \leq d(x, y)$ follows from the definition. Conversly, fix x in X and define

$$f_x(y) = d(x, y) - d(x, e).$$

We have $f_x \in B_{\mathrm{Lip}_0(X,d)}$ because $f_x(e) = 0$ and $\mathrm{Lip}(f_x) = 1$. Indeed,

$$\begin{aligned} \mathrm{Lip(f_x)} &= \sup_{y_1 \neq y_2} \frac{|f_x(y_1) - f_x(y_2)|}{d(y_1, y_2)}, \\ &\geq \sup_{y \neq x} \frac{|f_x(y) - f_x(x)|}{d(x, y)}, \\ &\geq \frac{d(x, y)}{d(x, y)} = 1. \end{aligned}$$

And

$$\begin{aligned} \mathrm{Lip(f_x)} &= \sup_{y_1 \neq y_2} \frac{|f_x(y_1) - f_x(y_2)|}{d(y_1, y_2)}, \\ &\leq \sup_{y_1 \neq y_2} \frac{|d(x, y_1) - d(x, y_2)|}{d(y_1, y_2)}, \\ &\leq \frac{d(y_1, y_2)}{d(y_1, y_2)} = 1. \end{aligned}$$

By part (1), we have

$$\begin{aligned} \left\| \mathbb{1}_{\{x\}} - \mathbb{1}_{\{y\}} \right\|_{Æ(X)} &\geq |< m_{xy}, f_x >|, \\ &\geq |m(x) f_x(x) - m(y) f_x(y)|, \\ &\geq |m(y) d(x, y)|, \\ &\geq d(x, y). \end{aligned}$$

3. Let $\|.\|_0$ be any semi norm such that

$$\left\| \mathbb{1}_{\{x\}} - \mathbb{1}_{\{y\}} \right\|_0 \leq d(x, y).$$

for all $x, y \in X$. Let $m = \sum\limits_{i}^{n} \lambda_i m_{x_i y_i}$ be a molecule. We have

$$\begin{aligned} \|m\|_0 &= \left\| \sum_{i}^{n} \lambda_i m_{x_i y_i} \right\|_0, \\ &\leq \sum_{i}^{n} |\lambda_i| \, \|m_{x_i y_i}\|_0, \\ &\leq \sum_{i}^{n} |\lambda_i| \, d(x_i, y_i). \end{aligned}$$

Taking the infimum of all such representation of m yields $\|m\|_0 \leq \|m\|_{Æ}$. $\square$

Corollary 2.2. *The application $i_X : X \longrightarrow \text{Æ}(X)$ defined by*

$$i_X(x) = \mathbb{1}_{\{x\}} - \mathbb{1}_{\{e\}},$$

is an isometric embedding of X into Æ(X).

Proof. We have by (2) in the precedente proposition

$$\begin{aligned} \|i(x) - i(y)\|_{\text{Æ}(X)} &= \left\|\mathbb{1}_{\{x\}} - \mathbb{1}_{\{y\}}\right\|_{\text{Æ}(X)} \\ &= d(x, y). \end{aligned}$$

For all $x, y \in X$. So i_X is an isometry. □

2.3 Universal property of Arens Eells space

The following theorem due to Weaver is very important. It is known as the linearization of Lipschitz operators.

Theorem 2.4. *[14] Let (X, e, d) be a pointed metric space. Let E be a Banach space and let $T : X \longrightarrow E$ be a Lipschitz map which preserves base point $(ie., T(e) = 0)$. Then there is a unique bounded linear operator $u : \text{Æ}(X) \longrightarrow E$, such that $T = u \circ i$ and $\|u\| = \mathrm{Lip}(T)$.*

$$\begin{tikzcd} \text{Æ}(X) \arrow[dr, "U"] & \\ X \arrow[u, "i_X"] \arrow[r, "T"'] & E \end{tikzcd}$$

Proof. Every molecule m is uniquely expressible in the form

$$m = \sum_{j=1}^{l} \lambda_j (\mathbb{1}_{x_{\{j\}}} - \mathbb{1}_{\{e\}}).$$

where the points x_j are all distinct and none equals to e, We then defined u by

$$u(m) = \sum_{j=1}^{l} \lambda_j T(x_j),$$

we have

$$\begin{aligned} \mathrm{Lip}(T) = \mathrm{Lip}(u \circ i) &\leq \mathrm{Lip}(u)\mathrm{Lip}(i), \\ &\leq \|u\|. \end{aligned}$$

So

$$\mathrm{Lip}(T) \leq \|u\|.$$

For the converse inequality, on define a semi norm on the space of molecules by setting $\|.\|_0$ on $\mathcal{M}(X)$ by

$$\|m\|_0 = \frac{\|u(m)\|}{\mathrm{Lip}(T)}.$$

Then

$$\begin{aligned} \left\|\mathbb{1}_{\{x\}} - \mathbb{1}_{\{y\}}\right\|_0 &= \frac{\|T(x) - T(y)\|}{\mathrm{Lip}(T)}, \\ &\leq d(x, y). \end{aligned}$$

For all $x, y \in X$. This implies that $\|.\|_0 \leq \|.\|_{Æ(X)}$. Thus $\|u(m)\| \leq \mathrm{Lip}(T)\, \|m\|_{Æ(X)}$ which shows that $\|u\| \leq \mathrm{Lip}(T)$ as desired. □

Notation : The operator u is denoted by T_L.

Sawashima [12] defined the Lipschitz adjoint (or dual) of T as a continuous linear operator.

Definition 2.4. *Consider X, Y in $\mathcal{M}_0$ and let $T : X \longrightarrow Y$ be a Lipschitz map which preserves base point. We define*

$$T : Y^{\#} \longrightarrow X^{\#},$$

by

$$T^{\#}(g) =< g, T(x) >= g \circ T(x).$$

The definition make sens by the property of composition maps.

Proposition 2.6. *Consider X, Y in $\mathcal{M}_0$ and let $T : X \longrightarrow Y$ be a Lipschitz map which preserves base point. Then $T^{\#}$is a bounded linear map and $\left\|T^{\#}\right\| = \mathrm{Lip}(T)$. The map $T^{\#}$ is compatible with products and preserves order.*

Proof. We have

$$\mathrm{Lip}(T^{\#}(g)) = \mathrm{Lip}(g \circ T) \leq \mathrm{Lip}(g)\mathrm{Lip}(T),$$

so $\left\|T^{\#}\right\| \leq \operatorname{Lip}(T)$ For the converse inequality. Fix $p, q \in Y$ soit $g = d(., q) - d(e_Y - q)$, then $\operatorname{Lip}(g) = 1$ and

$$\begin{aligned}
\left\|T^{\#}\right\| &\geq \operatorname{Lip}(T^{\#})(g), \\
&\geq \frac{\left|T^{\#}(g)(x) - T^{\#}(g)(y)\right|}{d_X(x,y)}, \\
&\geq \frac{|g(T(x) - g(T(y))|}{d_X(x,y)}, \\
&\geq \frac{|g(T(x)) - g(T(y))d_Y(T(x),T(y))|}{d_Y(T(x),T(y))d_X(x,y)}.
\end{aligned}$$

Taking the supremum over x and y, we find $\left\|T^{\#}\right\| \geq \operatorname{Lip}(T)$. □

Chapter 3

Lipschitz p-convex and q-concave maps

In this chapter we define Lipschitz p-convex and q-concave maps, and we also offer prove of some results, concerning Lipschitz versions of the Maurey/Nikishin and Krivine factorization theorems [2].

3.1 Lipschitz p-convex maps

We give the notion of p-convex operator which is inspired from the linear case.

Definition 3.1 (Lipschitz p-convex maps)**.** *Let $1 \leq p \leq \infty$. Let X be a metric space and E a Banach lattice. A Lipschitz map $T : X \longrightarrow E$ is called Lipschitz p-convex if there exists a constant $C \geq 0$ such that for any $x_j, x'_j \in X$ and $\lambda_j \geq 0$*

$$\left\| \left(\sum_{j=1}^{n} \lambda_j \left| Tx_j - Tx'_j \right|^p \right)^{\frac{1}{p}} \right\|_E \leq C \left(\sum_{j=1}^{n} \lambda_j d(x_j - x'_j)^p \right)^{\frac{1}{p}},$$

(with the obvious adjustment if $p = \infty$). The smallest such constant C is called the Lipschitz p-convexity constant of T and is denoted by $M^{(p)}_{\mathrm{Lip}}(T)$.

Theorem 3.1. *[2] Let X be a metric space and E a Banach lattice. A Lipschitz map $T : X \longrightarrow E$ is Lipschitz p-convex if, and only if, $T_L : \AE(X) \longrightarrow E$ is p-convex. Moreover, in this case the p-convexity constants are the same.*

Proof. The "if" part is trivial: p-convexity of T_L clearly implies Lipschitz p-convexity of T with no increment in the constant, since $\|m_{xx'}\|_{\AE(X)} = d(x, x')$ and $T_L m_{xx'} = Tx - Tx'$.

Now suppose that T is Lipschitz p-convex. The strategy of the proof will be to show that $T^*_L : E^* \longrightarrow \AE(X)^* = \mathrm{Lip}_0(X)$ is p'-concave. Let $\varphi^*_j \in E^*$ be arbitrary. For any $x_j, x'_j \in X$ with $x_j \neq x'_j$ we have

$$\left(\sum_j \left| \frac{< \varphi^*_j, Tx_j - Tx'_j >}{d(x_j, x'_j)} \right|^{p'} \right)^{\frac{1}{p'}} = \sup_{\sum_j |\alpha_j|^p \leq 1} \sum_j \alpha_j \frac{< \varphi^*_j, Tx_j - Tx'_j >}{d(x_j, x'_j)}.$$

Using Pro 1.2, the latter is bounded by

$$\sup_{\sum_j |\alpha_j|^p \leq 1} \left(\left(\sum_j \left| \varphi^*_j \right|^{p'} \right)^{\frac{1}{p'}} \right) \left(\left(\sum_j |\alpha_j|^p \frac{\left| Tx_j - Tx'_j \right|^p}{d(x_j, x'_j)^p} \right)^{\frac{1}{p}} \right) \leq$$
$$\left\| \left(\sum_j \left| \varphi^*_j \right|^{p'} \right)^{\frac{1}{p'}} \right\|_{E^*} \sup_{\sum_j |\alpha_j|^p \leq 1} \left\| \left(\sum_j |\alpha_j|^p \frac{\left| Tx_j - Tx'_j \right|^p}{d(x_j, x'_j)^p} \right)^{\frac{1}{p}} \right\|_E .$$

The Lipschitz p-convexity of T allows us to bound this by

$$\left\| \left(\sum_j \left|\varphi_j^*\right|^{p'} \right)^{\frac{1}{p'}} \right\|_{E^*} M_{\mathrm{Lip}}^{(p)}(T) \sup_{\sum_j |\alpha_j|^p \leq 1} \left(\sum_j |\alpha_j|^p \frac{d(x_j, x_j')^p}{d(x_j, x_j')^p} \right)^{\frac{1}{p}} = M_{\mathrm{Lip}}^{(p)}(T) \left\| \left(\sum_j \left|\varphi_j^*\right|^{p'} \right)^{\frac{1}{p'}} \right\|_{E^*}.$$

Therefore,

$$\left(\sum_j \left| \frac{(T_L^*\varphi_j^*)(x_j) - (T_L^*\varphi_j^*)(x_j')}{d(x_j, x_j')} \right|^{p'} \right)^{\frac{1}{p'}} \leq M_{\mathrm{Lip}}^{(p)}(T) \left\| \left(\sum_j \left|\varphi_j^*\right|^{p'} \right)^{\frac{1}{p'}} \right\|_{E^*},$$

so taking the supremum over all pairs $x_j, x_j' \in X$ with $x_j \neq x_j'$ we conclude

$$\left(\sum_j \left\|T_L^*\varphi_j^*\right\|^{p'} \right)^{\frac{1}{p'}} \leq M_{\mathrm{Lip}}^{(p)}(T) \left\| \left(\sum_j \left|\varphi_j^*\right|^{p'} \right)^{\frac{1}{p'}} \right\|_{E^*}.$$

Since the $\varphi_j^* \in E^*$ were arbitrary, this means that $T_L^* : E^* \longrightarrow \mathrm{Lip}_0(X)$ is p'-concave with $M_{(p')}(T_L^*) \leq M_{Lip(T)}^{(p)}$, and by duality $T_L : Æ(X) \longrightarrow E$ is p-convex with $M^{(p)}(T_L) \leq M_{Lip}^{(p)}(T)$. □

Remark 3.1. *[2] For a moment, one could think that in particular we have a result in the spirit of the Godefroy/Kalton theorem for the BAP, that is, for a Banach lattice E*

$$E \text{ is } p\text{-convex} \Longleftrightarrow Æ\,(E) \text{ is } p\text{-convex.}$$

However, what we do have is

$$id_E : E \longrightarrow E \text{ is } p\text{-convex} \Longleftrightarrow id_{EL} : Æ(E) \longrightarrow E \text{ is } p\text{-convex,}$$

and id_{EL} is the linearization of id_E.

3.2 Lipschitz q-concave maps

Like Lipschitz p-convex operator, we give the Lipschitz q-concave operator.

Definition 3.2 (Lipschitz q-concave maps)**.** *Let X be a metric space and E a Banach lattice. A Lipschitz map $T : E \longrightarrow X$ is called Lipschitz q-concave if there exists a constant $C \geq 0$ such that for any $v_j, v_j' \in E$ and any $\lambda_j \geq 0$.*

$$\left(\sum_{j=1}^n \lambda_j d(Tv_j, Tv_j')^q \right)^{\frac{1}{q}} \leq C \left\| \left(\sum_{j=1}^n \lambda_j \left|v_j - v_j'\right|^q \right)^{\frac{1}{q}} \right\|_E.$$

The smallest such constant C is the Lipschitz p-concavity constant of T and is denoted by $M_{(q)}^{\mathrm{Lip}}(T)$.

3.3 Factorization through L_p

The following factorization theorem and its proof are inspired by the corresponding result for linear maps proved by Krivine. The presentation follows from that of Theorem 1.2.

Theorem 3.2. *[2] Let X, Y be metric spaces with Y complete and E a Banach lattice. Suppose that $T : X \longrightarrow E$ Lipschitz p-convex and $S : E \longrightarrow Y$ is Lipschitz p-concave. Then the operator ST can be factorized through an $L_P(\mu)$ space. Moreover, we may arrange to have $ST = S_1T_1$ with $T_1 : X \longrightarrow L_p(\mu)$, $S_1 : L_p(\mu) \longrightarrow Y$, $\mathrm{Lip}(T_1) \leq M^{(p)}_{\mathrm{Lip}}(T)$ and $\mathrm{Lip}(S_1) \leq M^{\mathrm{Lip}}_{(p)}(S)$.*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[dr, "T_1"'] & E \arrow[r, "S"] & Y \\ & L_p(\mu) \arrow[ur, "S_1"'] & \end{tikzcd}$$

3.4 Factorization of Lipschitz p-convex maps

We start by proving the following characterization of Lipschitz p-convex maps, which is a generalization of Theorem 1.4.

Theorem 3.3. *[2] Let E be a Banach lattice, X a metric space and $1 \leq p \leq \infty$. A Lipschitz map $T : X \longrightarrow E$ is Lipschitz p-convex if, and only if, there exist a p-convex Banach lattice W, a positive operator (in fact, an injective interval preserving lattice homomorphism) $\psi : W \longrightarrow E$ and another Lipschitz operator $R : X \longrightarrow W$ such that $T = \psi \circ R$*

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "R"'] & & E \\ & W \arrow[ur, "\psi"'] & \end{tikzcd}$$

Moreover $M^{(p)}_{\mathrm{Lip}}(T) = \inf \mathrm{Lip}(R).M^p(I_W).\,\|\psi\|$ where the infimum is taken over all such factorizations.

Proof. We will assume that $1 \leq p < \infty$, the proof of the case $p = \infty$ can be obtained via the usual changes.

If we do have such a factorization, consider $x_j, x'_j \in X$. By the positivity of ψ, using Proposition

1.7 and the q-convexity of W,

$$\begin{aligned}
\left\| \left(\sum_{j=1}^{k} \left| Tx_j - Tx_j' \right|^p \right)^{\frac{1}{p}} \right\|_E &= \left\| \left(\sum_{j=1}^{k} \left| \psi(Rx_j - Rx_j') \right|^p \right)^{\frac{1}{p}} \right\|_E \\
&\leq \|\psi\| \left\| \left(\sum_{j=1}^{k} \left| \psi(Rx_j - Rx_j') \right|^p \right)^{\frac{1}{p}} \right\|_W \\
&\leq \|\psi\| \, M^{(p)}(I_W) \left(\sum_{j=1}^{k} \left\| (Rx_j - Rx_j') \right\|_W^p \right)^{\frac{1}{p}} \\
&\leq \|\psi\| \, M^{(p)}(I_W) \mathrm{Lip}(R) \left(\sum_{j=1}^{k} d(x_j - x_j')^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Showing that T is Lipschitz p-convex with $M_{\mathrm{Lip}}^{(p)}(T) \leq \|\psi\| \, M^{(p)}(I_W)\mathrm{Lip}(R)$.

Now let $T : X \longrightarrow E$ be Lipschitz p-convex. Consider the set

$$A := \left\{ u \in E : |u| \leq \left(\sum_{j=1}^{k} \lambda_j \left| Tx_j - Tx_j' \right|^p \right)^{\frac{1}{p}} \text{ where } \lambda_j \geq 0, \text{ and } \sum_{j=1}^{k} \lambda_j d(x_j, x_j')^p \leq 1 \right\}.$$

We can consider the Minkowski functional defined by its closure $\bar{A}$ in E,

$$\|z\|_W = \inf \left\{ \mu > 0 : \mu \bar{A} \right\}.$$

Clearly $\bar{A}$ is solid, and since T is Lipschitz p-convex, it is also a bounded subset of E. Let us consider the space $W = \{z \in E : \|z\|_W < \infty\}$. We claim that for any $z_1, ..., z_n$ in W it follows that $(\sum_{i=1}^{n} |z_i|^p)^{\frac{1}{p}}$ is in W and moreover

$$\left\| \left(\sum_{i=1}^{n} |z_i|^p \right)^{\frac{1}{p}} \right\|_W \leq \left(\sum_{i=1}^{n} \|z_i\|_W^p \right)^{\frac{1}{p}},$$

Given $\epsilon > 0$, for each $i = 1, ..., n$ there exist μ_i with $z_i \in \mu_i \bar{A}$ such that $\mu_i^p \leq \|z_i\|_W + \frac{\epsilon^p}{n}$. Thus, for every $\delta > 0$ there exist $\left\{ z_j^\delta \right\}_{i=1}^{n}$ in E such that $\left\| z_i - z_i^\delta \right\| \leq \delta$ and

$$\left| z_i^\delta \right| \leq \left(\sum_{j=1}^{m_{i,\delta}} \lambda_j^{i,\delta} \left| Tx_j^{i,\delta} - Ty_j^{i,\delta} \right|^p \right)^{\frac{1}{p}}.$$

Where the nonnegative numbers $\left\{ \lambda_j^{i,\delta} \right\}_{j=1}^{m_{i,\delta}}$ and the points $\left\{ x_j^{i,\delta}, y_j^{i,\delta} \right\}_{j=1}^{m_{i,\delta}}$ in X satisfy

$$\left(\sum_{j=1}^{m_{i,\delta}} \lambda_j^{i,\delta} d(x_j^{i,\delta}, y_j^{i,\delta})^p \right)^{\frac{1}{p}} \leq \mu_i,$$

for each $i = 1, ..., n$ and each $\delta > 0$. For each $\delta > 0$, define

$$w_\delta = \left(\sum_{i=1}^{n} \left|z_i^\delta\right|^p\right)^{\frac{1}{p}}.$$

Then

$$\left\| \left(\sum_{i=1}^{n} |z_i|^p\right)^{\frac{1}{p}} - w_\delta \right\|_E \leq \left\| \left(\sum_{i=1}^{n} \left|z_i - z_i^\delta\right|^p\right)^{\frac{1}{p}} \right\|_E \leq \left(\sum_{i=1}^{n} \left\|z_i - z_i^\delta\right\|_E^p\right)^{\frac{1}{p}} \leq n\delta.$$

Moreover, we will show that for every $\delta > 0$, w_δ belongs to $(\sum_{i=1}^{n} \mu_i^p)^{\frac{1}{p}}$ A.

Indeed,

$$|w_\delta| = \left(\sum_{i=1}^{n} \left|z_i^\delta\right|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} \sum_{j=1}^{m_{i,\delta}} \lambda_j^{i,\delta} \left|Tx_j^{i,\delta} - Ty_j^{i,\delta}\right|^p\right)^{\frac{1}{p}},$$

and

$$\left(\sum_{i=1}^{n} \sum_{j=1}^{m_{i,\delta}} \lambda_j^{i,\delta} d(x_j^{i,\delta}, y_j^{i,\delta})^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} \mu_i^p\right)^{\frac{1}{p}}.$$

Hence , $(\sum_{i=1}^{n} |z_i|^p)^{\frac{1}{p}}$ belongs to $\left(\sum_{i=1}^{n} \mu_i^p\right)^{\frac{1}{p}} \bar{A}$ Therefore, from the definition of $\|.\|_W$ and the choice of μ_i,

$$\begin{aligned} \left\| \left(\sum_{i=1}^{n} |z_i|^p\right)^{\frac{1}{p}} \right\|_W &\leq \left(\sum_{i=1}^{n} \mu_i^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{i=1}^{n} \|z_i\|_W^p + \frac{\epsilon^p}{n}\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} \|z_i\|_W^p\right)^{\frac{1}{p}} + \epsilon. \end{aligned}$$

Letting ϵ go to 0, we conclude

$$\left\| \left(\sum_{i=1}^{n} |z_i|^p\right)^{\frac{1}{p}} \right\|_W \leq \left(\sum_{i=1}^{n} \|z_i\|_W^p\right)^{\frac{1}{p}}.$$

It follows that the Minkowski functional $\|.\|_W$ is in fact a norm on W. Since $\bar{A}$ is bounded, $\|z\|_W = 0$ implies that $z = 0$. Moreover if $u, v \in W$ are not zero, set

$$\tilde{u} = \frac{|u|}{\|u\|_W}, \tilde{v} = \frac{|v|}{\|v\|_W}, \alpha = \frac{\|u\|_W}{\|u\|_W + \|v\|_W}, \beta = \frac{\|v\|_W}{\|u\|_W + \|v\|_W}.$$

Since $\|\tilde{u}\|_W = \|\tilde{v}\| = 1, \alpha, \beta \geq 0$ and $\alpha + \beta = 1$ we have

$$\begin{aligned} \|u+v\|_W &\leq \||u|+|v|\|_W = (\|u\|_W + \|v\|_W)\,\|\alpha\tilde{u} + \beta\tilde{v}\|_W \\ &\leq (\|u\|_W + \|v\|_W)\left\|(\alpha\tilde{u}^p + \beta\tilde{v}^p)^{\frac{1}{p}}\right\|_W \\ &\leq (\|u\|_W + \|v\|_W)\,(\alpha\,\|\tilde{u}\|_W^p + \beta\,\|\tilde{v}\|_W^p)^{\frac{1}{p}} \\ &\leq \|u\|_W + \|v\|_W\,. \end{aligned}$$

Thus, $(W, \|.\|_W)$ is a p-convex normed lattice with constant 1. Let $(w_i)_{i=1}^{\infty}$ be a Cauchy sequence in W. Since for every $z \in E$ it holds that $\|z\|_E \leq M_{\mathrm{Lip}}^{(p)}(T)\,\|z\|_W$, it follows that $(w_i)_{i=1}^{\infty}$ is also a Cauchy sequence in E and this has a limit w in E. Notice that since the w_i are bounded in W, there exists some finite μ such that $w_i \in \mu\bar{S}$ for every $i \in \mathbb{N}$ and since $\bar{S}$ is closed in E, we must have $w \in \mu\bar{S}$. Thus, w belongs to W, and we will show that $(w_i)_{i=1}^{\infty}$ converges to w also in W. To this end, let $\epsilon > 0$. Since $(w_i)_{i=1}^{\infty}$ is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that $w_i - w_j \in \epsilon\bar{S}$ whenever $i, j \geq N$. Thus, if $i \geq N$ we can write

$$w - w_i = (w - w_j) + (w_j - w_i),$$

for every $j \geq N$, and letting $j \longrightarrow \infty$ we obtain that $w - w_i \in \epsilon\bar{S}$. This shows that $(w_i)_{i=1}^{\infty}$ converges to w in W, hence W is complete and therefore a Banach lattice.

Let us observe that from the definition of W we clearly have $\|Tx - Tx'\|_W \leq d(x, x')$ for every $x, x' \in X$, so the map $R : X \longrightarrow W$ given by $Rx = Tx$ is Lipschitz with $\mathrm{Lip}(R) \leq 1$. Moreover, as noticed above it also holds that $\|z\|_E \leq M_{\mathrm{Lip}}^{(p)}(T)\,\|z\|_W$ for each $z \in E$, hence the formal inclusion $\psi : W \longrightarrow E$ is clearly and injective interval preserving lattice homomorphism with norm at most $M(p)_{\mathrm{Lip}}^{(p)}(T)$, and $T = \psi R$. □

3.5 Factorization of Lipschitz q-concave maps

The following characterization of Lipschitz q-concave maps is a nonlinear generalization of Theorem 1.3.

Theorem 3.4. *[2] Let E be a Banach lattice, X a complete metric space and $1 \leq p \leq \infty$. A Lipschitz map $T : E \longrightarrow X$ is Lipschitz q-concave if, and only if, there exist a q-concave Banach lattice V, a* *positive operator $\phi : E \longrightarrow V$(fact, a lattice homomorphism with dense image), and another Lipschitz map $S : V \longrightarrow X$ such that $T = S \circ \phi$.*

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "\phi"'] & & X \\ & V \arrow[ur, "S"'] & \end{tikzcd}$$

Moreover $M_{(q)}^{\mathrm{Lip}}(T) = \inf \|\phi\| .M_{(q)}(I_V).\mathrm{Lip}(S)$.

Proof. The case $q = \infty$ is trivial because every Banach lattice is ∞-concave, so let us assume that $1 \le q < \infty$. The specific construction carried out below, however, has an analogue in the case $q = \infty$. First, let us assume that such a factorization exists. Consider $u_j, u'_j \in E$. By the positivity of ϕ, using Proposition 1.7 and the q-concavity of V,

$$\begin{aligned}
\left(\sum_{j=1}^{n} d(Tu_j, Tu'_j)^q\right)^{\frac{1}{q}} &\le \left(\sum_{j=1}^{n} d(S\phi u_j, S\phi u'_j)^q\right)^{\frac{1}{q}} \\
&\le \mathrm{Lip}(S) \left(\sum_{j=1}^{n} \left\|\phi u_j - \phi u'_j\right\|^q\right)^{\frac{1}{q}} \\
&\le \mathrm{Lip}(S).M_{(q)}(I_V). \left\|\left(\sum_{j=1}^{n} \left|\phi(v_j - v'_j)\right|^q\right)^{\frac{1}{q}}\right\|_V \\
&\le \mathrm{Lip}(S).M_{(q)}(I_V). \|\phi\| \left\|\left(\sum_{j=1}^{n} \left|v_j - v'_j\right|^q\right)^{\frac{1}{q}}\right\|_E .
\end{aligned}$$

Showing that T is Lipschitz q-concave with $M_{(q)}^{\mathrm{Lip}}(T) \le \mathrm{Lip}(S).M_{(q)}(I_V). \|\phi\|$. Now suppose that T is Lipschitz q-concave. Given $u \in E$, define

$$\rho(u) = \sup\left\{ \left(\sum_{j=1}^{n} \lambda_j d(Tu_j, Tu'_j)^q\right)^{\frac{1}{q}} : \lambda_j \ge 0, \left(\sum_{j=1}^{n} \lambda_j \left|u_j - u'_j\right|^q\right)^{\frac{1}{q}} \le |u| \right\}.$$

Note that since

$$\left(\sum_{j=1}^{n} \lambda_j d(Tu_j, Tu'_j)^q\right)^{\frac{1}{q}} \le M_{(q)}^{\mathrm{Lip}}(T) \left\|\left(\sum_{j=1}^{n} \lambda_j \left|u_j - u'_j\right|^q\right)^{\frac{1}{q}}\right\|_E ,$$

we have for all $u, v \in E$

$$d(Tu, Tv) \le \rho(u - v) \le M_{(q)}^{\mathrm{Lip}}(T) \|u - v\|.$$

Moreover, we claim that ρ is a lattice seminorm on E. First notice that for any $u, v \in E$ and $\lambda \geq 0$, it is clear that $\rho(\lambda u) = \lambda \rho(u)$ and that $|v| \leq |u|$ implies $\rho(v) \leq \rho(u)$. To prove the triangle inequality, let $u, v \in E$ and $w = |u| + |v|$, and let $I_w \subset E$ be the ideal generated by w in E, which is identified with a space $C(K)$ in which w corresponds to the function identically equal to 1 [13, II.7]. Given $\epsilon > 0$, find $w_1, w_1', ..., w_n, w_n' \in E$ and $\lambda_1, ..., \lambda_n \geq 0$ such that

$$\left(\sum_{j=1}^{n} \lambda_j \left|w_j - w_j'\right|^q\right)^{\frac{1}{q}} \leq |w| \quad \text{and} \quad \rho(w) \leq \left(\sum_{j=1}^{n} \lambda_j d\left(Tw_j, Tw_j'\right)^q\right)^{\frac{1}{q}} + \epsilon,$$

Since $u, v \in I_w$, they correspond to functions $f, g \in C(K)$ such that $|f(t)| + |g(t)| = 1$for all $t \in K$. Similarly, each $w_j - w_j'$ corresponds to $h_j \in C(K)$ with $\left(\sum_{j=1}^{n} \lambda_j |h_j(t)|^q\right)^{\frac{1}{q}}$ for all $t \in K$. Let us now consider

$$f_j(t) = h_j(t) f(t), \quad g_j(t) = h_j(t) g(t).$$

Which belong to $C(K)$ and satisfy

$$\left(\sum_{j=1}^{n} \lambda_j |f_j(t)|^q\right)^{\frac{1}{q}} \leq |f(t)| \quad \text{and} \quad \left(\sum_{j=1}^{n} \lambda_j |g_j(t)|^q\right)^{\frac{1}{q}} \leq |g(t)|.$$

This means there are $u_j, v_j \in I_w \subset E$ with $u_j + v_j = w_j - w_j'$ for each $1 \leq j \leq n$ and satisfying

$$\left(\sum_{j=1}^{n} \lambda_j |u_j|^q\right)^{\frac{1}{q}} \leq |u| \quad \text{and} \quad \left(\sum_{j=1}^{n} \lambda_j |v_j|^q\right)^{\frac{1}{q}} \leq |v|\,.$$

Notice that $w_j - u_j = v_j + w_j'$, and hence

$$d(Tw_j, Tw_j') \leq d(Tw_j, T(w_j - u_j)) + d(T(v_j + w_j'), Tw_j').$$

Then,

$$\rho(u+v) \leq \rho(w) \leq \rho(w) \leq \left(\sum_{j=1}^{n} \lambda_j d\left(Tw_j, Tw_j'\right)^q\right)^{\frac{1}{q}} + \epsilon \leq$$
$$\left(\sum_{j=1}^{n} \lambda_j d\left(Tw_j, T(w_j - u_j)\right)^q\right)^{\frac{1}{q}} + \left(\sum_{j=1}^{n} \lambda_j d\left(T(v_j + w_j'), Tw_j'\right)^q\right)^{\frac{1}{q}} + \epsilon.$$

Since $w_j - (w_j - u_j) = u_j$ and $(v_j + w_j') - w_j' = v_j$, it follows from the definition of ρ that

$$\left(\sum_{j=1}^{n} \lambda_j d\left(Tw_j, T(w_j - u_j)\right)^q\right)^{\frac{1}{q}} \leq \rho(u) \quad \text{and} \quad \left(\sum_{j=1}^{n} \lambda_j d\left(T(v_j + w_j'), Tw_j'\right)^q\right)^{\frac{1}{q}} \leq \rho(v).$$

and thus $\rho(u+v) \leq \rho(u)+\rho(v)+\epsilon$. Letting ϵ go to 0, we have proved that ρ satisfies the triangle inequality. Let V denote the Banach lattice obtained by completing $E/\rho^{-1}(0)$ with the norm induced by ρ, and let φ be the quotient map $E \longrightarrow E/\rho^{-1}(0)$ seen as a map to V. For $u \in E$ let us define $S(\varphi(u)) = T(u)$. Since $d(Tu, Tv) \leq \rho(u-v)$ for any $u, v \in E$ the map S is well defined on $E/\rho^{-1}(0)$. Moreover, since X is complete we can extend S to a Lipschitz map $S : V \longrightarrow X$ such that $\mathrm{Lip}(S) \leq 1$ and $T = S\varphi$.

Now consider $\{u_i\}_{i=1}^n$ in E, and let $\epsilon > 0$. For each $i = 1, ..., n$ there exist $\left\{v_j^i, w_j^i\right\}_{j=1}^{k_i}$ in E and nonnegative numbers $\left\{\lambda_j^i\right\}_{j=1}^{k_i}$ such that

$$\left(\sum_{j=1}^{k_i} \lambda_j^i \left|v_j^i - w_j^i\right|^q\right)^{\frac{1}{q}} \leq |u_i| \text{ and } \rho(u_i)^q \leq \sum_{j=1}^{k_i} \lambda_j^i d(Tv_j^i, Tw_j^i)^q + \frac{\epsilon^q}{n}.$$

Adding up, we have that

$$\left(\sum_{i=1}^n \rho(u_i)^q\right)^{\frac{1}{q}} \leq \rho\left(\left(\sum_{i=1}^n |u_i|^q\right)^{\frac{1}{q}}\right) + \epsilon.$$

Letting ϵ go to 0 we conclude that the normed lattice $E/\rho^{-1}(0)$ is q-concave with constant 1, and thus so is its completion V. Finally, note that

$$\|\phi\| .M_{(q)}(I_V).\mathrm{Lip}(S) \leq M_{(q)}^{\mathrm{Lip}}(T).$$

□

Once again taking Chapiter 1 as a model, the nonlinear Krivine factorization theorem (Theorem 3.2) follows easily from the factorization Theorems 3.3 and 3.4.

Proposition 3.1. *[2] Let X, Y be metric spaces with Y complete and E a Banach lattice. Suppose that $T : X \longrightarrow E$ is Lipschitz p-convex and $S : E \longrightarrow Y$ is Lipschitz p-concave. Apply Theorems 3.3 and 3.4 (more specifically, their proofs) to obtain factorizations*

$$\begin{tikzcd}
X \arrow[rr, "T_1"] \arrow[dr, "\tau"'] & & E \arrow[rr, "T_2"] \arrow[dr, "\phi"'] & & Y \\
& W \arrow[ur, "\psi"'] & & V \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

where W (resp. V) is a p-convex (resp. p-concave) Banach lattice with constant 1, τ and σ are Lipschitz maps with constant at most 1, and ψ(resp. ϕ) is a lattice homomorphism of norm at most $M_{\mathrm{Lip}}^{(p)}(T)$ (resp. $M_{(p)}^{Lip}(S)$) From Lemma 1.8 , $\phi \circ \psi$ factors through an $L_p(\mu)$ space with the factors being lattice homomorphisms whose norms have product at most $M_{\mathrm{Lip}}^{(p)}(T)$. $M_{(p)}^{Lip}(S)$. The conclusion of Theorem 3.2 is now clear.

Similarly, Theorems 3.3 and 3.4 allow us to reduce the following two Lipschitz results to their linear counterparts (Proposition 1.9 and Corollary 1.2).

Proposition 3.2. *Let X, Y be metric spaces with Y complete and E a Banach lattice. Suppose that $T : X \longrightarrow E$ is Lipschitz p-convex and $S : E \longrightarrow Y$ is Lipschitz q-concave, with $1 \leq q < p < \infty$. Then ST factors through a canonical inclusion $i_{p,q} : L_p(\mu) \longrightarrow L_q(\mu)$. In fact, there is a factorization*

$$\begin{array}{ccccc} X & \xrightarrow{T} & E & \xrightarrow{S} & Y \\ {\scriptstyle T_1}\downarrow & & & & \uparrow{\scriptstyle S_1} \\ L_p(\mu) & & \xrightarrow[i_{p,q}]{\hspace{3em}} & & L_q(\mu) \end{array}$$

with $\mathrm{Lip}(T_1) \leq M_{\mathrm{Lip}}^{(p)}(T)$ *and* $\mathrm{Lip}(S_1) \leq M_{(q)}^{\mathrm{Lip}}(S)$.

Proposition 3.3. *Let X, Y be metric spaces with Y complete and E a Banach lattice, and $1 \leq p, q \leq \infty$. Suppose that $T : X \longrightarrow E$ is Lipschitz p-convex and $S : E \longrightarrow Y$ is Lipschitz q-concave. Then for every $\theta \in (0,1)$, ST factors through a Banach lattice U_θ that is $\dfrac{p}{p(1-\theta)+\theta}$-convex and $\dfrac{q}{1-\theta}$-concave.*

Corollary 3.1. *Let E be a Banach lattice, $1 \leq p, q \leq \infty$, and assume that $T : E \longrightarrow E$ is both Lipschitz p-convex and Lipschitz q-concave. Then for each $\theta \in (0,1)$, T^2 factors through a $\dfrac{p}{p(1-\theta)+\theta}$-convex and $\dfrac{q}{1-\theta}$-concave. Banach lattice. In particular, if $p > 1$ and $q < \infty$ then T factors through a super reflexive Banach lattice.*

A natural question in this context is: if a linear map $T : X \longrightarrow Y$ between Banach spaces can be factored as a Lipschitz p-convex map followed by a Lipschitz q-convex one, is there a factorization where the factor maps are in addition linear ? Under certain conditions, the answer is yes.

Theorem 3.5. *Let $T : X \longrightarrow Y$ be a linear map between a Banach space X and a dual Banach space Y, and assume that T admits a factorization $T = T_2T_1$ where T_1 is Lipschitz p-convex and T_2 is Lipschitz q-concave, with $1 \leq q < p < \infty$. Then there is also a factorization $T = \tau_2\tau_1$ where τ_1 is p-convex and τ_2 is q-concave, and moreover $M^{(p)}(\tau_1) \leq M^{(p)}_{\mathrm{Lip}}(T_1)$ and $M_{(q)}(\tau_2) \leq M^{\mathrm{Lip}}_{(q)}(T_2)$.*

Theorem 3.6. *Let $T : E \longrightarrow F$ be a linear map between Banach lattices E and F, and assume that T admits a factorization $T = T_2T_1$ where T_1 is Lipschitz q-concave and T_2 is Lipschitz p-convex. Then there is also a factorization $T = \tau_2\tau_1$ where τ_1 is q-concave and τ_2 is p-convex, and moreover $M^{(p)}(\tau_2) \leq M^{(p)}_{\mathrm{Lip}}(T_2)$ and $M_{(q)}(\tau_1) \leq M^{\mathrm{Lip}}_{(q)}(T_1)$.*

Even though the naive factorization scheme for linear maps that are both p-convex and q-concave did not work out, Raynaud and Tradacete were able to prove that if one "gives up" a little on the exponents of convexity and concavity involved, one still gets such a factorization Theorem 1.5. The following would be a Lipschitz version of that result.

Question 3.1. *Suppose that a Lipschitz map $T : E \longrightarrow F$ between Banach lattices is Lipschitz p-convex and Lipschitz q-concave, with $1 < p \leq \infty$ and $1 \leq q < \infty$. Can we find $1 < p_0 < p$ and $q < q_0 < \infty$ so that there is a factorization of T as*

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle \phi}\big\downarrow & & \big\uparrow{\scriptstyle \varphi} \\ E_0 & \xrightarrow[R]{} & F_0 \end{array}$$

where ϕ and φ are positive linear maps, E_0 is q_0-concave, F_0 is p_0-convex and R is a Lipschitz map? Moreover: given $\theta \in (0,1)$, can we have $p_0 = \dfrac{p}{\theta + (1-\theta)p}$ and $q_0 = \dfrac{q}{1-\theta}$?

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Abstract

سندرس تعميم المؤثرات p-محدب و q-مقعر في الحالة الخطية الى الحالة غير الخطية للتحديد اكثر الحالة الليبشتزية، كما سنهتم في هذه الدراسة برؤية بعض نظريات تحليل العوامل التي بدورها تسمح لتسهيل دراستهم.
الكلمات المفتاحية: مؤثر ليبشيز، مشابك بناخ، نظريات تحليل العوامل، مؤثر موجب، مؤثر p-محدب، مؤثر q-مقعر.

Nous étudierons la généralisation des opérateurs p-convexes et q-concaves du cas linéaire au cas non linéaire. Plus précisément le cas lipschitzien. Entre outre, on s'intéresse á quelques théorémes de factorisation ce qui facilite l'étude de ces opérateurs.
Mots-Clés: opérateur de Lipschitz, Banach réticulé, Théorémes de factorisation, opérateur positif, opérateur p -convexe, opérateur q -concave.

We will study the generalization of the p-convex and q-concave maps of the linear case to the nonlinear case. More precisely the Lipschitz case. Among others, we are interested in some factorization theorems for a good study these operators.
Key-words: Lipschitz operator, Banach lattice, Factorization theorems, Positive operator, p-convex operator , q-concave operator.