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On the two dimensional wave equation in time-dependent domains.

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Contents

Introduction **1**

1 Preliminaries **3**

1.1 Functional spaces 3

1.1.1 Distribution space 3

1.1.2 Sobolev spaces 6

1.1.3 Spaces involving time 7

1.2 Weak and weak* convergences 9

1.2.1 Weak convergence 9

1.2.2 Weak* convergence 9

1.3 The 2–D wave equation with weak damping in cylindrical domains 10

1.3.1 Physical assumptions 10

1.3.2 Derivation of the PDE model 10

1.4 An existence result for The wave equation in cylindrical domains 12

2 The wave equation in non-cylindrical domains **14**

2.1 Notation and assumptions 14

2.2 Energy decay and uniqueness 15

2.3 Penalization method 17

2.3.1 Penalized problem 17

2.3.2 A priori estimates 18

2.3.3 Passage to the limit 20

3 Numerical analysis of the problem **22**

3.1 Finite difference scheme 22

3.2 Numerical representation of the solution 24

3.2.1 Algorithm 25

3.2.2 Graphic representation 28

A Analytics and geometrics notions 35

Bibliography 36

Introduction

To study certain natural processes, we use mathematical models that describe all the essential characteristics of these processes. In many cases, the obtained model is a partial differential equation that needs to be solved by some analytical or numerical method. If the PDE is difficult to solve, we may simplify the model by assuming for example that the dependence of some data on one or several variables can be neglected. However, the accuracy of the obtained model may be compromised.

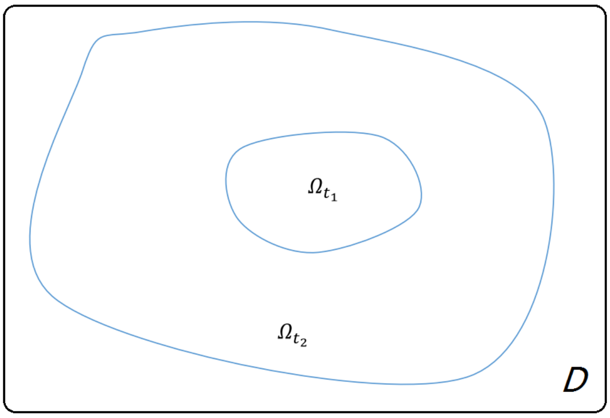
In this work, we deal with small vibrations of a membrane when the variation of its shape with respect to time can not be neglected. To be more precise, we assume that the transversal movements of this membrane are described by the wave equation

$$u_{tt}(x,t) - \Delta u(x,t) = 0, \quad x \in \Omega_t,\ t \in [0,T] \text{ for some } T > 0. \tag{1}$$

where $x = (x_1, x_2) \in \Omega_t \subset \mathbb{R}^2$ and Ω_t is a bounded planar domain depending on time. Although the problem is linear, the dependence of the domain on time need to be handled carefully in order to obtain existence and uniqueness results. One approach consists in using the penalization method proposed by Lions in [7]. The idea is to approximate the wave equation in (1) by the restriction on Ω_t of the solution of the following wave equation

$$u_{tt}(x,t) - \Delta u(x,t) + \frac{\chi_{\Omega_t^c}}{\varepsilon} u_t(x,t) = 0, \quad x \in D,\ t \in [0,T] \text{ for some } T > 0. \tag{2}$$

where D is a fixed bounded domain in $\mathbb{R}^2$ such that $\Omega_t \subset D$, $\forall t \in [0,T]$ and $\chi_{\Omega_t^c}$ is the characteristic function of $\Omega_t^c := D \backslash \Omega_t$ and ε is a parameter (of penalization) that will tend to 0. The term $\frac{\chi_{\Omega_t^c}}{\varepsilon} u_t(x,t)$ represents an artificial damping that forces the solution to be approximately zero outside Ω_t. The equation (2), associated with Dirichlet boundary conditions and initial data, is a classical wave equation that can be solved by a Faedo-Galerkin method, for each $\varepsilon > 0$. Thus, we obtain a sequence of solutions that depends on ε.



Assuming that the domain Ω_t is increasing, i.e. $\Omega_{t_1} \subset \Omega_{t_2}, \forall t_1 < t_2$ and passing to the limit $\varepsilon \to 0$, we deduce the existence of a solution for the wave equation (1) associated with Dirichlet boundary conditions. We also combine this penalization technique with an explicit finite differences method to obtain a numerical solution of Problem (1).

After the present introduction, we recall in chapter 1 some elements of functional analysis needed in the sequel. We also state a general existence and uniqueness result for the damped wave equation in a bounded domain that does not depend on time. In chapter 2, we establish the existence of a unique solution for the wave equation (1). In the final chapter, we apply the finite difference method to obtain a numerical solution for (1) . The results are illustrated by graphical representation for different initial data and different shapes of increasing domains.

Finally, some references related to this work are given at the ends of the manuscript.

Chapter 1

Preliminaries

1.1 Functional spaces

In this section, we recall some functional spaces and theorems that we need in the remaining of this work. We can find them with more details in [3, 9].

1.1.1 Distribution space

We denote by

$$D^{\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n}, \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$$

the derivative of order $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Definition 1.1 *We define the support of a function $f : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^n$ as:*

$$supp(f) = \overline{\{x \in \Omega : f(x) \neq 0\}} \ .$$

Then, f is compactly supported in Ω, if $supp(f)$ is a compact subset of Ω

Definition 1.2 *The space of test functions is given by:*

$$\mathcal{D}(\Omega) = \{f \in C^{\infty}(\Omega) : supp(f) \subset \Omega\} \ .$$

Definition 1.3 *Let $\{\varphi_k\} \subset \mathcal{D}(\Omega)$ and $\varphi \subset \mathcal{D}(\Omega)$. So:*

$$\varphi_k \to \varphi \quad in\ \mathcal{D}(\Omega), \quad as\ k \to +\infty$$

if the following two properties hold:

1. *There exists a compact set $K \subset \Omega$ containing the support of every φ_k.*

2. *$\forall \alpha = (\alpha_1, \cdots, \alpha_n), |\alpha| \geq 0, D^{\alpha}\varphi_k \to D^{\alpha}\varphi$ uniformly in Ω.*

Definition 1.4 *A distribution is a linear continuous functional in $\mathcal{D}(\Omega)$, and the set of all distributions denoted by:*

$$\mathcal{D}'(\Omega) = \mathcal{L}(\mathcal{D}(\Omega), \mathbb{R}) \ .$$

with:

1. $\forall \varphi_1, \varphi_2 \in \mathcal{D}(\Omega), \forall \alpha, \beta \in \mathbb{R} : \langle f, \alpha\varphi_1 + \beta\varphi_2 \rangle = \alpha \langle f, \varphi_1 \rangle + \beta \langle f, \varphi_2 \rangle$,
2. *If* $\varphi_k \overset{\mathcal{D}(\Omega)}{\to} \varphi$, *then* $\langle f, \varphi_k \rangle \to \langle f, \varphi \rangle$.

A central concept in the theory of distributions, due to Schwartz, is the notion of weak or distributional derivative. Clearly, we have to abandon the classical definition since, for instance, we are going to define the derivative for a function $f \in L^1(\Omega)$.

Definition 1.5 *If $\varphi \in \mathcal{D}(\Omega)$, denoting by $\nu = (\nu_1, \cdots, \nu_n)$ the outward normal unit vector to $\partial\Omega$, we have:*

$$\int_\Omega \partial_{x_i} f \varphi dx = \int_{\partial\Omega} f \varphi \nu_i dx - \int_\Omega \partial_{x_i} \varphi f dx,$$

since $\varphi = 0$ on $\partial\Omega$. The equation

$$\int_\Omega \partial_{x_i} f \varphi dx = -\int_\Omega \partial_{x_i} \varphi f dx,$$

interpreted in $\mathcal{D}'(\Omega)$, becomes

$$\langle \partial_{x_i} f, \varphi \rangle = -\langle \partial_{x_i} \varphi, f \rangle .$$

Definition 1.6 *Let Ω be an open set in $\mathbb{R}^n$.*
Denoted by $L^p(\Omega)$ the set of functions $f : \Omega \longrightarrow \mathbb{R}$, such that:

$$L^p(\Omega) = \left\{ f : \int_\Omega |f|^p \, dx < \infty \right\}, \ 1 \le p < +\infty.$$

equipped with the next norm, $L^p(\Omega)$ becomes a Banach space

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \, dx \right)^{1/p} .$$

When $p = +\infty$, the space $L^p(\Omega)$ will defined by the followig way:

$$L^\infty(\Omega) = \{ M : |f| < M \} .$$

The infimum of all numbers M, is called the essential supremum of f, and denoted by

$$\|f\|_{L^\infty(\Omega)} = esssup_\Omega(|f|) .$$

which is a norm in $L^\infty(\Omega)$.
Endowing with the above norm, $L^\infty(\Omega)$ becomes a Banach space.

we can see an special case of L^p spaces, which is the L^2 space.

Definition 1.7 *The space $L^2(\Omega)$ is the set of all functions squared integrable over Ω, i.e:*

$$L^2(\Omega) = \left\{ f : \Omega \to \mathbb{R} : \int_\Omega f^2 dx < \infty \right\}.$$

The space $L^2(\Omega)$ is a Hilbert space endowed by the scalar product:

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx, \quad \forall f, g \in L^2(\Omega)$$

and its associated norm:

$$\|f\|_{L^2(\Omega)} = \left(\int_\Omega f^2 dx \right)^{1/2} .$$

Theorem 1.8 (Fubini) *Let Ω_1 and Ω_2 be two open set from $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Assume that $F \in L^1(\Omega_1 \times \Omega_2)$. Then, for a.e. $x \in \Omega_1$,*

$$F(x, y) \in L^1_y(\Omega_2) \quad \text{and} \quad \int_{\Omega_2} F(x, y)dy \in L^1_x(\Omega_1).$$

Similarly, for a.e. $y \in \Omega_2$,

$$F(x, y) \in L^1_x(\Omega_1) \quad \text{and} \quad \int_{\Omega_1} F(x, y)dx \in L^1_y(\Omega_2).$$

Moreover, one has

$$\int_{\Omega_1} \left(\int_{\Omega_2} F(x, y)dy \right) dx = \int_{\Omega_2} \left(\int_{\Omega_1} F(x, y)dx \right) dy = \iint_{\Omega_1 \times \Omega_2} F(x, y)dxdy.$$

Theorem 1.9 (Dominated convergence theorem) *Let (f_n) be a sequence of functions in $L^p(\Omega)$ that satisfy*

- $f_n(x) \to f(x)$ *a.e. on* Ω,
- $\exists g \in L^p(\Omega)$ *such that,* $\forall n, |f_n(x)| \le g(x)$ *a.e. on* Ω,

Then $f \in L^p(\Omega)$ and $\|f_n \to f\|_{L^p(\Omega)} \to 0$.

Theorem 1.10 *Let $(f_n)_{n\ge 0}$ be a sequence from $L^p(\Omega), 1 \le p \le +\infty$, and $f \in L^p(\Omega)$ such that $\|f_n - f\|_{L^p(\Omega)} \to 0$ Then there exist a subsequence (f_{n_k}) and a function $h \in L^p(\Omega)$ such that*

- $f_{n_k}(x) \to f(x)$ *a.e. on* Ω,
- $|f_{n_k}(x)| \le h(x)\ \forall\ k$, *a.e. on* Ω.

1.1.2 Sobolev spaces

Definition 1.11 *The space $H^1(\Omega)$ is the space of functions in $L^2(\Omega)$, with first derivatives in the sense of distributions are functions in $L^2(\Omega)$. Thus:*

$$H^1(\Omega) = \left\{ f \in L^2(\Omega) : \nabla f \in L^2(\Omega; \mathbb{R}^n) \right\}.$$

The inner product and the norm in $H^1(\Omega)$ are given, respectively, by

$$(f, g)_{H^1(\Omega)} = \int_\Omega f(x)g(x)dx + \int_\Omega \nabla f \cdot \nabla g dx, \forall f, g \in H^1(\Omega)$$

and

$$\|f\|_{H^1(\Omega)} = \left(\|f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Proposition 1.12 *$H^1(\Omega)$ is a separable Hilbert space, and compactly embeded in $L^2(\Omega)$.*

We introduce now an important subspace of $H^1(\Omega)$.

Definition 1.13 *We denote by $H^1_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$, and its dual is denoted by $H^{-1}(\Omega)$.*

Thus $f \in H^1_0(\Omega)$ if and only if there exists a sequence $f_k \in \mathcal{D}(\Omega)$ such that $f_k \to f$ in $H^1(\Omega)$.

Theorem 1.14 (Poincaré's inequality) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. There exists a positive constant C_P (a Poincaré 's constant) depending only on n and diam(Ω), such that, for every $f \in H^1_0(\Omega)$,*

$$\|f\|_{L^2(\Omega)} \leq C_P \|\nabla f\|_{L^2(\Omega)}.$$

We will choose in $H^1_0(\Omega)$

$$(f, g)_{H^1_0(\Omega)} = (\nabla f, \nabla g)_{L^2(\Omega)} \quad \text{and} \quad \|f\|_{H^1_0(\Omega)} = \|\nabla f\|_{L^2(\Omega)}$$

as inner product and norm, respectively.

Theorem 1.15 (Trace) *Let Ω be an open set from $\mathbb{R}^n$ with smooth boundary Γ. Then, we define the trace function γ_0 by:*

$$\begin{aligned} \gamma_0 : \quad & \mathcal{D}(\bar{\Omega}) \longrightarrow C(\Gamma), \\ & u \longrightarrow \gamma_0(u) = u|_\Gamma. \end{aligned}$$

Theorem 1.16 (Green's formula) *Let Ω be a bounded open set from $\mathbb{R}^n$ with smooth boundary Γ, and $u, v \in H^1(\Omega)$. Then, the Green's formula is given by:*

$$\int_\Omega \partial_{x_i} uvdx_i = \int_\Omega uvdx_i - \int_\Gamma u\partial_{x_i} v \ \nu_i d\sigma \ i = 1, \cdots, n$$

with ν in the outward unit vector of Ω, and η_i its compenents for $i = 1, \cdots, n$.

The space $H_0^1(\Omega)$ is equivalent to the space of functions from $H^1(\Omega)$ with zero trace on the boundary Γ i.e.

$$H_0^1(\Omega) = \left\{ f \in H^1(\Omega) \quad \text{with} \quad f|_{\partial\Omega} = 0 \right\}.$$

Theorem 1.17 (Gauss-Green)

- *Suppose that $u \in C^1\left(\overline{\Omega}\right)$. then*

$$\int_\Omega u_{x_i} dx = \int_{\partial\Omega} u\nu^i dS \quad i = (1, \cdots, n).$$

  with $\nu = (\nu^1, \nu^2, \cdots, \nu^n)$ is the outward unit vector of Ω.

- *We have*

$$\int_\Omega div\mathbf{u}dx = \int_{\partial\Omega} \mathbf{u} \cdot \nu dS.$$

for each vector field $\mathbf{u} \in C^1\left(\overline{\Omega}; \mathbb{R}^n\right)$.

1.1.3 Spaces involving time

In evolution problems, it is convinient to separate the two variables t and x. The variable of time is dependent only on $t \in (0; T)$ and the variable of space is dependent on $x \in \Omega \subset \mathbb{R}^n$. Hence we need to consider some new spaces.

In all of this subsection we assume that V is a Banach space and V' denotes its dual.

Definition 1.18 *We define the space $C(0, T; V)$ as the set of continuous functions $f : (0, T) \to V$, in which is endowed by the norm:*

$$\|f\|_{C(0,T;V)} = sup_{0 \leq t \leq T} \|f\|_V \ ,$$

it is clear that this space is a Banach space.

Definition 1.19 *We define the spaces $L^p(0,T;V)$, with $1 \leq p \leq +\infty$, as the set of measurable functions $f : [0,T] \to V$, such that:*
if $1 \leq p < +\infty$:

$$\|f\|_{L^p(0,T;V)} = \left(\int_0^T \|f\|_V^p \, dt\right)^{1/p} < \infty,$$

while if $p = +\infty$:

$$\|f\|_{L^\infty(0,T;V)} = sup_{0\leq t\leq T}\left(\|f\|_V\right).$$

Endowed with the above norms, $L^p(0,T;V)$ becomes a Banach space for $1 \leq p \leq +\infty$.

In the case of $p = 2$ we get $L^2(0,T;V)$ which is a Hilbert space if V is a Hilbert space, and its norm defined by:

$$\|f\|_{L^2(0,T;V)} = \left(\int_0^T \|f\|_V^2 \, dt\right)^{1/2}.$$

Definition 1.20 *We defined the space $L^p(0,T;L^q(\Omega_t))$ where $1 \leq p,q \leq +\infty$, and Ω_t is a moving domain in time, by*

$$L^p(0,T;L^q(\Omega_t)) = \left\{u(t) \in L^q(\Omega_t) \text{ with } g(t) = \|u(t)\|_{L^q(\Omega_t)} \in L^p(0,T)\right\},$$

is a Banach space endowed with the norm

$$\|u\|_{L^p(0,T;L^q(\Omega_t))} = \|g(t)\|_{L^p(0,T)}, \tag{1.1}$$

if $q = 2$, the space $L^p(0,T;L^q(\Omega_t))$ will be

$$L^p(0,T;L^2(\Omega_t)) = \left\{u(t) \in L^2(\Omega_t) \text{ with } g(t) = \|u(t)\|_{L^2(\Omega_t)} \in L^p(0,T)\right\},$$

endowed with the norm

$$\|u\|_{L^p(0,T;L^2(\Omega_t))} = \|g(t)\|_{L^p(0,T)},$$

Definition 1.21 *The space $L^p(0,T;H^1(\Omega_t))$ is defined by*

$$L^p(0,T;H^1(\Omega_t)) = \left\{u(t) \in H^1(\Omega_t) \text{ with } h(t) = \|u(t)\|_{H^1(\Omega_t)} \in L^p(0,T)\right\},$$

is a Banach space, endowed by the norm

$$\|u\|_{L^p(0,T;H^1(\Omega_t))} = \|h(t)\|_{L^p(0,T)}.$$

Definition 1.22 *The first order Sobolev space $H^1(0,T;V)$ is the space of the functions $f \in L^2(0,T;V)$ such that $\partial_t f \in L^2(0,T;V)$.*

If V is a Hilbert space, $H^1(0,T;V)$ is also a Hilbert space endowed by the inner product:

$$(f,g)_{H^1(0,T;V)} = \int_0^T (f,g)_V + (\partial_t f, \partial_t g)_V \, dt.$$

1.2 Weak and weak* convergences

We recall that the dual V' is endowed with the dual norm:

$$\|f\|_{V'} = sup_{x\in V, \|x\|\leq 1} |\langle f, x\rangle| .$$

1.2.1 Weak convergence

Definition 1.23 *Let $(x_n)_{n\geq 0}$ be a sequence from V, $(x_n)_{n\geq 0}$ converges to x for the weak topology $\sigma(V, V')$, if:*

$$\langle f, x_n\rangle \longrightarrow \langle f, x\rangle , \forall f \in V',$$

which denoted by

$$x_n \rightharpoonup x \text{ as } n \to +\infty.$$

Proposition 1.24 *Let $(x_n)_{n\geq 0}$ be a sequence from V, we have*

- $x_n \rightharpoonup x \Leftrightarrow \langle f, x_n\rangle \longrightarrow \langle f, x\rangle , \forall f \in V'$
- $x_n \to x$ *strongly* $\Rightarrow x_n \rightharpoonup x$
- *if* $x_n \rightharpoonup x$ *and if* $f_n \to f$ *strongly in* V', *then* $\langle f_n, x_n\rangle \longrightarrow \langle f, x\rangle$.

1.2.2 Weak* convergence

Definition 1.25 *Let $(f_n)_{n\geq 0}$ be a sequence from V', $(f_n)_{n\geq 0}$ converges to f for the weak* topology $\sigma(V', V)$, if:*

$$\langle f_n, x\rangle \longrightarrow \langle f, x\rangle , \forall x \in V$$

which denoted by:

$$f_n \overset{*}{\rightharpoonup} f \quad \text{as } n \to +\infty.$$

Proposition 1.26 *Let $(f_n)_{n\geq 0}$ be a sequence from V', we have*

- $f_n \overset{*}{\rightharpoonup} f \Leftrightarrow \langle f_n, x\rangle \longrightarrow \langle f, x\rangle , \forall x \in V,$
- $f_n \to f$ *strongly* $\Rightarrow f_n \overset{*}{\rightharpoonup} f,$
- *if* $f_n \overset{*}{\rightharpoonup} f$, *and* $x_n \to x$ *strongly in* V *then* $\langle f_n, x_n\rangle \longrightarrow \langle f, x\rangle$.

1.3 The 2–D wave equation with weak damping in cylindrical domains

Before solving the wave equation, we need first to know the origin of this problem and the phenomena it represents. In this subsection we are going to recall the mathematical derivation of the wave equation by returning to the physical meaning and phenomena [5].

1.3.1 Physical assumptions

- The mass of the membrane per unit area is constant (homogeneous membrane). The membrane is perfectly flexible and offers no resistance to bending.
- The membrane is stretched and then fixed along its entire boundary in the xy-plane. The tension per unit length $\mathbf{T}$ caused by stretching the membrane is the same at all points and in all directions and does not change during the motion.
- The deflection $u(x, y, t)$ of the membrane during the motion is small compared to the size of the membrane, and all angles of inclination are small.
- There is an external factors of dissipation, like friction, given as a linear law expressing a force proportional to the speed of vibration.

1.3.2 Derivation of the PDE model

Let us consider the forces acting on a small portion of the physical system, the membrane as it is moving up and down. Since the deflections of the membrane and the angles of inclination are small, the sides of the portion are approximately equal to Δ_x and Δ_y. The tension $\mathbf{T}$ is the force per unit length. Hence the forces acting on the sides of the portion are approximately $\mathbf{T}\Delta_x$ and $\mathbf{T}\Delta_y$. Since the membrane is perfectly flexible, these forces are tangent to the moving membrane at every instant.

Horizontal components of the forces.

These components are obtained by multiplying the forces by the cosines of the angles of inclination. Since these angles are small, their cosines are close to 1. Hence the horizontal components of the forces at opposite sides are approximately equal. we conclude that we may regard the motion of the membrane as transversal, that is, each particle moves vertically.

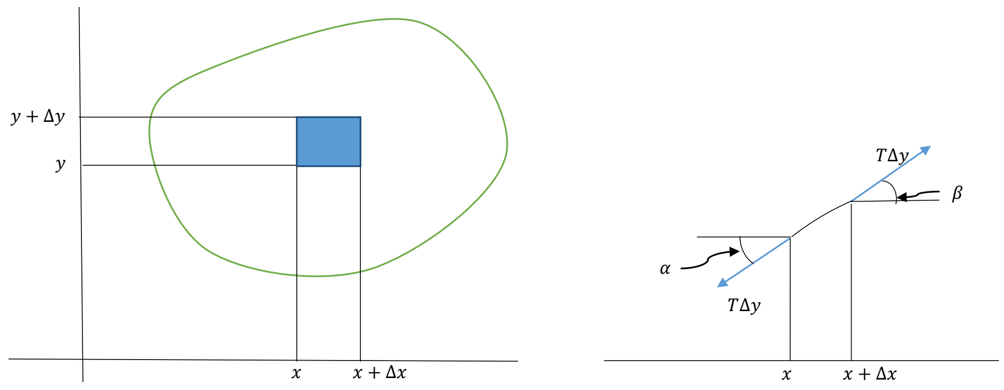


Figure 1.1: Vibrating membrane

Vertical components of the forces.

These components along the right side and the left side are , respectively,

$$\mathbf{T}\Delta_y \sin\beta \quad \text{and} \quad -\mathbf{T}\Delta_y \sin\alpha.$$

Here α and β are the values of the angle of inclination in the middle of the edges, and the "-" sign appears because the force on the left side is directed downward. Since the angles are small, we may write the two vertical components as

$$\begin{aligned} \mathbf{T}\Delta_y(\sin\beta - \sin\alpha) &\approx \mathbf{T}\Delta_y(\tan\beta - \tan\alpha) \\ &= \mathbf{T}\Delta_y[u_x(x+\Delta_x, y_1) - u_x(x, y_2)], \end{aligned}$$

where y_1 and y_2 are values between y and $y + \Delta_y$. Similarly, the resultant of the vertical components of the forces acting on the other two sides of the portion is

$$\mathbf{T}\Delta_x[u_y(x_1, y + \Delta_y) - u_y(x_2, y)]$$

where x_1 and x_2 are values between x and $x + \Delta_x$.

The friction force is given by a linear law proportional to the speed of vibration,

$$-d\,(x, y, t)\,\frac{\partial u}{\partial t}$$

where $d\,(x, y, t) \in L^\infty\,(\Omega \times (0, +\infty))$ is a positive friction coefficient, assumed here to depend on x, y and t.

The PDE model.

By Newton's second law, the sum of the forces is equal to the mass $\rho\Delta A$ of that small portion times the acceleration $\frac{\partial^2 u}{\partial t^2}$. Here ρ is the mass of the inflected membrane per unit area, and $\Delta A = \Delta_x\Delta_y$ is the area of that portion when it is inflected. Thus

$$\rho\Delta_x\Delta_y\frac{\partial^2 u}{\partial t^2} = \mathbf{T}\Delta_y[u_x(x+\Delta_x, y_1) - u_x(x, y_2)] + \mathbf{T}\Delta_x[u_y(x_1, y+\Delta_y) - u_y(x_2, y)] - d\,(t, x, y)\,\frac{\partial u}{\partial t}.$$

Division by $\rho\Delta_x\Delta_y$ gives

$$\frac{\partial^2 u}{\partial t^2} = \frac{\mathbf{T}}{\rho}\left[\frac{u_x(x+\Delta_x, y_1) - u_x(x, y_2)}{\Delta_x} + \frac{u_y(x_1, y+\Delta_y) - u_y(x_2, y)}{\Delta_y}\right] - d\,(t,x,y)\,\frac{\partial u}{\partial t}.$$

If we let Δ_x and Δ_y approach zero, we obtain the PDE of the model

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \gamma\frac{\partial u}{\partial t},\ \text{where } c^2 = \frac{\mathbf{T}}{\rho} > 0 \text{ and } \gamma\,(x,y,t) = \frac{d\,(x,y,t)}{\rho} > 0.$$

Hence, it can be written as

$$\frac{\partial^2 u}{\partial t^2} = c^2\Delta u - \gamma\frac{\partial u}{\partial t}.$$

This PDE is called the two-dimensional wave equation with a weak damping.

1.4 An existence result for The wave equation in cylindrical domains

Let Ω be a bounded domain in $\mathbb{R}^n$. The above mathematical model can be generalized to several dimensions to the following one

$$\begin{cases} u_{tt} = c^2\Delta u - \gamma u_t, & \text{in } \Omega\times(0,T),\\ u = 0, & \text{on } \partial\Omega\times(0,T),\\ u(x,0) = u_0, u_t(x,0) = u_1, & \text{in } \Omega. \end{cases} \tag{DWP}$$

where u_0 and u_1 are the initial shape and speed of the membrane respectively.

The existence of a unique solution can be established by a Faedo-Galerkin method, As special case form Theorem 9.1 in [1], we have the following result.

Theorem 1.27 *Assume that*

$$u_0 \in H_0^1(\Omega),\ \textit{and},\ u_1 \in L^2(\Omega)$$

Then there exists a unique $u \in L^2(0,T;H_0^1(\Omega))$, *with* $u_t \in L^2(0,T;L^2(\Omega))$, *that solves the problem*

$$\begin{cases} u_{tt} - \Delta u + \gamma u_t = 0, & \textit{in } \Omega\times(0,T),\\ u = 0, & \textit{on } \partial\Omega\times(0,T),\\ u(0) = u_0, u_t(0) = u_1 & \textit{in } \Omega, t = 0 \end{cases}$$

in the weak sense

$$\int_\Omega u_{tt}v + \nabla u\cdot\nabla v + \gamma u_t v\ dx = 0,\ \forall v \in H_0^1(\Omega)\ \textit{and a.e. } t\in(0,T)\,. \tag{1.2}$$

Moreover, we have

$$u \in C(0,T;H_0^1(\Omega)),\ \ u_t \in C(0,T;L^2(\Omega)),$$

and we have the estimate

$$esssup_{0\leq t\leq T}(\|u\|_{H_0^1(\Omega)} + \|u_t\|_{L^2(\Omega)}) \leq C(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)}).$$

We define the energy of the solution of (DWP),as

$$E(t) = \int_\Omega |u_t|^2 + |\nabla u|^2\, dx$$

is decaying in time. Indeed, for smooth solution of (DWP), we can take u_t as a test function in (1.2), then we obtain

$$\textstyle\int_\Omega u_{tt}u_t + \nabla u \cdot \nabla u_t + \gamma u_t u_t \ dx = 0,$$

$$\textstyle\int_\Omega \frac{d|u_t|^2}{dt} + \frac{d|\nabla u|^2}{dt} + \gamma\, |u_t|^2\, dx = 0,$$

$$\textstyle\frac{d}{dt}\int_\Omega |u_t|^2 + |\nabla u|^2\, dx = -\int_\Omega \gamma\, |u_t|^2\, dx,$$

Hence,

$$\frac{d}{dt}E(t) = -\int_\Omega \gamma\, |u_t|^2\, dx \leq 0,\ \text{since } \gamma \geq 0.$$

In particular, if $\gamma = 0$ the energy will be conserved in time.

Chapter 2

The wave equation in non-cylindrical domains

In this chapter, we are going to study the homogenous problem of the wave equation in a 2-D non-cylindrical domain

$$\Omega_t \in \mathbb{R}^n,$$

i.e. a domain that depends on time. Thus, some difficulties will arise if we use a Galerkin approximation. So, we include these domains in such a fixed domain D, i.e.

$$\forall t \in (0, T) : \Omega_t \subseteq D$$

and introduce a viscous damping (depending on ε) as a penalization term [7] then we pass to the limit $\varepsilon \to 0$.

2.1 Notation and assumptions

Let $T > 0$ and consider Ω_t a bounded open set, then let

$$Q_T = \bigcup_{t=0}^{t=T} \Omega_t \times \{t\} \ \text{ and } \Sigma_T = \bigcup_{t=0}^{t=T} \partial\Omega_t \times \{t\}, \ t \in (0, T)$$

denote the space-time domain and the lateral boundary and

$$\nu = (\nu_x, \nu_t)$$

be the normal outward unit vector to (x, t) from Σ_T.
We assume that Q_T is a time-like domain, i.e.

$$|\nu_t| \leq |\nu_x| \,. \tag{2.1}$$

This means in particular that the speed of the moving boundary is less then the speed of propagation of the wave (normalized here to $c = 1$).

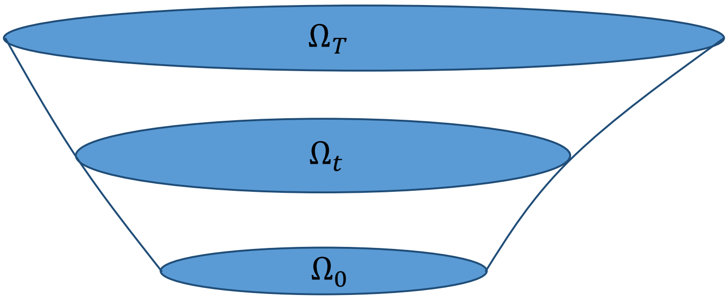


Figure 2.1: Moving domain Ω_t

We denote

$$\nabla_x = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n}) \text{ and } \nabla_{x,t} = (\nabla_x, \partial_t),$$
$$\nabla_{x,t} u = \frac{\partial u}{\partial \nu}\nu + \frac{\partial u}{\partial \tau}\tau,$$

with τ is the tangential unit vector at $(x,t) \in \Sigma_T$. But $u = 0$ on Σ_T, hence the tangential derivative vanishes on Σ_T, then $\frac{\partial u}{\partial \tau} = 0$ and

$$\nabla_{x,t} u = \frac{\partial u}{\partial \nu}\nu.$$

it gives:

$$\nabla_x u = \frac{\partial u}{\partial \nu}\nu_x, \text{ and, } \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \nu}\nu_t. \tag{2.2}$$

2.2 Energy decay and uniqueness

The homogeneous problem of the wave equation is given by:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } Q_T \\ u = 0, & \text{on } \Sigma_T \\ u(0) = u_0, u_t(0) = u_1, & \Omega_0 \end{cases} \tag{2.3}$$

where

$$u_0 \in H_0^1(\Omega_0) \text{ and } u_1 \in L^2(\Omega_0). \tag{2.4}$$

Now we will assume that Ω_t is an increasing domain. i.e
Let $\Omega^*_{t_1}$ and $\Omega^*_{t_2}$ be the projection of Ω_{t_1} and Ω_{t_1} on the hyperplane of the equation $t = 0$ respectively, then

$$\forall t_1 < t_2 : \Omega^*_{t_1} \subset \Omega^*_{t_2}. \tag{H1}$$

We can prove that the energy of the solution, defined by

$$E(t) = \frac{1}{2}\int_{\Omega_t} (u_t)^2 + |\nabla u|^2 \, dx \tag{2.5}$$

is decaying if the domain Ω_t expands.

As in [2]. Let us consider that u is a smooth function, then we estimate the following:

$$u_t (u_{tt} - \Delta u) = u_t u_{tt} - u_t \Delta u \tag{2.6}$$

then, as we know:

$$-u_t \Delta u = \frac{1}{2}\frac{d}{dt} |\nabla u|^2 - \operatorname{div}(u_t \nabla u) \tag{2.7}$$

and

$$u_t (u_{tt}) = \frac{1}{2}\frac{d}{dt}(u_t)^2 . \tag{2.8}$$

After a substitution of (2.7) and (2.8) in (2.6), we get:

$$\begin{aligned}\int_{Q_T} \frac{1}{2}\frac{d}{dt}(u_t)^2 + \frac{1}{2}\frac{d}{dt}|\nabla u|^2 - \operatorname{div}(u_t\nabla u)\, dxds &= \int_{Q_T} \frac{1}{2}\frac{d}{dt}\left[(u_t)^2 + |\nabla u|^2\right] - \operatorname{div}(u_t \nabla u)\, dxds \\ &= \int_{Q_T} \operatorname{div}\begin{pmatrix} \frac{1}{2}\left[(u_t)^2 + |\nabla u|^2\right] \\ -u_t \nabla u \end{pmatrix} dxds.\end{aligned}$$

Thanks to Gauss's theorem, the last integral equals

$$\begin{aligned}-\int_{\Omega_0} \frac{1}{2}\left[(u_t)^2 + |\nabla u|^2\right] dx + \int_{\Omega_t} \frac{1}{2}\left[(u_t)^2 + |\nabla u|^2\right] dx \\ + \int_{\Sigma_T} (-u_t \nabla u)\, \nu_x + \frac{1}{2}\left((u_t)^2 + |\nabla u|^2\right) \nu_t d\sigma.\end{aligned}$$

Then, we infer that:

$$\begin{aligned}\frac{1}{2}\int_{\Omega_t} (u_t)^2 + |\nabla u|^2 \, dx = \frac{1}{2}\int_{\Omega_0} (u_t)^2 + |\nabla u|^2 \, dx \\ - \int_{\Sigma_T} (-u_t \nabla u)\, \nu_x + \frac{1}{2}\left((u_t)^2 + |\nabla u|^2\right) \nu_t d\sigma.\end{aligned}$$

So,

$$E(t) = E(0) - \int_{\Sigma_T} (-u_t \nabla u)\, \nu_x + \frac{1}{2}\left((u_t)^2 + |\nabla u|^2\right) \nu_t d\sigma$$

by substitution of (2.2)and simplifying, it gives:

$$E(t) = E(0) + \frac{1}{2}\int_{\Sigma_T} \left(\frac{\partial u}{\partial \nu}\right)^2 \nu_t \left(|\nu_x|^2 - (\nu_t)^2\right) d\sigma$$

then:

$$E(t) - E(0) = \frac{1}{2}\int_{\Sigma_T} \left(\frac{\partial u}{\partial \nu}\right)^2 \nu_t \left(|\nu_x|^2 - (\nu_t)^2\right) d\sigma$$

and here $|\nu_x|^2 - |\nu_t|^2 \geq 0$, because we supposed that $|\nu_x| \geq |\nu_t|$, and ν_t is negative when the domain is expanding. So, the sign of the integral over Σ_T is negative, hence the energy is decaying in time. In particular, we have

$$E\left(t\right) \leq E\left(0\right), \forall t \geq 0.$$

Lemma 2.1 *Under the assumptions (2.1),(H1) and (2.4), the energy $E\left(t\right)$ of the solution of Problem (2.3) is deacreasing in time.*

Proof. Rewriting the above arguments between arbitrary values of time t_1 and $t_2, t_1 \geq t_2 \geq 0$, we obtain

$$E\left(t_2\right) - E\left(t_1\right) = \frac{1}{2}\int_{\Sigma_{t_1,t_2}} \left(\frac{\partial u}{\partial \nu}\right)^2 \nu_t \left(\left(\nu_x\right)^2 - \left(\nu_t\right)^2\right) d\sigma \leq 0,$$

where $\Sigma_{t_2,t_1} = \bigcup_{t=t_1}^{t=t_2} \partial\Omega_s \times \{s\}$. ■

Corollary 2.2 *Under the assumptions (2.1),(H1) and (2.4), the solution of Problem (2.3) is unique.*

Proof. Let us suppose that the problem admit two different solutions u_1 and u_2. Then,

$$U = u_1 - u_2$$

also solves the wave equation (2.3) with zero initial condions. This means that the initial energy of U is $E_U\left(0\right) = 0$. Since $E_U\left(t\right)$ is decaying, then

$$0 \leq E_U\left(t\right) \leq E_U\left(0\right) = 0, \forall t \geq 0.$$

hence,

$$U\left(t, x\right) = 0, \forall t \geq 0 \text{ and } x \in \Omega_t.$$

This gives the uniqueness of the solution. ■

2.3 Penalization method

2.3.1 Penalized problem

We use the Galerkin approximation method when the domain Ω is fixed, by considering an increasing sequence of linear subspace

$$V_m = span\left\{w_m\right\}_{m\geq 0} \subset H_0^1(\Omega)$$

since the space $H_0^1(\Omega)$ is separable. By supposing the solution as a sequence

$$u_m = \sum_{i=0}^{i=m} w_i g(t),$$

then we show that

$$u_m \longrightarrow u \ \text{ in } H_0^1(\Omega), \text{as } m \longrightarrow +\infty.$$

But in the case of moving domain Ω_t, the basis ω_i dependes also on time.i.e. $\omega_i = w_i\,(x,t)$ and the precedent passage to the limit is more difficult to justify.

To overcome the above difficulty, we include all of the domains Ω_t in a fixed domain D, i.e.

$$\forall t \in (0,T) : \Omega_t \subset D.$$

and we assume that the solution u verifies the next property

$$\text{If } u \in H_0^1(D) \text{ and that } u = 0, \text{ a.e. in } D - \Omega_t, \text{ then } u \in H_0^1(\Omega_t) \text{ for } t \in [0,T] \tag{H2}$$

By adding a dumping viscus term, as in [7], we introduce the following prnalized problem

$$\begin{cases} u_{tt}^{\varepsilon} - \Delta u^{\varepsilon} + \frac{\chi_{\Omega_t^c}}{\varepsilon} u_t^{\varepsilon} = 0, & \text{in } D \times (0,T) \\ u^{\varepsilon} = 0, & \text{on } \partial D \times (0,T) \\ u^{\varepsilon}(0) = \tilde{u}_0, \ u_t^{\varepsilon}(0) = \tilde{u}_1, & \text{in } \Omega_0 \end{cases} \tag{2.9}$$

where $\varepsilon > 0$ is a small parameter that will tends to 0 and the initial conditions $\tilde{u_1}$ and $\tilde{u_0}$ are the extension in D by zero of u_1 and u_0 respectively, i.e.

$$\tilde{u}_0 = \begin{cases} u_0, & t \in \Omega_t, \\ 0, & t \notin \Omega_t, \end{cases}, \quad \tilde{u}_1 = \begin{cases} u_1, & t \in \Omega_t, \\ 0, & t \notin \Omega_t. \end{cases} \tag{2.10}$$

The new problem is already considered in section (1.4), and the uniqueness of the solution is ensured by Theorem (1.27) .

2.3.2 A priori estimates

As $\varepsilon \to 0$, we obtain a sequence $(u^{\varepsilon})_{\varepsilon>0}$ of solution for (2.9). Then, we have

$$u_t^{\varepsilon}\left(u_{tt}^{\varepsilon} - \Delta u^{\varepsilon} + \frac{\chi_{\Omega_t^c}}{\varepsilon} u_t^{\varepsilon}\right) = u_t^{\varepsilon} u_{tt}^{\varepsilon} - u_t^{\varepsilon}\Delta u^{\varepsilon} + \frac{\chi_{\Omega_t^c}}{\varepsilon}\left(u_t^{\varepsilon}\right)^2. \tag{2.11}$$

As we know:

$$-u_t^{\varepsilon}\Delta u^{\varepsilon} = \frac{1}{2}\frac{d}{dt}\left|\nabla u^{\varepsilon}\right|^2 - \operatorname{div}\left(u_t^{\varepsilon}\nabla u^{\varepsilon}\right) \tag{2.12}$$

and:

$$u_t^{\varepsilon}\left(u_{tt}^{\varepsilon}\right) = \frac{1}{2}\frac{d}{dt}\left(u_t^{\varepsilon}\right)^2. \tag{2.13}$$

After a substitution of (2.12) and (2.13) in (2.11), we get:

$$\frac{1}{2}\int_{D\times(0,t)}\frac{d}{dt}\left[(u_t^\varepsilon)^2+|\nabla u^\varepsilon|^2\right]-\operatorname{div}(u_t^\varepsilon\nabla u^\varepsilon)+\frac{\chi_{\Omega_t^c}}{\varepsilon}(u_t^\varepsilon)^2dxds$$

$$=\int_{D\times(0,t)}\operatorname{div}\begin{pmatrix}\frac{1}{2}\left[(u_t^\varepsilon)^2+|\nabla u^\varepsilon|^2\right]\\ -u_t^\varepsilon\nabla u^\varepsilon\end{pmatrix}dxds+\frac{1}{\varepsilon}\int_0^t\int_{\Omega_t^c}(u_t^\varepsilon)^2\,dxds.$$

Thanks to Gauss's theorem, this is equal to

$$-\frac{1}{2}\int_{\Omega_0}(u_t^\varepsilon)^2+|\nabla u^\varepsilon|^2\,dx+\frac{1}{2}\int_{\Omega_t}(u_t^\varepsilon)^2+|\nabla u^\varepsilon|^2\,dx$$

$$+\int_{\partial D}(-u_t^\varepsilon\nabla u^\varepsilon)\,\nu_x+\frac{1}{2}\left((u_t^\varepsilon)^2+|\nabla u^\varepsilon|^2\right)\nu_t d\sigma+\frac{1}{\varepsilon}\int_0^t\int_{\Omega_t^c}(u_t^\varepsilon)^2\,dxds.$$

Then, since $u_t^\varepsilon(t)=0$ and $\nu_t=0$ on ∂D, we infer that:

$$\frac{1}{2}\int_{\Omega_t}(u_t^\varepsilon)^2+|\nabla u^\varepsilon|^2\,dx=\frac{1}{2}\int_{\Omega_0}(u_1)^2+|\nabla u^\varepsilon|^2\,dx-\frac{1}{\varepsilon}\int_0^t\int_{\Omega_t^c}(u_t^\varepsilon)^2\,dxds.$$

We denote the energy of the solution by

$$E^\varepsilon(t)=\frac{1}{2}\int_{\Omega_t}(u_t^\varepsilon)^2+|\nabla u^\varepsilon|^2\,dx.$$

So,

$$E^\varepsilon(t)=E^\varepsilon(0)-\frac{1}{\varepsilon}\int_{\Omega_t^c}\int_0^t(u_t^\varepsilon)^2\,dtdx\le E^\varepsilon(0).$$

Thanks to (2.10), we have

$$E^\varepsilon(0)=\int_D\frac{1}{2}\left[(\tilde{u}_0)^2+(\nabla\tilde{u}_1)^2\right]dx=\int_D\frac{1}{2}\left[(u_0)^2+(\nabla u_1)^2\right]dx=E(0),$$

Then

$$E^\varepsilon(t)\le E(0),\ \text{ for } t\in[0,T] \tag{2.14}$$

and $\forall\varepsilon>0$:

$$\max_{0\le t\le T}E^\varepsilon(t)\ \le\ E(0), \tag{2.15}$$

$$\frac{1}{\varepsilon}\int_0^T\int_{\Omega_t^c}(u_t^\varepsilon)^2dtdx\ \le\ E(0). \tag{2.16}$$

From (2.15) and (2.16), we obtain:

$$\|u_t^\varepsilon\|_{L^\infty(0,T;L^2(D))},\|\nabla u^\varepsilon\|_{L^\infty(0,T;L^2(D))}\le E(0),$$

where the right hand side is independent of ε.

2.3.3 Passage to the limit

So, we can find a subsequence denoted also by u^{ε}, and the following weak convergences:

$$\begin{cases} u^{\varepsilon} \overset{*}{\rightharpoonup} u, & \text{in } L^{\infty}\left(0,T;H_0^1\left(D\right)\right), \\ \nabla u^{\varepsilon} \overset{*}{\rightharpoonup} \nabla u, & \text{in } L^{\infty}\left(0,T;L^2\left(D\right)\right), \\ u_t^{\varepsilon} \overset{*}{\rightharpoonup} u_t, & \text{in } L^{\infty}\left(0,T;L^2\left(D\right)\right). \end{cases}$$

These convergeces means compenent by compenent convergences. The first and the last convergences implies

$$u^{\varepsilon} \rightharpoonup u, \quad \text{in } L^2\left(0,T;L^2\left(D\right)\right),$$
$$\nabla_{x,t} u^{\varepsilon} \rightharpoonup \nabla_{x,t} u, \quad \text{in } L^2\left(0,T;L^2\left(D\right)\right).$$

Due to Fubini's theorem, we can consider that $L^2\left(0,T;L^2\left(D\right)\right) = L^2\left(D\times(0,T)\right)$, we infer that

$$u^{\varepsilon} \rightharpoonup u \ , \ \text{in } H^1\left(D\times(0,T)\right)$$

and because of the compact embedding

$$H^1\left(D\times(0,T)\right) \hookrightarrow_{compact} L^2\left(D\times(0,T)\right),$$

we deduce that

$$u^{\varepsilon} \longrightarrow u \ \text{ strongly in } L^2\left(D\times(0,T)\right).$$

Thanks to Theorem (1.10), we have –up to a new subsequence–

$$u^{\varepsilon} \longrightarrow u, \quad \text{a.e. } D\times(0,T).$$

Now, it remains to characterize the limit element u .
Going back to (2.16), we get

$$\int_{D\times(0,T)} \left(\chi_{\Omega_t^c} u_t^{\varepsilon}\right)^2 dx \le \varepsilon E\left(0\right) \to 0,$$

whene $\varepsilon \to 0$. The viscous dumping term will vanish on the region Ω_t^c:

$$\chi_{\Omega_t^c} u_t^{\varepsilon} \longrightarrow 0, \quad \text{in } L^2\left(D\times(0,T)\right)$$

i.e.

$$u_t = 0, \ \text{a.e. in } D\times(0,T)\setminus Q_T.$$

According to [4, Theorem 2.1], we deduce that

$$u_t(x,t) = 0 \text{ a.e. in } D\setminus\Omega_t, \quad \text{for almost all } t\in(0,T).$$

which gives

$$u(x,t) = c \ \text{ a.e. in } D\setminus\Omega_t, \text{for a.e.} t\in(0,T).$$

and, as we have supposed in (2.10):

$$u(x,0) = \tilde{u}_0(x) = 0 \text{ in } D\backslash\Omega_0.$$

we can see that

$$u(x,t) = 0 \text{ a.e. in } D\backslash\Omega_t \text{ for a.e.} t \in (0,T). \tag{2.17}$$

Since $u\,(t) \in H_0^1(D)$, we deduce for the property (H2) that

$$u\,(t) \in H_0^1(\Omega_t), \text{ a.e. } t \in (0,T)\,.$$

Consequently,

$$u \in L^2\left(0,T;H_0^1\left(\Omega_t\right)\right), u_t \in L^2\left(0,T;L^2\left(\Omega_t\right)\right).$$

Going back to (2.14), we deduce that

$$u \in L^\infty(0,T;H_0^1(\Omega_t)),\ u_t \in L^\infty(0,T;L^2(\Omega_t)).$$

To summurize the results of this chapter, we have the following existence ad uniqueness result.

Theorem 2.3 *Assume that*

$$u_0 \in H_0^1(\Omega_0),\ \textit{and} u_1 \in L^2(\Omega_0)$$

Then there exists a unique $u \in L^2(0,T;H_0^1(\Omega_t))$*, with* $u_t \in L^2(0,T;L^2(\Omega_t))$ *that solves the problem*

$$\begin{cases} u_{tt} - \Delta u = 0, & \textit{in } Q_T, \\ u = 0, & \textit{on } \Sigma_T, \\ u(0) = u_0, u_t(0) = u_1 & \textit{in } \Omega_0 \end{cases}$$

in the weak sense

$$\int_{\Omega_t} u_{tt}v + \nabla u \cdot \nabla v dx = 0,\ \forall v \in H_0^1(\Omega_t) \textit{ and a.e. } t \in (0,T)$$

and we have the estimate

$$esssup_{0\le t\le T}(\|u\|_{H_0^1(\Omega_t)} + \|u_t\|_{L^2(\Omega_t)}) \le C(\|u_0\|_{H_0^1(\Omega_0)} + \|u_1\|_{L^2(\Omega_0)}).$$

Chapter 3

Numerical analysis of the problem

In this chapter we are going to use Finite difference method, for more details about this method we can see [6, 8].

3.1 Finite difference scheme

Let us consider the penalized problem of the precedent chapter in two dimensional domain

$$\begin{cases} u_{tt} - \Delta u + \frac{\chi_{\Omega_t^c}}{\varepsilon} u_t = 0, & \text{in } D \times (0,T) \\ u = 0, & \text{on } \partial D \times (0,T) \\ u(0) = f, u_t(0) = g, & \text{in } \Omega_0 \end{cases}$$

The discretization of the domain D gives

$$x_i = i\Delta_x, \text{and } y_j = j\Delta_y \text{ and } i,j = 1,2,\cdots$$

with Δ_x and Δ_y are the steps of descretization in the axis x and y respectively. The discretization of the time, gives

$$t_k = k\Delta_t, \text{with } \Delta_t \text{ is the step of time and } k = 1,2,\cdots.$$

since the domain is moving in time,we consider that

$$\chi_{\Omega_t^c}(x_i, y_j) \approx \chi_{\Omega_k}^{i,j},\ i,j,k = 1,2,\cdots$$

Recall that since Ω_k is increasing, we have:

$$1 \geq \chi_{\Omega_{k+1}}^{i,j} \geq \chi_{\Omega_k}^{i,j} \geq 0,\ i,j,k = 1,2,\cdots$$

after that we can notice

$$u(x_i, y_j, t_k) = u_{i,j}^k$$

and the following expressions

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u^k_{i+1,j} - 2u^k_{i,j} + u^k_{i-1,j}}{\Delta_x^2}$$
$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u^k_{i,j+1} - 2u^k_{i,j} + u^k_{i,j-1}}{\Delta_y^2}$$
$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u^{k+1}_{i,j} - 2u^k_{i,j} + u^{k-1}_{i,j}}{\Delta_t^2}$$
$$\frac{\partial u}{\partial t} \approx \frac{u^k_{i,j} - u^{k-1}_{i,j}}{\Delta_t}$$

then after a substitution in the equation we obtain

$$\frac{u^{k+1}_{i,j} - 2u^k_{i,j} + u^{k-1}_{i,j}}{\Delta_t^2} - \frac{u^k_{i+1,j} - 2u^k_{i,j} + u^k_{i-1,j}}{\Delta_x^2} - \frac{u^k_{i,j+1} - 2u^k_{i,j} + u^k_{i,j-1}}{\Delta_y^2} + \frac{\chi^{i,j}_{\Omega_k}}{\varepsilon}\frac{u^k_{i,j} - u^{k-1}_{i,j}}{\Delta_t} = 0,$$

choosing $\Delta_x = \Delta_y$ then we get

$$\frac{u^{k+1}_{i,j} - 2u^k_{i,j} + u^{k-1}_{i,j}}{\Delta_t^2} = \frac{u^k_{i+1,j} + u^k_{i-1,j} + u^k_{i,j+1} + u^k_{i,j-1} - 4u^k_{i,j}}{\Delta_x^2} - \frac{\chi^{i,j}_{\Omega_k}}{\varepsilon}\frac{u^k_{i,j} - u^{k-1}_{i,j}}{\Delta_t}.$$

Thus

$$u^{k+1}_{i,j} = 2u^k_{i,j} - u^{k-1}_{i,j} + \frac{\Delta_t^2}{\Delta_x^2}\left(u^k_{i+1,j} + u^k_{i-1,j} + u^k_{i,j+1} + u^k_{i,j-1} - 4u^k_{i,j}\right) - \frac{\chi^{i,j}_{\Omega_k}}{\varepsilon}\Delta_t\left(u^k_{i,j} - u^{k-1}_{i,j}\right)$$

Putting $\lambda = \left(\frac{\Delta_t}{\Delta_x}\right)^2$, then the final scheme is given by

$$u^{k+1}_{i,j} = 2u^k_{i,j} - u^{k-1}_{i,j} + \lambda\left(u^k_{i+1,j} + u^k_{i-1,j} + u^k_{i,j+1} + u^k_{i,j-1} - 4u^k_{i,j}\right) - \frac{\chi^{i,j}_{\Omega_k}}{\varepsilon}\Delta_t\left(u^k_{i,j} - u^{k-1}_{i,j}\right)$$

Now, for the stability of the final scheme we need to write it as follow

$$u^{k+1}_{i,j} = -u^{k-1}_{i,j} + \lambda\left(u^k_{i+1,j}\right) + \lambda\left(u^k_{i-1,j}\right) + \lambda\left(u^k_{i,j+1}\right) + \lambda\left(u^k_{i,j-1}\right) + (2-4\lambda)\left(u^k_{i,j}\right) - \frac{\chi^{i,j}_{\Omega_k}}{\varepsilon}\Delta_t\left(u^k_{i,j} - u^{k-1}_{i,j}\right)$$

By the convex combination method it is clear that the sum of all factors is 1. After that we should to have

$$2 - 4\lambda > 0$$

i.e.

$$\lambda < \frac{1}{2}$$

then,

$$\frac{\Delta_x}{\sqrt{2}} > \Delta_t$$

which is called the CFL condition.
At the final, we get

$$u_{i,j}^{k+1} = \begin{cases} 2u_{i,j}^k - u_{i,j}^{k-1} + \lambda\left(u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k - 4u_{i,j}^k\right) & ,(x_i, y_j) \notin \Omega_k^c \\ 2u_{i,j}^k - u_{i,j}^{k-1} + \lambda\left(u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k - 4u_{i,j}^k\right) & \\ -\frac{1}{\varepsilon}\Delta_t\left(u_{i,j}^k - u_{i,j}^{k-1}\right) & ,(x_i, y_j) \in \Omega_k^c \end{cases}$$

with

- The initial conditions

$$u_{i,j}^0 = f_{i,j} \text{ and } u_{i,j}^{-1} = u_{i,j}^0 - 2\Delta_t g_{i,j}$$

- The boundary conditions

$$u_{0,j}^k = u_{n,j}^k = u_{0,j}^k = u_{m,j}^k = 0$$

3.2 Numerical representation of the solution

In this section we are going to see how the wave moves when the domain increases, by tracing the graph of the solution u. Let us consider the following problem.
Let Ω_t defined as below

$$\Omega_t = \{(x, y) \in \mathbb{R} : a_1(t) \leq x \leq a(t), b_1(t) \leq y \leq (t)\}$$

And, we have taken

$$D = \{(x, y) \in \mathbb{R} : a_1(T) \leq x \leq a(T), b_1(T) \leq y \leq b(T)\}$$

With T is the maximal time.

3.2.1 Algorithm

Algorithm: solution's representation

Variables:

T, nx, ny, c, n, n1, n2, n3, np1, np2, np3, np4: integer

Lx, dx, Ly, dy, cfl, t, dt, eps, l, l1, l2, l3, p1, p2, p3, p4: double

wn, wnm1, wnp1, M: matrix

x ,y: vector

a, a1, b, b1, f, g: function

Begin:

- Maximal time
  T $\longleftarrow$ 10

- Boundaries functions
  a $\longleftarrow$ $a(t)$
  a1 $\longleftarrow$ $a1(t)$
  b $\longleftarrow$ $b(t)$
  b1 $\longleftarrow$ $b1(t)$

- Initial data
  f $\longleftarrow$ $f(x,y)$
  g $\longleftarrow$ $g(x,y)$

- Variables of space
  Lx $\longleftarrow$ $a(T) - a1(T)$
  dx $\longleftarrow$ 0.1
  nx $\longleftarrow$ $ceil(Lx/dx)$
  x $\longleftarrow$ linspace(a1(T),a(T),nx)
  Ly $\longleftarrow$ $b(T) - b1(T)$
  dy $\longleftarrow$ dx
  ny $\longleftarrow$ $ceil(Ly/dy)$
  y $\longleftarrow$ linspace(b1(T),b(T),ny)

- Parameters
  cfl $\longleftarrow$ 0.5
  c $\longleftarrow$ 1
  dt $\longleftarrow$ $(cfl * dx)/(c * sqrt(2))$
  eps $\longleftarrow$ 0.05

- Matrices

  wn $\longleftarrow$ zeros(nx,ny)

  wnm1 $\longleftarrow$ wn

  wnp1 $\longleftarrow$ wn

  M $\longleftarrow$ ones(nx,ny)

- Initial data

  L $\longleftarrow$ $a(T) - a(0)$

  n $\longleftarrow$ $ceil(L/dx)$

  L1 $\longleftarrow$ $b(T) - b(0)$

  n1 $\longleftarrow$ $ceil(L1/dx)$

  L2 $\longleftarrow$ $a1(0) - a1(T)$

  n2 $\longleftarrow$ $ceil(L2/dx)$

  L3 $\longleftarrow$ $b1(0) - b1(T)$

  n3 $\longleftarrow$ $ceil(L1/dx)$

  for i starts from $n2 + 1$ to $nx - n - 1$:

  for j start from $n3 + 1$ to $ny - n1 - 1$:

  wn(i,j) $\longleftarrow$ f(x(i),y(j))

  end of for

  end of for

  for i starts from $n2 + 1$ to $nx - n - 1$:

  for j starts from $n3 + 1$ to $ny - n1 - 1$:

  wnm1(i,j) $\longleftarrow$ $-dt * g(x(i), y(j)) + wn(i, j)$

  end of for

  end of for

  t $\longleftarrow$ 0

  while (t < T)

  t $\longleftarrow$ $t + dt$

- Current lengths

  p1 $\longleftarrow$ $a(T) - a(t)$

  np1 $\longleftarrow$ $ceil(p1/dx)$

  p2 $\longleftarrow$ $b(T) - b(t)$

  np2 $\longleftarrow$ $ceil(p2/dy)$

  p3 $\longleftarrow$ $a1(t) - a1(T)$

  np3 $\longleftarrow$ $ceil(p3/dx)$

p4 $\longleftarrow$ $b1(t) - b1(T)$

np4 $\longleftarrow$ $ceil(p2/dy)$

- Boundaries condition

  wn$(:,[1end])$ $\longleftarrow$ 0

  wn$([1end],:)$ $\longleftarrow$ 0

- calculate of Wn+1

  for i starts from 2 to $nx - 1$

  for j starts from 2 to $ny - 1$

  if(($np3 < i$ and $i < nx - np1$)and ($j < ny - np2$ and $j > np4$))

  M(i,j) $\longleftarrow$ 0

  else

  M(i,j) $\longleftarrow$ 1

  end of if

  wnp1(i,j) $\longleftarrow$ $2 * wn(i,j) - wnm1(i,j) + CFL^2 * (wn(i+1,j) + wn(i,j+1) - 4 * wn(i,j) + wn(i-1,j) + wn(i,j-1)) - (M(i,j) * dt/eps) * (wn(i,j) - wnm1(i,j))$

  end of for

  end of for

  wnm1 $\longleftarrow$ wn

  wn $\longleftarrow$ wnp1

  end of while.

End.

3.2.2 Graphic representation

In the graphic representation we are going to take some examples for better viewing the manner of the solution's variation when the domain expands.

Example 1: A square membrane with two moving sides

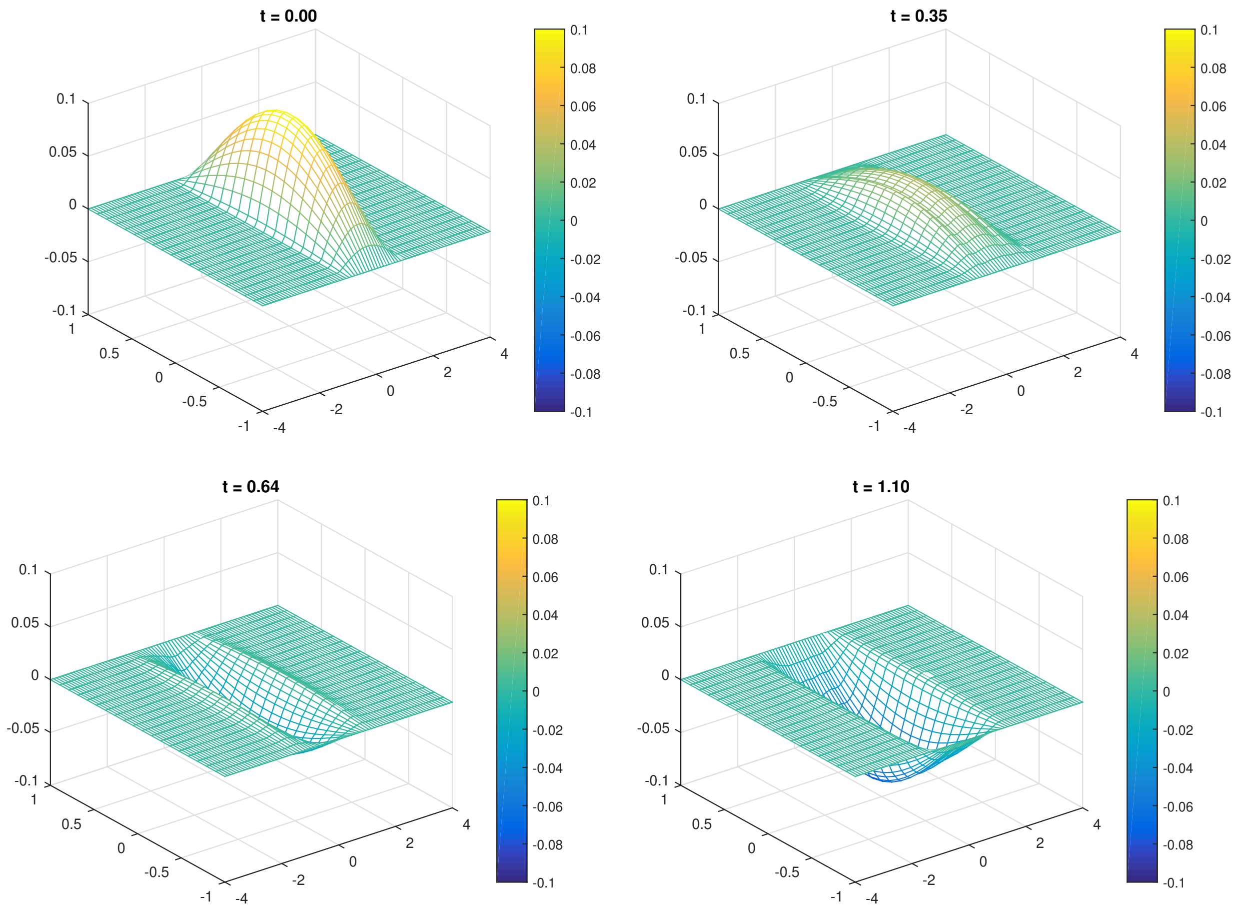
By taking

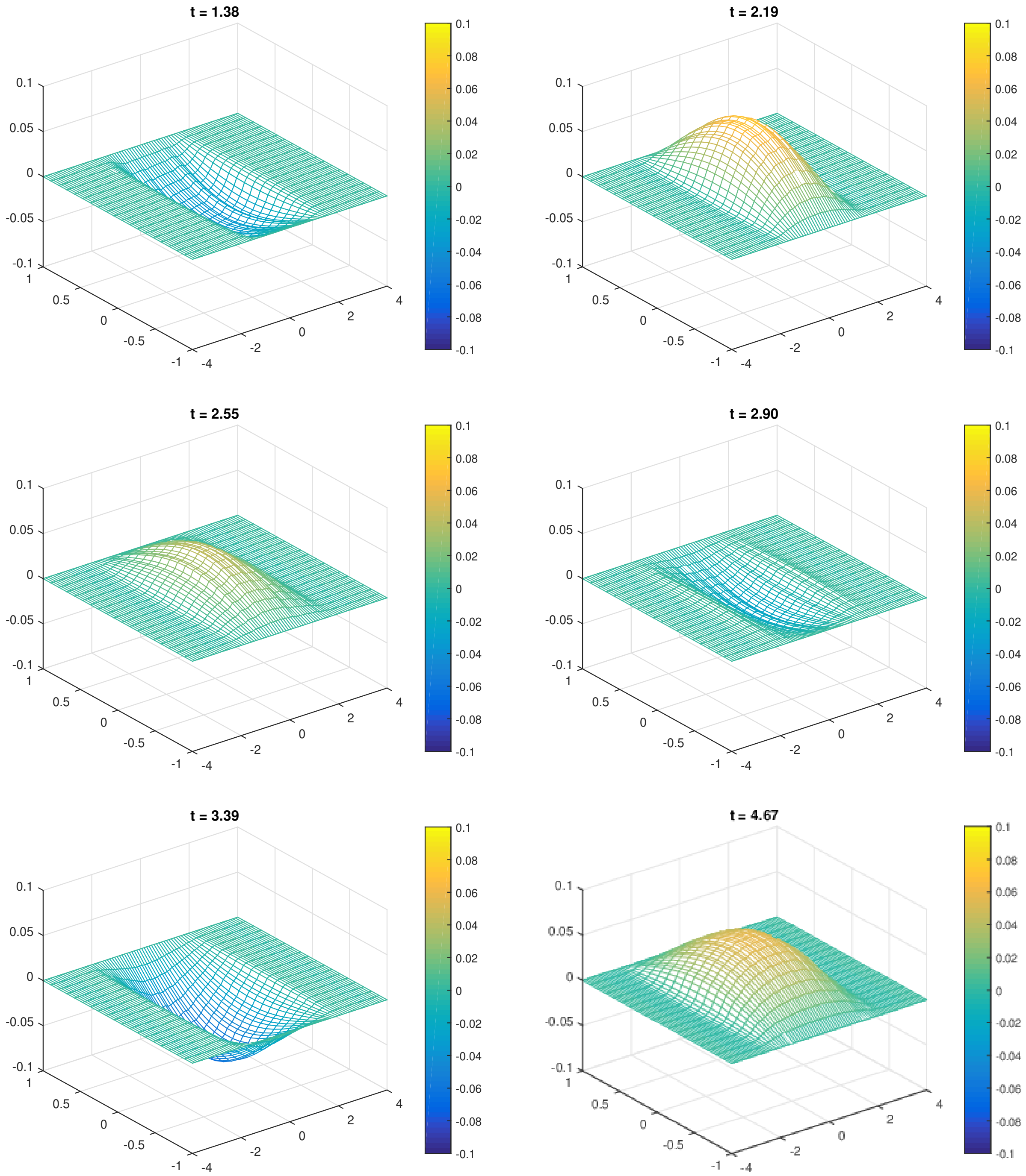
$$a(t) = 1 + 0.3 * t \quad a_1(t) = -1 - 0.3 * t \quad b(t) = 1 \quad b_1(t) = -1$$

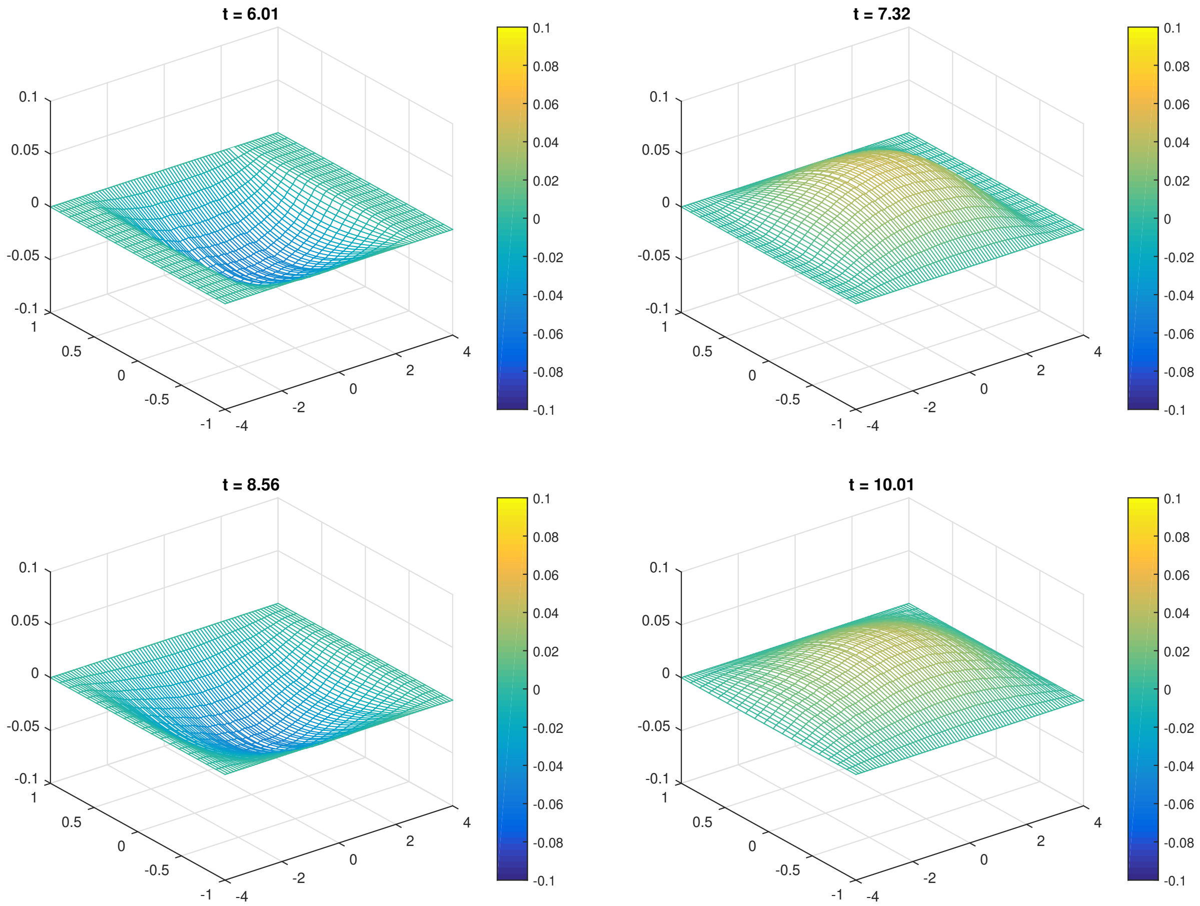
With

$$f(x, y) = \frac{\cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y)}{10} \quad \text{and, } g(x, y) = 0$$

and $T = 10$. After running the above algorithm we can obtain the next results for different times t.




t = 1.38
t = 2.19
t = 2.55
t = 2.90
t = 3.39
t = 4.67
0.1
0.05
0
-0.05
-0.1
1
0.5
0
-0.5
-1
-4
-2
0
2
4
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1


t = 6.01
t = 7.32
t = 8.56
t = 10.01
0.1
0.05
0
-0.05
-0.1
1
0.5
0
-0.5
-1
-4
-2
0
2
4
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1

Example 2: A square membrane with four moving sides

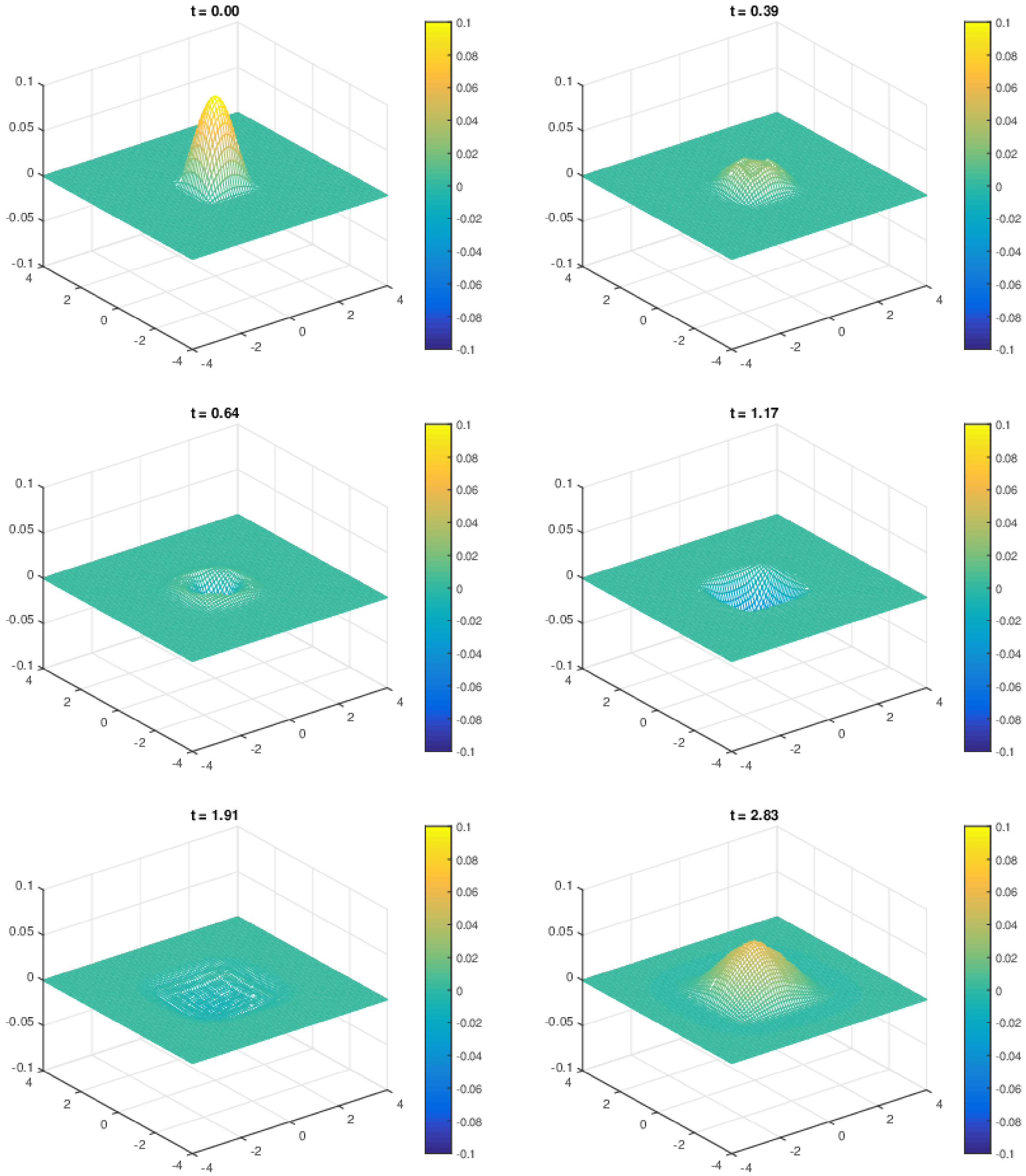
By taking

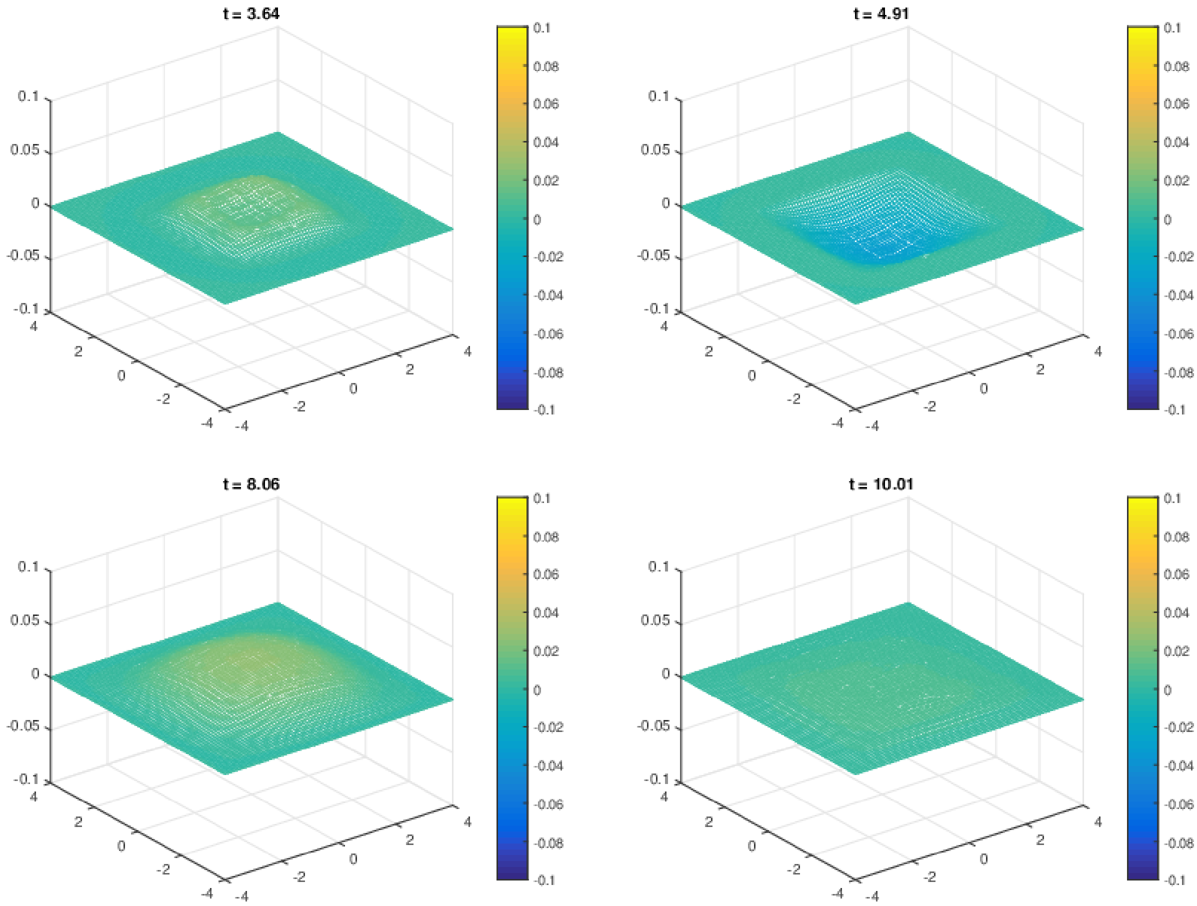
$$a(t) = 1 + 0.3 * t \quad a_1(t) = -1 - 0.3 * t \quad b(t) = 1 + 0.3 * t \quad b_1(t) = -1 - 0.3 * t$$

With the same initial data.

$$f(x, y) = \frac{\cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y)}{10} \quad \text{and, } g(x, y) = 0$$

and $T = 10$. we obtain the following graphes.




t = 3.64
t = 4.91
t = 8.06
t = 10.01

Example 3: A circular membrane with variable rayon

By taking Ω_t as follow

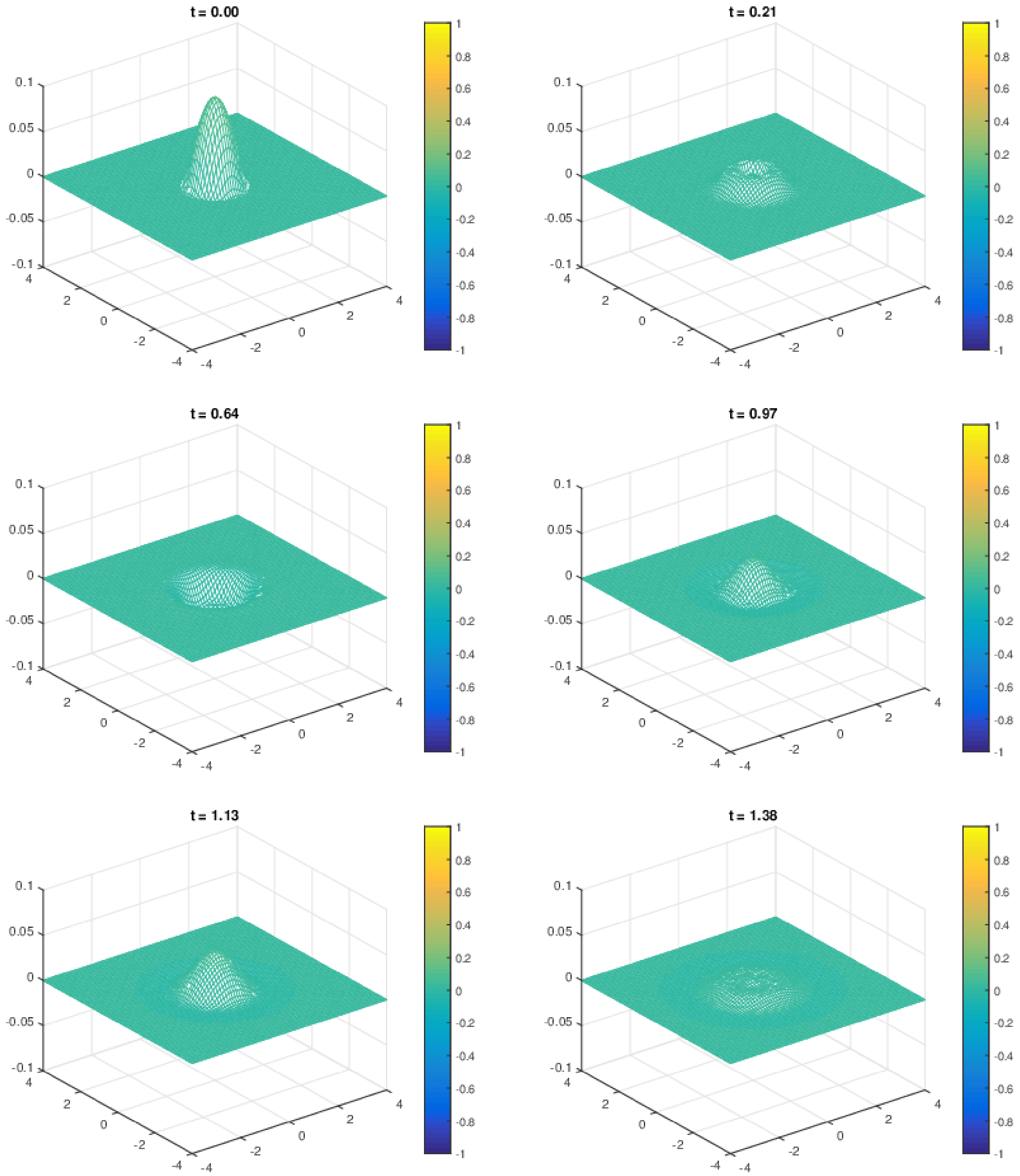
$$\Omega_t = \left\{(x,y) \in \mathbb{R}^2 \ : x^2 + y^2 \leq r(t)\right\}$$

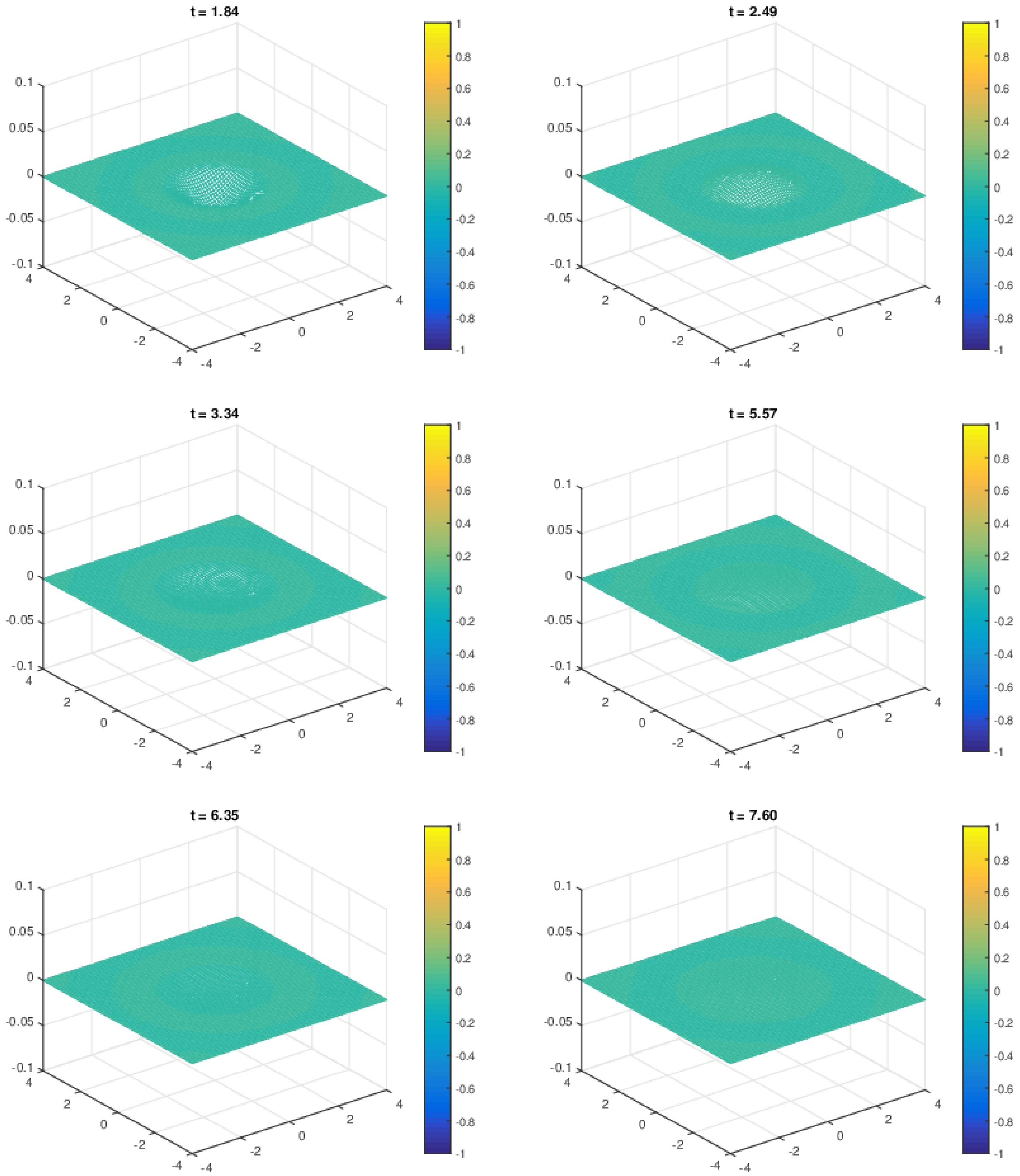
Where r is a continuous increasing function dependent on t.

When we take $r(t) = 1 + 0.3 * t$, with the initial data:

$$f(x,y) = \frac{\cos(\sqrt{x^2+y^2}\frac{\pi}{2})}{10} \quad \text{and, } g(x,y) = 0$$

and $T = 10$ we obtain the next results.




t = 1.84
t = 2.49
t = 3.34
t = 5.57
t = 6.35
t = 7.60

Appendix A

Analytics and geometrics notions

- Ω_t is a bounded open set of $\mathbb{R}^2$ with moving boundaries
- Non cylindrical domain Q $(t_1, t_2) = \bigcup_{t=t_1}^{t=t_2} \Omega_t \times \{t\}$
- when $t_1 = 0$ so: Q $(T) = \bigcup_{t=0}^{t=T} \Omega_t \times \{t\}$
- The lateral boundary $\Sigma(t_1, t_2) = \bigcup_{t=t_1}^{t=t_2} \partial\Omega_t \times \{t\}$
- when $t_1 = 0$ so: $\Sigma(T) = \bigcup_{t=0}^{t=T} \partial\Omega_t \times \{t\}$
- $x = (x_1, x_2) \in \mathbb{R}^2$ such that: $|x|^2 = \sum_{i=1}^{i=2} x_i^2$
- $\nabla_x = (\partial_{x_1}, \partial_{x_2})$ then $\nabla_{x,t} = (\nabla_x, \partial_t)$
- $\Delta = \sum_{i=1}^{i=2} \partial_{x_i}^2$
- $\nu = (\nu_x, \nu_t)$ is the unit outward normal vector at $(x, t) \in \Sigma$ with assuming that: $|\nu_t|^2 \leq |\nu_x|^2$

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ملخص:

ندرس معادلة الموجة ثنائية الأبعاد في ميدان غير أسطواني، أي ميدان متغير مع الزمن. أولا نستخدم طريقة إعاقة تعتمد على معامل صغيرε. نتحصل بذلك على مسألة لمعادلة الأمواج في مجال أسطواني، مع اضافة جزء تخميد كعامل إعاقة .إذا كان الميدان يتوسع بمرور الزمن، فإننا نبين أن حل المسألة المعاقة يتقارب إلى حل المشكلة الأصلية، لما $\varepsilon \longrightarrow 0$. للحصول على تقريب عددي للحلول، نقوم بدمج طريقة الإعاقة هذه مع طريقة الفروق المنتهية.

كلمات المفتاحية: معادلة الموجة، الميادين غير الأسطوانية، طريقة الإعاقة، الفروق المنتهية.

Abstract

We consider the 2-dimensional wave equation in non-cylindrical domains, i.e. domains depending on time. We first use a penalization method depending on a parameter ε. The obtained problem is a wave equation, posed in a cylindrical domain, with a penalized weak damping term. If the domain is expanding with time, we show that the solution of the penalized problem converges to the solution of the original problem, as $\varepsilon \to 0$. Finally, to obtain a numerical approximation of solutions, we combine this penalization method with an explicit finite differences approach.

keywords: Wave equation, non-cylindrical domains, penalization method, finite differences.

Résumé:

Nous considérons l'équation d'onde à 2 dimensions dans un domaine non cylindrique, c'est-à-dire un domaine dépendant du temps. Nous utilisons d'abord une méthode de pénalisation dépendant d'un paramètre ε. Le problème obtenu est une équation d'onde, posée dans un domaine cylindrique, avec un terme d'amortissement faible pénalisé. Si le domaine s'étend avec le temps, nous montrons que la solution du problème pénalisé converge vers la solution du problème original, comme $\varepsilon \to 0$. Enfin, pour obtenir une approximation numérique des solutions, nous combinons cette méthode de pénalisation avec une approximation par la methode de différences finies.

mots-clés : Equation des ondes, domaines non cylindriques, méthode de pénalisation, différences finies.