PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf university of M'sila
Faculty of Mathematics and Informatics
Department of Mathematics

Master memory

Field : Mathematics and Informatics
Branch : Mathematics
Option : Functional Analysis

Theme

INJECTIVE POLYNOMIAL IDEAL AND APPLICATION

Presented by :
BOUSSAG Yassmina

Publicly supported on : 28/06/2021.

In front of the jury composed of :

MEZRAG Lahcène	prof,	University of M'sila	**President**.
ALOUANI Ahlem	M.C.B,	University of M'sila	**Supervisor.**
ACHOUR Dahmane	Prof,	University of M'sila	**Examiner.**

University year 2020/2021

Dedications

In the name of ALLAH the most gracious the most merciful

I dedicate this work to :

The deare

their virtue

My dear sisters and their children
All my friends and family

All the student

2020/2021.

Acknowledgments

First, I am so grateful to " **Allah** " for giving us protection and ability to start and finish this work.

I would like to express my most sincere thanks to my supervisor **Ahlem Alouani** for his precious advice and guidance.

I am also grateful to the president of jury professor **Lahcène Mezrag** and the examiner **Achour Dahmane** .

I do not forget to thank my teachers of our department of mathematics at M'sila University for their unceasing encouragement, support and help.

Notations

$\mathbb{K}$	The field of real or complex numbers.
$\mathbb{R}$	The reel space.
B_X	The closed unit ball of X.
$\overset{\circ}{B}$	The open unit ball of X.
$\mathcal{L}(E, F)$	The set of all bonded linear operators from E into F.
$\mathcal{L}_f$	The set of all finite rank linear operators.
$\mathcal{I}$	The deal of all linear operator.
$\mathcal{P}(^mE; F)$	The space of all continuous m-homogeneous polynomials from E into F.
$\mathcal{P}_{\mathcal{F}}$	The finite rank polynomials.
$\mathcal{P}_{\mathcal{K}}$	The compacte polynomials.
$\mathcal{P}_{\mathcal{W}}$	The weakly compacte polynomials.
$\mathcal{K}$	The closed ideal of compact operators.
$\mathcal{Q}$	A polynomial ideal.

Introduction

The theory of linear operators ideals between normed (Banach) spaces have been developed by Albert Pietech in the excellent monography [23], this theory has proved to be a powerful tool for the investigation and classification of linear operators. Nowadays it has become a usual tool for analyzing some classical problems of the functional analysis as summability of series, but has also become the starting point to understand and solve new problems related to nonlinear operators.

The linear theory has spread to multilinear operators, polynomials, and Lipschitz operators, the first outline of such a multilinear theory was given by A.Pietsch in 1983 see [24]. These new multi-ideals could also be considered to be the starting point for the study of some specific properties of nonlinear operators, several authors see

([1],[2],[3],[5],[12],[13],[19]) have invested in this new area (nonlinear functional analysis), where did they try to transferring linear summability properties to its nonlinear counterpart, also point out that this process is not an obvious task. Certain multi-ideals the generalization is elementary for example, this is the case of complete, compact and weakly compact multilinear operators see ([4],[17]).

In the same path and according to the paper of G.Botelho and L.Torres [9] we present our work which reserved for the class of injective polynomial ideals. Inspired by the fact that injective operators are characterized by a domination property, we investigate the situation in the polynomial case. In order to achieve this goal, the memory was organized into three chapters. In the first chapter, we recall some basic notions, properties, and terminologies needed in the work as linear ideal, ideals of m-homogenous polynomials, and tensor product. In the second chapter, we present and study the injective linear operator ideals. We outline the basic theory of injective polynomials ideals, and give the definition, provide illustrative examples, characterize injective polynomial ideals by means of the injective hull. The main results appear in chapter three, where we answer the following question:

Can injective polynomial ideals be characterized by some related domination property ?

Also, we give a characterization of the injective polynomial duals.

Contents

CHAPTER 1

MULTILINEAR MAPPINGS AND POLYNOMIALS

In this chapter we present a collection of some definitions, properties and basic notions, of operator on Banach spaces, multilinear mappings, ideals of m-homogeneous polynomials and Tensor products of Banach spaces.

1.1 Operators on Banach spaces

We will write $\mathbb{K}$ for the real numbers field $\mathbb{R}$ or the complex numbers field $\mathbb{C}$. The set of all natural numbers $\{0, 1, ...\}$is denoted by $\mathbb{N}$. Along this work the letters E and F denotes Banach spaces with the norm $\|.\|$. The open unit ball of E is denoted by $\overset{\circ}{B}_E$ that is the set $\{x \in E : ||x|| < 1\}$ and the closed unit ball of E is denoted by B_E that is the set $\{x \in E : ||x|| \leq 1\}$. The set of all functionals of a normed space E (that is the continuous linear mapping from E into the scalars) is a Banach space denoted by E^* and called the topological dual of E.
We define the following operators :

1) A linear map $T : E \to F$ is continuous if and only if

$$||T|| = sup\{||T(x)|| : ||x|| \leq \infty, x \in E\} < \infty, \tag{1.1}$$

this value is a norm on the vector space $L(E; F)$ of all linear operators from E into F .
We denoted by $\mathcal{L}(E; F)$ The space of continuous linear operators, endowed by the norme (1.1).

2) A linear operator $T : E \to F$ is invertible if there exist a linear operator noted

$$T^{-1} : F \to E$$

such that, $T^{-1} \circ T = id_E$, and $T \circ T^{-1} = id_F$, where id_E (or id_F) is identity operator on E (or on F).

3) Let $T : E \to F$ be continuous linear operator. Then the continuous linear operator defined by

$$T^* : F^* \to E^*$$

$$T^*(y^*)(x) = y^*T(x)$$

for every $y^* \in F^*$ and $x \in E$ is called the adjoint of T and he have the property

$$||T|| = ||T^*||.$$

linear ideal

An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}(E, F)$ of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components

$$\mathcal{I}(E; F) := \mathcal{L}(E; F) \cap \mathcal{I}$$

satisfy:

(i) $\mathcal{I}(E; F)$ is a linear subspace of $\mathcal{L}(E; F)$ which contains the finite rank operators.

(ii) The ideal property: if $u \in \mathcal{L}(E; Z), v \in \mathcal{I}(Z; K)$ and $w \in \mathcal{L}(K; F)$, then the composition $w \circ v \circ u$ is in $\mathcal{I}(E; F)$.

If $||.||_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies:

(i') $(\mathcal{I}(E; F); ||.||_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces E and F,

(ii') $||id_{\mathbb{K}}||_{\mathcal{I}} = 1$, (iii') If $u \in \mathcal{L}(E;Z), v \in \mathcal{I}(Z;K)$ and $w \in \mathcal{L}(K;F)$,

$$\|w \circ v \circ u\|_{\mathcal{I}} \leq \|w\|\|v\|_{\mathcal{I}}\|u\|,$$

then $(\mathcal{I}; ||.||_{\mathcal{I}})$is called a normed (Banach) operator ideal. Then $||u|| \leq ||u||_{\mathcal{I}}$ for all $u \in \mathcal{I}$.

As an example, the set of all absolutely summing operators for all $1 \leq p < \infty$ (absolutely p-summing or just p-summing if it takes weakly p-summable sequences $(x_i)_{i=1}^{\infty}$ of E to absolutely $p-$summing sequences $(u(x)_i)_{i=1}^{\infty}$ of F) define by :

$$\left(\sum_{i=1}^{n} ||u(x_i)||^p\right)^{\frac{1}{p}} \leq C \sup_{||\xi||_{X^*}\leq 1} \left(\sum_{i=1}^{n} |\xi(x_i)|^p\right)^{\frac{1}{p}} \tag{1.2}$$

for every finite family $(x_i)_{i=1}^{n} \in E$.

is denoted by $\Pi_p(E,F)$, which constitute a Banach ideal under the ideal norm

$$\pi_p(u) := \{C, \text{for all C verifying the inequality(1.2)}\}.$$

Dual of an operator ideal

Definition 1.1. *Let $\mathcal{I}$ is a normed operator ideal. A linear mapping $u \in \mathcal{L}(E,F)$ belongs to $\mathcal{I}^{dual}$ if $u^* \in \mathcal{I}(F^*,E^*)$, where u^* is the adjoint of the operator u. In this case we define*

$$||u||_{\mathcal{I}^{dual}} = ||u^*||_{\mathcal{I}}.$$

Proposition 1.2. *([15], page 114) If $\mathcal{I}$ is a normed (Banach) operator ideal, then $(\mathcal{I}^{dual}; ||.||_{\mathcal{I}^{dual}})$ is as well. This normed (Banach) ideal is called the dual of $\mathcal{I}$.*

Pietsch has shown that the identity mapping from l_1 into l_2 is 2-summing but the adjoint operator is not 2-summing, for this reason the concept of strongly p-summing operators $1 < p \leq \infty$, appears as characterization of absolutely p^*-summing operators (see [10]).
We denoted the class of all strongly p-summing linear operators by $\mathcal{D}_p(E;F)$, therefore

$$\mathcal{D}_p = \Pi_{p^*}^{dual},$$

thus and according to proposition (1.2), $(\mathcal{D}_p, d_p)$ is Banach operator ideal.

1.2 Isomorphisms, injections

Isomorphisms

$U \in \mathcal{L}(E,F)$ is said to be an isomorphism if U admits an inverse $U^{-1} \in \mathcal{L}(E,F)$, or

$$a||x|| \leq ||U(x)|| \leq b||x|| \quad \text{for all} \quad x \in E$$

where a and b are positive constants. Banach spaces E and F are called isomorphic if there exists an isomorphism $U \in \mathcal{L}(E,F)$. Then we write $E \cong F$.

If $||U|| = ||U^{-1}||$, then U is said to be a metric isomorphism.

One refers to a linear map U from E onto F as an isometry if it preserves the norm:

$$||U(x)|| = ||x|| \quad \text{for all} \quad x \in E$$

Injections

An continuous linear operator $J : E \to F$ is said to be an injection (is an isomorphism) if there exists a constant $C > 0$ such that

$$||J(x)|| \geq C||x||$$

for all $x \in E$.

In this case that $||J(x)|| = ||x||$, we use the term metric injection (isometric isomorphism).

For every closed subspace M of E , the canonical embedding J_M^x from M into E has this property.

If there exists an injection from E into F , then we say that F contains an isomorphic copy of E.

A linear operator T is an embedding of E into F if T is an isomorphism onto its image $T(E)$. In this case we say that E embeds in F .

If

$$T : E \to F$$

is an embedding such that

$$||T(x)|| = ||x||$$

for all $x \in E$, then T is said to be an isometric embedding.

A metric injection is a linear operator

$$j : E \to F$$

such that

$$\|j(x)\| = \|x\|$$

for every $x \in E$. The metric injection

$$I_E : E \to l_\infty(B_{E^*}), I_E(x) = (\varphi(x))_{\varphi \in B_{E^*}},$$

is called the canonical metric injection.

1.3 Multilinear mappings and m-homogeneous polynomials

Let $m \in \mathbb{N}$, $E_1, ..., E_m; F$ be Banach spaces. An application $T : E_1 \times ... \times E_m \to F$ is called multilinear (or m-linear) if the mappings

$$\begin{aligned} T_j : E_j &\to F \\ x^j &\to T(x^1, ..., x^j, ..., x^m) \end{aligned}$$

are linear for each $x^k \in E_k, K \neq j$, in other words

$$T(x^1, ..., \lambda x^j + y^j, ..., x^m) = \lambda T(x^1, ..., x^j, ..., x^m) + T(x^1, ..., y^j, ..., x^m)$$

for all $\lambda \in \mathbb{K}$ and $x^j, y^j \in E_j (1 \leq j \leq m)$, we denoted by $L(E_1, ..., E_m; F)$ the space of all applications m-linear T from $E_1 \times ... \times E_m$ into F.

The set $\mathcal{S}$ of all vectors in F of the form $T(x^1, ..., x^m), x^j \in E_j (1 \leq j \leq m)$ is not in general vector subspace of F (see ([14], Section1.1)).

Now we define the following linear operations

$$(S + T)(x^1, ..., x^m) := S(x^1, ..., x^m) + T(x^1, ..., x^m)$$

$$(\lambda T)(x^1, ..., x^m) := \lambda T(x^1, ..., x^m), \lambda \in \mathbb{K}$$

which gives to $L(E_1, ..., E_m; F)$ a structure of a vector space. If $F = \mathbb{K}$, we write $L(E_1, ..., E_m)$.

Definition 1.3. *The multilinear application $T : E_1 \times ... \times E_m \to F$ is continuous if it is continuous as a function between two normed spaces.*

As a consequence of this definition, and the following equality

$$T(x^1, ..., x^m) - T(y^1, ..., y^m) = T(x^1 - y^1, ..., x^m) + T(x^1, x^2 - y^2, ..., x^m) + ... + T(x^1, ..., x^m - y^m),$$

we have a result that gives a characterization of continuous m-linear mapping.

Proposition 1.4. *Let $E_1, ..., E_m, F$ be normed spaces. For all $T \in L(E_1 ... E_m; F)$, the following statements are equivalent*

(1) T is continuous.

(2) T is continuous in $(0, ..., 0)$

(3)There exists a constant $C > 0$ such that

$$||T(x^1, ..., x^m)|| \leq C||x^1||...||x^m||$$

for all $x^j \in E_j, (\forall 1 \leq j \leq m)$.

If $E_j, (\forall 1 \leq j \leq m)$ and F are a normed spaces, then, we provide the space $\prod_{j=1}^m E_j$ of topology of product vector spaces, and we denote by $\mathcal{L}(E_1, ..., E_m; F)$ the vector space of all the continuous m-linear applications of $\prod_{j=1}^m E_j$ into F. We write by $\mathcal{L}(^m E; F)$ for $E_1 = ... = E_m = E$, and by $\mathcal{L}(E_1, ..., E_m)$ for $F = \mathbb{K}$. It is obviously if $m = 1$ and $F = \mathbb{K}$, then $\mathcal{L}(E; \mathbb{K}) = \mathcal{L}(E) = E^*$ the topological dual of E.

Proposition 1.5. *Let F be a Banach space, the vector space $\mathcal{L}(E_1, ..., E_m; F)$ is a Banach space endowed with the norm $||T||$ define by:*

$$\begin{aligned} ||T|| &= \sup_{||x^j|| \leq 1, j=1,..,m} ||T(x^1, ..., x^m)|| \\ &= \sup_{x^j \neq 1, j=1,..,m} \frac{||T(x^1, ..., x^m)||}{||x^1||...||x^m||} \\ &= \inf \left\{ C : ||T(x^1, ..., x^m)|| \leq C||x^1||...||x^m||. \right\} \end{aligned}$$

The symmetric multilinear application will play a prominent role in the study of homogeneous polynomials. In order to begin to study these applications, we will consider from now on the particular case of multilinear applications where:

$$E_1 = E_2 = ... = E_m = E,$$

in this case, the vector spaces of multilinear maps and continuous multilinear maps $A : E^m \to F$ will be denoted by $L(^mE; F)$ and $\mathcal{L}(^mE; F)$, respectively. In addition, we will adopt the following simplified notations:

$$L(^mE; \mathbb{K}) = L(^mE), \mathcal{L}(^mE; \mathbb{K}) = \mathcal{L}(^mE).$$

Definition 1.6. *A multilinear application $A : E^m \to F$ is symmetric if*

$$A(x_1, ..., x_m) = A(x_{\sigma(1)}, ..., x_{\sigma(m)})$$

for all $(x_1, ..., x_m) \in E^m$ and $\sigma \in S_m$ where S_m denotes the set of permutations m of the first natural numbers.

$\mathcal{L}_s(^mE; F)$ will denote the subspace of $\mathcal{L}(^mE; F)$ consisting of all the symmetric multilinear mappings from E^m into F.

Proposition 1.7. *For all $T \in \mathcal{L}(^mE; F)$. Let T_s be the associate symmetric operator of T. Then, the following are checked*

(1) $T_s \in \mathcal{L}_s(^mE; F)$

(2) $T_s = T$ if and only if $T \in \mathcal{L}_s(^mE; F)$

(3) $(T_s)_s = T_s$.

(4) If $x \in E$, then $T(x, ..., x) = T_s(x, ..., x)$.

A map $P : E \to F$ is called m-homogeneous polynomial, if there exists a symmetric m-linear mapping $\check{P}$, such that

$$P(x) = \check{P}(x, \overset{m}{\cdots}, x)$$

for all $x \in E$. In that case, we say that $\check{P}$ is the multilinear mapping associated with P.

Both mappings (m-homogeneous polynomial P and symmetric multilinear operator $\check{P}$) are related by polarization formula (for more details see [20]).

$$\check{P}(x^1, ..., x^m) = \frac{1}{2^m m!} \sum_{\xi=\pm 1} \xi_1 ... \xi_m P(\xi_1 x^1 + ... + \xi_m x^m), (x^j)_{j=1}^m \subset E$$

where $x^1, ..., x^m \in E$.

The m-homogeneous polynomial, $P : E \to F$ is continuous if there is a constant $C > 0$ with

$$||P(x)|| \leq C||x||^m,$$

we put

$$||P|| = \sup_{x \in B_X} ||P(x)||. \tag{1.3}$$

Moreover, if P is bounded, then $\check{P}$ is also bounded, and

$$||P|| \leq ||\check{P}|| \leq \frac{m^m}{m!} ||P||.$$

Or equivalently the next diagram

$$\begin{tikzcd} E \arrow[r, "\triangle m"] \arrow[dr, "P"'] & E \times \overset{m}{\cdots} \times E \arrow[d, "\check{P}"] \\ & F \end{tikzcd}$$

is commutative, where $\triangle_m$ is the natural embedding called diagonal mapping of E into $E \times \overset{m}{\cdots} \times E$ which is defined as:

$$\triangle_m : E \to E \times \overset{m}{\cdots} \times E$$

$$x \to (x, \overset{m}{\cdots}, x).$$

The space of all continuous m-homogeneous polynomials from E into F is denoted by $\mathcal{P}(^m E; F)(\mathcal{P}(^m E) \quad when \quad F = \mathbb{K})$, and becomes a Banach space when endowed with the norm $||P||$ (1.3).

When $m = 1$, the space $\mathcal{P}(^1 E; F)$ is nothing by $\mathcal{L}(E; F)$, the space of all bounded linear operators from E to F.

Example : [16]

Let E, F be a normed spaces, $\varphi \in E^*$ and let $u \in \mathcal{L}(E, F), K \in \mathbb{K}$. We define the mapping P by

$$P : E \to F, P(x) = (\varphi(x))^{m-1}u(x).$$

Then P is a continuous m-homogeneous polynomial, and $||P|| \leq ||\varphi||^{m-1}||u||$.

ideals of m-homogeneous polynomials

A polynomial $\mathcal{P}(^mE; F)$ is said to be of finite rank, if there exists $K \in \mathbb{N}$ such that

$$P(x) = \sum_{j=1}^{K}(\varphi_j(x))^m b_j.$$

where $\varphi_j \in E^*, b_j \in F$ and $j = 1, ..., k$.

It is easy to see that the subset of all m-homogeneous continuous polynomials of finite rank is a vector subspace of $\mathcal{P}(^mE; F)$, which will be denoted by $\mathcal{P}_f(^mE; F)$.

An ideal of m-homogeneous polynomials (or polynomial ideal) is a subclass $\mathcal{Q}$ of the class of every continuous m-homogeneous polynomials between Banach spaces such that, for all $m \in \mathbb{N}$ and any Banach spaces E and F the components

$$\mathcal{Q}(^mE; F) = \mathcal{P}(^mE; F) \cap \mathcal{Q}$$

satisfy:

(i) $\mathcal{Q}(^mE; F)$ contains the m-homogeneous polynomials of finite type,

(ii) $\mathcal{Q}$ has the ideal property: if $u \in \mathcal{L}(X; E), P \in \mathcal{Q}(^mE; F)$ and $t \in \mathcal{L}(F; Y)$, then $t \circ P \circ u$ belongs to $\mathcal{Q}(^mE; F)$.

$(\mathcal{Q}; \|.\|_{\mathcal{Q}})$ is a normed (Banach) polynomial ideal if

(i') $(\mathcal{Q}(^mE; F); \|.\|_{\mathcal{Q}})$ is a normed (Banach) space for all E, F ,

(ii') $||id_{\mathbb{K}^m} : \mathbb{K} \to \mathbb{K} : id_{K^m}(x) = x^m||_{\mathcal{Q}} = 1$ for all m,

(iii') If $u \in \mathcal{L}(X; E), P \in \mathcal{Q}(^mE; F)$ and $t \in \mathcal{L}(F; Y)$, then

$$||t \circ P \circ u||_{\mathcal{Q}} \leq \|t\|\|P\|_{\mathcal{Q}}\|u\|^m.$$

Now draw attention towards tensor products .

Tensor products of Banach spaces

The m-fold tensor product $E_1 \otimes ... \otimes E_m$ of the vector spaces $E_1, ..., E_m$ can be constructed from the elements of the space $L(E_1, ..., E_m)^*$. For $x^j \in E_j (j = 1, ..., m)$ we define the linear mapping

$$x^1 \otimes ... \otimes x^m : L(E_1, ..., E_m) \to \mathbb{K},$$

by

$$x^1 \otimes ... \otimes x^m(\phi) := \phi(x^1 \otimes ... \otimes x^m),$$

for each m-linear form ϕ on $L(E_1, ..., E_m)$. The functional $x^1 \otimes ... \otimes x^m$ is called an elementary tensor.

So a typical tensor $u \in E_1 \otimes ... \otimes E_m$ has the form

$$u = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes ... \otimes x_i^m,$$

where $(\lambda_i)_{i=1}^n \in \mathbb{K}, (x_i^j)_{i=1}^n \in E_j (j = 1, ..., m)$ and $n \in \mathbb{N}$ is arbitrary.

The projective norm π on $E_1 \otimes ... \otimes E_m$ is defined by

$$\pi(u) := \inf \left\{ \sum_{i=1}^{n} |\lambda| ||x_i^1|| ... ||x_i^m|| : u = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes ... \otimes x_i^m \right\},$$

where the infimum is taken over all representations of $u \in E_1 \otimes ... \otimes E_m$. The completion of $E_1 \otimes ... \otimes E_m$ when endowed with π is denoted by $E_1 \hat{\otimes}_\pi ... \hat{\otimes}_\pi E_m$.

It is well know that, if

$$T : E_1 \times ... \times E_m \to F$$

is a bounded m-linear operator then, there exists only one bounded linear operator T_L called linearization of T

$$T_L : E_1 \hat{\otimes}_\pi ... \hat{\otimes}_\pi E_m \to F,$$

such that

$$T_L(x_i^1 \otimes ... \otimes x_i^m) = T(x_i^1, ..., x_i^m)$$

for all $x^j \in E_j, j = 1, ..., m$. Moreover, $||T|| = ||T_L||$.

In other words, the mapping $T \to T_L$ from $\mathcal{L}(E_1, ..., E_m; F)$ to $\mathcal{L}(E_1 \hat{\otimes}_\pi ... \hat{\otimes}_\pi E_m; F)$ is an isometric isomorphism.

In particular, the vector spaces $\mathcal{L}(E_1 ... E_m; F)$ and $\mathcal{L}(E_1 \hat{\otimes}_\pi ... \hat{\otimes}_\pi E_m; F)^*$ are isometrically isomorphic. For more details refer to ([11],[22],[23]).

By definition homogeneous polynomials are the restriction to the diagonal of multilinear applications. One can shows that different multilinear forms may define the same polynomial. To obtain a one to one correspondence we restrict ourselves to symmetric multilinear applicatons and symmetric tensors. Consider now a Banach space E and the symmetric m-fold tensor product $\otimes^{m,s} E$ defined as the linear span of the elements in $\otimes^m E$ of the form $x \otimes ... \otimes x, x \in E$. The projective s-tensor norm π_s given by

$$\pi_s(z) = \inf \left\{ \sum_{i=1}^{K} |\lambda_i| ||x_i||^m : z = \sum_{i=1}^{K} \lambda_i x_i \otimes ... \otimes x_i \right\},$$

for $z \in \otimes^{m,s} E$.

We associated to all m-homogeneous polynomial $P \in \mathcal{P}(^m E; F)$ an unique linear operator $P_L \in \mathcal{L}(\hat{\otimes}_{\pi_s}^{m,s} E, F)$ such that:

$$P(x) = P_L(x \otimes ... \otimes x), \quad for\ all \quad x \in E.$$

The linear operator P_L called linearization of P.

Consider the canonical m-homogeneous polynomial $\delta_m : E \to \otimes_{\pi_s}^{m,s} E$ defined by $\delta_m = x \otimes ... \otimes x$

Then, the diagram

$$\begin{tikzcd}
E \arrow[r, "P"] \arrow[d, "\delta_m"'] & F \\
\hat{\otimes}_{\pi_s}^{m,s} E \arrow[ur, "P_L"'] &
\end{tikzcd}$$

is commutative, that is $P = P_L \circ \delta_m$.

The scalar case deserves special attention: when $F = \mathbb{K}$ we call

$$\Delta_m : \mathcal{P}(^m E) \to (\hat{\otimes}_{\pi_s}^{m,s} E)^*$$

the isometric isomorphism $\Delta_m(P) = P_L$ for all $P \in \mathcal{P}(^m E)$. Note that $\Delta_m^{-1} = \delta_m^*$.

We can associated to P other operator P^* the adjoint of P define from F^* into $\mathcal{P}(^m E)$ given by

$$P^*(y^*)(x) := y^*(P(x)),$$

for $y^* \in F^*$ and $x \in E$. It is well known that

$$(u \circ P)^* = P^* \circ u^*,$$

for all $u \in \mathcal{L}(F; Z)$, where Z is a Banach space.

The next proposition give a direct connection between m-homogenuous polynpmial, his linearization and adjont operator.

Proposition 1.8. *([21],Lemma2.1)*

The following are equivalent for apolynomial $P \in \mathcal{P}(^m E; F)$:

(a) P is a finite rank polynomial.

(b)P_L is a finite rank linear operator.

(c)P^* is a finite rank linear operator.

Proof. It is easy to check that P has finite rank if and only if there are $K \in \mathbb{N}$, polynomials $Q_1, ..., Q_K \in \mathcal{P}(^m E)$ and vectors $b_1, ..., b_K \in F$ such that

$$P(x) = \sum_{j=1}^{K} Q_j(x) b_j$$

for every $x \in E$.

To prove equivalence $(a) \Longleftrightarrow (b)$, let's start with the first implication

$(a) \implies (b)$ write

$$P = \sum_{j=1}^{K} Q_j \otimes b_j$$

where $Q_1, ..., Q_K \in \mathcal{P}(^m E)$ and $b_1, ..., b_K \in F$.

Then

$$(Q_j)_L \in (\hat{\otimes}_n^{s,\pi} E)^*$$

for $j = 1, ..., K$, and

$$\begin{aligned} P_L(x \otimes ... \otimes x) &= P(x) \\ &= \sum_{j=1}^{K} Q_j(x) b_j \\ &= \sum_{j=1}^{K} (Q_j)_L(x \otimes ... \otimes x) b_j \\ &= \left(\sum_{j=1}^{K} (Q_j)_L \otimes b_j \right) (x \otimes ... \otimes x) \end{aligned}$$

for every $x \in E$. Since both P_L and $\sum_{j=1}^{K} (Q_j)_L \otimes b_j$ are linear and continuous, we have

$$P_L = \sum_{j=1}^{K} (Q_j)_L \otimes b_j.$$

$(b) \Longrightarrow (a)$ let

$$\delta_m : E \to \hat{\otimes}_n^{s,\pi} E,$$

be the canonical m-homogeneous polynomial given by

$$\delta_m(x) = x \otimes ... \otimes x.$$

Since

$$P = P_L \circ \delta_m,$$

the range of P is contained in the range of P_L, so P has finite rank.

$(b) \Longrightarrow (c)$ from

$$P = P_L \circ \delta_m$$

we conclude that

$$P^* = \delta_m^* \circ P_L^*.$$

So the range of P^* is the image under δ_m^* of a finite dimensional space, thus P^* has finite rank.

$(c) \Longrightarrow (b)$ it is well known that δ_m^* is an isometric isomorphism, so

$$P_L^* = (\delta_m^*)^{-1} \circ P^*.$$

In particular, the range of P_L^* is contained in the range of P^*, so P_L^* has finite rank. $\square$

CHAPTER 2

INJECTIVE IDEAL POLYNOMIAL

In this chapter, we present and study the Injective linear operator ideals. We outline the basic theory of injective polynomials ideals, we give the definition, provide illustrative, example and characterize injective polynomial ideals by names of the injective hull.

2.1 injective linear operator ideal

Definition 2.1. *[15] A normed operator ideal $(\mathcal{I}; ||.||_{\mathcal{I}})$ is said to be injective if for every metric injective*

$$i : F \hookrightarrow G$$

and every $u \in \mathcal{L}(E, F)$ it follows from:

$$i \circ u \in \mathcal{I}(E, G)$$

that $u \in \mathcal{I}(E, F)$, moreover

$$||i \circ u||_{\mathcal{I}} = ||u||_{\mathcal{I}},$$

i.e, the ideal does not depend on the image space.

Notation 2.2. *([18],page 421) Let us introduce the Banach spaces*

$$E^{\infty} := l_{\infty}(B_{E'}) \quad and \quad E^{1} := l_{1}(B_{E}).$$

Recall that there is a canonical linear and isometric embedding :

$$J_E : E \to E^{\infty}$$

and also E^∞ has the extension property that is, if F be a subspace of the normed space $(E, ||.||)$, then every continuous linear operators

$$T : F \to E^\infty$$

extend to a continuous linear operators

$$\hat{T} : E \to E^\infty$$

such that:

$$||\hat{T}|| = ||T||$$

Definition 2.3. *Let $\mathcal{I}$ be an ideal . For arbitrary Banach spaces E and F, the injective of an operator ideal denoted by $\mathcal{I}^{inj}$ is defined by*

$$\mathcal{I}^{inj}(E, F) := \{S \in \mathcal{L}(E, F); J_F \circ S \in \mathcal{I}(E, F^\infty)\}.$$

$\mathcal{I}^{inj}$ be an ideal and , $\mathcal{I}$ is contained in it .

Here we can mention the most important and famous applications which are injective .
Example:

- The operator ideals $\mathcal{L}_{\mathcal{F}}$ of finite rank opertors (the range of the opertors generates a finite-dimensional subspace of the target space) is injective
- The operator ideals $\mathcal{L}_{\mathcal{K}}$ of compact opertors (bounded sets are sent to relatively compact sets) and $\mathcal{L}_{\mathcal{W}}$ of weakly compact opertors (bounded sets are sent to relatively weakly compact sets) are injective.
- All ideals of opertors of summing type (absolutely summing, dominated, strongly summing, multiple summing, etc) are injective.

Proposition 2.4. *$\mathcal{I}^{inj}$ is the smallest injective ideal. Consequently, $\mathcal{I}$ is injective if $\mathcal{I}^{inj} = \mathcal{I}$.*
It is for this reason that named $\mathcal{I}^{inj}$ is the injective hull of $\mathcal{I}$.

Proof. Let E, F, G be Banach spaces, and let $J \in (F, G)$ be a homeomorphic embedding .

Let $S \in \mathcal{L}(E, F)$ be such that :

$$J \circ S \in \mathcal{I}^{inj}(E, G).$$

This means

$$J_G \circ J \circ S \in \mathcal{I}(E, G^{\infty}).$$

By extension property of F^{∞}, there is a

$$\tilde{J}_F \in (G^{\infty}, F^{\infty})$$

such that:

$$\tilde{J}_F \circ J_G \circ J = J_F.$$

From

$$J_F \circ S = \tilde{J}_F \circ J_G \circ J \circ S \in \mathcal{I}(E, F^{\infty})$$

we conclude that S is in $\mathcal{I}^{inj}(E, F)$. Thus $\mathcal{I}^{inj}$ is injective .

If $\mathcal{I}_1$ is injective ideal containing $\mathcal{I}$, then $S \in \mathcal{I}^{inj}(E, F)$ implies

$$J_F \circ S \in \mathcal{I}(E, F^{\infty}) \subset \mathcal{I}_1(E, F^{\infty}),$$

hence $S \in \mathcal{I}_1(E, F)$ by injectivity of $\mathcal{I}_1$.

This proves

$$\mathcal{I}^{inj} \subset \mathcal{I}_1$$

□

Similar to the linear case, we definded this concept in the nonlinear case (m-homogeneous polynomials).

2.2 Injective polynomail ideal

A polynomial ideal is injective if the containment of a polynomial in the class depends on the norm of the target space rather than on the space itself.

Definition 2.5. *Let $\mathcal{Q}$ be a subclass of the class of homogeneous polynomials between Banach spaces such that, for every m and any Banach spaces E and F, the component*

$$\mathcal{Q}(^m E; F) := \mathcal{P}(^m E; F) \cap \mathcal{Q}$$

is a linear subspace of $\mathcal{P}(^m E; F)$ containing the polynomials of finite type. The class $\mathcal{Q}$ is said to be:
(a) A polynomial ideal if :

$$t \circ P \circ u \in \mathcal{Q}(^m E; H).$$

Whenever $t \in \mathcal{L}(G; H)$ *,* $P \in \mathcal{Q}(^m F; G)$ *and* $u \in \mathcal{L}(E; F)$.
(b) A polynomial hyper-ideal if:

$$t \circ P \circ Q \in \mathcal{Q}(^{mn} E; H).$$

Whenever $t \in \mathcal{L}(G, H)$ *,* $P \in \mathcal{Q}(^m F; G)$ *and* $Q \in \mathcal{P}(^n E; F)$.
Suppose that there is a function :

$$||.||_{\mathcal{Q}} : \mathcal{Q} \to \mathbb{R}$$

whose restriction to each component $\mathcal{Q}(^m E; F)$ is a norm such that :

$$||\lambda \in \mathbb{K} \to \lambda^m||_{\mathcal{Q}} = 1$$

for every m. $(\mathcal{Q}, ||.||_{\mathcal{Q}})$ is said to be:
(a′) A normed polynomial ideal if, in (a)

$$||t \circ P \circ u||_{\mathcal{Q}} \leq ||t||.||P||_{\mathcal{Q}}.||u||.$$

(b′) A normed polynomial hyper-ideal ideal if, in (b)

$$||t \circ P \circ Q||_{\mathcal{Q}} \leq ||t||.||P||_{\mathcal{Q}}.||Q||^{m}.$$

If each component $(\mathcal{Q}(^{m}E; F), ||.||_{\mathcal{Q}})$ *is a Banach space, then* $(\mathcal{Q}, ||.||_{\mathcal{Q}})$ *is said to be a Banach polynomial ideal or a Banach polynomial hyper-ideal. If the norm is the usual sup norm , we speak of a closed polynomial ideal or a closed polynomial hyper-ideal.*
The mth-component of a polynomial ideal $\mathcal{Q}$ *, defined as*

$$\mathcal{Q}_{m} = \bigcup_{E,F} \mathcal{Q}(^{m}E, F),$$

is called a (normed, Banach, closed) ideal of m-homogeneous polynomials. Of course, its linear component $\mathcal{Q}_1$ *is an operator ideal.*

Remark 2.6. *[9] Just to mention a few illustrative examples, the class of nuclear polynomials is a Banach polynomial ideal that fails to be a hyper-ideal and the classes of compact and weakly compact polynomials are closed hyper-ideals.*

Definition 2.7. *A polynomial ideal* $\mathcal{Q}$ *is said to be injective if* $P \in \mathcal{Q}(^{m}E, F)$ *whenever*
$P \in \mathcal{P}(^{m}E, F)$ *and :*

$$j : F \to G$$

is a metric injection such that:

$$j \circ P \in \mathcal{Q}(^{m}E, G).$$

A normed polynomial ideal $(\mathcal{Q}, ||.||_{\mathcal{Q}})$ *is injective if* $\mathcal{Q}$ *is an injective polynomial ideal and, in the situation above:*

$$||P||_{\mathcal{Q}} = ||j \circ P||_{\mathcal{Q}}.$$

We can mention a few examples which succeed to be injective

Example 2.8. *[9]*

- *The ideals* $\mathcal{P}_{\mathcal{F}}$ *of finite rank polynomials is injective.*

- *The ideals $\mathcal{P}_{\mathcal{K}}$ of compact polynomials and $\mathcal{P}_{\mathcal{W}}$ of weakly compact polynomials are injective.*
- *All ideals of polynomials of summing type (absolutely summing, dominated, strongly summing, multiple summing, etc) are injective.*

2.3 characterization by injective hull

In this part we aim to characterization injective polynomial ideal by means of the injective hull and establish the properties of a hull procedure.

Proposition 2.9. *(a) Let $\mathcal{Q}$ be a polynomial ideal . Then there exists a unique smallest injective polynomial ideal $\mathcal{Q}^{inj}$ containing $\mathcal{Q}$. For $P \in \mathcal{P}(^mE;F)$*

$$P \in \mathcal{Q}^{inj}(^mE;F) \Longleftrightarrow I_F \circ P \in \mathcal{Q}(^mE;l_\infty(B_{F^*})),$$

where I_F is the canonical metric injection.
(b) Let $(\mathcal{Q},||.||_{\mathcal{Q}})$ be normed (Banach) polynomial ideal . Then there exists a unique smallest normed (Banach) injective polynomial ideal $(\mathcal{Q}^{inj},||.||_{\mathcal{Q}^{inj}})$ containing $\mathcal{Q}$ and such that :

$$||.||_{\mathcal{Q}^{inj}} \leq ||.||_{\mathcal{Q}}.$$

For $P \in \mathcal{P}(^mE;F)$,

$$P \in \mathcal{Q}^{inj}(^mE;F) \Longleftrightarrow I_F \circ P \in \mathcal{Q}(^mE;l_\infty(B_{F^*}),$$

and

$$||P||_{\mathcal{Q}^{inj}} := ||I_F \circ P||_{\mathcal{Q}}. \tag{2.1}$$

The ideal $\mathcal{Q}^{inj}$ (normed, Banach ideal$(\mathcal{Q}^{inj},||.||_{\mathcal{Q}^{inj}})$) is called the injective hull of the ideal $\mathcal{Q}$ (normed, Banach ideal$(\mathcal{Q},||.||_{\mathcal{Q}})$).

Proof. $\mathcal{Q}^{inj}$ and $||.||_{\mathcal{Q}^{inj}}$ are defined according to (2.1).
We check only the hyper-ideal property, the other statements follow from standard arguments.
Let $Q \in \mathcal{P}(^nE;F)$, $P \in \mathcal{Q}^{inj}(^mF;G)$ and $t \in \mathcal{L}(G;H)$ be given.
By the definition of $\mathcal{Q}^{inj}(^mF;G)$:

$$I_G \circ P \in \mathcal{Q}(^mF;l_\infty(B_{G^*})).$$

Since I_G is a metric injection, an application of the metric extension property of $l_\infty(B_{H^*})$ the operator $I_H \circ t$ gives rise to an operator $S \in \mathcal{L}(l_\infty(B_{G^*}); l_\infty(B_{H^*}))$ such that :

$$I_H \circ t = S \circ I_G,$$

and

$$||S|| = ||I_H \circ t||.$$

$$\begin{tikzcd}
E \arrow[r, "Q"] & F \arrow[r, "P"] & G \arrow[r, "t"] \arrow[d, "I_G"] & H \arrow[r, "I_H"] & l_\infty(B_{H^*}) \\
 & & l_\infty(B_{H^*}) \arrow[urr, "s"] & &
\end{tikzcd}$$

Therefore,

$$\begin{aligned}
I_H \circ t \circ P \circ Q &= (S \circ I_G) \circ P \circ Q \\
&= S \circ (I_G \circ P) \circ Q \in \mathcal{Q}(^{mn}E; l_\infty(B_{H^*})),
\end{aligned}$$

that is:

$$t \circ P \circ Q \in \mathcal{Q}^{inj}(^{mn}E; H),$$

and

$$\begin{aligned}
||t \circ P \circ Q||_{\mathcal{Q}^{inj}} &= ||I_H \circ t \circ P \circ Q||_{\mathcal{Q}} \\
&= ||S \circ I_G \circ P \circ Q||_{\mathcal{Q}} \\
&\leq ||S||.||I_G \circ P||_{\mathcal{Q}}.||Q||^m \\
&= ||I_H \circ t||.||I_G \circ P||_{\mathcal{Q}}.||Q||^m \\
&\leq ||t||.||P||_{\mathcal{Q}^{inj}}.||Q||^m.
\end{aligned}$$

□

Corollary 2.10. *(a) A polynomial ideal $\mathcal{Q}$ is injective if and only if $\mathcal{Q} = \mathcal{Q}^{inj}$.*

(b) A normed (Banach) polynomial ideal $(\mathcal{Q}, ||.||_{\mathcal{Q}})$ is injective if and only if $\mathcal{Q} = \mathcal{Q}^{inj}$ isometrically.

Next we check that the correspondence $\mathcal{Q} \to \mathcal{Q}^{inj}$ is a hull procedure in the sense of [23]. We state only the case of normed/Banach polynomial ideals/hyper-ideals.

Proposition 2.11. *Let* $(\mathcal{Q}, ||.||_{\mathcal{Q}})$ *and* $(\mathcal{R}, ||.||_{\mathcal{R}})$ *be normed (Banach) polynomial ideals. Then:*

(a) $(\mathcal{Q}^{inj}, ||.||_{\mathcal{Q}^{inj}})$ *is a normed (Banach) polynomial ideal .*

(b) If $\mathcal{Q} \subseteq \mathcal{R}$ *and* $||.||_{\mathcal{R}} \leq ||.||_{\mathcal{Q}}$*, then :* $\quad \mathcal{Q}^{inj} \subseteq \mathcal{R}^{inj}$ *and* $\quad ||.||_{\mathcal{R}^{inj}} \leq ||.||_{\mathcal{Q}^{inj}}$.

(c)$(\mathcal{Q}^{inj})^{inj} = \mathcal{Q}^{inj}$ *and* $\quad ||.||_{(\mathcal{Q}^{inj})^{inj}} = ||.||_{\mathcal{Q}^{inj}}$.

(d)$\mathcal{Q} \subseteq \mathcal{Q}^{inj}$ *and* $\quad ||.||_{\mathcal{Q}^{inj}} \leq ||.||_{\mathcal{Q}}$.

Proof. (a) and (d) follow from Proposition 2.10 and (c) follows from Corollary 2.11.

To prove (b), let $P \in \mathcal{Q}^{inj}({}^m E; F)$ be given.

Thus

$$I_F \circ P \in \mathcal{Q}({}^m E; l_\infty(B_{F^*})) \subseteq \mathcal{R}({}^m E; l_\infty(B_{F^*})),$$

what gives

$$P \in \mathcal{R}^{inj}({}^m E; F).$$

Moreover,

$$\begin{aligned} ||P||_{\mathcal{R}^{inj}} &= ||I_F \circ P||_{\mathcal{R}} \\ &\leq ||I_F \circ P||_{\mathcal{Q}} \\ &= ||P||_{\mathcal{Q}^{inj}}. \end{aligned}$$

□

CHAPTER 3

CHARACTERIZATION OF INJECTIVE m-HOMGENEOUS POLYNOMIALS

Through this chapter we will answer the question:

Can injective polynomial ideals be characterized by some related domination property ?

And we charecterize the the polynomial duals.

3.1 The Domination Property

The next propostion is left as an exercice in ([15], EX9.10(6)). We give a short proof the sake of completeness.

Proposition 3.1. *[7] An operator ideal $\mathcal{I}$ is injective if and only if the following condition holds: if $u \in \mathcal{I}(E;F), v \in \mathcal{L}(E;G)$ and there exists a constant $C > 0$ (eventually depending on E, F, G, u, v) such that*

$$||v(x)|| \leq C||u(x)||$$

for every $x \in E$ then $v \in \mathcal{I}(E;G)$.

Proof. Assume that $\mathcal{I}$ is injective and let $u \in \mathcal{I}(E;F), v \in \mathcal{L}(E;G)$ be such that:

$$||v(x)|| \leq C||u(x)||$$

for every $x \in E$. This inequality guarantees that the map

$$W : u(E) \subseteq F \to G, W(u(x)) = v(x),$$

is a well defined continuous linear operator.

Considering the canonical metric injection

$$J_G : G \to l_\infty(B_{G^*}),$$

by the extension property of $l_\infty(B_{G^*})$, there is an extension $\tilde{w} \in \mathcal{L}(F; l_\infty(B_{G^*}))$ of $J_G \circ w$ to the whole of F.

From

$$\tilde{w} \circ u = J_G \circ v,$$

we conclude that $J_G \circ v$ belongs to $\mathcal{I}$, and the injectivity of $\mathcal{I}$ gives $v \in \mathcal{I}(E; G)$. The converse is obvious.

□

Transposing the linear domination property above literally to the polynomial case, we end up with the following:

Definition 3.2. *A polynomial ideal $\mathcal{Q}$ is said to have the strong domination property if given polynomials $P \in \mathcal{Q}(^mE; F)$ and $Q \in \mathcal{P}(^mE; G)$ such that*

$$||Q(x)|| \leq C.||P(x)||,$$

for every $x \in E$ and some constant $C \geq 0$ (eventually depending on E, F, G, P, Q, m), then $Q \in \mathcal{Q}(^mE; G)$.

Given a polynomial $P \in \mathcal{P}(^mE; F)$ and a metric injection

$$J : F \to G,$$

we have

$$\begin{aligned} ||P(x)|| &= ||j(P(x))|| \\ &= ||(j \circ P)(x)||, \end{aligned}$$

for every $x \in E$.

Remark 3.3. *The strong domination property is sufficient for a polynomial ideal to be injective: every polynomial ideal with the strong domination property is injective.*

In the linear case, the proof that every injective operator ideal has the domination property depends heavily on the linearity of the underlying operators, so it is not expected that every injective polynomial ideal has the strong domination property.
We also note that there is an example of an injective polynomial ideal failing the strong domination property, which establishes that this property does not characterize injective polynomial ideals (For more see [9]). This poses two questions: Can injective polynomial ideals be characterized by some related domination property? If yes, is this characterization useful? Next we answer these two questions affirmatively.

Definition 3.4. *A polynomial ideal $\mathcal{Q}$ is said to have the weak domination property if given polynomials $P \in \mathcal{Q}(^m E; F)$ and $Q \in \mathcal{P}(^m E; G)$ such that*

$$\left\| \sum_{i=1}^{K} \lambda_i Q(x_i) \right\| \leq C. \left\| \sum_{i=1}^{K} \lambda_i P(x_i) \right\|$$

for all $K \in \mathbb{N}, x_1, ..., x_K \in E, \lambda_1, ..., \lambda_K \in \mathbb{K}$ and some constant $C \geq 0$ (eventually depending on E, F, G, P, Q, m), then $Q \in \mathcal{Q}(^m E; G)$.

Theorem 3.5. *A polynomial ideal is injective if and only if it has the weak domination property.*

Proof. Suppose that $\mathcal{Q}$ is an injective polynomial ideal and let $P \in \mathcal{Q}(^m E; F)$ and $Q \in \mathcal{P}(^m E; G)$ be such that

$$\left\| \sum_{i=1}^{K} \lambda_i Q(x_i) \right\| \leq C. \left\| \sum_{i=1}^{K} \lambda_i P(x_i) \right\| \tag{3.1}$$

for all $K \in \mathbb{N}, x_1, ..., x_K \in E$, $\lambda_1, ..., \lambda_K \in \mathbb{K}$ and some constant C. Let us see that the operator

$$\begin{aligned} &W : span\,\{P(E)\} \subseteq F \to G, \\ &W\left(\sum_{i=1}^{K} \lambda_i P(x_i)\right) = \sum_{i=1}^{K} \lambda_i Q(x_i), \end{aligned}$$

is well defined: indeed,

$$\begin{aligned}
\sum_{i=1}^{K} \lambda_i P(x_i) = \sum_{j=1}^{l} \alpha_j P(x_j) & \\
&\Longrightarrow \left\| \sum_{i=1}^{K} \lambda_i P(x_i) - \sum_{j=1}^{l} \alpha_j P(x_j) \right\| = 0 \\
&\overset{(3.1)}{\Longrightarrow} \left\| \sum_{i=1}^{K} \lambda_i Q(x_i) - \sum_{j=1}^{l} \alpha_j Q(x_j) \right\| = 0 \\
&\Longrightarrow \sum_{i=1}^{K} \lambda_i Q(x_i) = \sum_{j=1}^{l} \alpha_j Q(x_j).
\end{aligned}$$

The linearity of W is clear and its continuity follows from

$$\begin{aligned}
\left\| W\left(\sum_{i=1}^{K} \lambda_i P(x_i) \right) \right\| &= \left\| \sum_{i=1}^{K} \lambda_i Q(x_i) \right\| \\
&\overset{(3.1)}{\leq} C. \left\| \sum_{i=1}^{K} \lambda_i P(x_i) \right\|.
\end{aligned}$$

Then there exists a (unique) bounded linear operator

$$W_1 : \overline{span\left\{P(E)\right\}} \subseteq F \to G,$$

such that

$$W_1|_{span\{P(E)\}} = W.$$

Denoting by

$$i : \overline{span\left\{P(E)\right\}} \to F,$$

the formal inclusion operator and by

$$P_0 : E \to span\left\{P(E)\right\},$$

the obvious polynomial, we have the diagram

$$\begin{tikzcd}
E \arrow[r, "P_0"] \arrow[d, "Q"'] & \overline{span\{P(E)\}} \arrow[r, "i"] \arrow[dl, "W_1"'] \arrow[dr, dashed, "J_G \circ W_1"] & F \arrow[d, dashed, "\tilde{W}_1"] \\
G \arrow[rr, "J_G"'] & & l_\infty(B_{G}*)
\end{tikzcd}$$

As i is a metric injection, from the metric approximation property of $l_\infty(B_{G^*})$ there exists an operator $\tilde{W}_1 \in \mathcal{L}(F; l_\infty(B_{G^*}))$ such that

$$J_G \circ W_1 = \tilde{W}_1 \circ i,$$

and

$$||\tilde{W}_1|| = ||J_G \circ W_1||.$$

From

$$\begin{aligned}(W_1 \circ P_0)(x) &= W_1(P(x)) \\ &= W(P(x)) \\ &= Q(x),\end{aligned}$$

for every $x \in E$, that is

$$Q = W_1 \circ P_0$$

we conclude that

$$\begin{aligned}J_G \circ Q &= J_G \circ W_1 \circ P_0 \\ &= \tilde{W}_1 \circ i \circ P_0 \\ &= \tilde{W}_1 \circ P.\end{aligned}$$

Since $P \in \mathcal{Q}(^m E; F)$, the ideal property of $\mathcal{Q}$ gives

$$J_G \circ Q \in \mathcal{Q}(^m E; l_\infty(B_{G^*})).$$

The injectivity of $\mathcal{Q}$ and the fact that J_G is a metric injection give $Q \in \mathcal{Q}(^m E; G)$, showing that $\mathcal{Q}$ has the weak domination property.

Conversely, suppose that $\mathcal{Q}$ is a polynomial ideal with the weak domination property.

Given $P \in \mathcal{P}(^m E; F)$ and a metric injection

$$j : F \to G,$$

such that

$$(j \circ P) \in \mathcal{Q}(^m E; F),$$

we have

$$\begin{aligned}\left\| \sum_{i=1}^{K} \lambda_i P(x_i) \right\| &= \left\| j \left(\sum_{i=1}^{K} \lambda_i P(x_i) \right) \right\| \\ &= \left\| \sum_{i=1}^{K} \lambda_i (j \circ P)(x_i) \right\|\end{aligned}$$

for all $K \in \mathbb{N}$, $x_1, ..., x_k \in E$ and $\lambda_1, ..., \lambda_K \in \mathbb{K}$. The weak domination property of $\mathcal{Q}$ gives $P \in \mathcal{Q}(^m E; F)$, proving that $\mathcal{Q}$ is injective.

□

Now we apply the characterization above to establish a quite useful formula regarding composition polynomial ideals, whose definition goes back to Pietsch [24], and we recall now: Given an operator ideal $\mathcal{I}$, a polynomial $P \in \mathcal{P}(^m E; F)$ belongs to $\mathcal{I} \circ \mathcal{P}(^m E; F)$ if there exist a Banach space G, a polynomial $Q \in \mathcal{P}(^m E; G)$ and an operator $u \in \mathcal{I}(G; F)$ such that

$$P = u \circ Q.$$

It is well known that $\mathcal{I} \circ \mathcal{P}$ is a polynomial hyper-ideal.

Proposition 3.6. *([8],Proposition 3.2) Let $\mathcal{I}$ be an operator ideal.*

(a) The following are equivalent for $A \in \mathcal{L}(E_1, ..., E_m; F)$:

(a1) $A \in \mathcal{I} \circ \mathcal{L}(E_1, ..., E_m; F)$

(a2) $A_L \in \mathcal{I}(E_1 \hat{\otimes}_\pi ... \hat{\otimes}_\pi E_m; F)$.

(b) The following are equivalent for $P \in \mathcal{P}(^m E; F)$:

(b1) $P \in \mathcal{I} \circ \mathcal{P}(^m E; F)$.

(b2) $P_L \in \mathcal{I}(\hat{\otimes}_\pi^{n,s} E; F)$.

(b3) $P_{L,s} \in \mathcal{I}(\hat{\otimes}_\pi^{n,s} E; F)$.

(b4) $\check{P} \in \mathcal{I} \circ \mathcal{L}(^m E; F)$.

(b5) There is $A \in \mathcal{I} \circ \mathcal{L}(^m E; F)$ such that $\hat{A} = P$.

Theorem 3.7. *For every operator ideal $\mathcal{I}$,*

$$\mathcal{I}^{inj} \circ \mathcal{P} = (\mathcal{I} \circ \mathcal{P})^{inj}.$$

In particular, the polynomial hyper-ideal $\mathcal{I}^{inj} \circ \mathcal{P}$ is injective.

Proof. Given

$$P \in \mathcal{I}^{inj} \circ \mathcal{P}(^m E; F), \quad P = u \circ Q,$$

for some Banach space G, $Q \in \mathcal{P}(^m E; G)$ and $u \in \mathcal{I}^{inj}(G; F)$. Then the factorization

$$I_F \circ P = I_F \circ u \circ Q,$$

with

$$I_F \circ u \in \mathcal{I}(G; l_\infty(B_{G^*})),$$

shows that $I_F \circ P$ belongs to $\mathcal{I} \circ \mathcal{P}$. This proves that

$$\mathcal{I}^{inj} \circ \mathcal{P} \subseteq (\mathcal{I} \circ \mathcal{P})^{inj}. \tag{3.2}$$

Let us prove that $\mathcal{I}^{inj} \circ \mathcal{P}$ is injective. Let $P \in \mathcal{I}^{inj} \circ \mathcal{P}(^m E; F)$ and $Q \in \mathcal{P}(^m E; G)$ be such that

$$\left\| \sum_{i=1}^{K} \lambda_i Q(x_i) \right\| \leq C. \left\| \sum_{i=1}^{K} \lambda_i P(x_i) \right\|$$

for all $K \in \mathbb{N}, x_1, ..., x_K \in E, \lambda_1, ..., \lambda_K \in \mathbb{K}$ and some constant $C \geq 0$. Call P_L and Q_L the linearizations of P and Q on the (completed) projective symmetric tensor product, that is

$$P_L : \hat{\otimes}_\pi^{m,s} E \to F,$$

and

$$Q_L : \hat{\otimes}_\pi^{m,s} E \to G,$$

are bounded linear operators such that

$$P_L(\otimes^m x) = P(x),$$

and

$$Q_L(\otimes^m x) = Q(x),$$

for every $x \in E$. Given

$$z = \sum_{i=1}^{K} \lambda_i \otimes^m x_i,$$

in the (incomplete) symmetric tensor product $\otimes_\pi^{m,s} E$, we have

$$\begin{aligned} ||Q_L(z)|| &= \left\| \sum_{i=1}^{K} \lambda_i Q(x_i) \right\| \\ &\leq C. \left\| \sum_{i=1}^{K} \lambda_i P(x_i) \right\| \\ &= C.||P_L(z)||. \end{aligned}$$

The continuity of P, of Q and of the norm give that

$$||Q_L(z) \leq C.||P_L(z)||,$$

for every $z \in \hat{\otimes}_\pi^{m,s} E$.

Since

$$P \in \mathcal{I}^{inj} \circ \mathcal{P}(^m E; F),$$

that

$$P_L \in \mathcal{I}^{inj}(\hat{\otimes}_\pi^{m,s} E; F).$$

Now the injectivity of $\mathcal{I}^{inj}$ and Proposition 3.1 give

$$Q_L \in \mathcal{I}^{inj}(\hat{\otimes}_\pi^{m,s} G; F).$$

We get

$$Q \in \mathcal{I}^{inj} \circ \mathcal{P}(^m G; F),$$

proving that $\mathcal{I}^{inj} \circ \mathcal{P}$ has the weak domination property, hence it is injective by Theorem 3.5.

Combining this with (3.2), with $\mathcal{I} \subseteq \mathcal{I}^{inj}$ and with Proposition 2.11(b), we get

$$\begin{aligned} (\mathcal{I}^{inj} \circ \mathcal{P})^{inj} &= \mathcal{I}^{inj} \circ \mathcal{P} \\ &\subseteq (\mathcal{I} \circ \mathcal{P})^{inj} \\ &\subseteq (\mathcal{I}^{inj} \circ \mathcal{P})^{inj}, \end{aligned}$$

which gives the desired formula. The second assertion follows from Proposition 2.9.

□

Lemma 3.8. *([**?**],Lemma 3.4) Let $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}$ be operator ideals, $m \in \mathbb{N}$ and $E, E_1, ..., E_m, F$ be Banach spaces.*

(a) If

$$\mathcal{I}_1 \circ \mathcal{L}(E_1, ..., E_m; F) \subseteq \mathcal{I}_2 \circ \mathcal{L}(E_1, ..., E_m; F) \quad then$$

$$\mathcal{I}_1(E_j; F) \subseteq \mathcal{I}_2(E_j; F)$$

for every $j = 1, ..., m$. In particular, If

$$\mathcal{I} \circ \mathcal{L}(E_1, ..., E_m; F) = \mathcal{L}(E_1, ..., E_m; F), \quad then$$

$$\mathcal{I}(E_j; F) = \mathcal{L}(E_j; F)$$

for every $j = 1, ..., m$.

(b) If

$$\mathcal{I}_1 \circ \mathcal{P}(^m E; F) \subseteq \mathcal{I}_2 \circ \mathcal{P}(^m E; F) \quad then$$

$$\mathcal{I}_1(E; F) \subseteq \mathcal{I}_2(E; F).$$

In particular, if

$$\mathcal{I} \circ \mathcal{P}(^m E; F) = \mathcal{P}(^m E; F) \quad then$$

$$\mathcal{I}(E; F) = \mathcal{L}(E; F).$$

Corollary 3.9. *The following are equivalent for an operator ideal $\mathcal{I}$:*

(a) $\mathcal{I}$ is an injective operator ideal.

(b)$\mathcal{I} \circ \mathcal{P}$ is an injective polynomial hyper-ideal.

(c) $(\mathcal{I} \circ \mathcal{P})_m$ is an injective ideal of m-homogeneous polynomials for some $m \in \mathbb{N}$.

Proof. $(a) \implies (b)$ follows from Theorem 3.5 and Corollary 2.10, $(b) \implies (c)$ is obvious and $(c) \implies (a)$ follows from [From the lemma 3.8]. □

Remark 3.10. *The corollary above gives another proof that the polynomial ideals of finite rank, compact and weakly compact polynomials are injective.*

3.2 injective polynomial dual

Next result asserts that the polynomials belonging to the polynomial dual of an operator ideal $\mathcal{I}$ are exactly those polynomials that admit a factorization through $\mathcal{I}^{dual}$:

Theorem 3.11. *([6],Theorem 2.2) Let $\mathcal{I}$ be an operator ideal. Then the adjoint of a continuous homogeneous polynomial P belongs to $\mathcal{I}$ if and only if P admits a factorization*

$$P = u \circ Q$$

where the adjoint of the linear operator u belongs to $\mathcal{I}$. That is,

$$\mathcal{I}^{\mathcal{P}-dual} = \mathcal{I}^{dual} \circ \mathcal{P}$$

If $(\mathcal{I}, ||.||_{\mathcal{I}})$ is a normed operator ideal, then

$$||.||_{\mathcal{I}^{\mathcal{P}-dual}} = ||.||_{\mathcal{I}^{dual} \circ \mathcal{P}}.$$

Proof. Let E and F be Banach spaces, $m \in \mathbb{N}$ and $P \in \mathcal{P}(^m E; F)$. Assume that

$$P \in \mathcal{I}^{\mathcal{P}-dual}(^m E; F),$$

that is

$$P^* \in \mathcal{I}(F^*; \mathcal{P}(^m E)).$$

By $\hat{\otimes}_n^{\pi s}$, we denote the completed symmetric n-fold tensor product of E endowed with the s-projective tensor norm π_s, by P_L we denote the linearization of P, that is,

$$P_L \in \mathcal{L}(\hat{\otimes}_n^{\pi s} E; F), \quad P_L(x \otimes ... \otimes x) = P(x).$$

Consider the canonical isometric isomorphism

$$J_n^E : \mathcal{P}(^m E) \to (\hat{\otimes}_n^{\pi s})^*,$$

$$J_n^E(Q)(x \otimes ... \otimes x) = Q(x).$$

For $\varphi \in F^*$ and $x \in E$,

$$\begin{aligned}\left(J_n^E \circ P^*\right)(\varphi)(x \otimes ... \otimes x) &= J_n^E(P^*(\varphi))(x \otimes ... \otimes x) \\ &= P^*(\varphi)(x) \\ &= \varphi(P(x)) \\ &= \varphi(P_L(x \otimes ... \otimes x)) \\ &= (P_L)^*(\varphi)(x \otimes ... \otimes x),\end{aligned}$$

and since $J_n^E \circ P^*$ and $(P_L)^*$ are both linear and continuous, it follows that

$$(P_L)^* = J_n^E \circ P^*.$$

So

$$(P_L)^* \in \mathcal{I}(F^*; (\hat{\otimes}_n^{\pi_s} E)^*),$$

that is

$$P_L \in \mathcal{I}^{dual}(\hat{\otimes}_m^{\pi_s} E; F).$$

We have

$$P \in \mathcal{I}^{dual} \circ \mathcal{P}(^m E; F),$$

and

$$\begin{aligned}||P||_{\mathcal{I}^{dual} \circ \mathcal{P}} &= ||P_L||_{\mathcal{I}^{dual}} \\ &= ||(P_L)^*||_{\mathcal{I}} \\ &= \left\| J_n^E \circ P^* \right\|_{\mathcal{I}} \\ &\leq \left\| J_n^E \right\| .||P^*||_{\mathcal{I}} \\ &= ||P||_{\mathcal{I}^{\mathcal{P}-dual}}.\end{aligned}$$

Conversely, assume that

$$P \in \mathcal{I}^{dual} \circ \mathcal{P}(^m E; F).$$

Then

$$P = u \circ Q \quad with \quad u^* \in \mathcal{I}(G; F),$$

and $Q \in \mathcal{P}(^mE;G)$ for some Banach space G. It follows that

$$P^* = Q^* \circ u^* \in \mathcal{I}(F^*;\mathcal{P}(^mE)),$$

by the ideal property of $\mathcal{I}$, hence

$$P \in \mathcal{I}^{\mathcal{P}-dual}(^mE;F).$$

In the normed case, given $\varepsilon > 0$,G, u and Q can be chosen such that

$$||u||_{\mathcal{I}^{dual}}.||Q|| \leq (1+\varepsilon)||P||_{\mathcal{I}\circ\mathcal{P}}.$$

So

$$\begin{aligned}
||P||_{\mathcal{I}^{\mathcal{P}-dual}} &= ||P^*||_{\mathcal{I}} \\
&= ||Q^* \circ u^*||_{\mathcal{I}} \\
&\leq ||Q^*||.||u^*||_{\mathcal{I}} \\
&= ||u||_{\mathcal{I}^{dual}}.||Q|| \\
&\leq (1+\varepsilon)||P||_{\mathcal{I}^{dual}\circ\mathcal{P}}.
\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain $||P||_{\mathcal{I}^{\mathcal{P}-dual}} \leq ||P||_{\mathcal{I}^{dual}\circ\mathcal{P}}$. □

The next application concerns the polynomial dual $\mathcal{I}^{\mathcal{P}-dual}$ of a given operator ideal $\mathcal{I}$ defined in [6] as

$$\mathcal{I}^{\mathcal{P}-dual}(^mE;F) = \{P \in \mathcal{P}(^mE;F) : P^* \in \mathcal{I}(F^*;\mathcal{P}(^mE))\},$$

where P^* is the Aron-Schottenloher adjoint of P, that is,

$$P^*(\varphi)(x) = \varphi(P(x)).$$

Definition 3.12. *An operator ideal is said to be symmetric if*

$$\mathcal{I} = \mathcal{I}^{dual}$$

Corollary 3.13. *A symmetric operator ideal is injective if and only if its polynomial dual is an injective polynomial ideal.*

Proof. Since

$$\mathcal{I}^{\mathcal{P}-dual} = \mathcal{I}^{dual} \circ \mathcal{P}$$

for every operator ideal $\mathcal{I}$ [From Theorem 3.11], the result follows from Corollary 3.9. □

Now we characterize the polynomial duals of the ideal $\mathcal{D}_p$ of Cohen strongly p-summing operators (see [[7],[10]]). and $\mathfrak{J}_p$ for the ideal of p-integral operators (see [23]).

Corollary 3.14. $\mathcal{D}_p^{\mathcal{P}-dual} = (\mathfrak{J}_{p^*} \circ \mathcal{P})^{inj}$.

Proof. In

$$\mathcal{D}_P^{\mathcal{P}-dual} = \mathcal{D}_P^{dual} \circ \mathcal{P} = \Pi_{p^*} \circ \mathcal{P} = \mathfrak{J}_{P^*}^{inj} \circ \mathcal{P} = (\mathfrak{J}_{P^*} \circ \mathcal{P})^{inj}$$

Π_{p^*} denotes the ideal of absolutely p^*-summing operators. the first equality follows from Theorem 3.11, the second from ([10]), the third from [[23], Theorem 19.2.7] and the fourth from Theorem 3.7. □

To complete the answer to the question ask at the beginning, (see[9] Example 3.10) give an example of an injective polynomial ideal failing the strong domination property, which establishes, in particular, that the weak and the strong domination properties are not equivalent.

Bibliography

[1] D. ACHOUR, Multilinear extensions of absolutely (p ,q ,r)-summing operators, Rend. Circ. Mat. Palermo. 60 (2011), 337-350.

[2] D. ACHOUR and A. ALOUANI, On the multilinear generalizations of the concept of nuclear operators, Colloquium Math. 120 (2010), 85-102.

[3] D. ACHOUR and L. MEZRAG, On the Cohen strongly p-summing multilinear operators, J. Math. Anal. Appl. 327 (2007), 550-563.

[4] R.M. ARON, G. GALINDO, Weakly compact multilinear mappings, Proc. Edinburgh Math. Soc. 40 (1997), 181-192.

[5] G. BOTELHO, H.-A. BRAUNSS and H. JUNEK, Almost p-summing polynomials and multilinear mappings, Arch. Math. 76 (2001), 109-118.

[6] G.BOTELHO , E.ÇALIŞKAN , G. MORAES, The polynomial dual of an operator ideal. Monatsh. Math. 173, 161–174 (2014).

[7] G.BOTELHO, J. CAMPOS, J. SANTOS, Operator ideals related to absolutely summing and Cohen strongly summing operators. Pac. J. Math. 287(1), 1–17 (2017)

[8] G.BOTELHO, D.PELLEGRINO, P. RUEDA, : On composition ideals of multilinear operators and homogeneous polynomials. Publ. Res. Inst. Math. Sci. 43, 1139– 1155 (2007).

[9] G.BOTELHO, LA.TORRES : Injective polynomial ideals and the dominationproperty, Results in Mathematics. (2020), 10.1007/s00025-019-1150-4.

[10] J.S. COHEN, Absolutely p-summing, p-nuclear operators and their conjugates, Math. Ann. 201 (1973), 177–200.

[11] E. ÇALISKAN, D. PELLEGRINO, On the multilinear generalizations of the concept of absolutely summing operators, Rocky Mountain J. Math. 37 (2007), 1137–1154.

[12] E. DAHIA, D. ACHOUR and E.A. SÁNCHEZ-PÉREZ, Absolutely continuous multilinear operators, J. Math. Anal. Appl. 397 (2013), 205-224.

[13] V. DIMANT, Strongly p-summing multilinear operators, J. Math. Anal. Appl. 278 (2003), 182-193.

[14] S.DINEEN , Complex analysis on infinite dimensional spaces. Springer-Verlag, London,1999.

[15] K. FLORET, D. GARCÍA, On ideals of polynomials and multilinear mappings between Banach spaces. Arch. Math. 81 , 300-308(2003)

[16] L. GARCIA, adjunto de um polinômio homogêneo contínuo entre espaços de Banach, Tese de Doutorado, Universidade federal de Uberlândia, 2013.

[17] J. Gutiérrez and I. VILLANUEVA, Extensions of multilinear operators and Banach space properties, Proc. Royal. Soc. Edinburgh Series A 133 (2003).

[18] H.JARCHOW, Locally convex spaces, B. G. Teubner, Stuttgart, 1981.

[19] M. C. MATOS, On multilinear mappings of nuclear type, Rev. Mat. Comput, 6 (1993), 61-81.

[20] J. MUJICA, $\mathcal{T}(p;q)$-summing mappings and the domination theorem, Port. Math. (2) 65 (2008), 211–226.

[21] J.MUJICA, Linearization of bounded holomorphic mappinson Banach spaces, Trans.Amer. Math.Soc.324(1991)867–887.

[22] J. MUJICA, Complex analysis in Banach spaces, Dover Publications, Dover, 2010.

[23] A.PIETSCH, Operator Ideals. North-Holland, Amsterdam(1980).

[24] A.PIETSCH, Ideals of multilinear functionals (designs of a theory). Proceedings of the Second International Conference on Operator Algebras, Ideals, and their Applications in Theoretical Physics (Leibzig). Teubner-Texte. (1983), 185-199.

[25] A.PIETSCH, The ideal of p-compact operators and its maximal hull. Proc. Am. Math. Soc. 142, 519–530 (2013)

[26] R.RYAN, Introduction to Tensor Products of Banach Spaces. Springer, London, (2002)

ملخص

في هذا العمل سنقوم بدراسة النظرية الأساسية للمؤثر المثالي المتباين لكثيرات الحدود المتجانسة .

كما نتطرق الى دراسة خاصية الهيمنة وهل تميز المؤثر المثالي المتباين لكثيرات الحدود المتجانسة ؟

ونميز أيضا ثنائيات الحدود ونقدم مثالا على ذلك.

الكلمات المفتاحية :

المؤثر المثالي , نظرية الهيمنة , فضاء بناخ , الجداء الموتر , كثيرات الحدود المتجانسة .

مؤثر مثالي متباين.

Abstract

In this study, we will discuss, the basic theory of injective ideals of m-homogeneous polynomials, we characterize injective polynomial ideals by means of a domination property, and also we characterize the polynomial duals an example is given.

Key-words: Banach spaces, polynomial ideals, domination property, operator ideal , m- homogeneous polynomials, injective operator ideal, Tensor products .

Résumé

Dans cette étude, nous discuterons de la théorie de base des idéaux injectifs de polynômes m-homogènes, nous caractérisons les idéaux polynomiaux injectifs par la propriété de domination et que nous caractérisant également le dual polynomiaux par un exemple est donné.

Mots clés : Espace de Banach , idéaux polynomiaux, domination Propriété,

Polynômes m-homogènes, Opérateur injectif Idéal, Produits Tensor.