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جامعة محمد بوضياف بالمسيلة

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هلال عبد العزيز المعنونة بـ:

Introduction to Topology

كمرجع للدروس لطلبة السنة الثانية ليسانس رياضيات.

وهذا بعد الاطلاع على التقارير الإيجابية للأستاذ الخبير المكلف بالمطبوعة.

رئيس اللجنة العلمية

جامعة محمد بوضياف بالمسيلة
كلية الرياضيات والإعلام الآلي
رئيس اللجنة العلمية لقسم الرياضيات

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Second Class License Academic (LMD)

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Introduction to Topology

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Contents

Symbols

In the follow up, we will employ the coming symbols:

X	or $Y,\ A,\ B$ .... Sets.
(X, d)	Metric space with metric d.
$B(x, r)$	Open ball centred at x with radius r.
$\mathbf{V}(x)$	or $\mathbf{N}(x)$ Set of neighbourhood of x.
$\mathcal{V}(x)$	The family of neighbourhoods of x.
$(x_n)_n$	Sequence of points.
$\rightarrow$	Limit of a sequence.
(X, τ)	Topological space.
$\mathcal{O} \subseteq X$	Open subset of topological space.
$\mathcal{F} \subseteq X$	Closed subset of topological space.
D°	Open interior of a subset $D \subseteq X$.
$\overline{D}$	Closure of a subset $D \subseteq X$.
∂D	$:= \overline{D} - D^\circ$ Boundary of a subset $D \subseteq X$.
$D^{'}$	Accumulation points of a subset $D \subseteq X$.
$\mathcal{B} \subseteq X$	Sub-basis or basis of a topology.
$\Delta = \{(x, x)\} \subset X \times X$	The diagonal of a topological space X.
$f : X \rightarrow Y$	A map between two topological spaces.
$f^{-1}(D)$	The pre-image.
$\{\mathcal{O}_i\}$	Open cover of X.
$X^{'}$	The dual space of X.
$C(X)$	Space of continuous functions on X.
$L(X, Y)$	Space of continuous maps from X to Y.
$\mathcal{L}(X, Y)$	Space of continuous bounded maps from X to Y.
$C^1(X, Y)$	Space of differentiable maps from X to Y.
$\\| \cdot \\|$	Norm of a function.

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Introduction

This course is designed for second class license mathematics academic (LMD) students for the third semester, introducing the fundamental concepts of topology necessary for advanced mathematical study.

This handout is the result of ten academic years of teaching experience in the Department of Mathematics at the University of M'sila. It is composed of four chapters. At the end of each chapter, there are exercises, with solutions provided at the end of the course. Additionally, some exams with their keys answers are included.

The first chapter presents preliminary concepts on metric spaces.

The second chapter explores the structure of topological spaces and their properties.

The third chapter covers compactness and connectedness.

Finally, the fourth chapter introduces the concepts and properties of normed vector spaces and continuous linear applications, along with key theorems commonly used in functional analysis and PDEs, including their applications.

We would like to express our gratitude to Pr. Saadi Abderachid for his valuable feedback, the teaching team of the Topology course, as well as Dr. Abdelhamid Tallab, Dr. Gagui Bachir, and Dr. Mechter Rabah for their assistance and insightful advice, which helped refine this handout.

Historical overview

The foundations of modern topology emerged through successive generalizations of classical analysis. Metric spaces were first introduced by Maurice Fr chet in his 1906 thesis *Sur quelques points du calcul fonctionnel*, building upon earlier work on function spaces by Ascoli (1883), Arzel (1889), and Volterra (1887). Felix Hausdorff's 1914 *Grundz ge der Mengenlehre* formalized the abstract notion of topological spaces, while separability (Fr chet, 1906) and density (du Bois-Reymond, 1882) emerged as fundamental properties.

The modern theory of limits evolved through three stages: Cauchy's *Cours d'Analyse* (1821) established the ϵ-δ framework, Abel (1826) developed uniform convergence, and E.H. Moore's 1915 general limit concept (formalized by his student H.L. Moore in 1922) abstracted these ideas to topological spaces. Compactness first appeared in Fr chet's 1904 notion of *compactness via sequential convergence* (now called sequential compactness), with the modern covering definition due to Alexandroff-Urysohn (1923).

Connectedness originated in Cantor's 1883 study of perfect sets in $\mathbb{R}^n$, with Jordan's 1892 *Cours d'Analyse* introducing the separation property. The standard definition emerged through Lennes (1911) and Hausdorff (1914), while path-connectednessimplicit in Weierstrass' 1880 curve workwas formalized by Hahn (1914). These developments collectively transformed analysis into modern topology.

Mathematics is the most beautiful and most powerful creation of the human spirit.- Stefan Banach, Polish mathematician

Chapter 1

Metric Spaces

This chapter develops the theory of metric spaces, providing the essential framework for modern analysis and serving as the foundation for subsequent topics in topology and functional analysis. For a thorough treatment of these concepts, we direct the reader to classic references [1, 2, 6, 7], among other authoritative sources.

1.1 The Space with Distance

Definition 1.1 (Metric) *A metric or distance on a set X is a function d that assigns a real number to each pair of elements of X such that:*

1. ***Non-negativity**: $d(x, y) \geq 0$ with equality if and only if $x = y$.*
2. ***Symmetry**: $d(x, y) = d(y, x)$ for all $x, y \in X$.*
3. ***Triangle inequality**: $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

Example 1.1

- ***Euclidean metric**: $d_2(x, y) = \left[\sum_{i=1}^{N} |x_i - y_i|^2\right]^{1/2}$.*

  ***Proof** Properties non-negativity and symmetry are clear. For the triangle inequality, use Cauchy-Schwarz:*

  $$\sum |x_i - z_i|^2 \leq \sum |x_i - y_i|^2 + \sum |y_i - z_i|^2 + 2\left(\sum |x_i - y_i|^2\right)^{1/2}\left(\sum |y_i - z_i|^2\right)^{1/2}$$

  Taking square roots gives the result. ■

- ***Metric** d_1: $d_1(x, y) = \sum_{i=1}^{N} |x_i - y_i|$.*

Proof *Non-negativity and symmetry are immediate. For the triangle inequality:*

$$\sum |x_i - z_i| \leq \sum(|x_i - y_i| + |y_i - z_i|) = \sum |x_i - y_i| + \sum |y_i - z_i|$$

by the regular triangle inequality. ■

- ***Metric*** d_p*: For* $p \geq 1$*,* $d_p(x, y) = \left[\sum_{i=1}^{N} |x_i - y_i|^p\right]^{1/p}$.

Proof *The Minkowski inequality proves the triangle inequality:*

$$\left(\sum |x_i + y_i|^p\right)^{1/p} \leq \left(\sum |x_i|^p\right)^{1/p} + \left(\sum |y_i|^p\right)^{1/p}$$

■

- ***Metric*** d_∞*:* $d_\infty(x, y) = \max_{1 \leq i \leq N} |x_i - y_i|$.

Proof *For any* i*,* $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i| \leq d_\infty(x, y) + d_\infty(y, z)$*. Taking max over* i *gives the result.* ■

- ***Discrete metric****:* $d(x, y) = \begin{cases} 1 & \textit{if } x \neq y \\ 0 & \textit{if } x = y \end{cases}$.

Proof *The only non-trivial case is when* $x \neq z$*. Then either* $x \neq y$ *or* $y \neq z$ *(or both), so* $1 \leq 1 + d(y, z)$ *and* $1 \leq d(x, y) + 1$*.* ■

- ***Metric on*** $C[a, b]$*:* $d_1(f, g) = \int_a^b |f(x) - g(x)|\, dx$*.*

Proof *Follows from properties of integrals and absolute value:*

$$\int |f - h| \leq \int (|f - g| + |g - h|) = \int |f - g| + \int |g - h|$$

■

- ***Supremum metric****:* $d_\infty(f, g) = \sup_{a \leq x \leq b} |f(x) - g(x)|$.

Proof *For any* x*,* $|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)| \leq d_\infty(f, g) + d_\infty(g, h)$*. Taking supremum preserves the inequality.* ■

Definition 1.2

- *A* ***metric space*** *is a pair* (X, d)*, where* X *is a set and* d *is a metric on* X*.*

- *The* ***usual metric*** *on* $\mathbb{R}$ *is* $d(x,y) = |x-y|$.
- *Metrics can define limits and continuity, generalizing the* ε-δ *definition.*

Proposition 1.1 (Technical inequalities) *Let* $x, y \geq 0$ *and* $a, b, c \in \mathbb{R}^N$*:*

1. ***Young's inequality****: For* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*,* $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$.
2. ***H lder's inequality****: For* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*,*
$$\sum_{i=1}^{N} |a_i| \cdot |b_i| \leq \left(\sum_{i=1}^{N} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{N} |b_i|^q\right)^{1/q}.$$
3. ***Minkowski's inequality****: For* $p > 1$*,*
$$\left(\sum_{i=1}^{N} |a_i - b_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{N} |a_i - c_i|^p\right)^{1/p} + \left(\sum_{i=1}^{N} |c_i - b_i|^p\right)^{1/p}.$$

Proof (1) Young's Inequality: Let $x, y \geq 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The inequality:
$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$
is proved using convexity of the functions $t \mapsto t^p$ and $s \mapsto s^q$ on $[0, \infty)$. Alternatively, it follows from the inequality:
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{for all } a, b \geq 0,$$
which is tight when $a^p = b^q$.

(2) H lder's Inequality: Let $a, b \in \mathbb{R}^N$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then by applying Young's inequality componentwise:
$$|a_i||b_i| \leq \frac{|a_i|^p}{p} + \frac{|b_i|^q}{q}.$$
Summing both sides from $i = 1$ to N gives:
$$\sum_{i=1}^{N} |a_i||b_i| \leq \frac{1}{p}\sum_{i=1}^{N} |a_i|^p + \frac{1}{q}\sum_{i=1}^{N} |b_i|^q.$$
To sharpen this, define:
$$A = \left(\sum_{i=1}^{N} |a_i|^p\right)^{1/p}, \quad B = \left(\sum_{i=1}^{N} |b_i|^q\right)^{1/q}.$$

If $A, B > 0$, define $\alpha_i = |a_i|/A$, $\beta_i = |b_i|/B$, so that:

$$\sum_{i=1}^{N} \alpha_i^p = 1, \quad \sum_{i=1}^{N} \beta_i^q = 1.$$

Then using Young's inequality on $\alpha_i \beta_i$ and summing:

$$\sum_{i=1}^{N} \alpha_i \beta_i \leq 1,$$

which implies:

$$\sum_{i=1}^{N} |a_i||b_i| \leq AB = \left(\sum_{i=1}^{N} |a_i|^p \right)^{1/p} \left(\sum_{i=1}^{N} |b_i|^q \right)^{1/q}.$$

(3) Minkowski's Inequality: Let $a, b \in \mathbb{R}^N$, and define the ℓ^p-distance:

$$\|a - b\|_p = \left(\sum_{i=1}^{N} |a_i - b_i|^p \right)^{1/p}.$$

We want to prove:

$$\|a - b\|_p \leq \|a - c\|_p + \|c - b\|_p.$$

Set $u_i = |a_i - c_i|$, $v_i = |c_i - b_i|$. Then by the triangle inequality in $\mathbb{R}$:

$$|a_i - b_i| \leq u_i + v_i.$$

Hence,

$$\sum_{i=1}^{N} |a_i - b_i|^p \leq \sum_{i=1}^{N} (u_i + v_i)^p.$$

By the convexity of $t \mapsto t^p$ and the generalized Minkowski inequality for sequences (proved via H lder's inequality), we have:

$$\left(\sum_{i=1}^{N} (u_i + v_i)^p \right)^{1/p} \leq \left(\sum_{i=1}^{N} u_i^p \right)^{1/p} + \left(\sum_{i=1}^{N} v_i^p \right)^{1/p},$$

which gives:

$$\|a - b\|_p \leq \|a - c\|_p + \|c - b\|_p.$$

■

1.2 Balls, Interior, and Open sets

Definition 1.3 (Open ball) *In a metric space (X, d), the open ball with center $x \in X$ and radius $r > 0$ is:*

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

- In $\mathbb{R}$, $B(x, r) = (x - r, x + r)$.
- For the discrete metric, $B(x, 1) = \{x\}$.
- In $X = [0, 2]$, $B(0, 1) = [0, 1)$.

Definition 1.4 (Bounded) *A subset $A \subset X$ is **bounded** if $A \subset B(x, r)$ for some $x \in X$ and $r > 0$.*

Definition 1.5 (Open set) *A subset $U \subset X$ is **open** if for each $x \in U$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$.*

Proposition 1.2 (Properties of open sets) *Let (X, d) be a metric space. Then:*

1. *$\emptyset$ and X are open.*
2. *Arbitrary unions of open sets are open.*
3. *Finite intersections of open sets are open.*
4. *Every open ball is open.*
5. *A set is open if and only if it is a union of open balls.*

Proof (1) *$\emptyset$ and X are open:* By definition of the topology induced by a metric, $\emptyset$ and X are always included in the collection of open sets.

(2) *Arbitrary unions of open sets are open:* Let $\{U_\alpha\}_{\alpha \in A}$ be a family of open subsets of X. Let $U = \bigcup_{\alpha \in A} U_\alpha$. Let $x \in U$. Then $x \in U_{\alpha_0}$ for some $\alpha_0 \in A$, and since U_{α_0} is open, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U_{\alpha_0} \subset U$. Hence, x is an interior point of U, so U is open.

(3) *Finite intersections of open sets are open:* Let $U_1, U_2, \ldots, U_n$ be open subsets of X, and set $U = \bigcap_{i=1}^{n} U_i$. Let $x \in U$. Then $x \in U_i$ for all $i = 1, \ldots, n$, and since each U_i is open, there exists $\varepsilon_i > 0$ such that $B(x, \varepsilon_i) \subset U_i$. Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$, then $B(x, \varepsilon) \subset U_i$ for all i, so $B(x, \varepsilon) \subset U$. Thus U is open.

(4) *Every open ball is open:* Let $B(a,r) = \{x \in X : d(x,a) < r\}$ be an open ball. Let $x \in B(a,r)$. Set $\delta = r - d(x,a) > 0$. Then for any $y \in B(x,\delta)$,

$$d(a,y) \leq d(a,x) + d(x,y) < d(a,x) + \delta = r.$$

So $y \in B(a,r)$, and hence $B(x,\delta) \subset B(a,r)$. Thus $B(a,r)$ is open.

(5) *A set is open if and only if it is a union of open balls:*

(⇒) Suppose $U \subset X$ is open. Then for every $x \in U$, there exists $\varepsilon_x > 0$ such that $B(x,\varepsilon_x) \subset U$. Hence,

$$U = \bigcup_{x \in U} B(x,\varepsilon_x),$$

i.e., U is a union of open balls.

(⇐) A union of open balls is open by property (2). Hence, if $U = \bigcup_{\alpha \in A} B(x_\alpha, r_\alpha)$, with each ball open, then U is open.

■

Definition 1.6 (Closed set) *A subset $A \subset X$ is* ***closed*** *if its complement $X - A$ is open.*

Proposition 1.3 (Properties of closed sets) *Let (X,d) be a metric space. Then:*

1. *Closed sets contain limits of their convergent sequences.*
2. *$\emptyset$ and X are closed.*
3. *Finite unions of closed sets are closed.*
4. *Arbitrary intersections of closed sets are closed.*

Proof (1) Let $F \subset X$ be closed and let (x_n) be a sequence in F that converges to $x \in X$. We want to show that $x \in F$.

Since F is closed, $X \setminus F$ is open. Suppose by contradiction that $x \notin F$, i.e., $x \in X \setminus F$. Since $X \setminus F$ is open, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subset X \setminus F$. But since $x_n \to x$, there exists N such that $n \geq N$ implies $x_n \in B(x,\varepsilon) \subset X \setminus F$, contradicting $x_n \in F$. Hence, $x \in F$.

(2) The complement of X is $\emptyset$ which is open, so X is closed. The complement of $\emptyset$ is X, which is open, so $\emptyset$ is closed.

(3) Let $F_1, F_2, \ldots, F_n$ be closed subsets of X. Then for each i, $X \setminus F_i$ is open. The union $\bigcup_{i=1}^{n}(X \setminus F_i)$ is open (finite unions of open sets are open). Its complement is

$$X \setminus \bigcup_{i=1}^{n}(X \setminus F_i) = \bigcap_{i=1}^{n} F_i,$$

which is closed. Hence, finite unions of closed sets are closed.

(4) Let $\{F_\alpha\}_{\alpha\in A}$ be a collection of closed sets. Then for each α, $X \setminus F_\alpha$ is open. So $\bigcup_{\alpha\in A}(X \setminus F_\alpha)$ is open. Its complement is

$$X \setminus \bigcup_{\alpha\in A}(X \setminus F_\alpha) = \bigcap_{\alpha\in A} F_\alpha,$$

which is closed. So arbitrary intersections of closed sets are closed. ∎

Definition 1.7 *Let (X, d) be a metric space, $A \subset X$ a non-empty subset, and $x \in X$ a point. The **distance from** x **to** A is defined as:*

$$d(x, A) := \inf_{a\in A} d(x, a)$$

Proposition 1.4 *For any metric space (X, d) and non-empty subset $A \subset X$:*

1. $d(x, A) = 0 \iff x \in \overline{A}$ *(the closure of A)*
2. *The function $x \mapsto d(x, A)$ is Lipschitz continuous with constant 1:*

$$|d(x, A) - d(y, A)| \le d(x, y) \quad \forall x, y \in X$$

3. *If A is compact, the infimum is attained: $\exists a_0 \in A$ such that $d(x, A) = d(x, a_0)$*

Example 1.2 *In $\mathbb{R}^2$ with standard metric:*

- *Let $A = \{(x, y)|x^2 + y^2 = 1\}$ (unit circle)*
- *For point $p = (2, 0)$: $d(p, A) = 1$ (distance to $(1, 0)$)*
- *For point $q = (0.5, 0.5)$: $d(q, A) = 1 - \sqrt{0.5^2 + 0.5^2} \approx 0.293$*

Example 1.3 *Let X be any set with discrete metric:*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

For any $A \subset X$ and $x \notin A$:

$$d(x, A) = \inf_{a\in A} d(x, a) = 1$$

Example 1.4 *Consider $C[0, 1]$ with supremum norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$:*

- *Let $A = \{f \in C[0, 1]|f(0) = 0\}$*

- *For $g(x) = 1 + x$, $d(g, A) = 1$ (achieved by $f(x) = x$)*

Definition 1.8 *For two non-empty subsets $A, B \subset X$:*

$$d(A, B) := \inf_{a \in A, b \in B} d(a, b)$$

Example 1.5 *In $\mathbb{R}$ with usual metric:*

- *$A = (0, 1)$, $B = (2, 3)$: $d(A, B) = 1$*
- *$A = \mathbb{N}$, $B = \{n + \frac{1}{n} | n \in \mathbb{N}\}$: $d(A, B) = 0$ but $AB = \emptyset$*

Definition 1.9 *Two metrics d_1 and d_2 on a set X are* ***equivalent*** *(or* ***topologically equivalent****) if they induce the same topology, i.e., the identity map $(X, d_1) \to (X, d_2)$ is a homeomorphism.*

Proposition 1.5 (Metric Equivalence Criterion) *For metrics d_1, d_2 on X, the following are equivalent:*

(i) d_1 and d_2 are topologically equivalent

(ii) They have the same convergent sequences: $x_n \xrightarrow{d_1} x \iff x_n \xrightarrow{d_2} x$

(iii) $\forall x \in X$, every d_1-ball contains a d_2-ball and vice versa

Example 1.6 (Non-Equivalent Metrics) *On $\mathbb{R}$, consider:*

- *$d_1(x, y) = |x - y|$ (usual metric)*
- *$d_2(x, y) = |\arctan x - \arctan y|$*

These induce different topologies since $x_n = n$ converges in d_2 but not in d_1.

Definition 1.10 *Metrics d_1 and d_2 are* ***uniformly equivalent*** *if $\exists c, C > 0$ such that:*

$$c\, d_1(x, y) \leq d_2(x, y) \leq C\, d_1(x, y) \quad \forall x, y \in X$$

1.3 Topologies Induced by a Metric

Definition 1.11 *Let (X, d) be a metric space. Define τ_d as follows:*

$$\tau_d = \{O \subset X : O = \emptyset \text{ or } \forall x \in O, \exists r_x > 0 \text{ such that } B(x, r_x) \subset O\}$$

The elements of τ_d are called open sets *in X with respect to the distance d.*

The collection τ_d is called the topology associated with the metric d.

Proposition 1.6 *The topology τ_d satisfies the following properties:*

1. *$X, \emptyset \in \tau_d$.*
2. *τ_d is closed under finite intersections.*
3. *τ_d is closed under arbitrary unions.*

Proof 1. Obvious.

2. Let $O_1, \ldots, O_n \in \tau_d$ and let $x \in \bigcap_{i=1}^n O_i$. Then $x \in O_i$ for each i, so there exist $r_i > 0$ such that $B(x, r_i) \subset O_i$. Let $r = \min\{r_1, \ldots, r_n\}$, then:

$$B(x, r) \subset \bigcap_{i=1}^{n} B(x, r_i) \subset \bigcap_{i=1}^{n} O_i.$$

So $\bigcap_{i=1}^n O_i \in \tau_d$.

3. Let $\{O_i\}_{i \in I} \subset \tau_d$ and $x \in \bigcup_{i \in I} O_i$. Then there exists i_0 such that $x \in O_{i_0}$. Hence, there exists $r_{i_0} > 0$ such that $B(x, r_{i_0}) \subset O_{i_0} \subset \bigcup_{i \in I} O_i$. So x has a neighborhood contained in the union, which implies the union is in τ_d. ■

Proposition 1.7 *In the topology associated with a distance d:*

1. *Every open ball is an open set.*
2. *Every closed ball is a closed set.*
3. *Every sphere is a closed set.*

Proof Let $a \in X$ and $r > 0$.

1. Let $x \in B(a, r)$ and set $r_x = d(a, x)$ and choose $\rho_x < \min(r_x, r - r_x)$. For any $y \in B(x, \rho_x)$,

$$d(a, y) \leq d(a, x) + d(x, y) < r_x + (r - r_x) = r,$$

so $B(x, \rho_x) \subset B(a, r)$, hence $B(a, r)$ is open.

2. Let $x \notin \overline{B}(a, r)$. Then $d(a, x) > r$, so $d(a, x) = r + \rho_x$ for some $\rho_x > 0$. For all $y \in B(x, \rho_x)$,

$$d(a, y) \geq d(a, x) - d(x, y) > r + \rho_x - \rho_x = r,$$

so $y \notin \overline{B}(a, r)$ and hence $B(x, \rho_x) \subset E \setminus \overline{B}(a, r)$. So the complement is open and $\overline{B}(a, r)$ is closed.

3. A sphere is the set $\{x \in X : d(a, x) = r\}$. This is the complement of the union of the open ball $B(a, r)$ and the exterior $\{x : d(a, x) > r\}$, both of which are open, hence the sphere is closed. ∎

Definition 1.12 *A topological space (X, τ) is **metrizable** if $\exists$ a metric d on X that induces τ.*

Theorem 1.1 (Urysohn's Metrization Theorem) *Every second-countable, regular, Hausdorff space is metrizable.*

Example 1.7 (Non-Metrizable Spaces)

- *The Sorgenfrey line (lower limit topology on $\mathbb{R}$)*
- *Zariski topology on algebraic varieties*
- *Any non-Hausdorff space like the trivial topology*

Definition 1.13 *A metric space (X, d) is **separable** if it contains a countable dense subset, i.e., $\exists \{x_n\}_{n=1}^{\infty} \subseteq X$ such that $\overline{\{x_n\}} = X$.*

Theorem 1.2 *For metric spaces, separability implies:*

- *Second-countability (has countable base)*
- *Lindel f property (every open cover has countable subcover)*

Proof Let $D = \{d_n\}$ be a countable dense set. Then $\{B_{1/k}(d_n) : n, k \in \mathbb{N}\}$ forms a countable base. For Lindel f: Given any open cover, select for each $x \in X$ a set containing some $B_{1/k}(d_n) \ni x$. ∎

Example 1.8

- *$\mathbb{R}^n$: $\mathbb{Q}^n$ is countable dense*
- *ℓ^p spaces ($1 \leq p < \infty$): Sequences with rational entries and finite support are dense*
- *$C[0, 1]$ with sup norm: Polynomials with rational coefficients are dense (Weierstrass)*

1.4 Cluster, Accumulation

Definition 1.14 (Cluster Point / Accumulation Point) *Let (X, d) be a metric space and $A \subset X$. A point $x \in X$ is called a* cluster point *(or* accumulation point*) of A if for every $\varepsilon > 0$, the punctured ball $B(x, \varepsilon) \setminus \{x\}$ contains at least one point of A, i.e.,*

$$\forall \varepsilon > 0, \quad (B(x, \varepsilon) \setminus \{x\})\, A \neq \emptyset.$$

Remark 1.1 *The terms* cluster point *and* accumulation point *are often used interchangeably in metric topology. Sometimes a distinction is made when discussing sequences, but in general topological terms, they refer to the same concept.*

Definition 1.15 (Isolated Point) *A point $x \in A$ is called an* isolated point *of A if there exists $\varepsilon > 0$ such that $B(x, \varepsilon) A = \{x\}$.*

Definition 1.16 (Closure of a Set) *The* closure *of a subset $A \subset X$, denoted by $\overline{A}$, is the set of all points $x \in X$ such that every open ball centered at x intersects A. Equivalently,*

$$\overline{A} = A \cup A',$$

where A' is the set of all cluster points of A.

Proposition 1.8 *A subset $F \subset X$ is closed if and only if it contains all of its cluster points.*

Proof ($\Rightarrow$) Suppose F is closed, and let x be a cluster point of F. Assume by contradiction that $x \notin F$. Since F is closed, its complement is open, so there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset X \setminus F$. But this contradicts the definition of x being a cluster point, since $B(x, \varepsilon)$ contains no point of F. Thus $x \in F$.

($\Leftarrow$) Suppose F contains all its cluster points. Then $X \setminus F$ contains no cluster point of F, meaning it is open (since around every point in $X \setminus F$ we can find a ball not intersecting F). Hence, F is closed. ■

Example 1.9 *Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subset \mathbb{R}$. The point 0 is a cluster point of A, although $0 \notin A$. So $\overline{A} = A \cup \{0\}$.*

1.5 Continuous Mappings

Definition 1.17 (Continuity at a Point) *Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. The function f is said to be* continuous at a point *$x_0 \in X$ if for* *every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$d_X(x, x_0) < \delta \quad \Rightarrow \quad d_Y(f(x), f(x_0)) < \varepsilon.$$

Definition 1.18 (Continuity on a Set) *The function* $f : X \to Y$ *is* continuous on X *if it is continuous at every point* $x \in X$.

Definition 1.19 (Sequential Continuity) *A function* $f : X \to Y$ *is continuous if and only if for every sequence* (x_n) *in* X *that converges to* x*, the sequence* $(f(x_n))$ *converges to* $f(x)$*, i.e.,*

$$x_n \to x \quad \Rightarrow \quad f(x_n) \to f(x).$$

Proposition 1.9 (Characterization via Preimages) *Let* $f : X \to Y$ *be a function between metric spaces. Then* f *is continuous if and only if for every open set* $V \subset Y$*, the preimage* $f^{-1}(V)$ *is open in* X.

Proof We prove both directions of the equivalence.

(⇒) Suppose f is continuous.

Let $V \subset Y$ be an open set. We want to show that $f^{-1}(V)$ is open in X.

Take any point $x \in f^{-1}(V)$. Then $f(x) \in V$. Since V is open in Y, there exists $\varepsilon > 0$ such that the open ball $B_Y(f(x), \varepsilon) \subset V$.

Since f is continuous at x, there exists $\delta > 0$ such that for all $x' \in X$,

$$d_X(x, x') < \delta \quad \Rightarrow \quad d_Y(f(x), f(x')) < \varepsilon,$$

which means $f(x') \in B_Y(f(x), \varepsilon) \subset V$.

Hence, $x' \in f^{-1}(V)$ for all $x' \in B_X(x, \delta)$, i.e.,

$$B_X(x, \delta) \subset f^{-1}(V).$$

This shows that every point $x \in f^{-1}(V)$ has a neighborhood entirely contained in $f^{-1}(V)$, so $f^{-1}(V)$ is open in X.

(⇐) Suppose that for every open set $V \subset Y$, the preimage $f^{-1}(V)$ is open in X.

We want to prove that f is continuous at every point $x \in X$.

Let $x \in X$ and let $\varepsilon > 0$. Then the open ball $B_Y(f(x), \varepsilon)$ is open in Y. By assumption, the preimage

$$U = f^{-1}(B_Y(f(x), \varepsilon))$$

is open in X. Since $x \in U$, and U is open, there exists $\delta > 0$ such that $B_X(x, \delta) \subset U$.

So for all $x' \in B_X(x, \delta)$, we have $f(x') \in B_Y(f(x), \varepsilon)$, i.e.,

$$d_Y(f(x'), f(x)) < \varepsilon.$$

This proves that f is continuous at x. Since x was arbitrary, f is continuous on X. ■

Example 1.10 (Linear Function) *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 1$. Then f is continuous on $\mathbb{R}$, since for every $x \in \mathbb{R}$ and every $\varepsilon > 0$, choosing $\delta = \varepsilon/2$ ensures that*

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| = 2|x - x_0| < \varepsilon.$$

Example 1.11 (Non-Continuous Function) *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is not continuous at $x = 0$, since any neighborhood of 0 contains points $x \neq 0$ with $f(x) = 1$, and thus $\lim_{x \to 0} f(x)$ does not exist.

Example 1.12 (Absolute Value) *The function $f(x) = |x|$ is continuous on $\mathbb{R}$. This can be shown directly using the definition:*

$$\big|\,|x| - |x_0|\,\big| \leq |x - x_0| \quad \Rightarrow \quad \text{continuity follows by choosing } \delta = \varepsilon.$$

Example 1.13 (Distance Function) *Let (X, d) be a metric space and fix $a \in X$. Define $f_a(x) = d(x, a)$. Then $f_a : X \to \mathbb{R}$ is continuous.*

1.6 Sequences and Cauchy Sequences

Definition 1.20 (Convergence) *A sequence $\{x_n\}$ in (X, d)* ***converges*** *to $x \in X$ if for every $\varepsilon > 0$, there exists N such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.*

Proposition 1.10 (Uniqueness of limits) *The limit of a convergent sequence in a metric space is unique.*

Proof Let (X, d) be a metric space and suppose a sequence (x_n) in X converges to both $x \in X$ and $y \in X$. We want to show that $x = y$.

Assume, for contradiction, that $x \neq y$. Then $d(x, y) = \varepsilon > 0$.

Since $x_n \to x$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$d(x_n, x) < \varepsilon/3.$$

Likewise, since $x_n \to y$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$d(x_n, y) < \varepsilon/3.$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, by the triangle inequality:

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3.$$

But this contradicts the assumption that $d(x, y) = \varepsilon > 2\varepsilon/3$.

Therefore, our assumption is false, and we must have $x = y$. Hence, the limit is unique.

■

Definition 1.21 (Cauchy sequence) *A sequence $\{x_n\}$ in (X, d) is* ***Cauchy*** *if for every $\varepsilon > 0$, there exists N such that:*

$$d(x_m, x_n) < \varepsilon \quad \textit{for all } m, n \geq N.$$

Proposition 1.11 (Properties of Cauchy sequences)

1. *Every convergent sequence is Cauchy.*
2. *Every Cauchy sequence is bounded.*

Proof

1. Let (x_n) be a convergent sequence with limit x. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - x| < \epsilon/2$. Then, for any $m, n \geq N$, we have:

$$|x_n - x_m| \leq |x_n - x| + |x - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, (x_n) is a Cauchy sequence.

2. Let (x_n) be a Cauchy sequence. Taking $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|x_n - x_m| < 1$. In particular, for all $n \geq N$, $|x_n - x_N| < 1$. By the triangle inequality, this implies:

$$|x_n| \leq |x_N| + |x_n - x_N| < |x_N| + 1.$$

Let $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + 1\}$. Then, for all $n \in \mathbb{N}$, $|x_n| \leq M$, showing that (x_n) is bounded.

■

- Not all Cauchy sequences converge. Example: $x_n = 1/n$ in $(0, 2)$.

1.7 Complete Metric Space

Proposition 1.12 (Completion) *Every metric space (X, d) has a completion (X', d') with an isometry $f : X \to X'$ such that X' is complete.*

Proof The construction proceeds in several steps:

Step 1: Define the space X'. Let $\mathcal{C}$ be the set of all Cauchy sequences in (X, d). Define an equivalence relation $\sim$ on $\mathcal{C}$ by:

$$(x_n) \sim (y_n) \quad \text{if and only if} \quad \lim_{n\to\infty} d(x_n, y_n) = 0.$$

The completion X' is defined as the quotient space $\mathcal{C}/\sim$, i.e., the set of equivalence classes of Cauchy sequences under $\sim$.

Step 2: Define the metric d' on X'. For two equivalence classes $\overline{x} = [(x_n)]$ and $\overline{y} = [(y_n)]$, define:

$$d'(\overline{x}, \overline{y}) = \lim_{n\to\infty} d(x_n, y_n).$$

This limit exists because $(d(x_n, y_n))$ is a Cauchy sequence in $\mathbb{R}$ (since (x_n) and (y_n) are Cauchy in X) and $\mathbb{R}$ is complete. The metric d' is well-defined because if $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$, then:

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n).$$

Step 3: Define the isometry $f : X \to X'$. For each $x \in X$, let $f(x)$ be the equivalence class of the constant sequence $(x, x, x, \dots)$. Then, for $x, y \in X$:

$$d'(f(x), f(y)) = \lim_{n\to\infty} d(x, y) = d(x, y),$$

so f is an isometry.

Step 4: Show X' is complete. Let $(\overline{x}^{(k)})$ be a Cauchy sequence in X', where each $\overline{x}^{(k)} = [(x_n^{(k)})]$. For each k, choose N_k such that for $m, n \geq N_k$, $d(x_m^{(k)}, x_n^{(k)}) < \frac{1}{k}$. Construct a diagonal sequence $(x_{N_k}^{(k)})$.

We claim $(\overline{x}^{(k)})$ converges to the equivalence class of $(x_{N_k}^{(k)})$. Indeed, for k large enough:

$$d'\left(\overline{x}^{(k)}, [(x_{N_k}^{(k)})]\right) = \lim_{n\to\infty} d(x_n^{(k)}, x_{N_k}^{(k)}) \leq \frac{1}{k} \to 0.$$

Thus, X' is complete.

Step 5: Show $f(X)$ is dense in X'. For any $\overline{x} = [(x_n)] \in X'$, the sequence $(f(x_n))$ converges to $\overline{x}$ in X' because:

$$d'(f(x_n), \overline{x}) = \lim_{m\to\infty} d(x_n, x_m) \to 0 \quad \text{as} \quad n \to \infty,$$

since (x_n) is Cauchy. Hence, X' is the completion of X. ■

Definition 1.22 (Complete metric space) *A metric space (X, d) is* ***complete*** *if every Cauchy sequence converges to a point in X.*

Proposition 1.13 (Properties of completeness)

1. *If a Cauchy sequence has a convergent subsequence, then it converges.*
2. *(Bolzano-Weierstrass) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*
3. *$\mathbb{R}$ is complete.*

Proof

1. Let (x_n) be a Cauchy sequence in a metric space (X, d), and suppose (x_{n_k}) is a convergent subsequence with limit $x \in X$. We show that $x_n \to x$.

   Given $\epsilon > 0$, since (x_n) is Cauchy, there exists N_1 such that for all $m, n \geq N_1$,

   $$d(x_m, x_n) < \frac{\epsilon}{2}.$$

   Since $x_{n_k} \to x$, there exists N_2 such that for all $k \geq N_2$,

   $$d(x_{n_k}, x) < \frac{\epsilon}{2}.$$

   Let $N = \max\{N_1, N_2\}$. For any $n \geq N$, choose k such that $n_k \geq N$ (which is possible since $n_k \to \infty$). Then,

   $$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

   Hence, $x_n \to x$.

2. Let (x_n) be a bounded sequence in $\mathbb{R}$. By the Bolzano-Weierstrass Theorem, there exists a monotonic subsequence (x_{n_k}). Since (x_n) is bounded, (x_{n_k}) is also bounded. By the Monotone Convergence Theorem, (x_{n_k}) converges.

   Alternative Proof (using nested intervals): Since (x_n) is bounded, there exists $[a_1, b_1]$ containing all x_n. Bisect $[a_1, b_1]$ into two subintervals. At least one subinterval contains infinitely many terms of (x_n); call it $[a_2, b_2]$. Repeat this process to obtain a nested sequence of intervals $[a_k, b_k]$ with $b_k - a_k \to 0$. By the Nested Interval Property, $\bigcap_{k=1}^{\infty}[a_k, b_k] = \{x\}$ for some $x \in \mathbb{R}$. Choosing $x_{n_k} \in [a_k, b_k]$ gives a subsequence converging to x.

3. Let (x_n) be a Cauchy sequence in $\mathbb{R}$. Since Cauchy sequences are bounded (by a previous proposition), by the Bolzano-Weierstrass Theorem (part 2), (x_n) has a convergent subsequence. Then, by part 1, (x_n) itself converges. Hence, every Cauchy sequence in $\mathbb{R}$ converges, so $\mathbb{R}$ is complete.

■

Proposition 1.14 (Examples of complete spaces) *1. $\mathbb{R}^N$ is complete with respect to any d_p metric.*

2. $C[a,b]$ is complete with respect to d_∞.

3. $C[a,b]$ is not complete with respect to d_1.

Proof

1. Let $\mathbf{x}^{(k)} = (x_1^{(k)}, \dots, x_N^{(k)})$ be a Cauchy sequence in $\mathbb{R}^N$ with respect to the d_p metric (for $1 \le p \le \infty$).

   For any $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $k, m \ge K$,

   $$d_p(\mathbf{x}^{(k)}, \mathbf{x}^{(m)}) = \left(\sum_{i=1}^{N} |x_i^{(k)} - x_i^{(m)}|^p \right)^{1/p} < \epsilon \quad (\text{or } \max_i |x_i^{(k)} - x_i^{(m)}| < \epsilon \text{ if } p = \infty).$$

   This implies that for each coordinate i, the sequence $(x_i^{(k)})$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $x_i^{(k)} \to x_i$ for some $x_i \in \mathbb{R}$.

   Let $\mathbf{x} = (x_1, \dots, x_N)$. Then, as $k \to \infty$,

   $$d_p(\mathbf{x}^{(k)}, \mathbf{x}) \to 0,$$

   proving that $\mathbb{R}^N$ is complete under d_p.

2. Let (f_n) be a Cauchy sequence in $(C[a,b], d_\infty)$. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \ge N$,

   $$d_\infty(f_n, f_m) = \sup_{x \in [a,b]} |f_n(x) - f_m(x)| < \epsilon.$$

   This implies (f_n) is uniformly Cauchy. By the uniform limit theorem, (f_n) converges uniformly to some $f \in C[a,b]$. Hence, $C[a,b]$ is complete under d_∞.

3. Consider the sequence (f_n) in $C[0,1]$ defined by:

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2}, \\ n\left(x - \frac{1}{2}\right) & \text{if } \frac{1}{2} < x \le \frac{1}{2} + \frac{1}{n}, \\ 1 & \text{if } \frac{1}{2} + \frac{1}{n} < x \le 1. \end{cases}$$

This sequence is Cauchy in $(C[0,1], d_1)$ since for $m > n$,

$$d_1(f_n, f_m) = \int_0^1 |f_n(x) - f_m(x)|\, dx \le \frac{1}{n} - \frac{1}{m} \to 0.$$

However, (f_n) converges pointwise (and in L^1) to the discontinuous function:

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

Since the limit $f \notin C[0,1]$, the space $(C[0,1], d_1)$ is not complete.

■

Proposition 1.15 (Complete subsets) *A subset A of a complete metric space (X,d) is complete if and only if A is closed in X.*

Proof We prove both directions:
($\Rightarrow$) [Complete $\Rightarrow$ Closed]
Assume A is complete. Let (x_n) be a sequence in A that converges to some $x \in X$. Since convergent sequences are Cauchy and A is complete, (x_n) must converge to a limit in A. Thus $x \in A$, showing A contains all its limit points. Therefore, A is closed.
($\Leftarrow$) [Closed $\Rightarrow$ Complete]
Assume A is closed in X. Let (x_n) be a Cauchy sequence in A. Since X is complete, (x_n) converges to some $x \in X$. But A is closed, so it contains all its limit points, hence $x \in A$. Therefore, every Cauchy sequence in A converges in A, proving A is complete. ■

Definition 1.23 (Continuity) *A function $f : X \to Y$ is **continuous** at $x \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that:*

$$d_X(x,y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon.$$

Definition 1.24 (Lipschitz continuous) *A function $f : X \to Y$ is **Lipschitz continuous** if there exists $L \ge 0$ such that:*

$$d_Y(f(x), f(y)) \le L \cdot d_X(x,y) \quad \text{for all } x, y \in X.$$

Proposition 1.16 (Properties of Lipschitz continuity) *1. Every Lipschitz continuous function is continuous.*

2. If $f : [a, b] \to \mathbb{R}$ is differentiable with bounded derivative, then f is Lipschitz continuous.

Proof

1. Let $f : X \to Y$ be Lipschitz continuous between metric spaces (X, d_X) and (Y, d_Y). This means there exists a constant $L > 0$ such that for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2).$$

To show f is continuous at any $x_0 \in X$, let $\epsilon > 0$ and take $\delta = \epsilon/L$. Then for all $x \in X$ with $d_X(x, x_0) < \delta$, we have

$$d_Y(f(x), f(x_0)) \leq L d_X(x, x_0) < L \cdot \frac{\epsilon}{L} = \epsilon.$$

Hence, f is (uniformly) continuous.

2. Suppose f is differentiable on $[a, b]$ with $|f'(x)| \leq M$ for some $M > 0$ and all $x \in [a, b]$. By the Mean Value Theorem, for any $x, y \in [a, b]$ with $x \neq y$, there exists c between x and y such that

$$\frac{f(x) - f(y)}{x - y} = f'(c).$$

Taking absolute values and multiplying by $|x - y|$ gives

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|.$$

Thus, f is Lipschitz continuous with constant M.

■

Example 1.14

- *The function $f(x) = x^2$ is Lipschitz continuous on $[0, 1]$.*
- *The function $f(x) = \sqrt{x}$ is not Lipschitz continuous on $[0, 1]$.*

1.8 Convergence of Functions

Definition 1.25 (Pointwise and uniform convergence) *Let $\{f_n(x)\}$ be a sequence of functions $f_n : X \to \mathbb{R}$, where X is a metric space.*

- *$f_n(x)$ converges **pointwise** to $f(x)$ if for every $\varepsilon > 0$ and $x \in X$, there exists N such that:*

$$|f_n(x) - f(x)| < \varepsilon \quad \textit{for all } n \geq N.$$

- *f_n converges **uniformly** to f on X if for every $\varepsilon > 0$, there exists N such that:*

$$|f_n(x) - f(x)| < \varepsilon \quad \textit{for all } n \geq N \textit{ and all } x \in X.$$

- For pointwise convergence, N may depend on x.
- For uniform convergence, N must work for all x.
- Uniform convergence implies pointwise convergence.

Proposition 1.17 (Characterizations of convergence) *1. $f_n(x) \to f(x)$ pointwise $\iff$ $|f_n(x) - f(x)| \to 0$ as $n \to \infty$ for each x.*

2. $f_n \to f$ uniformly on X $\iff$ $\sup_{x \in X} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

Proof

1. Pointwise convergence characterization:

   ($\Rightarrow$) If $f_n \to f$ pointwise, then by definition for every $x \in X$ and every $\epsilon > 0$, there exists $N(x, \epsilon) \in \mathbb{N}$ such that for all $n \geq N(x, \epsilon)$,

$$|f_n(x) - f(x)| < \epsilon.$$

   This is exactly the statement that $|f_n(x) - f(x)| \to 0$ as $n \to \infty$ for each fixed x.

   ($\Leftarrow$) Conversely, if $|f_n(x) - f(x)| \to 0$ for each x, then for any $\epsilon > 0$ and each $x \in X$, there exists $N(x, \epsilon)$ such that the above inequality holds. This establishes pointwise convergence.

2. Uniform convergence characterization:

   ($\Rightarrow$) Suppose $f_n \to f$ uniformly on X. Then for every $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ (independent of x) such that for all $n \geq N(\epsilon)$ and all $x \in X$,

$$|f_n(x) - f(x)| < \epsilon.$$

   Taking supremum over $x \in X$ gives:

$$\sup_{x \in X} |f_n(x) - f(x)| \leq \epsilon \quad \text{for all } n \geq N(\epsilon).$$

This shows $\sup_{x\in X} |f_n(x) - f(x)| \to 0$.

($\Leftarrow$) If $\sup_{x\in X} |f_n(x) - f(x)| \to 0$, then for any $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $n \geq N(\epsilon)$,

$$\sup_{x\in X} |f_n(x) - f(x)| < \epsilon.$$

This implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$ simultaneously, which is exactly uniform convergence.

■

Proposition 1.18 (Uniform limit of continuous functions) *If each f_n is continuous and $f_n \to f$ uniformly on X, then f is continuous on X.*

Proof Let $x_0 \in X$ be arbitrary. We will show f is continuous at x_0.

Given $\epsilon > 0$, by uniform convergence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in X$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}.$$

Fix such an $n \geq N$. Since f_n is continuous at x_0, there exists $\delta > 0$ such that for all $x \in X$ with $|x - x_0| < \delta$,

$$|f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}.$$

Now, for any $x \in X$ with $|x - x_0| < \delta$, we have:

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus, f is continuous at x_0. Since x_0 was arbitrary, f is continuous on X. ■

- The pointwise limit of continuous functions need not be continuous. Example: $f_n(x) = x^n$ on $[0, 1]$.

Definition 1.26 (Nowhere Dense Set) *A subset $A \subset X$ is said to be* nowhere dense *in a metric space X if the interior of its closure is empty:*

$$\textit{int}(\overline{A}) = \emptyset.$$

Definition 1.27 (First Category / Meagre Set) *A subset $A \subset X$ is said to be of* first category *(or* meagre*) if it is a countable union of nowhere dense sets:*

$$A = \bigcup_{n=1}^{\infty} A_n, \quad \textit{with each } A_n \textit{ nowhere dense}.$$

Definition 1.28 (Baire Space) *A metric space X is a* Baire space *if the countable union of nowhere dense sets cannot cover the whole space, i.e., every countable union of nowhere dense sets has empty interior.*

Theorem 1.3 (Baire Category Theorem) *Every complete metric space is a Baire space.*

Idea of Proof Assume (X, d) is a complete metric space and $X = \bigcup_{n=1}^{\infty} A_n$, where each A_n is closed and nowhere dense. The goal is to show that this union cannot be dense in X. One constructs a nested sequence of non-empty open sets with diameters tending to zero and applies completeness to find a point in all A_n^c, contradicting the assumption. See standard texts for details. ■

Example 1.15

- *The real line $\mathbb{R}$ with the usual metric is a Baire space.*
- *The rational numbers $\mathbb{Q}$ are not a Baire space (they are of first category in $\mathbb{R}$).*

Definition 1.29 (First Countable Space) *A metric space X is said to be* first countable *if every point $x \in X$ has a countable local base. That is, for each $x \in X$, there exists a countable collection $\{U_n\}$ of open neighborhoods such that any open neighborhood of x contains some U_n.*

Example 1.16 *Every metric space is first countable. Indeed, for $x \in X$, the family of open balls $B(x, 1/n)$ for $n \in \mathbb{N}$ forms a countable local base.*

Definition 1.30 (Second Countable Space) *A metric space X is said to be* second countable *if there exists a countable base for the topology of X, i.e., a countable collection $\mathcal{B}$ of open sets such that every open set is a union of elements of $\mathcal{B}$.*

Example 1.17 *The real line $\mathbb{R}$ is second countable: the collection of open intervals with rational endpoints forms a countable base.*

Remark 1.2

- *Second countable $\Rightarrow$ first countable, but not vice versa.*
- *Every separable metric space is second countable.*

Definition 1.31 (Contraction) *A function $f : X \to X$ is a* ***contraction*** *if there exists $0 \leq \alpha < 1$ such that:*

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y) \quad \text{for all } x, y \in X.$$

Theorem 1.4 (Banach's fixed point theorem) *Every contraction on a complete metric space has a unique fixed point.*

- Contractions are Lipschitz continuous.
- Completeness is essential: $f(x) = x/2$ on $(0, 1)$ has no fixed point.

1.9 Application in Differential Equations

Theorem 1.5 (Existence and uniqueness) *For the IVP $y'(t) = f(t, y(t))$, $y(0) = y_0$, if f is continuous in t and Lipschitz in y, then there exists a unique solution on $[0, \varepsilon]$ for some $\varepsilon > 0$.*

- Solutions may not exist for all t. Example: $y' = y^2$, $y(0) = 1$ has solution $y(t) = 1/(1 - t)$, undefined at $t = 1$.

1.10 Exercises:

Exercise 01

Let $(\mathbb{R}, |\cdot|)$ be the standard topology. Find the interior and closure of the following subsets:

$$A = \left\{-1 + \frac{1}{n} : n \in \mathbb{N}^*\right\}, \quad B = (-1, 1) \cup \{2\} \cup [3, 4), \quad C = \{x \in \mathbb{R} : x^2 \leq 4\} \bigcap [1, 5),$$

$$D = \mathbb{Q} \bigcap [-1, 1]$$

Exercise 02

Let $E = C([0, 1], \mathbb{R})$ and define:

$$d(f, g) = |f(0) - g(0)| + \int_0^1 |f(t) - g(t)|\, dt$$

1. Show that d is a metric on E.
2. Find the open ball in (E, d).

Exercise 03

Let $\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}$, and define:

$$f(x) = \begin{cases} -1 & \text{if } x = -\infty \\ \frac{x}{1+|x|} & \text{if } x \in \mathbb{R} \\ 1 & \text{if } x = +\infty \end{cases}$$

Show that $d(x, y) = |f(x) - f(y)|$ defines a metric on $\mathbb{R}$ and find $B(0, 1)$.

Exercise 04

Let $U \subset \mathbb{R}$ be nonempty with the standard topology. Define:

$$-U = \{-x : x \in U\}, \quad \lambda U = \{\lambda x : x \in U\}, \quad a + U = \{a + x : x \in U\}$$

1. U open $\Leftrightarrow -U$ open
2. U open $\Leftrightarrow \lambda U$ open
3. U open $\Leftrightarrow a + U$ open

Exercise 05

Let E be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A norm is a map:

$$\|\cdot\| : E \to \mathbb{R}_+, \quad x \mapsto \|x\|$$

such that:

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\alpha x\| = |\alpha|\|x\| \quad \forall \alpha \in \mathbb{K}, x \in E$
- $\|x + y\| \leq \|x\| + \|y\|$

Define $d(x, y) = \|x - y\|$. Show that d is a metric on E.

Exercise 06

Let $X = \{a, b, c, d\}$ be a set equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$, $Y = \{1, 2, 3, 4\}$ equipped with $\sigma = \{\emptyset, Y, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$, and suppose that

$$f : (X, \tau) \to (Y, \sigma)$$

is a map defined by: $f(a) = f(b) = 1, f(c) = 2, f(d) = 4$.

1. Calculate $\mathcal{V}(a), \mathcal{V}(b), \mathcal{V}(c), \mathcal{V}(d)$?
2. Calculate $\mathcal{V}(1), \mathcal{V}(2), \mathcal{V}(3), \mathcal{V}(4)$?
3. Calculate $f^{-1}(\mathcal{V}(1)), f^{-1}(\mathcal{V}(2)), f^{-1}(\mathcal{V}(4))$?
4. Study the continuity of f at a, b, c, d?

Exercise 07

Let (X, τ) be a topological space and $A \subset X$. We define the indicator map of A (noted χ_A) from (X, τ) to $(\mathbb{R}, |.|)$ by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Give a necessary and sufficient condition for the indicator application χ_A to be continuous.

Exercise 08

Let $E = C([0, 1]; \mathbb{R})$ be equipped with the following distances

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| dx \quad d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

and assume that the map H from E to $(\mathbb{R}, |.|)$ is defined by $H(f) = \int_0^1 |f(x)| dx$.

1. Prove that H is Lipschitz function from (E, d_1) to $(\mathbb{R}, |.|)$?
2. Prove that H is Lipschitz function from (E, d_∞) to $(\mathbb{R}, |.|)$?
3. Is H a bijective map?

Exercise 09

Is the following sets closed in $(\mathbb{R}^2, d_2)$?

1. $A = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$
2. $B = \{(x, y) \in \mathbb{R}^2 : y \leq x^2\}$
3. $C = \{(x, y, z) \in \mathbb{R}^3 : z \leq x^2 - y^2 + 5\}$

Exercise 10

Build homeomorphisms between:

1. Two intervals of the form $[a; b]$ and $[c; d]$ with $a < b$ and $c < d$.
2. The interval $]-1; 1[$ and $\mathbb{R}$.
3. The circle $C(0, 1)$ and $\mathbb{R}$.
4. The sphere $S(0, 1)$ and $\mathbb{R}^2$.

Exercise 11

Let $d : \mathbb{Q}^* \times \mathbb{Q}^* \longrightarrow \mathbb{R}^+$ be a map such that

$$d(p, q) = \begin{cases} 0, & \text{if } p = q \\ \dfrac{1}{|p|} + \dfrac{1}{|q|}, & \text{if } p \neq q \end{cases}$$

1. Show that d is a distance on $\mathbb{Q}^*$?
2. Are the two sequences $u_n = \dfrac{1}{n}, v_n = n$ Cauchy sequences?
3. Show that $(\mathbb{Q}^*, d)$ is not complete?

Exercise 12

Let $E = \mathbb{N}^*$ be a set. We put for all $n, m \in E$:

$$d(m, n) = \begin{cases} 0, & \text{if } m = n \\ 10 + \dfrac{1}{m} + \dfrac{1}{n}, & \text{if } m \neq n \end{cases}$$

1. Prove that d is a distance on E?

2. Show that (E, d) is not complete?

3. Let $f : E \to E$ be a map with $f(n) = n + 1$. Prove that for all $n, m \in E(n \neq m)$ we have $d(f(n), f(m)) < d(n, m)$, but f is not a contraction.

Task:

Part 01: (Contraction Theorem)

Let (X, d) be a metric space. A function $f : X \longrightarrow X$ is called a contraction (mapping) if there exists a real number $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha\, d(x, y) \text{ for every } x, y \in X.$$

1. Prove that if (X, d) is complete, non-empty, and $f : X \longrightarrow X$ is a contraction, there exists a unique point $c \in X$ such that $f(c) = c$.

2. Show that for every $x \in X$ the sequence $(f_n(x))$ converges to c?

Part 02:

1. Find a map $f : \mathbb{R} \longrightarrow \mathbb{R}$ without fixed points and such that

$$|f(x) - f(y)| < |x - y| \text{ for every } x, y \in \mathbb{R}.$$

2. Let (X, d) be complete and $U \subset X$ closed. Every Cauchy sequence in U is Cauchy in X, so it converges to a limit $x \in \overline{U}$; therefore, (A, d) is complete provided U is closed.

Chapter 2

Topological Spaces

This chapter presents the fundamental concepts and properties of topological spaces, which serve as a natural extension of the theory of metric spaces discussed in the previous chapter. For comprehensive treatments of this subject, we refer to [3, 4, 9] among other excellent references.

2.1 Topology, Open and Closed

Definition 2.1 (Topology) *A topology τ on a set X is a collection of subsets of X, called open sets, satisfying:*

1. *$\emptyset$ and X are open sets ($\emptyset, X \in \tau$)*
2. *The intersection of two open sets is open ($O_1 O_2 \in \tau$)*
3. *The union of any number of open sets is open ($\bigcup_i O_i \in \tau$)*

- Any finite intersection of open sets is open
- A **topological space** (X, τ) consists of a set X with a topology τ

Example 2.1
- ***Trivial topology***: $\tau = \{\emptyset, X\}$
- ***Discrete topology***: $\tau = \mathcal{P}(X)$ *(all subsets)*
- ***Euclidean topology*** *on* $\mathbb{R}$*: Open sets are unions of open intervals*
- ***Upper topology*** *on* $\mathbb{R}$*: Non-empty open sets are* $]-\infty, \alpha[$ *for* $\alpha \in \mathbb{R} \cup \{+\infty\}$

Definition 2.2 (Closed set) *A subset $Z \subset X$ is **closed** if $X - Z$ is open.*

Theorem 2.1 (Properties of closed sets)

1. *$\emptyset$ and X are closed*
2. *Any intersection of closed sets is closed*
3. *The union of two closed sets is closed*
4. *Closed sets contain limits of their convergent sequences*

Definition 2.3 (Closure) *The **closure** $\overline{D}$ of $D \subset X$ is the smallest closed set containing D:*

$$\overline{D} = \bigcap \{Z \subset X : Z \text{ closed}, D \subset Z\}$$

Theorem 2.2 (Properties of closure)

1. *$D \subset \overline{D}$*
2. *If $D \subset Z$, then $\overline{D} \subset \overline{Z}$*
3. *D is closed $\iff$ $\overline{D} = D$*
4. *$\overline{\overline{D}} = \overline{D}$*

- Examples:
  - $[-1, 2)$ has closure $[-1, 2]$
  - $(-1, 2)$ has closure $[-1, 2]$

Definition 2.4 (Interior) *The **interior** D° of $D \subset X$ is the largest open set contained in D:*

$$D^\circ = \bigcup \{O \subset X : O \text{ open}, O \subset D\}$$

Theorem 2.3 (Properties of interior)

1. *$D^\circ \subset D$*
2. *If $D \subset Z$, then $D^\circ \subset Z^\circ$*
3. *D is open $\iff$ $D^\circ = D$*
4. *$(D^\circ)^\circ = D^\circ$*

- Example: $[0, 1]$ has interior $(0, 1)$

Definition 2.5 (Boundary) *The **boundary** of $D \subset X$ is:*

$$\partial D = \overline{D} - D^\circ = \overline{DX - D}$$

- Examples in $\mathbb{R}$ with Euclidean topology:
  - $\partial[0,1] = \{0,1\}$
  - A set and its complement have the same boundary

Definition 2.6 (Neighbourhood) *A **neighbourhood** of $x \in X$ is any open set containing x.*

Theorem 2.4 (Characterizations) *For $D \subset X$:*

1. *$x \in \overline{D} \iff$ Every neighbourhood of x intersects D*
2. *$x \in D^\circ \iff$ Some neighbourhood of x lies within D*
3. *$x \in \partial D \iff$ Every neighbourhood of x intersects both D and $X - D$*
4. *$D^\circ \bigcap \partial D = \emptyset$ and $D^\circ \cup \partial D = \overline{D}$*

Example 2.2

Set	***Interior***	***Closure***	***Boundary***
$\{3\}$	$\emptyset$	$\{3\}$	$\{3\}$
$[1,4)$	$(1,4)$	$[1,4]$	$\{1,4\}$
$(-1,2)\cup(2,3)$	$(-1,2)\cup(2,3)$	$[-1,3]$	$\{-1,2,3\}$
$[-1,2]\cup\{3\}$	$(-1,2)$	$[-1,2]\cup\{3\}$	$\{-1,2,3\}$
$\mathbb{Z}$	$\emptyset$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Q}$	$\emptyset$	$\mathbb{R}$	$\mathbb{R}$
$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\emptyset$

2.2 Base and Subbase

Definition 2.7 (Subspace Topologies) *Given (X,τ) and $D \subset X$, the set*

$$\tau' = \{U \bigcap D \mid U \in \tau\}$$

is the subspace topology on D.

Theorem 2.5 (Inclusion maps are continuous) *The inclusion map $I : D \to X$ defined by $I(x) = x$ is continuous.*

Theorem 2.6 (Restriction maps are continuous) *Let $f : X \to Y$ be continuous and $D \subset X$. Then the restriction $h : D \to Y$ defined by $h(x) = f(x)$ (often $h = f|_D$) is continuous.*

2.3 Topological Products

Definition 2.8 (Product topology) *Given topological spaces (X, τ) and (Y, τ'), the product topology on $X \times Y$ consists of unions $\bigcup_{i,j}(O_i \times O_j)$, where $O_i \in \tau$ and $O_j \in \tau'$.*

Theorem 2.7 (Restriction maps) *For (X, τ), (Y, τ'), (Z, τ''), a function $f : Z \to X \times Y$ is continuous if and only if $p_1 \circ f$ and $p_2 \circ f$ are continuous, where $p_1(x, y) = x$ and $p_2(x, y) = y$ are projection maps.*

2.4 Hausdorff Spaces

Definition 2.9 *A topological space X is* ***Hausdorff*** *(or T_2) if for any two distinct points $x, y \in X$, there exist disjoint open sets $U \ni x$ and $V \ni y$.*

Remark 2.1

- *If X is equipped with the discrete topology, then X is Hausdorff.*
- *If X has the indiscrete topology and contains two or more elements, then X is not Hausdorff.*

Theorem 2.8 (Main facts about Hausdorff spaces)

1. *A convergent sequence in a Hausdorff space has a unique limit.*
2. *Every subset of a Hausdorff space is Hausdorff.*
3. *Every finite subset of a Hausdorff space is closed.*
4. *The product of two Hausdorff spaces is Hausdorff.*

Proposition 2.1 (Limit Points)

- *A set is closed if and only if it contains its limit points.*
- *A sequence accumulates at x means that x is a limit point. If a sequence converges to x, then it also accumulates to x, so x is a limit point; the converse is generally false.*
- *Intuitively, limit points of D are limits of sequences of points of D.*
- *Every point of $D = (0, 3)$ is a limit point of D, while $D' = [0, 3]$.*
- *The set $D = \left\{\frac{1}{n^2}, n \in \mathbb{N}\right\}$ has only one limit point, namely $x = 0$.*

Theorem 2.9 *Every metric space (X, d) is Hausdorff.*

Proof Let $x \neq y$ in X and set $\epsilon = d(x, y)/2 > 0$. Then the open balls $B_\epsilon(x)$ and $B_\epsilon(y)$ are disjoint: If $z \in B_\epsilon(x) B_\epsilon(y)$, then by the triangle inequality

$$d(x, y) \leq d(x, z) + d(z, y) < \epsilon + \epsilon = d(x, y),$$

a contradiction. ∎

Theorem 2.10 *Metric spaces satisfy all these separation axioms:*

- T_1*: Single points are closed*
- T_2*: Hausdorff (distinct points have disjoint neighborhoods)*
- T_3*: Regular (points and closed sets separated)*
- $T_{3\frac{1}{2}}$*: Completely regular (points and closed sets separated by continuous functions)*
- T_4*: Normal (disjoint closed sets separated)*

Proposition 2.2 *In Hausdorff spaces (hence in metric spaces):*

1. *Limits of sequences are unique*
2. *The diagonal $\Delta = \{(x, x) | x \in X\}$ is closed in $X \times X$*
3. *Compact subsets are closed*

Example 2.3 (Non-Hausdorff Space) *Consider $\mathbb{R}$ with the cofinite topology:*

- *Open sets: Complements of finite sets*
- *Any two non-empty open sets intersect*
- *Sequence limits are not unique (e.g., $(1/n)$ converges to every point)*

Theorem 2.11 *For a metric space (X, d):*

- *The metric $d : X \times X \to \mathbb{R}$ is continuous*
- *The topology is first-countable (has countable local bases)*
- *Hausdorff dimension is well-defined*

Example 2.4 (Hausdorff but Non-Metrizable)

- *The Sorgenfrey line (lower limit topology) is Hausdorff but not metrizable*
- *$\mathbb{R}^\omega$ (countable product of $\mathbb{R}$) in box topology*
- *Any infinite-dimensional Banach space in weak topology*

Proposition 2.3 *For a Hausdorff space Y, the space $C(X,Y)$ of continuous functions is Hausdorff in:*

- *Compact-open topology*
- *Topology of pointwise convergence*
- *Topology of uniform convergence (when X is metric)*

Proof For $f \neq g$ in $C(X,Y)$, choose x with $f(x) \neq g(x)$. Take disjoint neighborhoods $U \ni f(x)$, $V \ni g(x)$ in Y. Then

$$W(f,x,U)W(g,x,V) = \emptyset$$

in the pointwise topology, where $W(h,x,A) = \{k|k(x) \in A\}$. ■

Definition 2.10 (Continuous Maps) *A function $f : X \to Y$ between topological spaces is continuous if $f^{-1}(U)$ is open in X for each open set $U \subset Y$.*

Theorem 2.12 (Continuous Maps) *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

Theorem 2.13 (Continuity and sequences) *If $f : X \to Y$ is continuous and (x_n) is a sequence converging to x, then $(f(x_n))$ converges to $f(x)$.*

Definition 2.11 (Homeomorphism) *A function $f : X \to Y$ is a homeomorphism if it is bijective, continuous, and f^{-1} is continuous. In this case, X and Y are homeomorphic.*

2.5 Exercises:

Exercise 01

Let $E = \{a,b,c,d\}$ be a set.

1. Determine if the following families are topologies:

$$\tau_1 = \{\emptyset, E, \{a\}, \{c, d\}, \{a, c, d\}\}$$

$$\tau_2 = \{\emptyset, E, \{a\}, \{c, d\}, \{b, c, d\}\}$$

$$\tau_3 = \{\emptyset, E, \{a\}, \{a, b\}, \{a, b, c\}\}$$

2. In cases where the family is a topology, find the closed sets.

Exercise 02

Let $\alpha \in \mathbb{R}$, $I_\alpha = (\alpha, +\infty)$, and $\tau = \{\emptyset, \mathbb{R}, I_\alpha\}_{\alpha \in \mathbb{R}}$.

1. Prove that $(\mathbb{R}, \tau)$ is a topological space.

2. Compare τ with the Euclidean topology of $\mathbb{R}$.

Exercise 03

Let $E = \{a, b, c, d\}$ and $\tau = \{\emptyset, E, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$.

1. Show that τ is a topology on E.

2. Are $\{a\}$ and $\{a, b\}$ closed sets?

Exercise 04

Let $\mathbb{R}$ be equipped with the topology:

$$\tau = \{\emptyset, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}_{\mathbb{Q}} \cup \mathbb{R}, \mathbb{C}_{\mathbb{Q}} \cup \mathbb{R} \cup \mathbb{N}, \mathbb{C}_{\mathbb{Q}} \cup \mathbb{R} \cup \mathbb{Z}, \mathbb{R}\}$$

Let $D = \{3, \sqrt{3}\}$.

1. Find the neighbourhoods $V(D)$, accumulation points D', boundary Fr(D), and exterior Ext(D).

2. Prove that D is dense in $\mathbb{R}$. Conclude.

3. Compare the induced topology $\tau_{\mathbb{Z}}$ with the trivial topology on $\mathbb{Z}$.

Exercise 05

Let (E, τ) be a topological space and equip E^2 with the product topology. Show:

$$E \text{ is Hausdorff } \Leftrightarrow \Delta = \{(x,x) : x \in E\} \text{ is closed in } E^2$$

Task:

1. Give examples of subsets $A, B \subset \mathbb{R}$ such that:

$$AB = \emptyset, \quad A\overline{B} \neq \emptyset, \quad \overline{A}B \neq \emptyset$$

2. Let A, B be subsets in a topological space. Prove:

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

3. In $\mathbb{R}^2$, let $\tau = \{\emptyset, \mathbb{R}^2, \{x^2 + y^2 < r^2\}, r > 0\}$. Show that τ is a topology and find the closure of the hyperbola $xy = 1$.

Chapter 3

Compactness and Connectedness

This chapter is devoted to the highlight of two pivotal topological properties invariant under homeomorphisms: connectedness and compactness. We refer to [5, 8].

3.1 Covers, Compact Spaces, and Sets

Definition 3.1 *A **cover** of a space X is a family of subsets $\{U_i\}_{i\in I}$ such that $X = \bigcup_{i\in I} U_i$. It is an **open cover** if each U_i is open.*

Definition 3.2 *A **subcover** of a cover $\{U_i\}_{i\in I}$ is a cover $\{U_j\}_{j\in J}$ where $J \subset I$.*

3.1.1 Compact Spaces

Definition 3.3 *A space X is **compact** if every open cover has a finite subcover.*

Theorem 3.1 *Closed subsets of compact spaces are compact.*

Theorem 3.2 *The continuous image of a compact space is compact.*

Theorem 3.3 (Heine-Borel) *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Theorem 3.4 (Wallace) *Let A and B be compact subspaces in topological spaces X and Y respectively. For any open set $W \subset X \times Y$ containing $A \times B$, there exist open sets $U \subset X$ and $V \subset Y$ such that $A \times B \subset U \times V \subset W$.*

Definition 3.4 *A space X is **exhausted by compact sets** if there exists a sequence $\{K_n\}$ of compact subsets with $K_n \subset K_{n+1}^\circ$ and $X = \bigcup_{n=1}^{\infty} K_n$.*

Example 3.1 *$\mathbb{R}^n$ is exhausted by the closed balls $\overline{B}(0, n)$ of radius n.*

3.2 Disconnected and Connected

Definition 3.5 *A topological space X is called **connected** if it cannot be written as the union of two disjoint, non-empty open sets. Equivalently, X is connected if the only subsets that are both open and closed are $\emptyset$ and X itself.*

Example 3.2 *The interval $[0, 1]$ is connected in the Euclidean topology, while $[0, 1] \cup [2, 3]$ is not connected.*

Theorem 3.5 *The continuous image of a connected space is connected. That is, if $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.*

Theorem 3.6 (Intermediate Value Theorem) *Let $f : X \to \mathbb{R}$ be continuous with X connected. If $a, b \in f(X)$ and c lies between a and b, then $c \in f(X)$.*

Definition 3.6 *A **connected component** of a space X is a maximal connected subset, i.e., a connected subset that is not properly contained in any other connected subset.*

Proposition 3.1 *Every topological space can be partitioned into its connected components. The connected components are closed sets.*

Example 3.3 *In $\mathbb{Q}$ with the Euclidean topology, the connected components are the singleton sets $\{q\}$ for each $q \in \mathbb{Q}$.*

3.3 Exercises:

Exercise 01

Let $I =]0, 1[\subset \mathbb{R}$ be an open interval equipped with the Euclidean distance $|\cdot|$ and let the family $\{O_x\}_{x \in I}$ such that $O_x = \left]\frac{x}{2}, 2x\right[$.

(1) Show that $\{O_x\}_{x \in I}$ covers I. i.e. is a family of subsets contained in I such that:

$$I = \cup \{O_x\}_{x \in I}$$

(2) Prove that I is not compact.

Exercise 02: Alexandroff's compactness

Let Ω be a locally compact space, $\omega \notin \Omega$ and $X = \Omega \cup \{\omega\}$. We say open of X either an open of Ω, or a subset of the form $\Gamma \cup \{\omega\}$, or $\Gamma \subset \Omega$ is the complement of a compact of Ω. Prove that:

(1) We have defined a Topology on X.

(2) This Topology induced on Ω the initial Topology of Ω.

(3) X is Hausdorff (separated).

(4) X is compact.

Application: $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ the completed real line.

Exercise 03

Let $(\mathbb{R}, |\cdot|)$ be a topological space. Which of the following subsets are compact?

(1) $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

(2) $B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 2x = 1\}$

(3) $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 2y > 1\}$

Hint: Check the exercise 05 Chapter 01.

Exercise 04

Let X be a discrete topological space. Under what condition is the space X connected?

Exercise 05

Study the connectedness of $\mathbb{Q}$ and $\mathbb{R} \backslash \mathbb{Q}$.

Exercise 06

(1) Prove that $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected in $\mathbb{R}^2$.

(2) Study the connectedness of $A = \{(x, y) \in \mathbb{R}^2 : x, y > 1\}$.

Task:

(1) Consider the metric space $\mathbb{Q}$ of rational numbers with the Euclidean distance. Prove that

$$A = \{x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2}\}$$

is closed and bounded, yet not compact.

(2) Let $X \subset \mathbb{R}^2$ denote the union of the segment $\{x = 0,\ |y| \leq 1\}$ with the range of the function $f :]0, +\infty[\to \mathbb{R}^2$ given by

$$f(t) = \left(\frac{1}{t}, \cos(t)\right)$$

Prove that X is closed in $\mathbb{R}^2$, connected and not path connected.

Hint: A topological space X is path(wise)-connected if, given any two points $x, y \in X$, there is a continuous mapping $\alpha : [0, 1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. Such an α is called a path from x to y.

Chapter 4

Normed Vector Spaces

In Chapter 1, we introduced metric spaces as a particular class of topological spaces where the topology is induced by a distance function. We examined several fundamental properties of these structures, emphasizing how the metric topology arises from the distance $d\colon X \times X \to \mathbb{R}$.

A special feature of normed spaces is that their metric structure is naturally induced by the norm $\|\cdot\|\colon X \to \mathbb{R}$ through the relation $d(x, y) = \|x - y\|$. This connection between norms and metrics provides important examples of metric spaces in functional analysis.

4.1 Norms and Induced Distances

Definition 4.1 *Suppose X is a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A norm on X is a real-valued function $\|x\|$ with the following properties:*

1. ***Zero vector:*** $\|x\| = 0$ *if and only if* $x = 0$.
2. ***Scalar factors:*** $\|\lambda x\| = |\lambda| \cdot \|x\|$ *for all* $\lambda \in \mathbb{F}$, $x \in X$.
3. ***Triangle inequality:*** $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$.

A *normed vector space* $(X, \|\cdot\|)$ consists of a vector space X and a norm $\|x\|$. One generally thinks of $\|x\|$ as the length of x.

It is easy to check that every norm satisfies $\|x\| \geq 0$ for all $x \in X$.

Every normed vector space $(X, \|\cdot\|)$ is also a metric space (X, d), as one may define a metric d using the formula:

$$d(x, y) = \|x - y\|$$

This particular metric is said to be *induced by the norm*.

Example 4.1 *For any $p \geq 1$, define a norm on $\mathbb{R}^k$ by:*

$$\|x\|_p = \left(\sum_{i=1}^{k} |x_i|^p \right)^{1/p}$$

On $C[a,b]$, a similar norm is:

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}$$

For $p = \infty$, the norms become:

$$\|x\|_\infty = \max_{1 \leq i \leq k} |x_i|, \quad \|f\|_\infty = \max_{a \leq x \leq b} |f(x)|$$

The space ℓ^p consists of sequences $x = \{x_n\}$ such that:

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

with norm:

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

The space ℓ^∞ consists of all bounded real sequences:

$$\|x\|_\infty = \sup_{n \geq 1} |x_n|$$

The space c_0 consists of sequences converging to zero.

4.2 Equivalent Norms

Definition 4.2 *Norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent if $\exists C_1, C_2 > 0$ such that:*

$$C_1 \|x\|_a \leq \|x\|_b \leq C_2 \|x\|_a$$

Theorem 4.1 *All norms on a finite-dimensional space are equivalent. In $C[a,b]$, norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent. For $f_n(x) = x^n$, the ratio $\|f_n\|_q / \|f_n\|_p$ is unbounded when $p < q$.*

4.3 Banach Spaces

Definition 4.3 *A Banach space is a normed vector space that is complete with respect to the metric induced by its norm.*

Theorem 4.2

1. *Every finite-dimensional vector space is a Banach space.*
2. ℓ^p *is a Banach space for* $1 \leq p \leq \infty$.
3. c_0 *is a Banach space with the* $\|\cdot\|_\infty$ *norm.*
4. *If* Y *is Banach, then* $L(X, Y)$ *is Banach.*
5. $C[a, b]$ *is Banach with* $\|\cdot\|_\infty$*, but not with* $\|\cdot\|_p$ *when* $1 \leq p < \infty$.

A subspace $Y \subset X$ *is Banach if and only if it is closed.*

Definition 4.4 *A series* $\sum_{n=1}^{\infty} x_n$ *in a normed space* X *converges if the sequence of partial sums converges:*

$$s_N = \sum_{n=1}^{N} x_n \quad \text{and} \quad \lim_{N\to\infty} s_N \text{ exists}$$

The series converges absolutely if $\sum_{n=1}^{\infty} \|x_n\|$ *converges.*

Theorem 4.3 *In a Banach space, absolute convergence implies convergence.*

Theorem 4.4 *For normed spaces* X, Y*, define* $\|(x, y)\| = \|x\| + \|y\|$.

Theorem 4.5 *The following functions are continuous:*

1. $f(x) = \|x\|$
2. $g(x, y) = x + y$
3. $h(\lambda, x) = \lambda x$

Definition 4.5

- $T : X \to Y$ *is linear if* $T(x + y) = T(x) + T(y)$ *and* $T(\lambda x) = \lambda T(x)$.
- T *is bounded if* $\exists M > 0$ *such that* $\|T(x)\| \leq M\|x\|$.
- T *is continuous if it is continuous with respect to the induced metrics.*

Theorem 4.6 *A linear map* $T : X \to Y$ *is continuous iff it is bounded.*

Theorem 4.7 *The space $L(X, Y)$ of bounded linear operators has norm:*

$$\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|}$$

Then $\|T(x)\| \leq \|T\| \cdot \|x\|$, and $\|S \circ T\| \leq \|S\| \cdot \|T\|$.

Example 4.2

$$R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots), \quad \|R\| = 1$$

$$L(x_1, x_2, \ldots) = (x_2, x_3, \ldots), \quad \|L\| = 1$$

Example 4.3

$$\|T(x)\|_\infty \leq \max_{i,j} |a_{ij}| \cdot \|x\|_1 \Rightarrow \|T\| = \max_{i,j} |a_{ij}|$$

Theorem 4.8 *Let X have basis $\{x_1, \ldots, x_k\}$, so:*

$$x = \sum_{i=1}^{k} c_i x_i, \quad \|x\|_2 = \left(\sum_{i=1}^{k} |c_i|^2 \right)^{1/2}$$

Suppose that X is a vector space with basis $x_1, x_2, \ldots, x_k$. Then every element $x \in X$ can be expressed as a linear combination

$$x = c_1 x_1 + c_2 x_2 + \cdots + c_k x_k$$

for some uniquely determined coefficients $c_1, c_2, \ldots, c_k \in \mathbb{F}$.

Theorem 4.9 *Let X be a vector space with basis $x_1, \ldots, x_k$. Define:*

$$x = \sum_{i=1}^{k} c_i x_i \quad \Rightarrow \quad \|x\|_2 = \sqrt{\sum_{i=1}^{k} |c_i|^2}$$

This norm is known as the Euclidean or standard norm on X.

Definition 4.6 *A bounded linear operator $T : X \to X$ is invertible if there exists a bounded linear $S : X \to X$ such that $ST = TS = I$.*

Theorem 4.10 *If T is bounded with $\|T\| < 1$, then $I - T$ is invertible with inverse:*

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$$

Theorem 4.11 *The set of all invertible bounded linear operators on a Banach space X is open in $L(X, X)$.*

4.4 Dual space

Definition 4.7 *The dual space X^* of a normed space X over $\mathbb{R}$ is the set $L(X, \mathbb{R})$ of all bounded linear functionals.*

Theorem 4.12 *There exists a bijection $T : \mathbb{R}^k \to (\mathbb{R}^k)^*$ mapping each a to the linear operator $T_a(x) = \sum_{i=1}^{k} a_i x_i$.*

Theorem 4.13 *For $1 < p < \infty$, let $q = \frac{p}{p-1}$. Then $\ell^q \cong (\ell^p)^*$ via:*

$$T_a(x) = \sum_{i=1}^{\infty} a_i x_i$$

4.5 Exercises

Exercise 01

Consider the function

$$N : \mathbb{R}^2 \to \mathbb{R}$$

defined by

$$N(x, y) = \sup_{t \in [0,1]} |x + ty|$$

(1) Show that N is a norm on $\mathbb{R}^2$ (check the positivity, homogeneity, and the triangle inequality).

(2) Draw the unit sphere $\mathcal{S}(0, 1)$.

Exercise 02

Let $E = \mathcal{C}([0, 1], \mathbb{R})$ denote the $\mathbb{R}$-vector space of continuous functions on $[0, 1]$ with values in $\mathbb{R}$. We equip E with the supremum norm $\| \cdot \|_\infty$ and consider

$$D = \left\{ f \in E \text{ with } f(0) = 0 \text{ and } \int_0^1 f(x)\, dx \geq 1 \right\}$$

Calculate the distance from the point 0 to the set D,

$$d(0, D) = \inf_{f \in D} d(0, f)$$

Exercise 03

Let $E = \mathcal{C}([0,1],\mathbb{R})$ denote the $\mathbb{R}$-vector space of continuous functions on $[0,1]$ with values in $\mathbb{R}$, equipped with the norm defined by:

$$\|f\|_1 = \int_0^1 |f(x)|\, dx$$

(1) Verify that $\|\cdot\|_1$ defines a norm on E (check the positivity, homogeneity, and the triangle inequality).

(2) Let $T : E \to E$ be the operator defined as follows:

$$\forall f \in E : (Tf)(x) = \int_0^x f(t)\, dt, \forall x \in [0,1]$$

Prove that T is a bounded (continuous) linear operator.

(3) Let $|||T|||$ denote the operator norm of T in $\mathcal{L}(E,E)$.

i) Show that $|||T||| \leq 1$.

ii) We define the sequence of functions (f_n) by

$$f_n(x) = (1-x)^n, \forall x \in [0,1]$$

Determine $\|Tf_n\|$, where $\|\cdot\|$ denotes the norm defined on E.

iii) Deduce that $|||T||| = 1$.

Exercise 04

Let $E = \mathcal{C}([0,1],\mathbb{R})$ denote the $\mathbb{R}$-vector space of continuous functions on $[0,1]$ with values in $\mathbb{R}$, equipped with the norm defined by:

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$$

(1) Show that $\|\cdot\|_\infty$ is a norm on E.

(2) Is $(E, \|\cdot\|_\infty)$ a Banach space? Justify your answer.

(3) We denote by $\mathcal{L}(E)$ the space of continuous linear maps from E to E. Define the norm $\|\cdot\|_{\mathcal{L}(E)}$ on $\mathcal{L}(E)$.

Chapter 5

Exams and Key Answers

This concluding chapter provides comprehensive solutions to the exercises presented in the previous chapters, as well as some exams, along with valuable suggestions and problem-solving techniques.

5.1 Key Answers' Chapter 1

Exercise 01:

Determine the interior and the closure in $(\mathbb{R}, |\cdot|)$

- $A = \{-1 + \frac{1}{n}, n \in \mathbb{N}^*\}$: $\mathring{A} = \emptyset$, $\quad \overline{A} = A \cup \{-1\}$.
- $B = [-1, 1] \cup \{2\} \cup [3, 4]$; $\quad \mathring{B} = (-1, 1) \cup (3, 4)$, $\quad \overline{B} = [-1, 1] \cup \{2\} \cup [3, 4]$.
- $C = \{x \in \mathbb{R} : x^2 \leq 4\}[1, 5]$; $\quad \mathring{C} = (1, 2)$, $\quad \overline{C} = [1, 2]$.
- $D = \mathbb{Q}[-1, 1]$, $\quad \mathring{D} = \emptyset$, $\quad \overline{D} = [-1, 1]$.

Exercise 02:

For all $f, g \in E$:

$$d(f, g) = |f(0) - g(0)| + \int_0^1 |f(t) - g(t)|dt$$

1. Show that d is a distance on E:

- Positive: $d \geq 0$ (Obvious). Let $f, g \in E$.

$$d(f,g) = 0 \iff |f(0)-g(0)| = 0 \wedge \int_0^1 |f(t)-g(t)|dt = 0 \iff |f(t)-g(t)| = 0, \forall t \in [0,1].$$

$$\iff f(t) = g(t), \forall t \in [0,1]. \quad \Leftrightarrow f \equiv g$$

- Symmetry: Obvious.
- Triangle inequality: Let $f, g, h \in \mathbb{R}$. Then:

$$d(f,h) = |f(0) - h(0)| + \int_0^1 |f(t) - h(t)|dt$$

$$\leq |f(0) - g(0)| + \int_0^1 |f(t) - g(t)|dt + |g(0) - h(0)| + \int_0^1 |g(t) - h(t)|dt$$
$$= d(f,g) + d(g,h)$$

So, d is a distance on E.

2. Elements of the unit ball in $(E,d) : x, x^2, \sqrt{x}, x^\alpha (\alpha \in \mathbb{Q}^+)$.

Exercise 03:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}, f : \overline{\mathbb{R}} \to [-1,1] \text{ such that:}$$

$$f(x) = \begin{cases} -1 & : x = -\infty \\ \frac{x}{1+|x|} & : x \in \mathbb{R} \\ 1 & : x = +\infty \end{cases}$$

1. Let us show that $d(x,y) = |f(x) - f(y)|$ defines a distance on $\overline{\mathbb{R}}$: We must show that f is bijective.

- Positive: $d \geq 0$ (Obvious). Let $x, y \in \overline{\mathbb{R}}$.

$$d(x,y) = 0 \iff |f(x) - f(y)| = 0$$

$$\iff f(x) = f(y)$$

$$\iff x = y \text{ (since } f \text{ is bijective)}$$

- Symmetry: Obvious.

- Triangle inequality: Obvious.

2. $B(0,1) = \{x \in \overline{\mathbb{R}} : d(0,x) < 1\} = \mathbb{R}$.

3. $\overline{B}(0,1) = \{x \in \overline{\mathbb{R}} : d(0,x) \leq 1\} = \overline{\mathbb{R}}$.

Exercise 04:

$$-U = \{-x : x \in U\} \quad \lambda U = \{\lambda x : x \in U\}(\lambda \in \mathbb{R}^*) \quad a + U = \{a + x : x \in U\}(a \in \mathbb{R})$$

Show that:

1. $\Rightarrow$ Let $x \in -U$. Hence: $-x \in U$. Then, there exists $r > 0$ such that $B(-x,r) =]-x-r, -x+r[\subset U$. Thus; $]x-r, x+r[\subset -U$, i.e. $-U$ is open.

   $\Leftarrow$ $-U$ open $\Rightarrow U = -(-U)$ open.

2. $\Rightarrow$ Assume that $\lambda > 0$ and let $x \in \lambda U$. Thus, $\frac{x}{\lambda} \in U$. Then, there exists $r > 0$ such that $]\frac{x}{\lambda} - r, \frac{x}{\lambda} + r[\subset U$. Hence; $]x - \lambda r, x + \lambda r[\subset U$, i.e. λU is open. For all $\lambda < 0 : U$ open $\Rightarrow -\lambda U$ open $\Rightarrow \lambda U$ open. $\Leftarrow$ λU open $\Rightarrow U = \frac{1}{\lambda}(\lambda U)$ open.

3. $\Rightarrow$ Let $x \in a + U$. Then, $x - a \in U$. Then, there exists $r > 0$ such that $]x - a - r, x - a + r[\subset U$. Hence, $]x - r, x + r[\subset a + U$, i.e. $a + U$ is open. $\Leftarrow$ $a + U$ open $\Rightarrow U = -a + (a + U)$ open.

Exercise 05:

Let $x, y, z \in E$. We have:

1. $d(x,y) = 0 \Longleftrightarrow \|x - y\| = 0 \Longleftrightarrow x - y = 0 \Longleftrightarrow x = y$,

2. $d(x,y) = \|x - y\| = |-1|\|y - x\| = d(y,x)$,

3. $d(x,z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x,y) + d(y,z)$,

Hence, d is a distance on E.

Exercise 06:

Let $X = \{a, b, c, d\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$, and $Y = \{1, 2, 3, 4\}$ with the topology $\sigma = \{\emptyset, Y, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$. Define $f : (X, \tau) \to (Y, \sigma)$ as:

$$f(a) = f(b) = 1, \quad f(c) = 2, \quad f(d) = 4.$$

1. Neighborhoods:
   - $\mathcal{V}(a) = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$
   - $\mathcal{V}(b) = \{\{a\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$
   - $\mathcal{V}(c) = \{\{b, c, d\}, X\}$
   - $\mathcal{V}(d) = \{\{b, c, d\}, X\}$
2. Neighborhoods in Y:
   - $\mathcal{V}(1) = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, Y\}$
   - $\mathcal{V}(2) = \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, Y\}$
   - $\mathcal{V}(3) = \{\{1, 2, 3\}, Y\}$
   - $\mathcal{V}(4) = \{Y\}$
3. Preimages:
   - $f^{-1}(\mathcal{V}(1)) = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$
   - $f^{-1}(\mathcal{V}(2)) = \{\{a, b, c\}, X\}$
   - $f^{-1}(\mathcal{V}(4)) = \{X\}$
4. Continuity:
   - f is continuous at a because $f^{-1}(\mathcal{V}(1)) \subset \mathcal{V}(a)$.
   - f is continuous at b because $f^{-1}(\mathcal{V}(1)) \subset \mathcal{V}(b)$.
   - f is not continuous at c because $f^{-1}(\mathcal{V}(2)) \not\subset \mathcal{V}(c)$.
   - f is continuous at d because $f^{-1}(\mathcal{V}(4)) \subset \mathcal{V}(d)$.

Exercise 07:

Define the indicator function $\chi_A : (X, \tau) \to (\mathbb{R}, |\cdot|)$ as:

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Let O be an open set in $(\mathbb{R}, |\cdot|)$. We have the following cases:

- If $0, 1 \in O$, then $f^{-1}(O) = X$.
- If $0 \notin O$ and $1 \in O$, then $f^{-1}(O) = A$.
- If $0 \in O$ and $1 \notin O$, then $f^{-1}(O) = X \setminus A$.

The necessary and sufficient condition for the indicator function χ_A to be continuous is that the set A is both open and closed.

Exercise 08:

Let $f, g \in E$. We have:

1. $$|H(f) - H(g)| = \left| \int_0^1 (|f(x)| - |g(x)|)\, dx \right| \le \int_0^1 |f(x) - g(x)|\, dx = d_1(f, g)$$
   Thus, H is Lipschitz from (E, d_1) to $(\mathbb{R}, |\cdot|)$.

2. $$|H(f)-H(g)| \le \int_0^1 |f(x)-g(x)|\, dx \le \sup_{x\in[0,1]} |f(x)-g(x)| \int_0^1 dx = \sup_{x\in[0,1]} |f(x)-g(x)| = d_\infty(f, g)$$
   Thus, H is Lipschitz from (E, d_∞) to $(\mathbb{R}, |\cdot|)$.

3. Since $H(x) = H(1)$ for $x, 1 \in E$, the function H is not injective.

Exercise 09:

1. $A = \{(x, y) \in \mathbb{R}^2 : y = x^2\} = f^{-1}(0)$, where $f(x, y) = y - x^2$. Since $\{0\}$ is closed and f is continuous, A is closed.

2. $B = \{(x, y) \in \mathbb{R}^2 : y \le x^2\} = f^{-1}((-\infty, 0])$, where $f(x, y) = y - x^2$. Since $(-\infty, 0]$ is closed and f is continuous, B is closed.

3. $C = \{(x, y, z) \in \mathbb{R}^3 : z \leq x^2 - y^2 + 5\} = f^{-1}((-\infty, 5])$, where $f(x, y, z) = y^2 - x^2 + z$. Since $(-\infty, 5]$ is closed and f is continuous, C is closed.

Exercise 10:

1. $f : [a, b] \to [c, d]$, $f(x) = \frac{d-c}{b-a}x + \frac{bc-ad}{b-a}$.
2. $f : (-1, 1) \to \mathbb{R}$, $f(x) = \frac{x}{1-|x|}$.
3. $f : \mathbb{R} \to C(0, 1)$, $f(x) = \left(\frac{1-x^2}{1+x^2}, \frac{2x}{1+x^2}\right)$.
4. $f : \mathbb{R}^2 \to S(0, 1)$, $f(x, y) = \left(\frac{(1-x^2)(1-y^2)}{(1+x^2)(1+y^2)}, \frac{2y(1-x^2)}{(1+x^2)(1+y^2)}, \frac{2x}{1+x^2}\right)$.

Exercise 11:

Define $d : \mathbb{Q}^* \times \mathbb{Q}^* \to \mathbb{R}^+$ as:

$$d(p, q) = \begin{cases} 0, & p = q \\ \frac{1}{|p|} + \frac{1}{|q|}, & p \neq q \end{cases}$$

1. Verify that d is a metric on $\mathbb{Q}^*$:
   - Positivity (obvious).
   - Symmetry (obvious).
   - Triangle inequality: For $p, q, r \in \mathbb{Q}^*$, $d(p, r) \leq d(p, q) + d(q, r)$.
2. Sequences:
   - For $x_n = n$, $d(x_n, x_m) = n + m > 1$. Thus, (x_n) is not Cauchy.
   - For $y_n = \frac{1}{n}$, $d(y_n, y_m) = \frac{1}{n} + \frac{1}{m} < \varepsilon$ for $m \geq n_0 = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1$. Thus, (y_n) is Cauchy.
3. Assume $(\mathbb{Q}^*, d)$ is complete. Then, (y_n) converges to some $\ell \in \mathbb{Q}^*$. However, $\frac{1}{|\ell|} < \varepsilon$ for all $\varepsilon > 0$ implies $\frac{1}{|\ell|} = 0$, which is impossible. Thus, $(\mathbb{Q}^*, d)$ is not complete.

Exercise 12:

Let $E = \mathbb{N}^*$, and define the metric d as:

$$d(m,n) = \begin{cases} 0, & m = n \\ 10 + \frac{1}{m} + \frac{1}{n}, & m \neq n \end{cases}$$

1. Verify that d is a metric:
   - Positivity (obvious).
   - Symmetry (obvious).
   - Triangle inequality: For $n, m, p \in E$, $d(n,p) \leq d(n,m) + d(m,p)$.
2. Let (u_n) be a Cauchy sequence in (E,d). For $\varepsilon = 1$, $u_n = u_m$ for all $n, m \geq n_0$. Thus, (u_n) is stationary and convergent. Hence, (E,d) is complete.
3. Define $f : E \to E$, $f(n) = n + 1$. If f were contractive, it would have a fixed point, which is impossible. Thus, f is not contractive.

5.2 Key Answers' Chapter 2

Exercise 01:

$E = \{a, b, c, d\}$

1. We determine the topologies among the following families:

   i) $\tau_1 = \{\emptyset, E, \{a\}, \{c,d\}, \{a,c,d\}\}$:
   - $\emptyset, E \in \tau_1$,
   - $\forall O \in \tau_1 : \emptyset O = \emptyset \in \tau_1, EO = O \in \tau_1$,
   - $\{a\}\{c,d\} = \emptyset \in \tau_1, \{a\}\{a,c,d\} = \{a\} \in \tau_1, \{c,d\}\{a,c,d\} = \{c,d\} \in \tau_1$,
   - $\forall O \in \tau_1 : \emptyset \cup O = O \in \tau_1, E \cup O = E \in \tau_1, \{a\} \cup \{c,d\} = \{a,c,d\} \in \tau_1$,
   - $\{a\} \cup \{a,c,d\} = \{a,c,d\} \in \tau_1, \{c,d\} \cup \{a,c,d\} = \{a,c,d\} \in \tau_1$.

   So, τ_1 is a topology.

   ii) $\tau_2 = \{\emptyset, E, \{a\}, \{c,d\}, \{b,c,d\}\}$: We have $\{a\} \cup \{c,d\} = \{a,c,d\} \notin \tau_1$. Thus τ_2 is not a topology.

iii) $\tau_3 = \{\emptyset, E, \{a\}, \{a,b\}, \{a,b,c\}\}$:

- $\emptyset, E \in \tau_3$,
- $\forall O \in \tau_3 : \emptyset O = \emptyset \in \tau_3, EO = O \in \tau_3$,
- $\{a\}\{a,b\} = \{a\} \in \tau_3, \{a\}\{a,b,c\} = \{a\} \in \tau_3, \{a,b\}\{a,b,c\} = \{a,b\} \in \tau_3$,
- $\forall O \in \tau_3 : \emptyset \cup O = O \in \tau_3, E \cup O = E \in \tau_3, \{a\} \cup \{a,b\} = \{a,b\} \in \tau_3$,
- $\{a\} \cup \{a,b,c\} = \{a,b,c\} \in \tau_3, \{a,b\} \cup \{a,b,c\} = \{a,b,c\} \in \tau_3$.

Hence, τ_3 is a topology.

2. The closed sets of the topology τ_1 are $\{\emptyset, E, \{b,c,d\}, \{a,b\}, \{a\}\}$.

   The closed sets of the topology τ_3 are $\{\emptyset, E, \{b,c,d\}, \{c,d\}, \{d\}\}$.

Exercise 02:

$a \in \mathbb{R}, I_\alpha = |\alpha, +\infty|, \tau = \{\emptyset, \mathbb{R}, I_\alpha(\alpha \in \mathbb{R})\}$

1. Show that $(\mathbb{R}, \tau)$ is a topological space:

   - $\emptyset, \mathbb{R} \in \tau$,
   - $\forall O \in \tau : \emptyset O = \emptyset \in \tau, \mathbb{R}O = O \in \tau$,
   - Let $\{I_{\alpha_i}\}_{1\leq i\leq n} \subset \tau$. We put $\alpha = \max_{1\leq i\leq n} \alpha_i$. We have: $\bigcap_{i=1}^n I_{\alpha_i} = I_\alpha \in \tau$.
   - $\forall O \in \tau : \emptyset \cup O = O \in \tau, \mathbb{R} \cup O = \mathbb{R} \in \tau$,
   - Let $\{I_{\alpha_i}\}_{i\in I} \subset \tau$. We set $\alpha = \min_{i\in I} \alpha_i$.
     - If $\alpha = -\infty$ then we find $\bigcup_{i\in I} I_{\alpha_i} = \mathbb{R} \in \tau$.
     - If $\alpha > -\infty$ then we get $\bigcup_{i=1}^n I_{\alpha_i} = I_\alpha \in \tau$.

   Hence $(\mathbb{R}, \tau)$ is a topological space.

2. We have $\emptyset \in (\mathbb{R}, |.|), \mathbb{R} \in (\mathbb{R}, |.|)$, and for $I_\alpha \in \tau$ is an interval, then $I_\alpha \in (\mathbb{R}, |.|)$. We obtain $\tau \subset (\mathbb{R}, |.|)$. Therefore τ is coarser, weaker, or smaller than $\mathbb{R}$.

Exercise 03:

$E = \{a,b,c\}, \tau = \{\emptyset, E, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$

1. Show that τ is a topology:

- $\emptyset, E \in \tau$,
- $\forall O \in \tau : \emptyset O = \emptyset \in \tau, EO = O \in \tau$,
- $\{a\}\{b\} = \emptyset \in \tau, \{a\}\{a,b\} = \{a\} \in \tau, \{a\}\{a,c\} = \{a\} \in \tau$,
- $\{b\}\{a,b\} = \{b\} \in \tau, \{b\}\{a,c\} = \emptyset \in \tau, \{a,b\}\{a,c\} = \{a\} \in \tau$,
- $\forall O \in \tau : \emptyset \cup O = O \in \tau, E \cup O = E \in \tau$,
- $\{a\} \cup \{b\} = \{a,b\} \in \tau, \{a\} \cup \{a,b\} = \{a,b\} \in \tau, \{a\} \cup \{a,c\} = \{a,c\} \in \tau$,
- $\{b\} \cup \{a,b\} = \{a,b\} \in \tau, \{b\} \cup \{a,c\} = E \in \tau, \{a,b\} \cup \{a,c\} = E \in \tau$.

Thus τ is a topology.

2. $C_E^{\{a\}} = \{b,c\} \notin \tau, C_E^{\{a,b\}} = \{c\} \notin \tau$. Hence $\{a\}, \{a,b\}$ are not closed sets.

Exercise 04:

$\tau = \{\emptyset, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, C_{\mathbb{R}}^{\mathbb{Q}}, C_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{N}, C_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{Z}, \mathbb{R}\}, D = \{3, \sqrt{3}\}$

1. i) $\mathcal{V}(D) = \{V \subset \mathbb{R}; \exists O \in \tau : D \subset O \subset V\}$. The open sets that containing D are: $C_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{N}, C_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{Z}, \mathbb{R}$, but, we have: $C_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{N} \subset C_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{Z} \subset \mathbb{R}$. Hence, $\mathcal{V}(D) = \{V \subset \mathbb{R} : C_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{N} \subset V\}$.

   ii) $D' = \{x \in D, \forall V \in \mathcal{V}(x); V(D \setminus \{x\}) \neq \emptyset\}$.

   - If $x = 3$, one has $D \setminus \{x\} = \{\sqrt{3}\}$, and one has $\mathbb{N} \in \mathcal{V}(x)$ but, $\mathbb{N}(D \setminus \{x\}) = \emptyset$. Thus, $3 \notin D'$.
   - If $x = \sqrt{3}$, we have $D \setminus \{x\} = \{3\}$, in addition we have $\mathbb{Q} \in \mathcal{V}(x)$ but, $\mathbb{Q}(A \setminus \{x\}) = \emptyset$. Then $\sqrt{3} \notin D'$.
   - If $x \in \mathbb{Q} \setminus \{3\}$, one has $D \setminus \{x\} = D$, and $\forall v \in \mathcal{V}(x); (D \setminus \{x\}) = \{3\}$. Then, $x \in D'$.
   - If $x \in \mathbb{R} \setminus (\mathbb{Q} \cup \{\sqrt{3}\})$, we get $D \setminus \{x\} = D$, and $\forall v \in \mathcal{V}(x); (D \setminus \{x\}) = \{\sqrt{3}\}$. We find $x \in D'$.

   Moreover; $D' = \mathbb{R} \setminus D$.

   iii) $\overline{D} = D \cup D' = \mathbb{R}$, and $\overline{D} = \emptyset$, hence, $\mathcal{F}_T(D) = \overline{D} \setminus \overline{D} = \mathbb{R}$, and $\mathcal{E}_{xt}(D) = \emptyset$.

2. $\overline{D} = \mathbb{R}$. Thus, D is dense in $\mathbb{R}$. Conclusion: D is countable and everywhere dense in $\mathbb{R}$. Hence; $(\mathbb{R}, \tau)$ is separable.

3. $\tau_{\mathbb{Z}} = \{\emptyset, \mathbb{N}, \mathbb{Z}\}$ and $Trivial_{\mathbb{Z}} = \{\emptyset, \mathbb{Z}\}$. So, the topology $\tau_{\mathbb{Z}}$ is finer than $Trivial_{\mathbb{Z}}$.

Exercise 05:

$\Delta = \{(x;x) : x \in E\}$

$\Longrightarrow$ Suppose that Δ is a closed set of E^2, so $\Omega = C_E^2$ is an open set of E^2. Let $x, y \in E^2$ be such that $x \neq y$, then $(x,y) \in \Omega$ and Ω is a neighborhood of (x,y), $\exists V_x \in \mathcal{V}(x), \exists V_y \in \mathcal{V}(y)$: $V_x \times V_y \subset \Omega$, hence $V_x V_y = \emptyset$.

$\Longleftarrow$ Let $(x,y) \in \Omega$, we have $x \neq y$, then $\exists V_x \in \mathcal{V}(x), \exists V_y \in \mathcal{V}(y)$: $V_x V_y = \emptyset$. So $V_x \times V_y \subset \Omega$. i.e. $\Omega \in \mathcal{V}(x,y)$, $\forall (x,y) \in \Omega$, we deduce $\Omega = C_E^2$ is an open of E^2. i.e. Δ is closed.

5.3 Key Answers' Chapter 3

Exercise 01

Let $I = (0,1) \subset (\mathbb{R}, |\cdot|)$ with $O_x = \left(\frac{x}{2}, 2x\right)$ for $x \in I$.

1. Show that $\{O_x\}_{x \in I}$ forms a cover of I:

   For any $y \in I$, we have $y \in O_y \subset \bigcup_{x \in I} O_x$. Thus, $\{O_x\}_{x \in I}$ covers I.

2. Show that I is not compact:

   Suppose I is compact. Then there exists a finite subcover $\{O_{x_i}\}_{i=1}^n$ such that $I \subset \bigcup_{i=1}^n O_{x_i}$. Let $x_{i_0} = \min_{1 \leq i \leq n}\{x_i\}$. For any $x \in \left(0, \frac{x_{i_0}}{2}\right)$, $x \notin O_{x_i}$ for all i, which contradicts $I \subset \bigcup_{i=1}^n O_{x_i}$. Hence, I is not compact.

Exercise 02

Let Ω be a locally compact space, $\omega \notin \Omega$, and $X = \Omega \cup \{\omega\}$. Define the topology on X such that $\mathcal{O}$ is open in X if and only if $\mathcal{O}$ is open in Ω, or $\mathcal{O}$ is of the form $\Gamma \cup \{\omega\}$, where $\Gamma = \Omega \setminus K$ for some compact K in Ω. Show that:

1. This defines a topology on X:

   - $\emptyset$ and X are open by definition.
   - The union of any family of open sets is open.
   - The intersection of finitely many open sets is open.

2. The induced topology on Ω coincides with its original topology:

   For any open U in Ω, U is open in X. If $\mathcal{O} = \Gamma \cup \{\omega\}$, then $\mathcal{O}\Omega = \Gamma$, which is open in Ω.

3. X is Hausdorff:

   For distinct $x, y \in X$:

   - If $x, y \in \Omega$, disjoint neighborhoods exist since Ω is locally compact (hence Hausdorff).
   - If $x \in \Omega$ and $y = \omega$, there exists a compact neighborhood K of x. Then $\Omega \setminus K$ and $(\Omega \setminus K) \cup \{\omega\}$ are disjoint neighborhoods of x and ω, respectively.

4. X is compact:

   Any open cover of X must include a set of the form $\Gamma \cup \{\omega\}$, where $\Gamma = \Omega \setminus K$ for some compact K. Since K is covered by finitely many other sets, X is compact.

Application: The extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is compact.

Exercise 03

1. $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$:

   Let $f(x, y) = x^2 + y^2$. Then $A = f^{-1}([0, 1])$, which is closed and bounded. Thus, A is compact.

2. $B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 2x = 1\}$:

   Let $f(x, y, z) = x^2 + y^2 + z^2 - 2x$. Then $B = f^{-1}(\{1\})$, which is closed and bounded. Thus, B is compact.

3. $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 2y > 1\}$:

   C is not bounded, hence not compact.

Exercise 04

- If $\text{Card}(X) = 1$, then X is trivially connected.
- If $\text{Card}(X) \geq 2$, then X has non-trivial clopen sets (e.g., $\{x\}$), making it disconnected.

Thus, X is connected if and only if it is a singleton.

Exercise 05

- $\mathbb{Q} \subset (-\infty, \sqrt{2}) \cup (\sqrt{2}, +\infty)$, where both sets are open and non-empty. Hence, $\mathbb{Q}$ is disconnected.
- $\mathbb{R} \setminus \mathbb{Q} \subset (-\infty, 0) \cup (0, +\infty)$, where both sets are open and non-empty. Hence, $\mathbb{R} \setminus \mathbb{Q}$ is disconnected.

5.4 Key Answers Chapter 4

Exercise 01

Define the function $N : \mathbb{R}^2 \to \mathbb{R}$ by:

$$N(x, y) = \sup_{t \in [0,1]} |x + ty|$$

1. Show that N is a norm on $\mathbb{R}^2$:
   - **Positive definiteness**: $N(x, y) = 0 \iff x = y = 0$.
   - **Homogeneity**: For $\lambda \in \mathbb{R}$, $N(\lambda(x, y)) = |\lambda| N(x, y)$.
   - **Triangle inequality**: For $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$,
   $$N((x_1, y_1) + (x_2, y_2)) \leq N(x_1, y_1) + N(x_2, y_2).$$
2. The unit sphere $S(0, 1) = \{(x, y) \in \mathbb{R}^2 : N(x, y) = 1\}$ can be decomposed into six cases based on the signs of x and y:
   - $S_1 = \{(x, y) : x \geq 0, y \geq 0, x + y = 1\}$
   - $S_2 = \{(x, y) : x \leq 0, y \leq 0, x + y = -1\}$
   - $S_3 = \{(x, y) : x \geq 0, y \leq 0, 2x + y \geq 0, x = 1\}$
   - $S_4 = \{(x, y) : x \geq 0, y \leq 0, 2x + y \leq 0, x + y = -1\}$
   - $S_5 = \{(x, y) : x \leq 0, y \geq 0, 2x + y \geq 0, x + y = 1\}$
   - $S_6 = \{(x, y) : x \leq 0, y \geq 0, 2x + y \leq 0, x = -1\}$

Exercise 02

Let $E = \mathcal{C}([0,1],\mathbb{R})$ with the supremum norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Define:

$$D = \{f \in E : f \text{ is differentiable}\}, \quad P = E\mathbb{R}[X].$$

1. **Interior of** D: For any $f \in D$ and $r > 0$, the function $g = f + \frac{|2x-1|}{2} r$ satisfies $\|g - f\|_\infty = \frac{r}{2} < r$ but $g \notin D$. Thus, $\overset{\circ}{D} = \emptyset$.
2. **Interior of** P: For any $f \in P$ and $r > 0$, the function $g = f + \frac{r}{2} e^{x-1}$ satisfies $\|g - f\|_\infty = \frac{r}{2} < r$ but $g \notin P$. Thus, $\overset{\circ}{P} = \emptyset$.

Exercise 03

Let $E = \mathcal{C}([0,1],\mathbb{R})$ with the supremum norm, and define:

$$A = \left\{ f \in E : f(0) = 0 \text{ and } \int_0^1 f(t)dt \geq 1 \right\}.$$

Compute the distance $d(0, A) = \inf_{f\in A} \|f\|_\infty$.

- For any $f \in A$, $\int_0^1 f(t)dt \geq 1$ implies $\|f\|_\infty \geq 1$, so $d(0, A) \geq 1$.
- Define $f_n(x) = \begin{cases} \frac{2n^2}{2n-1} x & \text{if } x \in [0, \frac{1}{n}], \\ \frac{2n}{2n-1} & \text{if } x \in [\frac{1}{n}, 1]. \end{cases}$ Then $f_n \in A$ and $\|f_n\|_\infty = \frac{2n}{2n-1} \to 1$. Thus, $d(0, A) = 1$.

Exercise 04

Let $E = \mathbb{R}[X]$ with the norm $\|P\| = \max(|a_k|)$, and define $P_n = 1 + X + \frac{X^2}{2} + \cdots + \frac{X^n}{n}$.

1. Show that (P_n) is a Cauchy sequence: For $n > m$, $\|P_n - P_m\| = \frac{1}{m+1} \to 0$ as $m \to \infty$.
2. (P_n) does not converge in E: Assume $P_n \to P$. For $n \geq \deg(P)$, $\|P_n - P\| \geq \frac{1}{\deg(P)+1}$, a contradiction.

Exercise 05

Let $E = \mathbb{R}[X]$ with the norm $\|P\| = \sum |a_i|$.

1. The map $\varphi : P \mapsto P(x+1)$ is **not continuous**: Take $P_n = \frac{x^n}{n}$. Then $P_n \to 0$, but $\|\varphi(P_n)\| = 1 \not\to 0$.
2. For fixed $A \in E$, the map $\psi : P \mapsto AP$ is **continuous**: $\|\psi(P)\| \leq \|A\| \cdot \|P\|$.

Exercise 06

Let $E = \mathcal{C}([0,1], \mathbb{R})$ with the norm $\|f\| = \int_0^1 |f(x)|dx$.

1. Verify $\|\cdot\|$ is a norm (direct verification).

2. The operator $T : E \to E$, $Tf(x) = \int_0^x f(t)dt$, is continuous: $\|Tf\| \leq \|f\|$.

3. The operator norm $\|T\| = 1$:

   - $\|T\| \leq 1$ by (2).
   - For $f_n(x) = (1-x)^n$, $\frac{\|Tf_n\|}{\|f_n\|} = \frac{n+1}{n+2} \to 1$, so $\|T\| \geq 1$.

Exercise 07

A hyperplane H in a normed space E is either closed or dense in E.

5.5 Final Exam-(2024-2025)

Task 01 (03 Marks)

- "Course questions"- True or False?

1. A convergent sequence in a Hausdorff space has a unique limit.

2. The space $\mathbb{R}^N$ is compact whereas $\mathbb{R}$ is not compact.

3. If Y is a Banach space, then $\mathcal{L}(X, Y)$ is a Banach space.

Task 02 (07 Marks)

- Justify each step in your proof. Let $\mathbb{Z}$ be a set of integers numbers, and let

$$\tau = \{\emptyset, \mathbb{Z}, \{-2\}, \{-2, 1\}, \{-2, 1, 3\}, \{-4, -2, 1, 3\}\}$$

be a collection of subsets of $\mathbb{Z}$.

1. Show that $(\mathbb{Z}, \tau)$ is a topological space by verifying the topology properties?

2. Determine all the closed sets in the topological space $(\mathbb{Z}, \tau)$?

3. Is it true that $(\mathbb{Z}, \tau)$ is a connected space?

4. Assume that $D = \{-2, 0\}$. Find the interior of D, the closure of D, and the boundary of D in the given topological space?

5. Find the induced topology $\tau_{\mathbb{N}}$ on $\mathbb{N}$ in $(\mathbb{Z}, \tau)$, where $\tau_{\mathbb{N}} = \{\mathcal{O}\mathbb{N} \mid \mathcal{O} \in \tau\}$

6. Suppose that $f : (\mathbb{N}, \tau_{\mathbb{N}}) \to (\mathbb{Z}, \tau)$ is a map defined by $\forall n \in \mathbb{N} : f(n) = n + 1$. Study the continuity of f at 0?

Task 03 (06 Marks)

Consider the function $d :]0, +\infty[\times]0, +\infty[\longrightarrow \mathbb{R}^+$ defined by $d(x, y) = |\ln x - \ln y|$.

1. Show that d is a distance on $]0, +\infty[$?

2. Is $(]0, +\infty[, d)$ a complete metric space? Justify your answer?

3. Draw the circle $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ on the coordinate axes (the Euclidean plane), and prove that it is both compact and connected in $(\mathbb{R}^2, d_2)$?

Task 04 (04 Marks)

Let $E = \mathcal{C}([0, 1], \mathbb{R})$ denote the $\mathbb{R}$-vector space of continuous functions on $[0, 1]$ with values in $\mathbb{R}$, endowed with a norm defined by $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$.

1. Show that $\|\cdot\|_\infty$ is a norm on E?

2. Is $(E, \|\cdot\|_\infty)$ a Banach space? Justify your answer?

3. We denote by $\mathcal{L}(E, \mathbb{R})$ the space of continuous (bounded) linear maps from E to $\mathbb{R}$. Define the norm $\|\cdot\|_{\mathcal{L}(E,\mathbb{R})}$ on $\mathcal{L}(E, \mathbb{R})$?

"*Pure mathematics is, in its way, the poetry of logical ideas.*" - **Albert Einstein**.

Key Answers Final Exam-(2024-2025)

Task 01: (03 Marks)

"Course questions"- True or False?

1. A convergent sequence in a Hausdorff space has a unique limit. **True** (01 Mark)

2. The space $\mathbb{R}^N$ is compact whereas $\mathbb{R}$ is not compact. **False** (01 Mark)

3. If Y is a Banach space, then $\mathcal{L}(X, Y)$ is a Banach space. **True** (01 Mark)

Task 02: (07 Marks)

Given $\mathbb{Z}$ with $\tau = \{\emptyset, \mathbb{Z}, \{-2\}, \{-2,1\}, \{-2,1,3\}, \{-4,-2,1,3\}\}$.

1. To demonstrate that $(\mathbb{Z}, \tau)$ is a topological space, we show that τ satisfies the conditions of a topology on $\mathbb{Z}$.

   First Condition: $\emptyset, \mathbb{Z} \in \tau$,
   Second Condition: for all $O \in \tau : \emptyset O = \emptyset \in \tau, \mathbb{Z} \bigcap O = O \in \tau, \{-2\}\{-2,1\} = \{-2\} \in \tau, \{-2\}\{-2,1,3\} = \{-2\} \in \tau, \{-2,1\}\{-2,1,3\} = \{-2,1\} \in \tau, \{-2,1,3\}\{-4,-2,1,3\} = \{-2,1,3\} \in \tau$.
   Third Condition: for all $O \in \tau : \emptyset \cup O = O \in \tau, \mathbb{Z} \cup O = \mathbb{Z} \in \tau, \{-2\} \cup \{-2,1\} = \{-2,1\} \in \tau, \{-2\} \cup \{-2,1,3\} = \{-2,1,3\} \in \tau, \{-2,1,3\} \cup \{-4,-2,1,3\} = \{-4,-2,1,3\} \in \tau$.
   Hence, τ is a topology on $\mathbb{Z}$. (01 Mark)

2. The closed sets in the topological space $(\mathbb{Z}, \tau)$ are

$$C = \{\emptyset, \mathbb{Z}, \mathbb{Z} - \{-2\}, \mathbb{Z} - \{-2,1\}, \mathbb{Z} - \{-2,1,3\}, \mathbb{Z} - \{-4,-2,1,3\}\}$$

   (01 Mark)

3. The answer is Yes. (0.5 Mark)
   $(\mathbb{Z}, \tau)$ is a connected space because the only subsets that are both open and closed are $\emptyset$ and $\mathbb{Z}$. (0.5 Mark)

4. Given $D = \{-2, 0\}$, knowing that $D \subset D \subset \overline{D}$ with D is open and $\overline{D}$ is closed.

   The interior of D is $D = \{-2\}$. (0.5 Mark)
   The closure of D is $\overline{D} = \mathbb{Z}$, thus D is everywhere dense in $\mathbb{Z}$. (0.5 Mark + 0.5 Mark)
   The boundary of D is $\partial D = \overline{D} - D = \mathbb{Z} - \{-2\} \in C$. (0.5 Mark)

5. The induced topology $\tau_{\mathbb{N}}$ on $\mathbb{N}$ in $(\mathbb{Z}, \tau)$ is (01 Mark)

$$\tau_{\mathbb{N}} = \{\emptyset, \mathbb{N}, \{1\}, \{1,3\}\}$$

6. Given $f : (\mathbb{N}, \tau_{\mathbb{N}}) \to (\mathbb{Z}, \tau)$ with $\forall n \in \mathbb{N} : f(n) = n + 1$. We study the continuity of f at 0. Since $f(0) = 1$ we have $\mathbf{V}_{\tau_{\mathbb{N}}}(0) = \mathbb{N}$ and $\{-2, 1\} \in \mathbf{V}_\tau(1)$, but

$$f^{-1}(\{-2,1\}) = \{n \in \mathbb{N} : f(n) \in \{-2,1\}\} = \{0\} \notin \mathbf{V}_{\tau_{\mathbb{N}}}(0)$$

We deduce that f is not continuous at 0. (01 Mark)

Task 03: (06 Marks)

Given $d :]0, +\infty[\times]0, +\infty[\longrightarrow \mathbb{R}^+$ with $d(x, y) = |\ln x - \ln y|$.

1. We show that d is a distance on $]0, +\infty[$ using the properties of the Euclidean distance on $\mathbb{R}$ (i.e., $(\mathbb{R}, |\cdot|)$ is a metric space).

   i) Non-negativity: Let $x, y \in]0, +\infty[$; $d(x, y) \geq 0$ means $|\ln x - \ln y| \geq 0$ because $|z| \geq 0, \forall z \in \mathbb{R}$. On the other hand, $d(x, y) = 0 \iff |\ln x - \ln y| = 0 \iff \ln x = \ln y \iff x = y$ (because the logarithm is injective on $]0, +\infty[$). (0.5 Mark)

   ii) Symmetry: Let $x, y \in]0, +\infty[$:

   $$d(x, y) = |\ln x - \ln y| = |-(\ln y - \ln x)| = |\ln y - \ln x| = d(y, x)$$

   (because $|-z| = |z|, \forall z \in \mathbb{R}$). (0.5 Mark)

   iii) Triangle inequality: Let $x, y, z \in]0, +\infty[$:

   $$d(x, z) = |\ln x - \ln z| = |\ln x - \ln y + \ln y - \ln z| \leq |\ln x - \ln y| + |\ln y - \ln z| = d(x, y) + d(y, z)$$

   (because $|u + v| \leq |u| + |v|, \forall u, v \in \mathbb{R}$). (0.5 Mark)

2. The answer is Yes. $(]0, +\infty[, d)$ is a complete metric space. (0.5 Mark)
   Justification: Let $(u_n)_n$ be a Cauchy sequence in $(]0, +\infty[, d)$. We have $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N}, (p \geq q \geq n_0 \implies |\ln u_p - \ln u_q| < \varepsilon)$, meaning $(\ln u_n)_n$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$ which converges to $l \in \mathbb{R}$ (because $(\mathbb{R}, |\cdot|)$ is complete). Consequently, $(u_n)_n$ converges to $e^l \in]0, +\infty[$. (01 Mark)

3. Given $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

   Step 1: Drawing the circle S on the Euclidean plane. (0.5 Mark)
   i) It is centered at $(0, 0)$.
   ii) It has a radius of 1.

Step 2: Prove that S is compact in $(\mathbb{R}^2, d_2)$, where d_2 is the Euclidean metric:

i) Closedness: Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x, y) = x^2 + y^2$. Then $S = f^{-1}(\{1\})$, f is continuous on $\mathbb{R}^2$, and $\{1\}$ is closed in $\mathbb{R}$. Therefore, S is closed in $(\mathbb{R}^2, d_2)$. (01 Mark)

ii) Boundedness: S is bounded because $S \subseteq \overline{S} = D(0, 1) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. (0.5 Mark)

Step 3: Prove that S is connected in $(\mathbb{R}^2, d_2)$:

1^{st} Method: Let $h :]-\pi, +\pi] \longrightarrow \mathbb{R}^2$ be defined by $\forall x \in]-\pi, +\pi] : h(x) = (\cos x, \sin x)$. Then $S = h(] - \pi, +\pi])$. Since h is continuous and $] - \pi, +\pi]$ is connected (as an interval), S is connected. (01 Mark)

2^{nd} Method: The circle S is path-connected (can be parameterized by $x = \cos t, y = \sin t, t \in]0, 2\pi]$). Any two points on S can be connected by a path within S. Path-connectedness implies connectedness. Hence, S is connected. (01 Mark)

Task 04: (04 Marks)

Given $E = \mathcal{C}([0, 1], \mathbb{R})$ with the map

$$\| \cdot \| : E \longrightarrow \mathbb{R}^+, \quad f \longmapsto \|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$$

1. Showing $\| \cdot \|_\infty$ is a norm on E:

   It is easy to check that $\|f\|_\infty \geq 0$ for all $f \in E$ because $|f(x)| \geq 0, \forall x \in [0, 1]$. Moreover, since f is continuous on $[0, 1]$, the Bolzano-Weierstrass theorem ensures that $\sup_{x\in[0,1]} |f(x)|$ exists. (0.5 Mark)

   i) Zero vector: $\|f\|_\infty = 0 \Longleftrightarrow \sup_{x\in[0,1]} |f(x)| = 0 \Longleftrightarrow f \equiv 0$. (0.5 Mark)

   ii) Scalar factors: For $\alpha \in \mathbb{R}$ and $f \in E$,

   $$\|\alpha f\|_\infty = \sup_{x\in[0,1]} |\alpha f(x)| = |\alpha| \sup_{x\in[0,1]} |f(x)| = |\alpha|\|f\|_\infty.$$

   (0.5 Mark)

iii) Triangle inequality: For $f, g \in E$,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|, \forall x \in [0, 1].$$

Taking supremum gives $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. (0.5 Mark)

2. The answer is Yes. $(E, \|\cdot\|_\infty)$ is a Banach space. (0.5 Mark)
   Justification: Let $(f_n)_n$ be a Cauchy sequence in E. Then $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N}, (p \geq q \geq n_0 \Longrightarrow \sup_{x\in[0,1]} |f_p(x) - f_q(x)| < \varepsilon)$. For fixed $x \in [0, 1]$, $(f_n(x))$ is Cauchy in $(\mathbb{R}, |\cdot|)$, so it converges to $f(x)$. Thus, $(f_n)_n$ converges to $f \in E$. (0.5 Mark)

3. Given $\mathcal{L}(E, \mathbb{R})$ the space of bounded linear maps from E to $\mathbb{R}$. $\big(\mathcal{L}(E, \mathbb{R}), \|\cdot\|_{\mathcal{L}(E,\mathbb{R})}\big)$ is a normed vector space. (0.5 Mark)
   The norm is defined by
   $$\|T\|_{\mathcal{L}(E,\mathbb{R})} = \sup_{f\neq 0} \frac{|T(f)|}{\|f\|_\infty},$$
   and it satisfies $|T(f)| \leq \|T\|_{\mathcal{L}(E,\mathbb{R})} \cdot \|f\|_\infty$. (0.5 Mark)

"*Mathematics is a game played according to certain simple rules with meaningless marks on paper.*" - **David Hilbert**.

5.6 Final Exam-(2023-2024)

Task 01: (07 Marks)

Let $E = \{1, 2, 3, 4, 5\}$ and the family $\tau = \{\emptyset, E, \{1\}, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$.

1. Show that (E, τ) is a topological space. (2 points)

2. Give the set F of all closed sets in (E, τ), the neighborhood sets $V(2)$, and $V(3)$. (2 points)

3. Let $H = \{1, 4, 5\}$:

- Determine the interior of H.
- Show that H is dense in E. (1.5 points)

4. Consider the function $f : (E, \tau) \to (E, \tau)$ defined by:

$$f(1) = 5, \quad f(2) = 3, \quad f(3) = 4, \quad f(4) = 2, \quad f(5) = 1.$$

Study the continuity of f at the point $x_0 = 2$. (1.5 points)

Task 02: (06 Marks)

Let $E = (0, +\infty)$ and the function $d : E \times E \to \mathbb{R}^+$ defined by:

$$d(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|.$$

1. Verify that d is a metric on E. (1.5 points)
2. For $r \geq 1$, describe the open ball $B(1, r)$ in terms of r. (1.5 points)
3. Let the sequence $(u_n)_{n\in\mathbb{N}}$ be defined by $u_n = n^2$. Show that (u_n) is a Cauchy sequence in (E, d). (1.5 points)
4. Is (E, d) complete? Justify your answer. (1.5 points)

Task 03: (04 Marks)

Consider the metric space $(\mathbb{R}^2, d_2)$ and the following sets:

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}, \quad B = \{(x, y) \in \mathbb{R}^2 : x^2 y^2 \geq 1\}.$$

1. Plot A and B in an orthonormal coordinate system. (1 point)
2. Study the compactness of A and B. (1.5 points)
3. Study the connectedness of A. (1.5 points)

Task 04: (03 Marks)

Let $E = C([0, 1], \mathbb{R})$ equipped with the norm:

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

1. Verify that $\| \cdot \|_\infty$ is a norm on E. (1 point)
2. Is $(E, \| \cdot \|_\infty)$ a Banach space? Justify your answer. (1 point)
3. Let $L(E, F)$ be the space of continuous linear maps from E to F. Define the norm $\| \cdot \|_{L(E,F)}$ on $L(E, F)$ in the case where $E = C([0, 1], \mathbb{R})$ and $F = \mathbb{R}$. (1 point)

Key Answers Final Exam-(2023-2024)

Task 01: (07 Marks)

Let $E = \{1, 2, 3, 4, 5\}$ and $\tau = \{\emptyset, E, \{1\}, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$.

1. Show that (E, τ) is a topological space. (01 point)

- $\emptyset, E \in \tau$,
- For all $A \in \tau$:
  - $\emptyset A = \emptyset \in \tau$,
  - $EA = A \in \tau$,
  - All pairwise intersections of sets in τ yield results in τ.
- For all $A \in \tau$:
  - $\emptyset \cup A = A \in \tau$,
  - $E \cup A = E \in \tau$,
  - All pairwise unions of sets in τ yield results in τ.

Thus, τ is a topology and (E, τ) is a topological space.

2. The set F of all closed sets in (E, τ) is:

$$F = \{\emptyset, E, \{2,3,4,5\}, \{2,4,5\}, \{2,5\}, \{5\}\}.(01 point)$$

The neighborhood sets are:

- $V(2) = \{E, \{1,2,3,4\}\}$. (0.5 points)
- $V(3) = \{E, \{1,3\}, \{1,2,3\}, \{1,3,4\}, \{1,3,5\}, \{1,2,3,4\}, \{1,3,4,5\}, \{1,2,3,5\}\}$. (0.5 points)

3. For $H = \{1,4,5\}$:

- The interior of H is $H^\circ = \{1\}$. (01 point)
- Show that H is dense in E, i.e., $\overline{H} = E$. (0.5 points)
  - $1,4,5 \in H \implies 1,4,5 \in \overline{H}$.
  - For all $V \in V(2)$, $VH \neq \emptyset \implies 2 \in \overline{H}$.
  - For all $V \in V(3)$, $VH \neq \emptyset \implies 3 \in \overline{H}$.

  Thus, $\overline{H} = E$. (0.5 points)

4. The function $f : (E, \tau) \to (E, \tau)$ is defined by:

$$f(1) = 5, \quad f(2) = 3, \quad f(3) = 4, \quad f(4) = 2, \quad f(5) = 1.$$

Study the continuity of f at $x_0 = 2$:

- $\{2,5\} \in f^{-1}(V(3))$, but $\{2,5\} \notin V(2)$.
- Thus, f is not continuous at $x_0 = 2$. (01.5 points)

Task 02: (06 Marks)

Let $E = (0, +\infty)$ and $d : E \times E \to \mathbb{R}^+$ defined by:

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|.$$

1. Verify that d is a metric on E. (02 points)

- Positivity: $\forall x, y \in E, d(x,y) \geq 0$, and $d(x,y) = 0 \iff x = y$.
- Symmetry: $\forall x, y \in E, d(x,y) = d(y,x)$.
- Triangle inequality: For $x, y, z \in E$,

$$d(x,z) \leq d(x,y) + d(y,z).$$

2. For $r \geq 1$, describe the open ball $B(1,r)$:

$$B(1,r) = \left(\frac{1}{1+r}, +\infty\right).(01 point)$$

3. Show that the sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n = n^2$ is Cauchy in (E,d):

$$d(u_n, u_m) = \left|\frac{1}{n^2} - \frac{1}{m^2}\right| \leq \frac{1}{n^2} + \frac{1}{m^2} < \varepsilon.(01 point)$$

4. (E,d) is not complete. (02 points)

- Assume (E,d) is complete. The Cauchy sequence $u_n = n^2$ would converge to some $l \in E$.
- However, $\frac{1}{l} = 0$ leads to a contradiction.

Task 03: (04 Marks)

In $(\mathbb{R}^2, d_2)$, consider the sets:

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}, \quad B = \{(x,y) \in \mathbb{R}^2 : x^2y^2 \geq 1\}.$$

1. Graphical representation of A and B. (01 point)
2. Study the compactness of A and B:

- A is closed and bounded, hence compact. (01 point)
- B is closed but not bounded, hence not compact. (01 point)

3. Study the connectedness of A:

- A is the image of the connected interval $(-\pi, \pi]$ under the continuous map $\psi(x) = (\cos x, \sin x)$.
- Thus, A is connected. (01 point)

Task 04: (03 Marks)

Let $E = C([0,1], \mathbb{R})$ with the norm:

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

1. Verify that $\|\cdot\|_\infty$ is a norm on E. (01.5 points)

- $\|f\|_\infty = 0 \iff f = 0$.
- Homogeneity: $\|\alpha f\|_\infty = |\alpha|\|f\|_\infty$.
- Triangle inequality: $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

2. $(E, \|\cdot\|_\infty)$ is a Banach space. (01 point)

- Every Cauchy sequence in E converges to a limit in E.

3. The norm $\|\cdot\|_{\mathcal{L}(E,\mathbb{R})}$ on $\mathcal{L}(E,\mathbb{R})$ is defined by:

$$\|H\|_{\mathcal{L}(E,\mathbb{R})} = \sup_{\|f\|_\infty=1} |H(f)|.(0.5points)$$

5.7 Final Exam-(2022-2023)

Task 1: (05 Marks)

Let $E = \{a, b, c\}$, and the family $\tau = \{\emptyset, E, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$.

1. Show that (E, τ) is a topological space?
2. Give the set F of all closed sets of the space (E, τ)?

3. Let $D = \{b, c\}$. Determine the interior of D, the closure of D, and τ_D the trace topology of (E, τ) on D?

4. Let the family $\sigma = \{\emptyset, E, \{a\}, \{b, c\}\}$. Show that the topological space (E, σ) is not connected?

5. Let $f : (E, \tau) \to (E, \sigma)$ be defined by: $\forall x \in E : f(x) = x$. Study the continuity of the function f at the point $\alpha = b$?

Task 2: (05 Marks)

Let $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ be defined by:

$$d(n, m) = \begin{cases} 0 & \text{if } n = m \\ 1 + \frac{1}{n+m} & \text{if } n \neq m \end{cases}$$

1. Verify that d is a distance on $\mathbb{N}$?

2. For $r > 0$, describe the open ball $B(0, r)$ in terms of r?

3. What are the Cauchy sequences in $(\mathbb{N}, d)$?

4. Is $(\mathbb{N}, d)$ complete?

Task 3: (05 Marks)

Consider the metric space $(\mathbb{R}^2, d_2)$, and the following four sets:

$$A = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2\}$$
$$B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2\}$$
$$C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$
$$D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2\}$$

1. Draw A, B, C, and D in an orthonormal coordinate system?

2. Study the compactness of A, B, and D?

3. Study the connectedness of C?

Task 4: (05 Marks)

Let $E = C^1([0,1], \mathbb{R})$, equipped with the norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$.

1. Is $(E, \|\cdot\|)$ a Banach space? Justify your answer?

2. Let $T : E \to E$ be defined by:

$$\forall f \in E : Tf(x) = T(f)(x) = f'(x), \forall x \in [0,1]$$

   Is T continuous? Justify your answer?

   Note: f' denotes the derivative of f.

Key Answers Final Exam-(2022-2023)

Task 01: (05 Marks)

Let $E = \{a, b, c\}$ and $\tau = \{\emptyset, E, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$.

1. Show that (E, τ) is a topological space. (01 point)

- $\emptyset, E \in \tau$,
- For all $O \in \tau$: $\emptyset O = \emptyset \in \tau$, $EO = O \in \tau$, $\{a\}\{c\} = \emptyset \in \tau$, $\{a\}\{a, b\} = \{a\} \in \tau$, $\{a\}\{a, c\} = \{a\} \in \tau$, $\{c\}\{a, b\} = \emptyset \in \tau$, $\{c\}\{a, c\} = \{c\} \in \tau$, $\{a, b\}\{a, c\} = \{a\} \in \tau$.
- For all $O \in \tau$: $\emptyset \cup O = O \in \tau$, $E \cup O = E \in \tau$, $\{a\} \cup \{c\} = \{a, c\} \in \tau$, $\{a\} \cup \{a, b\} = \{a, b\} \in \tau$, $\{a\} \cup \{a, c\} = \{a, c\} \in \tau$, $\{c\} \cup \{a, b\} = E \in \tau$, $\{c\} \cup \{a, c\} = \{a, c\} \in \tau$, $\{a, b\} \cup \{a, c\} = E \in \tau$.

Thus, τ is a topology.

2. The set F of all closed sets in (E, τ) is $\{\emptyset, E, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. (01 point)

3. For $D = \{b, c\}$:

- The interior of D is $D^\circ = \{c\}$. (0.5 points)
- The closure of D is $\overline{D} = D$ because D is closed. (0.5 points)
- The trace topology of (E, τ) on D is $\tau_D = \{\emptyset, D, \{b\}, \{c\}\}$. (0.5 points)

4. Let $\sigma = \{\emptyset, E, \{a\}, \{b, c\}\}$. Show that the topological space (E, σ) is not connected. Since $\{a\}$ is both open and closed in (E, σ), but $\{a\} \neq \emptyset$ and $\{a\} \neq E$, (E, σ) is not connected. (0.5 points)

5. $\{b, c\} \in V_\sigma(b)$, but $\{b, c\} \notin V_\tau(b)$. Therefore, f is not continuous at the point b. (01 point)

Task 02: (05 Marks)

Define the metric $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ as:

$$d(n,m) = \begin{cases} 0 & \text{if } n = m, \\ 1 + \frac{1}{n+m} & \text{if } n \neq m. \end{cases}$$

1. Verify that d is a metric on $\mathbb{N}$. (01 point)

- Positivity: $\forall n, m \in \mathbb{N}, d(n,m) \geq 0$, and $d(n,m) = 0 \iff n = m$.
- Symmetry: $\forall n, m \in \mathbb{N}, d(n,m) = d(m,n)$.
- Triangle inequality: For $n, m, p \in \mathbb{N}$,

$$d(n,p) = 1 + \frac{1}{n+p} \leq 1 + \frac{1}{n+m} + 1 + \frac{1}{m+p} = d(n,m) + d(m,p).$$

2. Open ball $B(0,r)$:

- If $r \leq 1$, then $B(0,r) = \{0\}$. (0.5 points)
- If $r > 1$, then $B(0,r) = \{0, n_0+1, n_0+2, \ldots\}$, where $n_0 = \lfloor \frac{1}{r-1} \rfloor$. (0.5 points)

3. Let (u_n) be a non-stationary Cauchy sequence. For some $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, for $n, m > n_0$, $d(u_n, u_m) < \varepsilon$. If $\varepsilon < 1$, this leads to $1 + \frac{1}{u_n+u_m} < 0$, a contradiction. Thus, (u_n) is stationary. (01 point)

4. Every Cauchy sequence is stationary and hence convergent. Therefore, $(\mathbb{N}, d)$ is complete. (01 point)

Task 03: (05 Marks)

In $(\mathbb{R}^2, d_2)$, consider the sets:

$$A = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2\}, \quad B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2\},$$

$$C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}, \quad D = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2\}.$$

1. Graphical representation of A, B, C, and D. (01 point)
2. Study the compactness of A, B, and D:

- The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x^2+y^2$ is continuous. $A = f^{-1}([1, +\infty[)$ and $B = f^{-1}([0,2])$ are closed. (01 point)

- A is not bounded, hence not compact. (0.5 points)
- B is bounded and closed, hence compact. (0.5 points)
- $D = AB$ is closed and bounded, hence compact. (01 point)

3. Study the connectedness of C: The function $f : [-\pi, \pi] \to \mathbb{R}^2$ defined by $f(t) = (\cos t, \sin t)$ is continuous, and $f([-\pi, \pi]) = C$. Since $[-\pi, \pi]$ is connected, C is connected. (01 point)

Task 04: (05 Marks)

Let $E = \mathcal{C}^1([0, 1], \mathbb{R})$ equipped with the norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$.
1. $(E, \|\cdot\|)$ is a Banach space. (0.5 points)

- $\|\cdot\|_\infty$ is a norm on E:
  - $\|f\|_\infty = 0 \iff f = 0$. (0.5 points)
  - Homogeneity: $\|\alpha f\|_\infty = |\alpha|\|f\|_\infty$. (0.5 points)
  - Triangle inequality: $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. (0.5 points)
- Every Cauchy sequence in E converges to a limit in E. (01 point)

2. The operator $T : E \to E$ defined by $T(f)(x) = f'(x)$ is not continuous. (0.5 points)

- Consider the sequence $f_n(x) = \frac{\sin(nx)}{n}$, which converges to 0. However, $T(f_n)(x) = \cos(nx)$ does not converge to $T(0)(x) = 0$. (01 point)

Hint: Theorem

Let (E, d) and (F, δ) be metric spaces. A mapping $T : (E, d) \to (F, \delta)$ is continuous at θ if and only if for every sequence (u_n) in E converging to θ, the image sequence $T(u_n)$ converges to $T(\theta)$.

5.8 Replacement Exam-(2024-2025)

Exercise 01 (07 Marks)

'Course questions' - True or False? Justify your answer?

(1) Two topological spaces are said homeomorphic if there is a homeomorphism between them.

(2) There exists a discrete subspace of a metric space that is not closed.

(3) The closed interval $[0, 1]$ is compact in $\mathbb{R}$.

(4) Any path connected space is not connected.

(5) Any convex subset of $\mathbb{R}^N$ is path connected and hence connected.

(6) A discrete topological space is compact if and only if it is infinite.

(7) $(C([0, 1], \|\cdot\|_\infty))$ is a Banach space.

Exercise 02 (07 Marks)

Justify each step in your proof.

Let $X = \{1, 2, 3, 4, 5\}$ be a set with $\tau = \{\emptyset, X, \{1\}, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$ are collection of subsets of X.

(1) Show that (X, τ) is a topological space by verifying the topology properties?

(2) Determine all the closed sets in the topological space (X, τ)?

(3) Assume that $D = \{2, 5\}$ Find the interior of D, the closure of D, and the boundary of D in the given topological space?

(4) Find the induced topology τ_Y on $Y = \{2, 3\}$ in (X, τ), where $\tau_Y = \{O \bigcap Y, O \in \tau\}$

(5) Suppose that $f : (Y, \tau_Y) \to (X, \tau)$ is a map defined by $\forall n \in \mathbb{N} : f(n) = n$. Study the continuity of f at 3?

Exercise 03 (06 Marks)

Consider the function $d : [0, +\infty[\times[0, +\infty[\to \mathbb{R}^+$ defined by $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$

(1) Show that d is a distance on $[0, +\infty[$?

(2) Is $([0, +\infty[, d)$ a complete metric space? Justify your answer?

(3) Draw the disc $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ on the coordinate axes (the Euclidean plane), and prove that it is compact in $(\mathbb{R}^2, d_2)$?

"In mathematics, the path is often more important than the destination." - G.H. Hardy.

Key Answers Replacement Exam-(2024-2025)

Exercise 01 (07 Marks)

"Course questions" - True or False?

(1) Two topological spaces are said homeomorphic if there is a homeomorphism between them. **True** (01 Mark)

(2) There exists a discrete subspace of a metric space that is not closed. **False** (01 Mark)

(3) The closed interval $[0, 1]$ is compact in $\mathbb{R}$. **True** (01 Mark)

(4) Any path connected space is not connected. **False** (01 Mark)

(5) Any convex subset of $\mathbb{R}^N$ is path connected and hence connected. **True** (01 Mark)

(6) A discrete topological space is compact if and only if it is infinite. **False** (01 Mark)

(7) $(\mathcal{C}([0, 1], \| \cdot \|_\infty)$ is a Banach space. **True** (01 Mark)

Exercise 02 (07 Marks)

Include justification for every step in the proof.
Given $X = \{1, 2, 3, 4, 5\}$ with $\tau = \{\emptyset, X, \{1\}, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$.

(1) Show that (X, τ) is a topological space (02 Mark)

Solution:

- First Axiom: $\emptyset, X \in \tau$.
- Second Axiom: τ is closed under arbitrary unions.

- Third Axiom: τ is closed under finite intersections.

(2) Determine all the closed sets in (X, τ): (01 Mark)

$$\mathrm{C} = \{\emptyset, X, \{2,3,4,5\}, \{2,4,5\}, \{2,5\}, \{5\}\}$$

(3) For $D = \{2,5\}$, find:

- The interior of D is $\overset{\circ}{D} = \emptyset$ (0.5 Mark)
- The closure of D is $\overline{D} = D$ because D is closed in X (0.5 Mark + 0.5 Mark)
- The boundary of D is $\partial D = \overline{D} - \overset{\circ}{D} = D - \emptyset = D \in \mathrm{C}$ (0.5 Mark)

(4) The induced topology τ_Y on $Y = \{2,3\}$ in (X, τ) is: (01 Mark)

(5) Given $f : (Y, \tau_Y) \to (X, \tau)$ with $\forall n \in Y : f(n) = n$. Study the continuity of f at 3. (01 Mark)

Solution: Since $f(3) = 3$ we have the preimages:

$$f^{-1}\Big(\{1,3\}\Big) = \{3\} \in \tau_Y$$

$$f^{-1}\Big(\{1,3,4\}\Big) = \{3\} \in \tau_Y$$

$$f^{-1}\Big(\{1,2,3,4\}\Big) = \{2,3\} \in \tau_Y$$

Since the preimages of all open sets containing 3 are open in τ_Y, f is continuous at 3.

Exercise 03 (06 Marks)

Given $d :]0, +\infty[\times]0, +\infty[\to \mathbb{R}^+$ with $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$.

(1) Show that d is a distance on $]0, +\infty[$ (1.5 Marks)

Solution:

i) Non-negativity: Let $x, y \in]0, +\infty[$; $d(x, y) \geq 0$ because $|z| \geq 0,\ \forall z \in \mathbb{R}$. Also, $d(x, y) = 0 \Leftrightarrow x = y$ (since $x \mapsto 1/x$ is injective). (0.5 Mark)

ii) Symmetry: $d(x, y) = d(y, x)$ because $|-z| = |z|$. (0.5 Mark)

iii) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ follows from $|u + v| \leq |u| + |v|$. (0.5 Mark)

(2) Is $(]0, +\infty[, d)$ a complete metric space? (1.5 Marks)

Answer: No. (0.5 Mark)

Justification: There exist Cauchy sequences $(u_n)_n$ that diverge to infinity, which is outside $]0, +\infty[$. (01 Mark)

(3) Given $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$:

- Step 1: Drawing the disc S on the Euclidean plane (centered at $(0, 0)$ with radius 1). (01 Mark)
- Step 2: Prove S is compact in $(\mathbb{R}^2, d_2)$:
  - i) Closedness: $S = f^{-1}(] - \infty, 1])$ where $f(x, y) = x^2 + y^2$ is continuous. (01 Mark)
  - ii) Boundedness: $S \subseteq D(0, 1)$. (01 Mark)

"Mathematics is a game played according to certain simple rules with meaningless marks on paper." - David Hilbert.

5.9 Resit Exam-(2022-2023)

Exercise 01: (06 Marks)

Let $E = \{1, 2, 3\}$, and the family $\tau = \{\emptyset, E, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}\}$

1. Show that (E, τ) is a topological space.
2. Give the set F of all closed sets of the space (E, τ).
3. Let $D = \{1, 3\}$. Determine the interior of D, the closure of D, and τ_D, the trace of the topology (E, τ) on D.
4. Let $\sigma = \{\emptyset, E, \{1\}, \{2, 3\}\}$. Show that the topological space (E, σ) is not connected.
5. Let the function $f : (E, \tau) \to (E, \sigma)$ be defined by: $\forall x \in E : f(x) = x$. Study the continuity of the function f at the point $\alpha = 2$.

Exercise 02: (04 Marks)

Let the function $d : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+$, defined by:

$$d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$$

1. Verify that d is a metric on $\mathbb{R}_+^*$.
2. Define an open ball centered at 1 with radius r.
3. Let $x_n = \sqrt{n},\ n \in \mathbb{N}^*$. Is the sequence (x_n) Cauchy with respect to this metric?

Exercise 03: (06 Marks)

Consider the metric space $(\mathbb{R}^2, d_2)$, and the following four sets:

$$A = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2\}$$
$$B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$
$$C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$
$$D = \{(x, y) \in \mathbb{R}^2 : xy > 1\}$$

1. Draw sets A, B, C, D in an orthonormal coordinate system.
2. Study the compactness of sets A and B.
3. Show that C is a connected subset of $\mathbb{R}^2$.
4. Study the connectedness of the set D.

Exercise 04: (04 Marks)

Let $E = \mathcal{C}^1([0, 1], \mathbb{R})$. For any $f \in E$, define the functional

$$N(f) = |f(0)| + \int_0^1 |f'(x)|\, dx$$

Is (E, N) a normed vector space? Justify your answer.

Remark: f' denotes the derivative of f.

5.10 Resit Exam-(2023-2024)

Exercise 01: (08 Marks)

Let $X = \{a, b, c, d, e\}$ and the family $\tau = \{\emptyset, X, \{a\}, \{a, c\}, \{a, c, d\}, \{a, b, c, d\}\}$

1. Show that (X, τ) is a topological space.
2. Give the set F of all closed sets in the space (X, τ), and find the neighborhoods $V(b)$ and $V(c)$.
3. Let $A = \{a, d, e\}$. Determine the interior of A and show that A is dense in X.
4. Let the function $g : (X, \tau) \to (X, \tau)$ be defined by:

$$g(a) = e, \quad g(b) = c, \quad g(c) = d, \quad g(d) = b, \quad g(e) = a$$

Study the continuity of the function g at the point $x_0 = b$.

Exercise 02: (04 Marks)

Let $\mathbb{R}$ be equipped with the metric:

$$d(x, y) = |\arctan x - \arctan y|$$

1. Compare this distance to $d_1(x, y) = |x - y|$.
2. Show that $(\mathbb{R}, d)$ is not complete (consider the sequence $u_n = n$).

Exercise 03: (04 Marks)

1. Show that $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a connected subset of $\mathbb{R}^2$.
2. Study the connectedness of the set $H = \{(x, y) \in \mathbb{R}^2 : x \cdot y > 1\}$.

Exercise 04: (04 Marks)[1]

Define on the Banach space $(E, \|\cdot\|_\infty)$, with $E = \mathcal{C}([0, 1], \mathbb{R})$, the following application:

$$\varphi : E \to E \quad \text{by} \quad \varphi(f)(x) = \int_0^x \frac{f(t)}{4 + t^2}\, dt$$

[1] "Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them. - Joseph Fourier.

1. Show that for all $x \in [0, 1]$, $\arctan x \leq x$.
2. Knowing that $\arctan x = \int_0^x \frac{1}{1+t^2} dt$, show that the equation $\varphi(f) - f = 0$ admits a unique solution in E.