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**Université Mohamed Boudiaf de M'sila**
**Faculté des Mathématiques et de l'Informatique**
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# *Mémoire de Master*

**Domaine** : Mathématiques et Informatique
**Filière** : Mathématiques
**Option** : Analyse Fonctionnelle

# Thème

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*Majorizimg Lipschitz Operators*

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**Devant le jury composé de**

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# Dedications

**To my grandfather and my parents**

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# Notations

| | |
|---|---|
| $e$ | Neutral element |
| $\lVert . \rVert$ | Norm |
| $(E, \lVert . \rVert)$ | Normed linear space |
| $\mathcal{B}_E$ | Unit ball of $E$ |
| $E^*$ | Topological dual of $E$ |
| $\mathcal{B}(E, F)$ | The set of all bounded linear maps from $E$ to $F$ |
| $T^*$ | Adjoint of linear operator |
| $\mathcal{K}(E, F)$ | The set of all linear compact operators from $E$ to $F$ |
| $\mathcal{W}(E, F)$ | The set of all weakly compact operators from $E$ to $F$ |
| $d(.,.)$ | Distance |
| $(X, d)$ | Metric space |
| $(X, d, e)$ | Pointed metric space |
| $\mathcal{M}_0(X)$ | Class of complete pointed metric spaces |
| $\text{Lip}_0(X, Y)$ | Space of all Lipschitz functions between $X$ and $Y$ such that $T(e_X) = e_Y$ |
| $\text{Lip}(.)$ | Norm on $\text{Lip}_0(X, Y)$ |
| $X^\# = \text{Lip}_0(X, \mathbb{R}) = \text{Lip}_0(X)$ | Lipschitz dual of the pointed metric space $X$ |
| $\mathcal{F}(X)$ | Lipschitz free space of $X$ |
| $T^\#$ | Adjoint of Lipschitz operator |
| $\text{Lip}_{0k}(X, E)$ | The set of all Lipschitz compact operators from $X$ to $E$ |
| $\text{Lip}_{0w}(X, E)$ | The set of all Lipschitz weakly compact operators from $X$ to $E$ |
| $C(K)$ | The space of all continuous operators from$K$ to $\mathbb{R}$ |
| $L_p$ | Lebesgue space |
| $\Pi_p^L(X, E)$ | The space of all Lipschitz p-summing operators $T : X \longrightarrow E$ |
| $\pi_p^L(\cdot)$ | Norm on $\Pi_p^L(X, E)$ |

# Introduction

Douglas was the first one who described the notion of majorization for bounded linear operators in his book [5] for all $T$ and $S$ in $\mathcal{B}(H)$ such that $H$ is a Hilbert space. Embray generalized Douglas's result in his book [7] for all $T$ and $S$ in $\mathcal{B}(E)$ such that $E$ is a general Banach space. Harte in his book [10] considers these concepts in the general context where $S$ and $T$ are bounded linear operators with possibly different domain and range spaces. We say that $T \in \mathcal{B}(E, F)$ majorizes $S \in \mathcal{B}(E, G)$ if there exists $M > 0$ such that

$$\|S(a)\| \leq M \|T(a)\|, \text{ for all } a \in E.$$

In this work, we try to generalize this concept from the linear theory to the non linear theory, which is Lipschitz case.

This memory has been organized as follows. The first chapter is an overview of notions and basic concepts and results needed in the following chapters, these include the Lipschitz operators, Free Banach space and also we describe the Lipschitz compact and weakly compact operators.

In the second chapter, we interested in Barnes article, we present the definition of majorization for bounded linear operators, some proprieties and their proofs. Next we will see the notion of factorization. Furthermore, we investigate the relationships between the concepts , majorization, factorization and range inclusion.

In the last chapter, we try to generalize the notion of majorization for Lipschitz operators between metric spaces and Banach spaces. We prove that a Lipschitz map $S$ is majorized by a Lipschitz map $T$ if and only if there exists $V \in \mathrm{Lip}_0\left(\overline{T(X)}, F\right)$ such that $S = V \circ T$, where $T \in \mathrm{Lip}_0(X, E)$ and $S \in \mathrm{Lip}_0(X, F)$.

# Chapter 1

# Basic properties for Lipschitz operators

In this chapter, we recall some basic informations and concepts those used in chapter 2 and chapter 3. We start with some preliminaries on linear bounded operators we can consult [6]. We also need the Lipschitz operators and some proprieties based on Weaver's book [14], the definition of the compact (resp, weakly compact) Lipschitz operators which is mainly based in [11], and the Banach free space we used the paper of Godefroy [9].

## 1.1 Preliminaries on bounded linear operators

Let $E, F$ be Banach spaces over the same field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We recall that a Banach space is a complete normed space (i.e., all Cauchy sequence on $E$ converges on $E$), we denote by $\mathcal{B}(E, F)$ the Banach space of all bounded linear operators $T$ from $E$ to $F$, under the norm

$$\|T\| = \inf \{C > 0 : \|T(a)\|_F \leq C \|a\|_E ; \quad \forall a \in E\}.$$

When $F = \mathbb{K}$, $\mathcal{B}(E, F)$ is denoted by $E^*$ and it called the topological dual of $E$, and we denoted by $E^{**}$ the second dual of $E$. The normed space $E$ can be embedded isometrically in $E^{**}$ in a natural way.

$$\begin{array}{cccc} i : & E & \longrightarrow & E^{**} \\ & a & \longrightarrow & i(a) \end{array}$$

via the formula

$$i(a)(a^*) = a^*(a) = \langle a^*, a \rangle \text{ for each } a^* \in E^*.$$

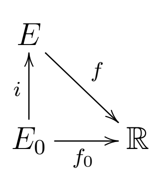
When the linear isometry $a \longmapsto i(a)$ from a Banach $E$ into $E^{**}$ is surjective, the Banach space $E$ is called reflexive.

For $T \in \mathcal{B}(E, F)$, we will consider the adjoint (dual) of $T$, the linear bounded operator $T^*$ from $F^*$ to $E^*$ given by

$$T^*(b^*)(a) = \langle T^*(b^*), a \rangle = \langle b^*, T(a) \rangle.$$

The Hahn-Banach theorem is one of the most important and fundamental result in Functional Analysis.

**Theorem 1.1.1 (Hahn Banach Analytic Form)** *Let $E$ be a Banach space and $E_0$ be a subspace of $E$ . Let $f_0 : E_0 \longrightarrow \mathbb{R}$ be a bounded linear operator. Then $f_0$ can be extended to a bounded linear operator $f : E \longrightarrow \mathbb{R}$ such that $\|f_0\| = \|f\|$.*




(*$i$ is the canonical injection from $E$ to $E_0$ and* $f_0 = f \circ i$)

**Corollary 1.1.1** *For every $a$ in $E$, we have*

$$\|a\|_E = \sup_{\|f\|_{E^*}=1} |\langle f, a\rangle| .$$

### 1.1.1 Compact operators

Before we give the notion of compact operators, we will remind the concept of weak and weak* topology.
Let $E$ be a Banach space and $E^*$ its topological dual.

1. The weak-topology $\sigma(E, E^*)$ on $E$, is the weakest topology such that each map $a^* \in E^*$ is continuous.

2. The weak*-topology $\sigma(E^*, E)$ on $E^*$, is the weakest topology such that each linear operator
$$\begin{array}{rccc} J(a): & E^* & \longrightarrow & \mathbb{K} \\ & f & \longrightarrow & J(a)(f) = f(a) \end{array}$$

is continuous.

**Theorem 1.1.2 (Banach-Alaoglu theorem)** *The closed unit ball $\mathcal{B}_{E^*}$ is compact in the weak*-topology $\sigma(E^*, E)$ .*

**Definition 1.1.1** *An operator $T : E \longrightarrow F$ is said to be compact (weakly compact) if $\overline{T(\mathcal{B}_E)}$ is compact (resp. weakly compact) in $F$. We denote by $\mathcal{K}(E, F)$ (resp. $\mathcal{W}(E, F)$) the set of all compact (resp. weakly compact) operators $T : E \to F$.*

**Theorem 1.1.3 (Schauder's theorem)** *Let $T \in \mathcal{K}(E, F)$, so the following statements are equivalent.*

1. *The operator $T$ is on $\mathcal{K}(E, F)$.*

2. *The operator $T^*$ is on $\mathcal{K}(F^*, E^*)$.*

**Proof.** We find the proof in [6] page 485. ∎

**Theorem 1.1.4 (Gantmacher's theorem)** *[6] Let $T \in \mathcal{W}(E,F)$, so the following statements are equivalent.*

1. *The operator $T$ is on $\mathcal{W}(E,F)$.*
2. *The operator $T^*$ is on $\mathcal{W}(F^*,E^*)$.*

**Corollary 1.1.2** *Let $T: E \longrightarrow F$ be a bounded linear operator. If $E$ or $F$ is reflexive then, $T$ is weakly compact (i.e., $\overline{T(\mathcal{B}_E)}$ is $\sigma(E^*,E)$ compact).*

### 1.1.2 Linear p-summing operators

A linear map $T: E \longrightarrow F$ is called p-summing operator $(1 \leq p < \infty)$, if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$, $a_1, a_2 ... a_n \in E$

$$\sum_{i=1}^{n} \|T(a_i)\| \leq C^P \sup_{f \in \mathcal{B}_{E^*}} \sum_{i=1}^{n} |f(a_i)|^p . \tag{1.1.1}$$

We denoted by $\Pi_p(E,F)$ the space of all linear p-summing operators $T: E \longrightarrow F$ and by $\pi_p(T)$ the smallest $C$ verifying (1.1.1).

**Remark 1.1.1** $(\Pi_p(E,F), \pi_p(\cdot))$ *is a Banach space.*

**Remark 1.1.2** *Every Banach space $E$ is isometric to a subspace of $C(K)$ such that $K$is compact set. $K = (\mathcal{B}_{E^*}, \sigma(E^*,E))$. Indeed,*
*Define*

$$\begin{array}{rccc} i: & E & \longrightarrow & C(K) \\ & a & \longmapsto & i(a) = \widetilde{a} = \langle a, \cdot \rangle \end{array}$$

*such that, $\widetilde{a}(f) = \langle a, f \rangle$, for all $f \in K$.*
*We have*

$$\begin{aligned} \|i(a)\|_{C(K)} &= \|\widetilde{a}\|_{C(K)} \\ &= \sup_{f \in K} |\widetilde{a}(f)| \\ &= \sup_{f \in K} |f(a)| \\ (Hahn\ Banach) &= \|a\| \end{aligned}$$

*And, we put*

$$J : C(K) \longrightarrow L_p(K,\mu)$$

*($J$ is the canonical injection from $C(K)$ to $L_p(K,\mu)$).*

**Theorem 1.1.5** *Let $1 \leq p < \infty$ the following statements are equivalents for a linear mapping $T : E \longrightarrow F$ :*

1. *$T \in \Pi_p(E,F)$.*

2. *There is a positive constant $C$ and Radon probability $\mu$ on $K$ such that*

$$\|T(a)\| \leq C \left( \int_K |\langle a, f\rangle|^p \, d\mu \right)^{\frac{1}{p}}. \tag{1.1.2}$$

   *for all $a \in E$.*

3. *The following Diagram commute*

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ \downarrow i & & \uparrow \widetilde{T} \\ S_\infty & \xrightarrow{J_p} & S_p \\ \cap & & \cap \\ C(K) & \xrightarrow{J} & L_p(K,\mu) \end{array}$$

## 1.2 Lipschitz operators

The aim of this section is to give the necessary properties for Lipschitz operators and we will define the notion of compacity in the Lipschitz case. We refer to reader Weaver's book [14] for more information on Lipschitz operators and to A. Jimènez-Vargas, J. M. Sepulcre, M. Villegas-Vallecillos [11] for Lipschitz compact operators.

**Definition 1.2.1** *We say that $X$ is a metric space if it is non empty set equipped with a function $d$ from $X^2$ into $\mathbb{R}$ such that for all $x, y, z$ in $X$, we have*

1. $d(x,y) = 0$ if $x = y$ *(separation)*,

2. $d(x,y) = d(y,x)$ *(symmetry)*,

3. $d(x,y) \leq d(x,z) + d(z,y)$ *(triangular inequality).*

**Definition 1.2.2** *A pointed metric space $(X,d,e)$, is a metric space $(X,d)$ with a distinguish element $e \in X$. We denoted by*

$$\mathcal{M}_0(X) = \{\textit{The space of all complete pointed metric spaces}\}.$$

**Definition 1.2.3** *A map $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is called Lipschitz if there is a positive constant $C$ such that*

$$\forall x, y \in X,\ d_Y(f(x), f(y)) \leq C d_X(x,y). \tag{1.2.1}$$

*For a Lipschitz map $f$, we define its Lipschitz constant by*

$$\begin{aligned} \|f\|_{\mathrm{Lip}} = \mathrm{Lip}(f) &= \sup_{x \neq y} \left\{ \frac{d_Y(f(x), f(y))}{d_X(x,y)} \right\} \\ &= \inf\{C,\ \textit{verifying}\ (1.2.1)\}. \end{aligned}$$

Let $(X, e_X, d_X), (Y, e_Y, d_Y)$ be pointed metric spaces. We say a map $f : (X, e_X, d_X) \to (Y, e_Y, d_Y)$ preserves distinguished point if $f(e_X) = e_Y$.
We denoted by $\mathrm{Lip}_0(X,Y)$ the Banach space of Lipschitz functions from $X$ to $Y$ which preserves distinguished point equipped with the Lipschitz norm $\mathrm{Lip}(f)$. If $Y = \mathbb{R}$ then $\mathrm{Lip}_0(X,\mathbb{R}) = \mathrm{Lip}_0(X) = X^{\#}$ is called Lipschitz dual.

**Definition 1.2.4** *A map $f : (X, d_X) \longrightarrow (Y, d_Y)$ is called bi-Lipschitz or quasi-isometry, if $f$ is bijective and both $f$, $f^{-1}$ are Lipschitz. In this case $X$ and $Y$ are called quasi-isometric (Nik Weaver).*
*A bi-Lipschitz function $f$ is an isometry if*

$$\forall x, y \in X,\ d_Y(f(x), f(y)) = d_X(x,y).$$

**Proposition 1.2.1** *Let $X, Y, Z$ be pointed metric space and $f : X \longrightarrow Y$ $g : Y \longrightarrow Z$ be Lipschitz maps. Then $g \circ f : X \longrightarrow Z$ is Lipschitz map and* $\mathrm{Lip}(g \circ f) \leq \mathrm{Lip}(g)\,\mathrm{Lip}(f)$.

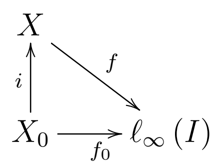
**Proof.** For $x$, $y$ in $X$, we have

$$\begin{aligned} d_Z(g \circ f(x), g \circ f(y)) &\leq \mathrm{Lip}(g)\, d_Y(f(x), f(y)) \\ &\leq \mathrm{Lip}(g)\,\mathrm{Lip}(f)\, d_X(x,y). \end{aligned}$$

and this shows the proposition. ■

The following theorem can be viewed in [14].

**Theorem 1.2.1 (Nonlinear Hahn-Banach theoram)** *Let $X_0$ be a subset of a metric space $(X, d)$ and let $f_0 : X_0 \to \ell_\infty(I)$ be a Lipschitz operator. Then $f_0$ can be extended to a Lipschitz operator $f : X \to \ell_\infty(I)$ such that* $\mathrm{Lip}(f_0) = \mathrm{Lip}(f)$ *(we say that $\ell_\infty(I)$ is a 1-injective).*




*($i$ is the canonical injection from $X_0$ to $X$ and* $f_0 = f \circ i$*).*

**Proof.** By considering each coordinate separately, it suffices to prove that for $\mathbb{R}$ instead of $l_\infty(I)$. Fix $z$ in $X - X_0$. We must find a value for $\widetilde{f}(z)$ such that for all $x$ in $X_0$

$$\left|\widetilde{f}(z) - f(x)\right| \leq \mathrm{Lip}(f)\mathrm{d}(x, z), \quad \forall x \in X_0$$

or equivalently

$$f(y) - \mathrm{Lip}(f)\mathrm{d}(y, z) \leq \widetilde{f}(z) \leq f(x) + \mathrm{Lip}(f)\mathrm{d}(x, z), \quad \forall y \in X_0$$

hence

$$\sup_{y \in X_0} \left(f(y) - \mathrm{Lip}(f)\mathrm{d}(y, z)\right) \leq \widetilde{f}(z) \leq \inf_{x \in E} \left(f(x) + \mathrm{Lip}(f)\mathrm{d}(x, z)\right)$$

It is possible because for all $x$,$y$ in $X_0$, we have

$$f(x) - f(y) \leq \mathrm{Lip}(f)\mathrm{d}(x, y) \leq \mathrm{Lip}(f)(\mathrm{d}(x, z) + \mathrm{d}(y, z)).$$

Define the function $\widetilde{f} : X \longrightarrow \mathbb{R}$ by the formula

$$\widetilde{f}(z) = \inf_{x \in X_0} \left(f(x) + \mathrm{Lip}(f)\mathrm{d}(x, z)\right),$$

To see that this function satisfies the results, fix an arbitrary $x_0 \in X_0$. Then, for any $x \in X_0$

$$\begin{aligned} f(x_0) - f(x) &\leq \text{Lip}(f)\text{d}(x_0, x), \\ &\leq \text{Lip}(f)\left(\text{d}(x_0, z) + \text{d}(z, x)\right). \end{aligned}$$

This implies (that $f(x) + \text{Lip}(f)\text{d}(x, z)$ is bounded below)

$$f(x_0) - \text{Lip}(f)\text{d}(x_0, z) \leq f(x) + \text{Lip}(f)\text{d}(x, z).$$

So $\widetilde{f}(z)$ is well-defined. Also, if $z \in X_0$, the above shows that $\widetilde{f}(z) = f(z)$. Finally (by definition of the inf), for $z, y \in X$ and $\epsilon > 0$, choose $x_z \in X_0$ such that

$$\widetilde{f}(z) \geq f(x_z) + \text{Lip}(f)\text{d}(z, x_z) - \epsilon$$
$$-\widetilde{f}(z) \leq -f(x_z) - \text{Lip}(f)\text{d}(z, x_z) + \epsilon$$

Then

$$\begin{aligned} \widetilde{f}(y) - \widetilde{f}(z) &\leq f(x_z) + \text{Lip}(f)\text{d}(y, x_z) - f(x_z) - \text{Lip}(f)d(z, x_z) + \epsilon \\ &\leq \text{Lip}(f)\text{d}(y, z) + \epsilon. \end{aligned}$$

Thus, we see that $\widetilde{f}$ is indeed $\text{Lip}(f)$-Lipschitz. ■

### 1.2.1 Lipschitz free space

It is proved without any reference to molecules (see the definition of molecules in [1]) that the closed linear subspace of $\left(X^{\#}\right)^*$ spanned by the evaluation function $\delta_x : X \longrightarrow \mathbb{K}$, given by

$$\delta_x(f) = f(x), \text{ for all } x \in X$$

is a predual of $X^{\#}$ (we note that any weak*-closed linear subspace $M$ of a conjugate space $E^*$ is itself a conjugate space. This follows from the observation that $\mathcal{B}_M$ is compact in the weak*-topology). This space was called Lipschitz-free space and denoted $\mathcal{F}(X)$ by Godefroy and Kalton in [9].

**Definition 1.2.5** *The Lipschitz free space on $X$ is*

$$\mathcal{F}(X, d_X) = \overline{span\{\delta_x, x \in X\}}^{\text{Lip}_0(X)^*}.$$

Put now

$$\|m\|_{\mathcal{F}(X,d_X)} = \inf\left\{\sum_{j=1}^{n} |\alpha_j|\, d(x_j, y_j)\right\}$$

over all representation of $m = \sum_{j=1}^{n} \alpha_j \left(\delta_{x_j} - \delta_{y_j}\right)$.

We notice that Lipschitz free space is introduced by Godefroy and Kalton in 2003 (see [9]), although this space was presented at the first time by Arens-Eells on 1956 (they used the molecules).

**Proposition 1.2.2** *For any metric space* $X$, $\mathcal{F}(X, \mathrm{d})^* \overset{isometrically}{\equiv} \mathrm{Lip}_0(X)$.

**Proof.** We define a linear surjective isometry $J$ on $\mathrm{Lip}_0(X)$ with values in $\mathcal{F}(X, \mathrm{d})^*$ by $J(f)(\delta_x) = f(x)$ and we extend by continuity to $\mathcal{F}(X, \mathrm{d})$. Consider $f$ in $\mathrm{Lip}_0(X)$ and $m$ in $\mathrm{span}\{\delta_x, \quad x \in X\}$ such that $m = \sum_{i=1}^{n} a_i \delta_{x_i}$. $J(f)(m) = \sum_{i=1}^{n} a_i f(x_i)$. We show that $J$ is a surjective isometry.

a) Consider $f$ in $\mathrm{Lip}_0(X)$ and $m$ in $\mathcal{F}(X, \mathrm{d})$. We have

$$\begin{aligned} |J(f)(m)| &= \left|\langle f, m\rangle_{(\mathrm{Lip}_0(X), \mathcal{F}(X))}\right| \\ &= \left|\langle f, m\rangle_{(\mathrm{Lip}_0(X), \mathrm{Lip}_0(X)^*)}\right| \\ &\leq \mathrm{Lip}(f)\, \|m\|_{\mathcal{F}(X)} \end{aligned}$$

and we obtain $\|J(f)\| \leq \mathrm{Lip}(f)$.

b) Let $(x, y)$ be in $\widetilde{X}$ and put $m = \dfrac{\delta_x - \delta_y}{\mathrm{d}(x, y)}$. We have $\|m\|_{\mathcal{F}(X)} = 1$ because $\delta_X$ is an isometry see Proposition 1.2.4 below and

$$\begin{aligned} \|J(f)\|_{\mathcal{F}(X,\mathrm{d})^*} &\geq |J(f)(m)| \\ &\geq \left|\frac{f(x) - f(y)}{\mathrm{d}(x, y)}\right| \\ \text{(we take the sup)} &\geq \mathrm{Lip}(f). \end{aligned}$$

*c*) Consider $\varphi \in \mathcal{F}(X, \mathrm{d})^*$. Then $\varphi$ is determinate by $\delta_x$ for every $x$ in $X$. We put for every $x$ in $X$, $f(x) = \varphi(\delta_x)$ and we prove that $f$ is Lipschitz and $J(f) = \varphi$.

$(i)$ We show that $f \in \mathrm{Lip}_0(X)$.

- $f(0) = \varphi(\delta_0) = \varphi(0) = 0$.

- Let $x, y$ be in $X$

$$\begin{aligned} |f(x) - f(y)| &= |\varphi(\delta_x) - \varphi(\delta_y)| \\ &= |\langle \varphi, \delta_x - \delta_y \rangle| \\ &\leq \|\varphi\|_{\mathcal{F}(X,\mathrm{d})^*} \|\delta_x - \delta_y\|_{(\mathrm{Lip}_0(X))^*} \\ &\leq \|\varphi\|_{\mathcal{F}(X,\mathrm{d})^*} \mathrm{d}(x, y). \end{aligned}$$

$(ii)$ Let $m = \sum_{i=1}^{n} a_i \delta_{x_i}$ be in span $\{\delta_x : x \in X\}$. Then, $\varphi(m) = \sum_{i=1}^{n} a_i f(x_i) = J(f)(m)$. This ends the proof. ■

To demonstrate the following proposition, we need the following definitions. Let $E$ be a Banach space.

**Definition 1.2.6 (convex part)** *A non-empty set $C$ of $E$ is convex if, for all $x, y \in C$ and $\theta \in [0, 1]$, we have $\theta x + (1 - \theta) y \in C$, it is clear that every subspace of $E$ is convex and any non-empty intersection of the convex parts of $E$ is convex.*

**Definition 1.2.7 (Convex hull)** *The convex hull of a non-empty set $C$ of $E$ is the intersection of all convex parts of $E$ containing $C$, it is the smallest convex part of $E$ containing $C$, it is noted by $co(C)$ and we have*

$$co(C) = \left\{ \sum_{i=1}^{n} \theta_i x_i : n \in \mathbb{N}, x_i \in C, \theta_i > 0, \sum_{j=1}^{n} \theta_i = 1 \right\}.$$

**Definition 1.2.8 (Absolutely convex)** *A non-empty set $C$ of $E$ is absolutely convex if, for all $x, y \in C$ and $\theta_1, \theta_2 \in \mathbb{K}$ such that $|\theta_1| + |\theta_2| < 1$, we have $\theta_1 x + \theta_2 y \in C$. Such parts are always contains the element $0$.*

**Definition 1.2.9 (Absolutely convex hull)** *The absolutely convex envelope of a non-empty set $C$ of $E$ is the intersection of all absolutely convex parts of $E$ containing $C$, it is noted by $\Gamma(C)$ and we have*

$$\Gamma(C) = \left\{ \sum_{i=1}^{n} \theta_i x_i : n \in \mathbb{N}, x_i \in C, \theta_i \in \Bbbk, \sum_{j=1}^{n} \theta_i \leq 1 \right\}.$$

We can consult the proof of the following proposition in [11]. By applying the bipolar theorem, we give a precise description of $\mathcal{B}_{\mathcal{F}(X)}$ by means of the Lipschitz evaluation functionals $\delta_{(x,y)} = \dfrac{\delta_x - \delta_y}{d(x,y)}$ defined on $X^{\#}$, where $(x, y)$ runs through $\widetilde{X} = \{(x, y) \in X^2 : x \neq y\}$

**Proposition 1.2.3** *The closed unit ball of $\mathcal{F}(X)$ is the closed absolutely convex hull of the set $\left\{\delta_{(x,y)} : (x,y) \in \widetilde{X}\right\}$ in $\left(X^{\#}\right)^{*}$ (i.e., $\mathcal{B}_{\mathcal{F}(X)} = \overline{\Gamma}\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)$.*

**Proposition 1.2.4** *Define*

$$\begin{array}{rcccc} \delta_X & : & X & \longrightarrow & \mathcal{F}(X) \\ & & x & \longmapsto & \delta_X(x) = \delta_x \end{array}$$

*The application $\delta_X$ is an isometry, i.e., for every $x_1, x_2$ in $X$ , one have $\|\delta_X(x_1) - \delta_X(x_2)\| = d(x_1, x_2)$.*

**Proof.** For $x_1, x_2 \in X$ , and for all $g \in X^{\#}$ we have in the first part

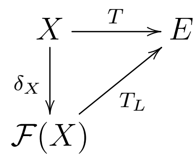
$$\begin{aligned} \|\delta_X(x_1) - \delta_X(x_2)\|_{\mathcal{F}(X)} &= \|\delta_{x_1} - \delta_{x_2}\|_{\mathcal{F}(X)} \\ &= \sup_{\varphi \in \mathcal{B}_{\mathcal{F}(X)^*}} |\delta_{x_1}(\varphi) - \delta_{x_2}(\varphi)| \\ (J(g) = \varphi) \quad &= \sup_{J(g) \in \mathcal{B}_{\mathcal{F}(X)^*}} |\langle \delta_{x_2} - \delta_{x_2}, J(g)\rangle| \\ &= \sup_{g \in \mathcal{B}_{X^{\#}}} |\langle \delta_{x_1} - \delta_{x_2}, g\rangle| \\ &\leq \sup_{g \in \mathcal{B}_{X^{\#}}} |\langle g(x_1) - g(x_2)| \\ &\leq \sup_{g \in \mathcal{B}_{X^{\#}}} \operatorname{Lip}(g)\, d(x_1, x_2) \leq d(x_1, x_2). \end{aligned}$$

It implies $\|\delta_{x_1} - \delta_{x_2}\|_{\mathcal{F}(X)} \leq d(x_1, x_2)$. In the second part, we put $f_z = d(\cdot, z) - d(e, z)$ such that $f_z \in X^{\#}$ and $\operatorname{Lip}(f_z) = 1$.
For all $x_1, x_2 \in X$ we have We have

$$\begin{aligned} \|\delta_{x_1} - \delta_{x_2}\|_{\mathcal{F}(X)} &= \sup_{f \in \mathcal{B}_{X^{\#}}} |\langle \delta_{x_1} - \delta_{x_2}, f\rangle| \\ &\geq |\langle \delta_{x_1} - \delta_{x_2}, f_{x_1}\rangle| \\ &\geq |f_{x_1}(x_1) - f_{x_1}(x_2)| \\ &\geq d(x_1, x_2). \end{aligned}$$

This ends the proof. ■

**Theorem 1.2.2** *[14] Let $E$ be a Banach space. For any Lipschitz operator $T \in \mathrm{Lip}_0(X,E)$, there is a unique linear map $T_L : \mathcal{F}(X) \longrightarrow E$ such that $T = T_L \circ \delta_X$, with $\|T_L\| = \mathrm{Lip}(T)$ and such that the following diagram commutes*




**Definition 1.2.10** *Consider $X, Y$ in $\mathcal{M}_0$ and let $T : X \longrightarrow Y$ be a Lipschitz map which preserves base point, Sawashima [13],[3] define the Lipschitz adjoint (dual) of $T$ which is a bounded linear operator noted by $T^{\#}$ from $Y^{\#}$ to $X^{\#}$ given by*

$$T^{\#}(g)(x) = (g \circ T)(x) = g(T(x)), \ \forall g \in \mathrm{Lip}_0(Y).$$

*If $Y = E$ is a Banach space, The restriction of $T^{\#}$ to $E^*$ is called the Lipschitz transpose map of $T$ and denoted by $T^t$. Such that $\left\|T^{\#}\right\| = \mathrm{Lip}(T) = \|T^t\|$*

### 1.2.2 Lipschitz compact operators

If $X$ is a metric space and $E$ is a Banach space, by the Lipschitz image of a mapping $T : X \longrightarrow E$ we mean the set

$$\left\{ \frac{T(x) - T(y)}{d(x,y)} : x, y \in X, \ x \neq y \right\}.$$

It is immediate that $T : X \longrightarrow E$ is a Lipschitz mapping if its Lipschitz image is a bounded subset of $E$. This motivates the following definition.

**Definition 1.2.11** *[11] Let $X$ be a pointed metric space and $E$ a Banach space. We say that a base-point preserving map $T : X \longrightarrow E$ is Lipschitz compact (resp, Lipschitz weakly compact) if its Lipschitz image is relatively compact (resp, relatively weakly compact) in $E$.*

We denote by $\mathrm{Lip}_{0k}(X,E)$ and $\mathrm{Lip}_{0w}(X,E)$ the sets of Lipschitz compact and weakly compact operators from $X$ *to* $E$, respectively. Plainly,

$$\mathrm{Lip}_{0k}(X,E) \subset \mathrm{Lip}_{0w}(X,E) \subset \mathrm{Lip}_0(X,E).$$

Observe that $\mathrm{Lip}_{0k}(X,E)$ *and* $\mathrm{Lip}_{0w}(X,E)$ are linear subspaces of $\mathrm{Lip}_0(X,E)$.

Next we study the relation between the compactness of a Lipschitz operator $f\in\mathrm{Lip}_0(X,E)$ and the compactness of its linearization $T_L\in\mathcal{B}(\mathcal{F}(X),E)$.

**Lemma 1.2.1** *[11] Let $T\in\mathrm{Lip}_0(X,E)$, we define the following application*

$$\begin{array}{rcl}\delta_{\widetilde{X}}\ :\widetilde{X} & \longrightarrow & \left(X^{\#}\right)^{*}\\ (x,y) & \longrightarrow & \delta_{\widetilde{X}}(x,y)=\delta_{(x,y)}\end{array}$$

*Then*

$$T_L\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\subset T_L\left(\overline{\Gamma}\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\right)\subset\overline{\Gamma}\left(T_L\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\right).$$

**Proof.** We show that $T_L\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\subset T_L\left(\overline{\Gamma}\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\right)$. We have

$$\delta_{\widetilde{X}}\left(\widetilde{X}\right)\subset\overline{\Gamma}\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right),\ \text{( with definition)}$$

then

$$T_L\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\subset T_L\left(\overline{\Gamma}\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\right).$$

For the second inclusion we take $z_1\in T_L\left(\overline{\Gamma}\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\right)$ implies $\exists z_2\in\overline{\Gamma}\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)$, such that $z_1=T_L(z_2)$. This implies that there is $b_n\in\Gamma\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)$ such that, $z_2=\lim_n b_n$, and $b_n=\sum_{i=1}^{n}\lambda_i^n\delta_{\left(x_i^{(n)},y_i^{(n)}\right)}$. We have

$$\begin{aligned}z_1&=T_L(z_2),\\&=T_L\left(\lim_n b_n\right),\\&=T_L\left(\lim_n\sum_{i=1}^{n}\lambda_i^n\delta_{\left(x_i^{(n)},y_i^{(n)}\right)}\right),\\&=\lim_n\sum_{i=1}^{n}\lambda_i^nT_L\left(\delta_{\left(x_i^{(n)},y_i^{(n)}\right)}\right).\end{aligned}$$

Where $z_1\in\overline{\Gamma}\left(T_L\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\right)$. We finally get

$$T_L\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\subset T_L\left(\overline{\Gamma}\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\right)\subset\overline{\Gamma}\left(T_L\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\right).$$

And this ends the proof. ■

**Proposition 1.2.5** *Let $X$ be a pointed metric space, $E$ be a Banach space and $T \in \mathrm{Lip}_0(X, E)$. Then $T$ is a Lipschitz compact if, and only if, $T_L$ is compact.*

**Proof.** We have

$$\begin{aligned} T_L\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right) &= \left\{T_L\left(\delta_{\widetilde{X}}(x, y);\ x, y \in X,\ x \neq y\right)\right\} \\ &= \left\{T_L\left(\frac{\delta_x - \delta_y}{d(x, y)}\right);\ x, y \in X,\ x \neq y\right\} \\ &= \left\{\frac{T_L \circ \delta_x - T_L \circ \delta_y}{d(x, y)};\ x, y \in X,\ x \neq y\right\} \\ &= \left\{\frac{T(x) - T(y)}{d(x, y)};\ x, y \in X,\ x \neq y\right\}. \end{aligned}$$

According to the previous Lemma 1.2.1 and the Proposition 1.2.3, we have

$$T_L\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right) \subset T_L\left(\mathcal{B}_{\mathcal{F}(X)}\right) \subset \overline{\Gamma}\left(T_L\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\right).$$

It is mean $\overline{\left\{\frac{T(x)-T(y)}{d(x,y)};\ x, y \in X,\ x \neq y\right\}} \subset \overline{T_L\left(\mathcal{B}_{\mathcal{F}(X)}\right)} \subset \overline{\Gamma}\left(\left\{\frac{T(x)-T(y)}{d(x,y)};\ x, y \in X,\ x \neq y\right\}\right)$.
We assume that $T_L$ is compact, therefore $\overline{T_L\left(\mathcal{B}_{\mathcal{F}(X)}\right)}$ is compact, then

$$\overline{\left\{\frac{T(x) - T(y)}{d(x, y)};\ x, y \in X,\ x \neq y\right\}},$$

is compact ( closed on a compact is compact), which implies that $T$ is Lipschitz compact.
Now we assume that $T$ is Lipschitz compact, therefore

$$\overline{\left\{\frac{T(x) - T(y)}{d(x, y)};\ x, y \in X,\ x \neq y\right\}},$$

is compact consequently

$$\overline{\Gamma}\left\{\frac{T(x) - T(y)}{d(x, y)};\ x, y \in X,\ x \neq y\right\},$$

is compact ( if $A$ is relatively compact$\Longrightarrow \overline{\Gamma}(A)$ is compact). We also have

$$\overline{T_L\left(\mathcal{B}_{\mathcal{F}(X)}\right)} \subset \overline{\Gamma}\left(T_L\left(\delta_{\widetilde{X}}\left(\widetilde{X}\right)\right)\right).$$

Hence the proposition ensure that $\overline{T_L\left(\mathcal{B}_{\mathcal{F}(X)}\right)}$ is compact implies that $T_L$ is compact. ■

**Proposition 1.2.6** *[11] Let $X$ be a pointed metric space and $E$ is a Banach space. The following are equivalent.*

1. *The Lipschitz operator $T$ is Lipschitz weakly compact.*

2. *The linearization $T_L$ in $\mathcal{B}(\mathcal{F}(X), E)$ is weakly compact.*

3. *There exist a reflexive Banach space $F$, a bounded linear operator $f \in \mathcal{B}(F, E)$ and Lipschitz operator $g \in \mathrm{Lip}_0(X, F)$ such that $T = f \circ g$ .*

**Proof.** The proof of previous proposition is valid to show the equivalence between $(i)$ and $(ii)$. If $(ii)$ holds, applying the Davis, Figiel, Johnson and Pełczyński theorem, there exists a reflexive Banach space $F$ and operators $f \in \mathcal{B}(F, E)$ and $S \in \mathcal{B}(\mathcal{F}(X), F)$ such that $T_L = f \circ S$. Let $g = S \circ \delta_X$. Clearly, $g \in \mathrm{Lip}_0(X, F)$ and $T = T_L \circ \delta_X = f \circ S \circ \delta_X = f \circ g$, and this proves $(iii)$. Finally, $(iii)$ implies $(ii)$ is trivial. ■

**Theorem 1.2.3** *Let $X$ be a pointed metric space and $E$ be Banach space. Consider $T$ in* $\mathrm{Lip}_0(X, E)$. *The following properties are equivalent.*

1. *$T$ is a Lipschitz compact (weakly compact).*

2. *$T^t$ is compact (resp, weakly compact) from $E^*$ into $X^\#$.*

**Proof.** We can find the proof in [11]. ■

# Chapter 2

# Majorization and factorization for bounded linear operators

In this chapter, we will describe the notion of majorization, factorization, some properties and their proofs. We are interested in Barnes article [2].

## 2.1 Majorization

Let $E$,$F$, $G$ and $Q$ be Banach spaces. For $T \in \mathcal{B}(E,F)$, let $\mathrm{R}(T) = \{T(a) : a \in E\}$ be the range space of $T$ and $\mathrm{N}(T) = \{a \in E : T(a) = 0\}$ be the null space of $T$.

**Definition 2.1.1** *Assume that $T \in \mathcal{B}(E,F)$ and $S \in \mathcal{B}(E,G)$. Then we say $T$ majorizes $S$ if there exists $M > 0$ such that*

$$\|S(a)\| \leq M \|T(a)\|$$

*for all $a \in E$.*

**Remark 2.1.1** *Assume that $T \in \mathcal{B}(E,F)$, it is easy to proof these following statements.*

1. *If $S_1, S_2 \in B(E,G)$ and $T$ majorizes $S_1$ and $S_2$, then $T$ majorizes $S_1 + S_2$.*
2. *If $S \in B(E,G)$, $R \in B(G,Q)$ and $T$ majorizes $S$, then $T$ majorizes $RS$.*

**Proposition 2.1.1** *Let $T \in \mathcal{B}(E,F)$ and $S \in \mathcal{B}(E,G)$. The following are equivalent:*

1. *$T$ majoriozes $S$.*
2. *There exists $V \in B\left(\overline{\mathrm{R}(T)}, G\right)$ such that $S = V \circ T$.*
3. *Whenever $\{a_n\}_{n\in\mathbb{N}} \subseteq E$ with $\|T(a_n)\| \longrightarrow 0$, then $\|S(a_n)\| \longrightarrow 0$.*

**Proof.** Assume that $T$ *majoriozes* $S$ and we will verify that (2) holds. Define

$$\begin{array}{rccc} V: & \mathrm{R}(T) & \longrightarrow & Z \\ & T(a) & \longmapsto & V(T(a)) = S(a) \end{array}$$

The map $V$ is well defined since $\mathrm{N}(T) \subseteq \mathrm{N}(S)$ it is clear that $V$ is linear. Now

$$\begin{aligned} \|V(T(a))\| &= \|S(a)\| \\ &\leq M \|T(a)\|. \end{aligned}$$

Thus, $V$ has a bounded extension, which we also denoted by $V$, on $\overline{\mathrm{R}(T)}$. From the definition of $V$, $S = V \circ T$.

Now we want to prove (2) $\Longrightarrow$ (1) . Suppose (2) holds, then there is $V \in \mathcal{B}\left(\overline{\mathrm{R}(T)}, G\right)$ such that $S = V \circ T$, this implies that for all $a \in E$, we have

$$\begin{aligned} \|S(a)\| &= \|V \circ T(a)\| \\ &\leq \|V\| \, \|T(a)\| . \end{aligned}$$

This means that $T$ majorizes $S$.

Suppose that $T$ majorizes $S$. So whenever $\{a_n\}_{n\in\mathbb{N}} \subseteq X$,

$$\|S(a_n)\| \leq M \|T(a_n)\| .$$

If $\|T(a_n)\| \longrightarrow 0$ then we have $\|S(a_n)\| \longrightarrow 0$. this shows that (3) holds.

Now, suppose that (2) holds. So whenever $\{a_n\}_{n\in\mathbb{N}} \subseteq E$,

$$\begin{aligned} \|S(a_n)\| &= \|V(T(a_n))\| \\ &\leq \|V\| \, \|T(a_n)\| . \end{aligned}$$

If $\|T(a_n)\| \longrightarrow 0$ then we have $\|S(a_n)\| \longrightarrow 0$. this shows that (3) holds.

Assume that the property in (3) holds. Note that this property implies that $\mathrm{N}(T) \subseteq \mathrm{N}(S)$. As above, define $V$ by

$$\begin{array}{rccc} V: & \mathrm{R}(T) & \longrightarrow & Z \\ & T(a) & \longmapsto & V(T(a)) = S(a) \end{array}$$

It is clear that $V$ is linear map. As a consequence of the assumption in (3), $V$ is a continuous map. This verifies that (2) holds.

This ends the proof. ■

**Proposition 2.1.2** *Let $T \in \mathcal{B}(E, F)$, $S \in \mathcal{B}(E, G)$, and that $T$ majorizes $S$.*

1. *If $T$ is compact, then $S$ is compact.*

2. *If $T$ is weakly compact, then $S$ is weakly compact.*

**Proof.** Since $T$ majorizes $S$, we have from Proposition 2.1.1, there exists $V \in \mathcal{B}\left(\overline{\mathrm{R}(T)}, G\right)$ such that $S = V \circ T$. Assume that $T$ is compact operator. Since the composition of a bounded operator with compact operator is a compact operator ([6], Theorem 4, p.486). This implies that (1) holds.

The proof of (2) , is the same using ([6], Theorem 5, p.484). When $T$ is weakly compact operator. ■

**Theorem 2.1.1** *1. Assume that $T \in \mathcal{B}(E,F)$ andS $\in \mathcal{B}(E,G)$. Suppose that $T$ majorizes $S$. Then $R(S^*) \subseteq R(T^*)$.*

*2. Consider $T \in \mathcal{B}(E,F)$ and $S \in \mathcal{B}(E,G)$. Suppose that $R(S^*) \subseteq R(T^*)$. Then $T$ majorizes $S$.*

*3. Take $T$ in $\mathcal{B}(E,F)$ and $S$ in $\mathcal{B}(G,F)$, and that $R(S) \subseteq R(T)$ . Then $T^*$ majorizes $S^*$.*

*4. Suppose that $E$ is reflexive. Assume that $T \in \mathcal{B}(E,F)$ and $S \in \mathcal{B}(G,F)$ , and that $T^*$ majorizes $S^*$. Then $R(S) \subseteq R(T)$.*

**Proof.**

1. Suppose that $T$ majorizes $S$. So by Proposition 2.1.1, there exists $V \in \mathcal{B}\left(\overline{\mathrm{R}(T)}, G\right)$ such that $S = V \circ T$. Let $c^* \in G^*$ and for all $a \in E$,

$$\begin{aligned} \langle a, S^*(c^*) \rangle &= \langle S(a), c^* \rangle \\ &= \langle V \circ T(a), c^* \rangle \\ &= \langle T(a), V^*(c^*) \rangle \end{aligned}$$

where $V^*(c^*)$ is a continuous linear functional on $\overline{\mathrm{R}(T)}$. Let $\widetilde{V^*(c^*)}$ be any extension of $V^*(c^*)$ to $F^*$ (Hahn-Banach theorem). Then for all $x \in X$,

$$\begin{aligned} \langle a, S^*(c^*) \rangle &= \langle T(a), \widetilde{V^*(c^*)} \rangle \\ &= \langle a, T^*\left(\widetilde{V^*(c^*)}\right) \rangle. \end{aligned}$$

Thus, $S^*(c^*) = T^*\left(\widetilde{V^*(c^*)}\right)$. This shows $\mathrm{R}(S^*) \subseteq \mathrm{R}(T^*)$.

2. Assume the hypotheses in (2). Note that $\mathrm{N}(T) \subseteq \mathrm{N}(S)$. Then the linear map

$$\begin{array}{cccc} V: & \mathrm{R}(T) & \longrightarrow & G \\ & T(a) & \longmapsto & V(T(a)) = S(a) \end{array}$$

is well defined, for all $a \in E$. Suppose $V$ is unbounded. Then there exists a sequence $\{a_n\} \subseteq E$ with $\|T(a_n)\| = 1$ for all $n \in \mathbb{N}$, and $\|S(a_n)\| \longrightarrow +\infty$. Let $c^* \in G^*$ be arbitrary, and choose $\widetilde{V^*(c^*)} \in F^*$ such that $S^*(c^*) = T^*\left(\widetilde{V^*(c^*)}\right)$.

Then

$$\begin{aligned} |\langle S(a_n), c^*\rangle| &= |\langle a_n, S^*(c^*)\rangle| \\ &= \left|\langle a_n, T^*\left(\widetilde{V^*(c^*)}\right)\rangle\right| \\ &= \left|\langle T(a_n), \left(\widetilde{V^*(c^*)}\right)\rangle\right| \\ &\leq \left\|\left(\widetilde{V^*(c^*)}\right)\rangle\right\| \text{ for all } n \geq 1. \end{aligned}$$

It follows from the uniform boundedness principale that $\|S(a_n)\|$ is bounded, a contradiction. Thus we have that $V$ is bounded on $\mathrm{R}(T)$ and $S = V \circ T$, so $T$ majorizes $S$.

3. Now Take $T$ and $S$ are as in (3) with $\mathrm{R}(S) \subseteq \mathrm{R}(T)$. This implies that $\mathrm{N}(T^*) \subseteq \mathrm{N}(S^*)$. Indeed, let $b \in \mathrm{N}(T^*) \subseteq F^*$ and for all $a \in E$ we have

$$\begin{aligned} \langle T^*(b), a\rangle &= \langle b, T(a)\rangle \\ &= 0. \end{aligned}$$

This implies $b \perp \mathrm{R}(T)$. As $\mathrm{R}(S) \subseteq \mathrm{R}(T)$ then we have $b \perp \mathrm{R}(S)$. So for all $c \in G$

$$\langle b, S(c)\rangle = 0$$

this implies

$$\langle S^*(b), c\rangle = 0$$

This mean $b \in \mathrm{N}(S^*)$.

Define

$$\begin{aligned} U: \quad \mathrm{R}(T^*) &\longrightarrow \qquad Z^* \\ T^*(b^*) &\longmapsto U(T^*(b^*)) = S^*(b^*) \end{aligned}$$

for all $b^* \in F^*$ it is clear that $U$ is well defined. If $U$ is unbounded, then there exists $(b_n^*) \subseteq F^*$ such that $\|T^*(b_n^*)\| = 1$ for all $n \in \mathbb{N}$, while $\|S^*(b_n^*)\| = \|U(T^*(b_n^*))\| \longrightarrow +\infty$. For an arbitrary $c \in G$, choose $a \in E$ such that $S(c) = T(a)$. Then

$$\begin{aligned} |\langle c, S^*(b_n^*)\rangle| &= |\langle S(c), b_n^*\rangle| \\ &= |\langle T(a), c_n^*\rangle| \\ &= |\langle a, T^*(b_n^*)\rangle| \\ &\leq \|a\|. \end{aligned}$$

From the Uniform Boundedness Principale, $\|S^*(b_n^*)\|$ is a bounded sequence. This contradiction proves that $U$ is bounded. Therefore, since $S^* = U \circ T^*$, $T^*$ majorizes $S^*$.

4. Consider the hypotheses in (4). By Proposition 2.1.1, $S^* = V \circ T^*$ where $V : \overline{\mathrm{R}(T^*)} \longrightarrow G^*$ is a bounded linear map. Then $V^* : G^{**} \to \overline{\mathrm{R}(T^*)}^*$. Assume that $c \in G$ is arbitrary. Then $c \in G^{**}$ and $V^*(c) \in \overline{\mathrm{R}(T^*)}^*$. Let $\widetilde{V^*(c)}$ be any extension of $V^*(c)$ to $E^{**}$ (Hahn Banach theorem)

$$\begin{array}{ccc} \mathrm{R}(T^*) & \xrightarrow{V^*(c)} & \mathbb{R} \\ {\scriptstyle i}\big\downarrow & \nearrow_{\widetilde{V^*(c)}} & \\ E^* & & \end{array}$$

Since $E$ is reflexive, $\exists a \in E$ such that $\langle a, a^*\rangle = \left\langle \widetilde{V^*(c)}, a^* \right\rangle$ for all $a^* \in E^*$. For all $b^* \in F^*$,

$$\begin{aligned} \langle S(c), b^*\rangle &= \langle c, S^*(b^*)\rangle \\ &= \langle c, V \circ T^*(b^*)\rangle \\ &= \langle V^*(c), T^*(b^*)\rangle \\ &= \langle b, T^*(b^*)\rangle \\ &= \langle T(b), b^*\rangle. \end{aligned}$$

Therefore $S(c) = T(b)$, so $\mathrm{R}(S) \subseteq \mathrm{R}(T)$.

This ends the proof. ■

The dual properties in the theorem above combined with Proposition 2.1.1 yield the following results concerning range inclusion.

**Proposition 2.1.3** *Consider* $T \in \mathcal{B}(E,F)$ *and* $S \in \mathcal{B}(G,F)$.

1. *If* $\mathrm{R}(S) \subseteq \mathrm{R}(T)$ *and* $T$ *is compact, then* $S$ *is compact.*

2. *If* $\mathrm{R}(S) \subseteq \mathrm{R}(T)$ *and* $T$ *is weakly compact, then* $S$ *is weakly compact.*

**Proof.** Assume that the hypotheses in (1), Since $\mathrm{R}(S) \subseteq \mathrm{R}(T)$, by Theorem 2.1.1 part 3, $T^*$ majorizes $S^*$. Now $T^*$ is compact, and so by Proposition 2.1.2, $S^*$ is compact. Therefore $S$ is compact (Schauder's theorem 1.1.3).
Assume that $\mathrm{R}(S) \subseteq \mathrm{R}(T)$, from Theorem 2.1.1 part 3 , $T^*$ majorizes $S^*$. Now $T^*$ is weakly compact, and by Proposition 2.1.2, $S^*$ is weakly compact. Therefore $S$ is weakly compact (Gantmacher's theorem 1.1.4). ■

## 2.2 Factorization

Assume that $T \in B(E,F)$ and $S \in B(E,G)$. Then we say that $S$ is left multiple of $T$ if there exists $V \in \mathcal{B}(F,G)$ such that $S = V \circ T$. There is a similar terminology for when $S$ is a right multiple of $T$, $S = T \circ U$. In either case we say that $S$ factors with respect to $T$. Recall that a closed subspace $M$ of $E$ is complemented if there exists a closed subspace $N$ of $E$ such that $E = M \oplus N$.

**Theorem 2.2.1** *Assume* $T \in \mathcal{B}(E,F)$.

1. *If* $S \in \mathcal{B}(E,G)$ *is majorized by* $T$ *and* $\overline{\mathrm{R}(T)}$ *is complemented, then there exists* $V \in \mathcal{B}(F,G)$ *such that* $S = V \circ T$.

2. *If* $S \in \mathcal{B}(G,F)$ *with* $\mathrm{R}(S) \subseteq \mathrm{R}(T)$ *and* $\mathrm{N}(T)$ *is complemented, then there exists* $U \in \mathcal{B}(G,E)$ *such that* $S = T \circ U$.

3. *If* $S \in \mathcal{B}(E,G)$ *with* $\mathrm{R}(S^*) \subseteq \mathrm{R}(T^*)$ *and* $\overline{\mathrm{R}(T)}$ *is complemented, then there exists* $V \in \mathcal{B}(F,G)$ *such that* $S = V \circ T$.

4. *Assume that* $E$ *is reflexive. If* $S \in \mathcal{B}(G,F)$, $T^*$ *majorizes* $S^*$, *and* $\mathrm{N}(T)$ *is complemented, then there exists* $U \in S \in \mathcal{B}(G,E)$ *such that* $S = V \circ T$.

**Proof.**

1. $\overline{\mathrm{R}(T)}$ is complemented that is mean there exists a closed subspace $N$ of $F$ such that $F = \overline{\mathrm{R}(T)} \oplus N$. from Proposition 1.2.3, there exist $V \in \mathcal{B}\left(\overline{\mathrm{R}(T)}, G\right)$ such that $S = V \circ T$, (Simply extend $V$ to be the zero operator on $N$). (1) holds.

2. Assume the hypotheses in (2) . Let $W$ be closed subspace of $E$ with $E = \mathrm{N}(T) \oplus W$. Let $\widetilde{T} : W \longrightarrow F$ be the restriction of $T$ to $W$. Now $\widetilde{T}^{-1}$ is closed linear map on $\mathrm{R}(T)$, and since $\mathrm{R}(S) \subseteq \mathrm{R}(T)$, $\widetilde{T}^{-1} \circ S : G \longrightarrow W$ is a closed, hence bounded operator by the Closed Graph Theorem. We may consider $U \equiv \widetilde{T}^{-1} \circ S$ as an operator in $\mathcal{B}(G, E)$. Clearly, $S = T \circ \widetilde{T}^{-1} \circ S = T \circ U$.

3. Assume the hypotheses in (2). We have $\mathrm{R}(S^*) \subseteq \mathrm{R}(T^*)$, so from Theorem 2.1.1 (2), $T$ majorizes $S$ and by part (1), then there exists $V \in \mathcal{B}(F, G)$ such that $S = V \circ T$. (3) holds.

4. (4) verifier from Theorem 2.1.1 (4) and part (2) .

This ends the proof. ■

# Chapter 3

# Majorization and factorization for Lipschitz operators

In this chapter, we will define the concept of majorization for Lipschitz operators, and we study some proprieties concerning this concept. Finally, we interested in particular in Pietsch factorization theorem and some application of this notion.

## 3.1 Majorization

Let $X$, $Y$ be pointed metric spaces and $E$, $F$ and $G$ be Banach spaces. Recall that the image of Lipschitz operator $T \in \mathrm{Lip}_0(X, E)$ is defined and noted by

$$Im_{Lip}(T) = \left\{ \frac{(T(x) - T(y))}{d(x, y)} : x, y \in X, x \neq y \right\}.$$

We assume the base point of $T(X)$ is 0.

**Definition 3.1.1** *Assume that $T \in \mathrm{Lip}_0(X, E)$ and $S \in \mathrm{Lip}_0(X, F)$. Then $T$ majorizes $S$ if there exists a constant $C > 0$ such that*

$$\|S(x_1) - S(x_2)\| \leq C \|T(x_1) - T(x_2)\|,$$

*for all $x_1, x_2 \in X$.*

**Remark 3.1.1** *Let $T \in \mathrm{Lip}_0(X, E)$. The proof of the two following statements is easy:*

1. *If $S_1, S_2 \in \mathrm{Lip}_0(X, F)$ and $T$ majorizes $S_1$ and $S_2$, then $T$ majorizes $S_1 + S_2$.*
2. *If $S \in \mathrm{Lip}_0(X, F)$, $R \in \mathrm{Lip}_0(F, G)$ and $T$ majorizes $S$, then $T$ majorizes $RS$.*

**Proposition 3.1.1** *Consider two Lipschitz maps $T \in \mathrm{Lip}_0(X, E)$ and $S \in \mathrm{Lip}_0(X, F)$. Then $T$ majorizes $S$ if, and only if, there exists $V \in \mathrm{Lip}_0\left(\overline{T(X)}, F\right)$ such that $S = V \circ T$ and $\mathrm{Lip}(V) \leq C$.*

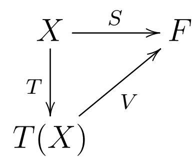
**Proof.** Assume that $T$ majorizes $S$. Define

$$\begin{array}{rccc} V: & T(X) & \longrightarrow & F \\ & T(x) & \longmapsto & V(T(x)) = S(x) \end{array}$$

The Lipschitz operator $V$ is well defined. Indeed, if $T(x_1) = T(x_2) = z$, we have $V(T(x_1)) = S(x_1)$ and $V(T(x_2)) = S(x_2)$. So,

$$\begin{aligned} \|V(T(x_1)) - V(T(x_2))\| &= \|S(x_1) - S(x_2)\| \\ &\leq C \|T(x_1) - T(x_2)\| \\ &= 0. \end{aligned}$$

and this implies that $V$ is well defined. From the definition of $V$, we have $T = V \circ S$ such that $\mathrm{Lip}(V) \le C$. Since the base point of $T(X)$ is 0, $V$ preserving the base point.




We end the proof by extended $V$ by density to $\overline{T(X)}$. Thus, $V \in \mathrm{Lip}_0\left(\overline{T(X)}, F\right)$.

The converse is clear. ■

**Proposition 3.1.2** *Let $T \in \mathrm{Lip}_0(X, E)$, $S \in \mathrm{Lip}_0(X, F)$. If the linearization of $T$ ( i,e. $T_L$) majorizes the linearization of $S$ (i,e. $S_L$) then $T$ majorizes $S$.*

**Proof.** Assume that $T_L$ majorizes $S_L$ then there exists a constant $C > 0$ such that

$$\|S_L(m)\| \le C \|T_L(m)\| \text{ for all } m \in \mathcal{F}(X).$$

Such that $m = \sum_{i=1}^{n} \alpha_i \delta_{x_i y_i}$. Take $i = 1$ and $\alpha_i = 1$ then we have

$$\|S_L(\delta_{xy})\| \le C \|T_L(\delta_{xy})\|.$$

This implies,

$$\|S(x) - S(y)\| \le C \|T(x) - T(y)\|.$$

for all $x, y \in X$. Thus, $T$ majorizes $S$. ■

**Proposition 3.1.3** *Assume that $T \in \mathrm{Lip}_0(X, E)$, $S \in \mathrm{Lip}_0(X, F)$ and $T_L$ majorizes $S_L$.*

1. *If $T$ is Lipschitz compact, then $S$ is Lipschitz compact.*

2. *If $T$ is Lipschitz weakly compact, then $S$ is Lipschitz weakly compact.*

**Proof.**

1. Suppose that $T$ is Lipschitz compact operator, then we have from Proposition 1.2.5, $T_L$ is compact operator. Since the Proposition 2.1.2 we find that $S_L$ is compact operator. Thus, $S$ is Lipschitz compact operator by Proposition 1.2.5.

2. The proof is the same using Proposition 1.2.5 and 2.1.2 when $T$ is weakly compact operator.

This ends the proof. ■

**Theorem 3.1.1** *1. Let $T \in \mathrm{Lip}_0(X, E)$ and $S \in \mathrm{Lip}_0(X, F)$. Assume that $T$ majorizes $S$. Then $R(S^t) \subseteq R(T^t)$.*

*2. Suppose that $T \in \mathrm{Lip}_0(X, E)$, $S \in \mathrm{Lip}_0(X, F)$ such that $R(S^t) \subseteq R(T^t)$. Assume that $T$ is injective, then $T$ majorizes $S$.*

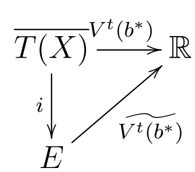
*3. Assume that $T \in \mathrm{Lip}_0(X, E)$ and $S \in \mathrm{Lip}_0(Y, E)$, and that $Im_{Lip}(S) \subseteq Im_{Lip}(T)$. Then $T^t$ majorizes $S^t$.*

**Proof.**

1. Suppose that $T$ majorizes $S$, so by Proposition 3.1.1, there exists $V \in \mathrm{Lip}_0\left(\overline{T(X)}, F\right)$ such that $S = V \circ T$. Now $T^t \in \mathcal{B}\left(E^*, X^\#\right)$ and $S^t \in \mathcal{B}\left(F^*, X^\#\right)$. Consider $b^* \in F^*$. For all $x \in X$ we have

$$\begin{aligned} \left|S^t(b^*)(x)\right| &= |b^*(S(x))| \\ &= |b^*(V(T(x)))| \\ &= \left|V^t(b^*)(T(x))\right|. \end{aligned}$$

Where $V^t(b^*)$ in $\overline{T(X)}^\#$. Let $\widetilde{V^t(b^*)}$ be any extension of $V^t(b^*)$ to $E^*$ ( nonlinear Hahn Banach theorem),




Then for all $x \in X$ we have

$$\begin{aligned} \left|\left(S^t(b^*)\right)(x)\right| &= \left|\widetilde{V^t(b^*)}(T(x))\right| \\ &= \left|\left(T^t\left(\widetilde{V^t(b^*)}\right)\right)(x)\right|. \end{aligned}$$

Thus, $S^t(b^*) = T^t\left(\widetilde{V^t(b^*)}\right)$. This shows $\mathrm{R}(S^t) \subseteq \mathrm{R}(T^t)$.

2. Let $T \in \mathrm{Lip}_0(X,E)$, $S \in \mathrm{Lip}_0(X,F)$ such *that* $\mathrm{R}(S^t) \subseteq \mathrm{R}(T^t)$, suppose that $T$ is injective. Let $b^* \in F^*$ be arbitrary and choose $\widetilde{V^t(b^*)} \in E^*$ such that

$$S^t(b^*) = T^t\left(\widetilde{V^t(b^*)}\right). \tag{3.1.1}$$

Then

$$\begin{aligned} |b^*(V(T(x)) - V(T(y)))| &= |b^*(S(x) - S(y))| \\ &= |b^*(S(x)) - b^*(S(y))| \\ &= \left|S^t(b^*)(x) - S^t(b^*)(y)\right| \\ \text{(By equality 3.1.1)} &= \left|T^t\left(\widetilde{V^t(b^*)}\right)(x) - T^t\left(\widetilde{V^t(b^*)}\right)(y)\right| \\ &= \left|\widetilde{V^t(b^*)}(T(x)) - \widetilde{V^t(b^*)}(T(y))\right| \\ &\leq \mathrm{Lip}\left(\widetilde{V^t(b^*)}\right)\|T(x) - T(y)\|. \end{aligned}$$

Using the Theorem of Hahn Banach, we find

$$V(T(x)) - V(T(y)) \leq \left\|V^t\right\|\|T(x) - T(y)\|.$$

Thus, we have that $V$ is Lipschitz operator on $T(X)$ and $S = V \circ T$, so $T$ majorizes $S$.

3. Now assume $T \in \mathrm{Lip}_0(X,E)$ *and* $S \in \mathrm{Lip}_0(Y,E)$. Since $Im_{Lip}(S) \subseteq Im_{Lip}(T)$ we find $\mathrm{N}(T^t) \subseteq \mathrm{N}(S^t)$. Indeed, for all $a^* \in \mathrm{N}(T^t) \subset E^*$, and for all $x, x' \in X$ we have

$$T^t \circ a^*(x) - T^t \circ a^*(x') = 0 - 0 = 0,$$

this implies,

$$a^*(T(x) - T(x')) = 0.$$

Thus,

$$a^*(Im_{Lip}(T)) = \{0\}.$$

As $Im_{Lip}(S) \subseteq Im_{Lip}(T)$, then we have $a^*(Im_{Lip}(S)) = \{0\}$ this implies for all $z, z' \in Y$

$$a^*(S(z) - S(z')) = 0.$$

This implies,

$$S^t \circ a^*(z) - S^t \circ a^*(z') = 0.$$

Take $z' = e_Y$, this mean $S^t \circ a^*(z) = 0$ for all $z \in Y$. Thus $a^* \in \mathrm{N}(S^t)$.

Now Define

$$\begin{array}{rccc} U: & \mathrm{R}(T^t) & \longrightarrow & Y^{\#} \\ & T^t(a^*) & \longmapsto & U(T^t(a^*)) = S^t(a^*) \end{array}$$

for all $a^* \in E^*$ it is clear that $U$ is well defined. Now, we will prove that $U$ is bounded. For an arbitrary $y, y' \in Y$, choose $x_0, x_0' \in X$ such that $S(y) - S(y') = T(x_0) - T(x_0')$. Let $m \in \mathcal{B}_{\mathcal{F}(Y)}$, then $m = \sum_{j=1}^{n} \alpha_j \delta_{y_j y_j'}$

$$\begin{aligned}
\left|\left(S^t(\alpha^*)\right)_L(m)\right| &= \left|\left(S^t(\alpha^*)\right)_L\left(\sum_{j=1}^{n}\alpha_j\delta_{y_jy_j'}\right)\right| \\
&= \left|\sum_{j=1}^{n}\alpha_j\left(S^t(\alpha^*)\right)_L\left(\delta_{y_jy_j'}\right)\right| \\
&= \left|\sum_{j=1}^{n}\alpha_j\left(S^t(\alpha^*)(y_j) - S^t(\alpha^*)\left(y_j'\right)\right)\right| \\
&= \left|\sum_{j=1}^{n}\alpha_j\left(\alpha^*\left(S(y_j) - S\left(y_j'\right)\right)\right)\right| \\
&= \left|\sum_{j=1}^{n}\alpha_j\left(\alpha^*\left(T(x_{0j}) - T\left(x_{0j}'\right)\right)\right)\right| \\
&= \left|\sum_{j=1}^{n}\alpha_j\left(T^t(\alpha^*)\right)_L\left(\delta_{x_{0j}x_{0j}'}\right)\right| \\
&\leq \sup_{m' \in \mathcal{B}_{\mathcal{F}(X)}}\left|\left(T^t \circ \alpha^*\right)_L(m')\right| \\
&\leq \left\|\left(T^t \circ \alpha^*\right)_L\right\| \leq \mathrm{Lip}\left(T^t(\alpha^*)\right).
\end{aligned}$$

Taking the supremum over $m \in \mathcal{B}_{\mathcal{F}(Y)}$. Thus,

$$\mathrm{Lip}\left(U\left(T^t(a^*)\right)\right) = \mathrm{Lip}\left(S^t(a^*)\right) \leq \mathrm{Lip}\left(T^t(a^*)\right),$$

$U$ is bounded. Therefore, since $S^t = U \circ T^t$, then $T^t$ majorizes $S^t$.

This ends the proof. ■

**Proposition 3.1.4** *Assume that* $T \in \mathrm{Lip}_0(X,E)$ *and* $S \in \mathrm{Lip}_0(Y,E)$.

1. *If* $Im_{Lip}(S) \subseteq Im_{Lip}(T)$ *and* $T$ *is Lipschitz compact, then* $S$ *is Lipschitz compact.*

2. *If* $Im_{Lip}(S) \subseteq Im_{Lip}(T)$ *and* $T$ *is Lipschitz weakly compact, then* $S$ *is Lipschitz weakly compact.*

**Proof.**

1. Assume the hypotheses in $(1)$. Since $Im_{Lip}(S) \subseteq Im_{Lip}(T)$ by Theorem 3.1.1 part $(3)$, $T^t$ majorizes $S^t$. As $T$ is compact then $T^t$ is compact, By Proposition 2.1.2 part $(1)$, $S^t$ is compact. Therefore, $S$ is Lipschitz compact from Proposition 1.2.3.

2. The proof of (2) is the same as the proof of $(1)$, using the fact that $T$ is Lipschitz weakly compact if and only if $T^t$ is weakly compact (see Proposition 1.2.3).

The proof is complete. ■

## 3.2 Applications

Let $(1 \leq p < \infty)$, Farmer and W-B Johnson introduces the class of Lipschitz p-summing operator $T : X \longrightarrow E$ in [8]. Recall that a Lipschitz map $T$ is p-summing if there exists $C \geq 0$ such that for all $\{x_i\}_{1\leq i \leq\infty}$, $\{y_i\}_{1\leq i \leq\infty}$ in $X$ and all $\{a_i\}_{1\leq i \leq\infty} \subset \mathbb{R}_+$, we have

$$\sum_{i=1}^{n} a_i \left\| T(x_i) - T(y_i) \right\|^p \leq C^P \sup_{f \in \mathcal{B}_{X^\#}} \sum_{i=1}^{n} a_i \left| f(x_i) - f(y_i) \right|^p . \tag{3.2.1}$$

The space $\Pi_p^L(X,E)$ of Lipschitz p-summing from any pointed metric space into Banach space is a Banach space under the norm $\pi_p^L(\cdot)$ such that

$$\pi_p^L(T) = \inf \{C,\ C \text{ verifying } 3.2.1\}.$$

**Remark 3.2.1** *Every pointed metric space* $(X,d)$ *is isometric to a subspace of* $C(\mathcal{B}_{X^\#})$.
*Indeed,*
*Define*

$$\begin{array}{rccc} i_X : & X & \longrightarrow & C(\mathcal{B}_{X^\#}) \\ & x & \longmapsto & i_X(x) \end{array}$$

*such that,* $i_X(f) = f(x)$, *for all* $f \in \mathcal{B}_{X^\#}$.

*We have*

$$\begin{aligned} \|i_X(x_1) - i_X(x_2)\| &= \sup_{f \in \mathcal{B}_{X^\#}} |i_X(x_1)(f) - i_X(x_2)(f)| \\ &= \sup_{f \in \mathcal{B}_{X^\#}} |f(x_1) - f(x_2)| \\ &= \sup_{f \in \mathcal{B}_{X^\#}} \frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)} d(x_1, x_2) \\ &\leq d(x_1, x_2). \end{aligned}$$

*In other side, take* $f_0(\cdot) = d(\cdot, x_2) - d(0, x_2)$ *is in* $X^\#$, *has Lipschitz constant* 1 *and satisfies* $|f_0(x_1) - f_0(x_2)| = d(x_1, x_2)$. *This implies that* $\|i_X(x_1) - i_X(x_2)\| = d(x_1, x_2)$ *and hence* $\delta$ *is isometry.*

**Theorem 3.2.1 (Domination, Factorisation )** *Let* $1 \leq p < \infty$. *The following properties are equivalent for a Lipschitz map* $T : X \longrightarrow E$.

**a)** *The mapping* $T$ *is Lipschitz p-summing .*

**b)** *there is a positive constant* $C$ *and a probability* $\mu$ *on* $\mathcal{B}_{X^\#}$ *such that for all* $x, y \in X$ *we have*

$$\|T(x) - T(y)\| \leq C \left( \int_{\mathcal{B}_{X^\#}} |f(x) - f(y)|^p \, d\mu \right)^{\frac{1}{p}}.$$

*(Pietsch domination)*

**c)** *For the canonical injection* $J$ *of* $C(\mathcal{B}_{X^\#})$ *into* $L_p(\mathcal{B}_{X^\#}, \mu)$, *the following diagram commute*

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ \downarrow i_X & & \uparrow \widetilde{T} \\ i_X(X) = S_\infty & \xrightarrow{J_p} & S_p = \left(\overline{J_p \circ i_X(X)}\right) \\ \cap & & \cap \\ C(\mathcal{B}_{X^\#}) & \xrightarrow{J} & L_P(\mathcal{B}_{X^\#}, \mu) \end{array}$$

**Proof.** we find the proof $(a) \Longleftrightarrow (b)$ in [8].

$(b) \Longrightarrow (c)$. Assume that $T \in \text{Lip}_0(X, E)$ is Lipschitz p-summing $(1 \leq p < \infty)$ then there is a positive constant $C$ and probability $\mu$ on $\mathcal{B}_{X^\#}$ such that

$$\|T(x) - T(y)\| \leq C \left( \int_{\mathcal{B}_{X^\#}} |f(x) - f(y)|^p \, d\mu \right)^{\frac{1}{p}}.$$

For all $x, y \in X$.

Then put $f = J_p \circ i_X$. Then we have

$$\|T(x) - T(y)\| \leq C \, \|J_p \circ i_X(x) - J_p \circ i_X(y)\|_{L_P(\mathcal{B}_{X^\#}, \mu)}$$

This mean $J_p \circ i_X$ majorizes $T$. From proposition 3.1.1 there is $\widetilde{T} \in \text{Lip}_0\left(\overline{J_p \circ i_X(X)}, E\right)$ such that $T = \widetilde{T}(J_p \circ i_X)$.

$(c) \Longrightarrow (a)$ obvious. ■

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# ملخص

في هذا العمل درسنا محدودية المؤثرات الخطية المعرفة بواسطة بارنس. قمنا بتعميم هذا المفهوم إلى الحالة الليبشيزية. و أخيرا درسنا كحالة خاصة نظرية تحليل العوامل لبيتش و بعض التطبيقات لهذا المفهوم.

الكلمات المفتاحية : مؤثر ليبشيتز, مؤثر ليبشيتز متراص(ضعيف متراص), مؤثر ليبشيتز ب-التجميعية, تحليل العوامل, المحدودية.

# Abstract

In this work we studied the majorizing of linear operators introduced by Barnes. We generalized this concept to the Lipschitz case. Finally we studied as a particular case the Pietsch factorization theorem and some applications of this notion.

Key-Words: Lipschitz operators, Lipschitz compact (weakly compact) operators, Lipchitz p-summing operators, majorization, factorization.

# Résumé

Dans ce travail nous avons étudié la majoration des operateurs linéaires introduits par Barnes. Nous avons généralisé ce concept au cas Lipchitzien, finalement nous avons étudié comme cas particulier le théorème de factorisation de Pietsch et certaine applications de cette notion.

Mots-Clés : opérateurs Lipchitzien, opérateurs Lipchitzien compact (faiblement compact), opérateurs Lipchitzien p-sommable, majoration, factorisation.