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People's Democratic Republic of Algeria
وزارة التعليم العالي والبحث العلمي
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كـلية التكنـــولـــوجيـــا
نيابة العمادة لما بعد التدرج والبحث العلمي والعلاقات الخارجية

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شهادة ادارية

المصادقة على تقارير خبرة للموافقة على مطبوعة بيداغوجية

بعد الإطلاع على تقارير لجنة الخبراء للموافقة على المطبوعة البيداغوجية للأستاذ : **حرايز توفيق– أستاذ محاضر قسم ب** ، بالقاعدة المشتركة بكلية التكنولوجيا بجامعة محمد بوضياف بالمسيلة والتي كانت كلها ايجابية ، تمّ تقرير التّالي:

1–المصادقة على تقارير لجنة الخبراء للموافقة المطبوعة البيداغوجية والمعنونة بـــ:

Analyse 2
Engineering cycle

2– تمّ تشكيل لجنة الخبراء المكونة من السادة الآتية أسماؤهم:

–**يطو مراد،** أستاذ محاضر أ ، جامعة محمد بوضياف– المسيلة

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وتمت الموافقة بالاجماع على هذه المطبوعة.

رئيس المجلس العلمي للكلية

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Lecture Notes on Analysis 2

Domain: Engineering

Field: Electrical Eng., Mechanical Eng., Civil Eng.

Level: First Year (L1)

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Lecture Notes on Analysis 2 for First-Year Engineering Students

Electrical Eng., Mechanical Eng., and Civil Eng.

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Abstract

These lecture notes are specifically designed for first-year undergraduate students pursuing degrees in Engineering such that Electrical, Mechanical or Civil Engineering and Natual Sciences. The material covers foundational concepts essential for Engineering disciplines, with a focus on building both theoretical understanding and practical problem-solving skills.

These notes are primarily intended for:

- Freshmen in Computer Science and Information Technology
- First-year Engineering and Technology students
- Natural Sciences undergraduates

Contents

Introduction

Mathematics is the universal language of engineering, providing the analytical tools to model, optimize, and innovate across disciplines from designing electrical circuits to predicting mechanical stresses or analyzing fluid dynamics. These lecture notes are crafted to equip first year engineering students in electrical, mechanical, and civil engineering with the mathematical foundations essential for tackling real world challenges. Over four interconnected chapters, we will explore Taylor polynomials and finite expansions, integration of real functions, ordinary differential equations (ODEs), and multivariable functions. Each topic bridges theoretical rigor with practical application, empowering you to translate abstract concepts into solutions for systems you will encounter in your studies and career.

We begin with Taylor polynomials and finite expansions, powerful tools for approximating complex functions using simpler polynomial forms. Whether analyzing signal distortions in electrical circuits, predicting material behavior under stress, or optimizing structural loads, Taylor series enable engineers to simplify nonlinear phenomena into manageable models. You will learn to construct these approximations, assess their accuracy, and apply them to predict system behavior near equilibrium points a skill critical for control systems, numerical simulations, and error estimation in design processes.

Next, we delve into the integration of real functions, a cornerstone of engineering mathematics. Integration quantifies accumulated quantities—from calculating the work done by a variable force in mechanical systems to determining the total charge in an electrical circuit or the volume of materials in civil infrastructure. Through techniques like substitution, integration by parts, and partial fractions, you will master methods to solve definite and indefinite integrals, enabling you to compute areas, solve differential equations, and evaluate probabilities in statistical models. These skills are indispensable for analyzing continuous systems and interpreting real-world data.

The course then advances to ordinary differential equations (ODEs), the language of dynamic systems. ODEs model phenomena such as circuit transients, heat transfer in mechanical components, or population growth in environmental engineering. You will study first-order separable equations, linear ODEs, and second order linear differential equations, learning to predict system evolution over time. Finally, multivariable functions extend these principles to higher dimensions, where variables like temperature, pressure, and load interact in complex ways. By exploring partial derivatives, gradients, and optimization, you will gain the tools to design multi-parameter systems, such as optimizing energy efficiency in electrical grids or stress distribution in civil structures. Together, these chapters forge a mathematical toolkit that transforms theoretical insight into engineering innovation.

Chapter 1

Taylor Formula and Finite Expansion

In the case of a polynomial P of degree n, knowing the value of P and its derivatives up to order n at a point a allows us to determine P at any point $x = a + h$ in $\mathbb{R}$. However, this property no longer holds for arbitrary functions. For a general function f, knowing its value and derivatives up to order n at a point a provides only an n^{th} order approximation of $f(a + h)$ near $a$achieved by equating it to the value of a polynomial P_n, called the "regular part" of its Taylor expansion. In analysis, the Taylor-Lagrange formula, named after mathematician Brook Taylor (1685–1731), who established it in 1712, approximates a sufficiently differentiable function near a point using a polynomial whose coefficients depend solely on the function's derivatives at that point. This chapter begins with the theory of function comparison near a point and Landau notation, followed by a detailed presentation of Taylor's formula and its core concepts, including finite expansions and the Taylor-Young theorem. We will explore Finite expansions of common elementary functions and conclude with practical applications of these expansions, demonstrating their utility in simplifying limits, asymptotic analysis, and error estimation.

1.1 Taylor's Formula for a Polynomial:

Let $P(X)$ be a polynomial of degree n expressed in terms of increasing powers of X:

$$P(X) = a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n$$

We want to express this polynomial in terms of powers of $(X - x_0)$, where $x_0 \in \mathbb{R}$. For example, $X^2 + 2X + 5 = 4 + (X + 1)^2$. To do this, we set:

$$P(X) = b_0 + b_1(X - x_0) + b_2(X - x_0)^2 + \cdots + b_n(X - x_0)^n \quad (1)$$

where the coefficients $b_0, b_1, \ldots, b_n$ are unknowns. We have:

$$X = x_0 \Rightarrow P(x_0) = b_0$$

$$P'(X) = b_1 + 2b_2(X - x_0) + 3b_3(X - x_0)^2 + \cdots + nb_n(X - x_0)^{n-1} \Rightarrow P'(x_0) = b_1$$

$$P''(X) = 2b_2 + 6b_3(X - x_0) + \cdots + n(n-1)(X - x_0)^{n-2} \Rightarrow P''(x_0) = 2b_2 \Rightarrow b_2 = \frac{P''(x_0)}{2!}$$

$$\vdots$$

Similarly, we obtain:

$$b_n = \frac{P^{(n)}(x_0)}{n!}$$

Substituting into equation (1), we get: Taylor's formula for a polynomial is given by:

$$P(X) = P(x_0) + \frac{P'(x_0)}{1!}(X - x_0) + \frac{P''(x_0)}{2!}(X - x_0)^2 + \cdots + \frac{P^{(n)}(x_0)}{n!}(X - x_0)^n \quad \ldots (2)$$

1.2 Taylor's Formula for a Function

Let f be a function n times differentiable at the point x_0. We aim to construct a polynomial of degree n in increasing powers of x_0 satisfying the following conditions:

$$P(x_0) = f(x_0)$$

$$P'(x_0) = f'(x_0)$$
$$P''(x_0) = f''(x_0)$$
$$P^{(n)}(x_0) = f^{(n)}(x_0)$$

From the previous formula, it follows that:

$$P(X) = f(x_0) + \frac{f'(x_0)}{1!}(X - x_0) + \frac{f''(x_0)}{2!}(X - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(X - x_0)^n \quad \ldots (3)$$

Let:

$$f(x) - P(x) = R_n(x, x_0)$$

Then we have:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x, x_0) \quad \ldots (4)$$

Expression (4) is called Taylor's formula for the function f in the neighborhood of the point x_0 with the remainder $R_n(x, x_0)$.

1.3 Estimation of the Remainder

Theorem 1.1. *If the function $r = r(x)$ is differentiable up to order n at the point x_0 and moreover*

$$r(x_0) = r'(x_0) = r''(x_0) = \cdots = r^{(n)}(x_0) = 0$$

Then $r(x)$ is a function negligible compared to $(x - x_0)^n$ in the neighborhood of x_0, $c - a - d$:

$$r(x) = o((x - x_0)^n) \quad x \to x_0$$

Proof. We prove that

$$\lim_{x\to x_0}\frac{r(x)}{(x-x_0)^n}=0$$

By applying L'Hôpital's rule n times, we obtain:

$$\lim_{x\to x_0}\frac{r(x)}{(x-x_0)^n}=\lim_{x\to x_0}\frac{r'(x)}{n(x-x_0)^{n-1}}=\lim_{x\to x_0}\frac{r''(x)}{n(n-1)(x-x_0)^{n-2}}=\cdots=\lim_{x\to x_0}\frac{r^{(n)}(x)}{n!(x-x_0)^0}=0$$

□

1.3.1 Taylor's Formula with Peano's Remainder

Let f be a function differentiable up to order n at the point x_0. Then there exists a neighborhood of the point x_0 in which the following formula holds:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o((x-x_0)^n)$$

Proof. We have

$$f(x)-P_n(x)=R_n(x,x_0)$$

where

$$P_n(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Hence, it follows that $R_n(x,x_0)$ satisfies the conditions of the previous theorem and

$$R_n(x,x_0)=o((x-x_0)^n)\quad x\to x_0$$

□

1.3.2 Taylor's Formula with General Remainder

Theorem 1.2. *Let f be a function differentiable up to order $n+1$ in the neighborhood of the point x_0. Then, for all $\delta>0$ and for all $x\in V(x_0)$ with $[x_0,x]\subseteq V(x_0)$ (or $[x,x_0]\subseteq V(x_0)$), there exists $c\in[x_0,x]$ such that:*

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(c)}{n!}\left(\frac{x-x_0}{x-c}\right)^{\delta}\frac{(x-c)^{n+1}}{n!\delta}$$

1.3.3 Taylor's Formula with Lagrange Remainder

Remark 1.3. *In the general formula, if we take $\delta=n+1$, we find:*

$$R_n(x,x_0)=f^{(n+1)}(c)\frac{(x-x_0)^{n+1}}{(x-c)^{n+1}}\frac{(x-c)^{n+1}}{n!(n+1)}=f^{(n+1)}(c)\frac{(x-x_0)^{n+1}}{n!(n+1)}=f^{(n+1)}(c)\frac{(x-x_0)^{n+1}}{(n+1)}$$

Hence, Taylor's formula with Lagrange remainder is given by:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

1.3.4 Determining an Approximate Value of c

Let:

$$\frac{c-x_0}{x-x_0}=\theta, \quad 0<\theta<1$$

$$c-x_0=\theta(x-x_0) \Rightarrow c=\theta(x-x_0)+x_0$$

Thus, Taylor's formula with Lagrange remainder can be written as:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(x_0+\theta(x-x_0))}{(n+1)!}(x-x_0)^{n+1}$$

1.4 Taylor's Formula with Cauchy Remainder

In the general formula, if we take $\delta=1$, we find:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(x_0+\theta(x-x_0))}{(n+1)!}(x-x_0)^{n+1}(1-\theta)^n$$

Because we have:

$$c-x_0=\theta(x-x_0) \Leftrightarrow x-c+x_0=x-\theta(x-x_0) \Leftrightarrow x-c=(x-x_0)-\theta(x-x_0)$$

$$\Leftrightarrow (x-c)^n=(x-x_0)^n(1-\theta)^n$$

$$\Rightarrow (x-x_0)(x-c)^n=(x-x_0)^{n+1}(1-\theta)^n$$

1.5 Maclaurin Formulas

Maclaurin formulas are obtained from Taylor's formulas for the special case $x_0=0$. thus, by sunstitution, we can get the following formulas

1. Maclaurin with Peano Remainder

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\cdots+\frac{f^{(n)}(0)}{n!}x^n+o(x^n)$$

2. Maclaurin with Lagrange Remainder

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

3. Maclaurin with Cauchy Remainder

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(\alpha x)}{n!}x^{n+1}(1-\theta)^n$$

1.6 Application of Maclaurin's Formula for Common Functions

1.6.1 Maclaurin Series for e^x

For $f(x) = e^x$, we have $f^{(n)}(x) = e^x$ and thus $f^{(n)}(0) = 1$. Also, $f^{(n+1)}(\theta x) = e^{\theta x}$.

Therefore, the Maclaurin series for e^x is:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^{\theta x}}{(n+1)!}x^{n+1}$$

1.6.2 Maclaurin Series for $\sin x$

For $f(x) = \sin x$, we have:

$$f^{(n)}(x) = \sin\left(x + n\frac{\pi}{2}\right) \Rightarrow f^{(n)}(0) = \sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & \text{if } n = 2p \\ (-1)^p & \text{if } n = 2p+1 \end{cases}$$

Also:

$$f^{(2p+2)}(\theta x) = \sin\left(\theta x + (2p+2)\frac{\pi}{2}\right) = \sin(\theta x + (p+1)\pi) = (-1)^{p+1}\sin(\theta x)$$

Therefore, the Maclaurin series for $\sin x$ is:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^p x^{2p+1}}{(2p+1)!} + \frac{(-1)^{p+1}\sin(\theta x)}{(2p+2)!}x^{2p+2}$$

1.6.3 Maclaurin Series for $\cos x$

For $f(x) = \cos x$, we have:

$$f^{(n)}(x) = \cos\left(x + n\frac{\pi}{2}\right) \Rightarrow f^{(n)}(0) = \cos\left(n\frac{\pi}{2}\right)$$

Similarly, we obtain the Maclaurin series for $\cos x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^p x^{2p}}{(2p)!} + \frac{(-1)^{p+1}\sin(\theta x)}{(2p+1)!}x^{2p+1}$$

1.6.4 Maclaurin Series for $\log(1+x)$

For $f(x) = \log(1+x)$ with $x > -1$, we have:

$$f'(x) = \frac{1}{1+x}; \quad f''(x) = \frac{-1}{(1+x)^2}; \quad f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$$

Thus:

$$f^{(n)}(0) = (-1)^{n+1}(n-1)!$$

Therefore, the Maclaurin series for $\log(1+x)$ is:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \frac{(-1)^n x^{n+1}}{(n+1)(1+\theta x)^{n+1}}$$

Chapter 2

Integrals

Integration, the inverse operation of differentiation, lies at the heart of calculus, unifying the study of accumulation, area, and volume. This chapter begins with simple integrals, exploring the concept of primitives (indefinite integrals) as functions that reverse differentiation, enabling the reconstruction of original functions from their derivatives. We then transition to defined integrals, which quantify accumulated quantities such as the area under a curve or the volume of a solid of revolution using limits of Riemann sums. Through the Fundamental Theorem of Calculus, we bridge these two ideas, revealing how primitives facilitate the exact computation of defined integrals. Applications span geometry (calculating planar regions and 3D volumes) and physics (work, displacement), showcasing integration's power to transform abstract antiderivatives into tangible solutions. Prepare to master techniques like substitution, integration by parts, and numerical approximations, while deepening your intuition for continuous summation.

2.1 Indefinite Integral

2.1.1 General Concepts

Definition 2.1. *Let f be a continuous function on an interval I. A primitive of f is any differentiable function F satisfying $F' = f$ on I.*

Example 2.2. *The function f defined on R by $f(x) = 2x$ has a primitive F defined on R by $F(x) = x^2$.*

Proposition 2.3. *Let f be a continuous function with a primitive F. Then any function $F + c$, where c is a constant, is also a primitive of f.*

Definition 2.4. *Let f be a continuous function on $I \subseteq R$. The* ***Indefinite Integral*** *of f is the set of all primitives of f.*

- The indefinite integral of f is denoted by $f(x)dx$, where is the integral sign, $f(x)$ is the integrand, and dx is the differential notation.

Note that x is the variable of integration. Thus, we write:

$$f(x)dx = F(x) + c, \text{ where } F \text{ is a particular primitive and } c \in R.$$

Example 2.5. $2xdx = x^2 + c$, *where* $c \in R$.

Remark 2.6. *The variable x is a dummy variable. Replacing x with t, for example, does not change the result.*

$$\cos x dx = \sin x + c, \text{ and } \cos t dt = \sin t + c,\ c \in R.$$

Properties Let f and g be continuous functions:

1. $f'(x)dx = f(x) + c$, $c \in R$
2. $(f(x)dx)' = f(x)$
3. $(f(x) + g(x))\, dx = f(x)dx + g(x)dx$
4. $\lambda f(x)dx = \lambda f(x)dx$ for any scalar λ

Properties 3 and 4 establish the linearity of the integral operator.

2.1.2 Techniques for Computing Primitives

Integration by Recognition

Recognize the integrand as the derivative of a known function (or composite function). In other words:

$$\text{If } f(x) = F'(x), \text{ then } \int f(x)\, dx = F(x) + c.$$

Example 2.7. ***Example 2***

1) $\int \cos(x)dx = ?$
We know $(\sin(x))' = \cos(x)$.
Thus, $\int \cos(x)dx = \int (\sin(x))'\, dx = \sin(x) + c$, $\forall c \in \mathbb{R}$.
2) $\int \cos(x^2)2xdx = ?$
We know $(\sin(x^2))' = (\sin(u))' = u' \cos(u)$.
Hence, $(\sin(x^2))' = 2x\cos(x^2)$, so $\int \cos(x^2)2xdx = \sin(x^2) + c$.

Integration by Parts

Proposition 2.8. *Let $f, g : I \to \mathbb{R}$ be two C^1 functions on I. Then:*

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Proof. Consider the derivative of the product $f(x)g(x)$:
$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.
Integrate both sides:
$\int (f(x)g(x))'\, dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx$.
Therefore:
$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$. □

The idea is to choose f' and g such that f' and g' are easier to integrate.

Example 2.9. $I = \int x \sin(x) dx$

Apply integration by parts:
Let: $f(x) = x \quad f'(x) = 1$
$g'(x) = \sin(x) \quad g(x) = -\cos(x)$
Then:
$I = -x\cos(x) + \int \cos(x) dx$
$I = -x\cos(x) + \sin(x) + c$, where c is the constant of integration.

Remark 2.10. *Integration by parts is frequently used for integrals of the form:*

$$\int x^k \sin(x) dx,\ \int x^k \cos(x) dx,\ \int x^k e^{\alpha x} dx,\ \int x^k \ln(x) dx.$$

Integration by Substitution

If computing $\int f(x)dx$ is difficult, replace x with $\varphi(t)$, a differentiable function. Then $dx = \varphi'(t)dt$, leading to:

$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt$$

Example 2.11. $I = \int \sin(x)\cos(x)dx$

Let $t = \sin(x)$, so $dt = \cos(x)dx$.
Then $I = \int t dt = \frac{1}{2}t^2 + c = \frac{1}{2}\cos^2(x) + c$.

Remark 2.12. $\int \dfrac{g'(x)}{g(x)} dx = \int \dfrac{dt}{t} = \ln|t| + c = \ln|g(x)| + c$

Here, set $t = g(x) \Rightarrow dt = g'(x)dx$. Success depends on choosing the right substitution to simplify calculations.

2.1.3 Integration of Rational Functions

Definition 2.13. *A rational function is of the form* $f(x) = \dfrac{p(x)}{q(x)}$, *where* $p(x)$ *and* $q(x)$ *are polynomials with* $q(x) \neq 0$.

2.1.4 Integration of Rational Functions

Let $f(x) = \dfrac{\alpha x + \beta}{ax^2 + bx + c}$, where $\alpha, \beta, a, b, c \in \mathbb{R}$ and $a \neq 0$. Three cases arise:

First case: If $\Delta = b^2 - 4ac > 0$, the denominator $ax^2 + bx + c$ has two distinct real roots x_1, x_2.

Then $f(x)$ decomposes as:

$$f(x) = \frac{A}{x - x_1} + \frac{B}{x - x_2}$$

Thus:

$$\int f(x)dx = A\ln|x - x_1| + B\ln|x - x_2| + c$$

Example 2.14. $\int \frac{2x+3}{x^2-x-2}dx = \int \frac{2x+3}{(x+1)(x-2)}dx = \int \frac{A}{x+1}dx + \int \frac{B}{x-2}dx$

Determine A and B:
For A, multiply by $x+1$ and set $x=-1$:
$A = \frac{2(-1)+3}{-1-2} = -\frac{1}{3}$.
For B, multiply by $x-2$ and set $x=2$:
$B = \frac{2(2)+3}{2+1} = \frac{7}{3}$.
Thus:
$\int \frac{2x+3}{x^2-x-2}dx = -\frac{1}{3}\ln|x+1| + \frac{7}{3}\ln|x-2| + c$.
Second case: If $\Delta = 0$, the denominator has a repeated root x_0.
Then $f(x)$ decomposes as:

$$f(x) = \frac{A}{(x-x_0)^2} + \frac{B}{x-x_0}$$

Thus:

$$\int f(x)dx = -\frac{A}{x-x_0} + B\ln|x-x_0| + c$$

Example 2.15. *Compute* $\int \frac{5x+6}{x^2+2x+1}dx$

Decompose:
$\int \frac{5x+6}{(x+1)^2}dx = \int \frac{1}{(x+1)^2}dx + \int \frac{5}{x+1}dx = -\frac{1}{x+1} + 5\ln|x+1| + c$.
Third case: If $\Delta < 0$, the denominator has no real roots. Complete the square:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)$$

Example 2.16. $I = \int \frac{x+4}{x^2+2x+5}dx$

Rewrite:
$I = \frac{1}{2}\int \frac{2x+8}{x^2+2x+5}dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}dx + 3\int \frac{dx}{x^2+2x+5}$.
Compute $I_2 = \int \frac{dx}{(x+1)^2+4} = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right)$.
Thus:

$$I = \frac{1}{2}\ln|x^2+2x+5| + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + c$$

Partial Fraction Decomposition

Proposition 2.17. *A rational function* $\frac{p(x)}{q(x)}$ *decomposes into a polynomial part (by performing polynomial division if* $\deg p \geq \deg q$*) and partial fractions such that:*

- Each irreducible linear factor $(ax+b)^k$ *in the denominator generates k partial fractions of the form:*

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}, \quad \text{with } A_i \in \mathbb{R}.$$

- Each irreducible quadratic factor $(ax^2+bx+c)^k$ *in the denominator generates k partial fractions of the form:*

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}, \quad \text{with } A_i, B_i \in \mathbb{R}.$$

Example 2.18. $\frac{x}{x(x-1)^2} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2}$

$\frac{2x}{x^2+2x+1} = \frac{2x}{(x+1)^2} = \frac{a}{x+1} + \frac{b}{(x+1)^2}$

$\frac{x-3}{x(x^2+1)^2} = \frac{a}{x} + \frac{bx+c}{x^2+1} + \frac{dx+e}{(x^2+1)^2}$

$$\frac{x^5}{(x^2-1)(x^2-4x+5)} = \frac{x^5}{(x-1)(x+1)(x^2-4x+5)} = x+4+\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2-4x+5}$$

A) Integration of Simple Elements $\frac{A}{(x-x_0)^n}$

- If $n = 1$:

$$\int \frac{A}{x-x_0}dx = A\ln|x-x_0| + c$$

- If $n \geq 2$:

$$\int \frac{A}{(x-x_0)^n}dx = \frac{A}{(n-1)(x-x_0)^{n-1}} + c$$

B) Integration of Simple Elements $\frac{Bx+c}{(ax^2+bx+c)^n}$

Rewrite the integrand as:

$$\frac{Bx+c}{(ax^2+bx+c)^n} = \alpha\frac{2ax+b}{(ax^2+bx+c)^n} + \delta\frac{1}{(ax^2+bx+c)^n}$$

Integrate using substitution $u = ax^2+bx+c$:

$$\int \frac{2ax+b}{(ax^2+bx+c)^n}dx = \int \frac{u'}{u^n}dx = \frac{1}{(n-1)u^{n-1}} + c$$

For $\int \frac{dx}{(ax^2+bx+c)^n}$:

- If $n = 1$, complete the square:

$$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{a} \int \frac{dx}{\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)} = \frac{1}{a} \arctan\left(\frac{x + \frac{b}{2a}}{\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}}\right) + c$$

- If $n \geq 2$, use reduction formulas via integration by parts to relate I_n to I_{n-1}.

Integrating $p(x)e^{\alpha x}$

To compute $\int p(x)e^{\alpha x}dx$, where p is a polynomial and α is a scalar:
- Use successive integration by parts if $\deg p(x)$ is small. - Alternatively, assume a primitive of the form $\Phi(x)e^{\alpha x}$ with $\deg p(x) = \deg \Phi(x)$ and solve for coefficients.

2.1.5 Integration of Trigonometric Functions

Integrals of the form $\int \sin(\alpha x)\cos(\beta x)dx$ Use trigonometric identities:

$$\sin(\alpha x)\cos(\beta x) = \frac{1}{2}\left[\sin((\alpha + \beta)x) + \sin((\alpha - \beta)x)\right]$$

Integrals of the form: $\int \cos^p(x)\sin^q(x)dx$, $p, q \in \mathbb{N}$

Case 1: If p is odd, substitute $t = \sin(x)$.
Case 2: If q is odd, substitute $t = \cos(x)$.
Case 3: If both are even, use power-reduction formulas.

Bioche's Rules

For $\int f(\sin x, \cos x)dx$, use substitution based on symmetry:
- $u = \cos x$ if f is even in $\cos x$. - $u = \sin x$ if f is even in $\sin x$. - $u = \tan x$ if periodic in π.

Example 2.19. $\int \frac{\cos x}{2 - \cos^2 x}dx = \arctan(\sin x) + c$

2.2 Definite Integral

Let $f : I \to \mathbb{R}$ and $a, b \in I$. Then:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Example 2.20. $\int_0^{\frac{3}{2}} x^2 dx = \left[\frac{x^3}{3}\right]_0^{\frac{3}{2}} = \frac{9}{8}$

2.2.1 Riemann Integral

In the presentation of the Riemann integral, step functions play a pivotal role. We begin by outlining their properties and defining their integrals.

Step Functions

Definition 2.21. *A* ***subdivision*** *of a compact interval (i.e., closed and bounded)* $[a, b]$ *in* $\mathbb{R}$ *is a finite set of points* $x_0, x_1, x_2, \ldots, x_n$ *such that:*

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Note that $x_i - x_{i-1} > 0$ for all $i = 1, \ldots, n$. The **mesh** of the subdivision is the real number $\max_{1 \leq i \leq n}(x_i - x_{i-1})$.

Definition 2.22. *A function* $f : [a, b] \to \mathbb{R}$ *is called a* ***step function*** *if there exists a subdivision* $\{x_0, x_1, \ldots, x_n\}$ *of* $[a, b]$ *such that* f *is constant on each subinterval* (x_{i-1}, x_i) *for* $i = 1, 2, \ldots, n$.

A function is called a step function on $\mathbb{R}$ *if there exists an interval* $[a, b]$ *such that* f *is zero outside* $[a, b]$ *and a step function on* $[a, b]$.

Example 2.23. *1) The function* f *defined on* $[0, 1]$ *by:*

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x \in \left(0, \frac{1}{2}\right) \\ 3 & \text{if } x = \frac{1}{2} \\ 1 & \text{if } x \in \left(\frac{1}{2}, 1\right) \end{cases}$$

is a step function on $[0, 1]$.

2) A constant function on $[a, b]$ *is a step function on* $[a, b]$.

Integral of Step Functions

Definition 2.24. *Let* f *be a step function on* $[a, b]$*, where* $f(x) = c_i$ *for* $x \in (x_{i-1}, x_i)$ *and* $i = 1, 2, \ldots, n$. *The* ***integral of*** f *over* $[a, b]$ *is defined as:*

$$I(f) = \sum_{i=1}^{n} c_i(x_i - x_{i-1})$$

This sum S_n is called the Riemann sum of f over $[a, b]$, and we denote it by $I(f) = \int_a^b f(x)dx$.

Remark 2.25. *1) The value* $I(f)$ *depends only on* f *and not on the choice of subdivision.*

2) $I(f)$ *is independent of* x.

3) If f *is a step function on* $[a, b]$ *with* $f \geq 0$*, then* $\int_a^b f(x)dx = 0 \nRightarrow f = 0$.

4) The integral $\int_a^b f(x)dx$ *does not depend on the values of* f *at the subdivision points.*

Example 2.26. *1) Let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = c$. Then:*

$$\int_a^b f(x)dx = (b - a)c$$

2) For the function f defined on $[0, 1]$ by:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x \in \left(0, \frac{1}{2}\right) \\ 3 & \text{if } x = \frac{1}{2} \\ 1 & \text{if } x \in \left(\frac{1}{2}, 1\right) \end{cases}$$

the integral is:

$$\int_0^1 f(x)dx = \frac{1}{2}\left(\frac{1}{2} - 0\right) + 1\left(1 - \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

2.2.2 Properties

Proposition 2.27 (Linearity of the Integral)**.** *Let f and g be step functions on $[a, b]$, and let λ be a real constant. Then:*

1. $\int_a^b (\lambda f)(x)dx = \lambda \int_a^b f(x)dx$
2. $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

Proposition 2.28 (Monotonicity of the Integral)**.** *Let f and g be step functions on $[a, b]$. Then:*

1. *If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \geq 0$.*
2. *If $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.*
3. *For any $u \in [a, b]$, $\int_u^u f(x)dx = 0 \nRightarrow f = 0$.*
4. *For $u, v \in [a, b]$ with $u < v$, $\int_u^v f(x)dx = -\int_v^u f(x)dx$.*
5. *If $c \in (a, b)$, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.*

Chapter 3

Ordinary differential Equations

Ordinary differential equations (ODEs) are the cornerstone of modeling dynamic systems, describing relationships between functions and their derivatives to capture phenomena such as growth, motion, decay, and equilibrium. These equations classified by order, linearity, and solution techniques serve as governing principles in physics, engineering, biology, and economics. This chapter introduces the foundational theory of ODEs, focusing on first-order equations, linear second-order equations, and methods like separation of variables, integrating factors, and characteristic equations. We explore initial and boundary value problems, emphasizing analytical solutions. By linking theoretical frameworks to real world applications from population dynamics to mechanical oscillations we unravel how ODEs translate abstract mathematics into predictive tools for understanding continuous change.

3.1 Definitions and Properties

Definition 3.1. *An ordinary differential equation, denoted (ODE), of order n is a relation between the real variable x, an unknown function $x \mapsto y(x)$, and its derivatives $y', y'', \ldots, y^{(n)}$ at x, defined by:*

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

where: $y' = \dfrac{dy}{dx}$ and $y^{(n)} = \dfrac{d^n y}{dx^n}$.

If $n = 1$, the function $F(x, y, y') = 0$ is called a **first-order differential equation**.

If $n = 2$, the function $F(x, y, y', y'') = 0$ is called a **second-order differential equation**.

$$\begin{aligned} y'(x) - x = 0 : &\quad \text{first-order differential equation.} \\ y''(x) - y'(x) = 2x\sin(x) : &\quad \text{second-order differential equation.} \\ y^{(4)}(x) + 2y''(x) - y(x) = x : &\quad \text{fourth-order differential equation} \end{aligned}$$

Notation: We write y instead of $y(x)$, and y' instead of $y'(x)$.

For example: $y' = \cos x$ means $y'(x) = \cos x$.

3.2 First-Order Differential Equations

Definition: An equation of the form $F(x, y, y') = 0$, where y is a function of x, is called a first-order differential equation.

3.2.1 Separable Differential Equations (SDE)

They are of the form:

$$y' f(y) = g(x)$$

where f and g are given functions with known antiderivatives F and G. We have:

$$\int y' f(y)\, dy = \int g(x)\, dx \quad (y' = \tfrac{dy}{dx}) \quad \Longrightarrow \; F(y) = G(x) + C \quad \text{where } C \in \mathbb{R}.$$

Example 3.2. *1.*

Example 3.3. *Solve the equation:* $y' = x^2 + 1$.

We have $y' = x^2 + 1 \implies \int y'\, dx = \int (x^2 + 1)\, dx$.
$\implies \int dy = \int (x^2 + 1)\, dx$
$\implies y = \frac{x^3}{3} + x + C, \quad \forall C \in \mathbb{R}$.

2 Integrate the following equation: $y' = xy$.

We have $y' = xy \implies \dfrac{y'}{y} = x$
$\implies \int \frac{1}{y}\, dy = \int x\, dx$
$\implies \ln|y| = \frac{x^2}{2} + C$ with $C \in \mathbb{R}$.
Thus, any non-zero solution is of the form:

$$y(x) = K e^{\frac{x^2}{2}}, \quad \text{with } K \in \mathbb{R}^*.$$

3.2.2 Homogeneous Differential Equations (HDE)

They are of the form:

$$y' = F\left(\frac{y}{x}\right)$$

To solve this equation, we set $t = \frac{y}{x}$ (i.e., $y = xt$ and $y' = t'x + t$), where t is a function of x. This transforms the equation into a separable differential equation.

Example 3.4. $xy' = x + y$

$xy' = x + y$ is a homogeneous equation, as it can be written as $y' = 1 + \frac{y}{x}$.
Setting $\frac{y}{x} = t$ (i.e., $y = xt$), we obtain the equation $t'x + t = 1 + t$.
Hence, $t' = \frac{1}{x}$ (SDE).
The general solution is $t = \ln|x| + C, \quad C \in \mathbb{R}$.
Thus, the general solution of the homogeneous equation is:

$$y = x \ln|x| + K, \quad K \in \mathbb{R}.$$

3.2.3 Linear Differential Equations (LDE)

They are of the form:

$$y' + f(x)y = g(x). \tag{(E)}$$

where f and g are given functions.

The equation E is called homogeneous (HE) or without a right-hand side if $g = 0$, i.e.,

$$y' + f(x)y = 0. \tag{(E)}$$

The general solution of the complete equation E is of the form:

$$y_g = y_0 + y_p$$

where y_p is a particular solution of E and y_0 is the general solution of E_0.

Method of Variation of Constants

The method of variation of constants is used to determine solutions of a nonhomogeneous differential equation by using the solution of the homogeneous equation.

If y_0 is a solution of the homogeneous equation, we seek a particular solution in the form $y(x) = C(x)y_0(x)$.

Example: Solve:

$$xy' - y = x^2e^x \quad \text{on } (0, +\infty).$$

1. Solve the homogeneous equation $xy' - y = 0$:

$$\begin{aligned} x\tfrac{dy}{dx} - y = 0 \quad &\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \\ &\Rightarrow \ln y = \ln x + \ln C \\ &\Rightarrow y = Cx. \end{aligned}$$

2. Seek a particular solution of the form $y = C(x)x$. Then $y' = C'(x)x + C(x)$. Substituting into the complete equation:

$$C'(x)x^2 + C(x)x - C(x)x = x^2e^x \Leftrightarrow C'(x) = e^x$$

Thus, $C(x) = e^x + \lambda$, and the general solution is:

$$y = x(e^x + \lambda)$$

3.2.4 Bernoulli Differential Equation

A Bernoulli differential equation is any differential equation of the form:

$$y' + yf(x) = y^n g(x)$$

If $n = 0$, it becomes a complete linear equation. If $n = 1$, it becomes a homogeneous linear equation.

For $n \neq 0$ and $n \neq 1$, and $y \neq 0$, we set $z = y^{1-n}$. This transforms the equation into a linear form:

$$\frac{1}{1-n} z' + z f(x) = g(x).$$

Example 3.5. *Integrate the equation:* $y' + 2y - (x+1)\sqrt{y} = 0$.

Set $z = y^{1-\frac{1}{2}} = \sqrt{y}$. The equation becomes:

$$2z' + 2z = x + 1 \quad \text{(Linear differential equation).}$$

3.3 Second-Order Differential Equations

3.4 General Concepts

A second-order differential equation is any relation of the form:

$$F(x, y, y', y'') = 0$$

between the variable x, the function $y(x)$, and its first two derivatives.

Example 3.6. $y'' + 2y' + y = 0$; $y'' + 4y' + 3y = e^{-2x}$.

3.5 Incomplete Second-Order Differential Equations (Not Containing y)

These equations are of the form:

$$F(x, y', y'') = 0.$$

To solve, substitute $y' = t$, reducing the equation to a first-order differential equation:

$$F(x, t, t') = 0.$$

Example 3.7. $xy'' + 2y' = 0$ *(Equation E).*

Substitute $y' = t \implies xt' + 2t = 0$ (separable first-order equation):

$$\begin{aligned} &\implies x\tfrac{dt}{dx} = -2t \\ &\implies \textstyle\int \frac{dt}{t} = -2\int \frac{dx}{x} \\ &\implies \ln|t| = -2\ln|x| + \ln|C| \\ &\implies t = \tfrac{C}{x^2} = y'. \end{aligned}$$

Thus, the general solution of E is:

$$y = -\frac{C}{x} + k, \quad k \in \mathbb{R}.$$

3.5.1 Linear Second-Order Differential Equations with Constant Coefficients

These equations are of the form:

$$ay'' + by' + cy = f(x) \tag{(EC)}$$

where a, b, c are constants, and $f(x)$ is the nonhomogeneous term. The associated homogeneous equation is:

$$ay'' + by' + cy = 0 \quad (E_0).$$

The general solution of EC is:

$$y = y_H + y_P,$$

where y_H is the general solution of E_0, and y_P is a particular solution of EC.

Solution Method:

a) Finding y_H: Solve E_0 using the characteristic equation $ar^2 + br + c = 0$. The solution depends on the discriminant $\Delta = b^2 - 4ac$:

1. Case 1: $\Delta > 0$ Two distinct real roots r_1 and r_2:

$$y_H = C_1 e^{r_1 x} + C_2 e^{r_2 x}, \quad C_1, C_2 \in \mathbb{R}.$$

2. Case 2: $\Delta = 0$ One double root $r = -\frac{b}{2a}$:

$$y_H = (C_1 x + C_2)e^{rx}, \quad C_1, C_2 \in \mathbb{R}.$$

3.Case 3: $\Delta < 0$ Complex roots $\alpha \pm \beta i$:

$$y_H = e^{\alpha x}\left(C_1 \cos \beta x + C_2 \sin \beta x\right), \quad C_1, C_2 \in \mathbb{R}.$$

Example 3.8. *Solve the following differential equations:*

$$1)\ y'' + y' - 2y = 0, \quad 2)\ y'' + 2y' + y = 0, \quad 3)\ y'' + y = 0.$$

1. Equation 1: Characteristic equation: $r^2 + r - 2 = 0$ has roots $r_1 = 1$, $r_2 = -2$. General solution:

$$y = C_1 e^x + C_2 e^{-2x}, \quad C_1, C_2 \in \mathbb{R}.$$

2. Equation 2: Characteristic equation: $r^2 + 2r + 1 = 0$ has a double root $r = -1$. General solution:

$$y = e^{-x}(C_1 + C_2 x), \quad C_1, C_2 \in \mathbb{R}.$$

3. Equation 3: Characteristic equation: $r^2 + 1 = 0$ has roots $r = \pm i$. General solution:

$$y = C_1 \cos x + C_2 \sin x, \quad C_1, C_2 \in \mathbb{R}.$$

b) Finding y_P: Particular Solution of EC

$$ay'' + by' + cy = f(x).$$

b.1) Case where $f(x) = e^{\alpha x} P_n(x)$: Here, $P_n(x)$ is a polynomial of degree n, and $\alpha \in \mathbb{R}$. The particular solution depends on α:

- i. If α is not a root of E_0: $y_P = e^{\alpha x} Q_n(x)$, where $Q_n(x)$ is a polynomial of degree n.
- ii. If α is a simple root of E_0: $y_P = x e^{\alpha x} Q_n(x)$.
- iii. If α is a double root of E_0: $y_P = x^2 e^{\alpha x} Q_n(x)$.

Example 3.9. *Solve:* $y'' + 2y' + 4y = xe^x$.

1.Homogeneous solution y_H: Characteristic equation $r^2+2r+4 = 0$ has complex roots $r = -1 \pm \sqrt{3}i$. General solution:

$$y_H = e^{-x}\left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right).$$

2.Particular solution y_P: Here, $f(x) = xe^x$, $\alpha = 1$ (not a root of E_0). Assume $y_P = e^x(ax + b)$. Substituting into EC, solve for coefficients $a = \frac{1}{7}$, $b = -\frac{4}{49}$. Thus:

$$y_P = e^x\left(\frac{1}{7}x - \frac{4}{49}\right).$$

Final general solution:

$$y_G = e^{-x}\left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right) + e^x\left(\frac{1}{7}x - \frac{4}{49}\right).$$

b.2) Case where $f(x) = P_n(x)$: If $f(x)$ is a polynomial $P_n(x)$, the particular solution form depends on c:
- i. If $c \neq 0$: $y_P = Q_n(x)$. -ii. If $c = 0$, $b \neq 0$: $y_P = xQ_n(x)$. -iii.If $c = 0$, $b = 0$: $y_P = x^2Q_n(x)$.
Where $Q_n(x)$ is a polynomial of degree n.

3.5.2 General Method

Method of Variation of Parameters

We aim to solve a differential equation of the form:

$$ay'' + by' + cy = f(x) \quad \text{with } a \neq 0.$$

When the nonhomogeneous term $f(x)$ does not match standard forms, we use the method of variation of parameters. Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation:

$$ay'' + by' + cy = 0.$$

Assume a particular solution of the complete equation in the form:

$$y_P = C_1(x)y_1 + C_2(x)y_2,$$

where $C_1(x)$ and $C_2(x)$ are differentiable functions. Impose the condition:

$$C_1'(x)y_1 + C_2'(x)y_2 = 0.$$

Differentiating y_P, we find:

$$y_P' = C_1y_1' + C_2y_2',$$

and differentiating again:

$$y_P'' = C_1'y_1' + C_2'y_2' + C_1y_1'' + C_2y_2''.$$

Substituting into the original equation and simplifying (using y_1, y_2 as homogeneous solutions), we derive the system:

$$\begin{cases} C_1'y_1' + C_2'y_2' = \dfrac{f(x)}{a}, \\ C_1'y_1 + C_2'y_2 = 0. \end{cases}$$

Example 3.10. *Solve:* $y'' + y = \cos(x)$.

1. Homogeneous solution: The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$. General solution:

$$y_H = C_1\cos(x) + C_2\sin(x).$$

2. Particular solution via variation of parameters: Let $y_1 = \cos(x)$, $y_2 = \sin(x)$. Solve the system:

$$\begin{cases} -C_1'\sin(x) + C_2'\cos(x) = \cos(x), \\ C_1'\cos(x) + C_2'\sin(x) = 0. \end{cases}$$

Solving for C_1' and C_2':

$$C_1'(x) = -\frac{\sin(2x)}{2}, \quad C_2'(x) = \frac{\cos(2x)+1}{2}.$$

Integrate to find:

$$C_1(x) = \frac{\cos(2x)}{4}, \quad C_2(x) = \frac{\sin(2x)}{4} + \frac{x}{2}.$$

Thus, the particular solution is:

$$y_P = \frac{\cos(x)}{4} + \frac{x\sin(x)}{2}.$$

3. General solution:

$$y(x) = C_1\cos(x) + C_2\sin(x) + \frac{\cos(x)}{4} + \frac{x\sin(x)}{2}.$$

—

Example 3.11. *Solve:* $y'' - 3y' + 2y = 2x^2 - 5x + 2$.

1. Homogeneous solution: Characteristic equation: $r^2 - 3r + 2 = 0$ has roots $r_1 = 1$, $r_2 = 2$. General solution:

$$y_H = C_1e^x + C_2e^{2x}.$$

2. Particular solution: Assume $y_P = C_0 + C_1x + C_2x^2$. Substituting into the equation and solving for coefficients:

$$y_P = x^2 + \frac{1}{2}x + \frac{3}{4}.$$

3.General solution:

$$y_G = C_1e^x + C_2e^{2x} + x^2 + \frac{1}{2}x + \frac{3}{4}.$$

Chapter 4

Functions of Several Variables

Functions of several real variables, or multivaraible functions form the cornerstone of multivariable calculus, extending the principles of single-variable analysis to higher dimensions. These functions, which map inputs from $\mathbb{R}^{\ltimes}$ to $\mathbb{R}$, model complex phenomena where multiple interdependent factors coexist—from physical fields like temperature distributions to economic optimization problems. This chapter explores foundational concepts such as partial derivatives, gradients, and multiple integrals, which enable the study of continuity, differentiability.

4.1 Definitions and Properties

4.2 Review of Topological Concepts in $\mathbb{R}^n$

For a non-zero integer n, we define $\mathbb{R}^n$ as the Cartesian product of $\mathbb{R}$ with itself n times. Thus, $\mathbb{R}^n$ is the set of vectors with n real components. Whether in theoretical or numerical analysis, as well as in computer science, we primarily manipulate vectors and apply treatments analogous to those used with numbers. We need to measure the proximity of two vectors to discuss limits, continuity, and differentiability of functions whose domain is not an interval of $\mathbb{R}$ but a set of vectors.

Distance

In $\mathbb{R}$; the distance between two real numbers x and y is given by $d(x, y) = |x - y|$.

In $\mathbb{R}^2$; the distance between two points $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ is given by $d(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

In $\mathbb{R}^n$; the distance between two points $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ is given by $d(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$.

Norm

Definition 4.1. *Let $(E, +, .)$ be a $\mathbb{R}$-vector space. A norm on E is any application N from E to $\mathbb{R}^+$ such that:*

- $\forall x \in E, N(x) = 0 \Longleftrightarrow x = 0.$
- $\forall \alpha \in \mathbb{R}, \forall x \in E, N(\alpha x) = |\alpha| N(x).$
- $\forall x \in E, \forall y \in E, N(x + y) \leq N(x) + N(y)$ (Triangle inequality).

A normed space is a vector space equipped with a norm.

Remark 4.2. *The norm of an element X in $\mathbb{R}^n$ is its distance to the origin (Note that $\mathbb{R}^n$ is a normed $\mathbb{R}$-vector space).*

Usual Norms on $\mathbb{R}^n$ For any element $X = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$:

- **Euclidean norm** denoted $N_2(X) = \|X\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$.
- **Maximum norm** denoted $N_\infty(X) = \|X\|_\infty = \max_{1\le i\le n} |x_i|$.
- **Sum norm** denoted $N_1(X) = \|X\|_1 = \sum_{i=1}^n |x_i|$.

These three norms are equivalent. (Two norms N_1 and N_2 on a space E are equivalent if and only if there exist positive real numbers a and b such that $\forall x \in E, aN_1(x) \le N_2(x) \le bN_1(x)$).

Open and Closed Balls in $\mathbb{R}^n$

Definition 4.3. *The open ball centered at $X_0 \in \mathbb{R}^n$ with radius $r > 0$, denoted $B(X_0, r)$, is the set of points in $\mathbb{R}^n$ whose distance to X_0 is strictly less than r, i.e.,*

$$B(X_0, r) = \{X \in \mathbb{R}^n, \|X - X_0\| < r\}$$

- In $\mathbb{R}$, the usual norms coincide with the absolute value, so the open ball $B(x_0, r)$ in $\mathbb{R}$ is the open interval $(x_0 - r, x_0 + r)$.
- In $\mathbb{R}^n$ using the Euclidean norm, the ball $B(X_0, r) = \left\{X = (x, y) \in \mathbb{R}^2, \sqrt{(x - x_0)^2 + (y - y_0)^2} < r\right.$ is the open disk centered at X_0 with radius r.
- The open ball for the maximum norm in $\mathbb{R}^2$ is a square. Indeed,

$$\|X - X_0\|_\infty < r \Longleftrightarrow \max(|x - x_0|, |y - y_0|) < r \Longleftrightarrow |x - x_0| < r \text{ and } |y - y_0| < r$$

Here, $|x - x_0| < r$ characterizes a vertical band of width $2r$, bounded by the lines $x = x_0 - r$ and $x = x_0 + r$. The intersection of these bands is a square centered at X_0.

Definition 4.4. *The closed ball centered at $X_0 \in \mathbb{R}^n$ with radius $r > 0$, denoted $\overline{B}(X_0, r)$, is the set of points in $\mathbb{R}^n$ whose distance to X_0 is less than or equal to r, i.e.,*

$$\overline{B}(X_0, r) = \{X \in \mathbb{R}^n, \|X - X_0\| \le r\}$$

Neighborhood of a Point A neighborhood of a point $X_0 \in \mathbb{R}^n$ is any subset of $\mathbb{R}^n$ containing an open ball centered at X_0.

A neighborhood of infinity is a annulus $\{X \in \mathbb{R}^n, \|X\| > r \text{ with } r > 0 \text{ very large}\}$.

Open Set in $\mathbb{R}^n$ A subset U of $\mathbb{R}^n$ is called open if it is a neighborhood of each of its points. In other words,

$$U \text{ open } \Longleftrightarrow (\forall X_0 \in U; \exists r > 0, B(X_0, r) \subset U)$$

- By convention, the empty set $\emptyset$ and $\mathbb{R}^n$ are open sets in $\mathbb{R}^n$.
- The arbitrary union of open sets is open.
- The finite intersection of open sets is open.

Example 4.5. *The set $U = \{(x, y) \in \mathbb{R}^2, x > y\}$ is open; it is the half-plane below the first bisector (excluding the line).*

Closed Set in $\mathbb{R}^n$ A subset F of $\mathbb{R}^n$ is closed if its complement in $\mathbb{R}^n$ is open.

Example 4.6. *The set $F = \{(x, y) \in \mathbb{R}^2, x \leq y\} = \mathbb{R}^2 \setminus U$ is closed.*

- The empty set $\emptyset$ is open, so its complement $\mathbb{R}^n$ is closed. Similarly, since $\mathbb{R}^n$ is open, its complement $\emptyset$ is closed.
- $\mathbb{R}^n$ and $\emptyset$ are the only subsets of $\mathbb{R}^n$ that are both open and closed.
- A general subset of $\mathbb{R}^n$ need not be open or closed; for example, a half-open interval is neither open nor closed, and the set $\mathbb{Q} \times \mathbb{Q}$ is neither open nor closed in $\mathbb{R}^2$.

Bounded Set A subset A of $\mathbb{R}^n$ is bounded if it is contained within a ball. In other words, A is bounded $\Longleftrightarrow \exists r > 0, \forall x \in A, \|X\| \leq r$.

Compact Set in $\mathbb{R}^n$ Any non-empty, closed, and bounded subset K of $\mathbb{R}^n$ is called compact.

4.3 Real-Valued Functions of Several Variables

Definition 4.7. *A real-valued function of several variables is a mapping*

$$\begin{aligned} f:\ & D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R} \\ & X = (X_1, X_2, ..., X_n) \longmapsto f(X) \end{aligned}$$

D is the domain of definition of f.

- $f(x, y) = 2(x + y)$ is a function of two variables representing the perimeter of a rectangle with length x and width y, defined on $\mathbb{R}^2$.
- $f(x, y) = \dfrac{xy}{x^2 + y^2}$, a function of two variables defined on $\mathbb{R}^2 \setminus \{(0, 0)\}$.
- $f(P, V, T) = PV - nRT$ is a function of three variables representing the ideal gas law, where n is the amount of substance, R is a constant, V is the volume, P is the pressure, and T is the temperature.

Graphical Representation
For $n = 1$; $y = f(x)$ is represented by a curve in the plane $\mathbb{R}^2$.
For $n = 2$; $z = f(x, y)$ is represented by a surface in the space $\mathbb{R}^3$.
For $n \geq 3$; graphical representation is difficult to visualize.

Remark 4.8. *By fixing $z = f(x, y) = k$, we obtain curves called level lines (contour lines) of f, denoted $L_k = \{(x, y) \in \mathbb{R}^2, f(x, y) = k \ (k \text{ real})\}$. These reflect physical realities:*

- On a topographic map, they indicate altitude.
- On a nautical chart, they indicate depth (soundings).
- On a weather map, isobars connect points of equal atmospheric pressure.

Example 4.9. $z = \sin x + \cos(x + y)$

4.4 Continuity of a Real-Valued Function of Several Variables

Definition 4.10. *Let f be a function defined in a neighborhood of a point $A \in \mathbb{R}^n$. We say f is continuous at A if $\lim_{X \to A} f(X) = f(A)$, i.e., $\forall \varepsilon > 0; \forall X \in D_f; \exists \alpha > 0; \|X - A\| < \alpha \Longrightarrow |f(X) - f(A)| \leq \varepsilon$.*

Here, X and $A \in \mathbb{R}^n$, so $X = (x_1, x_2, ..., x_n)$ and $A = (a_1, a_2, ..., a_n)$. Then $X \to A$ means:

$$\begin{cases} x_1 \to a_1 \\ x_2 \to a_2 \\ \vdots \\ x_n \to a_n \end{cases}$$

Note that in $\mathbb{R}$, f is continuous at $A \Longleftrightarrow \lim_{X \overset{>}{\to} A} f(X) = \lim_{X \overset{<}{\to} A} f(X) = f(A)$; there is only one path to approach A. But in $\mathbb{R}^n$, $X \to A$ means all coordinates of X approach those of A simultaneously and independently; there are infinitely many paths to approach A, making the definition non-trivial to apply (e.g., $\lim_{(x,y)\to(a,b)} f(X) \neq \lim_{x\to a} \lim_{y \to b} f(X)$ in general).

- f is said to be continuous on a subset $D \subset \mathbb{R}^n$ if it is continuous at every point of D.

Practical Techniques

- Applying the definition: Let $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. Show f is continuous at $(0, 0)$. Let $M = (x, y)$, $\|\overrightarrow{OM}\| = \sqrt{x^2 + y^2}$. We have $|x| \leq \sqrt{x^2 + y^2} = \|\overrightarrow{OM}\|$ and $|y| \leq \sqrt{x^2 + y^2} = \|\overrightarrow{OM}\|$. Thus, $|f(x, y) - f(0, 0)| = \left|\dfrac{xy}{x^2 + y^2}\right| \leq \|\overrightarrow{OM}\|^2$, which tends to 0 as M approaches the origin. Hence, $\forall \varepsilon > 0; \exists \alpha = \sqrt{\varepsilon} > 0$ such that $\|\overrightarrow{OM}\| < \alpha \Longrightarrow |f(x, y) - f(0, 0)| \leq \varepsilon$, proving continuity at $(0, 0)$.
- Variable substitution using polar coordinates (for two variables).

Let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r > 0,\ \theta \in [0, 2\pi]$.

Example 4.11. $\lim\limits_{(x,y)\longrightarrow(0,0)} \dfrac{x^3y}{x^2+y^2} = \lim\limits_{r\longrightarrow 0} \dfrac{r^4\cos\theta\sin\theta}{r^2} = 0$

Example 4.12. $\lim\limits_{(x,y)\longrightarrow(0,0)} \dfrac{xy}{x^2+y^2} = \lim\limits_{r\longrightarrow 0} \cos\theta\sin\theta$, *depend on* θ. *Thus the limit does not exist.*

- **Using Taylor expansions:** Consider the function $f(x,y) = \frac{x\sin(y)-y}{x^2+y^2}$ defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. Near $y = 0$, we know $\sin(y) - y = \circ(y^2)$, i.e., there exists a function ε such that $\forall y \in V(0)$, $\sin(y) - y = y^2\varepsilon(y)$ with $\varepsilon(y) \to 0$ as $y \to 0$. Since $|xy| \leq \frac{1}{2}(x^2+y^2)$, we have:

$$|f(x,y)| = \left|\frac{xy^2\varepsilon(y)}{x^2+y^2}\right| \leq \frac{|y|\varepsilon(y)}{2} \underset{(x,y)\to(0,0)}{\longrightarrow} 0.$$

In this case, f is **extendable by continuity** at $(0,0)$.

- To show that a limit does not exist, it suffices to find two distinct paths yielding different limit values.

Example 4.13. *The limit* $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ *does not exist. Approaching along the* $y = 0$ *axis gives* $f(x,0) = 1 \to 1$, *while approaching along the* $x = 0$ *axis gives* $f(0,y) = -1 \to -1$. *Hence, the limit does not exist.*

Theorem 4.14. *Let* $f, g : D \subset \mathbb{R}^n \to \mathbb{R}$.

- If f and g are continuous at $A \in \mathbb{R}^n$, then $f + g$ and $f \cdot g$ are continuous at A.
- If f is continuous at A and $f(A) \neq 0$, then $\frac{1}{f}$ is continuous at A.

4.5 Partial Derivatives

Definition 4.15. *Let* $f : D \subset \mathbb{R}^n \to \mathbb{R}$ *and* $A = (a_1, a_2, \ldots, a_n)$ *be an interior point of* D_f. *Define:*

$$g_i : \mathbb{R} \to \mathbb{R}, \quad t \mapsto f(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n).$$

The first-order partial derivative of f *with respect to its* i^{th} *variable at* A, *denoted* $\frac{\partial f}{\partial x_i}(A)$, *is the derivative of* g_i *at* a_i:

$$\frac{\partial f}{\partial x_i}(A) = \lim_{h\to 0} \frac{f(a_1, \ldots, a_i + h, \ldots, a_n) - f(A)}{h}.$$

For a function of two variables, the partial derivatives (if they exist) are:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}, \quad \frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Example 4.16. *Consider* $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. *The partial derivatives at* $(0,0)$ *are:*

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0, \quad \frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = 0.$$

Practical Method: Compute the partial derivative with respect to the i^{th} variable by fixing the others and differentiating the resulting single-variable function.

Example 4.17. *Let* $f(x,y) = y^x$ *defined on* $\mathbb{R} \times \mathbb{R}_+^*$. *Then:*

$$\frac{\partial f}{\partial x}(x,y) = (\ln y)y^x, \quad \frac{\partial f}{\partial y}(x,y) = xy^{x-1}.$$

Remark 4.18. *For single-variable functions, differentiability implies continuity. For multivariable functions, the existence of partial derivatives does not guarantee continuity.*

Example 4.19. *The function* $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ *is not continuous at* $(0,0)$, *as* $\lim_{(x,y)\to(0,0)} f(x,y)$ *depends on* θ *in polar coordinates. However, both partial derivatives at* $(0,0)$ *exist and equal zero.*

Definition 4.20. *Let* $U \subset \mathbb{R}^n$ *be open. A function* $f : U \to \mathbb{R}$ *is of* ***class*** $\mathcal{C}^1(U)$ *if all first-order partial derivatives exist and are continuous on* U.

Corollary 4.21. *If* $f : U \subset \mathbb{R}^n \to \mathbb{R}$ *is of class* $\mathcal{C}^1(U)$, *then* f *is continuous on* U.

4.6 Higher-Order Partial Derivatives

For a function $f(x,y)$, the second-order partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

Generally, the number of partial derivatives equals (number of variables)$^{\text{order}}$.

Definition 4.22. *A function* f *is of* ***class*** $\mathcal{C}^2$ *if all its first-order partial derivatives are of class* $\mathcal{C}^1$.

Theorem 4.23 (Schwarz's Theorem)**.** *Let* $U \subset \mathbb{R}^n$ *be open and* $f : U \to \mathbb{R}$ *be of class* $\mathcal{C}^2(U)$. *Then for all* $i \neq j$:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

A function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is of **class** $\mathcal{C}^k(U)$ if all its partial derivatives up to order k exist and are continuous.

4.7 Partial Derivatives of Composite Functions

Theorem 4.24. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be of class $\mathcal{C}^1(U)$ and $g : \mathbb{R} \to U$, $t \mapsto (x_1(t), \ldots, x_n(t))$, be of class $\mathcal{C}^1$. The composite function $f \circ g(t) = f(x_1(t), \ldots, x_n(t))$ is of class $\mathcal{C}^1$, and its derivative is:*

$$(f \circ g)'(t) = \sum_{i=1}^{n} x_i'(t) \frac{\partial f}{\partial x_i}(x_1(t), \ldots, x_n(t)).$$

Example 4.25. *Let $F(t) = f(\cos t, \sin t)$ with $f(x, y) = x^2 - y^2$. Then:*

$$F'(t) = -\sin t \cdot 2x(t) + \cos t \cdot (-2y(t)) = -4 \cos t \sin t.$$

4.7.1 Change of Variables in $\mathbb{R}^2$

Let $f(x, y)$ be of class $\mathcal{C}^1$, and consider a change of variables $x = x(u, v)$, $y = y(u, v)$. The partial derivatives of $g(u, v) = f(x(u, v), y(u, v))$ are:

$$\begin{cases} \frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \\ \frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}. \end{cases}$$

Example 4.26. *For polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, simplify $y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}$. Let $f(x, y) = g(r, \theta)$. Then:*

$$y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = r\sin\theta\left(\frac{\partial g}{\partial r}\cos\theta - \frac{\partial g}{\partial \theta}\frac{\sin\theta}{r}\right) + r\cos\theta\left(\frac{\partial g}{\partial r}\sin\theta + \frac{\partial g}{\partial \theta}\frac{\cos\theta}{r}\right) = r\frac{\partial g}{\partial \theta}.$$

Solutions to $y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0$ satisfy $\frac{\partial g}{\partial \theta} = 0$, implying g depends only on r.

4.7.2 Directional Derivative

Definition 4.27. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined near $A = (a_1, \ldots, a_n)$, and let $\mathbf{u} = (u_1, \ldots, u_n)$. The* ***directional derivative*** *of f at A in the direction $\mathbf{u}$ is:*

$$f'_{\mathbf{u}}(A) = \lim_{t \to 0} \frac{f(a_1 + tu_1, \ldots, a_n + tu_n) - f(A)}{t}.$$

For $n = 2$:

$$f'_{\mathbf{u}}(A) = \lim_{t \to 0} \frac{f(a_1 + tu_1, a_2 + tu_2) - f(a_1, a_2)}{t}.$$

Remark 4.28. *Partial derivatives are directional derivatives along the canonical basis vectors.*

Example 4.29. *For $f(x, y) = x^2 + xy$ and direction $\mathbf{u} = (\cos\theta, \sin\theta)$ at $(1, -1)$:*

$$f'_{\mathbf{u}}(1, -1) = \cos\theta + \sin\theta.$$

The maximum slope occurs at $\theta = \frac{\pi}{4}$.

4.7.3 Differential

Definition 4.30. *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$, and $A \in U$. f is **differentiable** at A if:*

$$\lim_{\|\mathbf{h}\| \to 0} \frac{f(A + \mathbf{h}) - f(A) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(A) h_i}{\|\mathbf{h}\|} = 0.$$

For $n = 2$:

$$\lim_{\|(h_1,h_2)\| \to 0} \frac{f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) - \left(\frac{\partial f}{\partial x} h_1 + \frac{\partial f}{\partial y} h_2\right)}{\|(h_1, h_2)\|} = 0.$$

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