Quadratic numerical treatment for singular integral equations with logarithmic kernel

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Abstract: The goal of this paper is to present a direct method for an approximative solution of a weakly singular integral equations (WSIE) with logarithmic kernel on a piecewise smooth integration path using a modified quadratic spline approximation, we also show that this approximation gives an efficient approach to the analytical solution of WSIE.

Keywords: weakly singular integral; quadratic interpolation; holder space and holder condition.

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1 Introduction

In this work, we present a direct method for an approximative solution of a weakly singular integral equations (WSIE) with logarithmic kernel on a piecewise smooth integration path using an adapted quadratic spline, this approximation constructed by the author (Nadir, 2012) gives an efficient approach to the analytical solution of WSIE. A different quadrature method for solving WSIE for a closed curve, involving subtraction of the singularity, was analysed in Prossdorf et al. (1993) and Saranen (1991), this kind of equations arises, for example, in electrostatics, fluid dynamics and in simulation of cracks.

Also we note that, the solution of a large class of boundary-value problems in mathematical physics can be reduced to WSIE of the form:

$$a(t_0)\varphi(t_0)+\frac{b(t_0)}{\pi i}\int_\Gamma \ln(t-t_0)\varphi(t)dt+\frac{1}{i\pi}\int_\Gamma k(t,t_0)\varphi(t)dt=f(t_0), \tag{1}$$

where Γ is any piecewise smooth closed contour (Muskhelishvili, 1953), t_0 and t are points on Γ, the known functions $a(t)$, $b(t)$ and $k(t, t_0)$ are defined on Γ and satisfying the holder condition $H(\alpha)$, $0 < \alpha \le 1$ (Muskhelishvili, 1953). Further, anywhere on Γ we have:

$$a(t) \neq 0,\ a'(t) \neq 0 \text{ and } a'(t)+2b(t) \neq 0. \tag{2}$$

As it is known, the integral of the dominant part of the above equation (1) exists in the sense of a Cauchy principal value integral for all density φ that satisfies the holder condition $H(\alpha)$ and also exists for all function $\varphi \in L^2(\Gamma)$:

The present note is divided into two parts. In the first one, we present a formulation of the quadrature formula for the evaluation of weakly singular integral proposed by Nadir (2012), this quadrature formula is based on the adapted quadratic approximation of the density $\varphi(t)$:

In the second part, we present the numerical realisation of this approximation; also the estimate of the error of the approximation integral was established. Besides, pointwise convergence of the approximate solutions to an exact solution is obtained (Nadir, 2012; Prossdorf et al., 1993; Sanikidze, 1971; Saranen, 1991).

A method to proceed is to solve the WSIE by numerical means, like the reduction to a system of linear algebraic equations after the use of an appropriate quadrature rule.

2 Quadrature

We denote by t the parametric complex function $t(s)$ of the curve Γ defined by:

$$t(s)=x(s)+iy(s),\ a_1 \le s \le b_1,$$

where $x(s)$ and $y(s)$ are continuous functions on the finite interval of definition $[a_1, b_1]$ and have continuous first derivatives $x'(s)$ and $y'(s)$ never simultaneously null. Let N be an arbitrary natural number, generally we take it large enough and divide the interval $[a_1, b_1]$ into N equal subintervals $I_1, I_2, \ldots, I_N$ by the points:

$$s_\sigma = a_1+\sigma\frac{1}{N},\ l=b_1-a_1,\ \sigma=0,1,2,\ldots,N.$$

Further, we fix a natural number $M > 1$; and divide each of the segments $[s_\sigma, s_\sigma + 1]$ by the equidistant points:

$$[s_\sigma, s_\sigma+1]=\{s_\sigma=s_{\sigma 0}<s_{\sigma 1}<\ldots..<s_{\sigma 2M}=s_{\sigma+1}\}.$$

We introduce the notation:

$$t_\sigma=t(s_\sigma),\ t_{\sigma k}=(s_{\sigma k});\quad \sigma=0,1,2,\ldots.,N;\quad k=0,1,\ldots,2M.$$

Assuming that, for the indices σ, $v = 0, 1, 2, \ldots, N - 1$ the points t and t_0 belong respectively to the arcs $t_\sigma \widehat{t}_{\sigma+1}$ and $t_v \widehat{t}_{v+1}$ where $t_\mu \widehat{t}_{\mu+1}$ designates the smallest arc with ends t_μ and $t_\mu + 1$ (Nadir and Antidze, 2004; Nadir, 2010; 2012; Sanikidze, 1971).

For an arbitrary number $\sigma = 0, 1, 2, \ldots, N - 1$ we define the piecewise quadratic Lagrange interpolation polynomial $S_2(\varphi, t, \sigma)$ dependent on φ, t and σ which represents the quadratic approximation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ. As we know, the interval $[t_\sigma, t_{\sigma+1}]$ is divided into subintervals $[t_{\sigma k}, t_{\sigma(k+2)}]$ of length $(t_{\sigma(k+2)} - t_{\sigma k})$, $k = 2i$, $i = 0, 1, \ldots, M - 1$. We interpolate the function density $\varphi(t)$ with respect to the values $\varphi(t_{\sigma k})$, $\varphi(t_{\sigma(k+1)})$ and $\varphi(t_{\sigma(k+2)})$ at the points $t_{\sigma k}$, $t_{\sigma(k+1)}$ and $t_{\sigma(k+2)}$ respectively with a quadratic polynomial, given by the following formula.

For $t_{\sigma k} \leq t_{\sigma(k+2):}$

$$
\begin{aligned}
S_2(\varphi; t, \sigma) = {} & \frac{(t - t_{\sigma(k+1)})(t - t_{\sigma(k+2)})}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma k})} \varphi(t_{\sigma k}) \\
& - \frac{(t - t_{\sigma k})(t - t_{\sigma(k+2)})}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma(k+1)})} \varphi(t_{\sigma(k+1)}) \\
& + \frac{(t - t_{\sigma k})(t - t_{\sigma(k+1)})}{(t_{\sigma(k+2)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma(k+1)})} \varphi(t_{\sigma(k+2)}),
\end{aligned}
\tag{3}
$$

this piecewise quadratic interpolating polynomial exists and is unique.

We define for an arbitrary numbers σ and v, such that $0 \leq \sigma$, $v \leq N - 1$; the following continuous function $\beta_{\sigma v}(\varphi, t, t_0)$, dependents on φ, t and t_0:

$$
\beta_{\sigma v}(\varphi; t, t_0) = \begin{cases} U_\sigma(\varphi; t, t_0) - V_{\sigma v}(\varphi; t, t_0) & \text{for } t \neq t_0 \\ 0 & \text{for } t = t_0 \end{cases}
\tag{4}
$$

The function $U\sigma(\varphi;\ t,\ t_0)$ represents a modified quadratic interpolation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ.

Indeed, for $t_{\sigma k} \leq t \leq t_{\sigma(k+2)}$ and $t - t_0 \neq 1$, we put:

$$
\begin{aligned}
U_\sigma(\varphi; t, t_0) = {} & \frac{(t - t_{\sigma(k+1)})(t - t_{\sigma(k+2)})}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma k})} \varphi(t_{\sigma k}) \frac{\ln(t_{\sigma k} - t_0)}{\ln(t - t_0)} \\
& - \frac{(t - t_{\sigma k})(t - t_{\sigma(k+2)})}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma(k+1)})} \varphi(t_{\sigma(k+1)}) \frac{\ln(t_{\sigma(k+1)} - t_0)}{\ln(t - t_0)} \\
& + \frac{(t - t_{\sigma k})(t - t_{\sigma(k+1)})}{(t_{\sigma(k+2)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma(k+1)})} \varphi(t_{\sigma(k+2)}) \frac{\ln(t_{\sigma(k+2)} - t_0)}{\ln(t - t_0)},
\end{aligned}
$$

and the function $V_{\sigma v}(\varphi; t, t_0)$ is given by:

$$V_{\sigma v}\left(\varphi;t,t_0\right)=S_2\left(\varphi;t_0,v\right)\frac{\left(t-t_{\sigma(k+1)}\right)\left(t-t_{\sigma(k+2)}\right)}{\left(t_{\sigma(k+1)}-t_{\sigma k}\right)\left(t_{\sigma(k+2)}-t_{\sigma k}\right)}\frac{\ln\left(t_{\sigma k}-t_0\right)}{\ln\left(t-t_0\right)}$$
$$-S_2\left(\varphi;t_0,v\right)\frac{\left(t-t_{\sigma k}\right)\left(t-t_{\sigma(k+2)}\right)}{\left(t_{\sigma(k+1)}-t_{\sigma k}\right)\left(t_{\sigma(k+2)}-t_{\sigma(k+1)}\right)}\frac{\ln\left(t_{\sigma(k+1)}-t_0\right)}{\ln\left(t-t_0\right)}$$
$$+S_2\left(\varphi;t_0,v\right)\frac{\left(t-t_{\sigma k}\right)\left(t-t_{\sigma(k+1)}\right)}{\left(t_{\sigma(k+2)}-t_{\sigma k}\right)\left(t_{\sigma(k+2)}-t_{\sigma(k+1)}\right)}\frac{\ln\left(t_{\sigma(k+2)}-t_0\right)}{\ln\left(t-t_0\right)},$$

where the function φ represents a given function on the curve Γ and of the class $H(\alpha)$:

Denoting by $\psi_{\sigma v}(\varphi;\ t,\ t_0)$ the cubic approximation of the density $\varphi(t)$ at the point $t\in[t_\sigma,t_{\sigma+1}]$, $t_0\in[t_v,t_{v+1}]$ and $0\le\sigma, v\le N-1$ by:

$$\psi_{\sigma v}\left(\varphi;t,t_0\right)=\varphi\left(t_0\right)+\beta_{\sigma v}\left(\varphi;t,t_0\right). \tag{5}$$

Using the quadratic spline interpolation of the kernel $k(t,\ t_0)$ and of the density $\varphi(t)$, the regular part of the singular integral equation (1) will be obtained as:

$$K\varphi\left(t_0\right)=\frac{1}{\pi i}\int_\Gamma k\left(t,t_0\right)\varphi(t)dt$$
$$\simeq\frac{1}{\pi i}\int_\Gamma\tilde{k}\left(t,t_0\right)\tilde{\varphi}(t)dt$$
$$=\frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma 2k}}^{t_{\sigma(2k+2)}}\frac{\left(t-t_{\sigma(2k+1)}\right)\left(t-t_{\sigma(2k+2)}\right)}{\left(t_{\sigma(2k+1)}-t_{\sigma 2k}\right)\left(t_{\sigma(2k+2)}-t_{\sigma 2k}\right)}k\left(t_{\sigma 2k},t_0\right)\varphi\left(t_{\sigma 2k}\right)$$
$$-\frac{\left(t-t_{\sigma 2k}\right)\left(t-t_{\sigma(2k+2)}\right)}{\left(t_{\sigma(2k+1)}-t_{\sigma 2k}\right)\left(t_{\sigma(2k+2)}-t_{\sigma(2k+1)}\right)}k\left(t_{\sigma(2k+1)},t_0\right)\varphi\left(t_{\sigma(2k+1)}\right)dt$$
$$+\frac{\left(t-t_{\sigma k}\right)\left(t-t_{\sigma(k+1)}\right)}{\left(t_{\sigma(k+2)}-t_{\sigma k}\right)\left(t_{\sigma(k+2)}-t_{\sigma(k+1)}\right)}k\left(t_{\sigma(2k+2)},t_0\right)\varphi\left(t_{\sigma(2k+2)}\right)dt.$$
$$=\tilde{K}\tilde{\varphi}\left(t_0\right).$$

Let $A\varphi(t_0)$ denote the left side of the equation (1):

$$A\varphi\left(t_0\right)=\left(aI+bW+K\right)\varphi\left(t_0\right)$$
$$=a\left(t_0\right)\varphi\left(t_0\right)+\frac{b\left(t_0\right)}{\pi i}\int_\Gamma\ln\left(t-t_0\right)\varphi(t)dt+\frac{1}{\pi i}\int_\Gamma k\left(t,t_0\right)\varphi(t)dt$$
$$=a\left(t_0\right)\varphi\left(t_0\right)+\frac{b\left(t_0\right)}{\pi i}\int_\Gamma\ln\left(t-t_0\right)\left(\varphi(t)-\varphi\left(t_0\right)\right)dt+\frac{1}{\pi i}\int_\Gamma k\left(t,t_0\right)\varphi(t)dt$$
$$=\left(aI+bW_1+K\right)\varphi\left(t_0\right)$$

and $\tilde{A}\tilde{\varphi}(t_0)$ be the adapted quadrature interpolation formula for the operator $A\varphi(t)$ given by:

$$\begin{aligned}\tilde{A}\tilde{\varphi}(t_0) &= \left(aI + b\widetilde{W} + \tilde{K}\right)\tilde{\varphi}(t_0) \\ &= a(t_0)\tilde{\varphi}(t_0) + \frac{b(t_0)}{\pi i}\int_\Gamma \ln(t - t_0)\psi_{\sigma\nu}(\varphi; t, t_0)\,dt + \frac{1}{\pi i}\int_\Gamma \tilde{k}(t, t_0)\tilde{\varphi}(t)dt \\ &= a(t_0)\tilde{\varphi}(t_0) + \frac{b(t_0)}{\pi i}\int_\Gamma \ln(t - t_0)\beta_{\sigma\nu}(\varphi; t, t_0)\,dt + \frac{1}{\pi i}\int_\Gamma \tilde{k}(t, t_0)\tilde{\varphi}(t)dt \\ &= \left(aI + b\widetilde{W}_1 + \tilde{K}\right)\tilde{\varphi}(t_0).\end{aligned}$$

We denote by the function $\tilde{\varphi}(t)$ the approximate solution of (1) and find it from the equality of the functions e $\tilde{A}\tilde{\varphi}(t_0)$ and $f(t_0)$ at the points $t_{\sigma k}$, ($\sigma = 0, 1, \ldots, N - 1$, $k = 0, 1, \ldots, 2M$).

3 Main result

Theorem: The weakly singular integral equation of the form (1) with the condition (2) has a unique solution $\varphi(t)$ and an approximate solution $\tilde{\varphi}(t)$ converges to the solution $\varphi(t)$ with the following estimation:

$$\left|\varphi(t) - \tilde{\varphi}(t)\right| \le \frac{C_1 \ln(2MN)}{(2MN)^\alpha} = \frac{C_2}{(MN)^2}; \quad M, N > 1,$$

where the constants C_1 and C_2 depend only on the curve Γ and the holder constant of the function φ.

Proof: We can write the integral equation (1) as:

$$A\varphi = (aI + bW + K)\varphi = f,$$

while as an approximating equation in the space $H(\alpha)$ we consider:

$$\tilde{A}\tilde{\varphi} = \left(aI + b\tilde{W} + \tilde{K}\right)\tilde{\varphi} = f.$$

It follows from [6] that, for all $\varphi(t)$ in $H(\alpha)$ we have:

$$\left\|W\varphi - \tilde{W}\tilde{\varphi}\right\| \le \frac{C_1 \ln(2MN)}{(2MN)^\alpha},$$

and also it is known that:

$$\left\|K\varphi - \tilde{K}\tilde{\varphi}\right\| \le \frac{C_2}{(MN)^2},$$

for all K compact and $\varphi \in H(\alpha)$.

It is easily to see that:

$$
\begin{aligned}
|\varphi-\tilde{\varphi}| &= \left|\frac{1}{a(t)}\right| \left| b(t)\left(W\varphi-\tilde{W}\tilde{\varphi}\right)+\left(K\varphi-\tilde{K}\tilde{\varphi}\right)\right| \\
&\le \left|\frac{1}{a(t)}\right| \left(\left| b(t)\left(W\varphi-\tilde{W}\tilde{\varphi}\right)+\left(K\varphi-\tilde{K}\tilde{\varphi}\right)\right|\right) \\
&\le \left|\frac{b(t)}{a(t)}\right| \left|\left(W\varphi-\tilde{W}\tilde{\varphi}\right)\right| + \left|\frac{1}{a(t)}\right| \left|\left(K\varphi-\tilde{K}\tilde{\varphi}\right)\right| \\
&\le \frac{C_3 \ln(2MN)}{(2MN)^{\alpha}} + \frac{C_4}{(MN)^2}; \quad M, N > 1,
\end{aligned}
$$

where $C_3 = \sup_t \left|\frac{b(t)}{a(t)}\right| C_1$ and $C_4 = \sup_t \left|\frac{1}{a(t)}\right| C_2$.

4 Numerical experiments

In this section, we describe some of the numerical experiments performed in solving the WSIE (1). In all cases, the curve Γ designates the unit circle and we chose the right hand side *f*(*t*) in such way that we know the exact solution. This exact solution is used only to show that the numerical solution obtained with our method is correct.

We apply the algorithms described in Antidze (1975), Nadir and Antidze (2004) and Nadir (2012) to solve WSIE and we present results concerning the accuracy of the calculations. In this numerical experiments it is easily to see that the matrix of the system of algebraic equation given by our approximation is invertible, confirmed in Nadir and Antidze (2004) and Sanikidze (1971).

In each table, φ represents the exact solution given in the sense of the principal value of Cauchy and $\tilde{\varphi}$ corresponds to the approximate solution produced by our approximation at points values interpolation (Nadir and Antidze, 2004; Nadir, 2012).

Example 1: Consider the weakly singular integral equation:

$$
\cos t_0 \varphi(t_0) + \frac{1}{\pi i}\int_{\Gamma} \ln(t-t_0)\varphi(t)dt = 3t_0 \cos t_0 + t_0^2 \cos t_0,
$$

where the curve Γ designates the unit circle and the function density φ is given by the following expression:

$$
\varphi(t) = t^2 + 3t.
$$

The approximate solution $\tilde{\varphi}(t)$ of $\varphi(t)$ is obtained by our modified quadratic spline approximation for $N = 20$.

Table 1 Shows the exact and the approximate solution and the computed the error for the example 1

Values of points t	*Exact solution φ*	*Approximate solution* $\tilde{\varphi}$	*Error*
1.000e+00 +0.000e+00i	4.000e+00 +0.000e+00i	4.000e+00 –4.071e-14i	9.050e-14
7.071e-01 +7.071e-01i	2.121e+00 +3.121e+00i	2.121e+00 +3.121e+00i	1.030e-14
0.000e-00 +1.000e+00i	–1.000e+00 +3.000e+00i	–1.000e+00 +3.000e+00i	4.440e-15
–7.071e-01 +7.071e-01i	–2.121e+00 +1.121e+00i	–2.121e+00 +1.121e+00i	3.972e-15
–1.000e+00 +0.000e+00i	–2.000e+00 +1.224e–16i	–2.000e+00 –2.498e-15i	2.620e-15
–7.071e-01 –7.071e-01i	–2.121e+00 –1.121e+00i	–2.121e+00 –1.121e+00i	4.930e-15
0.000e-00 –1.000e+00i	–1.000e+00 –3.000e+00i	–1.000e+00 –3.000e+00i	3.179e-15
7.071e-01 –7.071e-01i	2.121e+00 –3.121e+00i	2.121e+00 –3.121e+00i	9.310e-14

Example 2: Consider the weakly singular integral equation:

$$\frac{1}{2}t_0\varphi(t_0)+\frac{1}{\pi i}\int_\Gamma \ln(t-t_0)\varphi(t)dt=\frac{1}{2}t_0\ln(t_0+3),$$

where the curve Γ designates the unit circle and the function density φ is given by the following expression:

$$\varphi(t)=\ln(t+3).$$

The approximate solution $\tilde{\varphi}(t)$ of $\varphi(t)$ is obtained by our modified quadratic spline approximation for $N = 20$.

Table 2 Shows the exact and the approximate solution and the computed the error for the example 2

Values of points t	*Exact solution φ*	*Approximate solution* $\tilde{\varphi}$	*Error*
1.000e+00 +0.000e+00i	1.386e+00 +0.000e+00i	1.386e+00 +5.880e-06i	5.901e-06
7.071e-01 +7.071e-01i	1.328e+00 +1.884e-01i	1.328e+00 +1.884e-01i	1.075e-05
0.000e-00 +1.000e+00i	1.151e+00 +3.217e-01i	1.151e+00 +3.217e-01i	1.315e-05
–7.071e-01 +7.071e-01i	8.752e-01 +2.991e-01i	8.752e-01 +2.991e-01i	3.908e-05
–1.000e+00 +0.000e+00i	6.931e-01 +6.123e-17i	6.931e-01 +6.123e-17i	5.402e-05
–7.071e-01 –7.071e-01i	8.752e-01 –2.991e-01i	8.752e-01 –2.991e-01i	4.362e-05
0.000e-00 –1.000e+00i	1.151e+00 –3.217e-01i	1.151e+00 –3.217e-01i	1.179e-05
7.071e-01 –7.071e-01i	1.328e+00 –1.884e-01i	1.328e+00 –1.884e-01i	1.137e-05

Example 3: Consider the weakly singular integral equation:

$$t_0^2\varphi(t_0)+\frac{1}{\pi i}\int_\Gamma \ln(t-t_0)\varphi(t)dt=t_0^2\sin t_0$$

where the curve Γ designates the unit circle and the function density φ is given by the following expression:

$$\varphi(t)=\sin(t).$$

The approximate solution $\tilde{\varphi}(t)$ of $\varphi(t)$ is obtained by our modified quadratic spline approximation for $N = 20$.

Table 3 Shows the exact and the approximate solution and the computed the error for the example 3

Values of points t	*Exact solution φ*	*Approximate solution $\tilde{\varphi}$*	*Error*
1.000e+00 +0.000e+00i	8.414e-01+0.000e+00i	8.414e-01 –1.986e-05i	4.512e-05
7.071e-01 +7.071e-01i	8.189e-01 +5.835e-01i	8.189e-01 +5.836e-01i	1.009e-04
0.000e-00 +1.000e+00i	9.448e-17 +1.175e+00i	1.403e-04 +1.175e+00i	2.836e-04
–7.071e-01 +7.071e-01i	–8.189e-01 +5.835e-01i	–8.187e-01 +5.833e-01i	1.856e-04
–1.000e+00 +0.000e+00i	–8.414e-01 +6.616e-17i	–8.415e-01 +1.355e-04i	1.430e-04
–7.071e-01 –7.071e-01i	–8.189e-01 –5.835e-01i	–8.188e-01 –5.834e-01i	1.692e-04
0.000e-00 –1.000e+00i	–2.834e-16 –1.175e+00i	1.019e-04 –1.175e+00i	1.060e-04
7.071e-01 –7.071e-01i	8.189e-01 –5.835e-01i	8.189e-01 –5.836e-01i	1.212e-04

Example 4: Consider the weakly singular integral equation:

$$2t_0\varphi(t_0)+\frac{1}{\pi i}\int_\Gamma \ln(t-t_0)\varphi(t)dt=\frac{3}{(t_0+2)}$$

where the curve Γ designates the unit circle and the function density φ is given by the following expression:

$$\varphi(t)=-\frac{1}{t+2}.$$

The approximate solution $\tilde{\varphi}(t)$ of $\varphi(t)$ is obtained by our modified quadratic spline approximation for $N = 20$.

Table 4 Shows the exact and the approximate solution and the computed the error for the example 4

Values of points t	*Exact solution φ*	*Approximate solution $\tilde{\varphi}$*	*Error*
1.000e+00 +0.000e+00i	–3.333e-01 +0.000e+00i	–3.333e-01 +2.789e-06i	6.058e-06
7.071e-01 +7.071e-01i	–3.458e-01 +9.032e-02i	–3.458e-01 +9.031e-02i	9.803e-06
0.000e-00 +1.000e+00i	–4.000e-01 +2.000e-01i	–4.000e-01 +1.999e-01i	2.404e-05
–7.071e-01 +7.071e-01i	–5.953e-01 +3.256e-01i	–5.954e-01 +3.257e-01i	1.099e-04
–1.000e+00 +0.000e+00i	–1.000e+00 +1.224e-16i	–1.000e+00 –4.160e-04i	4.571e-04
–7.071e-01 –7.071e-01i	–5.953e-01 –3.256e-01i	–5.954e-01 –3.257e-01i	1.298e-04
0.000e-00 –1.000e+00i	–4.000e-01 –2.000e-01i	–4.000e-01 –2.000e-01i	1.557e-05
7.071e-01 –7.071e-01i	–3.458e-01 –9.032e-02i	–3.458e-01 –9.031e-02i	1.018e-05

5 Conclusions

We have considered the numerical solution of WSIE and have presented an efficient scheme to compute this approximate solution. The essential idea is to find a combination of functions of approximation for the function density where we can be using it to remove integrable singularities. The regular part where it is the remaining integrands are well behaved and pose no serious numerical problem. The numerical solution of WSIE was verified by comparing the analytical and numerical solutions which agree well.

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