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Theme

On Pell and Pell-Lucas numbers

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Thanks

All praise is due to God, abundant and inexhaustible, and prayers and peace be upon the most honorable of creation, whom God illuminated with His light and chose. Based on the principle that he who does not thank people does not thank God, we extend our sincere thanks and appreciation to our supervising professor, "Professor A. Boudaoud," for all his guidance and advice, which we have never ceased to receive. We also extend our sincere thanks to our parents, who provided us with the ideal conditions to complete this work. Thanks also go to everyone who supported us in this endeavor, from near or far. We also thank all the professors and supervisors who offered us assistance, as well as all the colleagues and professors under whom we studied and from whom we learned so much.

Dedication

We extend our greetings and thanks for completing this humble project:

To our generous parents...To our brothers...

To our supervisor who spared no effort in providing us with his thoughtful guidance...

To the department professors and administrative staff who provided us with an atmosphere of study and research...

To everyone who helped us complete this project, whether from near or far...

Notations

- $\mathbb{N}$: Natural numbers.
- $\mathbb{Z}$: Integer numbers.
- $\mathbb{R}$: Real numbers.
- $\mid x \mid$: Absolute value of real number x.
- $[x]$: Greatest integer $\leq x$.
- LHS : Left-hand side.
- RHS : Right-hand side.
- P_n : nth Pell number.
- Q_n : nth Pell-Lucas number.
- $\mid D \mid$: Determinant of square matrix D.
- $\mid$: divide.
- $gcd(m, n)$: The greatest common divisor between m,n.
- $lcm(m, n)$: The least common multiple between m,n.
- PMI : Principle of mathematical induction.

Introduction

The Pell $(P_n)_{n\geq 0}$ (resp. Pell-Lucas $(Q_n)_{n\geq 0}$) sequence defined by the following recursive formula

$$P_n = 2P_{n-1} + P_{n-2}, \text{ for } n \geq 2 \text{ with } P_0 = 0 \text{ and } P_1 = 1 \tag{1}$$

and

$$Q_n = 2Q_{n-1} + Q_{n-2}, \text{ for } n \geq 2 \text{ with } Q_0 = 2 \text{ and } Q_1 = 2 \tag{2}$$

are considered twin sequences, i.e. each is a twin of the other, and feature prominently in number theory and combinatorics textbooks. They have many uses, and overlap widely with other mathematical disciplines.

These two sequences are still used today as research topics, frequently in the field of Diophantine equations; see [2, 4]. What characterizes these sequences is the large number of identities and laws that link the terms of these sequences together; see ([2, 4]).

The first few Pell and Pell-Lucas numbers, calculated using equations 1 and 2 , are presented in the following Tables.

n	0	1	2	3	4	5	6
P_n	0	1	2	5	12	29	70

Table 01 : first few Pell numbers

n	0	1	2	3	4	5	6
Q_n	2	2	6	14	34	82	198

Table 02 : first few Pell-Lucas numbers

The document is divided into four chapters :

Chapter 1 introduces the concept of linear homogeneous recurrence relations with constant coefficients, along with an overview of Pell and Pell-Lucas numbers.

Chapter 2 focuses on numbers of other families which are Pell or Pell-Lucas numbers.
Chapter 3 presents various Pell identities, including notable results such as Pell and Pell-Lucas hybridity, the identities obtained by using the Candido's identity and other algebraic identities and also by using matrices and determinants. This chapter also investigates the divisibility properties of Pell numbers.
Chapter 4 studies sums and products involving Pell and Pell-Lucas numbers.
In this work, studying definitions, identities, and sequences in both Pell and Pell-Lucas sequences.
We note that we have used several references, as indicated at the end of the memory. But the most used reference is the book : **Pell and Pell-Lucas Numbers with Applications, Thomas Koshy - 2014**, due to its completeness, importance, and simplicity of style.

Chapter 1

Preliminaries

1.1 Linear homogeneous recurrence with constant coefficients (LHRWCCs)

Definition 1.1. *A kth-order linear homogeneous recurrence with constant coefficients is a recurrence of the form:*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + .. + c_k a_{n-k} \tag{1.1}$$

where each c_i is a real number and $c_k \neq 0$.

The term linear means the power of every predecessor of an on the RHS1 of equation (1.1) is at most one. A recurrence is homogeneous if every ai has the same exponent. Since an depends on its k immediate predecessors, the order of the recurrence is k; consequently, we will need k initial conditions to solve the LHRWCCs.

Example 1.1. *The recurrence $b_n = 7b_{n-1} - b_{n-2}$ is linear of order 2.*

Example 1.2. *$b_n = 3b_{n-1} - 2b_{n-2} + b_{n-3}$ is linear of order 3.*

We will confine our discussion to the second-order LHRWCCs

$$a_n = \mathrm{a}a_{n-1} + \mathrm{b}a_{n-2} \tag{1.2}$$

Where a and b are nonzero real numbers . Suppose this recurrence has a nonzero solution of the form cr^n. Then $cr^n = acr^{n-1} + bcr^{n-2}$.since cr $\neq 0$,this implies that r must be a solution of the characteristic equation:

$$x^2 - ax - b = 0. \tag{1.3}$$

Theorem 1.1. *Let r and s be the distinct (real or complex) characteristic roots of recurrence 1.2. Then the general solution of the recurrence is of the form* $a_n = Ar^n + Bs^n$*, where A and B are constants.*

Example 1.3. *The first five terms are*

$$b_n = 6b_{n-1} - b_{n-2}, b_1 = 1, b_2 = 6.$$

The corresponding characteristic equation is

$$x^2 - 6\text{x} + 1 = 0$$

There are two solutions $r = 3 + 2\sqrt{2}$ and $s = 3 - 2\sqrt{2}$ the initial conditions $b_1 = 1$ and $b_2 = 6$. We get the following 2×2 linear system.

$$\begin{cases} \text{Ar} + \text{Bs} = 1 \\ \text{Ar}^2 + \text{B}s^2 = 6 \end{cases}$$

Solving this, we obtained $A = \frac{1}{4\sqrt{2}} = -B$, thus the general solution of the recurrence is $b_n = \frac{r^n - s^n}{4\sqrt{2}}$ where n ≥ 1. The next example illustrates how to solve a linear non homogeneous recurrence with constant coefficients LNHRWCCs.

Example 1.4. *Fibonacci numbres are definied by the recurence*

$$\begin{cases} F_1 = F_2 = 1 \\ F_n = F_{n-1} + F_{n-2}, n \geq 3 \end{cases}$$

The first five Fibonacci numbres are 1,1,2,3 and 5.

The cracatartic polynomial $x^2 - x - 1 = 0$.

Then the roots are $r = \frac{1+\sqrt{5}}{2}$, $s = \frac{1-\sqrt{5}}{2}$ *and*

$$\begin{aligned} F_n &= A.r^n + B.s^n \\ &= A(\frac{1+\sqrt{5}}{2})^n + B(\frac{1-\sqrt{5}}{2})^n \end{aligned}$$

By using initial conditions we calculated A and B as from we set the general solution.

1.2 Pell and Pell- Lucas numbers : Overview

The Pell numbers and the Pell-Lucas numbers are twin mathematical sequences that appear everywhere and share the same properties, which makes them belong to the same family. However, the Pell numbers are denoted by P_n , while the Pell-Lucas numbers are denoted by Q_n. Recursively, we find :

$$\begin{cases} P_0 = 0, \ P_1 = 1 \\ P_n = 2P_{n-1} + P_{n-2}, \ \text{n} \geq 2; \end{cases} \begin{cases} Q_0 = 2, \ Q_1 = 2 \\ Q_n = 2Q_{n-1} + Q_{n-2}, \ \text{n} \geq 2 \end{cases}$$

Both satisfy the same recurrence of Pell, $X_n = 2X_{n-1} + X_{n-2}$ but the only difference between the two recursive definitions lies in the initial conditions, which lead to $P_2 = 2$ and $Q_2 = 6$. The first six Pell numbers are 0, 1, 2, 5, 12, 29, and the first six Pell-Lucas numbers are 2, 2, 6, 14, 34, 82.

1.2.1 Binet-Like formulas

The characteristic equation of the Pell recurrence is $t^2 - 2\text{t} - 1 = 0$, and two distinct roots are $\gamma = 1 + \sqrt{2}$ or $\delta = 1 - \sqrt{2}$, notice that

$$\gamma + \delta = 1 + \sqrt{2} + 1 - \sqrt{2} = 2$$

$$\begin{aligned} \gamma - \delta &= (1 + \sqrt{2}) - (1 - \sqrt{2}) \\ &= 1 + \sqrt{2} - 1 + \sqrt{2} = 2\sqrt{2} \end{aligned}$$

And

$$\gamma \cdot \delta = (1 + \sqrt{2}) \cdot (1 - \sqrt{2}) = -1. \tag{1.4}$$

We will be using these facts frequently. By Theorem 1.1. The general solution of the Pell recurrence is $P_n = A\gamma^n + \text{B}\delta^n$ the initial conditions $P_1 = 1$ and $P_2 = 2$ yield the equations $\text{A}\gamma + \text{B}\delta = 1$ and $\text{A}\gamma^2 + \text{B}\delta^2 = 2$ solving these equations, we get:

$$\text{A} = -\text{B} = \frac{1}{2\sqrt{2}} = \frac{1}{\gamma - \delta} \text{ so } P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} = \frac{\gamma^n - \delta^n}{2\sqrt{2}}.$$

Similarly $Q_n = \frac{r^n+\delta^n}{2}$ thus we have the following Binet-like formulas for P_n and Q_n :

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \text{ and } Q_n = \frac{\gamma^n + \delta^n}{2}, \text{n} \geq 1.$$

For example:

$$P_4 = \frac{\gamma^4 - \delta^4}{\gamma - \delta} = \frac{\gamma^4 - \delta^4}{2\sqrt{2}},$$
$$Q_3 = \frac{\gamma^3 - \delta^3}{2}.$$

The Binet-like formulas that $Q_n + \sqrt{2}P_n = \gamma^n$ and $Q_n - \sqrt{2}P_n = \delta^n$ since $Q_n - \sqrt{2}P_n = \delta^n$,

$$\left| Q_n - \sqrt{2}P_n \right| = |\, \delta^n| < \frac{1}{2}, -\frac{1}{2} < \left| Q_n - \sqrt{2}P_n \right| < \frac{1}{2}.$$

So

$$-\frac{1}{2} + \sqrt{2}P_n < Q_n < \frac{1}{2} + \sqrt{2}P_n.$$

Since Q_n is an integer, it follows that $Q_n = \left[\sqrt{2}P_n + \frac{1}{2}\right]$, this gives an explicit formula for Q_n in terms of P_n. Likewise , $P_n = \frac{1}{2}\left[\sqrt{2}Q_n + 1\right]$.

1.3 Some identities

The Binet formulas satisfied by Pell and Pell-Lucas numbers are given by

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \tag{1.5}$$

$$Q_n = \gamma^n + \delta^n \tag{1.6}$$

1. Pell Primes:

   A Pell number which is a prime number is called Pell Prime. The first few Pell Primes are

$$2, 5, 29, 5741, 33461, \ldots\ldots.$$

The indices of the Pell Primes in the sequence of Pell numbers respectively are

$$2, 3, 5, 11, 13, \ldots\ldots.$$

That is, $P_2 = 2, P_3 = 5, P_5 = 29, P_{11} = 5741, \ldots\ldots$ etc. The indices of Pell Primes are also prime numbers.

2. Pell-Lucas Primes:

   The number $\frac{Q_n}{2}$ is called Pell-Lucas Prime. The Pell-Lucas Primes are

$$3, 7, 17, 41, 239, 577, \ldots\ldots \text{ etc.}$$

   The indices of the above Pell-Lucas numbers in Pell-Lucas sequence are

$$2, 3, 4, 5, 7, 8, \ldots\ldots\ldots\ldots \text{ etc.}$$

   That is, $\frac{Q_2}{2} = 3$, $\frac{Q_3}{2} = 7$, $\frac{Q_4}{2} = 17, \ldots\ldots$ etc.

3. The relation between Pell and Pell-Lucas numbers is given by

$$Q_n = \frac{P_{2n}}{P_n} \tag{1.7}$$

Proof. Using the Binet formulas 1.5 and 1.6 we get

$$\begin{aligned} P_nQ_n &= \left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right)(\gamma^n + \delta^n) \\ &= \frac{\gamma^{2n} - \delta^{2n}}{\gamma - \delta} \\ &= P_{2n} \\ \Rightarrow Q_n &= \frac{P_{2n}}{P_n} \end{aligned}$$

Thus 1.7 is proved.

For example, $Q_3 = \frac{P_6}{P_3} = \frac{70}{5} = 14$.

4. Simpson formula

   The Pell numbers satisfy Simpson's formula given by

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n \tag{1.8}$$

Proof. Using Binet formula 1.5, we get

$$\begin{aligned}
P_{n+1}P_{n-1} - P_n^2 &= \frac{\left(\gamma^{n+1} - \delta^{n+1}\right)\left(\gamma^{n-1} - \delta^{n-1}\right)}{(\gamma - \delta)^2} - \frac{\left(\gamma^n - \delta^n\right)^2}{(\gamma - \delta)^2} \\
&= \frac{\left(\gamma^{2n} - \gamma^{n+1}\delta^{n-1} - \delta^{n+1}\gamma^{n-1} + \delta^{2n}\right) - \left(\gamma^{2n} + \delta^{2n} - 2\gamma^n\delta^n\right)}{(\gamma - \delta)^2} \\
&= \frac{-\gamma^{n-1}\delta^{n-1}\left(\gamma^2 + \delta^2 - 2\gamma\delta\right)}{(\gamma - \delta)^2} \\
&= -(\gamma\delta)^{n-1} = -(-1)^{n-1} \quad \text{(by 1.4)} \\
&\Rightarrow P_{n+1}P_{n-1} - P_n^2 = (-1)^n
\end{aligned}$$

Thus Simpson formula 1.8 is proved.

5. As proved the sum of Pell numbers up to $(4n + 1)$ is a perfect square as given below.

$$\sum_{i=0}^{4n+1} P_i = \left(P_{2n} + P_{2n+1}\right)^2 \tag{1.9}$$

For example if $n = 1$, LHS $= \sum_{i=0}^{5} P_i = P_0 + P_1 + P_2 + P_3 + P_4 + P_5 = 0 + 1 + 2 + 5 + 12 + 29 = 49$ and RHS $= \left(P_2 + P_3\right)^2 = (2 + 5)^2 = 49$. Hence the above relation 1.9 is verified.

1.4 Pythagorean triples

Definition 1.2. *If A, B and C are the integer sides of a right angled triangle satisfying the Pythagoras theorem $A^2 + B^2 = C^2$, then the integers (A, B, C) are known as Pythagorean triples.*

It's know their. That these triples can be formed by Pell numbers.

$$(A, B, C) \equiv \left(2P_nP_{n+1}, P_{n+1}^2 - P_n^2, P_{2n+1}\right) \tag{1.10}$$

Example 1.5. *If $n = 2, A = 2P_2P_3 = 2 \times 2 \times 5 = 20, B = P_3^2 - P_2^2 = 5^2 - 2^2 = 21$ and $C = P_5 =$ 29. That is, the Pythagorean triple for $n = 2$ is $(20, 21, 29)$.*

The sequence of Pythagorean triples obtained by putting $n = 1, 2, 3, \ldots$ etc. in 1.10 is given by $(4, 3, 5), (20, 21, 29), (120, 119, 169), (696, 697, 985), \ldots\ldots\ldots\ldots$.etc.

Chapter 2

Pell numbers and other number families

2.1 Triangular Pell numbers

Definition 2.1. *The triangular numbers* $t_1, t_2, ..., t_k, ..$ *is the sequence where* t_k *is the number of dots in the triangular array of* k *rows with* j *dots in the* j*th row.*

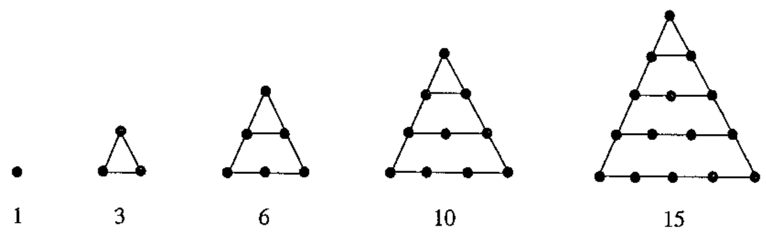

Figure 2.1: The Triangular Numbers

We recall the following Theorem of characteristic.

Theorem 2.1. *A positive integer* n *is a triangular number iff* $8\text{n}+1$ *it is a square.*

In 1996, the mathematician W.L. McDaniel of the University of Missouri at St. Louis proved that $P_1 = 1$ is the only triangular Pell number[6].

Recall that the equation $x^2 - 8y^2 = 1$ has been shown to have infinitely many solution, namely,

$$(x_n, y_n) = (Q_{2n}, \frac{1}{2}P_{2n}), n \geq 1$$

Then there are infinitely many square triangular number y_k^2 since $8y^2 + 1 = x^2$.
In the following table, we give three solution (x_k, y_k), the first three square triangular numbers y_k^2, and their corresponding subscripts

k	x_k	y_k	Square-Triangular Number y_k^2	Corresponding Subscript n_k in t_{n_k}
1	3	1	1	1
2	17	6	36	8
3	99	35	1225	49

The formula for the square-triangular number y_k^2 can be written in different ways :

$$\begin{aligned} y_k^2 &= \frac{1}{4}P_{2k}^2 = \frac{1}{32}\left((1+\sqrt{2})^{4k} + (1-\sqrt{2})^{4k} - 2\right) \\ &= \frac{1}{32}\left((3+2\sqrt{2})^{2k} + (3-2\sqrt{2})^{2k} - 2\right) = \frac{1}{16}\left(Q_{4k} - 1\right) \\ &= \frac{1}{32}\left((17+12\sqrt{2})^{k} + (17-12\sqrt{2})^{k} - 2\right). \end{aligned}$$

2.2 Pentagonal numbers

Definition 2.2. *Pentagonal numbers P_n are polygonal numbers which are positive integer they can be represented geometrically by regular pentagons. The first ten pentagonal numbers are :* 1, 5, 12, 22, 35, 51, 70, 92, 117 *and* 145.
Figure 2.2 show the geometric Representations of the first four pentagonal numbers :

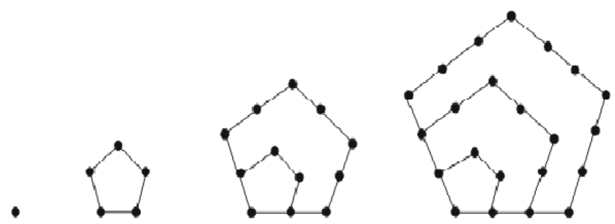

Figure 2.2

Pentagonal numbers can also be defined recursively:

$$\begin{cases} P_1 = 1, \\ P_n = P_{n-1} + 3(\text{n}-1) + 1, \text{n} \geq 2. \end{cases}$$

See figure 2.3. Clearly, this recurrence can be rewritten as $P_n = P_{n-1} + 3n - 2$. Explicitly, $P_n = \frac{\text{n}(3\text{n}-1)}{2}$, where n ≥ 1; this can be confirmed easily using PMI.

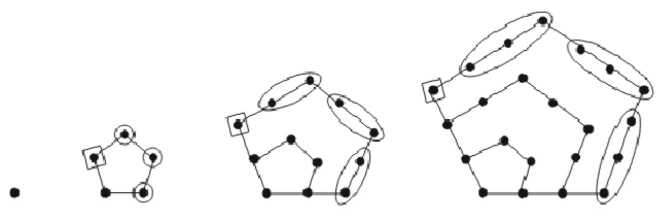

Figure 2.3

For example : $P_6 = P_5 + 19 = 35 + 16 = 51$. Likewise, $P_{10} = 145$.

2.3 Pentagonal Pell numbers

In 2002, V.S.R. Prasad and B.S. Rao at Osmania University, Hyderabad, India, proved that the numbers 1, 5, 12, 70 are the only pentagonal Pell numbers. Indeed, we have the follwing Theorem which her proof is long and complex.

Theorem 2.2 (Prasad and Rao, 2002 [6]).
P_n is a pentagonal number if and only if $n = 1, 3, 4,$ or 6.

2.4 Pentagonal Pell-Lucas numbers

Theorem 2.3. *[6] Q_1 is the only Pentagonal number of type Pell-Lucas*

The proof of this result necessity the following lemmas.

Lemma 2.1. *(Prasad and Rao, 2001[5]). Let n and k be nonnegative integers, and m an even integer. Then: $Q^m_{n+2km} \equiv (-1)^k Q_n \pmod{Q_m}$.*

Lemma 2.2. *Suppose $n \equiv 0$ or $1 \,(\mathrm{mod} 2^2 \cdot 3^2)$. Then $24Q_n + 1$ is a square if and only if $n = 0$ or 1.*

Lemma 2.3. *Suppose $n \equiv 3 \,(\mathrm{mod} 2^2 \cdot 3^2 \cdot 7)$. Then $24Q_n + 1$ is a square if and only if $n = 3$.*

Lemma 2.4. *Suppose $n \equiv 0, 1$ or $3 \,(\mathrm{mod} 2^3 \cdot 3^2 \cdot 5 \cdot 7)$. Then $24Q_n + 1$ is a square only if $n = 0, 1,$ or 3.*

Lemma 2.5. *$71824Q_n + 1$ is not a square if $n \not\equiv 0, 1,$ or $3 \,(\mathrm{mod} 2^3 \cdot 3^2 \cdot 5 \cdot 7)$.*

Proof. of Theorem 2.3
$24Q_n + 1$ is a square when n = 1 or 3 is a $Q_3 = 7$ is not a Pentagonal, and $Q_1 = 1$ is a Pentagonal , then Q_1 is the only Pentagonal number.

2.5 Heptagonal Pell numbers

Definition 2.3. *A heptagonal number is a positive integer of the form $\frac{m(5m-3)}{2}$, where m is a positive integer.*
The first six heptagonal numbers are 1, 7, 18, 34, 55 and 81. Like triangular and Pentagonal numbers, they also can be represented geometrically; see figure 2.4.

Figure 2.4

A positive integer N is heptagonal if and only if $N = \frac{m(5m-3)}{2}$. Then $2N = 5m^2.3m$; so $40N + 9 = 100m^2 - 60m + 9 = (10m - 3)^2$. Thus N is heptagonal if and only if $40N + 9$ is a positive square. Consequently, P_n is heptagonal if and only if $40P_n + 9$ is a square ≥ 1.

Clearly, $P_1 = 1$ is heptagonal.

Are there others? In 2005 Rao established that P_1 is the only such number . His proof employs three lemmas and the following fundamental properties, some of which we have already seen[7]

$Q_n^2 = 2P_n^2 + (-1)^n$

$Q_{3n} = Q_n\,(Q_n^2 + 6P_n^2)$

$P_{m+n} = 2P_mQ_n - (-1)^n P_{m-n}$

$P_{n+2kt} \equiv (-1)^{t(k+1)} P_n\,(\text{mod} Q_k)$

$2 \mid P_n$ if and only if $2 \mid$ n, and $2 \mid Q_n$ for any n .

$3 \mid P_n$ if and only if $4 \mid$ n, and $3 \mid Q_n$ if and only if n $= 2(\text{mod}4)$

$5 \mid P_n$ if and only if $3 \mid$ n, and $5 \mid Q_n$ for any n .

$9 \mid P_n$ if and only if $12 \mid$ n, and $9 \mid Q_n$ if and only if n $= 6(\text{mod}12)$.

Let n be odd then

1. $Q_m = \pm1(\text{mod}4)$ according as m $= \pm1(\text{mod}4)$;
2. $P_m = 1(\text{mod}4)$;
3. $Q_m^2 + 6P_m^2 = 7$ (mod 8).

Lemma 2.6. *Suppose* $n \equiv \pm1\,(\text{mod}2^2 \cdot 5)$. *Then* $40P_n + 9$ *is a square if and only if* $n = \pm1$.

Lemma 2.7. *Suppose* $n \equiv 6\,(\bmod 2^2 \cdot 5^3 \cdot 7^2)$. *Then* $40P_n + 9$ *is a square if and only if* $n = 6$.

When $n = 6$, notice that $40P_n + 9 = 40P_6 + 9 = 40 \cdot 70 + 9 = 53^2$, a square.

Lemma 2.8. *Suppose* $n \equiv 0\,(\bmod 2 \cdot 7 \cdot 5^3)$. *Then* $40P_n + 9$ *is a square if and only if* $n = 0$.
When $n = 0$, *notice that* $40P_n + 9 = 40P_0 + 9 = 40 \cdot 0 + 9 = 3^2$, *again a square.*

The following result follows from these three lemmas.

Corollary 2.1. *Suppose* $n \equiv 0, \pm 1$, *or* $6\,(\bmod 2^2 \cdot 5^3 \cdot 7^2)$. *Then* $40P_n + 9$ *is a square if and only if* $n = 0, \pm 1$, *or 6 .*

We will need one more lemma.

Lemma 2.9. *If* $n \not\equiv 0, \pm 1$, *or* $6\,(\bmod 2^2 \cdot 5^3 \cdot 7^2)$. *Then* $40P_n + 9$ *is not a square.*
For example, let $n = 17$. *Then* $40P_n + 9 = 40P_{17} + 9 = 40 \cdot 1136689 + 9 = 45,467,569$ *is not a square.*

With these tools it was proven that P_1 is the only heptagonal Pell number.

Chapter 3

Pell Identities

3.1 Pell Identities

Let us assume that x_n satisfies the Pell recurrence. Then

$$x_{2n} = 6x_{2n-2} - x_{2n-4}.$$

Indeed

$$\begin{aligned} 6x_{2n-2} - x_{2n-4} &= 5x_{2n-2} + (x_{2n-2} - x_{2n-4}) \\ &= 5\text{x}_{2n-2} + 2x_{2n-3} \\ &= 5x_{2n-2} + (x_{2n-1} - 2x_{2n-2}) \\ &= x_{2n-2} + 2x_{2n-1} \\ &= x_{2n}. \end{aligned}$$

In particular, we have

1) $P_{2n} = 6P_{2n-2} - P_{2n-4}$.

2) $Q_{2n} = 6Q_{2n-2} - Q_{2n-4}$.

Numerical illustration

$$P_8 = 6P_6 - P_4 = 6(70) - 12 = 408.$$

And

$$Q_8 = 6Q_6 - Q_4 = 6 \times 198 - 34 = 1154.$$

3) $P_n^2 = 6P_{n-1}^2 - P_{n-2}^2 - 2(-1)^n$. Indeed

$$\begin{aligned}
P_n^2 &= (2P_{n-1} + P_{n-2})^2 \\
&= 4P_{n-1}^2 + P_{n-2}^2 + 4P_{n-1}P_{n-2} \\
&= 4P_{n-1}^2 + P_{n-2}^2 + 2P_{n-2}(P_n - P_{n-2}) \\
&= 4P_{n-1}^2 - P_{n-2}^2 + 2P_nP_{n-2} \\
&= 6P_{n-1}^2 - P_{n-2}^2 + 2\left(P_nP_{n-2} - P_{n-1}^2\right) \\
&= 6P_{n-1}^2 - P_{n-2}^2 + 2(-1)^{n-1} \\
&= 6P_{n-1}^2 - P_{n-2}^2 - 2(-1)^n. \qquad (3.1)
\end{aligned}$$

Numerical example

$$P_5^2 = 6P_4^2 - P_3^2 - 2(-1)^5 = 6 \times 12^2 - 5^2 + 2 = 841 = 29^2.$$

Formula 3.1 demonstrates that the sequences of Pell numbers can be defined through a recursive relation :

$$P_1^2 = 1 \cdot P_2^2 = 4$$
$$P_{n-1}^2 - P_{n-2}^2 - 2(-1)^n. \quad n \geq 3.$$

3.2 A Pell and Pell-Lucas Hybridity

In his book Pell's Equation (Springer-Verlag, New York, 2003), E.J. Barbeau presents an Intriguing numerical pattern, challenging readers to deduce its underlying formula

$$3^4 - 5 \cdot 4^2 = 1$$
$$7^4 - 2^4 \cdot 10^2 = 1$$
$$17^4 - 145 \cdot 24^2 = 1$$
$$41^4 - 840 \cdot 58^2 = 1.$$

These equality verifies the equation

$$Q_{n+1}^4 - P_n P_{n+2} \left(2P_{n+1}\right)^2 = 1$$

Proof.

$$\begin{aligned}
Q_{n+1}^4 - P_n P_{n+2} \left(2P_{n+1}\right)^2 &= \begin{vmatrix} Q_{n+1}^2 & 4P_n P_{n+2} \\ P_{n+1}^2 & Q_{n+1}^2 \end{vmatrix} \\
&= \begin{vmatrix} \mathrm{Q}_{\mathrm{n+1}}^2 + 0 & 4\left[\mathrm{P}_{\mathrm{n+1}}^2 + (-1)^{\mathrm{n+1}}\right] \\ \mathrm{p}_{\mathrm{n+1}}^2 & \mathrm{Q}_{\mathrm{n+1}}^2 \end{vmatrix} \\
&= \begin{vmatrix} \mathrm{Q}_{\mathrm{n+1}}^2 & 4\mathrm{P}_{\mathrm{n+1}}^2 \\ \mathrm{p}_{\mathrm{n+1}}^2 & \mathrm{Q}_{\mathrm{n+1}}^2 \end{vmatrix} + \begin{vmatrix} 0 & 4(-1)^{\mathrm{n+1}} \\ \mathrm{P}_{\mathrm{n+1}}^2 & \mathrm{Q}_{\mathrm{n+1}}^2 \end{vmatrix} \\
&= \left(\mathrm{Q}_{n+1}^4 - 4\mathrm{P}_{n+1}^4\right) - 4(-1)^{n+1}\mathrm{P}_{n+1}^2. \\
&= \left(\mathrm{Q}_{\mathrm{n+1}}^2 - 2\mathrm{P}_{\mathrm{n+1}}^2\right)\left(\mathrm{Q}_{\mathrm{n+1}}^2 + 2\mathrm{P}_{\mathrm{n+1}}^2\right) - 4(-1)^{\mathrm{n+1}}\mathrm{P}_{\mathrm{n+1}}^2 \\
&= (-1)^{\mathrm{n+1}}\left(\mathrm{Q}_{\mathrm{n+1}}^2 + 2\mathrm{P}_{\mathrm{n+1}}^2\right) - 4(-1)^{\mathrm{n+1}}\mathrm{P}_{\mathrm{n+1}}^2 \\
&= (-1)^{\mathrm{n+1}}\left(\mathrm{Q}_{\mathrm{n+1}}^2 - 2\mathrm{P}_{\mathrm{n+1}}^2\right) \\
&= (-1)^{\mathrm{n+1}}(-1)^{\mathrm{n+1}} \\
&= 1.
\end{aligned}$$

3.3 Matrices and Pell numbers

To generate Pell numbers, we use numerical matrices

$$\mathrm{P} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Then

$$P^2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = 2\text{P} + \text{I}$$

P check the equation $M^2 = 2M + I$, where

$$\text{P} = \begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix} \text{ and } P^2 = \begin{bmatrix} P_3 & P_2 \\ P_2 & P_1 \end{bmatrix}.$$

Therefore, the following result can be proven using PMI.

Theorem 3.1. *Let n be a positive integer, then*

$$P^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$$

Proof. When n = 1 is an integer, we assume it is an integer k $\geq$ 1.

$$P^k = \begin{bmatrix} P_{k+1} & P_k \\ P_k & P_{k-1} \end{bmatrix}$$

Then, using the Pell recurrence, we have:

$$\begin{aligned} P^{k+1} &= P^k \cdot \text{P} \\ &= \begin{bmatrix} P_{k+1} & P_k \\ P_k & P_{k-1} \end{bmatrix} \begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix} \\ &= \begin{bmatrix} 2P_{k+1} + P_k & P_{k+1} \\ 2P_k + P_{k-1} & P_k \end{bmatrix} \end{aligned}$$

So the

$$= \begin{bmatrix} P_{k+2} & P_{k+1} \\ P_{k+1} & P_k \end{bmatrix}.$$

□

We deduce from Theorem 3.1, the following result

Theorem 3.2 (Cassini-like formula for P_n)**.** *We have* $P_{n+1}P_{n-1} - P_n^2 = (-1)^2$.

Theorem 3.3. *We have*

$$P_{m+n} = P_m P_{n+1} + P_{m-1} P_n$$

Proof. Since $P^{m+n} = P^m P^n$, by Theorem 3.2, we have

$$\begin{aligned}
\begin{bmatrix} P_{m+n+1} & P_{m+n} \\ P_{m+n} & P_{m+n-1} \end{bmatrix} &= \begin{bmatrix} P_{m+1} & P_m \\ P_m & P_{m-1} \end{bmatrix} \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} \\
&= \begin{bmatrix} P_{m+1}P_{n+1} + P_m P_n & P_{m+1}P_n + P_m P_{n-1} \\ P_m P_{n+1} + P_{m-1} P_n & P_m P_n + P_{m-1} P_{n-1} \end{bmatrix}
\end{aligned}$$

The addition formula follows by equating the lower left-hand elements from both sides.

Some applications

1. We get the addition formula by equating the lower left elements of both sides, for example $P_{10} = P_{4+6} = P_4 P_7 + P_3 P_6 = 12.169 + 5.70 = 2378$.

   We noticed that the Pell iteration follows the addition formula, so the addition formula is a generalization of the Pell iteration and thus formula Produces three important profits .

2. Suppose we let m = n, then

$$\begin{aligned}
P_{2n} &= P_n P_{n+1} + P_{n-1} P_n = P_n \left(P_{n+1} + P_{n-1}\right) \\
&= P_n \left(2Q_n\right) = 2P_n Q_n
\end{aligned}$$

   A fact we already knew

3. assume that $13|P_{7n}$ for every integer n ≥ 1 we will prove this using PMI, firstly, we noticed that $P_7 = 169$ and $13|P_7$ this assumption applies to possitive integer n, we find :

   $P_{7(n+1)} = P_{7n+7} = P_{7n} P_8 + P_{7n-1} P_7$ as $13|P_{7n}$, according to the inductive hypothesis, $13 \mid P_{7(n+1)}$ and using the formula PMI the result is true for all $\mathrm{n} \geq 1$ in general.

Where q is a prime number, we assume $\mathrm{q} \mid p_n$ and then follow a similar formula that says that $\mathrm{q} \mid p_{mn}$ for every integer.

This is an example : $17 \mid P_8$ and $17 \mid P_{16}$

4. we make $\mathrm{m} = 2\mathrm{n}$ in the formula we find

$$\begin{aligned} P_{3n} &= P_{2n}P_{n+1} + P_{2n-1}P_n \\ &= 2P_nQ_nP_{n+1} + P_{2n-1}P_n \\ &= (2Q_nP_{n+1} + P_{2n-1})\,P_n \end{aligned}$$

so $P_n \mid P_{3n}$ for every positive integer n , for example $P_5 \mid P_{15}$ where $P_{15} = 195025$ and $P_5 = 29$

Secondly, in general, we have the following theorems which we get from using PMI.

Theorem 3.4. *When n is a positive integer , then* $P_m|P_{m\cdot n}$ *and within the limits of* $\mathrm{n} = 1$ *and* $\mathrm{n} \geq 1$ *we find :*

$$\begin{aligned} P_{m(n+1)} &= P_{mn+m} \\ &= P_{mn}P_{n+1} + P_{mn-1}, P_m \end{aligned}$$

Since $P_m|P_{mn}$ then $P_m|P_{m(n+1)}$ and using PMI when $\mathrm{n} \geq 1$ is an integer . The opposite is true as shown in the following theory.

Theorem 3.5. *If* $P_m|P_n$*, them* m|n *using the division algorithm and assuming that* $\mathrm{n} = \mathrm{mk} + \mathrm{r}$ *we find :* $0 < \mathrm{r} < \mathrm{m}$

$$\begin{aligned} P_n &= P_{mk+r} \\ &= P_{mk}P_{r+1} + P_{mk-1}P_r. \end{aligned}$$

Since $P_m|P_{mk}$ and $P_m|P_n$ then $P_m|P_{mk-1}P_r$ and since $(P_{mk}.P_{mk-1}) = 1$ Then $P_m \cdot P_{nk-1}$ it does not happen $P_m|P_r$ unless it is $\mathrm{r} = 0$ and therefore

$\mathrm{n} = \mathrm{mk}$ and m|n.

Theorem 3.6. $P_m|P_n \Leftrightarrow$ m|n.

3.4 Pell Divisibility Properties

We begin with the observation that $(P_m, P_{m-1}) = 1$. Using Theorem 3.5, we can generalize this property, as in the following lemmas.

Lemma 3.1. *For any positive integer q ,*

$$(P_{qn-1}, P_n) = 1$$

Proof. Let $d = (P_{qn-1}, P_n)$. Then d | P_{qn-1} and d | P_n. By Theorem 3.2, $P_n \mid P_{qn}$, so d | P_{qn}. Thus, d divides both P_{qn} and P_{qn-1}. However, since $(P_{qn}, P_{qn-1}) = 1$, it follows that d | 1. Hence, d = 1, proving the claim.
Before establishing the main result, we require an additional lemma concerning the greatest common divisor of Pell numbers.

Lemma 3.2. *Let* m = qn + r*, where* $0 \leq$ r < n*. Then,*

$$gcd\,(P_m, P_n) = gcd\,(P_n, P_r)\,.$$

Proof. Applying the recurrence relation and Lemma 3.1, we derive:

$$\begin{aligned}(P_m, P_n) &= (P_{qn+r}, P_n)\\ &= (P_{qn}P_{r+1} + P_{qn-1}P_r, P_n)\\ &= (P_{qn-1}P_r, P_n)\,(\text{ since } x_{y^2} \mid x_{y^2})\\ &= (P_r, P_n)\,(\text{ by } \text{ Lemma3.1 })\\ &= (P_n, P_r)\,.\end{aligned}$$

Example 3.1. *Let* m = 15 *and* n = 6*, so* q = 2 *and* r = 3*. Then,*

$$(P_{15}, P_6) = (195025, 70) = 5 = (70, 5) = (P_6, P_3)\,.$$

which confirms the lemma.

The proof of the desired property hinges on the Euclidean algorithm, leveraging the recursive structure of Pell numbers and their divisibility relations.

Theorem 3.7. *For any two positive integers m and n , the greatest common divisor of the Pell numbers P_m and P_n is equal to the Pell number indexed by $gcd(m, n)$. That is,*

$$gcd\,(P_m, P_n) = P_{gcd(m,n)}.$$

Proof. Without loss of generality, assume $m \geq n$. We proceed by applying the Euclidean algorithm to the pair (m, n), which yields the following sequence of divisions:

$$\begin{aligned} m &= q_1 n + r_1, && 0 \leq r_1 < n \\ n &= q_2 r_1 + r_2, && 0 \leq r_2 < r_1 \\ &\vdots \\ r_{k-1} &= q_{k+1} r_k + r_{k+1}, && 0 \leq r_{k+1} < r_k \\ r_k &= q_{k+2} r_{k+1} + 0 \end{aligned}$$

Here, $r_{k+1} = gcd(m, n)$. By Lemma 3.2: we have:

$$gcd\,(P_m, P_n) = gcd\,(P_n, P_{r_1}) = gcd\,(P_{r_1}, P_{r_2}) = \cdots = gcd\left(P_{r_k}, P_{r_{k+1}}\right)$$

Since r_{k+1} divides r_k, it follows from the properties of Pell numbers that divides P_{r_k}. Thus,

$$gcd\left(P_{r_k}, P_{r_{k+1}}\right).$$

Theorem 3.8. *Provides an efficient computational method for determining the greatest common divisor of two Pell numbers.*

Example 3.2 (Application). *For* P_{18} *and* P_{12} *:*

1. *First compute* $gcd(18, 12) = 6$.

2. *Then* $P_6 = 70$ *gives the result directly.*

This is markedly more efficient than computing $gcd(1031154, 13860)$ *directly.*

Corollary 3.1 ((LCM of Pell Numbers))**.** *Enables efficient computation of the least common multiple of two Pell numbers through the fundamental relation:*

$$lcm\left(P_m, P_n\right) = \frac{P_m P_n}{gcd\left(P_m, P_n\right)} = \frac{P_m P_n}{\mathrm{P}_{gcd(m,n)}}.$$

Example 3.3 ((Numerical Example))**.** *For* $P_8 = 408$ *and* $P_6 = 70$ *:*

$$lcm\left(P_8, P_6\right) = \frac{408 \times 70}{P_{gcd(8,6)}} = \frac{28560}{P_2} = \frac{28560}{2} = 14280.$$

Remark 3.1. *This result completes the parallel with integer arithmetic, showing that both GCD and LCM operations in Pell numbers reduce to their index computations.*

3.5 Some algebraic identities and Pell family

By using the following identity

$$(x+y)^3 - x^3 - y^3 = 3xy(x+y).$$

We obtain news identities for $(P_n)_{n\geq 1}$ and $(Q_n)_{n\geq 1}$.
But $x = 2P_n,\ y = P_{n-1}$ we obtain

$$P_{n+1}^3 - 8P_n^3 - P_{n-1}^3 = 6P_{n+1}P_nP_{n-1}.$$

Similarly, for the Pell-Lucas sequence, we have:

$$Q_{n+1}^3 - 8Q_n^3 - Q_{n-1}^3 = 6Q_{n+1}Q_nQ_{n-1}.$$

Example 3.4. *For* n = 5*,*

$$P_6^3 - 8P_5^3 - P_4^3 = 70^3 - 8 \times 29^3 - 10^3 = 146160 = 6 \times 70 \times 29 \times 12 = 6P_6P_5P_4.$$

and

$$\begin{aligned} Q_6^3 - 8Q_5^3 - Q_4^3 = 99^3 - 8 \times 41^3 - 17^3 = 414018 = 6 \times 99 \times 41 \times 17 \\ = 6Q_6Q_5Q_4. \end{aligned}$$

Also by using the following identities

$$(x+y)^5 - x^5 - y^5 = 5xy(x+y)\left(x^2+xy+y^2\right)$$
$$(x+y)^7 - x^7 - y^7 = 7xy(x+y)\left(x^2+xy+y^2\right)^2.$$

We find other formulas concerning $(P_n)_n$ and $(Q_n)_n$,

$$P_{n+1}^5 - 32P_n^5 - P_{n-1}^5 = \frac{5}{2}P_{n+1}P_nP_{n-1}\left(5Q_{2n} - (-1)^n\right),$$
$$Q_{n+1}^5 - 32Q_n^5 - Q_{n-1}^5 = 10Q_{n+1}Q_nQ_{n-1}\left(2Q_{2n} + (-1)^n\right)$$
$$P_{n+1}^7 - 128P_n^7 - P_{n-1}^7 = \frac{5}{8}P_{n+1}P_nP_{n-1}\left(5Q_{2n} - (-1)^n\right)^2$$
$$Q_{n+1}^7 - 128Q_n^7 - Q_{n-1}^7 = 10Q_{n+1}Q_nQ_{n-1}\left(5Q_{2n} + (-1)^n\right)^2.$$

In the same sense, the following algebraic identities,

$$(x+y)^2 + x^2 + y^2 = 2\left(x^2+y^2+xy\right)$$
$$(x+y)^4 + x^4 + y^4 = 2\left(x^2+y^2+xy\right)^2.$$

Give the following formulas

$$P_{n+1}^2 + 4P_n^2 + P_{n-1}^2 = \frac{1}{2}\left(2Q_{2n+1} + Q_{2n}^2 - (-1)^n\right)$$
$$Q_{n+1}^2 + 4Q_n^2 + Q_{n-1}^2 = 2Q_{2n+1} + Q_{2n}^2 + (-1)^n$$
$$P_{n+1}^4 + 16P_n^4 + P_{n-1}^4 = \frac{1}{8}\left(2Q_{2n+1} + Q_{2n}^2 - (-1)^n\right)^2$$
$$Q_{n+1}^4 + 16Q_n^4 - Q_{n-1}^4 = \frac{1}{2}\left(2Q_{2n+1} + Q_{2n}^2 + (-1)^n\right)^2$$

3.6 Candido's Identity and the Pell Family

Let x and y be any two real numbers. Consider the following square

where $AB = x^2, BC = y^2, CD = (x+y)^2$.

Candido observed that

$$\left(x^2+y^2+(x+y)^2\right)^2 = 2\left(x^4+y^4+(x+y)^4\right).$$

That is $(x^2+y^2+(x+y)^2)^2$ which equal to $2\,(x^4+y^4+(x+y)^4)$.

By substitution $x = P_{n-2},\ y = 2P_{n-1}$ into Candido's identity.

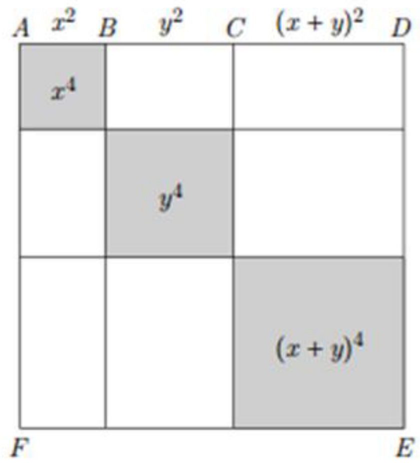

Figure 3.1: Pascalâ*es* Triangle

We give the following Pell numbers identity

$$(P_{(n-2)}^2 + 4P_{(n-1)}^2 + P_n^2)^2 = 2(P_{(n-2)}^4 + 16P_{(n-1)}^4 + P_n^4). \tag{3.2}$$

Numerical illustration

Let n=6. Then

$$\begin{aligned} LHS &= (P_3^2 + 4P_4^2 + P_5^2)^2 \\ &= (5^2 + 4(12)^2 + (29)^2)^2 \\ &= (25 + 576 + 841)^2 \\ &= (1442)^2 \\ &= 2079364. \end{aligned}$$

and

$$\begin{aligned} \text{RHS} &= 2\left(P_3^4 + 16P_4^4 + P_5^4\right) \\ &= 2\left(5^4 + 16 \cdot 12^4 + 29^4\right) \\ &= 2(625 + 16 \cdot 20736 + 707281) \\ &= 2(625 + 331776 + 707281) \\ &= 2 \cdot 1039682 \\ &= 2079364. \end{aligned}$$

Thus, LHS = RHS, confirming the identity.

Also by substitution $x = Q_{n-2},\ y = 2Q_{n-1}$ into Candido's identity.

We give

$$\left(Q_{n+2}^2 + 4Q_{n-1}^2 + Q_n^2\right)^2 = 2\left(Q_{n-2}^4 + 16Q_{n-1}^4 + Q_n^4\right).$$

Numerical Verification

For $n = 6$, we compute both sides of the identity:

$$\begin{aligned} LHS = \left(Q_3^2 + 4Q_4^2 + Q_5^2\right)^2 &= \left(7^2 + 4 \times 17^2 + 41^2\right)^2 \\ &= (49 + 4 \times 289 + 1681)^2 \\ &= (49 + 1156 + 1681)^2 \\ &= 2886^2 \\ &= 8328996. \end{aligned}$$

$$\begin{aligned} RHS = 2\left(Q_3^4 + 16Q_4^4 + Q_5^4\right) =& 2\left(7^4 + 16 \times 17^4 + 41^4\right) \\ =& 2(2401 + 16 \times 83521 + 2825761) \\ =& 2(2401 + 1,336336 + 2825761) \\ =& 2 \times 4164498 \\ =& 8328996. \end{aligned}$$

The equality LHS = RHS = 8328996 confirms the validity of identity for $n = 5$.

3.7 Pell Determinants

Determinants involving Pell and Pell-Lucas numbers can be evaluated algebraic techniques combined with Pell recurrence relations and iden Consider the following determinant:

$$D = \begin{vmatrix} p_{n+3} & P_{n+2} & P_{n+1} & p_n \\ P_{n+2} & p_{n+3} & p_n & P_{n+1} \\ P_{n+1} & p_n & p_{n+3} & P_{n+2} \\ p_n & P_{n+1} & P_{n+2} & p_{n+3} \end{vmatrix}.$$

This determinant resembles a special form first evaluated in 1866
Applying this formula to our Pell determinant, we obtain: $D = 16P_{2n+2}P_{2n+4}$

Proof.

$$D = (P_{n+3} \; + P_{n+2} + P_{n+1} + P_n)\,(P_{n+3} + P_{n+2} - P_{n+1} - P_n) \times (P_{n+3} -$$
$$+ P_{n+1} - P_n)\,(P_{n+3} - P_{n+2} - P_{n+1} + P_n)\,.$$

We now simplify each factor:

1.First Factor

$$P_{n+3} + P_{n+2} + P_{n+1} + P_n = (P_{n+3} + P_{n+1}) + (P_{n+2} + P_n$$
$$2P_{n+2} + 2P_{n+1} = 2Q_{n+2}.$$

2.Second Factor

$$P_{n+3} + P_{n+2} - P_{n+1} - P_n = 2P_{n+2} + 2P_{n+1} = 2Q_{n+2}.$$

3.Third Factor

$$P_{n+3} - P_{n+2} + P_{n+1} - P_n = 2P_{n+1} + 2P_n = 2Q_{n+1}.$$

4.Fourth Factor

$$P_{n+3} - P_{n+2} - P_{n+1} + P_n = 2P_n - 2P_{n-1} = 2\,(P_n - P_{n-1})\,.$$

Combining these simplified factors:

$$D = (2Q_{n+2})\,(2Q_{n+2})\,(2Q_{n+1})\,(2\,(P_n - P_{n-1}))\,.$$

Further simplification yields:

$$D = 16P_{2n+2}P_{2n+4}.$$

Similarly for

$$T = \begin{vmatrix} Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+2} & Q_{n+3} & Q_n & Q_{n+1} \\ Q_{n+1} & Q_n & Q_{n+3} & Q_{n+2} \\ Q_n & Q_{n+1} & Q_{n+2} & Q_{n+3} \end{vmatrix}.$$

We obtain $T = 64P_{2n+2}P_{2n+4}$.

Remark 3.2. *The determinant of the closed matrix by P equals the determinant by Q multiple by* $1|4$.

Chapter 4

Pell Sums and Products

4.1 Pell and Pell-Lucas Sum

Telescoping sums, along with fundamental identities, and the Principle of Mathematical Induction (PMI), can be employed to derive various summation formulas related to the Pell and Pell-Lucas sequences.

Several examples of these formulas include

$$\sum_{i=1}^{n} P_i = \frac{Q_{n+1}-1}{2} \tag{4.1}$$

$$\sum_{i=1}^{n} Q_i = P_{n+1}-1$$

$$\sum_{i=1}^{n} P_{2i-1} = \frac{P_{2n}}{2}$$

$$\sum_{i=1}^{n} P_{2i} = \frac{P_{2n+1}-1}{2}$$

$$\sum_{i=1}^{n} Q_{2i-1} = \frac{Q_{2n}-1}{2} \tag{4.2}$$

$$\sum_{i=1}^{n} Q_{2i} = \frac{Q_{2n+1}-1}{2}.$$

Example 4.1. *Let* $a_n = \left(2\sum_{i=0}^{n} Q_i\right)^2 - 2\sum_{i=0}^{n} Q_{2i+1}$, *where* i ≥ 0. *Evaluate* $\sum_{i=0}^{\infty} \frac{a_n}{n!}$.

Solution 4.1. *Using formula (4.1),* $\sum_{i=0}^{n} Q_i = P_{n+1}$. *By formula (4.2),*
$2\sum_{i=0}^{n} Q_{2i+1} = Q_{2n+2} - 1$.

We can rewrite this as $2\sum_{i=0}^{n} Q_{2i+1} = 4P_{n+1}^2 - 1 - (-1)^n$*. Thus*

$$\begin{aligned} a_n &= 4P_{n+1}^2 - \left[4P_{n+1}^2 - 1 - (-1)^n\right] \\ &= 1 + (-1)^n \\ \sum_{i=0}^{\infty} \frac{a_n}{n!} &= \sum_{i=0}^{n} \frac{1}{n!} + \sum_{\mathrm{i}=0}^{\infty} \frac{(-1)^{\mathrm{n}}}{\mathrm{n}!} \\ &= \mathrm{e} + \frac{1}{e} \\ &\approx 3.08616126963. \end{aligned}$$

4.2 Infinite Pell and Pell-Lucas sums

Using the identities we have developed thus far, we can evaluate infinite sums involving members of the Pell family . The next two examples illustrate this .

Example 4.2. *Evaluate the infinite sum* $\sum_{n=1}^{\infty} \frac{Q_n}{P_{n+1}P_n}$.

Solution 4.2. *Since* $Q_n = P_{n+1} - P_n$*, we have*

$$\begin{aligned} \frac{Q_n}{P_{n+1}P_n} &= \frac{P_{n+1} - P_n}{P_{n+1}P_n} \\ &= \frac{1}{P_n} - \frac{1}{P_{n+1}} \\ \sum_{n=1}^{k} \frac{Q_n}{P_{n+1}P_n} &= \sum_{n=1}^{k} \left(\frac{1}{P_n} - \frac{1}{P_{n+1}}\right) \\ &= 1 - \frac{1}{P_{k+1}} \\ \sum_{n=1}^{\infty} \frac{Q_n}{P_{n+1}P_n} &= \lim_{k\to\infty} \left(1 - \frac{1}{P_{k+1}}\right) \\ &= 1 - 0 \\ &= 1. \end{aligned}$$

Where we have used the telescoping sum $\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0$. Similarly, we can show that

$$\sum_{n=1}^{\infty} \frac{P_n}{Q_{n+1}Q_n} = \frac{1}{2}.$$

4.3 A Pell inequality

The next example feature a Pell inequality, studied by J.Diaz -Barrero and J. Egozcue of Barcelona, Spain in 2003. Although the inequality looks a bit overwhelming, the proof is a straightforward application of the binomial theorem, and the power series. $\frac{1}{(1-x)^{r+1}} = \sum_{n=0}^{\infty}\binom{n+r}{n}x^n$, which converges when |x| < 1.

Example 4.3. *Let m and n be positive integers. Prove that*

$$\sum_{k=0}^{n}\binom{m+k+1}{k+1}\left[\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}P_n^{j-k-1}\right] \leq P_n^{m+1}-1.$$

Proof. By the binomial theorem, we have

$$\begin{aligned}\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}P_n^{j-k-1} &= \frac{(-1)^{k+1}}{P_n^{k+1}}\sum_{j=0}^{k+1}\binom{k+1}{j}(-P_n)^j\\ &= \frac{(-1)^{k+1}}{P_n^{k+1}}(1-P_n)^{k+1}\\ &= (1-1|P_n)^{k+1}.\end{aligned}$$

Therefore, since $\frac{1}{P_n} \leq 1$, we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{m+k+1}{k+1}\left[\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}P_n^{j-k-1}\right] &= \sum_{k=0}^{n}\binom{m+k+1}{k+1}(1-1|P_n)^{k+1}\\ &= \sum_{r=1}^{n+1}\binom{m+r}{r}(1-1|P_n)^{r}\\ &\leq \sum_{r=1}^{\infty}\binom{m+r}{r}(1-1|P_n)^{r}\\ &= \frac{1}{[1-(1-1|P_n)]^{m+1}}-1\\ &= P_n^{m+1}-1, \text{ as claimed. When n = 1.}\end{aligned}$$

4.4 An Infinite Pell Product

The next example features an infinite Pell product, studied by M. Catalani of the university of turin, Italy , in 2004. The solution employs the Binet-like formula for

Q_n, and identity:

$$Q_n^2 = 2P_n^2 + (-1)^n; \text{ so } Q_{2^k}^2 = 2P_{2^k}^2 + 1, \text{ where k} \geq 1$$

Example 4.4. *evaluate* $\Pi_{k=1}^{\infty}\left(1+\frac{1}{\sqrt{2P_2^2k+1}}\right)$ *if it exists.*

Solution 4.3. *we have*

$$\begin{aligned}
\frac{1}{\sqrt{2P_{2^2}^2+1}} &= \frac{1}{\sqrt{Q_{2^2}^2}} \\
&= \frac{1}{Q_{2^k}} = \frac{2}{\gamma^{2^k+\delta^{2^k}}} \\
&= \frac{2}{\gamma^{2^k}\left[1+\delta^{2^k}(-\delta)^{2^k}\right]} \\
&= \frac{2(-\delta)^{2^k}}{1+\delta^{2^{k+1}}} \\
1+\frac{1}{\sqrt{2P_{2^k+1}^2}} &= 1+\frac{2\delta^{2^k}}{1+\delta^{2^{k+1}}} \\
&= \frac{\left(1+\delta^{2^k}\right)^2}{1+\delta^{2^{k+1}}} \\
\prod_{k=1}^{n}\left(1+\frac{1}{\sqrt{2P_{2^k}^2+1}}\right) &= \frac{\left(1+\delta^2\right)^2}{1+\delta^4}\cdot\frac{\left(1+\delta^4\right)^2}{1+\delta^8}\cdots\frac{\left(1+\delta^2\right)^2}{1+\delta^{2^{n+1}}} \\
&= \frac{1+\delta^2}{1+\delta^{2^{n+1}}}\cdot\left(1+\delta^2\right)\left(1+\delta^4\right)\ldots\left(1+\delta^{2^n}\right) \\
&= \frac{1+\delta^2}{1+\delta^{2^{n+1}}}\cdot\frac{1-\delta^{2^{n+1}}}{1-\delta^2} \\
&= \frac{1+\delta^2}{1-\delta^2}\cdot\frac{1-\delta^{2^{n+1}}}{1+\delta^{2^{n+1}}} \\
\prod_{k=1}^{\infty}\left(1+\frac{1}{\sqrt{2P_{2^k}^2+1}}\right) &= \frac{1+\delta^2}{1-\delta^2}\cdot\lim_{n\to\infty}\frac{1-\delta^{2^{n+1}}}{1+\delta^{2^{n+1}}}.
\end{aligned}$$

Since $|\delta| < 1, \lim_{n\to\infty}\frac{1-\delta^2 \;\; n+1}{1+\delta^2 \;\; n+1} = \frac{1-0}{1+0} = 1$; so the limit exists and hence the given infinite product converges. Consequently,

$$\begin{aligned}\prod_{k=1}^{\infty}\left(1+\frac{1}{\sqrt{2P_{2^2}^2+1}}\right) &= \frac{1+\delta^2}{1-\delta^2}\\ \frac{\delta^2(\delta-\gamma)}{\delta(-\gamma-\delta)} &= \frac{\gamma-\delta}{\gamma+\delta}\\ &= \frac{2\sqrt{2}}{2} = \sqrt{2}.\end{aligned}$$

Conclusion

After writing this modest memory, we came to the conclusion that Pell and Pell-Lucas numbers are a vast field characterized by an ocean of identities that are sometimes quite fascinating. It is always possible to find other identities in this field, simply by changing the tools used each time.

This work is therefore very interesting and promising, and we hope to continue our future research in this field.

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John Pell

John Pell (1 March 1611 â€“ 12 December 1685) was an English diplomat and mathematician.
John Pell was born in Southwick, Sussex. He entered Trinity College at the age of 13. By the age of 20, he knew 10 languages: English, German, Italian, Spanish, French, Latin, Greek, Hebrew, Arabic, and Dutch. His reputation and the influence of Sir William Boswell, the English con led to his election in 1644 to the chair of mathematics at the Athenaeum Illustre in Amsterdam, a post left vacant by the departure of Martin van den Hove to the University of Leiden. Between 1644 and 1646, he worked on a pamphlet against Longomontanus, seeking the help and testimony of Bonaventura Cavalieri, his patron Charles Cavendish, the Frenchman Descartes, Hobbes, Father Mersenne, Mydorge, and Roberval. This writing appeared under the title of Controversy with Longomontanus concerning the Quadrature of the Circle (1647). Pell arrived in Breda in 1646 at the invitation of Stadtholder Frederick Henry, and taught at the College of Orange in Breda until 1652. Pell then became aware of the imminence of a conflict between England and the United Provinces: he returned to England in July 1652. On his return, Oliver Cromwell appointed him professor of mathematics in London, before sending him to Switzerland, from 1654 to 1658, to negotiate the accession of the Confederation to a league of Protestant states in Europe, without success Upon his return to England, he entered the holy orders in 1661 and became rector of the grammar school at Fobbing in Essex.
In 1673, he met Leibnitz in London, to whom he informed him that some of the discoveries the German mathematician believed to be original had, in fact, already been published by FranÃ§ois Regnaud and Gabriel Mouton.

Edouard Pell-Lucas

Edouard Pell-Lucas, a French mathematician born in 1842 and died in 1891, is known for his. work on Lucas numbers and the Pell sequence, as well as for his primality test of Mersenne numbers. He also published books on mathematical recreations under a pseudonym Edouard Lucas introduced the Lucas sequence, which is a sequence of integers defined by a recurrence relation similar to the Fibonacci sequence, but with different initial conditions. Edouard Lucas also studied Pell numbers, which are a sequence of integers defined by a recurrence relation different from the Lucas sequence, and the Pell-Lucas sequence, which is a sequence associated with Pell numbers. Edouard Lucas also published mathematical recreation books, which were very popular in his time. These books were aimed at a wider audience and allowed people to explore mathematics in a fun and accessible way.

Leonardo Fibonacci

Leonardo Fibonacci lived between 1170 and 1250 AD. He was an Italian mathematician and one of the most prominent mathematicians of medieval Europe. Fibonacci is known for his contribution to the spread of Arabic numerals in Europe, particularly in his 13th-century book, The Book of Arithmetic In 1225 AD, he wrote a book considered one of his most important works, entitled "The Book of Squares," in which he selected many important mathematical problems, including obtaining the Pythagorean triple. In 1228 AD, he wrote a book in which he presented solutions to many mathematical problems.

ملخص

في وذا العمل، نتهاول المتتاليات الخطية المتكررة ذات المعاملات الثابتة من الرتبة ٢، مع التركيز على تلك التي تُع رف أعداد بيل وبيل لوكاس. وفي وذا السياق، درسها صيغ بيهيه الموافقة - وإمكانية أن يصبح عدد بيل أو بيل لوكاس عطواً في عائلة أخرى من الأعداد مثل عائلة الأعداد - المثلثية والخماسية والسباعية ... . استعرضها أيطاًا بعض المتطابقات التي تربط بين وذه الأعداد، وأنهيها عملها ببعض الصيغ التي تتطمن مجموع وذه الأعداد وضربها. .

كلمات مفتاحية

عدد بيل، عدد بيل لوكاس، صيغة بينيه، متطابقة.

Abstract

In this work, we consider the linear recurrent sequences with constant coefficients of order 2, focusing on those defining Pell and Pell-Lucas numbers. In this context, we have examined the corresponding Binet formulas and the possibilities for a Pell or Pell-Lucas number to become a member of another family such as the families of triangular, pentagonal, heptagonal numbers, ... We also reviewed some identities linking these numbers, and finished our work with some formulas involving the sum and multiplication of these numbers.

Key words

Pell number, Pell-Lucas number, Binet formulas, Identity.

Résumé

Dans ce travail, nous considérons les suites récurrentes linéaires à coefficients constants d'ordre 2, en nous concentrant sur celles définissant les nombres de Pell et de Pell-Lucas. Dans ce contexte, nous avons examiné les formules de Binet correspondantes et les possibilités pour qu'un nombre de Pell ou de Pell-Lucas deviennent membre d'autre famille tels que les familles des nombres triangulaires, pentagonales, heptagonales, ... Nous avons également passé en revue quelques identités reliant ces nombres et on a terminé notre travail par quelques formules faisant intervenir la somme et la multiplication de ces nombres.

Mot-clés

Nombre de Pell, Nombre de Pell-Lucas, Formule de Binet, Identité.