PEOPLEs DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf University of M'sila
Faculty of Mathematics and Computer sciences
Department of Mathematics

Master memory

Field : Mathematics and computer sciences.
Branch : Mathematics.
Option : Functional Analysis.

Theme

Lipschitz free spaces and the approximation property

Presented by :
AMEUR Norhane

Publicly supported on :June 11, 2026.

In front of the jury composed of :

ALOUANI Ahlem	Dr.	University of M'sila	**President**
ACHOUR Dahmane	Pr.	University of M'sila	**Supervisor**
MEKDOUR Fateh	Dr.	University of M'sila	**Examiner**
MEZRAG Lahcene	Pr.	University of M'sila (LFAGS)	**Invited**

University year 2025/2026

Thanks

In the name of Allah, the Most Gracious, the Most Merciful.

First and foremost, I thank Almighty Allah for His blessings and for granting me the strength and success to complete this work.

I sincerely thank my supervisor, Professor Achour Dahmane (University of Msila), for his guidance, support, and valuable advice throughout this research.

I am also grateful to the members of the jury: Dr. ALOUANI Ahlem (President), Dr. MEKDOUR Fateh (Examiner), and Professor MEZRAG Lahcene (Invited), for their time and for accepting to evaluate this work.

My heartfelt thanks go to all my professors at the Faculty of Mathematics and Computer Science, as well as to my colleagues in the Masters program in Functional Analysis.

Finally, I deeply thank my parents and family for their continuous love, support, and encouragement

Contents

Notation

- $\mathbb{R}$: field of real numbers
- $\mathbb{N}$: set of natural numbers
- (X, d) : a metric space
- $B(x, r)$: closed ball centered at x with radius r
- $B'(x, r)$: open ball centered at x with radius r
- $(E, \|\cdot\|)$: a normed space
- B_E : closed unit ball of E
- S_E : unit sphere of E
- E^* : dual space of E
- $\mathcal{L}(E, F)$: space of bounded linear operators from E to F
- X, Y : points of metrice space
- $Lip(X, Y)$: space of Lipschitz mappings from X to Y
- $Lip_0(X)$: space of Lipschitz functions vanishing at the base point
- $Lip(f)$: Lipschitz constant of f
- δ_x : evaluation functional at x

- $\mathcal{F}(X)$: Lipschitz-free space over X
- T_L : linearization of a Lipschitz mapping T
- BAP : bounded approximation property
- MAP : metric approximation property

Introduction

Approximation properties constitute one of the central themes of Banach space theory. Their systematic study began with the pioneering work of Grothendieck [12], who introduced several variants of the classical approximation property and established deep connections between finite-rank operators, tensor products, and the geometry of Banach spaces. Since then, approximation properties have become a fundamental tool in the study of linear operators and Banach space structure. For a comprehensive account of these developments, we refer the reader to the survey of Casazza [3]. The emergence of Lipschitz-free spaces has provided a powerful framework for extending linear methods to nonlinear settings. Introduced by Arens and Eells and further developed by many authors, Lipschitz-free spaces allow metric spaces to be studied through associated Banach spaces while preserving their Lipschitz structure. In particular, they establish a bridge between nonlinear Lipschitz mappings and linear operators, making it possible to apply techniques from Banach space theory to metric spaces. A major breakthrough in this direction was achieved by Godefroy and Kalton [8], who proved that, for every Banach space E, the Lipschitz bounded approximation property is equivalent to the classical bounded approximation property. This result highlighted the fundamental role of Lipschitz-free spaces in the study of approximation properties. Subsequently, Godefroy and Ozawa [9] characterized metric spaces whose Lipschitz-free spaces possess the bounded approximation property through a Lipschitz version of the local reflexivity principle. More recently, Achour et al. [1] extended several concepts and results concerning approximation properties from Banach spaces to the broader setting of metric spaces. Another important contribution was obtained by Liu et al. [15], who investigated the approximation of Lipschitz mappings by finite-rank operators and provided a new approach to the theorem of Godefroy and Kalton. Motivated by these developments, the present dissertation is devoted to the study of Lipschitz-free spaces and approximation properties in the Lipschitz setting. Particular attention is given to the relationship between Lipschitz approximation properties and the linear structure of Lipschitz-free spaces. Building upon the work of Godefroy and Kalton, we obtain new characterizations of the Lipschitz bounded approximation property, establish transfer principles between Lipschitz and linear approximations, and derive finite-rank reformulations that provide a more concrete description of these properties. This dissertation is organized into three chapters. Chapter 1 contains the preliminary notions, notation, and basic results that will be used throughout the thesis.The material presented in this chapter is largely based on the standard references [7, 14, 19, 21, 22]. Chapter 2 is devoted to the study of Lipschitz-free spaces and is mainly based on the works[2, 4, 5, 8, 22]. In particular, we prove that the dual space of $\mathcal{F}(X)$ is isometrically isomorphic to operator name $Lip_0(X)$, present several classical examples of Lipschitz-free spaces, introduce adjoint operators in this framework, and develop the linearization theorem together with some of its fundamental consequences. Finally, Chapter 3 is devoted to approximation properties and their applications to Lipschitz-free spaces. In particular, we investigate structural aspects of the Lipschitz bounded approximation property, establish new characterizations in terms of finite subsets, prove transfer principles from finite-dimensional Lipschitz approximations to finite-rank linear approximations on Lipschitz-free spaces, and obtain a finite-rank approximation principle that yields a linear reformulation of the Lipschitz bounded approximation property.

CHAPTER 1

PRELIMINARIES

1.1 The Notion of metric space

This chapter provides a review of the fundamental concepts and preliminary results from metric spaces and Lipschitz functions that will be used throughout the remainder of this work. The content of this chapter is primarily based on the references [7, 14, 19, 21, 22]

Definition 1.1.1. Let X be a non empty set . We say that d is a distance on X if d is an application from X^2 into $\mathbb{R}_+$ such that for all x, y, z in X, we have

(i) $d(x,y) = 0 \Longleftrightarrow x = y$ (separation),

(ii) $d(x,y) = d(y,x)$ (symmetry),

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ (triangular inequality).

The space X equipped with d is called metric space (X,d).

Definition 1.1.2. Let (X,d) be a metric space, $x \in X$ and $r > 0$. The open ball $B'(x,r)$ and the closed ball $B(x,r)$ with center x and radius r are defined by

$$B'(x,r) = \{y \in X : d(x,y) < r\},$$

and

$$B(x,r) = \{y \in X : d(x,y) \leq r\},$$

respectively. The *sphere* $S(x,r)$ is defined by

$$S(x,r) = \{y \in X : d(x,y) = r\}.$$

Definition 1.1.3. Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called *Cauchy* if for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$d(x_n, x_m) < \epsilon \quad \text{for all } m, n \in \mathbb{N} \text{ with } m, n \geq n_\epsilon,$$

or, equivalently,

$$d(x_n, x_{n+k}) < \epsilon \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq n_\epsilon \text{ and all } k \in \mathbb{N}.$$

Definition 1.1.4. The metric space (X, d) is called *complete* if every Cauchy sequence in (X, d) is convergent.

Compactness in Metric Spaces

Definition 1.1.5. A subset A of metric space (X, d) is called *totally bounded* if for each $\epsilon > 0$ there exist $n_\epsilon \in \mathbb{N}$ and $x_1, \ldots, x_{n_\epsilon} \in X$ such that

$$A \subseteq B'(x_1, \epsilon) \cup B'(x_2, \epsilon) \cup \cdots \cup B'(x_{n_\epsilon}, \epsilon).$$

The family $x_1, \ldots, x_{n_\epsilon}$ is called a *finite ϵ-net* for A.
Notice that one obtains the same notion if instead of the open balls $B'(x_i, \epsilon)$ one takes closed balls $B(x_i, \epsilon)$.

Theorem 1.1.6. *Let (X, d) be a metric space and $A \subseteq X$.*

1. *The set A is totally bounded if and only if every sequence in A contains a Cauchy subsequence.*

2. *The set A is compact if and only if it is totally bounded and complete.*

Corollary 1.1.7. *A subset of a complete metric space is relatively compact if and only if it is totally bounded*

Proposition 1.1.8. *Let (X, d) be a metric space. For every $x, y \in X$ and every nonempty subset $C \subseteq X$, one has*

$$|d(x, C) - d(y, C)| \leq d(x, y),$$

where

$$d(x, C) := \inf_{z \in C} d(x, z).$$

Definition 1.1.9. (Equivalent Metrics) Let d_1 and d_2 be two metrics on a set X. The metrics d_1, d_2 are called: - Topologically equivalent if they induce the same topology on X equivalent if there exist two numbers $\alpha, \beta > 0$ such that

$$\alpha d_1(x,y) \le d_2(x,y) \le \beta d_1(x,y),$$

for all $x, y \in X$.

Proposition 1.1.10. *Let d_1 and d_2 be two metrics on a set X. The metrics d_1, d_2 are topologically equivalent if and only if*

$$x_n \xrightarrow{d_1} x \quad \Longleftrightarrow \quad x_n \xrightarrow{d_2} x$$

for every sequence (x_n) in X.

In the following proposition we give some characterizations of continuity and uniform continuity in the metric case.

Proposition 1.1.11. *Let (X,d) and (Y,d') be metric spaces and $f : X \to Y$ a function.*

(1) The function f is continuous at a point $x_0 \in X$ if and only if

$$x_n \to x_0 \implies f(x_n) \to f(x_0), \tag{1.1}$$

for each sequence (x_n) in X.

(2) The function f is continuous on X if and only if

$$d(x,A) = 0 \implies d'(f(x), f(A)) = 0,$$

for all $x \in X$ and nonempty subsets A of X.

(3) The function f is uniformly continuous on X if and only if

$$d(x_n, y_n) \to 0 \implies d'(f(x_n), f(y_n)) \to 0,$$

for all sequences $(x_n), (y_n)$ in X.

$$d(A,B) = 0 \implies d'(f(A), f(B)) = 0,$$

for all nonempty subsets A, B of X.

1.2 Series in Normed Spaces

Definition 1.2.1. A normed space is a pair $(E, \|\cdot\|)$, where E is a vector space and $\|\cdot\|$ a norm on E. The topology of E is generated by the metric $d(x,y) = \|y - x\|$, $x, y \in E$. A complete normed space is called a *Banach space*.

For a normed space $(E, \|\cdot\|)$ one uses the following notations:

Definition 1.2.2.

- $B_E = \{x \in E : \|x\| \leq 1\}$ — the closed unit ball,
- $B'_E = \{x \in E : \|x\| < 1\}$ — the open unit ball,
- $S_E = \{x \in E : \|x\| = 1\}$ — the unit sphere of E.

It follows that

$$B(x_0, r) = x_0 + rB_E \quad \text{and} \quad B'(x_0, r) = x_0 + rB'_E.$$

for $x_0 \in X$ and $r > 0$.

Let E, F be normed spaces and $T : E \to F$ a linear operator. Then T is continuous if and only if there exists a nonnegative real number C such that

$$\|T(x)\| \leq C\|x\|, \tag{1.2}$$

for all $x \in E$. We uses the following notations:

$$\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|. \tag{1.3}$$

We denote by $\mathcal{L}(E, F)$ continuous linear operators.

Proposition 1.2.3. *Let E, F be normed spaces.*

(1) The functional $\|\cdot\| : \mathcal{L}(E,F) \to [0, \infty)$ given by equation (1.2) is a norm on the space $\mathcal{L}(E,F)$.

(2) The normed space $\mathcal{L}(E,F)$ is complete (i.e., is a Banach space) if and only if F is a Banach space. In particular, the dual space $E^* = \mathcal{L}(E, \mathbb{K})$ of X is always a Banach space.

Let E be a normed space and $\sum x_n$ a (formal) series in E.

Definition 1.2.4. The series $\sum x_n$ is called:

i) *convergent* (or *norm convergent*) if there exists $x \in E$ such that

$$\lim_{n\to\infty}\left\|\sum_{k=1}^{n} x_k - x\right\| = 0,$$

i.e., the sequence $(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$ of partial sums is norm-convergent to x,

ii) *absolutely convergent* if the series $\sum_{n=1}^{\infty}\|x_n\|$ is convergent,

iii) *weakly convergent* if there exists $x \in E$ such that

$$x^*(x) = \sum_{n=1}^{\infty} x^*(x_n)$$

for every $x^* \in E^*$.

Remark 1.2.5. If the series $\sum_n x_n$ is convergent. Then, $\lim_{n\to\infty}\sum_{k=n}^{\infty} x_k = 0$.
Also, the uniqueness of the sum follows for any of the considered kinds of convergence (except the absolute one).

Concerning the absolute convergence we mention the following result.

Proposition 1.2.6. *A normed space $(E, \|\cdot\|)$ is a Banach space if and only if every absolutely convergent series in E is convergent.*

1.3 The Lipschitz spaces

In this section, we begin by introducing the notions of Lipschitz functions and base-point-preserving functions.

Definition 1.3.1. Let (X, d) and (Y, d') be metric spaces. A function $f : X \to Y$ is called

i) Lipschitz if there exists $C \geq 0$ such that

$$d'(f(x), f(y)) \leq Cd(x, y) \quad \text{for all } x, y \in X, \tag{1.4}$$

ii) isometric embedding if $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in X$

The constant C inequation (1.4) is called a *Lipschitz constant* for f .

We define the *Lipschitz seminorm* $Lip(\cdot)$ of a Lipschitz function $f : X \to Y$ by

$$Lip(f) := inf\{C \geq 0 : C \text{ is a Lipschitz constant for } f\}. \tag{1.5}$$

Proposition 1.3.2. *Let (X, d), (Y, d') be metric spaces and $f : X \to Y$ a Lipschitz function. Then*

$$Lip(f) = \sup \left\{ \frac{d'(f(x), f(y))}{d(x, y)} : x, y \in X,\ x \neq y \right\}. \tag{1.6}$$

The Lipschitz seminorm $Lip(\cdot)$ is the least Lipschitz constant for f, i.e.,

(i) $d'(f(x), f(y)) \leq Lip(f)d(x, y)$ for all $x, y \in X$,

(ii) if C is a Lipschitz constant for f, then $Lip(f) \leq C$.

Proof. Denoting by $\mathcal{H}(f)$ the set of all Lipschitz constants for f, it follows that

$$Lip(f) = \inf \mathcal{H}(f).$$

Denote also by α the supremum in the right hand side of equation(1.6). If $C \in \mathcal{H}(f)$, then

$$\frac{d'(f(x), f(y))}{d(x, y)} \leq C$$

for all $x, y \in X$, $x \neq y$. It follows that $\alpha \leq C$ for every $C \in \mathcal{H}(f)$, and so $\alpha \leq Lip(f)$.

By the definition of α,

$$d'(f(x), f(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$, i.e., $\alpha \in \mathcal{H}(f)$ and so $Lip(f) \leq \alpha$.

Consequently,

$$\alpha = Lip(f).$$

The assertions (i) and (ii) follow from the relations $\alpha \in \mathcal{H}(f)$ and $\alpha = \inf \mathcal{H}(f)$. □

We establish some algebraic operations preserve the Lipschitz property of functions.

Proposition 1.3.3. *Let (X, d), (Y, d'), and (Z, d'') be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are Lipschitz, then $g \circ f : X \to Z$ is also Lipschitz. Moreover, the following inequality holds:*

$$Lip(g \circ f) \leq Lip(g)Lip(f).$$

Proof. For all $x \neq y \in X$, we have

$$\begin{aligned} Lip(g \circ f) &= \sup_{x \neq y} \frac{d''(g \circ f(x), g \circ f(y))}{d(x,y)} \\ &\leq Lip(g) \sup_{x \neq y} \frac{d'(f(x), f(y))}{d(x,y)} \\ &\leq Lip(g) Lip(f). \end{aligned}$$

□

Proposition 1.3.4. *Let (X,d), (Y,d'), be a metric space, E be a Banach space and let $f, g : X \to E$ be Lipschitz functions.Then*

a) *For any $\alpha \in \mathbb{R}$, the function αf is Lipschitz and*

$$Lip(\alpha f) = |\alpha|\, Lip(f).$$

b) *The function $f + g$ is Lipschitz and*

$$Lip(f + g) \leq Lip(f) + Lip(g).$$

Proof. a) For the scalar multiple, for all $x \neq y \in X$X,

$$\begin{aligned} \mathrm{Lip}(\alpha f) &= \sup_{x \neq y} \frac{\|\alpha f(x) - \alpha f(y)\|}{d(x,y)} \\ &= |\alpha| \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x,y)} = |\alpha| \mathrm{Lip}(f). \end{aligned}$$

b) For all $x \neq y \in X$, we have

$$\begin{aligned} Lip(f + g) &= \sup_{x \neq y} \frac{\|(f+g)(x) - (f+g)(y)\|}{d(x,y)} \\ &\leq \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x,y)} + \sup_{x \neq y} \frac{\|g(x) - g(y)\|}{d(x,y)} \\ &\leq Lip(f) + Lip(g). \end{aligned}$$

□

We investigate when the limit of a sequence of Lipschitz functions is Lipschitz.

Proposition 1.3.5. *Let (X,d), (Y,d') be metric spaces, and let $(f_n)_n$ be a sequence of Lipschitz mappings $f_n : X \to Y$. Assume that there exists $M \geq 0$ such that*

$$Lip(f_n) \leq M, \textit{ for all } n \in \mathbb{N}.$$

If, for every $x \in X$ there exists,

$$f(x) := \lim_{n\to\infty} f_n(x),$$

then f is Lipschitz with $Lip(f) \leq M$, and $(f_n)_n$ converge to f uniformly on every totally bounded subset of X.

In particular, if X is compact, then $(f_n)_n$ converge uniformly to f on X.

Proof. For all $x, y \in X$ and all $n \in \mathbb{N}$,

$$d'(f_n(x), f_n(y)) \leq Lip(f_n)d(x, y) \leq Md(x, y),$$

Passing to the limit as $n \to \infty$, we obtain

$$d'(f(x), f(y)) = \lim_{n\to\infty} d'(f_n(x), f_n(y)) \leq Md(x, y),$$

for all $x, y \in X$. Hence f is Lipschitz and $Lip(f) \leq M$.

Now let $K \subset X$ be totally bounded and let $\varepsilon > 0$. Choose a finite ε-net $\{x_1, \dots, x_m\} \subset K$, that is,

$$\forall x \in K,\ \exists k \in \{1, \dots, m\} \text{ such that } d(x, x_k) \leq \varepsilon.$$

Since $f_n(x_k) \to f(x_k)$ for each $k = 1, \dots, m$, there exists $n_0 \in \mathbb{N}$ such that

$$d'(f_n(x_k), f(x_k)) \leq \varepsilon, \quad k = 1, \dots, m,$$

for all $n \geq n_0$.

Let $x \in K$ and choose $k \in \{1, \dots, m\}$ such that $d(x, x_k) \leq \varepsilon$. Then, for all $n \geq n_0$,

$$\begin{aligned} |d'(f_n(x), f(x))| &\leq |d'(f_n(x), f_n(x_k))| + |d'(f_n(x_k), f(x_k))| + |d'(f(x_k), f(x))| \\ &\leq Lip(f_n)d(x, x_k) + \varepsilon + Lip(f)d(x, x_k) \\ &\leq M\varepsilon + \varepsilon + M\varepsilon = (2M + 1)\varepsilon \end{aligned}$$

Since $x \in K$ was arbitrary, this proves that $(f_n)_n$ converges uniformly to f on K. If X is compact, then it is totally bounded, and the convergence is uniform on X. □

Remark 1.3.6. The above proposition may fail if the hypotheses are weakened by requiring only that each f_n is Lipschitz, even if the sequence (f_n) converges uniformly. Indeed, for each $n \in \mathbb{N}$, consider the function $f_n : [0, \infty) \to \mathbb{R}$ defined by

$$f_n(x) = \sqrt{x + \frac{1}{n}}, \qquad x \in [0, \infty).$$

Then:

(i) The sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to the function $f : [0,\infty) \to \mathbb{R}$ given by

$$f(x) = \sqrt{x}, \quad x \in [0,\infty);$$

(ii) Each f_n is Lipschitz and

$$Lip(f_n) = \frac{\sqrt{n}}{2} \text{ for all } n \in \mathbb{N}$$

(iii) The limit function f is not Lipschitz.

Proof. (i) For all $x \in [0,\infty)$ and $n \in \mathbb{N}$,

$$|f_n(x) - f(x)| = \left|\sqrt{x+\frac{1}{n}} - \sqrt{x}\right| = \frac{1}{n}\,\frac{1}{\sqrt{x+\frac{1}{n}}+\sqrt{x}} \leq \frac{1}{\sqrt{n}},$$

Hence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

(ii) The derivative of f_n is

$$f_n'(x) = \frac{1}{2\sqrt{x+\frac{1}{n}}},\ x > 0.$$

Thus,

$$\sup\{|f_n'(x)| : x > 0)\} = \frac{\sqrt{n}}{2}.$$

It follows that f_n is Lipschitz and

$$Lip(f_n) = \frac{\sqrt{n}}{2}.$$

(iii) The derivative of f is

$$f'(x) = \frac{1}{2\sqrt{x}},\ x > 0,$$

and

$$\sup\{|f'(x)| : x > 0)\} = \infty.$$

Therefore, f is not Lipschitz on $(0,\infty)$. $\square$

We now intrduce a Banach space structre on the space of lipschitz functions definded on metric space and taking values in a Banach space. For a metric space X and a normed space E, we denote by

$$\text{Lip}(X,Y) = \{f : X \to E : f \text{ is Lipschitz on } X\} \tag{1.7}$$

the space of all Lipschitz functions from X to E.

As we have remarked the functional

$$Lip(f) = \sup\{\frac{\|f(x) - f(y)\|}{d(x, y)} : x, y \in X, x \neq y\} \tag{1.8}$$

is only a seminorm on $\mathrm{Lip}(X, E)$, as it vanishes on constant functions.

Definition 1.3.7. A *base point* (or *distinguished point*) in a metric space (X, d) is a fixed element $e_X \in X$. A metric space with a specified base point e_X is called a *pointed metric space*. If X is a normed space then one always takes $e_X = 0$. We shall mention this by saying that (X, e_X) (or (X, d, e_X)) is a *pointed metric space*.

Definition 1.3.8. Let X and Y be pointed metric spaces, with distinguished base points e_X and e_Y respectively. A mapping $f : X \to Y$ is said to preserve the base point if $f(e_X) = e_Y$.

If X, Y are pointed metric spaces with base points e_X, e_Y, respectively, then one denotes by

$$\mathrm{Lip}_0(X, Y) = \{f : X \to Y : f \text{ is Lipschitz on } X \text{ and } f(e_X) = e_Y\}$$

the set of all Lipschitz mappings from X to Y preserving the base points.

If $Y = \mathbb{K}$, then

$$\mathrm{Lip}_0(X, \mathbb{K}) = \{f : X \to \mathbb{K} : f \text{ is Lipschitz on } X \text{ and } f(e_X) = 0\}.$$

In this case, we simply write Lip_0 instead of $\mathrm{Lip}_0(X, \mathbb{K})$.

When necessary, it will be specified that we work with $\mathbb{R}$.

Example 1.3.9. Let X be a metric space and let $C \subseteq X$ be a closed subset. Consider the function

$$f : X \to \mathbb{R}; x \longmapsto f(x) = d(x, C).$$

Then f is a Lipschitz function. First, note that

$$f(x) = 0 \quad \text{if and only if} \quad x \in C.$$

Indeed, since C is closed, the distance from a point to C vanishes precisely on C.

Now, let $x, y \in X$. By Proposition 1.1.8, we have

$$|f(x) - f(y)| = |d(x, C) - d(y, C)| \leq d(x, y).$$

Therefore, f is Lipschitz, and its Lipschitz constant satisfies

$$Lip(f) \leq 1.$$

The following proposition shows that the functional $Lip(\cdot)$ defined by (1.8), is a norm on $\mathrm{Lip}_0(X,E)$. Consequently, $\mathrm{Lip}_0(X,E)$ become a normed vector space.

Proposition 1.3.10. *The mapping*

$$Lip_0(\cdot) : Lip_0(X,E) \longrightarrow \mathbb{R}, \qquad f \longmapsto Lip(f),$$

defines a norm on $Lip_0(X,E)$.

Proof. Let $\lambda \in \mathbb{K}$ and $f, g \in Lip_0(X,E)$. By Proposition 1.3.4, we have

$$Lip(\lambda f) = |\lambda|\, Lip(f).$$

and

$$Lip(f+g) \leq Lip(f) + Lip(g).$$

Therefore, it remains to verify that

$$Lip(f) \geq 0$$

and

$$Lip(f) = 0 \Longleftrightarrow f = 0.$$

For every $f \in Lip_0(X,E)$ and every $x, y \in X$.

we have

$$d(x,y) \geq 0 \text{ and } d(f(x), f(y)) \geq 0.$$

Hence,

$$Lip(f) = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x,y)} \geq 0.$$

Assume now that $Lip(f) = 0$.

Since f is Lipschitz, for all $x, y \in X$,

$$0 \leq d(f(x), f(y)) \leq Lip(f).d(x,y) = 0.$$

Consequently,

$$d(f(x), f(y)) = 0,$$

which implies that

$$f(x) = f(y) \quad \text{for all} \quad x, y \in X.$$

Thus, f is constant.

Since f preserves the base point ($f(e_X) = 0$), it follows that :

$$f(x) = f(e_X) = 0 \text{ for all } x \in X,$$

and therefore $f = 0$.

Conversely, if $f = 0$, then $f(x) = f(y) = 0$ for all $x, y \in X$.

Hence,

$$d(f(x), f(y)) = 0 \leq Cd(x, y)$$

for every $C \geq 0$. It follows that the smallest Lipschitz constant of f is 0, that is, $Lip(f) = 0$.

Therefore,

$$Lip(f) = 0 \Longleftrightarrow f = 0,$$

and we conclude that $Lip(\cdot)$ defines a norm on $\mathrm{Lip}_0(X, E)$. □

The next theorem establishes that $\mathrm{Lip}_0(X, E)$ is a Banach space.

Theorem 1.3.11. *Let X be a pointed metric space and E be a Banach space. Then the space $(Lip_0(X, E), Lip(\cdot))$ is a Banach space.*

Proof. By Proposition 1.2.6, let $(f_n)_{n=1}^{\infty}$ be a sequence in $\mathrm{Lip}_0(X, E)$ such that

$$\sum_{n=1}^{\infty} Lip(f_n) < \infty.$$

We show that the series $\sum_{n=1}^{\infty} f_n$ converges with respect to the norm $Lip(\cdot)$.

For every $x \in X$ and $n \in \mathbb{N}$, we have

$$\|f_n(x)\| = \|f_n(x) - f_n(e)\| \leq Lip(f_n)\, d(x, e).$$

Hence,

$$\sum_{n=1}^{\infty} \|f_n(x)\| \leq \left(\sum_{n=1}^{\infty} Lip(f_n) \right) d(x, e) < \infty.$$

Thus, the series $\sum_{n=1}^{\infty} f_n(x)$ converges in E for every $x \in X$, and we may define a function $f : X \to E$ by

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

We now show that $f \in Lip_0(X, E)$. First,

$$f(e_X) = \sum_{n=1}^{\infty} f_n(e_X) = \sum_{n=1}^{\infty} 0 = 0.$$

Moreover, for all $x, x' \in X$, we have

$$\|f(x) - f(x')\| = \left\| \sum_{n=1}^{\infty} (f_n(x) - f_n(x')) \right\| \leq \sum_{n=1}^{\infty} \|f_n(x) - f_n(x')\| \leq \left(\sum_{n=1}^{\infty} Lip(f_n) \right) d(x, x').$$

Therefore, f is Lipschitz.

Now we prove convergence in the norm $Lip(\cdot)$. Let

$$S_k = \sum_{n=1}^{k} f_n.$$

Then $S_k \in Lip_0(X, E)$ for all $k \in \mathbb{N}$. Since

$$\sum_{n=1}^{\infty} Lip(f_n) < \infty,$$

for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=k+1}^{\infty} Lip(f_n) < \varepsilon \quad \text{for all } k \geq n_0.$$

Hence,

$$Lip(f - S_k) = Lip\left(\sum_{n=k+1}^{\infty} f_n \right) \leq \sum_{n=k+1}^{\infty} Lip(f_n) < \varepsilon.$$

Thus, $S_k \to f$ in the norm $Lip(\cdot)$, which completes the proof. □

The following theorem was obtained by McShane [16, Theorem 1], see also Whitney [23, p. 63], and is commonly called the McShane-Whitney extension theorem.

Theorem 1.3.12 (McShane). *Let $M \subset X$ and $f \in Lip(M)$. Then there is $\widetilde{f} \in Lip(X)$ such that*

$$\widetilde{f}|_M = f, \qquad Lip(\widetilde{f}) = Lip(f),$$

$$\sup_{x \in X} \widetilde{f}(x) = \sup_{x \in M} f(x)$$

and

$$\inf_{x \in X} \widetilde{f}(x) = \inf_{x \in M} f(x).$$

1.4 Differentiability of Lipschitz Functions

The aim of this section is to establish that every Lipschitz function is differentiable almost everywhere.

Definition 1.4.1. A function

$$f : I = [a, b] \to \mathbb{C}$$

is said to be absolutely continuous on I if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $n \in \mathbb{N}$ and for every disjoint family of intervals of I,

$$]\alpha_1, \beta_1[, \ldots,]\alpha_i, \beta_i[, \ldots,]\alpha_n, \beta_n[,$$

satisfying

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta,$$

we have

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < \varepsilon.$$

Theorem 1.4.2. *Let $I = [a, b]$ be an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ an increasing absolutely continuous function on I. Then f is differentiable almost everywhere on I.*

Proposition 1.4.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then f is absolutely continuous.*

Proof. Let $Lip(f)$ be the Lipschitz constant of f.

Let $\varepsilon > 0$ and let $(]\alpha_i, \beta_i[)_{i \le n}$ be a disjoint family such that

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \frac{\varepsilon}{Lip(f)}.$$

Then,

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| \le \sum_{i=1}^{n} L(\beta_i - \alpha_i) < Lip(f) \frac{\varepsilon}{Lip(f)} = \varepsilon.$$

That is, f is absolutely continuous. □

Theorem 1.4.4. *(Rademacher's Theorem) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then f is differentiable almost every where.*

Proof. Let $Lip(f)$ be the Lipschitz constant of f.

We write, for all $x \in \mathbb{R}$,

$$f(x) = g(x) - h(x),$$

where

$$g(x) = Lip(f)x + f(x) \quad \text{and} \quad h(x) = Lip(f)x$$

First, we prove that g is differentiable almost everywhere; more precisely, we show that g is continuous and increasing.

Let $x < y$. Then

$$g(y) - g(x) = Lip(f)y + f(y) - Lip(f)x - f(x) = Lip(f)(y - x) + (f(y) - f(x)).$$

We have

$$Lip(f)(y - x) > 0 \quad \text{and} \quad |f(y) - f(x)| \leq Lip(f)(y - x),$$

hence

$$g(y) - g(x) \geq 0,$$

so g is increasing.

For all $x, y \in \mathbb{R}$,

$$|g(x) - g(y)| = |Lip(f)x + f(x) - Lip(f)y - f(y)| \leq Lip(f)|x - y| + Lip(f)|x - y| = 2Lip(f)|x - y|.$$

By Theorem 1.4.2, g is differentiable almost everywhere. Similarly, h is differentiable at every point.

Finally, f is differentiable almost everywhere as the sum of two functions differentiable almost everywhere. $\square$

CHAPTER 2

LIPSCHITZ FREE SPACES

This chapter is devoted to the study of Lipschitz-free spaces, introduced by Godefroy and Kalton, and their main properties. We begin with the space $Lip_0(M)$ and the evaluation functionals δ_x, which naturally lead to the definition of the Lipschitz-free space $\mathcal{F}(X)$, also known as the ArensEells space, constructed from molecules. We then introduce the Lipschitz adjoint operator and examine the linearization of Lipschitz mappings, highlighting some of its most significant consequences. The exposition of this chapter relies mainly on references chapter is mainly based on [2, 4, 5, 8, 17, 21, 22].

2.1 Lipschitz Free Banach Spaces

Definition 2.1.1. Let(X, e) be a point metric space, we define the function $\delta_x : Lip_0(X) \to \mathbb{R}$, for $x \in X$ by:

$$\delta_x(f) = f(x), \quad \text{for all} \quad f \in Lip_0(X)$$

Proposition 2.1.2. *Let (X, e) be a pointed metric space. Then:*

(i) For every $x \in X$, the evaluation functional δ_x belongs to $Lip_0(X)^*$.

(ii) The mapping

$$\delta_X : X \longrightarrow Lip_0(X)^*, \qquad x \longmapsto \delta_x,$$

is an isometric embedding. In general, δ_X is not linear, since X need not possess a linear structure.

Proof. (i) Let $x \in X$ and $f \in Lip_0(X)$. It is clear that$x$ is linear. Since $f(e) = 0$, we have

$$|\delta_x(f)| = |f(x)| = |f(x) - f(e)| \leq Lip(f)\, d(x, e).$$

Hence,

$$|\delta_x(f)| \leq d(x, e) Lip(f),$$

which shows that δ_x is a bounded linear functional on $Lip_0(X)$. Therefore,

$\delta_x \in Lip_0(X)^*$ and $\|\delta_x\| \leq d(x,e)$.

(ii) Let $x, y \in X$. Then

$$\|\delta_X(x) - \delta_X(y)\| = \|\delta_x - \delta_y\| = \sup_{f \in B_{Lip_0(X)}} |(\delta_x - \delta_y)(f)|$$

$$= \sup_{f \in B_{Lip_0(X)}} |f(x) - f(y)| \leq \sup_{f \in B_{Lip_0(X)}} Lip(f)\, d(x,y) \leq d(x,y),$$

where $B_{Lip_0(X)}$ denotes the closed unit ball of $Lip_0(X)$.

To prove the reverse inequality, consider the function

$$g(z) = d(z,y) - d(0,y), \qquad z \in X.$$

Then $g \in Lip_0(X)$ and $Lip(g) \leq 1$. Moreover,

$$|g(x) - g(y)| = |d(x,y) - 0| = d(x,y).$$

Therefore,

$$\|\delta_x - \delta_y\| \geq \frac{|g(x) - g(y)|}{Lip(g)} \geq d(x,y).$$

Combining the two inequalities, we obtain

$$\|\delta_x - \delta_y\| = d(x,y).$$

Hence, δ_X is an isometric embedding.

□

Definition 2.1.3. Let X be a pointed metric space, we define $\mathcal{F}(X)$, the Lipschitz-free space over X, by

$$\mathcal{F}(X) = \overline{span}\{\delta_x, x \in X\} \subset Lip_0(X)^*.$$

Proposition 2.1.4. *The dual space of $\mathcal{F}(X)$ is isometrically isomorphic to* $\mathrm{Lip}_0(X)$*.*

Proof. Consider the mapping

$$J : \mathrm{Lip}_0(X) \longrightarrow \mathcal{F}(X)^*,$$

defined by

$$J(f) = J_f,$$

where J_f acts on $\operatorname{span}\{\delta_x : x \in X\}$ by

$$J_f\left(\sum_{i=1}^{n} a_i \delta_{x_i}\right) = \sum_{i=1}^{n} a_i f(x_i).$$

Since J_f is bounded, it extends uniquely by continuity to the whole space $\mathcal{F}(X)$.

We show that J is a surjective isometry. Let $f \in \mathrm{Lip}_0(X)$. For every $\mu \in \mathcal{F}(X)$,

$$|J_f(\mu)| = |\mu(f)| \leq \|\mu\|_{\mathcal{F}(X)} \,\mathrm{Lip}(f).$$

Hence,

$$\|J_f\|_{\mathcal{F}(X)^*} \leq \mathrm{Lip}(f).$$

To prove the reverse inequality, let $x, y \in X$ with $x \neq y$, and define

$$\mu = \frac{\delta_x - \delta_y}{d(x,y)}.$$

Since the canonical embedding

$$\delta_X : X \longrightarrow Lip_0(X)^*, \qquad x \longmapsto \delta_x$$

is an isometry, we have

$$\|\mu\|_{Lip_0(X)^*} = 1.$$

Therefore,

$$\|J_f\|_{\mathcal{F}(X)^*} \geq |J_f(\mu)| = \left|\frac{f(x) - f(y)}{d(x,y)}\right|.$$

Taking the supremum over all $x \neq y$, we obtain

$$\|J_f\|_{\mathcal{F}(X)^*} \geq \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} = \mathrm{Lip}(f).$$

Consequently,

$$\|J_f\|_{\mathcal{F}(X)^*} = \mathrm{Lip}(f),$$

and J is an isometry.

Let us now prove the surjectivity of$J$. Let $\varphi \in \mathcal{F}(X)^*$. Define

$$f : X \longrightarrow \mathbb{K}, \qquad f(x) = \varphi(\delta_x),$$

Since $\delta_e = 0$, we have

$$f(e) = \varphi(\delta_e) = 0.$$

For every $x, y \in X$,

$$\begin{aligned}|f(x) - f(y)| &= |\varphi(\delta_x) - \varphi(\delta_y)| \\ &= |\varphi(\delta_x - \delta_y)| \\ &\leq \|\varphi\|_{\mathcal{F}(X)^*} \|\delta_x - \delta_y\|_{Lip_0(X)^*} \\ &= \|\varphi\|_{\mathcal{F}(X)^*} d(x, y).\end{aligned}$$

Thus $f \in \mathrm{Lip}_0(X)$ and

$$\mathrm{Lip}(f) \leq \|\varphi\|_{\mathcal{F}(X)^*}.$$

It remains to show that $J_f = \varphi$. Let

$$\mu = \sum_{i=1}^{n} a_i \delta_{x_i} \in \mathrm{span}\{\delta_x : x \in X\}.$$

Then

$$\begin{aligned}\varphi(\mu) &= \varphi\left(\sum_{i=1}^{n} a_i \delta_{x_i}\right) \\ &= \sum_{i=1}^{n} a_i \varphi(\delta_{x_i}) \\ &= \sum_{i=1}^{n} a_i f(x_i) \\ &= J_f(\mu).\end{aligned}$$

Hence J_f and φ coincide on the dense subspace $\mathrm{span}\{\delta_x : x \in X\}$. Since both are continuous linear functionals on $\mathcal{F}(X)$, they coincide on all of $\mathcal{F}(X)$. Therefore,

$$J_f = \varphi.$$

Thus J is surjective. We conclude that J is a surjective isometry. Therefore,

$$\mathcal{F}(X)^* \cong \mathrm{Lip}_0(X)$$

isometrically. □

2.2 Examples

In this section, we present some examples of Lipschiz free spaces.

Example 2.2.1. Let $X = \mathbb{N} \longrightarrow \mathbb{R}$ endowed with the usual metric inherited from $\mathbb{R}$. Then

a) $\mathrm{Lip}_0(X) \cong \ell_\infty$

b) $\mathcal{F}(X) \cong \ell_1$

To see this, a) consider the linear operator

$$T : \mathrm{Lip}_0(X) \longrightarrow \ell_\infty, \qquad Tf = (f(n) - f(n-1))_n, \quad n \geq 1.$$

We first show that T is an isometry. Let $f \in \mathrm{Lip}_0(X)$. Since

$$|f(n) - f(n-1)| \leq \mathrm{Lip}(f),$$

for every $n \in \mathbb{N}$, it follows that

$$\|Tf\|_\infty \leq \mathrm{Lip}(f).$$

Conversely, let $m < n$. Then

$$f(n) - f(m) = \sum_{k=m+1}^{n} \big(f(k) - f(k-1)\big),$$

and therefore

$$\frac{|f(n) - f(m)|}{n - m} \leq \max_{m<k\leq n} |f(k) - f(k-1)| \leq \|Tf\|_\infty.$$

Taking the supremum over all $m < n$ yields

$$\mathrm{Lip}(f) \leq \|Tf\|_\infty.$$

Hence

$$\|Tf\|_\infty = \mathrm{Lip}(f),$$

and T is an isometry.

To prove surjectivity, let $(a_n) \in \ell_\infty$ and define $f : X \to \mathbb{R}$ by

$$f(e) = 0, \qquad f(n) = \sum_{k=1}^{n} a_k, \quad n \geq 1.$$

Then

$$f(n) - f(n-1) = a_n,$$

so that $Tf = (a_n)_n$.

Moreover, for $m < n$,

$$\frac{|f(n) - f(m)|}{n - m} = \frac{1}{n - m} \left| \sum_{k=m+1}^{n} a_k \right| \leq \|(a_n)_n\|_\infty.$$

Thus

$$\mathrm{Lip}(f) \leq \|(a_n)_n\|_\infty,$$

and consequently $f \in \mathrm{Lip}_0(X)$.

Therefore T is a surjective isometry, and

$$\mathrm{Lip}_0(X) \cong \ell_\infty.$$

b) Since $\mathcal{F}(X)^* \cong \ell_\infty$ and ℓ_∞ has a unique predual (Godefroy's theorem [10]), it follows that

$$\mathcal{F}(X) \cong \ell_1.$$

The corresponding isometry

$$S : \mathcal{F}(X) \longrightarrow \ell_1$$

is the preadjoint of T $(i.e., S^* = T)$ and is given by

$$S(\delta_n) = \sum_{k=1}^{n} e_k, \qquad n \geq 1,$$

where (e_k) denotes the canonical basis of ℓ_1.

Example 2.2.2. The Lipschitz-free space over $\mathbb{R}$ is isometrically isomorphic to $L_1(\mathbb{R})$, that is,

$$\mathcal{F}(\mathbb{R}) \cong L_1(\mathbb{R}).$$

Proof. We first show that

$$\mathrm{Lip}_0(\mathbb{R}) \cong L_\infty(\mathbb{R}).$$

Consider the operator

$$T : \mathrm{Lip}_0(\mathbb{R}) \longrightarrow L_\infty(\mathbb{R}), \qquad T(f) = f'.$$

By Rademacher's Theorem1.4.4, every Lipschitz function on $\mathbb{R}$ is differentiable almost everywhere and

$$|f'(x)| = \left| \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right| \leq \mathrm{Lip}(f)$$

for almost every $x \in \mathbb{R}$. Hence

$$\|f'\|_\infty \leq \mathrm{Lip}(f).$$

Conversely, for any $x, y \in \mathbb{R}$,

$$f(x) - f(y) = \int_y^x f'(t)\, dt,$$

and therefore

$$|f(x) - f(y)| \leq |x - y| \, \|f'\|_\infty.$$

Taking the supremum over all $x \neq y$ gives

$$\mathrm{Lip}(f) \leq \|f'\|_\infty.$$

Thus

$$\mathrm{Lip}(f) = \|f'\|_\infty,$$

and T is an isometry.

To prove surjectivity, let $g \in L_\infty(\mathbb{R})$ and define

$$f(x) = \int_e^x g(t)\, dt.$$

Then $f(e) = 0$ and

$$|f(x) - f(y)| = \left| \int_y^x g(t)\, dt \right| \leq \|g\|_\infty |x - y|.$$

Hence $f \in \mathrm{Lip}_0(\mathbb{R})$.

Moreover,

$$T(f) = g$$

Therefore T is surjective, and

$$\mathrm{Lip}_0(\mathbb{R}) \cong L_\infty(\mathbb{R}).$$

Since

$$L_1(\mathbb{R})^* = L_\infty(\mathbb{R}),$$

it is natural to identify $\mathcal{F}(\mathbb{R})$ with $L_1(\mathbb{R})$. Define

$$S : \mathcal{F}(\mathbb{R}) \longrightarrow L_1(\mathbb{R})$$

by

$$S\left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) = \sum_{i=1}^n \alpha_i \, \mathbf{1}_{[0,x_i]}.$$

We claim that

$$S^* = T^{-1}.$$

Where $T^{-1} : L_\infty(\mathbb{R}) \to Lip_0(\mathbb{R}), \quad g \mapsto \left(x \mapsto \int_0^x g(t)\,dt\right)$.

Indeed, for every $x \in \mathbb{R}$ and $g \in L_\infty(\mathbb{R})$,

$$\langle S\delta_x, g\rangle = \int_e^x g(t)\,dt = (T^{-1}g)(x) = \langle \delta_x, T^{-1}g\rangle.$$

Since the span a dense subspace of $\mathcal{F}(\mathbb{R})$, it follows that $S^* = T^{-1}$.

Note that $S(e) = 0$, so it suffices to show that for all $\sum_{i=1}^n \alpha_i \delta_{x_i}$.

Finally, for every $\mu \in \operatorname{span}\{\delta_x : x \in \mathbb{R}\}$,

$$\|S\mu\|_{L_1} = \sup_{\|g\|_\infty \le 1} |\langle S\mu, g\rangle| = \sup_{\|f\|_{\mathrm{Lip}} \le 1} |\langle \mu, f\rangle| = \|\mu\|_{\mathcal{F}(\mathbb{R})}.$$

Hence S is an isometry. Since $S^* = T^{-1}$ is surjective, S is onto. Therefore,

$$\mathcal{F}(\mathbb{R}) \cong L_1(\mathbb{R}).$$

$\square$

2.3 The Lipschitz Conjugate Operator

In this subsection we shall present, following [4], the Lipschitz analog of the conjugate of a linear mapping. Compactness properties for Lipschitz operators and their conjugates will be presented in Sect. We consider now that E, F are Banach spaces with the origins as base points. Then the space $\mathcal{L}(E, F)$ of all continuous linear operators from E to F is contained in the space $\mathrm{Lip}_0(E, F)$ of all Lipschitz mappings $F : E \to F$ satisfying $F(0) = 0$ and the operator norm

$$\|A\| := \sup\{\|Ax\| : x \in E, \|x\| \le 1\}$$

of $A \in \mathcal{L}(E, F)$ agrees with its Lipschitz norm $Lip(A)$:

$$Lip(A) = \sup_{x \ne x'} \frac{\|Ax - Ax'\|}{\|x - x'\|} = \sup_{x \ne x'} \left\| A\left(\frac{x - x'}{\|x - x'\|}\right)\right\| = \sup\{\|Au\| : \|u\| = 1\} = \|A\|,$$

provided that $E \ne \{0\}$. If $E = \{0\}$.

Then $A = O$ (the null operator) and $Lip(A) = 0 = \|A\|$.

In particular, $X^* \subseteq \mathrm{Lip}_0(X)$ and

$$\|x^*\| = Lip(x^*)$$

for $x^* \in E^*$.

Let(X, Y)be a pointed metric space, for $F \in \mathrm{Lip}_0(X, Y)$, define $F^{\#} : \mathrm{Lip}_0(Y) \to \mathrm{Lip}_0(X)$ by:

$$F^{\#}g = g \circ F, \quad g \in \mathrm{Lip}_0(Y).$$

Proposition 2.3.1. *Under the above hypotheses, $F^{\#}$ is a continuous linear operator from $Lip_0(Y)$ to $Lip_0(X)$ with norm $\|F^{\#}\| = Lip(F)$.*

Proof. From

$$Lip(g \circ F) \leq Lip(g)Lip(F)$$

it follows that $F^{\#}$ is well-defined. Since the linearity is obvious, the above inequality shows its continuity and the inequality

$$\|F^{\#}\| = \sup_{F \in Lip_0(Y), Lip(f) \leqslant 1} Lip(F \circ g) \leqslant Lip(g)$$

So $\|F^{\#}\| \leqslant Lip(g)$.
Let $x \neq y \in X$. Define the mapping:

$$F : Y \to \mathbb{R}$$
$$z \mapsto d'(z, g(y)) - d'(0, g(y))$$

Then $Lip(F) = 1$ and $F(0) = 0$.
so, $F \in Lip_0(Y)$.
Moreover,

$$\begin{aligned} Lip(F^{\#}F) = Lip(F \circ g) &= \sup_{z \neq t \in X} \frac{|F \circ g(z) - F \circ g(t)|}{d(z,t)} \\ &\geqslant \frac{|F \circ g(x) - F \circ g(y)|}{d(x,y)} = \frac{d'(g(x), g(y))}{d(x,y)} \end{aligned}$$

□

Thus, for all $x, y \in X, x \neq y$,

$$\|F^{\#}\| \geqslant \frac{d'(g(x), g(y))}{d(x,y)}$$

.
By taking the supremum: $\|F^{\#}\| \geqslant Lip(F)$.
Therefore $\|F^{\#}\| = Lip(F)$.

Proposition 2.3.2. *Let E, F be Banach spaces and $A : E \to F$ a continuous linear operator. Then*

$$A^{\#}|_{F^*} = A^*.$$

Proof. Indeed, for any $y^* \in F^*$,

$$A^{\#}y^* = y^* \circ A = A^*y^*.$$

$\square$

We need following Linearization Theorem

2.4 Linearization Theorem

Lemma 2.4.1. *Let E and F be two Banach spaces. If $T : F^* \to E^*$ is a $w^* - w^*$ continuous operator, then there exists an operator $S : E \to F$ such that $T = S^*$. [5]*

Proposition 2.4.2. *Let (X, d) and (Y, d') be two metric spaces. Let $T : X \to Y$ be a Lipschitz mapping such that $T(0) = 0$. Then there exists a unique linear mapping $\hat{T} : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that $\hat{T}\delta_X = \delta_Y T$. Moreover, $\|\hat{T}\| = Lip(T)$.*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow \delta_X & & \downarrow \delta_Y \\ \mathcal{F}(X) & \xrightarrow{\hat{T}} & \mathcal{F}(Y) \end{array}$$

Proof. Consider the mapping:

$$T^{\#} : Lip_0(Y) = \mathcal{F}(Y)^* \to \mathcal{F}(X)^* = Lip_0(X)$$
$$F \mapsto F \circ T$$

1. Let us show that $T^{\#}$ is the adjoint of an operator $\hat{T} : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that $\hat{T}^* = T^{\#}$.
By the previous lemma, it suffices to show that $T^{\#}$ is $w^* - w^*$ continuous.
Let $(F_n)_{n\in\mathbb{N}} \subset Lip_0(N)$ be a sequence converging weak* to $F \in Lip_0(N)$.
That is, $\forall y \in N, (F_n(y))_{n\in\mathbb{N}}$ converges to $F(y)$, or simply $(F_n)_{n\in\mathbb{N}}$ converges pointwise to F.
Let us show that $F_n \circ T$ converges weak* to $F \circ T$:
For all $x \in X, T(x) \in Y$ and

$$F_n \circ T(x) = F_n(T(x)) \to F(T(x)) = F \circ T(x)$$

So $(F_n \circ T)_{n\in\mathbb{N}}$ converges pointwise to $F \circ T$. In other words, $(T^{\#}(F_n))_{n\in\mathbb{N}}$ converges weak* to $T^{\#}(F)$.

Thus, $T^{\#}$ is $w^* - w^*$ continuous.

so, there exists

$$\hat{T} : \mathcal{F}(X) \to \mathcal{F}(Y)$$

such that $\hat{T}^* = T^{\#}$.

2. We have seen that $\|T^{\#}\| = Lip(T)$.

Furthermore, $\hat{T}$ is the pre-adjoint of $T^{\#}$, so $\|\hat{T}\| = \|T^{\#}\|$.

Hence $\|\hat{T}\| = Lip(T)$.

3. Let us show that $\hat{T}\delta_X = \delta_Y T$:

Let $x \in X$. We want to show that $\hat{T}\delta_x = \delta_Y T(x)$.

Let $F \in Lip_0(Y) = \mathcal{F}(Y)^*$.

$$\hat{T}\delta_x(F) = \langle \hat{T}\delta_x, F \rangle = \langle \delta_x, F \circ T \rangle = F \circ T(x)$$

$$\delta_Y T(x)(F) = \delta_{T(x)}(F) = F(T(x)) = F \circ T(x)$$

Thus, $\hat{T}\delta_x = \delta_Y T(x), \forall x \in X$ and therefore $\hat{T}\delta_X = \delta_Y T$.

4. This equality allows us to prove the uniqueness of $\hat{T}$.

Indeed,

$$\forall \mu \in \overline{span}\{\delta_x, x \in X\}$$

$\hat{T}\delta_x$ is uniquely defined by $\delta_Y T(x)$.

Furthermore, $\mathcal{F}(X) = \overline{span}\{\delta_x, x \in X\}$

Now let X be a Banach space.

Consider the mapping β_X defined on $span\{\delta_x, x \in X\}$ by:

$$\forall \sum_{i=1}^{n} a_i \delta_{x_i} \in span\{\delta_x, x \in X\}, \beta_X \left(\sum_{i=1}^{n} a_i \delta_{x_i} \right) = \sum_{i=1}^{n} a_i x_i$$

Let $x^* \in X^* \subset Lip_0(X)$ with $\|x^*\|$ being its Lipschitz constant. Then we have:

$$\left\langle \beta_X \left(\sum_{i=1}^{n} a_i \delta_{x_i} \right), x^* \right\rangle_{X,X^*} = \left\langle \sum_{i=1}^{n} a_i x_i, x^* \right\rangle_{X,X^*} = \left\langle \sum_{i=1}^{n} a_i \delta_{x_i}, x^* \right\rangle_{\mathcal{F}(X),Lip_0(X)}$$

According to the Hahn-Banach theorem:

$$\begin{aligned} \left\| \sum_{i=1}^{n} a_i x_i \right\|_X &= \sup_{x^* \in S_{X^*}} \left\langle \sum_{i=1}^{n} a_i x_i, x^* \right\rangle_{X,X^*} \\ &\leqslant \sup_{f \in S_{Lip_0(X)}} \left\langle \sum_{i=1}^{n} a_i \delta_{x_i}, f \right\rangle_{\mathcal{F}(X),Lip_0(X)} = \left\| \sum_{i=1}^{n} a_i \delta_{x_i} \right\|_{\mathcal{F}(X)} \end{aligned}$$

Hence $\|\beta_X\| \leqslant 1$.

Moreover, let $x \in X$. Then $\beta_X(\delta_x) = x$ and thus;

$$\|\beta_X(\delta_x)\|_X = \|x\|_X = \|\delta_x\|_{\mathcal{F}(X)}.$$

From which it follows that $\|\beta_X\| \geqslant 1$.

Finally, $\|\beta_X\| = 1$ and it can be extended into a bounded linear mapping:

$$\beta : \mathcal{F}(X) \to X$$

Lemma 2.4.3. *β_X is a quotient mapping from $\mathcal{F}(X)$ onto X and its right inverse is δ_X:*

$$\beta_X \delta_X = Id_X$$

Definition 2.4.4. A quotient mapping is a continuous surjective linear mapping.

Proof. β_X is a bounded linear operator as an extension of a bounded linear operator. Furthermore, by the definition of β_X, for all $x \in X$,

$$\beta_X(\delta_X(x)) = \beta_X(\delta_x) = x.$$

thus β_X is surjective and its right inverse is δ_X. □

Proposition 2.4.5. *Let X and Y be two Banach spaces and $T : X \to Y$ be a Lipschitz mapping such that $T(0) = 0$. Then there exists a unique linear mapping $T_L : \mathcal{F}(X) \to Y$ such that:*

$$T_L \delta_X = T$$

and,

$$\|T_L\| = Lip(T).$$

Proof. According to Proposition (2.4.2), there exists a unique linear mapping $\hat{T} : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that

$$\hat{T}\delta_X = \delta_Y T.\text{and, } \|\hat{T}\| = Lip(T).$$

Let $T_L = \beta_Y \hat{T}$. Note that T_L is indeed linear. Furthermore, since $\beta_Y \delta_Y = Id_Y$:

$$T_L \delta_X = \beta_Y \hat{T} \delta_X = \beta_Y \delta_Y T = Id_Y T = T.$$

Recall that β_Y is an operator with norm less than or equal to 1, hence:

$$\|T_L\| = \|\beta_Y \hat{T}\| \leqslant \|\hat{T}\| = Lip(T)$$

Let $x \neq x' \in X$. Since δ_X is an isometry,

$$\|\delta_X(x) - \delta_X(x')\| = \|x - x'\|.$$

Moreover,

$$T_L(\delta_X(x) - \delta_X(x')) = T(x) - T(x').$$

Thus,

$$\|T_L\| \geqslant \frac{\|T(\delta_X(x)) - T(\delta_X(x'))\|}{\|\delta_X(x) - \delta_X(x')\|} = \frac{\|T(x) - T(x')\|}{\|x - x'\|}$$

That is to say, for all $x \neq x'$,

$$\|T(x) - T(x')\| \leqslant \|T_L\|\|x - x'\|.$$

Therefore

$$\|T\| \leqslant \|T_L\|$$

and finally,

$$\|T\| = \|T_L\|.$$

□

Theorem 2.4.6. *Let K be a closed subset of X that contains 0. Then $\mathcal{F}(K)$ is linearly isometric to*

$$\overline{\mathrm{span}}\delta_K \subset \mathcal{F}(X).$$

Proof. Apply Proposition(2.4.5)to the restricted mapping

$$\delta_X : K \to \mathcal{F}(X)$$

to obtain a linear operator

$$T : \mathcal{F}(K) \to \mathcal{F}(X)$$

such that the following diagram is commutative:

$$\begin{tikzcd} K \arrow[r, "\delta_M"] \arrow[d, "\delta_K"'] & \mathcal{F}(M) \\ \mathcal{F}(K) \arrow[ur, "T"'] & \end{tikzcd}$$

i.e. such that

$$T(\delta_K(x)) = \delta_X(x)$$

for any $x \in K$.

Now let

$$m \in \operatorname{span} \delta_K(K),$$

say

$$m = \sum_{i=1}^{n} a_i \delta_K(x_i)$$

where $a_i \in \mathbb{R}$ and $x_i \in K$. Then

$$T(m) = \sum_{i=1}^{n} a_i \delta_X(x_i),$$

and we have

$$\begin{aligned} \|m\| &= \sup_{f \in B_{\mathrm{Lip}_0(K)}} \langle m, f \rangle = \sup_{f \in B_{\mathrm{Lip}_0(K)}} \langle T(m), f \rangle \\ &= \sup_{f \in B_{\mathrm{Lip}_0(X)}} \sum_{i=1}^{n} a_i f(x_i) = \sup_{f \in B_{\mathrm{Lip}_0(X)}} \langle T(m), f \rangle. \end{aligned}$$

Notice that, by Corollary (2.4.7), $B_{\mathrm{Lip}_0(K)}$ consists precisely of the restrictions to K of the functions in $B_{\mathrm{Lip}_0(X)}$. Therefore both suprema have the same value, and

$$\|T(m)\| = \|m\|.$$

Since this is true for m in a dense subset of $\mathcal{F}(K)$, T must be an isometry, and it is clear that its range is

$$T(\overline{\operatorname{span}} \delta_K(K)) = \overline{T(\operatorname{span} \delta_K(K))} = \overline{\operatorname{span}} \delta_X(K).$$

□

We shall denote the subspace appearing in Theorem 2.2.6 as

$$\mathcal{F}_X(K) = \overline{\operatorname{span}} \delta(K) = \overline{\operatorname{span}} \{\delta(x) : x \in K\},$$

with the convention that

$$\overline{\operatorname{span}} \emptyset = \{0\}.$$

The requirement that $0 \in K$ in Theorem (2.4.6)is just a technical condition to ensure that K is pointed with the same base point as X, so that the identity mapping

$$K \to X$$

leaves the base point invariant.

Nevertheless, notice that

$$\mathcal{F}_X(K) = \mathcal{F}_X(K \cup \{0\})$$

by definition, so we may always attach the base point to K to get the same result.

The Lipschitz-free space $\mathcal{F}(K)$ can be completely identified with the subspace $\mathcal{F}_X(K)$ of $\mathcal{F}(X)$ and we will often do so, but certain arguments appearing in this thesis require keeping track of the particular overspace M under consideration and this notation will prove useful there.

Corollary 2.4.7. *If $Y \subset X$, then*

$$B_{\mathrm{Lip}_0(Y)} = \{f|_Y : f \in B_{\mathrm{Lip}_0(X)}\}.$$

Indeed, it is clear that

$$Lip(f|_Y) \leq Lip(f)$$

as the supremum defining the Lipschitz constant is taken over a smaller set of pairs of points in the left hand side. This yields inclusion $\supset$. The reverse inclusion is precisely McShanes theorem.

Definition 2.4.8. A Banach space X has the *(isometric) Lipschitz-lifting property* if there exists a (norm one) continuous linear map

$$T : X \longrightarrow \mathcal{F}(X)$$

such that

$$\beta T = \mathrm{Id}_X .$$

We shall refer to the Lipschitz-lifting property as the lifting property.

Theorem 2.4.9. *Every separable Banach space has the isometric lifting property.*

CHAPTER 3

APPROXIMATION PROPERTIES IN THE LIPSCHITZ SETTING

In this chapter, we study approximation properties in the Lipschitz setting, with particular emphasis on the bounded approximation property for Lipschitz-free spaces. The investigation of approximation properties has played a central role in Banach space theory since the pioneering work of Grothendieck [11], who introduced several variants of the classical approximation property. For a comprehensive survey of these notions, we refer the reader to the article of Casazza [3]

In recent years, considerable attention has been devoted to approximation properties involving Lipschitz mappings and Lipschitz-free spaces. A fundamental result due to Godefroy and Kalton [8] establishes that the Lipschitz bounded approximation property is equivalent to the classical bounded approximation property for every Banach space. Subsequently, Godefroy and Ozawa [9] characterized metric spaces whose Lipschitz-free spaces possess the bounded approximation property through a Lipschitz version of the local reflexivity principle. More recently, Achour et al. [1] extended several notions and results concerning approximation properties from Banach spaces to the broader framework of metric spaces.

Another important line of research concerns the approximation of Lipschitz operators by finite-rank mappings. In this direction, Liu et al. [15] proved that whenever a bounded linear operator can be approximated pointwise by a uniformly bounded net of finite-rank Lipschitz mappings, it can also be approximated, in the strong operator topology, by a uniformly bounded net of finite-rank linear operators. As a consequence, they recovered the theorem of Godefroy and Kalton from a new perspective.

3.1 The bounded approximation property

We begin by recalling the basic definitions and fundamental properties of approximation theory that will be needed throughout this chapter.

Recall that a linear operator $T \in \mathcal{L}(E, F)$ is said to be of finite-rank if $T(E)$ is a finite dimensional subspace of F. The class of all finite-rank linear operators from E into F is denoted by $\mathcal{F}(E, F)$. It is well known that $T \in \mathcal{L}(E, F)$ has finite-rank if and only if there exist $(x_i^*)_{i=1}^n \subset E^*$ and $(y_i)_{i=1}^n \subset F$ such that

$$T(x) = \sum_{i=1}^{n} x_i^*(x) y_i,$$

for every $x \in F$. In particular, we write

$$\mathcal{F}(E) = \mathcal{F}(E, E).$$

Definition 3.1.1 (Schauder Basis)**.** A sequence $(x_n)_{n=1}^\infty$ in a Banach space E is called a Schauder basis of E if for every $x \in E$ there exists a unique sequence of scalars $(a_n)_{n=1}^\infty$ such that

$$x = \sum_{n=1}^{\infty} a_n x_n,$$

where the series converges in the norm of X, that is,

$$\lim_{N \to \infty} \left\| x - \sum_{n=1}^{N} a_n x_n \right\| = 0.$$

A sequence $(x_n)_{n=1}^\infty$ which is a Schauder basis of its closed linear span is called a basic sequence.

It is important to notice that a Schauder basis is not only a set of vectors but also an ordered sequence.

For

$$x = \sum_{n=1}^{\infty} a_n x_n,$$

the maps $f_n : E \to \mathbb{K}$, defined by

$$f_n(x) = a_n,$$

are called the coefficient functionals associated with the basis. The projections $(P_n)_{n=1}^\infty$ defined by

$$P_n\left(\sum_{j=1}^{\infty} a_j x_j\right) = \sum_{j=1}^{n} a_j x_j,$$

are called the natural projections associated with $(x_n)_{n=1}^{\infty}$.

A basis $(x_n)_{n=1}^{\infty}$ is said to be normalized if $\|x_n\| = 1$ for every n. Clearly, whenever $(x_n)_{n=1}^{\infty}$ is a Schauder basis of E, the sequence

$$\left(\frac{x_n}{\|x_n\|}\right)_{n=1}^{\infty}$$

is a normalized basis of E.

An important property of a Schauder basis is that the coefficient functionals associated with the basis are continuous. This result was originally proved by Banach (see). We now present a few examples.

Example 3.1.2. If $E = c_0$ or ℓ_p with $1 \leq p < \infty$, then the sequence $(e_n)_{n=1}^{\infty}$ of standard unit vectors,

$$e_n = (\delta_{nj})_{j=1}^{\infty},$$

is a Schauder basis for E.

Indeed, if $x = (a_n) \in c_0$, then

$$\left\| x - \sum_{n=1}^{N} a_n e_n \right\| = \sup_{n>N} |a_n| \longrightarrow 0 \qquad (N \to \infty).$$

Similarly, if $x = (a_n) \in \ell_p$, then

$$\left\| x - \sum_{n=1}^{N} a_n e_n \right\| = \left(\sum_{n=N+1}^{\infty} |a_n|^p \right)^{1/p} \longrightarrow 0 \qquad (N \to \infty).$$

Definition 3.1.3 (Grothendieck Approximation Property)**.** A Banach space X is said to have the approximation property if, for every compact subset $K \subset X$ and every $\varepsilon > 0$, there exists a finite-rank operator $T : X \to X$ such that

$$\|Tx - x\| \leq \varepsilon \qquad \text{for all } x \in K.$$

Equivalently, the identity operator $I : X \to X$ can be approximated uniformly on every compact subset of X by finite-rank operators.

Theorem 3.1.4. *Let X be a Banach space with a Schauder basis $(x_n)_{n=1}^{\infty}$. Then X has the approximation property.*

Proof. Let $(S_n)_n$ denote the partial sum operators associated with the basis (x_n), that is,

$$S_n(x) = \sum_{k=1}^{n} \alpha_k x_k, \qquad x = \sum_{k=1}^{\infty} \alpha_k x_k \in X.$$

Each $S_n : X \to X$ is linear and has finite rank ($Im(S_n(X) \subset span\,\{x_1, ..., x_n\}$. Since

$$\lim_{n\to\infty} \|S_n x - x\| = 0 \qquad \text{for every } x \in X,$$

the Uniform Boundedness Principle implies that

$$s := \sup_{n\in\mathbb{N}} \|S_n\| < \infty.$$

Let $K \subset X$ be compact and let $\varepsilon > 0$. Since K is compact, there exist points $y_1, \ldots, y_m \in K$ such that

$$K \subset \bigcup_{j=1}^{m} B\left(y_j, \frac{\varepsilon}{2(1+s)}\right).$$

Since $S_n y_j \to y_j$ for each $j = 1, \ldots, m$, there exists $N \in \mathbb{N}$ such that

$$\|y_j - S_n y_j\| < \frac{\varepsilon}{2}, \qquad j = 1, \ldots, m,$$

whenever $n \geq N$.

Let $x \in K$. Choose y_j such that

$$\|x - y_j\| < \frac{\varepsilon}{2(1+s)}.$$

Then, for $n \geq N$,

$$\begin{aligned}
\|x - S_n x\| &\leq \|x - y_j\| + \|y_j - S_n y_j\| + \|S_n x - S_n y_j\| \\
&\leq (1 + \|S_n\|)\|x - y_j\| + \|y_j - S_n y_j\| \\
&\leq (1+s)\frac{\varepsilon}{2(1+s)} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \sup_{x\in K} \|x - S_n x\| = 0.$$

Hence the identity operator on X is the uniform limit on compact sets of finite-rank operators, and therefore X has the approximation property. □

Definition 3.1.5. Let $\lambda \geq 1$. A Banach space E is said to have the λ-bounded approximation property (λ-BAP) if, for every compact set $K \subset E$ and every $\varepsilon > 0$, there exists a finite-rank operator $T : E \to E$ such that

$$\|T\| \leq \lambda$$

and

$$\|Tx - x\| < \varepsilon, \qquad x \in K.$$

If $\lambda = 1$, this property is called the metric approximation property (MAP).

If E has the λ-BAP for some $\lambda \geq 1$, then E is said to have the bounded approximation property (BAP).

Remark 3.1.6. In the definition of the BAP, the compact set K may be replaced by a finite subset of E.

Indeed, let $K \subset E$ be compact and let $\varepsilon > 0$. Since K is compact, there exist points $x_1, \ldots, x_n \in K$ such that

$$K \subset \bigcup_{i=1}^{n} B\left(x_i, \frac{\varepsilon}{3(1+\lambda)}\right).$$

Suppose that $T : E \to E$ is a finite-rank operator satisfying

$$\|T\| \leq \lambda$$

and

$$\|Tx_i - x_i\| < \frac{\varepsilon}{3}, \qquad i = 1, \ldots, n.$$

Let $x \in K$. Choose $i \in \{1, \ldots, n\}$ such that

$$\|x - x_i\| < \frac{\varepsilon}{3(1+\lambda)}.$$

Then

$$\|Tx - x\| \leq \|T(x - x_i)\| + \|Tx_i - x_i\| + \|x_i - x\| \leq (\|T\| + 1)\|x - x_i\| + \frac{\varepsilon}{3} \leq \frac{2\varepsilon}{3} < \varepsilon.$$

Hence

$$\|Tx - x\| < \varepsilon \qquad (x \in K).$$

Therefore, in the definition of the BAP, it is enough to require approximation on finite subsets of E.

Proposition 3.1.7. *For a Banach space E, the following implications hold:*

$$E \textit{ has a Schauder basis} \Longrightarrow E \textit{ has the } BAP \Longrightarrow E \textit{ has the } AP.$$

Proof. Assume that E has a Schauder basis $(x_n)_{n=1}^{\infty}$ and let $(S_n)_{n=1}^{\infty}$ be the associated partial sum operators, that is,

$$S_n\left(\sum_{k=1}^{\infty} a_k x_k\right) = \sum_{k=1}^{n} a_k x_k.$$

Since $(x_n)_{n=1}^{\infty}$ is a Schauder basis, we have

$$S_n x \longrightarrow x \qquad (x \in E).$$

Moreover, by the Uniform Boundedness Principle,

$$M := \sup_{n\in\mathbb{N}} \|S_n\| < \infty.$$

Let $K \subset E$ be compact and let $\varepsilon > 0$. As in the proof of the approximation property for spaces with a Schauder basis, one obtains

$$\lim_{n\to\infty} \sup_{x\in K} \|S_n x - x\| = 0.$$

Hence there exists $N \in \mathbb{N}$ such that

$$\|S_N x - x\| < \varepsilon, \qquad x \in K.$$

Since S_N has finite rank and

$$\|S_N\| \leq M,$$

it follows that E has the M-BAP. Consequently, E has the BAP.

The implication

$$\text{BAP} \Longrightarrow \text{AP}$$

follows immediately from the definitions, since the AP does not require a uniform bound on the norms of the approximating finite-rank operators (In other words, BAP = AP + a uniform bound on the norms of the approximating finite-rank operators. Removing this extra requirement gives AP.). □

Remark 3.1.8. Neither converse implication is true in general. Indeed, Gilles Pisier [18] constructed a Banach space with the approximation property but without the bounded approximation property. Moreover, Stanisław Szarek [20] constructed a Banach space with the bounded approximation property that does not admit a Schauder basis. We also recall that the first counterexample to the basis problem was given by Per Enflo [6], who constructed a separable Banach space without the approximation property (AP). Since every Banach space with a Schauder basis has the AP, it follows that this space cannot admit a Schauder basis.

Proposition 3.1.9. *Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces and let*

$$E = \left(\bigoplus_{n=1}^{\infty} E_n\right)_{\ell^p} \qquad (1 \le p < \infty) \quad \textit{or} \quad E = \left(\bigoplus_{n=1}^{\infty} E_n\right)_{c_0}.$$

Then E has the λ-bounded approximation property if and only if the spaces E_n have the λ-bounded approximation property with a uniform bound on their BAP constants

Proof. $\Rightarrow$**)** Assume that X has the λ-BAP. Let $n \in \mathbb{N}$. We show that E_n has λ-BAP uniformly in n.

Let $K_n \subset E_n$ be compact and let $\varepsilon > 0$. Consider the compact set

$$\widetilde{K}_n = \{J_n(z) : z \in K_n\} \subset E,$$

where $J_n : E_n \to E, z \longmapsto (0, ..., 0, z, O, ...)$, is the canonical embedding.

Since E has the λ-BAP, there exists a finite-rank operator $T : E \to E$ such that $\|T\| \le \lambda$ and

$$\|TJ_n(z) - J_n(z)\| < \varepsilon \qquad (z \in K_n).$$

Define

$$S_n = \pi_n \circ T \circ J_n : E_n \to E_n,$$

where $\pi_n : E \to E_n, (x_k)_{k\ge1} \longmapsto x_n$, is the coordinate projection. Then S_n is finite-rank, $\|S_n\| \le \lambda$, and for all $z \in K_n$,

$$\|S_n(z) - z\| = \|\pi_n(TJ_n(z)) - J_n(z))\| \le \|TJ_n(z) - J_n(z)\| < \varepsilon.$$

Hence each E_n has λ-BAP, and the constant λ is independent of n.

$\Leftarrow$**)** Assume that there exists $\lambda \ge 1$ such that each E_n has λ-BAP.

Let $K \subset E$ be compact and $\varepsilon > 0$. We first show that the tails of elements of K are uniformly small.

For each $m \in \mathbb{N}$ define

$$K^{(m)} = \{(x_n)_n \in K : \ \|(x_n)_{n>m}\| \ge \varepsilon\}.$$

If all $K^{(m)}$ were non-empty, we could choose a sequence $(x^{(m)}) \subset K$ such that the tail norm beyond m is at least ε. By compactness of K, a subsequence converges to some $x \in K$, but this contradicts the fact that the tail norm of $x^{(m)}$ does not vanish uniformly. Hence there exists $N \in \mathbb{N}$ such that

$$\|(x_n)_{n>N}\| < \varepsilon \qquad \text{for all } x = (x_n) \in K.$$

Now fix $1 \leq n \leq N$. Let $P_n(K) \subset E_n$ denote the projection of K onto E_n. Since E_n has λ-BAP, there exists a finite-rank operator

$$T_n : E_n \to E_n, \qquad \|T_n\| \leq \lambda,$$

such that

$$\|T_n x_n - x_n\| < \varepsilon \qquad (x_n \in P_n(K)).$$

Define $T : E \to E$ by

$$T(x_1, x_2, \dots) = (T_1 x_1, \dots, T_N x_N, 0, 0, \dots).$$

Then T is finite-rank and $\|T\| \leq \lambda$.

Let $x = (x_n) \in K$. Using the ℓ^p-norm (or the c_0-norm), we estimate:

$$\|Tx - x\| \leq \left(\sum_{n=1}^{N} \|T_n x_n - x_n\|^p \right)^{1/p} + \|(x_n)_{n>N}\|.$$

The first term is bounded by ε (up to a constant depending only on N), and the second term is $< \varepsilon$ by the choice of N. Hence, after adjusting ε, we obtain

$$\|Tx - x\| < C\varepsilon \qquad (x \in K),$$

for a constant C independent of x. Rescaling ε shows that E has the λ-BAP. □

3.2 Lipschitz bounded approximation properties

The study of the bounded approximation property for Lipschitz-free spaces was initiated by Godefroy and Kalton in their seminal paper [13].

Definition 3.2.1. Let E be a Banach space and $\lambda \geq 1$. Then we say that E has the λ-Lipschitz bounded approximation property if for every compact set $K \subset F$ and $\varepsilon > 0$ there exists a

Lipschitz map $F : E \to E$ with finite-dimensional range such that

$$Lip(F) \leq \lambda \quad \text{and} \quad \|F(x) - x\| \leq \varepsilon \;\; \text{for } x \in K.$$

We remark that it is easy to see that we can assume $F(0) = 0$ in the above definition.

Remark 3.2.2. The condition $F(0) = 0$ is not restrictive: given any Lipschitz approximation F with finite-dimensional range satisfying the norm condition on K, the map $\widetilde{F}(x) := F(x) - F(0)$ still has finite-dimensional range, satisfies $\widetilde{F}(0) = 0$, and

$$\mathrm{Lip}(\widetilde{F}) = \mathrm{Lip}(F), \qquad \left\|\widetilde{F}(x) - x\right\| \leq \|F(x) - x\| + \|F(0)\| < \varepsilon + \|F(0)\|.$$

Adjusting ε accordingly yields the normalised form.

The fundamental equivalence between BAP and LBAP was established by Godefroy and Kalton [13].

Theorem 3.2.3. *Let E be a finite-dimensional Banach space. Then $\mathcal{F}(E)$ has the metric approximation property (MAP).*

This theorem plays a crucial role in establishing the following result.

Theorem 3.2.4. *Let E be an arbitrary Banach space. Then the following conditions are equivalent:*

1 *E has the λ-(BAP).*

2 *$\mathcal{F}(E)$ has the λ-(BAP).*

3 *E has the λ-Lipschitz bounded approximation property.*

Proof. It is trivial that (1) implies (3). Let us prove that (3) implies (2). To show $\mathcal{F}(E)$ has the λ-(BAP) it suffices by a density argument to show that if $\varepsilon > 0$ and $x_1, \ldots, x_n \in X$ then there exists a finite rank linear map $T : \mathcal{F}(E) \to \mathcal{F}(E)$ such that

$$\|T(\delta(x_j)) - \delta(x_j)\| < \varepsilon \quad \text{for } 1 \leq j \leq n$$

and $\|T\| \leq \lambda$.

To do this, note there is a finite-rank Lipschitz map $F : E \to E$ with

$$F(0) = 0,$$

$$Lip(F) \leq \lambda \quad \text{and} \quad \|F(x_j) - x_j\| < \varepsilon$$

for $1 \leq j \leq n$.

Let $\widehat{F}$ be the induced linear map as in Proposition(2.4.2). Then $\widehat{F}$ has range included in $\mathcal{F}(M)$ for some finite-dimensional subspace M of E. By Theorem 3.2.3this space has (MAP) and we can find a finite-rank linear operator

$$S : \mathcal{F}(M) \to \mathcal{F}(M)$$

so that $\|S\| \leq 1$ and

$$\|S\widehat{F}(\delta(x_j)) - \delta(x_j)\| < \varepsilon \quad \text{for } 1 \leq j \leq n.$$

Let $T = S\widehat{F}$ and we are done.That (2) implies (1) is trivial if E is separable by Theorem (2.4.9), which asserts that E is isometric to a contractively complemented subspace of $\mathcal{F}(E)$. In the nonseparable case we use Proposition (3.2.8).

Suppose $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$. Let $M = [x_1, \ldots, x_n]$ be their linear span. Then we can find a linear operator

$$W : E \to \mathcal{F}(E)^{**}$$

with $\|W\| = 1$.

so that;

$$\beta^{**}Wx = x \quad \text{for} \quad x \in E \quad \text{and} \quad W(M) \subset \mathcal{F}(E).$$

Let

$$T : \mathcal{F}(X) \to \mathcal{F}(X)$$

be a finite-rank operator with $\|T\| \leq \lambda$ and

$$\|TWx_j - Wx_j\| \leq \varepsilon \quad \text{for } 1 \leq j \leq n.$$

Then let $S = \beta^{**}T^{**}W$. Since T^{**} maps into $\mathcal{F}(E)$ we see that $S : E \to E$ is finite-rank and

$$\|Sx_j - x_j\| \leq \varepsilon \quad \text{for } j = 1, \ldots, n.$$

□

Theorem 3.2.5. *Let E be a Banach space with (BAP) and suppose Y is a Banach space which is Lipschitz isomorphic to E. Then Y has (BAP).*

Proof. The Lipschitz isomorphism induces a linear isomorphism between $\mathcal{F}(X)$ and $\mathcal{F}(Y)$. The conclusion follows from Theorem 3.2.4. $\square$

Definition 3.2.6. Let E be a metric space and let A be a subspace of E. A Lipschitz map $p : E \to A$ is called a Lipschitz retraction if $p|_A = Id$. In this case, we say that A is a Lipschitz retract of E. A metric space A is called an absolute Lipschitz retract if it is a Lipschitz retract of every metric space containing it.

Theorem 3.2.7. *Let E be a Banach space which is an absolute Lipschitz retract. Then E has the bounded approximation property.*

Proof. The space E is a Lipschitz retract of $G = \ell_\infty(B_{E^*})$. The space G has (MAP), and thus $\mathcal{F}(G)$ has (MAP) by Theorem 3.2.4. By applying proposition(2.4.5)to the retract of G onto E, the space $\mathcal{F}(E)$ is complemented in $\mathcal{F}(G)$ and thus it has (BAP). By Theorem (3.2.4), E also has (BAP). $\square$

Proposition 3.2.8. *Let E be a Banach space and suppose A is a finite-dimensional subspace of E. Then there exists a linear operator $W : E \to \mathcal{F}(E)^{**}$ such that $\|W\| = 1$, $\beta^{**}Wx = x$ for all $x \in E$, and $W(A) \subset \mathcal{F}(E)$.*

3.3 Structural results on the Lipschitz bounded approximation property

The first result shows that, in verifying the LBAP, it is sufficient to produce Lipschitz approximations on *finite* sets rather than on arbitrary compact sets. This is a significant practical reduction.

Proposition 3.3.1 (Finite-set characterisation of the LBAP)**.** *A Banach space E has the λ-LBAP if and only if for every finite set $\{x_1, \ldots, x_n\} \subset E$ and every $\varepsilon > 0$ there exists a Lipschitz map $T : E \to E$ such that $T(0) = 0$, $\mathrm{Lip}(T) \leq \lambda$, $T(E)$ is contained in a finite-dimensional subspace of E, and $\|T(x_j) - x_j\| < \varepsilon$ for all $1 \leq j \leq n$.*

Proof. Necessity. If E has the λ-LBAP, then by definition for every compact subset $K \subset E$ and every $\varepsilon > 0$ there exists a Lipschitz map $T : E \to E$ satisfying conditions of hypoteses. Since every finite subset of E is compact, the conclusion follows immediately.

Sufficiency**.** Let $K \subset E$ be compact and let $\varepsilon > 0$ be given. Since K is compact, it is totally bounded. Hence there exist points $x_1, \ldots, x_n \in K$ such that

$$K \subset \bigcup_{j=1}^{n} B\left(x_j, \frac{\varepsilon}{\lambda+2}\right).$$

By the finite-set assumption, there exists a Lipschitz map $T : E \to E$ such that $T(0) = 0$, $Lip(T) \leq \lambda$, $T(E)$ is contained in a finite-dimensional subspace of E, and

$$\|T(x_j) - x_j\| < \frac{\varepsilon}{\lambda+2}, \qquad j = 1, \ldots, n.$$

Let $x \in K$. Choose $j \in \{1, \ldots, n\}$ such that

$$\|x - x_j\| < \frac{\varepsilon}{\lambda+2}.$$

Then

$$\begin{aligned}
\|T(x) - x\| &\leq \|T(x) - T(x_j)\| + \|T(x_j) - x_j\| + \|x_j - x\| \\
&\leq Lip(T)\, \|x - x_j\| + \|T(x_j) - x_j\| + \|x_j - x\| \\
&< \lambda \frac{\varepsilon}{\lambda+2} + \frac{\varepsilon}{\lambda+2} + \frac{\varepsilon}{\lambda+2} \\
&= \varepsilon.
\end{aligned} \tag{3.1}$$

Since $x \in K$ was arbitrary, we obtain

$$\|T(x) - x\| < \varepsilon \qquad (x \in K).$$

Therefore E has the λ-LBAP. $\square$

Proposition 3.3.2 (Lipschitz-free transfer principle)**.** *Let $F : E \to E$ be a Lipschitz map with $F(0) = 0$, and let $M \subset E$ be a finite-dimensional subspace such that $F(E) \subset M$. Then the canonical linearisation $\widehat{F} : \mathcal{F}(E) \to \mathcal{F}(E)$ satisfies*

$$\widehat{F}(\mathcal{F}(E)) \subset \mathcal{F}(M).$$

Moreover, $\left\|\widehat{F}\right\| = \mathrm{Lip}(F)$.

Proof. By the universal property of the Lipschitz-free space, there exists a unique bounded linear operator

$$\widehat{F} : \mathcal{F}(E) \longrightarrow \mathcal{F}(E)$$

such that

$$\widehat{F}(\delta(x)) = \delta(F(x)) \qquad (x \in E).$$

Moreover,

$$\|\widehat{F}\| = Lip(F).$$

Since $F(E) \subset M$, for every $x \in E$ we have $F(x) \in M$, and hence

$$\widehat{F}(\delta(x)) = \delta(F(x)) \in \mathcal{F}(M).$$

Therefore,

$$\widehat{F}(\delta(E)) \subset \mathcal{F}(M).$$

Now $\mathcal{F}(M)$ is a closed linear subspace of $\mathcal{F}(E)$, and the linear span of $\delta_X(E)$ is dense in $\mathcal{F}(E)$. Since $\widehat{F}$ is linear and continuous, it follows that

$$\widehat{F}(\mathcal{F}(E)) \subset \mathcal{F}(M).$$

This proves the result. □

Corollary 3.3.3. *Under the hypotheses of Proposition 3.3.2, if M is finite-dimensional then $\mathcal{F}(M)$ has the* MAP, *and hence there exists a net (or sequence in the separable case) of finite-rank operators S_α : $\mathcal{F}(M) \to \mathcal{F}(M)$ with $\|S_\alpha\| \leq 1$ converging strongly to the identity on $\mathcal{F}(M)$.*

Proof. This follows immediately from Proposition 3.3.2 and the Theorem 3.2.4 applied to the finite-dimensional space M, which trivially has the MPA. □

The third and most central result translates the LBAP — defined in terms of Lipschitz maps on E — into a purely *linear* approximation condition on the free space $\mathcal{F}(E)$, involving finite-rank operators evaluated at Dirac elements.

Theorem 3.3.4. *A Banach space E has the λ-LBAP if and only if for every finite family $\delta_E(x_1), \ldots, \delta_E(x_n) \in \mathcal{F}(E)$ and every $\varepsilon > 0$ there exists a finite-rank operator $T : \mathcal{F}(E) \to \mathcal{F}(E)$ such that*

$$\|T\delta_E(x_j) - \delta_E(x_j)\| < \varepsilon \quad \textit{for } j = 1, \ldots, n, \qquad \textit{and} \qquad \|T\| \leq \lambda.$$

Proof. **⇒)** Suppose that E has the λ-LBAP. Let $x_1, \ldots, x_n \in E$ and $\varepsilon > 0$ be given. By Proposition 3.3.1, there exists a Lipschitz map $F : E \to E$ such that

$$F(0) = 0, \qquad Lip(F) \leq \lambda,$$

and

$$F(E) \subset M$$

for some finite-dimensional subspace $M \subset E$, with

$$\|F(x_j) - x_j\| < \frac{\varepsilon}{2}, \qquad j = 1, \ldots, n.$$

Let

$$\widehat{F} : \mathcal{F}(E) \to \mathcal{F}(E)$$

denote the linearisation of F. By Proposition 3.3.2,

$$\widehat{F}(\mathcal{F}(E)) \subset \mathcal{F}(M).$$

Since M is finite-dimensional, Corollary 3.3.3 yields that $\mathcal{F}(M)$ has the MAP. Therefore, there exists a finite-rank operator

$$S : \mathcal{F}(M) \to \mathcal{F}(M)$$

such that

$$\|S\| \leq 1$$

and

$$\|S(\widehat{F}(\delta(x_j))) - \widehat{F}(\delta(x_j))\| < \frac{\varepsilon}{2}, \qquad j = 1, \ldots, n.$$

Define

$$T := S \circ \widehat{F} : \mathcal{F}(E) \to \mathcal{F}(E).$$

Then T is finite-rank and

$$\|T\| \leq \|S\| \, \|\widehat{F}\| \leq \lambda.$$

Moreover, for each $j = 1, \ldots, n$,

$$\begin{aligned}
\|T\delta(x_j) - \delta(x_j)\| &\leq \|T\delta(x_j) - \widehat{F}(\delta(x_j))\| + \|\widehat{F}(\delta(x_j)) - \delta(x_j)\| \\
&= \|S(\widehat{F}(\delta(x_j))) - \widehat{F}(\delta(x_j))\| + \|\delta(F(x_j)) - \delta(x_j)\| \\
&< \frac{\varepsilon}{2} + \|F(x_j) - x_j\| \\
&< \varepsilon,
\end{aligned}$$

where we used the isometric embedding property

$$\|\delta_E(y) - \delta_E(z)\|_{\mathcal{F}(E)} = \|y - z\|_E.$$

Hence the finite-rank condition holds.

$\Rightarrow$) Assume that the finite-rank condition holds. Let $x_1, \dots, x_n \in E$ and $\varepsilon > 0$ be given. Then there exists a finite-rank operator

$$T : \mathcal{F}(E) \to \mathcal{F}(E)$$

such that

$$\|T\| \leq \lambda$$

and

$$\|T\delta_E(x_j) - \delta_E(x_j)\| < \varepsilon, \qquad j = 1, \dots, n.$$

Define

$$G : E \to E, \qquad G(x) := \beta(T(\delta_E(x))),$$

where

$$\beta : \mathcal{F}(E) \to E$$

is the barycenter map.

Since T has finite-dimensional range, so does G. Moreover,

$$G(0) = \beta(T(\delta_E(0))) = 0.$$

For $x, y \in E$,

$$\begin{aligned}
\|G(x) - G(y)\| &= \|\beta(T(\delta_E(x) - \delta_E(y)))\| \\
&\leq \|T(\delta_E(x) - \delta_E(y))\| \\
&\leq \|T\| \, \|\delta_E(x) - \delta_E(y)\| \\
&= \|T\| \, \|x - y\| \\
&\leq \lambda \|x - y\|.
\end{aligned}$$

Therefore,

$$Lip(G) \leq \lambda.$$

Finally,

$$\begin{aligned}
\|G(x_j) - x_j\| &= \|\beta(T\delta_E(x_j)) - \beta(\delta_E(x_j))\| \\
&\leq \|T\delta_E(x_j) - \delta_E(x_j)\| \\
&< \varepsilon,
\end{aligned}$$

since $\beta(\delta_E(x)) = x$ and $\|\beta\| \leq 1$.

Thus G satisfies the finite-set formulation of the λ-LBAP. By Proposition 3.3.1, E has the λ-LBAP. $\square$

Remark 3.3.5. The relationship between Theorem 3.3.4 and 3.2.4 deserves clarification. 3.2.4 asserts the equivalence of the λ-BAP for E, the λ-BAP for $\mathcal{F}(E)$, and the λ-LBAP for E. 3.3.4 should be understood as a *finite quantitative reformulation* of the middle condition (the λ-BAP of $\mathcal{F}(E)$), specialised to approximation at Dirac elements $\delta_E(x_j)$. It does not supersede 3.2.4 but rather makes its content more explicit and more directly applicable to computations in Lipschitz-free spaces.

Conclusion

The results of this dissertation build directly on the foundational work of Godefroy and Kalton, which showed that the λ-LBAP for a Banach space E is equivalent to the λ-BAP of its Lipschitz-free space $\mathcal{F}(E)$. The new results added here make this equivalence more concrete and practical: the finite-set characterisation shows it is enough to check the approximation condition on finite sets rather than all compact sets, the transfer principle explains how finite-dimensional Lipschitz approximation on E becomes finite-rank linear approximation inside $\mathcal{F}(E)$, and the finite-rank approximation principle gives a clean linear reformulation of the entire property. Running through all three results is the Lipschitz-free space $\mathcal{F}(E)$, whose role is to convert nonlinear Lipschitz problems on E into linear operator problems without losing any metric information. This is what makes $\mathcal{F}(E)$ so important: it is the right space in which to study Lipschitz approximation, because its linear structure fully reflects the Lipschitz geometry of E, and standard tools from Banach space theory — finite-rank operators, the MAP, and the canonical quotient map (or the linearisation of the identity) — become directly available.

Bibliography

[1] D. Achour, P. Rueda, E A. Sanchez-Perez and R. Yahi, Lipschitz operator ideals and the approximation property. J. Math Anal. Appl., 436 (2016) 217–236.

[2] R-F. Arens and J. Eells, On embedding uniform and topological spaces. Pacific .J. Math., **6** (1956) 397–403.

[3] P.G. Casazza, Approximation properties, Handbook of the geometry of Banach spaces. vol.1 (2001) 271-316.

[4] S. Cobza, Adjoints of Lipschitz mappings, Studia Univ. Babe-Bolyai Math. **48**(1),(2003) 49–54.

[5] A. Dalet. Linear and non linear isometries, Master thesis under the supervision of Gilles Lancien (2011)

[6] P. Enflo, A counterexample to the approximation property for Banach spaces, Acta Math. 130 (1973) 309- 317

[7] J. Heinonen, Lectures on Analysis on Metric Spaces. Universitext (Springer, New York.(2001).

[8] G. Godefroy and N J. Kalton, Lipschitz-free Banach spaces. Studia Math, 159 (2003) 121–141.

[9] G. Godefroy and N. Ozawa, Free Banach spaces and the approximation properties. Proc Amer Math Soc, 142 (2014) 1681–1687

[10] G. Godefroy, Existence and uniqueness of isometric preduals: a survey,in *Banach space theory (Iowa City, IA, 1987)*, ser. Contemporary Mathematics. Providence, RI: American Mathematical Society, 1989, no. 85, pp. 131–193.

[11] A. Grothendieck, Produits tensoriels topologiques et espaces nuclť eaires, Mem. Amer. Math. Soc. 16 (1955).

[12] A.Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Memoirs of the American Mathematical Society, (1955).

[13] N.-J. Kalton and G. Godefroy, Lipschitz-free Banach spaces, Studia Math. **159** (2003), 121–141.

[14] L. Mezreg, Non lineaire geometry of spaces, University of MSila. Mester's-level Course, 2023.

[15] R. Liu, J. Shen, B T.Zheng, Operators with the Lipschitz bounded approximation property. Sci China Math, (2022), 65, https://doi.org/10.1007/s11425-022-2000-y

[16] E. J. McShane, Extension of range of functions, Bull. Amer. Math. Soc. 40 (1934), no. 12, 837–842.

[17] V. G. Pestov, *Free Banach spaces and representations of topological groups*, Funct. Anal. Appl. **20**, 70–72 (1986).

[18] G. Pisier, Counterexamples to a conjecture of Grothendieck, Acta Mathematica, v151(1983) 181–208

[19] M. Ó Searcóid, *Metric Spaces*. (Springer, Berlin, 2006).

[20] Szarek, Stanis, A Banach space without a basis which has the bounded approximation property, Acta Mathematica, v159 (1987) 81–98

[21] Ş.CobzaŞ, R. Miculescu, A. Nicolae,Lipschitz function.Springer Nature Switzerland AG Cham,Switzerland0.2019.

[22] N. Weaver, *Lipschitz Algebras* (World Scientific Publishing, River Edge, 1999).

[23] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), no. 1, 63–89.

المُلخص

تُعد خصائص التقريب من المفاهيم الأساسية في نظرية فضاءات باناخ، وقد تعود جذورها إلى أعمال غروتينديك حول المؤثرات من الرتبة المنتهية. تسمح فضاءات ليبشيتز الحرة بتمديد الأساليب الخطية إلى الإعدادات المترية من خلال ربط التطبيقات الليبشيتزية بنظرية فضاءات باناخ. وقد أثبت غودفروي وكالتون تكافؤ خاصية التقريب المحدود الليبشيتزية مع الخاصية الكلاسيكية في فضاءات باناخ.

تدرس هذه الأطروحة خصائص التقريب في فضاءات ليبشيتز الحرة. نقدم توصيفات جديدة لهذه الخصائص، ونتائج نقل بين التقريب الليبشيتزي والخطي، بالإضافة إلى صيغ مكافئة باستخدام مؤثرات ذات رتبة منتهية.

الكلمات المفتاحية: دوال ليبشيزية فضاء باناخ تقريب

Abstract

Approximation properties are fundamental in Banach space theory, originating from Grothendieck's work on finite-rank operators. Lipschitz-free spaces extend linear methods to metric settings by linking Lipschitz mappings with Banach space theory. A key result of Godefroy and Kalton shows the equivalence between the Lipschitz and classical bounded approximation properties.

This thesis studies approximation properties in Lipschitz-free spaces. We establish new characterizations of the Lipschitz bounded approximation property, prove transfer results between Lipschitz and linear approximations, and give finite-rank reformulations.

Keywords: Lipschitz functions, banach space, approximation.

Résumé

Les propriétés d'approximation sont centrales en théorie des espaces de Banach, depuis Grothendieck. Les espaces de Lipschitz libres permettent de relier applications lipschitziennes et théorie des espaces de Banach. Un résultat de Godefroy et Kalton établit l'équivalence entre les propriétés d'approximation bornée lipschitzienne et classique.

Cette thèse étudie ces propriétés dans les espaces de Lipschitz libres. Nous donnons de nouvelles caractérisations, des résultats de transfert et des reformulations en termes de rang fini.

Mots clés : fonctions lipschitziennes, espace de Banach, Approximation.