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كمراجع للدروس لطلبة السنة الثانية ليسانس رياضيات.
وهذا بعد الاطلاع على التقارير الإيجابية للأستاذ الخبير المكلف بالمطبوعة.

رئيس اللجنة العلمية



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Preface

This handout is the result of several years of experience teaching the ALG4 module to second-year university students.

It is divided into three chapters, each containing relevant examples. The first two chapters explore the concepts of symmetric bilinear and quadratic forms, as well as Hermitian forms, followed by a chapter dedicated to applications of these concepts.

I hope that both students and instructors find this handout insightful and beneficial in enhancing their understanding and teaching of the material.

ALG 4

Chapter 1

Bilinear and Quadratic forms

1.1 Bilinear forms

Definition.

Bilinear form of a vector space V is a function φ of two variables on V , with values in the field F satisfying the bilinear axioms which are:

$$\begin{aligned}\varphi(u_1 + u_2, v) &= \varphi(u_1, v) + \varphi(u_2, v) \\ \varphi(u, v_1 + v_2) &= \varphi(u, v_1) + \varphi(u, v_2) \\ \varphi(\alpha u, v) &= \alpha\varphi(u, v) = \varphi(u, \alpha v)\end{aligned}$$

for all $u, v \in V$ and $\alpha \in F$

Bilinear form will be denoted by $\langle u, v \rangle$

Remark. $\langle 0, v \rangle = \langle v, 0 \rangle = 0$

Symmetric bilinear forms

Definition 1. A bilinear form is said to be symmetric if

$$\langle u, v \rangle = \langle v, u \rangle$$

and skew symmetric if

$$\langle u, v \rangle = -\langle v, u \rangle$$

Examples.

- $\varphi(x, y) = \langle x, y \rangle$ in \mathbb{R}^n is bilinear and symmetric for any scalar product.
- $\varphi((x_1, y_1), (x_2, y_2)) = x_1y_1 + 2x_1y_2 + 3y_1x_2 + 4y_1y_2$ is bilinear, but not symmetric.

Proprieties of a bilinear form. Let V be a vector space and let φ be a bilinear form on V . We have

$$\begin{aligned}\varphi(u+v, u+v) &= \varphi(u, u) + 2\varphi(u, v) + \varphi(v, v) \\ \varphi(u-v, u-v) &= \varphi(u, u) - 2\varphi(u, v) + \varphi(v, v) \\ \varphi(u, v) &= \frac{1}{2}(\varphi(u+v, u+v) - \varphi(u, u) - \varphi(v, v)) \\ \varphi(u, v) &= \frac{1}{4}(\varphi(u+v, u+v) - \varphi(u-v, u-v))\end{aligned}$$

Definition 2. A $n \times n$ matrix A is called symmetric if $A^t = A$

Theorem 1. Bilinear form given in above example is symmetric if and only if matrix A is symmetric.

Proof. Assume that A is symmetric. Since V^tAU is a 1×1 matrix, it is equal to its transpose: $V^tAU = (V^tAU)^t = U^tA^tV = U^tAV$ and hence $\langle V, U \rangle = \langle U, V \rangle$ and it follows that form is symmetric. Conversely, let the form is symmetric. Set $U = e_i$ and $V = e_j$ where e_i and e_j are elements of fixed basis. We find that $\langle e_i, e_j \rangle = e_i^t A e_j = a_{ij}$ while $\langle e_j, e_i \rangle = e_j^t A e_i = a_{ji}$ and as the form is symmetric we get that $a_{ij} = a_{ji}$ and the matrix A is symmetric. \square

Computation of the value of bilinear form. Let $u, v \in V$ and let U and V be their coordinates in the basis B so that $u = BU$ and $v = BV$. Then

$$\langle u, v \rangle = \left\langle \sum_i u_i x_i, \sum_j v_j y_j \right\rangle$$

This expands using bilinearity to $\sum_{i,j} x_i y_j \langle u_i, v_j \rangle = \sum_{i,j} x_i a_{ij} y_j = U^tAV$

$$\langle u, v \rangle = U^tAV$$

Thus if we identify V with F^n using basis B then bilinear form \langle, \rangle corresponds to U^tAV .

Matrix of a bilinear form

Definition. Let $B = (v_1, \dots, v_n)$ be a basis of V and let φ be a bilinear form on V . The matrix of φ with respect to B is

$$[\varphi]_B = \begin{bmatrix} \varphi(v_1, v_1) & \varphi(v_1, v_2) & \cdots & \varphi(v_1, v_n) \\ \varphi(v_2, v_1) & \varphi(v_2, v_2) & \cdots & \varphi(v_2, v_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(v_n, v_1) & \varphi(v_n, v_2) & \cdots & \varphi(v_n, v_n) \end{bmatrix}$$

Expression for a bilinear form on a basis.

Lemma 1. Let $B = (v_1, \dots, v_n)$ be a basis of V and let φ be a bilinear form on V . For any $u, v \in V$, we have

$$\varphi(u, v) = U^t AV$$

Proof. Let $u = (\alpha_1, \dots, \alpha_n)$ and $v = (\beta_1, \dots, \beta_n)$. We have

$$\begin{aligned} \varphi(u, v) &= \varphi(\alpha_1 v_1 + \dots + \alpha_n v_n, \beta_1 v_1 + \dots + \beta_n v_n) \\ &= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j \varphi(v_i v_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j ([\varphi]_C)_{i,j} \\ &= U^T AV \end{aligned}$$

□

If φ is a symmetric bilinear form, then

$$\varphi(u, v) = \sum_{1 \leq i \leq n} \alpha_i \beta_i \varphi(e_i, e_i) + 2 \sum_{1 \leq i < j \leq n} \alpha_i \beta_j \varphi(e_i, e_j)$$

Remark. $[\varphi]_C$ is the only matrix with this property. A bilinear form φ is symmetric if and only if $[\varphi]_C$ is a symmetric matrix.

Corollary. Let V be a vector space over a field F . Let $B = (v_1, \dots, v_n)$ be a basis of V . For every $n \times n$ matrix M over F , there exists a unique bilinear form $\varphi : V \times V \rightarrow F$ such that $\varphi(v_i, v_j) = M_{i,j}$ for $1 \leq i, j \leq n$.

Proof. Define $\varphi(u, v) = U^T AV$ and observe that φ is bilinear. No other bilinear form with this property exists, since any bilinear form satisfying the assumption has matrix M , which Lemma 1. uniquely determines the value of the bilinear form. □

Example. The bilinear form $\varphi((x_1, y_1), (x_2, y_2)) = x_1 x_2 + 2x_1 y_2 + 3y_1 x_2 + 4y_1 y_2$ has matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with respect to the standard basis

$$\varphi((x_1, y_1), (x_2, y_2)) = [x_1, y_1] \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

Change of basis.

Lemma 2. Let $B = (v_1, \dots, v_n)$ and $B' = (v'_1, \dots, v'_n)$ be two bases of V and let φ be a bilinear form on V . Let $P = [id]_{B, B'}$. Then

$$[\varphi]_{B'} = P^t[\varphi]_B P$$

Proof. We have

$$\begin{aligned} (P^t[\varphi]_B P)_{i,j} &= e_i^t P^t [\varphi]_{B'} P e_j \\ &= [v_i]_B P^t [\varphi]_{B'} P [v_j]_B^t \\ &= [v_i]_{B'} [\varphi]_{B'} [v_j]_{B'}^t \\ &= \varphi(v_i, v_j) = ([\varphi]_{B'})_{i,j} \end{aligned}$$

□

Application.

Let $\langle \cdot, \cdot \rangle$ be a bilinear form on \mathbb{R}^2 defined by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = 2x_1y_1 - 3x_1y_2 + x_2y_2$$

1. Find the matrix A of this bilinear form in the basis $\{u_1 = (1, 0)$ and $u_2 = (1, 1)\}$
2. Find the matrix B of given bilinear form in the basis $\{v_1 = (2, 1)$ and $v_2 = (1, -1)\}$
3. Find the transition matrix P from the basis $\{u_i\}$ to $\{v_i\}$ and verify that $B = P^t A P$

Solution.

1. Set $A = (a_{ij})$ where $a_{ij} = \langle u_i, u_j \rangle$
 $a_{11} = \langle u_1, u_1 \rangle = \langle (1, 0), (1, 0) \rangle = 2 - 0 + 0 = 2$
 Rest of the entries in the matrix are calculated using the following formula:

$$\begin{aligned} a_{12} &= \langle u_1, u_2 \rangle \\ a_{21} &= \langle u_2, u_1 \rangle \\ a_{22} &= \langle u_2, u_2 \rangle \end{aligned}$$

Thus the matrix A is as follows

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$$

2. Similarly matrix B is

$$B = \begin{bmatrix} 3 & 9 \\ 0 & 6 \end{bmatrix}$$

3. Now we write $E - 1$ and v_2 in terms of u_1 and u_2

$$\begin{aligned} (2, 1) &= u_1 + u_2 \\ (1, -1) &= 2u_1 - u_2 \end{aligned}$$

Thus $P = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ and so $P^t = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$. Thus $P^t A P = P = \begin{bmatrix} 3 & 9 \\ 0 & 6 \end{bmatrix} = B$

Example. Let A be an $n \times n$ matrix in F and define

$$\langle u, v \rangle = U^t A V$$

where U and V are coordinates of u and v respectively in some basis of V . Then we see that this defines a bilinear form on V . This coincides with usual inner product of V if $A = I$.

1.2 Quadratic forms

Definition.

A quadratic form q on $V = K^n$ is a function $q : K^n \rightarrow K$ given by

$$q(u) = q(x_1, x_2, \dots, x_n) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j$$

Property.

$$\forall c \in K, \quad q(cv) = c^2 q(v)$$

Symmetric bilinear form on K^n and Quadratic form on K^n

Definition 1. We can define a quadratic form q using a symmetric bilinear form

$$q(u) = \varphi(u, u)$$

$$q(u) = q(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_n) A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$q(u) = U^t A U$$

Example. Let q be a quadratic form defined on $F^3 : q(x, y, z) = x^2 + 4xy + 3y^2 - 6yz + xz - 2z^2$. The matrix A of q is

$$A = \begin{bmatrix} 1 & 2 & \frac{1}{2} \\ 2 & 3 & -3 \\ \frac{1}{2} & -3 & -2 \end{bmatrix}$$

Lemma. Let q be a quadratic form on $V = F^n$, $\text{char} F \neq 2$, that comes from a symmetric bilinear form $V, q(u) = \varphi(u, u)$. Then the bilinear form may be recovered from q :

$$\varphi(u, v) = \frac{1}{2}[q(u+v) - q(u) - q(v)]$$

Proof. $\frac{1}{2}(\varphi(u+v, u+v) - \varphi(u, u) - \varphi(v, v)) = \frac{1}{2}(\varphi(u, v) + \varphi(v, u)) = \varphi(u, v) \quad \square$

Remark 1. The correspondence between quadratic forms and symmetric matrices is one-to-one, when a basis is fixed. So quadratic forms are simply homogeneous polynomials in n variables, where each monomial has a degree 2.

Definition 2. Two matrices A and B are called congruent if $A = P^t B P$ for some non-singular P .

Theorem. Every real symmetric matrix A is congruent to a diagonal matrix

$$D = P^t A P$$

Proposition. Let q be a quadratic form, $q : V = F^n \rightarrow F$ and $\dim V = n$

$$q(u) = q(x_1, \dots, x_n) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j$$

Then $\exists \alpha_1, \alpha_2, \dots, \alpha_n \in F$ and l_1, l_2, \dots, l_n are linear forms such that:

$$q(u) = q(x_1, \dots, x_n) = \sum_{i=1}^n \alpha_i (l_i(x_1, \dots, x_n))^2$$

Proof. Using the proof by induction over the dimension of V , $\dim V = n$

$$\begin{cases} p(1) & \text{is true} \\ p(n-1) \Rightarrow p(n) & \text{is true} \end{cases}$$

For $n = 1$, $p(1)$ is true.

Let $n \geq 2$:

Case 1. $\exists i/a_{ii} \neq 0$, for example, $a_{11} \neq 0$, q has a pure square, the term $a_{11}x_1^2$. Consider all terms which contain x_1 and complete the square. We write all terms containing x_1 as:

$$\begin{aligned} & a_{11}x_1^2 + \sum_{j=2}^n a_{1j}x_1x_j \\ &= a_{11} \left[x_1^2 + 2x_1 \sum_{j=2}^n \frac{a_{1j}x_j}{2a_{11}} \right] \\ &= a_{11} \left[\left(\underbrace{x_1 + \sum_{j=2}^n \frac{a_{1j}x_j}{2a_{11}}}_{l_1} \right)^2 - \left(\sum_{j=2}^n \frac{a_{1j}x_j}{2a_{11}} \right)^2 \right] \\ &= a_{11} (l_1(x_1, \dots, x_n))^2 - a_{11} \left(\sum_{j=2}^n \frac{a_{1j}x_j}{2a_{11}} \right)^2 \end{aligned}$$

Then $q(x_1, \dots, x_n) = a_{11}(l_1(x_1, \dots, x_n))^2 + q'(x_1, \dots, x_n)$

q' is a quadratic form (Q.F) over F^{n-1} and using the inductive hypothesis, we obtain

$$q(x_1, \dots, x_n) = \sum_{k=1}^n \alpha_k (l_k(x_1, \dots, x_n))^2.$$

Case 2. If q has no terms $a_{ii}x_i^2$, but has a term of the form $a_{ij}x_ix_j$, for example $a_{12}x_1x_2$ ($a_{12} \neq 0$). Consider all terms which contain x_1 or x_2 :

$$\begin{aligned} a_{12}x_1x_2 + \sum_{j=3}^n a_{ij}x_ix_j + \sum_{j=3}^n a_{2j}x_2x_j \\ &= a_{12}x_1x_2 + Bx_1 + Cx_2 \\ &= a_{12} \left(x_1x_2 + \frac{B}{a_{12}} + \frac{C}{a_{12}}x_2 \right) \\ &= a_{12} \left(\underbrace{x_1 + \frac{C}{a_{12}}}_a \right) \left(\underbrace{x_2 + \frac{B}{a_{12}}}_b \right) - \frac{BC}{a_{12}} \end{aligned}$$

$$q(u) = a_{12}ab + q''(x_1, \dots, x_n)$$

q'' is a Q.F over F^{n-2} , since $ab = \frac{1}{4}(a+b)^2 - \frac{1}{4}(a-b)^2$ and by the inductive hypothesis we obtain

$$q(u) = \frac{a_{12}}{4}(x_1+x_2+l'_1(x_3, \dots, x_n))^2 - \frac{a_{12}}{4}(x_1-x_2+l'_2(x_3, \dots, x_n))^2 + \sum_{k=3}^n \alpha_k(l_k(x_3, \dots, x_n))^2$$

Finally, $q(u) = \sum_{k=1}^n \alpha_k(l_k(x_1, \dots, x_n))^2$. \square

Remark 2. This procedure is called Gauss method and is used to write a quadratic form as sum of squares.

Example.

$$V = \mathbb{R}^4,$$

$$q(u) = q(x_1, x_2, x_3, x_4) = x_1^2 + 9x_2^2 + 4x_3^2 + 6x_1x_2 + 4x_1x_3 + 16x_2x_3 + 4x_2x_4 + 8x_3x_4$$

The matrix A of q is $A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & 9 & 8 & 2 \\ 2 & 8 & 4 & 4 \\ 0 & 2 & 4 & 0 \end{bmatrix}$. A is symmetric because $A^t = A$.

We consider all terms which contain x_1 :

$$x_1^2 + 6x_1x_2 + 4x_1x_3 = x_1^2 + 2x_1(3x_2 + 2x_3) = (x_1 + 3x_2 + 2x_3)^2 - (3x_2 + 2x_3)^2 = (x_1 + 3x_2 + 2x_3)^2 - 9x_2^2 - 4x_3^2 - 12x_2x_3.$$

We can write q : $q(u) = q(x_1, x_2, x_3, x_4) = (x_1 + 3x_2 + 2x_3)^2 + 4x_2x_3 + 4x_2x_4 + 8x_3x_4$.

We consider all terms which contain x_2 or x_3 :

$$4x_2x_3 + 4x_2x_4 + 8x_3x_4 = 4(x_2x_3 + x_2x_4 + 2x_3x_4) = 4[(x_2 + 2x_4)(x_3 + x_4) - 2x_4^2] =$$

$4[\frac{1}{4}(x_2 + x_3 + 3x_4)^2 - \frac{1}{4}(x_2 - x_3 + x_4)^2 - 2x_4^2] = (x_2 + x_3 + 3x_4)^2 - (x_2 - x_3 + x_4)^2 - 8x_4^2$.
Finally,

$$q(u) = q(x_1, x_2, x_3, x_4) = (x_1 + 3x_2 + 2x_3)^2 + (x_2 + x_3 + 3x_4)^2 - (x_2 - x_3 + x_4)^2 - 8x_4^2$$

The matrix A is congruent to a diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -8 \end{bmatrix}$

and we can write $D = P^tAP$. We put $\begin{cases} x'_1 = x_1 + 3x_2 + 3x_3 \\ x'_2 = x_2 + x_3 + 3x_4 \\ x'_3 = x_2 - x_3 + x_4 \\ x'_4 = x_4 \end{cases}$, then $\begin{cases} x_1 = x'_1 - \frac{5}{2}x'_2 - \frac{1}{2}x'_3 + 8x'_4 \\ x_2 = \frac{1}{2}x'_2 + \frac{1}{2}x'_3 - 2x'_4 \\ x_3 = \frac{1}{2}x'_2 - \frac{1}{2}x'_3 - x'_4 \\ x_4 = x'_4 \end{cases}$

We obtain $P = \begin{bmatrix} 1 & -\frac{5}{2} & -\frac{1}{2} & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & -2 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The new basis is $B' = (e'_1, e'_2, e'_3, e'_4)$, where

$$e'_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, e'_2 = \begin{pmatrix} -\frac{5}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, e'_3 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} \text{ and } e'_4 = \begin{pmatrix} 8 \\ -2 \\ -1 \\ 1 \end{pmatrix}$$

Finally, $q(u') = q(x'_1, x'_2, x'_3, x'_4) = x_1'^2 + x_2'^2 - x_3'^2 - 8x_4'^2$

Positive definite forms

Definition.

A bilinear form φ on a real vector space V is positive definite, if $\varphi(u, u) > 0$, for all $u \neq 0$.

A real $n \times n$ matrix A is positive definite, if $U^tAU > 0$, for all $U \neq 0$.

Remark. A bilinear form on V is positive definite if and only if the matrix of the form with respect to some basis of V is positive definite.

Examples.

1. A positive definite form on \mathbb{R}^n is given by the dot product (\cdot) .
 $u \cdot v = \sum_{i=1}^n x_i y_i \implies u \cdot u = \sum_{i=1}^n x_i^2 > 0$, for all $u = (x_1, x_2, \dots, x_n) \neq 0$.

2. Consider the symmetric bilinear form on \mathbb{R}^n which is defined by $\varphi(u, v) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$. The quadratic form is $q(u) = \varphi(u, u) = x_1^2 - 4x_1x_2 + 5x_2^2 = (x_1 - 2x_2)^2 + x_2^2$. Using Gauss method, we can write: $q(u) = (x_1 - 2x_2)^2 + x_2^2$. Then the form φ is positive definite because $q(u) > 0$, for all $u \neq 0$.

Tests for positive definiteness.

Theorem. The following conditions are equivalent for a symmetric matrix A :

1. $\varphi(u, u) = U^tAU > 0$ for all $u \neq 0$.
2. The eigenvalues of A are all positive $\forall \lambda_i, \lambda_i > 0$.
3. One has $\det A_k > 0$ for all $k \times k$ upper left submatrices A_k (Sylvester's criterion).

Remark. We say that A is negative definite, if A has negative eigenvalues.

Example 1. $A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$A_1 = [2] \rightarrow \det A_1 = 2 > 0$$

$$A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \rightarrow \det A_2 = 5 > 0$$

$$A_3 = A \rightarrow \det A_3 = \det A > 0$$

Then A is positive definite.

Example 2. Let a be a real parameter and consider the matrix $A = \begin{bmatrix} a & 1 & 1 \\ 1 & 1 & a \\ 1 & a & 5 \end{bmatrix}$

By **Sylvester's criterion**, A is positive definite if and only if

$$a > 0, \quad \det \begin{bmatrix} a & 1 \\ 1 & 1 \end{bmatrix} > 0, \quad \det A > 0$$

The first two conditions give $a > 0$ and $a > 1$, while

$$\det A = -a^3 + 7a - 6 = -(a - 1)(a - 2)(a + 3)$$

It easily follows that A is positive definite if and only if $1 < a < 2$.

1.3 Orthogonality

Suppose that φ is a symmetric bilinear form on a real vector space V :

1. **Orthogonal vectors:** Two vector u, v are called orthogonal, if $\varphi(u, v) = 0$.
2. **Orthogonal basis:** A basis $B = (v_1, v_2, \dots, v_n)$ of V is called orthogonal, if $\varphi(v_i, v_j) = 0$ for all $i \neq j$ and it is called orthonormal, if it is orthogonal with $\varphi(v_i, v_i) = 1$ for all i .
3. If F is a subspace of V , the orthogonal of F is $F^\perp = \{u \in V / \varphi(u, v) = 0, \forall v \in F\}$, which is also a subspace of V .
4. **Isotropic vectors:** A vector $u (u \neq 0)$ is called isotropic, if $q(u) = \varphi(u, u) = 0$.
5. **Kernel, non-degenerate forms:** φ is called a non-degenerate form, if $E^\perp = \{u \in V / \varphi(u, v) = 0, \forall v \in V\} = \{0\}$. Otherwise, φ is called degenerate. The kernel of φ or q , $\ker\varphi = \ker q = E^\perp$.
6. The isotropic cone of a quadratic form q is the set of all isotrops of V under q . $C(q) = \{u \in V / q(u) = 0\}$
7. A subspace F of V is called isotropic, if $F \cap F^\perp \neq (0)$.

Proprieties.

- $\ker q \subset C(q)$
- $\dim V = \dim \ker(q) + \text{rg}(q)$
- $\dim V = \dim F + \dim F^\perp - \dim(F \cap \ker q)$, F is a subspace of V .
In particular, if q is non-degenerate, $\dim V = \dim F + \dim F^\perp$.
- $F^{\perp\perp} = F + \ker q$
- $V = F \oplus F^\perp \iff F$ is not isotropic ($F \cap F^\perp = 0$).

Gram-Schmidt procedure.

Suppose that (v_1, v_2, \dots, v_n) is a basis of a dot product space V , then we can find an orthogonal basis $(v'_1, v'_2, \dots, v'_n)$ as follows:

We put

$$v'_1 = v_1$$

$$v'_2 = v_2 - \frac{v_2 \cdot v'_1}{v'_1 \cdot v'_1} v'_1$$

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$$v'_n = v_n - \sum_{i=1}^{n-1} \frac{v_n \cdot v'_i}{v'_i \cdot v'_i} v'_i$$

then v'_1, v'_2, \dots, v'_n are orthogonal.

Example.

We find an orthogonal basis of \mathbb{R}^3 , starting with the basis

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

We define the first vector by $v'_1 = v_1$ and the second by

$$v'_2 = v_2 - \frac{v_2 \cdot v'_1}{v'_1 \cdot v'_1} v'_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Then v'_1, v'_2 are orthogonal and we may define the third vector by

$$v'_3 = v_3 - \frac{v_3 \cdot v'_1}{v'_1 \cdot v'_1} v'_1 - \frac{v_3 \cdot v'_2}{v'_2 \cdot v'_2} v'_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Theorem. The eigenvalues of a real symmetric matrix A are all real. i.e $\lambda_i \in \mathbb{R}$
The eigenvectors of a real symmetric matrix A corresponding to distinct eigenvalues are necessarily orthogonal to one another. i.e $\lambda_i \neq \lambda_j \Rightarrow v_i \cdot v_j = 0$

Orthogonal matrices.

Definition. A real $n \times n$ matrix P is called orthogonal, if $P^t P = I_n$ i.e $P^{-1} = P^t$.

Proprieties.

- To say that an $n \times n$ matrix is orthogonal is to say that the columns of P form an orthonormal basis of \mathbb{R}^n .
- The product of two $n \times n$ orthogonal matrices is orthogonal.

Example.
$$P = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Spectral theorem.

Every real symmetric matrix A is diagonalisable. In fact, there exists an orthogonal matrix P such that $P^{-1}AP = P^tAP$ is diagonal.

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = P^{-1}AP = P^tAP$$

Remarks.

- When the eigenvalues of A are distinct, the eigenvectors of A are orthogonal and we may simply divide each of them by its norm to obtain an orthonormal basis of \mathbb{R}^n .
- When the eigenvalues of A are not distinct, the eigenvectors of A may not be orthogonal. In that case, one may use the **Gram-Schmidt procedure** to replace eigenvectors that have the same eigenvalue with orthogonal eigenvectors that have the same eigenvalue.
- The converse of the spectral theorem is also true. That is, if P is an orthogonal matrix and P^tAP is diagonal, then A is symmetric.

1.4 Diagonalisation of quadratic forms

Theorem. Let $q(u) = U^tAU$ for some symmetric $n \times n$ matrix A . Then there exists an orthogonal change of variables $U = PU'$ such that:

$$q(u') = q(x'_1, x'_2, \dots, x'_n) = \sum_{i=1}^n \lambda_i x_i'^2$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix A .

Signature of a quadratic form.

Definition. The signature of a quadratic form $q(u) = U^tAU$ is defined as the pair of integers (n_+, n_-) , where n_+ is the number of positive eigenvalues of A and n_- is the number of negative eigenvalues of A .

Examples.

1. We diagonalise the quadratic form in \mathbb{R}^2 , $B = (e_1, e_2)$ the standard basis

$$q(u) = q(x_1, x_2) = 5x_1^2 + 4x_1x_2 + 2x_2^2$$

We have $A = M_B(q) = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$.

The eigenvalues $\lambda = 1, 6$ are distinct and one can easily check that

$$P = (e'_1 e'_2) = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}, \text{ then } D = M_{B'}(q) = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} = P^{-1}AP = P^tAP$$

As usual, the columns of P were obtained by finding the eigenvectors of A and by dividing each eigenvector by its norm.

Changing variables by $U = PU'$, we now get $U' = P^tU$ and also

$$q(u') = q(x'_1, x'_2) = x_1'^2 + 6x_2'^2 = \left(\frac{x_1 - 2x_2}{\sqrt{5}}\right)^2 + 6\left(\frac{2x_1 + x_2}{\sqrt{5}}\right)^2.$$

We can use the Gauss method to find the sum of squares of q .

$$\begin{aligned} \text{We take } (I) : 5x_1^2 + 4x_1x_2 &= 5\left(x_1^2 + \frac{4}{5}x_1x_2\right) \\ &= 5\left[x_1^2 + 2x_1\left(\frac{2}{5}x_2\right)\right] = 5\left[\left(x_1 + \frac{2}{5}x_2\right)^2 - \left(\frac{2}{5}x_2\right)^2\right] = 5\left[\left(x_1 + \frac{2}{5}x_2\right)^2 - \frac{4}{25}x_2^2\right] = \\ &5\left(x_1 + \frac{2}{5}x_2\right)^2 - \frac{4}{5}x_2^2 \end{aligned}$$

replace this in $q(u)$

$$q(u) = 5\left(x_1 + \frac{2}{5}x_2\right)^2 - \frac{4}{5}x_2^2 + 2x_2^2, \text{ we obtain}$$

$$q(u) = 5\left(x_1 + \frac{2}{5}x_2\right)^2 + \frac{6}{5}x_2^2$$

$$\text{We put } \begin{cases} x'_1 = x_1 + \frac{5}{2}x_2 \\ x'_2 = x_2 \end{cases} \Rightarrow \begin{cases} x_1 = x'_1 - \frac{5}{2}x'_2 \\ x_2 = x'_2 \end{cases}$$

We obtain the orthogonal matrix P and the new basis $B' = (e'_1, e'_2)$:

$$P = (e'_1 e'_2) = \begin{bmatrix} 1 & -\frac{5}{2} \\ 0 & 1 \end{bmatrix} \text{ and the formula } D = M_{B'}(q) = \begin{bmatrix} 5 & 0 \\ 0 & \frac{6}{5} \end{bmatrix} = P^tAP.$$

Finally,

$$q(u') = q(x'_1, x'_2) = 5x_1'^2 + \frac{6}{5}x_2'^2$$

The signature of q is $(2, 0)$, q is a non-degenerate form and the rank of q is 2.

2. We consider the quadratic form defined in \mathbb{R}^3 by the real symmetric matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$q(u) = U^t A U = (x, y, z) \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x, y, z) \begin{bmatrix} 2x + y + z \\ x + 2y + z \\ x + y + 2z \end{bmatrix}$$

$$\begin{aligned} &= x(2x + y + z) + y(x + 2y + z) + z(x + y + 2z) \\ &= 2x^2 + xy + xz + yx + 2y^2 + 2yz + zx + zy + 2z^2 \end{aligned}$$

$$\boxed{q(u) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz}$$

Diagonalisation of A:

$$\text{We have } P(\lambda) = -(\lambda - 1)^2(\lambda - 4), \quad P(\lambda) = 0 \Rightarrow \begin{cases} \lambda_1 = 1, m_1 = 2 \\ \lambda_2 = 4, m_2 = 1 \end{cases}$$

$$E(\lambda_1) = \text{span}\{v_1, v_2\}, \quad E(\lambda_2) = \text{span}\{v_3\}$$

$$\text{where } v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

In this case, use the Gram-Schmidt procedure to replace v_1, v_2 by two orthogonal eigenvectors v'_1, v'_2 , dividing each of v'_1, v'_2, v_3 by its norm, we then obtain the columns of the orthogonal matrix: we put

$$v'_1 = v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$v'_2 = v_2 - \frac{v_2 \cdot v'_1}{v'_1 \cdot v'_1} v'_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

$$v'_3 = v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We obtain the orthogonal matrix and the new basis (orthonormal basis)

$B' = (e'_1, e'_2, e'_3)$ and

$$P = (e'_1 e'_2 e'_3) = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

We obtain the formula $D = M_{B'}(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = P^{-1}AP = P^tAP$

we have $U = PU'$, then $U' = P^tU \Rightarrow \begin{cases} x'_1 = \frac{1}{\sqrt{2}}(-x_1 + x_3) \\ x'_2 = \frac{1}{\sqrt{6}}(-x_1 + 2x_2 - x_3) \\ x'_3 = \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3) \end{cases}$

Finally,

$$q(u') = (x'_1, x'_2, x'_3) = x_1'^2 + x_2'^2 + 4x_3'^2$$

The signature of q is $(3, 0)$ and the rank equals 3. q is a non-degenerate form.

Gauss Method:

$$q(u) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz$$

$$\begin{aligned} \text{We take (I): } 2x^2 + 2xy + 2xz &= 2 \left[x^2 + 2x \left(\frac{y+z}{2} \right) \right] \\ &= 2 \left[\left(x + \frac{1}{2}y + \frac{1}{2}z \right)^2 - \left(\frac{y+z}{2} \right)^2 \right] \\ &= 2 \left(x + \frac{1}{2}y + \frac{1}{2}z \right)^2 - \frac{1}{2}(y^2 + z^2 + 2yz) \\ &= 2 \left(x + \frac{1}{2}y + \frac{1}{2}z \right)^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 - yz \end{aligned}$$

$$q(u) = 2 \left(x + \frac{1}{2}y + \frac{1}{2}z \right)^2 + \frac{3}{2}y^2 + \frac{3}{2}z^2 + yz$$

$$\begin{aligned} \text{We take (II): } \frac{3}{2}y^2 + yz &= \frac{3}{2} \left(y^2 + \frac{2}{3}yz \right) = \frac{3}{2} \left[y^2 + 2y \left(\frac{z}{3} \right) \right] \\ &= \frac{3}{2} \left[\left(y + \frac{z}{3} \right)^2 - \left(\frac{z}{3} \right)^2 \right] \\ &= \frac{3}{2} \left(y + \frac{1}{3}z \right)^2 - \frac{1}{6}z^2 \end{aligned}$$

Then,

$$q(u) = q(x, y, z) = 2 \left(x + \frac{1}{2}y + \frac{1}{2}z \right)^2 + \frac{3}{2} \left(y + \frac{1}{3}z \right)^2 + \frac{4}{3}z^2$$

$$D = M_{B'}(q) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = P^t A P$$

$$\text{We put } \begin{cases} x' = x + \frac{1}{2}y + \frac{1}{2}z \\ y' = y + \frac{1}{3}z \\ z' = z \end{cases} \Rightarrow \begin{cases} x = x' - \frac{1}{2}(y' - \frac{1}{3}z') - \frac{1}{2}z' \\ y = y' - \frac{1}{3}z' \\ z = z' \end{cases} \Rightarrow \begin{cases} x = x' - \frac{1}{2}y' - \frac{1}{3}z' \\ y = y' - \frac{1}{3}z' \\ z = z' \end{cases}$$

$$\text{We obtain the orthogonal matrix } P = (e'_1 e'_2 e'_3) = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{The new basis is } B'(e'_1, e'_2, e'_3): e'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e'_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, e'_3 = \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Finally,

$$\boxed{q(u') = q(x', y', z') = 2x'^2 + \frac{3}{2}y'^2 + \frac{4}{3}z'^2}$$

Chapter 2

Hermitian and hermitian quadratic forms

Definition 1. Let V be a \mathbb{C} -vector space. A hermitian form is a function φ of V in \mathbb{C} , satisfying

$$\begin{aligned}\varphi(u_1 + u_2, v) &= \varphi(u_1, v) + \varphi(u_2, v) \\ \varphi(u, v_1 + v_2) &= \varphi(u, v_1) + \varphi(u, v_2) \\ \varphi(\alpha u, v) &= \alpha \varphi(u, v) \\ \varphi(u, \alpha v) &= \bar{\alpha} \varphi(u, v) \\ \varphi(u, u) &= \overline{\varphi(u, u)}\end{aligned}$$

Remark 1. Since $\varphi(u, u) = \overline{\varphi(u, u)}$, then $\varphi(u, u) \in \mathbb{R}$.

Definition 2. An hermitian quadratic form is a function $q : V \rightarrow \mathbb{R}$ given by

$$q(u) = \varphi(u, u)$$

Propriety.

$$q(\alpha u) = |\alpha|^2 q(u), \text{ for all } \alpha \in \mathbb{C}$$

Proposition 1.

Let φ be an hermitian form and q is the associated hermitian quadratic form of φ , then

$$\begin{aligned}Re(\varphi(u, v)) &= \frac{q(u+v) - q(u) - q(v)}{2} = \frac{q(u+v) - q(u-v)}{4} \\ Im(\varphi(u, v)) &= \frac{q(u+iv) - q(u) - q(v)}{2} = \frac{q(u+iv) - q(u-iv)}{4}\end{aligned}$$

and,

$$\varphi(u, v) = \frac{q(u+v) - q(u-v) + iq(u+iv) - iq(u-iv)}{4}$$

Examples.

1. The form $z \mapsto |z|^2$ is an hermitian quadratic form on $V = \mathbb{C}$, associated with the hermitian form

$$(z, w) \mapsto z\bar{w}$$

2. The form $(z_1, \dots, z_n) \mapsto |z_1|^2 + \dots + |z_n|^2$ is an hermitian quadratic form, associated with the hermitian form

$$((z_1, \dots, z_n), (w_1, \dots, w_n)) \mapsto z_1\bar{w}_1 + \dots + z_n\bar{w}_n$$

Definition 3. Let $A \in M_n(\mathbb{C})$. A is called an hermitian matrix if $A^t = \bar{A}$. If $A = (a_{ij})_{1 \leq i, j \leq n}$, then $a_{ij} = \overline{a_{ji}}$ for all i, j .

Proposition 2. Let $A \in M_n(\mathbb{C})$, then $\varphi : \begin{cases} \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C} \\ (u, v) \mapsto U^t A \bar{V} \end{cases}$ is an hermitian form over \mathbb{C}^n .

Proposition-Definition. Let φ be an hermitian form and q is the associated hermitian quadratic form of φ . The matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ is a matrix of φ (or of q) over the standard basis $B = (e_1, e_2, \dots, e_n)$, where

$$a_{ij} = \varphi(e_i, e_j)$$

1. The matrix A is hermitian.
2. Let $u = \sum_{i=1}^n x_i e_i$ and $v = \sum_{j=1}^n y_j e_j$, then

$$\varphi(u, v) = \sum_{1 \leq i, j \leq n} a_{ij} x_i \bar{y}_j = U^t A \bar{V}$$

3. If $B' = (e'_1, e'_2, \dots, e'_n)$ is another basis of V , then

$$A' = M_{B'}(\varphi) = P^t A \bar{P}$$

Remark 2. We have

$$q(u) = \sum_{i=1}^n a_{ii}|x_i|^2 + \sum_{1 \leq i < j \leq n} 2\operatorname{Re}(a_{ij}x_i\bar{x}_j)$$

The rank of φ is the rank of its matrix over all basis of V .

The form φ (or q) is called non-degenerate if φ is of the rank n .

Theorem. There exists an orthogonal basis of V for the hermitian quadratic form q and

$$q(u) = q(x_1, x_2, \dots, x_n) = \sum_{i=1}^k \alpha_i |l_i(x_1, x_2, \dots, x_n)|^2$$

where $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ and l_1, \dots, l_k are the linear forms over V .

Example. Let q be an hermitian quadratic form over \mathbb{C}^3 defined by

$$q(z_1, z_2, z_3) \mapsto z_1\bar{z}_1 + 3z_2\bar{z}_2 - z_3\bar{z}_3 + iz_1\bar{z}_2 - iz_2\bar{z}_1 - z_1\bar{z}_3 - z_3\bar{z}_1 + 2iz_2\bar{z}_3 - 2iz_3\bar{z}_2$$

The matrix of q is $A = M_B(q) = \begin{bmatrix} 1 & i & -1 \\ -i & 3 & 2i \\ -1 & -2i & 1 \end{bmatrix}$

$$\begin{aligned} q(z_1, z_2, z_3) &= (|z_1|^2 + 2\operatorname{Re}(iz_1\bar{z}_2) + 2\operatorname{Re}(-z_1\bar{z}_3)) + 3|z_2|^2 - |z_3|^2 + 2\operatorname{Re}(2iz_2\bar{z}_3) \\ &= (|z_1 - iz_2 - z_3|^2 - |z_2|^2 - |z_3|^2 - 2\operatorname{Re}(iz_2\bar{z}_3)) + 3|z_2|^2 - |z_3|^2 + 4\operatorname{Re}(iz_2\bar{z}_3) \\ &= |z_1 - iz_2 - z_3|^2 + 2\left|z_2 - \frac{iz_3}{2}\right|^2 - \frac{5|z_3|^2}{2} \end{aligned}$$

$$\begin{aligned} q(z_1, z_2, z_3) &= 2|w_1|^2 - 2|w_2|^2 + 2\operatorname{Re}((1+2i)w_1\bar{z}_3) - 2\operatorname{Re}(w_3\bar{z}_3) \\ &= 2\left|w_1 + \left(\frac{1}{2} - i\right)z_3\right|^2 - \frac{5}{2}|z_3|^2 - 2\left|w_2 + \frac{z_3}{2}\right|^2 + \frac{|z_3|^2}{2} \\ &= \frac{1}{2}|iz_1 + z_2 + (1-2i)z_3|^2 - \frac{1}{2}|iz_1 - z_2 + z_3|^2 - 2|z_3|^2 \end{aligned}$$

Chapter 3

Exercises and mock exam

Exercise 1. Find the matrix associated to each one of the following bilinear forms with respect to the canonical basis:

a. $f : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $f((x, y), (x', y')) = xx' + 3xy' - yx' + 2yy'$

b. $f : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $f((x_1, y_1), (y_1, y_2)) = x_1y_2 - x_2y_1$

c. $f : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$, $f((x, y, z), (x', y', z')) = xx' + 2xz' - 4yy' + 2zx' + 8zz'$

Exercise 2. Indicate whether the bilinear forms in **Exercise 1.** are symmetric, asymmetric or neither one thing nor the other.

Exercise 3.

a. Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a bilinear form whose associated matrix relative to the canonical basis is $F_C = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$. Obtain $f((1, 1), (3, 2))$.

b. Let $f : M_{2 \times 2}(\mathbb{R}) \times M_{2 \times 2}(\mathbb{R}) \rightarrow \mathbb{R}$ be a bilinear form whose associated matrix relative to the canonical basis is $F_C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 3 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$.

Obtain $f\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\right)$.

Exercise 4. Let B be the canonical basis of \mathbb{R}^2 and let $B' = \{(3, 2), (1, 1)\}$.

- a. $F_B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is the matrix associated to a certain bilinear form relative to the basis B . Obtain its associated matrix $F_{B'}$ relative to the basis B' .
- b. $F_{B'} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ is the matrix associated to a certain bilinear relative to the basis B' . Obtain its associated matrix F_B relative to the basis B .

Exercise 5. Let V be the space of real 2×2 matrices. Consider the bilinear form $\langle A, B \rangle = \text{trace}(AB)$.

- a. Compute the matrix of the form with respect to the standard basis $B = \{e_1, e_2\}$.
- b. Calculate the signature of this form. Is it a positive definite form?
- c. Find an orthogonal basis for V .
- d. Let W be the subspace of V of trace zero matrices. Determine the signature of the form restricted to W .

Exercise 6. For the bilinear form $\varphi(p, q) = \int_0^1 p(x)q(x)dx$ on $P_2(\mathbb{R})$, find $[\varphi]_B$ for the basis $B = \{1, x, x^2\}$.

Exercise 7. We consider the linear map $q : \mathbb{R}^3 \longrightarrow \mathbb{R}$, defined by

$$q(x, y, z) = x^2 - 2yz - xz$$

1. Is q a quadratic form?
2. Find the matrix A of q .
3. Determine the bilinear form φ associated to q .
4. Find the sum of squares, using the Gauss method.
5. Find the rank and signature of q .
6. Find an orthogonal basis B' for q .

Exercise 8. Let the linear map $q : \mathbb{R}^3 \rightarrow \mathbb{R}$, defined by

$$q(x, y, z) = xy + yz + zx$$

1. Find the matrix A of q .
2. Using Gauss method, deduce the sum of squares and an orthogonal basis B' for q .
3. Find the rank and signature of q .
4. Find whether q has isotropic vectors and in that case find them.

Exercise 9. Let q be a quadratic form over \mathbb{R}^4 defined by $\forall u = (x, y, z, t)$ in \mathbb{R}^4 ,

$$q(u) = 9x^2 + 5y^2 + 8yz - 4yt + 5z^2 + 4zt + 8t^2$$

1. Find the matrix associated to q .
2. Using Gauss method, define the sum of squares of q .
3. Find the rank and signature of q and whether it is degenerate or not.
4. Find $\ker(q)$, the kernel of q .
5. Find the isotropic vectors of q .

Exercise 10. Let q be a quadratic form over \mathbb{R}^4 defined by $\forall x = (x_1, x_2, x_3, x_4)$ in \mathbb{R}^4 ,

$$q(x) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$

1. Find the matrix associated to q .
2. Using Gauss method, find the sum of squares of q .
3. Find the rank and signature of q and whether it is degenerate or not.
4. Find an orthogonal basis B' for q , i.e a basis such that $M_{B'}(q)$ is a diagonal matrix.

Exercise 11. We consider the hermitian form φ over \mathbb{C}^2 defined by the matrix

$$A = \begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix}$$

1. Find the hermitian form φ .
2. Find the hermitian quadratic form q associated with φ .
3. Using Gauss method, find the sum of squares of q .
4. Find the rank and signature of q .
5. Find the orthonormal basis B' for q and the formula $D = \bar{P}^t AP$.

Exercise 12. We consider the hermitian form φ over \mathbb{C}^3 defined by the matrix

$$A = \begin{bmatrix} 1 & -i & i \\ i & 1 & 0 \\ -i & 0 & 1 \end{bmatrix}$$

1. Why is the matrix A hermitian?
2. Find the hermitian form φ .
3. Find the hermitian quadratic form q associated with φ .
4. Using Gauss method, find the sum of squares of q .
5. Find the rank and signature of q .
6. Find the orthonormal basis B' for q and the formula $D = \bar{P}^t AP$.

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Exercise 1.

We consider the quadratic form $q : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by

$$q(x, y, z) = x^2 + 2xy + 2xz + 2y^2 + 2z^2 + 2yz$$

1. Find the matrix A of q .
2. Find the symmetric bilinear form φ associated with q .
3. Using Gauss method, find the sum of squares of q and the orthogonal basis B' for q and the formula $D = P^t AP$.
4. Find the rank and signature of q .

Exercise 2.

We consider the quadratic form q over \mathbb{R}^3 defined by:

$$q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^3 + 2x_1x_2$$

1. Find the matrix A of q .
2. Using Gauss method, find the sum of squares of q .
3. Deduce the rank and the signature of q .
4. Find $\ker q$.
5. Find $C(q)$, the set of isotropic vectors and verify that $\ker q \subset C(q)$.

Exercise 3.

We consider the hermitian form φ over \mathbb{C}^2 defined by the matrix

$$A = \begin{bmatrix} 1 & 2+i \\ 2-i & 1 \end{bmatrix}$$

1. Find the hermitian form φ and the quadratic hermitian form q .
2. Using Gauss method, deduce the rank and the signature of q .

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