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Theme

Continuous Solutions of Integral Equations via Schauder's Theorem

Presented by :
LAKHAL Malika

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In front of the jury composed of :

M^r *NADIR* Mostafa	Prof.,	Mohamed Boudiaf University of M'sila	**President**.
M^s *KHIRANI* Amina	MCA,	Mohamed Boudiaf University of M'sila	**Supervisor**.
M^r *DJAIDJA* Naoui	MCB,	Mohamed Boudiaf University of M'sila	**Examiner**.

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بسم الله الرحمن الرحيم

"وَقُلِ اعْمَلُواْ فَسَيَرَى اللّهُ عَمَلَكُمْ وَرَسُولُهُ وَالْمُؤْمِنُونَ وَسَتُرَدُّونَ إِلَى عَالِمِ الْغَيْبِ وَالشَّهَادَةِ فَيُنَبِّئُكُم بِمَا كُنتُمْ تَعْمَلُونَ" [التوبة: 105]

ويقول العماد الأصفهاني

(إني رأيت أنه لا يكتب أحد كتابا في يومه إلا قال فيه عنه : لو غير هذا لكان أحسن ، ولو زيد لكان يستحسن ، ولو قدم هذا لكان أفضل ، ولو ترك هذا لكان أجمل ، وهذا من أعظم العبر ، وهو دليل على استيلاء النقص على جملة البشر).

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Dedication

I dedicate my work
to my family an many of my friends.

Aspecial feeling of gratitude to my mother,

many God pretect her and prolong her life.

I also dedicate my research work to my colleagues and for all who have

suppoited theonghout the process.

الملخص

في هذه المذكرة، ندرس تصنيف المعادلات التكاملية، و نركز على المعادلات التكاملية الغير خطية لفولتير و فريدهولم.

والهدف من هذا العمل هو اثبات وجود حلول بعض المعادلات التكاملية لغير خطية لفولتير وفريدهولم، وكذلك المعادلات التكاملية المتعلقة بتأخر الزمن في فضاءات بناخ باستعمال النقطة الصامدة لشودار.

الكلمات لمفتاحية : معادلات تكاملية لفولتيرا، معادلات تكاملية فريدهولم، نظرية النقطة الصامدة لشودار، نظرية ريزيس، المعادلات التكاملية المتعلقة بتأخر الزمن.

Abstract

In this memory, we study and classify integral equations, and focus on Volterra and Fredholm non-linear integral equations.

The aim of this paper is to prove the existence of solution of some non-linear Fredholm and Volterra integral equation, integral equation with Delay in Banach spaces, using Schauder's fixed point theorem.

The Key words: Fredholm's integral equations, Volterra's integral equations, Schauder's fixed point theorem, Riesz's theorem, integral equation with Delay.

Résumé

Dans ce mémoire, nous avons étudié et classé les équations intégrales, nous avons basé sur les équations intégrales non- linéaire de Fredholm et de Volterr.

L'objectif de cette recherche est de prouver l'existance de la résolution de quelques équations intégrales non-linéaire de Fredholm et de Volterra, et équation intégrale avec retard dans l'espace de Banach, En utilisant le théorème du point fixe de Schauder.

Les mots clés: *Équations intégrales de Fredholm, équations intégrales de Volterra, Théorème du point fixe de Schauder, Théorème de Riesz, équation intégrale avec retard.*

Notation

$\mathbb{R}^+$	set of all nonnegative real numbers.
$\mathbb{R}^n$	set of all $n-$tuples $x = (x_1, x_2, ..., x_n)$.
Ω	open bounded of $\mathbb{R}^n$.
$\overline{\Omega}$	Closed bounded of $\mathbb{R}^n$.
T	integral operator.
$C([a, b])$	the Banach space of all continuous functions from $[a, b]$ into $\mathbb{R}^n$.
$\mid . \mid$	norm in X, also denoted by $\mid . \mid_x$.
$\lvert\varphi\rvert_\infty$	$\max_{t\in\overline{\Omega}} \lvert\varphi(t)\rvert$ $\Omega \in \mathbb{R}^n$ bounded open, $\varphi \in C\left(\overline{\Omega}, \mathbb{R}^n\right)$.
$L^p(\Omega, \mathbb{R}^n)$	space of all measurable functions $\varphi : \Omega \to \mathbb{R}^n$ with $\int_\Omega \lvert\varphi(t)\rvert^p dt < \infty$ $(\Omega \subset \mathbb{R}^n$ open,of $1 \leq p < \infty)$.
$\mid . \mid_p$.	norm in $L^p(\Omega, \mathbb{R}^n)$, $\mid \varphi \mid_p = \left(\int_\Omega \lvert\varphi(t)\rvert^p dt\right)^{\frac{1}{p}}$.
$Conv\ A$	convex hull of A.
$\overline{Conv}\ A$	Closed convex hull of A.
χ_E	characteristic function of the set E.
$C(\Omega; \mathbb{R}^n)$	the Banach space of all continuous functions from Ω into $\mathbb{R}^n$.

Contents

Introduction

Integral equations are very useful mathematical tools in both pure and applied mathematics, appear in various fields of science and numerous applications such that elasticity, plasticity and mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics, medicine, etc.

An integral equation is defined as an equation in which the unknown function $\varphi(t)$ to be determined appear under the integral sign. Many initial and boundary value problems associated with ordinary differential equation(ODE) and partial differential equation(PDE) can be transformed into problems of solving some approximate integral equations.

A general from of an non linear integral equation in $\varphi(t)$ is of the from

$$\varphi(t) = f(t) + \lambda \int_{\alpha(t)}^{\beta(t)} K(t, s, \varphi(s))\, ds.$$

where $K(t, s)$ is called the kernel of the integral equation, $\alpha(t)$ and $\beta(t)$ are the limits of integration. It is to be noted here that both the kernel $K(t, s)$ and the function $f(t)$ in the integral equation are given functions, λ is a constant parameter.

If the lower limit of integration is constant and the upper one is variable we are in the case of Volterra integral equations, but if the two terms of integration are constant, then we are in the case of Fredholm integral equations.

The aim of this memory is to study the existence of the integral equations, and we focus more on the non-linear Fredholm and Volterra, integral equation with Delay, by using Schauder's fixed point theorem(1930). Our work is divided as follows.

In the introductory chapter we recall some notions Ascoli-Arzela's theorem, Hansdorff's theorem, complety continuous operator, Shaefer's fixed point theorem, Schauder's fixed point theorem.

In the second chapter we present integral equation and we illustrute different criteria of classification of these equations, we focus on Fredholm and Volterra integral equations. Finally we mention some theorems to prove existence of solutions of some kind of linear integral equation (Riesz's theorem and Fredholm's Alternative).

In the final chapter we study the existence solutions of non-linear Volterra and Fredholm,integral equation with Delay,and Hammerstein integral equation, using Schauder's fixed point theorem.

Chapter 1

Preliminary results and necessary definitions

In this section we provide some important definitions and theorems in the note. We also touch on the fixed point theorems (Schaefer's Fixed Point Theorem, Schauder's Fixed Point Theorem).

1.1 Definitions and Theorems

Proposition 1.1 (Hausdorff's theorem [5]) *Let* (X, d) *be a metric space. The following statements are equivalent :*

(i) Every sequence of elements of X has a convergent subsequence in X .

(ii) The space X is complete and for each $\varepsilon \succ 0$ it admits a finite covering by open balls of radius ε.

Proof. see [11]. ■

Definition 1.1 ([11]) *A metric space* X *is said to be compact if it satisfies condition* (i) *(equivalently* (ii)*) in proposition* 1.1.

1.1.1 Ascoli-Arzela's Theorem

Theorem 1.1 ([5]) *Let* X *be a compact Hausdorff space. If* $\mathcal{F}$ *is an equicontintinuous and pointwise bounded subset of* $C(X)$, *then* $\mathcal{F}$ *is totally bounded .*

Proof. see [11]. ■

1.1.2 completely Continuous Operator

Definition 1.2 ([5]) *Let X be a Banach space. An operator $T : X \to X$ is called totally bounded if for every bounded subset S of X, $T(S)$ is compact. Moreover, T is said to be completely continuous over X if it is continuous and totally bounded over X.*

Since our Banach space is $C([a,b])$, then the following version of the Ascoli-Arzelà theorem is very useful in proving the total boundedness of our proposed operator.

Theorem 1.2 ([11])

(i) If the operator $T_k : D \to Y$; $D \subset X$, $k = 1, 2, ...$are completely continuous and $T : D \to Y$ is such that

$$T(\varphi) = \lim_{k\to\infty} T_k(\varphi) \tag{1.1}$$

uniformly on any bounded subset of D, then T is completely continuous too.

(ii) Let $D \subset X$ be a bounded closed set and $T : D \to Y$ a completely continuous operator. Then there exists a sequence of continuous operator $T_k : D \to Y$ of finite rank such that (1.1) holds, uniformly on D, and

$$T_k(D) \subset conv(T(D))$$

for every k.

Proof. see [11]. ■

Theorem 1.3 (Brouwer 's theorem [11]) *Let $D \subset \mathbb{R}^n$ be a nonempty convex compact set and let $T : D \to D$ be a continuous mapping. Then there exists of at least one $\varphi \in D$ with $T(\varphi) = \varphi$.*

Proof. see [11]. ■

1.2 Fixed Point Theorem

Definition 1.3 ([3]) *Let T be a defined in a Banach space E in itself, then for $x \in E$, such that $\varphi = T(\varphi)$, is called a fixed point of the operator T.*

Definition 1.4 ([3]) *Let T be an operator of a Banach space E in itself, the operator T is said a contraction operator if it can be exist a positive constant k, such that $0 < k < 1$and*

$$\|T\varphi_1 - T\varphi_2\| \leq k \|\varphi_1 - \varphi_2\|, \ \forall \varphi_1, \varphi_2 \in E$$

Theorem 1.4 ([3]) *Let T be a contraction operator in a Banach space E, the equation $T\varphi = \varphi$ admits a unique solution φ in E, this solution is said to be a fixed point of the operator.*

1.2.1 Schaefer's Fixed Point Theorem

Theorem 1.5 (Schaefer's Fixed -point Theorem [5]) *Let X be a Banach space and let $T : X \to X$ be a completely continuous operator. Then either :*

1. The operator equation $x = \lambda T x$ has a solution for $\lambda = 1$.

2. The set $\varepsilon = \{x \in X; x = \lambda T x, \lambda \in]0, 1[\}$ is unbounded.

Proof. see [6]. ■

1.2.2 Schauder 's Fixed Point Theorem

Theorem 1.6 (Schauder[11]) *Let X be a Banach space, $K \subset X$ a nonempty convex compact set and let $T : K \to K$ be a continuous operator. Then T has at least one fixed point .*

Proof. obviously T is completely continuous. Consequently by Theorem 1.2 there exists a sequence of continuous operators $T_j : K \to K$ of finite rank such that

$$T(\varphi) = \lim_{j\to\infty} T_j(\varphi)$$

uniformly on K. Let $n = n(j)$ be the dimension of the subspace X_n generated by $T_j(K)$. We have

$$T_j : K \cap X_n \to K \cap X_n.$$

Consequently by Brouwer's theorem there exists $\varphi_j \in K \cap X_n$ with

$$T_j(\varphi_j) = \varphi_j.$$

Since K is compact there exists a subsequence of (φ_j)convergent to some element $\varphi \in K$. As in Step 1 in the proof of Brouwer's theorem we can conclude that $T(\varphi) = \varphi$. ■

The following variant of Schauder's theorem is most useful in applications.

Theorem 1.7 (Schauder [11]) *Let X be a Banach space, $D \subset X$ a nonempty convex bounded closed set and let $T : D \to D$ be a completely continuous operator. Then T has at least one fixed point.*

We can derive Theorem 1.7 from Theorem 1.6 via the following result.

Lemma 1.1 (Mazur [11]) *The convex hull of any relatively compact subset of a Banach space is relatively compact.*

Proof. Let Y be a relatively compact subset of a Banach space X.Then, given $\varepsilon \succ 0$ we find a finite number of elements of X, say $\varphi_1, \varphi_2, ..., \varphi_m$ such that

$$Y \subset \overset{m}{\underset{i=1}{U}} B_\varepsilon(\varphi_i). \tag{1.2}$$

Let

$$\Re = conv\{\varphi_1, \varphi_2, ..., \varphi_m\},$$

Our goal is to prove that $\Re$ is a relatively compact $\varepsilon - net$ for $conv(Y)$. Once this is proved, we may say that $conv(Y)$ is a relatively compact set in base of Hausdorff's theorem. To this end consider an arbitrary element$\varphi \in conv(Y)$.We have

$$\varphi = \sum_{j=1}^{n} \lambda_j \upsilon_j \ ,\lambda_j \succ 0, \sum_{j=1}^{n} \lambda_j = 1, \upsilon_j \in Y.$$

According to (1.2),for each υ_j there is an $i_j \in \{1, 2, ..., m\}$with $\upsilon_j \in B\left(\varphi_{i_j}\right)$.Then

$$\begin{aligned} \left| \varphi - \sum_{j=1}^{n} \lambda_j \varphi_{i_j} \right| &= \left| \sum_{j=1}^{n} \lambda_j \left(\upsilon_j - \varphi_{i_j} \right) \right| \\ &\leq \sum_{j=1}^{n} \lambda_j \left| \upsilon_j - \varphi_{i_j} \right| \prec \varepsilon \end{aligned}$$

and $\sum_{j=1}^{n} \lambda_j \varphi_{i_j} \in \Re$.This shows that $\Re$ is an $\varepsilon - net$ for $conv(Y)$. Finally ,the compactness of $\Re$ follows from the representation :

$$\Re = \left\{ \sum_{j=1}^{n} \lambda_j \varphi_j \ : \ 0 \leq \lambda_j \leq 1, \sum_{j=1}^{n} \lambda_j = 1 \right\}.$$

Thus the proof is complete . ■

Proof. (proof of Theorem 1.7) Since T is completery continuous and D is bounded the set $T(D)$is relatively compact. Mazur's lemma then implies that the set $K = \overline{conv}\,(T(D))$ is compact (and obviously convex). From $T(D) \subset D$ and D closed convex it follows that $K \subset D$. Now we apply Theorem 1.6 to the operator $T : K \to K$. ■

Chapter 2

Integral Equations

In this chapter we give the definitions and types of integal equations (classifications) that we will use in the following chapter, with an introduction to the theory of these equations (the Riesz theory and the alternative of Fredholm and Neumann series, ...).

2.1 Definition of Integral Equations

Definition 2.1 *A functional equation in which the unknown fonction is shown under the integration sign is called an integral equation* $\int$.

This is usually the equation with respect to the unknown φ of the from

$$\int_E K(t, s, \varphi(s))\, ds = \lambda\varphi(t) + f(t)\,,\ t \in E \tag{2.1}$$

Where E is a measure space, $f(t)$ a given measurabe fonction on E, λ a given scalar which can be real of complex, and $K(t, s, \varphi(s))$ a measurable fonction on E^3 called the kernel of the integral equation.

With all these date, our problem is to find the function φ that satisfies the equation (2.1).

1. For the study of the following integral equation

$$\int_E K(t, s, \varphi(s))\, dt = \lambda\varphi(t) + f(t)\,,\ t \in E \tag{2.2}$$

   We restrict ourselves to $L_P(E)$ avec $(1 \leq p \leq +\infty)$.

   Implicily, for function $f \in L_P(E)$, we are looking for function φ in $Lp(E)$ verifies this equation, this means that in this restriction, we use only the kernels $K(t, s)$ for which $T\varphi$ is in $L_P(E)$ when φ is.

2. if we have

$$K(t, s, \varphi(s)) = K(t, s)\,\varphi(s)$$

the equation (2.2) becomes linear, ie

$$f(t) = \int_E K(t, s)\,\varphi(s)\,ds - \lambda\varphi(t)$$

is otherwise becomes a non-linear integral equation.

3. Note that the equation (2.2) can be written as an operator

$$T\varphi = \lambda\varphi + f$$

or the operator T is written as

$$T\varphi(t) = \int_E K(t, s, \varphi(s))\,dt.$$

4. The most general type of integral equation

$$h(t)\,\varphi(t) = f(t) + \lambda\int_E K(t, s, \varphi(s))\,dt.$$

The function $h(t)$ determines the type of the equation.

2.2 Classifiction of Integral Equations

The Classifiction of Integral Equations is centred on three basic characteristics that describe their overall structure, it is useful to mention them before going into detail.

1. The type of an equation refers to the location of the unknown function. For first type equations, the unknown function only appears inside the integral sign. However, for second type equations, the unknown function also appears outside the integral sign.

2. The historical description Fredholm and Volterra concerns the bounds of integration. In a Fredholm equation, the bounds of integration are fixed, in the Volterra equation the bounds of integration are undefined.

3. The adjective singular is sometimes used on the one hand, when the integration is improprer, on the other hand if one or both of the bounds of integration is unbounded on the given range, obviously, an integral equation can be singular in both directions.

2.3 Linear Integral Equations

2.3.1 Fredholm Integral Equation

A Fredholm linear Integral Equation such that both limits of integration are constant is an equation with one unknown $\varphi(t)$, of the from

$$h(t)\,\varphi(t) = f(t) + \lambda \int_a^b K(t,s)\,\varphi(s)\,ds, \tag{2.3}$$

where $f(t)$, $K(t,s)$ are known functions and λ is a non-zero, real or complex parameter.

The function $h(t)$ determines the type of the integral equation.

1. If $h(t) = 0$, the equation (2.3) is written

$$f(t) + \lambda \int_a^b K(t,s)\,\varphi(s)\,ds = 0. \tag{2.4}$$

and is called the Fredholm integral equation of the first kind.

2. If $h(t) = c =$ constant, the equation (2.3) is written

$$c\,\varphi(t) = f(t) + \lambda \int_a^b K(t,s)\,\varphi(s)\,ds. \tag{2.5}$$

and is called the Fredholm integral equation of the second kind.

If $h(t) \neq 0$, Thus the from (2.3) is called the Fredholm integral equation of the third kind.

Remark 2.1 *We distinguish two cases :*

i- If $f(t) = 0$, the equation (2.3) is said to be homogeneous.

ii- If $f(t) \neq 0$, the equation (2.3) is said to be non-homogeneous.

2.3.2 Volterra Integral Equation

A volterra linear integral equation such that both limits of integration are variable, an equation of the form

$$h(t)\,\varphi(t) = f(t) + \lambda \int_a^t K(t,s)\,\varphi(s)\,ds. \tag{2.6}$$

1. we call the volterra integral equation of the first kind, if $h(t) = 0$, so the equation (2.6) is written

$$f(t) + \lambda \int_a^t K(t,s)\,\varphi(s)\,ds = 0 \tag{2.7}$$

2. we call the integral equation of volterra of the second kind, if $h(t) = c = constant \neq 0$, so the equation (2.6) is written

$$c\,\varphi(t) = f(t) + \lambda \int_a^t K(t,s)\,\varphi(s)\,ds. \tag{2.8}$$

If $h(t) \neq 0$,Thus the from (2.6) is called the volterra integral equation of the third kind.

Remark 2.2 *We distinguish two cases :*

i- If $f(t) = 0$, the equation (2.6) is said to be homogeneous.

ii- If $f(t) \neq 0$, the equation (2.6) is said to be non-homogeneous.

Remark 2.3 *The volterra integral equation is a special case of the fredholm integral equation, it is sufficient to take the kernel K verifies the condition.*

$$K(t,s) = 0, \ \ pour\ t < s$$

2.3.3 Abel Integral Equation

An equation of the from

$$\int_a^t \frac{\varphi(s)}{(t-s)^{\alpha}}\,ds = f(t)\,. \tag{2.9}$$

where α is a constant, $0 < \alpha < 1$.

2.4 Nonlinear Integral Equation

2.4.1 Fredholm Integral Equation

The nonlinear Fredholm integral equation of the first kind takes the form

$$f(t) + \lambda \int_a^b K(t,s,\varphi(s))\,ds = 0 \tag{2.10}$$

Is called Fredholm's integral equation of the second kind, the equation of the from.

$$c\,\varphi(t) = f(t) + \lambda \int_a^b K(t,s,\varphi(s))\,ds. \tag{2.11}$$

where c =constant$\neq 0$, and thirdly, of the from

$$h(t)\,\varphi(t) = f(t) + \lambda \int_a^b K(t, s, \varphi(s))\, ds. \tag{2.12}$$

Remark 2.4 *We distinguish two cases :*

i- If $f(t) = 0$, Thus the equation is said to be homogeneous.

ii- If $f(t) \neq 0$, Thus the equation is said to be non-homogeneous.

2.4.2 Volterra Integral Equation

The non-linear integral equation of volterra of the first kind takes the from

$$f(t) + \lambda \int_a^t K(t, s, \varphi(s))\, ds = 0. \tag{2.13}$$

Is called volterra's integral equation of the second kind,the equation of the from.

$$c\,\varphi(t) = f(t) + \lambda \int_a^t K(t, s, \varphi(s))\, ds. \tag{2.14}$$

where c =constant$\neq 0$, and thirdly, of the from

$$h(t)\,\varphi(t) = f(t) + \lambda \int_a^t K(t, s, \varphi(s))\, ds. \tag{2.15}$$

Remark 2.5 *We distinguish two cases :*

i- If $f(t) = 0$, Thus the equation is said to be homogeneous.

ii- If $f(t) \neq 0$, Thus the equation is said to be non-homogeneous.

2.4.3 Abel Integral Equation

An equation of the form

$$\varphi(t) = \int_{-\infty}^t (t-s)^{\alpha-1}\, g(\varphi(s))\, ds. \tag{2.16}$$

or $-\infty < t, 0 < \alpha < 1$, and $g : [0; +\infty[\to [0; +\infty[$ such as $g(0) = 0$ and $g(t) > 0$ For all $t > 0$.

For more information see [9].

2.4.4 Uryson and Hammerstein Integral Equation

1. An equation of the Uryson form is given by

$$\varphi(t) = f(t) + \int_{\Omega} K(t, s, \varphi(s))\, ds, \;\; s \in \Omega \tag{2.17}$$

   where K and f are arbitrary functions.

   or

$$\varphi(t) = f(t) + \int_{\Omega} K(t, s)\, F(\varphi(s))\, ds, \;\; s \in \Omega$$

   such that F is a non-linear function.

2. An equation of the Hammerstein form is given by

$$\varphi(t) = f(t) + \int_{\Omega} K(t, s)\, g(s, \varphi(s))\, ds, \;\; s \in \Omega \tag{2.18}$$

   For more information see[2] .

Remark 2.6 *Hammerstein's equation is a special case of Uryson's equation.*

2.5 Mixed Integral Equation

2.5.1 Fredholm-Volterra Integral Equation

An equation of the form

$$h(t)\, \varphi(t, s) + \lambda \int_a^b K(t, y)\, \varphi(y, s)\, dy + \lambda \int_a^b F(s, x)\, \varphi(t, x)\, dx = f(t, x), \;\; s \in [0, S], \;\; S < \infty. \tag{2.19}$$

The function h determines the type of the integral equation.

2.5.2 Volterra-Fredholm Integral Equation

An equation of the form

$$h(t)\, \varphi(t, s) + \int_0^s \int_a^b K(t, s)\, F(s, x)\, \varphi(y, x)\, dy dx = f(t, s), \;\; s \in [0, S], \;\; S < \infty. \tag{2.20}$$

2.6 Singular Integral Equation

An integral equation is said to be singular if one or both limits of integration are infinite, or the kernel becomes infinite in the neighbourhood of thelimits of integration.

Definition 2.2 *Consider the following integral equation.*

$$\varphi(t) = f(t) + \int_{\Omega} M(t,s) K(t,s) \varphi(s) ds. \tag{2.21}$$

we say that(2.21)is singular if $M(t,s)$ admits a singularity or the domain T is un bounded.

2.6.1 Volterra and Fredholm Type Singulaite

Consider the second kind integral equation of the form.

$$\varphi(t) = f(t) + \int_{a}^{t} M(t,s) K(t,s) \varphi(s) ds, \; a \leq t < \infty. \tag{2.22}$$

where $K(t,s)$ is weakly singular, in general $K(t,s)$ is given by

$$K(t,s) = \begin{cases} |t-s|^{-\alpha}, \; 0 < \alpha < 1 \\ \log|t-s| \end{cases}$$

So

1. The equation (2.22) is volterra.
2. If $t = b$, the equation (2.22) is Fredholm.
3. The case where $K(t,s) = |t-s|^{-\alpha}$, $0 < \alpha < 1$, is called an algebraic singularity.
4. The case where $K(t,s) = \log|t-s|$, is called logarithmic singularity.

Definition 2.3 (Carleman Integral Equation) *An equation of the form*

$$a(t) \frac{1}{\pi i} \int_{-1}^{1} \frac{\varphi(s)}{s-t} ds + \frac{1}{\pi i} \int_{-1}^{1} \frac{b(s)}{s-t} \varphi(s) ds = f(t). \tag{2.23}$$

where a, b and φ are continuous function.

For more information see [**?**].

2.6.2 Cauchy type singularity

Let D be a bounded and convex domain in a complex plane, then the Cauchy integral gives by the formula.

$$\frac{1}{2\pi i}\int_{\partial D}\frac{\varphi(s)}{s-t}ds = f(t), \; s \in C. \tag{2.24}$$

Definition 2.4 *A Cauchy integral equation is an equation of the form.*

$$a(t)\varphi(t) + b(t)\int_{\Gamma}\frac{\varphi(s)}{s-t}ds + \int_{\Gamma}K(t,s)\varphi(s)\,ds = f(t). \tag{2.25}$$

such as $\Gamma = \partial D$.

2.7 Integral Equation With Delay

We call integral equation with Delay, The equation of the from

$$\varphi(t) = \int_{t-\tau}^{t} f(s,\varphi(s))\,ds$$

In this equation $\varphi(t)$ is the proportion of infectives in a population at time, and $f(t,\varphi(t))$ is the proportion of new infectives per unit time.

2.8 Reduction of ODEs to the Volterra IE

Consider an ordinary differential equation (ODE) of the first order

$$\frac{dy}{dt} = f(t,y) \tag{2.26}$$

with the initial condition

$$y(t_0) = y_0, \tag{2.27}$$

where $f(t,y)$ is defined and continuous in a two-dimensional domain G which contains the point (t_0,y_0).

Integrating (2.26) subject to (2.27) we obtain

$$\varphi(t) = y_0 + \int_{t_0}^{t} f(s,\varphi(s))\,ds, \tag{2.28}$$

which is called the Volterra integral equation of the second kind with respect to the unknown function $\varphi(s)$. This equation is equivalent to the initial value problem(2.26) and (2.27). Note that this is generally a nonlinear integral equation with respect to $\varphi(s)$.

Consider now a linear ODE of the second order with variable coefficients

$$y'' + A(t)\, y' + B(t)\, y = g(t) \tag{2.29}$$

with the initial condition

$$y(t_0) = y_0,\ y'(t_0) = y_1, \tag{2.30}$$

where $A(t)$ and $B(t)$ are given functions continuous in an interval G which contains the point t_0. Integrating y''in (2.29) we obtain

$$y'(t) = -\int_{t_0}^{t} A(s)\, y'(t)\, dt - \int_{t_0}^{t} B(t)\, y(x)\, dx + \int_{x_0}^{x} g(x)\, dx + y_1. \tag{2.31}$$

Integrating the first integral on the right-hand side in (2.31) by parts yields

$$y'(t) = -A(t)\, y(t) - \int_{t_0}^{t} \left(B(t) - A'(t)\right) y(t)\, dt + \int_{t_0}^{t} g(t)\, dt + A(t_0)\, y_0 + y_1. \tag{2.32}$$

Integrating a second time we obtain

$$\begin{aligned} y(t) &= -\int_{t_0}^{t} A(t)\, y(t)\, dt - \int_{t_0}^{t}\int_{t_0}^{t} \left(B(s) - A'(s)\right) y(s)\, sdt + \int_{t_0}^{t}\int_{t_0}^{t} g(s)\, dsdt \\ &\quad + \left[A(t_0)\, y_0 + y_1\right](t - t_0) + y_0. \end{aligned} \tag{2.33}$$

Using the relationship

$$\int_{t_0}^{t}\int_{t_0}^{t} f(s)\, dsdt = \int_{t_0}^{t} (t - s)\, f(s)\, ds, \tag{2.34}$$

we transform (2.33) to obtain

$$y(t) = -\int_{t_0}^{t} \left\{A(s) + (t - s)\left[\left(B(s) - A'(s)\right)\right]\right\} y(s)\, ds + \int_{t_0}^{t} (t - s)\, g(s)\, ds + \tag{2.35}$$

$$+ \left[A(t_0)\, y_0 + y_1\right](t - t_0) + y_0. \tag{2.36}$$

Separate the known functions in (2.35) and introduce the notation for the kernel function

$$K(t, s) = -A(s) + (s - t)\left[\left(B(s) - A'(s)\right)\right], \tag{2.37}$$

$$f(t) = \int_{t_0}^{t} (t - s)\, g(s)\, ds + \left[A(t_0)\, y_0 + y_1\right](t - t_0) + y_0. \tag{2.38}$$

Then (2.35) becomes

$$y(t) = f(t) + \int_{t_0}^{t} K(t, s)\, y(s)\, ds, \tag{2.39}$$

which is the Volterra IE of the second kind with respect to the unknown function $\varphi(s)$. This equation is equivalent to the initial value problem (2.29) and (2.30). Note that here we obtain a linear integral equation with respect to $y(t)$.

Example 2.1 *Consider a homogeneous linear ODE of the second order with constant coefficients*

$$y^{''} + \omega^2 y = 0 \tag{2.40}$$

and the initial conditions (at $t_0 = 0$)

$$y(0) = 0, \; y^{'}(0) = 1. \tag{2.41}$$

We see that here

$$A(t) \equiv 0, \; B(t) \equiv \omega^2, \; y_0 = 0, \; y_1 = 1.$$

if we use the same notation as in (2.29) and (2.30).

Substituting into (2.35) and calculating (2.37) and (2.38),

$$\begin{aligned} K(t,s) &= \omega^2 (s-t), \\ f(t) &= t, \end{aligned}$$

we find that the IE (2.39), equivalent to the initial value problem (2.40) and (2.41), takes the form

$$y(t) = t + \int_0^t (s-t)\, y(s)\, ds. \tag{2.42}$$

2.9 Existence and Uniqueness of Solution for Linear Integral Equation

2.9.1 Riesz's Theorem

In this paragraph, we denote by $A : X \to X$ the compact linear operator norm space in itself.

we present the basic theorem for an equation

$$\varphi - A\varphi = f.$$

The operator T is defined as

$$T = I - A$$

Where I designate the identity operator.

Theorem 2.1 ([3]) *The null space of the oprator T, i.e.The kernel of the operator T.*

$$N(T) = \ker(T) = \{\varphi \in X; T\varphi = 0\}$$

is a finite dimensional subspace.

Proof. The kernel of the linear operator bounded T is a closed subspace of X.Since each sequence $\varphi_n \to \varphi, n \to \infty$ and $T\varphi_n = 0$, then we have $T\varphi = 0$, So $\varphi \in \ker(T)$ equivalent to a $A\varphi = \varphi$.

And so the restriction of A in $\ker(T)$ coincides with the identity operator on $\ker(T)$, the operator A is compact in X and so make $\ker(T)$ compact on $\ker(T)$, since $\ker(T)$ is closed. Therefore $\ker(T)$ is of finite dimension. ■

Theorem 2.2 ([3]) *The image of the operator $T, i.e$*

$$R(T) = \{T\varphi; \varphi \in X\}$$

is a closed and finite co- dimensional linear subspace.

Proof. The image of the operator T is a subspace. Let f be an element of $\overline{T(X)}$, then there exists a sequence (φ_n) in X such that $T\varphi_n \to f, n \to \infty$,we choose the best approximation $\chi_n, i.e$

$$\|\varphi_n - \chi_n\| = \inf_{\chi_n \in \mathrm{Im}(T)} \|\varphi_n - \chi_n\|,$$

we define the sequence

$$\widetilde{\varphi_n} = \varphi_n - \chi_n, n \in \mathbb{N},$$

who is bounded .

We suppose that the sequence $(\widetilde{\varphi_n})$ is not bounded, then we can extract a sub-sequence $(\widetilde{\varphi}_{n(k)})$, such that $\|\widetilde{\varphi}_{n(k)}\| \geq k$, for all $k \in \mathbb{N}$,now we pose

$$\psi_k = \frac{\widetilde{\varphi}_{n(k)}}{\|\widetilde{\varphi}_{n(k)}\|}, k \in \mathbb{N},$$

with $\|\psi_k\| = 1$ and A is compact, then there exictsts a subsequence $\psi_{k(j)}$ such that $A\psi_{k(j)} \to \psi$, $j \to \infty$ in addition

$$T\psi_k = \frac{\|T\widetilde{\varphi}_{n(k)}\|}{\|\widetilde{\varphi}_{n(k)}\|} \leq \frac{\|T\widetilde{\varphi}_{n(k)}\|}{k} \to 0,\ k \to \infty,$$

since the sequence $(T\varphi_n)$ is converged and therefore bounded. Therefore

$$T\psi_{k(j)} \to 0, j \to \infty,$$

then, we obtain

$$\psi_{k(j)} = T\psi_{k(j)} + A\psi_{k(j)} \to \psi,\ j \to \infty$$

and since T is bound, and by the two previous equations we conclude that $T\varphi = 0$.

But since

$$\chi_{n(k)} + \left\|\widetilde{\varphi}_{n(k)}\right\| \psi \in \operatorname{Im}(T), \ \forall\ k \in \mathbb{N},$$

There are

$$\begin{aligned}
\|\psi_n - \psi\| &= \frac{1}{\left\|\widetilde{\varphi}_{n(k)}\right\|} \left\|\varphi_{n(k)} - \left\{\chi_{n(k)} + \left\|\widetilde{\varphi}_{n(k)}\right\|\right\} \psi\right\| \\
&\geq \frac{1}{\left\|\widetilde{\varphi}_{n(k)}\right\|} \inf_{k \in \operatorname{Im}(L)} \left\|\varphi_{n(k)} - \chi_n\right\| \\
&= \frac{1}{\left\|\widetilde{\varphi}_{n(k)}\right\|} \left\|\varphi_{n(k)} - \chi_n\right\| = 1.
\end{aligned}$$

This contradicts the fact that it $\psi_{k(j)} \to \psi, j \to \infty$.Therefore $(\widetilde{\varphi}_n)$ is bounded, and since A is compact, we can extract a sub-sequence $\left(\widetilde{\varphi}_{n(k)}\right)$ such that $\left(A\widetilde{\varphi}_{n(k)}\right)$ converges for $k \to \infty$.Because $T\widetilde{\varphi}_{n(k)} \to f$, $k \to \infty$, and by

$$\widetilde{\varphi}_{n(k)} = T\widetilde{\varphi}_{n(k)} + A\widetilde{\varphi}_{n(k)}$$

It can be seen that $\widetilde{\varphi}_{n(k)} \to \varphi \in X$, $k \to \infty$, but $T\widetilde{\varphi}_{n(k)} \to T\varphi \in X$, $k \to \infty$. ■

Theorem 2.3 ([3]) *It exists a unique $r \in \mathbb{N}$ called the Riesz number of the operator T such that :*

$\{0\} = \ker(T^0) \subset \ker(T^1) \subset \ldots \subset \ker(T^r) \subset \ker(T^{r+1})$

$E = \operatorname{Im}(T^0) \supset \operatorname{Im}(T^1) \supset \ldots \supset \operatorname{Im}(T^r) \supset \operatorname{Im}(T^{r+1})$.

And we have the direct sum

$$E = \ker(T^r) \oplus \operatorname{Im}(T^r).$$

Proof. For evidence see [7]. ■

Theorem 2.4 ([3]) *Under the same conditions in the previous theorems, then*

1. $I - A$ is injective if and only if it is surjective.

2. If $I - A$ is injective, then the inverse operator $(I - A)^{-1} : X \to X$ is bounded.

Proof. By theorem 2.2 , the injectivity of the operator $I - A$ is equivalent to $r = 0$. And by theorem 2.3, the surjectivity of the operator $I - A$ is equivalent to $r = 0$, since the injective and the surjective of the operator $I - A$ are equivalent.

It remains to show that T^{-1} is bounded when $T = I - A$ is injective. Assume that $T^{-1}$is not bounded then there exists a sequence (f_n) of X with $\|f_n\| = 1$tell that $\|T^{-1}f_n\| \geq n$ for all $n \in \mathbb{N}$, we define

$$g_n = \frac{f_n}{\|T^{-1}f_n\|},\ \varphi_n = \frac{T^{-1}f_n}{\|T^{-1}f_n\|},\ \forall n \in \mathbb{N}$$

then $g_n \to 0, n \to \infty$, and$\|\varphi_k\| = 1$ for all n. Since A is compact, we can extract a subsequence $\left(\varphi_{n(k)}\right)$ such $A\varphi_{n(k)} \to \varphi \in X$, $k \to \infty$, then since

$$\varphi_n \to A\varphi_n = g_n$$

we observe that $\varphi_{n(k)} \to \varphi$, $k \to \infty$ and $\varphi \in \mathrm{Im}\,(T)$. Consequently $\varphi = 0$,contradiction .

Car

$$\|\varphi_n\| = 1 \text{ pourtout } n.$$

■

2.9.2 Fredholm's Alternative

1. If the homogeneous equation

$$\varphi - A\varphi = 0 \tag{2.43}$$

admits only the trivial solution $\varphi = 0$, there for all $f \in X$ the equation

$$\varphi - A\varphi = f \tag{2.44}$$

admits a unique solution $\varphi \in X$ and this solution depends on the continuity of f.

2. If the homogeneous equation (2.43) does not admit the trivial solution $\varphi = 0$, then it has only a finite number $m \in \mathbb{N}$ of solutions $\varphi_1, \varphi_2, ..., \varphi_n$ of X are linearly in dependend and the non-homogeneous equation (2.44) is unsolvable or its solution is of the general from.

$$\varphi = \widetilde{\varphi}\ + \sum_{k=1}^{m} \alpha_k \varphi_k,$$

Where $\alpha_1, \alpha_2, ..., \alpha_k$ are complex arbtrary numbers and $\widetilde{\varphi}$ a solution particular of the non- homogeneous equation.

Proof. For evidence see [7]. ■

Definition 2.5 ([3]) *Let $A : X \to X$ be compact linear operator in a norm space within itself.*

The complex number λ is called an eigenvalue, if there exists an element $\varphi \in X$, $\varphi \neq 0$ such that $A\varphi = \lambda\varphi$

1. The number λ is called a regular value of A.

2. If $(\lambda I - A)^{-1}$ exists and bound, then the set of all regular values of A is called the set of resolvers $\rho(A)$ and $R(\lambda; A) = (\lambda I - A)^{-1}$.

3. The complement of $\rho(A)$ in $\mathbb{C}$ is called the specter of A, note $\sigma(A)$ and $r(A) = \sup_{\lambda\in\sigma(a)} |\lambda|$ is called the spectral raduis of A.

2.9.3 Series Neumann

Theorem 2.5 ([3]) *Let A be a boindary linear operator of Banach E space in itself, with $\|A\| < \lambda$.*

Then $A_\lambda = A - \lambda I$ admits a boundary inverse operator given by the series

$$(A - \lambda I)^{-1} = -\sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}},$$

addition

$$(A - \lambda I)^{-1} \leq \frac{1}{\|\lambda\| - \|A\|}$$

Proof. Since $\|A/\lambda\| < 1$,we have

$$\sum_{k=0}^{\infty} \left\| \frac{A^k}{\lambda^k} \right\| \leq \sum_{k=0}^{\infty} \left\| \frac{A}{\lambda} \right\|^k < \infty.$$

Therefore, since the space $\mathcal{L}(E, F)$is complete, thenn there is a boundary operator B in E such that

$$B = \sum_{k=0}^{\infty} \frac{A^k}{\lambda^k}.$$

By the way

$$\begin{aligned}
(A - \lambda I) B &= (A - \lambda I) \left(\sum_{k=0}^{\infty} \frac{A^k}{\lambda^k} \right) \\
&= \sum_{k=0}^{\infty} (A - \lambda I) \frac{A^k}{\lambda^k} \\
&= \sum_{k=0}^{\infty} \frac{A^{k+1} - \lambda A^k}{\lambda^k} \\
&= \lambda \sum_{k=0}^{\infty} \left(\frac{A^{k+1}}{\lambda^{k+1}} - \frac{A^k}{\lambda^k} \right) = -\lambda I,
\end{aligned}$$

■

As well as

$$(A-\lambda I)^{-1} = -\frac{B}{\lambda} = -\sum_{k=0}^{\infty}\frac{A^k}{\lambda^k}.$$

To demonstrate the second relationship, we observe that

$$\begin{aligned} \|A_\lambda\| &\leq \frac{1}{|\lambda|}\sum_{k=0}^{\infty}\left\|\frac{A^k}{\lambda^k}\right\| = \frac{1}{|\lambda|}.\frac{1}{\|\lambda\|-\|A\|} \\ &= \frac{1}{\|\lambda\|-\|A\|}. \end{aligned}$$

Corollary 2.1 ([3]) *Let A be a bounded linear operator in a Banach space, then the equation*

$$x = x_0 + \lambda A x$$

admits a unique solution given by

$$x = \sum_{k=0}^{\infty}\lambda^k A^k x_0, \ \ avec\ \ |\lambda|\,\|A\| < 1.$$

Theorem 2.6 ([3]) *Under the assumptions of the theorem 2.5, the method of successive approximations*

$$\varphi_{n+1} = A\varphi_n + g, \ \ n = 0, 1, 2, ...$$

Converges to the unique solution φ of the equation $A\varphi - \varphi = g$, for all $g \in X$ and φ_0 is arbitrary in X.

Proof. On garlic la successive

$$\begin{aligned} \varphi_0 &= g \\ \varphi_1 &= A\varphi_0 + g \\ \varphi_2 &= A\varphi_1 + g = A^2\varphi_0 + Ag + g \\ &. \\ &. \\ &. \\ \varphi_n &= A\varphi_{n-1} + g, \end{aligned}$$

so

$$\varphi_{n+1} = A^n\varphi + \sum_{k=0}^{n-1}A^k g,$$

Hence ■

$$\lim_{n\to\infty}\varphi_n = \lim_{n\to\infty}\sum_{k=0}^{n-1} A^k g = \sum_{k=0}^{\infty} A^k g = (I - A)^{-1} g.$$

Chapter 3

Applications of the Schauder Fixed Point Theormes on Nonlinear Integral Equations

In this chapter we talk about the applications of Schauder's fixed-point theorem on nonlinear integral equations of Frodholm and Volterra type, integral equations with Delay.

3.1 Existence of solution for Nonlinear Fredholm Integral Equations

3.1.1 Homogeneous Nonlinear Fredholm Integral Equation

consider the non-linear Fredholm integral equation of the second kind

$$\varphi(x) = \int_0^1 k(x, y, \varphi(x))\, dy,\ \ x \in [0,1] \tag{3.1}$$

If we are looking for a continuous solution φ for equation (3.1) in the ball of centre 0 and radius R, $\overline{B}_R(0; C([0,1]); \mathbb{R})$, form the space $C([0,1]; \mathbb{R})$, then it can be reformulated as a fixed point problem under appropriate assumptions suppsu:

$k : [0,1] \times [0,1] \times [-R, R] \to \mathbb{R}$ is continuous

$$M = \max_{[0,1]\times[0,1]\times[-R,R]} \mid k(x, y, z) \mid \leq R$$

we consider the Banch space $E = C([0,1]; \| \, . \, \|_\infty)$ of continuous functions of $[0,1]$ in $\mathbb{R}$ with the norm $\| \varphi \|_\infty = \sup_{x \in [0,1]} | \varphi(x) |$.

We define the operator $T : \overline{B}_R(0; C([0,1]); \mathbb{R}) \to C([0,1]; \mathbb{R})$ given by

$$T(\varphi)(x) = \int_0^1 k(x, y, \varphi(x))\, dy, \;\; x \in [0,1] \tag{3.2}$$

The operator T applies the ball $\overline{B}_R(0; C([0,1]); \mathbb{R})$, on itself. Therefore, the integral equation (3.1) is equivalent to the fixed point problem $T\varphi = \varphi$ or T is defined by (3.2).

Remark 3.1 *The operator T is continuous and compact, and then, by the shauder fixed point theorem, T has at least one fixed point in the ball $\overline{B}_R(0; C([0,1]); \mathbb{R})$, which is a solution of the interal equation (3.1) in the ball $\overline{B}_R(0; C([0,1]); \mathbb{R})$.*

3.1.2 Non-Homogeneous Nonlinear Fredholm Integral Equation

Theorem 3.1 ([5]) *Consider the nonlinear Fredholm integral equation :*

$$\varphi(t) = f(t) + \int_a^b g(t, s, \varphi(t))\, ds \;, \quad -\infty < a \le t \le b < +\infty \tag{3.3}$$

Where $f(.) \in C([a,b])$. Assume that the function $g(t, s, \varphi)$ satisfies the following conditions :

$$\sup\left(|g(t,s,\varphi)|, \left| \frac{\partial g}{\partial t}(t,s,\varphi) \right| \right) \le V_1(t) V_2(t) \phi(|\varphi|), \left| \frac{\partial g}{\partial \varphi}(t,s,\varphi) \right| \le V_1(t) V_2(t) \psi(|\varphi|), \tag{3.4}$$

Where $V_1(.) \in C([a,b]), V_2(.) \in L^1([a,b])$, $\phi(\varphi)$ is positive and bounded over $[0,+\infty[$and $\psi(\varphi)$ is positive and continuous over $[0,+\infty[$. Under the above conditions, equation (3.3) has a solution in $C([a,b])$.

Proof. We first define the operator T on $C([a,b])$ by : ■

$$T\varphi(t) = f(t) + \int_a^b g(t, s, \varphi(s))\, ds.$$

The proof of the theorem is divided into two steps .

First step : In this step, we prove that $T : \; C([a,b]) \to C([a,b])$ is continuous . We first show that $T\varphi(t) \in C([a,b])$, whenever$\varphi(t) \in C([a,b])$. Let $(t_n)_n$be a sequence in $[a,b]$ converging to t . Since

$$\begin{aligned} |T\varphi(t_n) - T\varphi(t)| &\leq \left| f(t_n) + \int_a^b g(t_n, s, \varphi(s))\, ds - \left(f(t) - \int_a^b g(t, s, \varphi(s))\, ds \right) \right| \quad (3.5) \\ &\leq |f(t_n) - f(t)| + \int_a^b |g(t_n, s, \varphi(s)) - g(t, s, \varphi(s))|\, ds \end{aligned}$$

and

$$\begin{aligned} |g(t_n, s, \varphi(s)) - g(t, s, \varphi(s))| &\leq |g(t_n, s, \varphi(s))| + |g(t, s, \varphi(s))| \\ &\leq [V_1(t_n) + V_1(t)]\, V_2(s)\, \phi(|\ \varphi(s)\ |) \\ &\leq [|\ V_1(t_n)\ | + |\ V_1(t)\ |]\, V_2(s)\, \phi(|\ \varphi(s)\ |) \\ &\leq \left(\sup_{t_n \in [a,b]} |\ V_1(t_n)\ | + \sup_{t \in [a,b]} |\ V_1(t)\ | \right) V_2(s) \sup_{u \geq 0} |\ \phi(u)\ | \\ &\leq |\ (\|\ V_1\ \|_\infty + \|\ V_1\ \|_\infty)\, V_2(s) \sup_{u \geq 0} |\ \phi(u)\ | \\ &\leq 2\ \|\ V_1\ \|_\infty\, V_2(s) \sup_{u \geq 0} |\ \phi(u)\ | \\ &= 2\ \|\ V_1\ \|_\infty\, V_2(s) M_\phi \in L^1([a,b]). \end{aligned}$$

then using the continuily of $g(t, s, \varphi)$ w.r.t $.t$ and applying the dominated the convergence theorem to (3.5), one concludes that

$$\begin{aligned} \lim_{n\to\infty} |T\varphi(t_n) - T\varphi(t)| &\leq \lim_{n\to\infty} |f(t_n) - f(t)| + \int_a^b \lim_{n\to\infty} |g(t_n, s, \varphi(s)) - g(t, s, \varphi(s))|\, ds \\ &= 0 \end{aligned}$$

Hence $T\varphi(.) \in C([a,b])$. Next, we prove that the operator T is continuous over$C([a,b])$. Let $(\varphi_n)_n \in C([a,b])$be a sequence converging uniformly to $\varphi(.)$. Since $C([a,b])$is complete, then$\varphi(.) \in C([a,b])$. Next, $\forall t \in [a,b]$, we have

$$\begin{aligned} |T\varphi_n(t) - T\varphi(t)| &\leq \int_a^b |g(t, s, \varphi_n(s)) - g(t, s, \varphi(s))|\, ds \\ &\leq \int_a^b |\varphi_n(s) - \varphi(s)| \left| \frac{\partial g}{\partial \varphi}(t, s, \theta_s \varphi_n(s) + (1-\theta_s)\varphi(s)) \right| ds,\ 0 < \theta_S < 1, \\ &\leq \|\varphi_n - \varphi\|_\infty \|V_1\|_\infty \int_a^b V_2(s)\, \psi(|\theta_s \varphi_n(s) + (1-\theta_s)\varphi(s)|). \end{aligned}$$

Since $\varphi_n(.)$ converges uniformly to $\varphi(.) \in C([a,b])$, then $\forall t \in [a,b]$, $|\theta_s \varphi_n(s) + (1-\theta_s)\varphi(s)|$ is contained in a compact set K of $[0,+\infty[$, then there exists a constant M_ψ such that $\forall n \in N$ and $\forall s \in [0,1]$ we have $\psi(|\theta_s \varphi_n(s) + (1-\theta_s)\varphi(s)|) \leq M_\psi$. By combining the previous two inequalities, one concludes that

$$\|T\varphi_n - T\varphi\|_\infty \leq \|\varphi_n - \varphi\|_\infty \|V_1\|_\infty \|V_2\|_1 M_\psi,$$

or equivalently, the operator T is continous over the Banach space $C([a,b])$.

Second step : In this step, we prove that T has a fixed point in $C([a,b])$ by applying the Schaefer fixed-point theorem. We first prove that T is totally bounded, By Ascoli -Arzela's theorem, we need only prove that $\mathcal{F} = \{T\varphi_n; n \in \mathbb{N}\}$ is equicontinuous and bounded for every uniformly bounded sequence$(\varphi_n)_n$ of $C([a,b])$. Since

$$\begin{aligned} |T\varphi_n(t) - T\varphi_n(\tau)| &= \left| |f(t_n) - f(\tau)| + \int_a^b g(t,s,\varphi_t(s)) - g(\tau,s,\varphi_n(s)) \right| ds \\ &\leq |f(t_n) - f(\tau)| + \int_a^b |(t-\tau)| \left| \frac{\partial g}{\partial t}(t+\theta_t(t-\tau), s, \varphi_n(s)) \right| ds, \ 0<\theta_t<1 \\ &\leq |f(t_n) - f(\tau)| + |(t-\tau)| \|V_1\|_\infty \|V_2\|_1 M_\psi, \end{aligned}$$

then $\mathcal{F}$ is equicontinuous. Moreover, it is clear that $\mathcal{F}$ is bounded if $(\varphi_n)_n$ is bounded. Hence $\mathcal{F}$ is totally bounded and, consequently, T is totally bounded over $C([a,b])$. Since we have already shown that T is continuous, we conclude that T is completely continuous over $C([a,b])$. Finally, define the set ε by $\varepsilon = \{\varphi \in C([a,b]), \exists\lambda \in]0,1[; \varphi = \lambda T\varphi\}$. We prove that ε is bounded. Let $\varphi(.) \in \varepsilon$. Then $\forall t \in [a,b]$, we have

$$\begin{aligned} |\varphi(t)| &= |\lambda T\varphi(t)| \leq \|f\|_\infty + \int_a^b |g(t,s,\varphi(s))| ds \leq \|f\|_\infty + \|V_1\| \|V_2\| \sup_{y\geq 0} |\phi(y)| \\ &\leq \|f\|_\infty + \|V_1\| \|V_2\| M_\phi. \end{aligned}$$

Hence, the set ε is bounded. Finally, by applying Schaefer's fixed-point theorem, one concludes that T has a fixed point in $C([a,b])$ or, equivalently, (3.3) has a continuous solution over $C([a,b])$.

Condition (3.4) with bounded $\phi(.)$ is a limitation of Theorem 1.1 . Nonetheless, by using a convenient new norm $\|.\|_\mu$ and the Schauder fixed-point theorem, one can extend the result of the previous theorem to more general nonlinear integral equations under weaker conditions. This is the subject of the next theorem.

Theorem 3.2 ([5]) *Consider the nonlinear integral equation*

$$\varphi(t) = f(t) + \int_a^b g(t,s,\varphi(s)) ds, \quad -\infty < a \leq t \leq b < +\infty \tag{3.6}$$

Assume that $f(.)$ is bounded and $g(t,s,\varphi)$ satisfies the following conditions :

$$|g(t,s,\varphi)| \leq V_1(t)\, V_2(t)\, \phi(|\varphi|)\,, \quad \left|\frac{\partial g}{\partial \varphi}(t,s,\varphi)\right| \leq V_1(t)\, V_2(t)\, \psi(|\varphi|)\,, \tag{3.7}$$

where $V_1(t)$ is a measurable and bounded positive function, $\phi(.)$is positive and measurable function satisfying the condition

$$\sup_{x\,\geq\,0} \frac{\phi(\varphi)}{\varphi} = L < +\infty \tag{3.8}$$

and $\psi(.)$ is a positive and continuous function over $[0,+\infty[$. Moreover, assume that there exists a continuous and strictly positive function $\mu(.)$ satisfying the following condition :

$$\|V_1.\mu\|_\infty \left\|\frac{V_2}{\mu}\right\|_1 < \frac{1}{L}. \tag{3.9}$$

Under these conditions, the nonlinear integral equation (3.6) has a solution in $C([a,b])$.

Proof. We first note that $\|.\|_\mu$defined on $X = C([a,b])$ by $\|\varphi\|_\mu = \sup_{t\in[a,b]} |\mu(t)\,\varphi(t)|$ is a norm on X. Moreover, by using conditions on $\mu(.)$, one shows that $X = (C([a,b]), \|.\|_\mu)$ is a Banach space. Next, let $r \geq 0$ be a positive real number, to be fixed later on, and define a closed ball B_r of X by

$$B_r = \left\{\varphi \in C([a,b])\,;\ \|\varphi\|_\mu \leq r\right\}.$$

It is clear that B_r is a closed and convex subset of X. Then, prove that the operator T associated with (3.6) is continuous over X. Finally, show the inclusion

$$T(B_r) \subset B_r,\ \forall r \geq \frac{\|f\|_\mu}{1 - L\,\|V_1\|_\mu\,\|V_2/\mu\|_1} = r_0.$$

Hence by using Schauder 's fixed -point theorem, one concludes that (3.6) has a solution in B_{r_0} and consequently it has a solution in $C([a,b]\ )$. ■

Example 3.1 *Consider the following nonlinear integral equation :*

$$\varphi(t) = f(t) + \int_0^1 (t+\alpha)(s+\beta)\ln\left(1+\varphi^2(s)\right) ds,\ 0 \leq t \leq 1, \tag{3.10}$$

where $f \in C([a,b])$, α and β are two parameters satisfying $0 \leq \alpha \leq 1$ and $\beta \geq 0$. Using the notation of Theorem 3.2, one gets $V_1(t) = t+\alpha$, $V_2(s) = s+\beta$, $\phi(\varphi) = \lim(1+\varphi^2)$ and $\psi(\varphi) = \frac{2\varphi}{1+\varphi^2}$. It is clear that $\phi(.)$ satisfies condition (3.8) with $L = 1$and $\psi(.)$ is a continuous and positive function over $[0,+\infty[$. Next, consider the function $\mu(t) = e^{-t}$, $t \in [0,1]$. In this case, it is easy to check that

$$\|V_1\|_\mu \|V_1\|_1 = e^{\alpha-1}[\beta(e-1)+1].$$

Hence by the previous theorem, one concludes that (3.10) has a continuous solution whenever the parameters α, β satisfy the condition

$$e^{\alpha-1}\left[\beta\left(e-1\right)+1\right] < \frac{1}{L} = 1.$$

The application of schauder theorem in non-linear Fredholm integral equation, which required the operator is completly continuous, the non-linear Fredholm integral operator is completly continuous, in fact:

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.

Theorem 3.3 ([11]) *Let $h : \overline{\Omega}^2 \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping . Then the Fredholm operator associated to $h, T : C\left(\overline{\Omega}; \mathbb{R}^n\right) \to C\left(\overline{\Omega}; \mathbb{R}^n\right)$ given by*

$$T(\varphi)(x) = \int_\Omega h\left(x, y, \varphi(y)\right) dy, \quad x \in \overline{\Omega} \tag{3.11}$$

is completely continuous.

Proof. We first prove that T is continuous. Let $\varphi_0 \in C\left(\overline{\Omega}; \mathbb{R}^n\right)$ and choose any number $R > |\varphi_0|_\infty$. Let $\varepsilon > 0$. Since h is uniformly continuous on the compact set $\overline{\Omega}^2 \times \overline{B}_R\left(0; \mathbb{R}^n\right)$, there exists a $\delta_\varepsilon > 0$ such that for every $\varphi \in C\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfying $|\varphi - \varphi_0|_\infty \leq \delta_\varepsilon$ one has $\varphi\left(y\right) \in \overline{B}_R\left(0; \mathbb{R}^n\right)$ and

$$\left|h\left(x, y, \left(\varphi y\right)\right) - h\left(x, y, \varphi_0\left(y\right)\right)\right| \leq \varepsilon$$

for all $x, y \in \overline{\Omega}$. Then

$$\begin{aligned} \left|T\left(\varphi\right)\left(x\right) - T\left(\varphi_0\right)\left(x\right)\right| &\leq \int_\Omega \left|h\left(x, y, \varphi\left(y\right)\right) - h\left(x, y, \varphi_0\left(y\right)\right)\right| dy \\ &\leq \varepsilon\mu\left(\Omega\right) \end{aligned}$$

for every $x \in \overline{\Omega}$. Hence

$$\left|T\left(\varphi\right) - T\left(\varphi_0\right)\right|_\infty \leq \varepsilon\mu\left(\Omega\right)$$

whenever$|\varphi - \varphi_0|_\infty \leq \delta_\varepsilon$. Therefore T is continuous at φ_0. ■

Next, given a bounded subset Y of $C\left(\overline{\Omega}; \mathbb{R}^n\right)$, we shall prove that $T\left(Y\right)$is relatively compact in $C\left(\overline{\Omega}; \mathbb{R}^n\right)$. According to the Ascoli-Arzela theorem, we have to show that $T\left(Y\right)$ is bounded and equicontinuous.

Indeed, since Y is bounded there exists a constant $c > 0$ such that

$$|\varphi|_\infty \leq c \text{ for all } \varphi \in Y.$$

It follows that for any $\varphi \in Y$ we have

$$|T(\varphi)|_\infty \leq M\mu(\Omega),$$

where

$$M = \max_{\overline{\Omega}^2 \times \overline{B_c}(0;\mathbb{R}^n)} |h(x,y,z)|.$$

Hence the set $T(Y)$ is bounded in $C\left(\overline{\Omega};\mathbb{R}^n\right)$.

On the other hand, using the uniform continuity of h on the compact $\overline{\Omega}^2 \times \overline{B_c}(0;\mathbb{R}^n)$, for each $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that

$$\left|h(x,y,\varphi(y)) - h\left(x',y,\varphi(y)\right)\right| \leq \varepsilon$$

for all $x, x', y \in \overline{\Omega}$ with $\left|x - x'\right| \leq \delta_\varepsilon$ and $\varphi \in Y$. This immediately yields

$$\left|T(\varphi)(x) - T(\varphi)\left(x'\right)\right| \leq \varepsilon\mu(\Omega),$$

for all $x, x' \in \overline{\Omega}$ satisfying $\left|x - x'\right| \leq \delta_\varepsilon$ and $\varphi \in Y$. Thus $T(Y)$ is equicontinuous .

The next result is a local version of Theorem 3.3 .

Theorem 3.4 ([11]) *Let $R > 0$ and $h : \overline{\Omega}^2 \times B_R(0;\mathbb{R}^n) \to \mathbb{R}^n$ be a continuous mapping. Then the operator $T : \overline{B}_R\left(0; C\left(\overline{\Omega};\mathbb{R}^n\right)\right) \to C\left(\overline{\Omega};\mathbb{R}^n\right)$ given by (3.11) is completely continuous.*

Proof. Essentially the same reasoning as in the proof of Theorem 3.3 establishes the result.

Using Theorem 3.4 and Shauder's fixed point therem we shall prove the existence of continuouns solutions in a ball of $C\left(\overline{\Omega};\mathbb{R}^n\right)$, to the Fredholm Integral equation in $\mathbb{R}^n$

$$\varphi(x) = \int_\Omega h(x,y,\varphi(y))\,dy, \quad x \in \overline{\Omega}. \tag{3.12}$$

■

Theorem 3.5 ([11]) *Let $R > 0$ and $h : \overline{\Omega}^2 \times B_R(0;\mathbb{R}^n) \to \mathbb{R}^n$ be a continuous mapping. Assume*

$$M\mu(\Omega) \leq R, \tag{3.13}$$

where

$$M = \max_{\overline{\Omega}^2 \times \overline{B_R}(0;\mathbb{R}^n)} |h(x,y,z)|.$$

Then (3.12) has at least one solution $\varphi \in C\left(\overline{\Omega};\mathbb{R}^n\right)$ with $|\varphi|_\infty \leq R$.

Proof. According to Theorem 3.4, the operator $T : \overline{B}_R\left(0; C\left(\overline{\Omega}; \mathbb{R}^n\right)\right) \to C\left(\overline{\Omega}; \mathbb{R}^n\right)$ given by (3.11) is completely continuous. On the other hand, (3.13) guarantees that

$$T\left(\overline{B}_R\left(0; C\left(\overline{\Omega}; \mathbb{R}^n\right)\right)\right) \subset \overline{B}_R\left(0; C\left(\overline{\Omega}; \mathbb{R}^n\right)\right).$$

Thus the conclusion follows from Theorem 1.7 . ■

3.2 Existence of Solution for Nonlinear Volterra Integral Equations

3.2.1 Homogeneous Nonlinear Volterra Integral Equation

consider the nonlinear volterra integral equation of the second kind

$$\varphi(x) = \int_0^x k(x, y, \varphi(x))\, dy, \;\; x \in [0, 1] \tag{3.14}$$

If we are looking for a continuous solution φ for equation (3.14) in the ball of centre 0 and radius R, $\overline{B_R}(0; C([0,1]); \mathbb{R})$, form the space $C([0,1]; \mathbb{R})$, then it can be reformulated as a fixed point problem under appropriate assumptions suppsu: $k : [0,1] \times [0,1] \times [-R, R] \to \mathbb{R}$ is continuous and that there is $\alpha, \beta \in \mathbb{R}_+$ that

$$|k(x, y, z)| \leq \alpha |z| + \beta; \forall (x, y, z) \in [0,1] \times [0,1] \times [-R, R]$$

we consider the Banch space $E = C([0,1]; \|\ .\ \|_\infty)$ of continuous functions of $[0,1]$ in $\mathbb{R}$ with the norm $\|\varphi\|_\infty = \sup\limits_{x \in [0,1]} |\varphi(x)|$.

We define the operator $T : \overline{B_R}(0; C([0,1]); \mathbb{R}) \to C([0,1]; \mathbb{R})$ given by

$$(T\varphi(x)) = \int_0^x k(x, y, \varphi(x))\, dy, x \in [0,1] \tag{3.15}$$

Remark 3.2 *The operator T applies the ball $\overline{B}_R(0; C([0,1]); \mathbb{R})$, on itself. Therefore, the integral equation (3.14) is equivalent to the fixed point problem $T\varphi = \varphi$ or T is defined by (3.15).*

Remark 3.3 *The operator T is continuous and compact, and then, by the schauder fixed point theorem, T has at least one fixed point in the ball $\overline{B}_R(0; C([0,1]); \mathbb{R})$, which is a solution of the interal equation (3.14) in the ball $\overline{B}_R(0; C([0,1]); \mathbb{R})$.*

3.2.2 Non-Homogeneous Nonlinear Volterra Integral Equation

In this section, we study the existence as well as the uniqueness of the solution of a nonlinear Volterra Equations. If in the Fredholm integral equation (3.3) we replace the upper integration limit b by the variable t, we obtain a nonlinear Volterra equation. Under some conditions on the function $g(t,s,x)$, the following theorem ensures the existence of a solution of this nonlinear Volterra equation.

Theorem 3.6 ([5]) *Consider the nonlinear Volterra integral equation*

$$\varphi(t) = f(t) + \int_a^t g(t,s,\varphi(s))\,ds\ ,\ -\infty < a \le t \le b < +\infty \tag{3.16}$$

where f is continuous over $[a,b]$. Assume that $g(t,s,\varphi)$satisfies the following conditions:

$$|\, g(t,s,\varphi)\,| \le V_1(t)\,V_2(s)\,\phi(|\,\varphi\,|)\,,\, |\,\frac{\partial g}{\partial \varphi}(t,s,\varphi)\,| \le V_1(t)\,V_2(s)\,\psi(|\,\varphi\,|)\,,$$

where $V_1(.)$ is a positive and continuous function over $[a,b]$, $V_2(.)$is a positive and integrable function over $[a,b]$and $\psi(.)$ is a positive and continuous function over $[0,+\infty[$. Finally, assume that the function $\phi(.)$ is positive, continuous and satisfies the condition $\lim\limits_{y\to\infty} \frac{\phi(y)}{y} = L < +\infty$. Under these conditions, Eq. (3.16) has a continuous solution over $[a,b]$.

Proof. see [4] ■

The uniqueness of the solution of the nonlinear Volterra equation (3.16) is given by the following proposition.

Proposition 3.1 ([5]) *Consider the nonlinear Volterra integral equation (3.16) and assume that $g(t,s,x)$satisfies the conditions of Theorem* 3.6 *with $V_2(.) \in (L^1 \cap L^p)([a,b])$ for some $p > 1$.Then (3.16) has a unique solution .*

Proof. *The existence of a solution is ensured by Theorem* 3.6. *Also, we mention that, in the proof of Theorem* 3.6, *we have shown that the solutions of $x = Tx$ are uniformly bounded by the same constant M, and consequently they are contained in the closed ball B_M defined by $B_M = \{x \in C([a,b])\,;\, \|\, x\,\|_\infty \le M\}$. Hence, to prove the uniqueness of the solution of (3.16), it suffices to check that there exists an $n_0 \in \mathbb{N}$ such that $T^{n_0}$is a contraction in B_M. Next, show that $\forall n \in \mathbb{N}, \forall t \in [a,b]$and $\forall x(.)\,, y(.) \in B_M$,* ■

$$|T^n y(t) - T^n x(t)| \le C^n \,\|\, y - x\,\|_\infty \prod_{i=1}^{n-1} \frac{1}{q+i}\,(t-a)^{n-1+1/q}\,.$$

Consequently

$$\| T^n y(t) - T^n x(t) \|_\infty \leq C^n \| y - x \|_\infty \left[\prod_{i=1}^{n-1} \frac{1}{q+i} \right] (b-a)^{n-1+1/q}.$$

Since $\lim\limits_{n\to\infty} \left[\prod\limits_{i=1}^{n-1} \frac{1}{q+i} \right] C^n \quad (b-a)^{n-1+1/q} = 0$, then there exists an $n_0 \in \mathbb{N}$ such that T^{n_0} is a contraction over B_M and, consequently, the fixed point of T^{n_0} is unique. Since a fixed point of T is also fixed point of T^{n_0}, one concludes that the fixed point of T is also unique and, consequently, the solution of (3.16) is unique.

The application of schauder theorem in non-linear volterra integral equation, which required the operator is completly continuous, the non-linear volterra integral operator is completly continuous, in fact :

Theorem 3.7 ([11]) *Let* $h : [a,b]^2 \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous. Then the Volterra operator associated to* h*,* $T : C([a,b];\mathbb{R}^n) \to C([a,b];\mathbb{R}^n)$ *given by*

$$T(\varphi)(t) = \int_a^t h(t,s,\varphi(s))\,ds, \quad t \in [a,b] \tag{3.17}$$

is completely continuous.

Theorem 3.8 ([11]) *Let* $R > 0$ *and let* $h : [a,b]^2 \times \overline{B_R}(0;\mathbb{R}^n) \to \mathbb{R}^n$ *be continuous. Then the operator* $T : \overline{B}_R(0; C([a,b];\mathbb{R}^n)) \to C([a,b];\mathbb{R}^n)$ *given by (3.17) is completely continuous.*

As an application we present an existence theorem for the Volterra integral equation in $\mathbb{R}^n$

$$\varphi(t) = \int_a^t h(t,s,\varphi(s))\,ds + \upsilon(t), \quad t \in [a,b]. \tag{3.18}$$

Theorem 3.9 ([11]) *Let* $h : [a,b]^2 \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous and let* $\upsilon \in C([a,b];\mathbb{R}^n)$. *Assume that there exists constants* $\alpha, \beta \in \mathbb{R}_+$ *such that*

$$|h(t,s,z)| \leq \alpha |z| + \beta \tag{3.19}$$

for all $t, s \in [a,b]$, $z \in \mathbb{R}^n$. *Then (3.18) has at least one solution* $\varphi \in C([a,b];\mathbb{R}^n)$.

Proof. Let $T : C([a,b];\mathbb{R}^n) \to C([a,b];\mathbb{R}^n)$ be given by

$$T(\varphi)(t) = \int_a^t h(t,s,\varphi(s))\,ds + \upsilon(t).$$

According to Theorem 3.7, T is completely continuous. We now show that T is a self-mapping of a closed ball of the space $C([a,b];\mathbb{R}^n)$ endowed with a suitable norm, equivalent to the sup-norm $|.|_\infty$. Indeed, for any given number $\theta > 0$ we have

$$\begin{aligned}
|T(\varphi)(t)| &\leq \left|\int_a^t h(t,s,\varphi(s))\,ds + \upsilon(t)\right| \\
&\leq \int_a^t |\, h(t,s,\varphi(s))\,|\, ds + |\,\upsilon\,|_\infty \\
&\leq \alpha \int_a^t |\varphi(s)|\, ds + \beta(b-a) + |\,\upsilon\,|_\infty \\
&= \alpha \int_a^t |\varphi(s)|\, e^{-\theta(s-a)} e^{\theta(s-a)} ds + \beta(b-a) + |\,\upsilon\,|_\infty \\
&\leq \alpha \left|\varphi(.)\, e^{-\theta(.-a)}\right|_\infty \int_a^t e^{\theta(s-a)} ds + \beta(b-a) + |\,\upsilon\,|_\infty \\
&= \alpha\theta^{-1} \left|\varphi(.)\, e^{-\theta(.-a)}\right|_\infty \left(e^{\theta(s-a)} - 1\right) + \beta(b-a) + |\,\upsilon\,|_\infty \\
&\leq \alpha\theta^{-1} \left|\varphi(.)\, e^{-\theta(.-a)}\right|_\infty e^{\theta(s-a)} + \beta(b-a) + |\,\upsilon\,|_\infty
\end{aligned}$$

Since $e^{-\theta(t-a)} \leq 1$ on $[a,b]$, we deduce

$$|T(\varphi)(t)|\, e^{-\theta(t-a)} \leq \alpha\theta^{-1} \left|\varphi(.)\, e^{-\theta(.-a)}\right|_\infty + \beta(b-a) + |\,\upsilon\,|_\infty$$

and so

$$\left|T(\varphi)(.)\, e^{-\theta(.-a)}\right|_\infty \leq \alpha\theta^{-1} \left|\varphi(.)\, e^{-\theta(.-a)}\right|_\infty + \beta(b-a) + |\,\upsilon\,|_\infty\,. \tag{3.20}$$

Now fix any $\theta > \alpha$. Then we can find $R > 0$ such that

$$\alpha\theta^{-1}R + \beta(b-a) + |\,\upsilon\,|_\infty \leq R. \tag{3.21}$$

Consider a new norm on $C([a,b];\mathbb{R}^n)$, namely

$$\|\,\varphi\,\| = \left|\varphi(.)\, e^{-\theta(.-a)}\right|_\infty.$$

It is clear that the norm $|.|_\infty$ and $\|\,.\,\|$ are equivalent (thus T is also completely continuouns with respect to$\|\,.\,\|$). On the other hand, (3.20) and (3.21) show that T maps the closed ball of center 0 and radius R of the space $(C([a,b];\mathbb{R}^n), \|\,.\,\|)$, into itself. Now the conclusion follows from Theorem 1.7 .

The reader could try to give variants of Theorem 3.9 to the case where condition (3.19) is replaced by an inequality of the form

$$|h(t,s,z)| \leq \psi(|z|)$$

with other types of functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$. ■

3.3 Existence of solution for Nonlinear Integral Equation Hammerstein-Type

Assume that $K(t,s)$ is a continuous function for $0 \leq t,s \leq 1$and that $f(s,\varphi)$ is a bounded continuous function for $0 \leq s \leq 1$and $\varphi \in \mathbb{R}$. Then the equation

$$\varphi(t) = \int_0^1 K(t,s) f(s,\varphi(s))\, ds$$

has a continuous solution $\varphi(t)$.

We want to prove that $T(\varphi)$, $\varphi \in C([0,1])$, has a fixed point where $(T(\varphi))(t) = \int_0^1 K(t,s) f(s,\varphi(s))\, ds$. To show this we will apply the generalization of Schauder's fixed point theorem. We will choose a closed convex subset $S \subset C([0,1])$ such that the mapping $T : S \to C([0,1])$ is continuous and the image set $T(S)$ is relatively compact in $C([0,1])$.

First we observe that T maps continuous functions to continuous functions, i.e. that we have

$$T(C([0,1])) \subset C([0,1]).$$

This can be seen as follows : From the hypothesis there exists a $B > 0$ such that

$$|f(s,\varphi)| \leq B \text{ if } (s,\varphi) \in [0,1] \times \mathbb{R}.$$

Moreover $K(t,s)$ is continuous on the compact set $[0,1] \times [0,1]$ and hence K is

uniformly continuous on $[0,1] \times [0,1]$. Fix an $\epsilon > 0$.Then there exists a $\delta > 0$ such that

$$\left|K(t,s) - K\left(\widetilde{t},\widetilde{s}\right)\right| < \frac{\epsilon}{B} \text{ if } \left|(t,s) - \left(\widetilde{t},\widetilde{s}\right)\right| < \delta.$$

Consequently for arbitrary $\varphi \in C([0,1])$ we have

$$\begin{aligned}
\left|(T(\varphi))(t) - (T(\varphi))\left(\widetilde{t}\right)\right| &= \left|\int_0^1 \left(K(t,s) - K\left(\widetilde{t},s\right)\right) f(s,\varphi(s))\, ds\right| \\
&\leq \int_0^1 \left|K(t,s) - K\left(\widetilde{t},s\right)\right| |f(s,\varphi(s))|\, ds \\
&\leq B \int_0^1 \left|K(t,s) - K\left(\widetilde{t},s\right)\right| ds < \epsilon.
\end{aligned}$$

provided $\left|t - \widetilde{t}\right| < \delta$. This means that $T(\varphi) \in C([0,1])$.

A natural choice for the closed convex set S is as follows :

$$S = \{u \in C([0,1]) : \|u\| \leq D\},$$

where $D > 0$ is a constant that should be chosen such that $T(S) \subset S$. Here we note that since K is continuous on the compact set $[0,1] \times [0,1]$ there exists an $A > 0$ such that

$$|K(t,s)| \leq A \text{ if } (t,s) \in [0,1] \times [0,1].$$

This implies that

$$\begin{aligned} |(T(\varphi))(t)| &= \left| \int_0^1 K(t,s) f(s, \varphi(s)) \, ds \right| \\ &\leq \int_0^1 |K(t,s)| \, |f(s, \varphi(s))| \, ds \\ &\leq AB. \end{aligned}$$

We get

$$\|T(\varphi)\| \leq D$$

provided we choose $D \geq AB$. For instance set $D = AB$. With this choice for S we get

$$T(S) \subset S.$$

To apply Schauder's theorem we have to show that $T(S)$ is relatively compact in $C([0,1])$ and that T is continuous on S. The relatively compactness is consequence of Arzela-Ascoli theorem once we have shown that $T(S)$ is uniformly bounded and equicontinuous on S.

We have above verified that $T(C([0,1]))$ is uniformly bounded and equicontinuous on S. It remains to prove that $T : S \to T(S)$ is continuous. From the definition of S it follows that $|\varphi(t)| \leq D$ for all $t \in [0,1]$. The continuily of $f(s, \varphi)$ on the compact set $[0,1] \times [-D, D]$ implies that f is uniformly continuous on $[0,1] \times [-D, D]$. Fix an arbitrary $\epsilon > 0$. Then there exists a $\delta > 0$ such that

$$|f(s,\varphi) - f(\widetilde{s}, \widetilde{\varphi})| < \frac{\epsilon}{A} \text{ if } |(s,\varphi) - (\widetilde{s}, \widetilde{\varphi})| < \delta.$$

Hence for arbitrary $\varphi_1, \varphi_2 \in S$ we have

$$\begin{aligned} \|T(\varphi_1) - T(\varphi_2)\| &= \sup_{t \in [0,1]} \left| \int_0^1 K(t,s) \left(f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \right) ds \right| \\ &\leq \sup_{t \in [0,1]} \int_0^1 |K(t,s)| \, |(f(s, \varphi_1(s)) - f(s, \varphi_2(s)))| \, ds \\ &\leq A \int_0^1 |(f(s, \varphi_1(s)) - f(s, \varphi_2(s)))| \, ds \leq \epsilon. \end{aligned}$$

Now we have shown that T is continuous on S. Schauder's fixed point theorem implies that the equation $\varphi = T(\varphi)$ has at least one solution.

3.4 Existence of solution for Nonlinear Integral Equations with Delay

The following delay equation

$$\varphi(t) = \int_{t-\tau}^{t} f(s, \varphi(s))\, ds$$

can be interpreted as a model for the spread of certain infections diseases with a contact rate that varies seasonally. In this equation $\varphi(t)$ is the proportion of infectives in a population at time, and $f(t, \varphi(t))$ is the proportion of new infectives per unit time.

In this section we study the existence of continuous solutions on a given interval of time $[0, t_1]$, for the initial value problem

$$\begin{cases} \varphi(t) = \int_{t-\tau}^{t} f(s, \varphi(s))\, ds, & 0 \leq t \leq t_1 \\ \varphi(t) = \psi(t), & -\tau \leq t \leq 0. \end{cases} \tag{3.22}$$

We assume

$$f \in C([-\tau, t_1] \times \mathbb{R}^n; \mathbb{R}^n), \ \psi \in C([-\tau, 0]; \mathbb{R}^n)$$

and the following sewing condition holds

$$\psi(0) = \int_{-\tau}^{0} f(s, \varphi(s))\, ds. \tag{3.23}$$

By a solution of (3.22) we mean a function $\varphi \in C([-\tau, t_1]; \mathbb{R}^n)$ with $\varphi(t) = \psi(t)$ for all $t \in [-\tau, 0]$.

The initial value problem (3.22)was studied for the first time in Precup [10].

Theorem 3.10 ([11]) *Assume* $f \in C([-\tau, t_1] \times \mathbb{R}^n; \mathbb{R}^n)$, $\psi \in C([-\tau, 0]; \mathbb{R}^n)$ *and that (3.23) holds. Then the delay integral operator* $T : D(T) \to C([0, t_1]; \mathbb{R}^n)$ *given by*

$$T(\varphi)(t) = \int_{t-\tau}^{t} f(s, \widetilde{\varphi}(s))\, ds \quad t \in [0, t_1],$$

where

$$D(T) = \{\varphi \in C([0, t_1]; \mathbb{R}^n) : \varphi(0) = \psi(0)\}$$

and

$$\widetilde{\varphi}(t) = \begin{cases} \psi(t) \ \text{for} \ t \in [-\tau, 0], \\ \varphi(t) \ \text{for} \ t \in [0, t_1], \end{cases}$$

is completely continuous.

Proof. *Use the Ascoli-Arzela theorem and follow the same steps as in the proof of Theorem* 3.3. *We omit the details.* ■

Theorem 3.11 ([11]) *Assume* $f \in C([-\tau, t_1] \times \mathbb{R}^n; \mathbb{R}^n)$, $\psi \in C([-\tau, 0]; \mathbb{R}^n)$ *and that (3.23)holds. In addition assume that there exist* $\alpha, \beta \in \mathbb{R}_+$ *such that*

$$|f(t,z)| \leq \alpha |z| + \beta \tag{3.24}$$

for all $z \in \mathbb{R}^n$ *and* $t \in [0, t_1]$. *Then (3.22) has a solution* $\varphi \in C([-\tau, t_1]; \mathbb{R}^n)$.

Proof. The proof closely resembles the proof of Theorem 3.9. Let

$$\gamma = \max_{t \in [-\tau, 0]} |f(t, \varphi(t))|.$$

We have

$$\begin{aligned} |T(\varphi)(t)| &\leq \tau\gamma + \int_0^t |f(s, \varphi(s))| \, ds \\ &\leq \tau\gamma + \alpha \int_0^t |\varphi(s)| \, ds + \beta t_1 \\ &= \tau\gamma + \beta t_1 + \alpha \int_0^t |\varphi(s)| \, e^{-\theta s} e^{\theta s} ds \\ &\leq \tau\gamma + \beta t_1 + \alpha \left|\varphi(s) e^{-\theta s}\right|_\infty \int_0^t e^{\theta s} ds \\ &\leq \tau\gamma + \beta t_1 + \alpha \left|\varphi(s) e^{-\theta s}\right|_\infty \theta^{-1} e^{\theta s}. \end{aligned}$$

It follows that

$$\left|T(\varphi)(t) e^{-\theta t}\right|_\infty \leq \tau\gamma + \beta t_1 + \alpha\theta^{-1} \left|\varphi(s) e^{-\theta s}\right|_\infty.$$

Now choose $\theta > \alpha$ and a number $R > 0$ with

$$\tau\gamma + \beta t_1 + \alpha\theta^{-1} R \leq R.$$

Then $T(B) \subset B$, where

$$B = \left\{\varphi \in D(T) : \left|\varphi(t) e^{-\theta t}\right| \leq R \text{ for all } t \in [0, t_1]\right\}.$$

It is clear that B is a nonempty convex bounded closed subset of $C([0, t_1]; \mathbb{R}^n)$. The conclusion is now immediate from Theorem1.7 .

As in the previous section, the reader could try to obtain existence results for (3.22) assuming instead of (3.24) a condition of the form

$$|h(t,z)| \leq \omega(|z|)$$

with different types of functions $\omega : \mathbb{R}_+ \to \mathbb{R}_+$. ■

Conclusion

In this work we had studied the application of Schauder's fixed point theorem in integral equations theory, in order to proving the existence of solution of some non-linear integral equations, from the following figure :

$$\varphi(t) = f(t) + \int_{\Omega} K(t, s, \varphi(s))\, ds,$$

integral equation with Delay witch have the from :

$$\varphi(t) = \int_{t-\tau}^{t} f(s, \varphi(s))\, ds,$$

We had focus mention different versions of schauder's fixed point theorem in order to study differnet cases.

Finally, we gave some illustring examples.

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تــم بِحَمْدِ اللهِ

"الْحَمْدُ لِلَّهِ الَّذِي لَهُ مَا فِي السَّمَاوَاتِ وَمَا فِي الْأَرْضِ وَلَهُ الْحَمْدُ فِي الْآخِرَةِ وَهُوَ الْحَكِيمُ الْخَبِيرُ" [سبأ: 01]

الملخص:

في هذه المذكرة، ندرس تصنيف المعادلات التكاملية، ونركز على المعادلات التكاملية الغير خطية لفولتيرا وفريدهولم.

والهدف من هذا العمل هو إثبات وجود حلول بعض المعادلات التكاملية الغير خطية لفولتيرا و فريدهولم، وذلك المعادلات التكاملية المتعلقة بتأخر الزمن في فضاءات بناخ، باستعمال نظرية النقطة الصامدة لشودار.

الكلمات المفتاحية: معادلات تكاملية لفولتيرا، معادلات تكاملية فريدهولم، نظرية النقطة الصامدة لشوادر، نظرية ريزيس، المعادلات التكاملية المتعلقة بتأخر الزمن.

Résumé :

Dans ce mémoire, nous avons étudié et classé les équations intégrales, nous avons basé sur les équations intégrales non- linéaire de Fredholm et de Volterra.

L'objectif de cette recherche est de prouver l'existance de la résolution de quelques équations intégrales non-linéaire de Fredholm et de Volterra, et équation intégrale avec retard dans l'espace de Banach, En utilisant le théorème du point fixe de Schauder.

Les mots clés: *Équations intégrales de Fredholm, équations intégrales de Volterra, Théorème du point fixe de Schauder, Théorème de Riesz, équation intégrale avec retard.*

Abstract:

In this memory, we study and classify integral equations, and focus on Volterra and Fredholm non-linear integral equations.

The aim of this paper is to prove the existence of solution of some non-linear Fredholm and Volterra integral equation, integral equation with Delay in Banach spaces, using Schauder's fixed point theorem.

The Key words: *Fredholm's integral equations, Volterra's integral equations, Schauder's fixed point theorem, Riesz's theorem, integral equation with Delay.*