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Clonal sets of a binary relation: Theory and Applications

Hassane Bouremel

Thesis submitted in fulfillment of the
requirements for the degree of
Doctor of Mathematics
Academic year 2016-2017

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To the spirit of my father who passed away, God have mercy on him.

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Hassane Bouremel

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List of symbols

R^* : The closure transitive relation of a relation R on a set X (defined on page 4)

$\{\mathbf{x{,}y}\}^l$: The set of all lower bounds of x and y (defined on page 4)

$\{\mathbf{x{,}y}\}^u$: The set of all lower bounds of x and y (defined on page 5)

$\widehat{\ }$: The closure operator on a set X (defined on page 6)

$\mathcal{E}$: The set of all closed subsets of X under a given closure operator $\widehat{\ }$ (defined on page 7)

$\approx_R$: The clone relation of a binery relation R (defined on page 20)

δ_X : The smallest equivalence relation on a universe X (defined on page 20)

$\triangleleft_R$: The binary relation of the left comparable clones R (defined on page 31)

$\triangleright_R$: The binary relation of the right comparable clones of R(defined on page 31)

$\circ_R$: The binary symmetric relation of comparable clones of R (defined on page 31)

$\diamond_R$: The binary relation of incomparable clones of R (defined on page 31)

$(\mathbb{P} = (P, R_P))$: The set P equipped with the relation R_P on P. (defined on page 38)

$\mathbb{P}\overrightarrow{\cup}\mathbb{Q}$: The unidirectional disjoint union of $\mathbb{P}$ and $\mathbb{Q}$. (defined on page 39)

$\mathbb{P}\overleftrightarrow{\cup}\mathbb{Q}$: The bidirectional disjoint union of $\mathbb{P}$ and $\mathbb{Q}$. (defined on page 39)

$\mathcal{C}_R$: The set of all clonal sets of R on X. (defined on page 53)

$\mathcal{P}(X)$: The power set of the set X (defined on page 53)

$\widehat{A}$: The small clonal set contains A. (defined on page 61)

$\widehat{\cup}$: The sup operation of the complete lattice of the set of clonal sets (defined on page 62)

$\circ^r_R$: The reflexive related clones relation (defined on page 71)

$\diamond^i_R$: The irreflexive unrelated clones relation (defined on page 72)

$\triangledown$: The set of couples which their set of lower bound no empty (defined on page 89)

$\vartriangle$: The set of couples which their set of upper bound no empty (defined on page 89)

$\boxtimes$: The set of couples which their set of lower or upper bound no empty (defined on page 89)

Introduction

The clone relation of a strict order relation introduced by De Baets et al. [32]. This notion is based on how elements are related w.r.t. each other in a partially ordered set (poset, for short). Two elements of a poset are said to form a pair of clones (or to be clones, for short) if every other element that is greater (resp. smaller) than one of them is also greater (resp. smaller) than the other one. The clone relation of a strict order relation always is a tolerance relation and it is built up by two different types of pairs of clones: pairs of comparable clones (which constitute an antitransitive relation) and pairs of incomparable clones (which constitute a transitive relation). This partition of the clone relation played a key role in the characterization of the L-fuzzy tolerance relations and the L-fuzzy equivalence relations that a strict order relation is compatible with. Extending the definition of the clone relation of a strict order relation to an arbitrary binary relation is a trivial task. Nevertheless, when doing so, its properties significantly vary from these of the clone relation of a strict order relation. For instance, this extension leads to the distinction between two different types of pairs of comparable clones: pairs of clones in which one element is related to the other and not the other way around (which constitute an antitransitive relation) and pairs of clones in which both elements are related to each other (which constitute a transitive relation).

When restricting to a total order relation, the clone relation coincides with the covering relation, i.e., two elements are clones if and only if they are consecutive. This notion of consecutive elements in a totally ordered set was already independently considered in the field of social choice theory by Tideman under the same name: clones. Clones are important in the field of social choice theory since they can easily change the result of an election. Several methods have been proposed in order to guarantee the independence of clones (see [68, 71, 73]).

Outside the field of social choice theory, the notions of left and right trace of a binary relation were introduced by Doignon et al. [38] based on a concept similar to that of the clone relation. This notion played a key role in the characterization of the basic properties of a fuzzy relation and of the compatibility of fuzzy relations (see [2, 41, 53]).

The notion of compatibility of a given fuzzy relation with another one, extensively studied in [53], establishes an interesting relation on the set of fuzzy relations. It generalizes the notion of extensionality, introduced by Höhle and Blanchard [47], or the equivalent notion of compatibility, as it was coined by Bělohlávek [2], of a fuzzy relation w.r.t. a fuzzy equality relation. This notion appears, among others, in the study of fuzzy lattices [5, 57, 75, 74], in the study of fuzzy functions [33, 62, 63, 64], in the study of fuzzy order relations [9, 14, 13, 34] and in the lattice-theoretic approach to concept lattices [2].

Given the importance of fuzzy tolerance and fuzzy equivalence relations in the theory and applications of fuzzy sets, it is not surprising that compatibility has mainly been studied for a given fuzzy relation with the mentioned types of fuzzy relations. In this context, some of the present authors have focused their attention on the case where the given fuzzy relation is simply a crisp (strict) order relation, leading to surprising negative results as well as interesting representation theorems [32]. These results were obtained thanks to the introduction of the notion of clone relation associated with a strict order relation. Recently, we hav been shown how a clone relation can be associated with any crisp relation [20]. This clone relation allows us to take a step further in this work and aim at characterizing the fuzzy tolerance and fuzzy equivalence relations a given crisp relation is compatible with.

The main aim of this work is to analyse the properties of the clone relation of binary relation in order to solve the general problem of characterizing the L-fuzzy equivalence relations a given relation is compatible with. Also we aim to provide a representation of all L-fuzzy equivalence relations compatible with a given order relation.

This dissertation is structured as follows.

- Part I:
    1. In Chapter 1, we provide generalities on binary relations, ordered sets, lattice, complete lattice, residuated lattices and fuzzy relations that we need throughout this thesis.
    2. In Chapter 2, we focus on proprieties of clone relation of a binary relation. First, we extend the notion of clone relation of a strict order relation to an arbitrary binary relation. In particular, we introduce the partition of the clone relation in terms of three different types of pairs of clones. Second, we characterize the clone relation of the three different types of disjoint union. Finally, we analyse the properties of the clone relation of order n and the n-th power relation of the clone relation.
    3. In Chapter 3, we extend the notion of clone relation of two elements to a set of elements. In that way, we provide that the clonal set of a given relation is based on how any two elements of this set are related in same way w.r.t. to any other elements. We investigate the most important properties of the clonal sets of a given binary relation, paying particular attention to show that the set of all clonal sets of a binary relation is a complete lattice with the usual intersection and a clonal closure union.
- Part II:
    1. In Chapter 4, after recalling some basic definitions and properties on compatibility of two L-fuzzy relations on a residuated lattice. In par-

ticular related to the clone relation of a crisp relation, we study two auxiliary relations associated with this clone relation. These auxiliary relations respectively gather the reflexive related clones and the irreflexive unrelated clones. Also we study the compatibility of a given crisp relation with the latter auxiliary relations. The results are exploited to characterize the fuzzy tolerance and fuzzy equivalence relations a given crisp relation is compatible with. These characterizations turn out to be pleasingly elegant and insightful.

    2. In Chapter 5, we focus on other points related to this notion of compatibility. We study the compatibility of a fuzzy equivalence relation with an order relation, in that way we study the equivalent of the three type of compatibility of fuzzy equivalence relation with an order relation, and we provide a representation of all fuzzy equivalence relations compatible with a given order relation.

- Finally, general conclusions and future research are drawn.

Most of our work presented in this dissertation has already been published or submitted for publication in peer-reviewed international journals. Chapters 2, have been described in [20]. Chapters 4, have been described in [26].

PART I

THEORY: CLONAL SETS OF BINARY RELATION

1 Generalities on relations, residuated lattices and L-fuzzy relations

The purpose of this first chapter is to provide a basic introduction to the binary relations, posets, lattices, t-norm, residuated lattices. Next, we recall some basic notions of fuzzy logic, L-fuzzy sets and L-fuzzy relations.

1.1. Binary relations

A binary relation on a set X is a subset of X^2, i.e., it is a set of couples $(x, y) \in X^2$. For a relation $R \subseteq X^2$, we often write xRy instead of $(x, y) \in R$. Two elements x and y of a set X equipped with a relation R are called comparable elements, denoted by $x \nparallel y$, if it holds that xRy or yRx. Otherwise, they are called incomparable elements, denoted by $x \parallel_R y$, or simply $x \parallel y$ when no confusion can occur. We denote by R^c the complement of the relation R on X, i.e., for any $x, y \in X$, xR^cy denotes the fact that $(x, y) \notin R$. We denote by R^t the transpose of the relation R on X, i.e., for any $x, y \in X$, xR^ty denotes the fact that yRx. We denote by R^d the dual of the relation R on X, i.e., for any $x, y \in X$, xR^dy denotes the fact that yR^cx. A relation R on a set X is said to be included in a relation S on the same set X, denoted by $R \subseteq S$, if, for any $x, y \in X$, xRy implies that xSy. The union of two relations R and S on a set X is the relation $R \cup S$ on X defined as $R \cup S = \{(x, y) \in X^2 \mid xRy \vee xSy\}$. Similarly, the intersection of two relations R and S on a set X is the relation $R \cap S$ on X defined as $R \cap S = \{(x, y) \in X^2 \mid xRy \wedge xSy\}$. If $R \cap S = \emptyset$, then R and S are called disjoint relations. The composition of two relations R and S on a set X is the relation $R \circ S$ on X defined as $R \circ S = \{(x, z) \in X^2 \mid (\exists y \in X)(xRy \wedge ySz)\}$. For any $n \in \mathbb{N}^*$, the n-th power relation R^n of R is recursively defined as follows:

$$(R^1 = R) \wedge (\forall n \geqslant 1)(R^{n+1} = R^n \circ R)\,.$$

A binary relation R on a set X is called:

(i) reflexive, if, for any $x \in X$, it holds that xRx;

(ii) irreflexive, if, for any $x \in X$, it holds hat xR^cx;

(iii) symmetric, if, for any $x, y \in X$, it holds that xRy implies that yRx;

(iv) antisymmetric, if, for any $x, y \in X$, it holds that xRy and yRx imply that $x = y$;

(v) asymmetric, if, for any $x, y \in X$, it holds that xRy implies that $yR^c x$;

(vi) transitive, if, for any $x, y, z \in X$, it holds that xRy and yRz imply that xRz;

(vi) antitransitive, if, for any $x, y, z \in X$, it holds that xRy and yRz imply that $xR^c z$;

(vii) complete, if, for any $x, y \in X$, either xRy or yRx holds.

For a relation R on X, R^* denotes its transitive closure, i.e., the smallest transitive relation on X that contains R.

$$R^* = \bigcup_{k \geq 1} R^k \,,$$

where R^k is the k-th power of R.

In addition to the transitivity of the relation R^*, the following proposition shows other basic properties of R^* based on the properties of R.

Proposition 1.1. [27] *Let R be a relation on X and R^* be its transitive closure. Then it holds that*

(i) If R is reflexive, then R^* is reflexive.

(ii) If R is symmetric, then R^* is symmetric.

(iii) R is transitive if and only if $R = R^*$.

We recall here a well-known result concerning the n-th power relation. For more details, we refer to [27].

Proposition 1.2. *Let R be a relation on a set X. The following statements hold:*

(i) If R is reflexive, then it holds that $(\forall n \in \mathbb{N}^*)(R^n \subseteq R^{n+1})$.

(ii) If R is transitive, then it holds that $(\forall n \in \mathbb{N}^*)(R^{n+1} \subseteq R^n)$.

(iii) If R is reflexive and transitive, then it hold that $(\forall n \in \mathbb{N}^*)(R^n = R)$.

A binary relation R on a set X is called:

(i) a pseudo-order relation if it is reflexive and antisymmetric;

(ii) an order relation if it is reflexive, antisymmetric and transitive;

(iii) a strict order if it is irreflexive and transitive;

(iv) a total order relation if it is reflexive, antisymmetric, transitive and complete;

((v)) a tolerance relation if it is reflexive and symmetric;

(vi) an equivalence relation if it is reflexive, symmetric and transitive.

A set X equipped with an order relation $\leqslant$ is called a partially ordered set (poset, for short), denoted $(X, \leqslant)$. Further, $\{x, y\}^u$ denotes the set of all upper bounds of x and y, while $\{x,y\}^l$ denotes the set of all lower bounds of x and y, i.e., $\{x,y\}^u = \{z \in X \mid x \leqslant z \ \wedge \ y \leqslant z\}$ and $\{x,y\}^l = \{z \in X \mid z \leqslant x \ \wedge \ z \leqslant y\}$.

A strict order relation $<$ on a set X is a relation that is irreflexive (i.e., $x < x$ does not hold for any $x \in X$) and transitive, implying that it is asymmetric (i.e., $x < y$ implies $\neg(y < x)$, for any $x, y \in X$). To any order relation $\leqslant$ corresponds a strict order relation $<$ (its strict part or irreflexive kernel): $x < y$ if $x \leqslant y$ and $x \neq y$. Conversely, to any strict order relation $<$ corresponds an order relation $\leqslant$ (its reflexive closure): $x \leqslant y$ if $x < y$ or $x = y$.

For any tolerance/equivalence relation R on a set X, the tolerance/equivalence class of an element $x \in X$ is given by $[x]_R = \{y \in X \mid xRy\}$.

For more details on binary relations, we refer to [1, 24, 51, 67, 69].

1.2. Lattices and closure operators

1.2.1. Lattices

Many important proprieties of an order set $(L, \leq)$ are expressed in term of the existence of certain upper bounds or lower bounds of subsets of X. We will be particularly interested in two of the most important classes of ordered sets defined in this way are lattice and complete lattice. We often write $x \vee y$ instead of $\sup\{x,y\}$ when it exists and $x \wedge y$ instead of $\inf\{x,y\}$ when it exists. Similarly we write $\bigvee S$ (the join of S) and $\bigwedge S$ (the meet of S) instead of $\sup S$ and $\inf S$ when these exist.

Definition 1.1. *([24]) Let $(X, \leq)$ be an ordered set.*

(i) *If $x \vee y$ exists for all $x, y \in X$, then $(X, \leq)$ is called a $\vee$-semi-lattice.*

(ii) *If $x \wedge y$ exists for all $x, y \in X$, then $(X, \leq)$ is called a $\wedge$-semi-lattice.*

(iii) *$(X, \leq)$ is called a lattice if it is both a $\wedge$-semi-lattice and a $\vee$-semi-lattice.*

(iv) *If $\bigvee S$, $\bigwedge S$ exist for all $S \subseteq X$, then $(X, \leq)$ is called a complete lattice.*

A subset $M \neq \emptyset$ of a lattice $(L, \wedge, \vee)$ is called sub-lattice of L if, for any $a, b \in M$, it holds that $a \wedge b \in M$ and $a \vee b \in M$. A sub-lattice M of a lattice L is called a convex sub-lattice of L, if $x \leq z \leq y$ and $x, y \in M$ implies that $z \in M$, for any $x, y, z \in L$.

A bounded lattice is a lattice that additionally has a greatest element 1 and a smallest element 0, which satisfy $0 \leq x \leq 1$ for any x in X.

A lattice $(L, \leq, \wedge, \vee)$ is distributive if the following additional condition holds

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\,, \text{ for any } x, y, z \in L\,.$$

This means that the meet operation preserves non-empty finite joins. It is known that the above condition is equivalent to its dual

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\,, \text{ for any } x, y, z \in L\,.$$

A lattice $(L, \wedge, \vee)$ is modular if the following condition holds:

$$x \leq z, \text{ implies that } x \vee (y \wedge z) = (x \vee y) \wedge z\,, \text{ for any } x, y, z \in L\,.$$

This condition is also equivalent to its dual

$$z \leq x, \text{ implies that } x \wedge (y \vee z) = (x \wedge y) \vee z\,, \text{ for any } x, y, z \in L\,.$$

The following theorem characterizes the modular lattice L.

Theorem 1.1. [24] *Let be L a lattice.*

(i) L is modular if and only if it has no sub-lattice of the form $\mathcal{N}_5$,

(ii) If L is distributive, then it holds that L is modular.

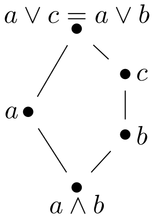


Figure 1.1: Hasse diagram of the lattice $\mathcal{N}_5$.

A complemented lattice is a bounded lattice $(L, \wedge, \vee, 0, 1)$, in which any element x has a complement, i.e., there exists an element $y \in L$ such that

$$x \vee y = 1 \text{ and } x \wedge y = 0\,.$$

An element may have more than one complement in general. However, if $(L, \wedge, \vee, 0, 1)$ is distributive then every element will have at most one complement.

1.2.2. Closure operator

Definition 1.2. *A closure operator on a set X is a mapping $\widehat{\ }: \mathcal{P}(X) \to \mathcal{P}(X)$ from the power set of X to itself which satisfies the following conditions:*

(i) $A \subseteq \widehat{A}$, for any $A \in \mathcal{P}(X)$;

(ii) $A \subseteq B \Rightarrow \widehat{A} \subseteq \widehat{B}$, for any $A, B \in \mathcal{P}(X)$,

(iii) $\widehat{\widehat{A}} = \widehat{A}$, *for any* $A \in \mathcal{P}(X)$.

A subset A of a set X is called closed under a given closure operator $\widehat{\ }$ if $\widehat{A} = A$. The set of all closed subsets of X is denoted by $\mathcal{E}$, i.e., $\mathcal{E} = \{A \subseteq X \mid \widehat{A} = A\}$.

In this work, we need the following well known result.

Theorem 1.2. [24] *Let be* X *a set with a closure operator* $\widehat{\ } : \mathcal{P}(X) \to \mathcal{P}(X)$. *Then the set* $\mathcal{E} = \{A \subseteq X \mid \widehat{A} = A\}$ *ordered by inclusion is a complete lattice, in which*

$$\underset{i\in I}{\wedge} A_i = \underset{i\in I}{\cap} A_i \text{ and } \underset{i\in I}{\vee} A_i = \widehat{(\underset{i\in I}{\cup} A_i)},$$

for any family $\{A_i\}_{i\in I} \subseteq \mathcal{E}$.

Remark 1.1. *If* (X, τ) *is a topological space, then the topological closure map* $^{-}$ *is a closure operator on* $\mathcal{P}(X)$, *and the set of all closed sets of* $\mathcal{P}(X)$ *is* $\tau(X)$.

For more information on lattices, complete lattices and closur operators can be found in, for instance,[19, 8, 24, 67].

1.3. T-norms

The history of triangular-norms(*t-norms*) started with Menger [56]. His main idea was to construct metric spaces where probability distributions are used to describe the distance between two elements. Schweizer and Sklar [66] provided the axioms of *t-norms*, as they are used today.

Definition 1.3. [58] *A t-norm* T *on* $[0, 1]$ *is a function* $T : [0, 1]^2 \to [0, 1]$ *satisfies the following four axioms:*

(T1) *Commutativity:* $(\forall x, y \in [0, 1])(T(x, y) = T(y, x))$;

(T2) *Associativity:* $(\forall x, y, z \in [0, 1])(T(x, T(y, z)) = T(T(x, y), z)))$;

(T3) *Monotonicity:* $(\forall x, y, z \in [0, 1)(x \leq y \Rightarrow T(x, z) \leq T(y, z))$;

(T4) *Boundary condition:* $(\forall x \in [0, 1])(T(x, 1) = x)$.

Conditions (T4) and (T3) imply that for any t-norm T it holds that $T(x, y) \leq x$, $T(x, y) \leq y$, $T(x, y) \leq Min(x, y)$ and $T(x, 0) = 0$.

The following definition of a t-norm on a bounded partially ordered set $(L, \leq)$ is analogous to the definition of a t-norm on the real unit interval $[0, 1]$.

Definition 1.4. [31] *A t-norm* T *on a bounded poset* $(L, \leq)$ *is a function* $T : L^2 \to L$ *satisfies the following four axioms:*

(i) *Commutativity:* $(\forall x, y \in L)(T(x, y) = T(y, x))$;

(ii) *Associativity:* $(\forall x, y, z \in L)(T(x, T(y, z)) = T(T(x, y), z)))$;

(iii) *Monotonicity:* $(\forall x, y, z \in L)(x \leq y \Rightarrow T(x, z) \leq T(y, z))$;

(iv) *Boundary condition* $(\forall x \in L)(T(x, 1) = x)$.

Example 1.1. *The following four operations are the most common t-norms:*

(T5) *Minimum:* $T_M(x, y) = \min\{x, y\}$

(T6) *Product:* $T_P(x, y) = x.y$

(T7) *Lukasiewicz:* $T_L(x, y) = \max\{x + y - 1, 0\}$

(T8) *Drastic product:*

$$T_D(x, y) = \begin{cases} x & \text{if } \ y = 1 \\ y & \text{if } \ x = 1 \\ 0 & \text{if } \ x, y < 1. \end{cases}$$

Let T be a t-norm on $[0, 1]$.
An element $a \in]0, 1[$ is called a zero divisor of T if there exists some $b > 0$ such that $T(a, b) = 0$.
An element $a \in [0, 1]$ is called an idempotent element of T if $T(a, a) = a$.
T is called Archimedean if $T(x, x) < x$ for every $x \in [0, 1]$.
Each $a \in [a, b]$ is an idempotent element of the Minimum t-norm T_M (Actually T_M is the only t-norm whose set of idempotent is equal $[0, 1]$), T_M has no zero divisor. Each $a \in]0, 1[$ is a zero divisor of the Lukasiewicz t-norm T_L as well of the Drastic product t-norm T_D. For two t-norms T_1 and T_2 on $[0, 1]$, we define:

$$T_1 \leq T_2 \Leftrightarrow (\forall x, y \in [0, 1])(T_1(x, y) \leq T_2(x, y)).$$

Let be T_1 and T_2 two t-norms. If $T_1 \leq T_2$, then then T_1 is called weaker than T_2 (or, equivalently, T_2 is called stronger than T_1). Note that T_D is the weakest t-norm, and T_M is the strongest t-norm, i.e. for any t-norm it holds: (T9) $\mathrm{T_D \leq T \leq T_M}$. Since $T_L \leq T_P$, it obviously holds: (T10) $\mathrm{T_D \leq T_L \leq T_P \leq T_M}$.

Definition 1.5. *[12] Let T_1 and T_2 be two t-norms. T_1 is said to dominate T_2 if and only if, for any $x, y, z, t \in [0, 1]$, it holds that:*

$$T_1(T_2(x, y), T_2(z, t)) \geq T_2(T1(x, z), T_1(y, t))$$

.

Lemma 1.1. *([30], [54])*

(i) *Any t-norm T dominates itself.*

(ii) *The minimum t-norm T_M dominates any other t-norm.*

(iii) *If a t-norm T_1 dominates another t-norm T_2, then T_1 is stronger than T_2.*

Lemma 1.1 particularly implies that dominance is a reflexive and antisymmetric relation on the set of t-norms. Note that it still remains an open problem whether it is transitive.

1.4. Residuated lattices

1.4.1. Basic Concept

Residuated lattices, introduced by Dilworth and Ward [37] and some related algebraic systems [60]play an important role because they provide an algebraic frameworks to fuzzy logic and fuzzy reasoning. In this chapter, we recall some important properties of residuated lattices which are related to our work in the two last chapters on compatibility of crisp relation with fuzzy equivalence relations and compatibility of order relation with fuzzy equivalence relations .

Definition 1.6. *[3] A residuated lattice is an algebra* $(L, \wedge, \vee, *, \rightarrow, 0, 1)$*, or simply,* $(L, *, \rightarrow)$ *where:*

- (i) $(L, \wedge, \vee, 0, 1)$ *is a lattice (the corresponding order will be denoted by* $\leq$*) with the least element* 0 *and the greatest element* 1*;*
- (ii) $(*, \rightarrow)$ *forms an adjoint couple on* L*, i.e. for any* $a, b, c \in L$*:*
  - $(R1)$ *If* $a \leq b$ *and* $c \leq d$ *then* $a * c \leq b * d$*;*
  - $(R2)$ *If* $b \leq c$ *then* $a \rightarrow b \leq a \rightarrow c$*;*
  - $(R3)$ *If* $a \leq b$ *then* $b \rightarrow c \leq a \rightarrow c$*;*
  - $(R4)$ $a * b \leq c \Leftrightarrow a \leq b \rightarrow c$ *(adjointness condition);*
- (iii) $(L, *, 1)$ *forms a commutative monoid, i.e. for any* $a, b, c \in L$*:*
  - $(R5)$ $(a * b) * c = a * (b * c)$*;*
  - $(R6)$ $a * b = b * a$*;*
  - $(R7)$ $1 * a = a$*.*

Residuated lattice L is called complete if $(L, \wedge, \vee, 0, 1)$ is a complete lattice.
$*$ and $\rightarrow$ called multiplication and residuum, respectively. Multiplication is isotone, residuum is isotone in the first and antitone in the second argument (w.r.t. lattice order $\leq$).

Example 1.2. *Consider* $L = (0, a, b, c, d, m, 1)$ *with* $0 < a < b < m < 1$ *and* $0 < c < d < m < 1$*, but elements* $\{a, c\}$ *and* $\{b, d\}$ *are pairwise incomparable, the Hass diagram of* L *is shown in Figure 1.2.*

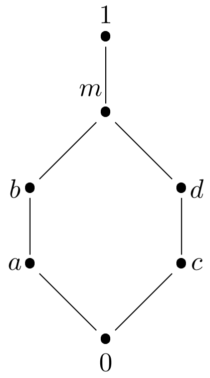


Figure 1.2: Hasse diagram of L.

Then ([52], page 23) $(L, \wedge, \vee, *, \to, 0, 1)$ *becomes a residuated lattice relative to the following operations:*

$\to$	0	a	b	c	d	m	1
0	1	1	1	1	1	1	1
a	d	1	1	d	d	1	1
b	d	m	1	d	d	1	1
c	b	b	b	1	1	1	1
d	b	b	b	m	1	1	1
m	0	b	b	d	d	1	1
1	0	a	b	c	d	m	1

$*$	0	a	b	c	d	m	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	a	a
b	0	a	a	0	0	a	b
c	0	0	0	c	c	c	c
d	0	0	0	c	c	c	d
m	0	a	a	c	c	m	m
1	0	a	b	c	d	m	1

1.4.2. Main properties

In the following, we use $\pounds$ to denote the class of all residuated lattices, and we always suppose that L is a bounded lattice with the smallest element 0 and the greatest element 1, $*$ and $\to$ are two binary operations on L. In addition, we often use the following derived operations: $a^0 = 1, a^n = a^{n-1} * a$, where $n \in \mathbb{N}$ and $a \in L$.

The following Proposition lists the fundamental properties of residuated lattices.

Proposition 1.3. *[60, 61] If* $(L, *, \to) \in \pounds$, *then the following statements hold:*

(R8) $a \leq b \to a * b$;

(R9) $(a \to b) * a \leq b$;

(*R10*) $f_a : L \to L, x \mapsto x * a$ *preserves all joins existing in* L, *i.e.*

$$(\bigvee_{i\in I} a_i) * a = \bigvee_{i\in I}(a_i * a);$$

(*R11*) $g_a : L \to L, x \mapsto a \to x$ *preserves all meets existing in* L, *i.e.*

$$a \to (\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I}(a \to a_i);$$

(*R12*) $h_a : L \to L, x \mapsto a * x$ *preserves all joins existing in* L, *i.e.*

$$a * (\bigvee_{i\in I} a_i) = \bigvee_{i\in I}(a * a_i);$$

(*R13*) $k_a : L \to L, x \mapsto x \to a$ *changes all joins existing in* L *to meets, i.e.*

$$(\bigvee_{i\in I} a_i) \to b = \bigwedge_{i\in I}(a_i \to b);$$

(*R14*) $b \to c \leq (a \to b) \to (a \to c)$ *and* $b \to a \leq (a \to c) \to (b \to c)$;

(*R15*) $a = 1 \to a$;

(*R16*) $a \leq b \Leftrightarrow a \to b = 1$;

(*R17*) $a \leq b \to c \Leftrightarrow b \leq a \to c$;

(*R18*) $a \to b \leq a * c \to b * c$ *and* $(a \to b) * (b \to c) \leq (a \to c)$;

(*R19*) $a * b \to c = a \to (b \to c)$;

(*R20*) $a \to (b \to c) = b \to (a \to c)$;

(*R21*) $a * b \leq a \wedge b$;

(*R22*) $a^n \leq a^m, n, m \in \mathbb{N}, m \leq n$.

Proposition 1.4. *[55] Let* $(L, *, \to)$ *be a residuated lattice. The following two statements hold:*

(i) $x * (y \wedge z) \leq (x * y) \wedge (x * z)$,

(ii) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

The following proposition shows that under suitable conditions, the operations $*$ and $\to$ are not independent.

Proposition 1.5. *[60, 61] Let* $(L, \wedge, \vee, 0, 1)$ *be a complete lattice.*

(i) *If* $*$ *is a binary operation on* L *satisfying conditions* $(R1)$ *and* $(R10)$*, then there exists a binary operation* $\to$ *satisfying conditions* $(R2), (R3)$ *and* $(R4)$*. Such operation is unique which is determined by the following formula:*

$$a \to b = \bigvee\{x \in L \mid x * a \leq b\},\ a, b \in L$$

(ii) *If* $\to$ *is a binary operation on* L *satisfying conditions* $(R2), (R3)$ *and* $(R11)$*, then there exists a binary operation* $*$ *satisfying conditions* $(R1)$ *and* $(R4)$*, and such operation is unique which is determined by the following formula:*

$$a * b = \bigwedge\{x \in L \mid a \leq b \to x\},\ a, b \in L$$

Now we characterize the residuated lattices by the following propositions.

Proposition 1.6. *[60] Let* $(L, \wedge, \vee, 0, 1)$ *be a lattice.* $(L, *, \to) \in \pounds$ *if and only if the following conditions hold, for all* $a, b, c \in L$*:*

(i) $(R4)$ $a * b \leq c \Leftrightarrow a \leq b \to c$*;*

(ii) $(R7)$ $1 * a = a$*;*

(iii) $(R20)$ $a \to (b \to c) = b \to (a \to c)$*.*

Proposition 1.7. *[60]Let* $(L, \wedge, \vee, 0, 1)$ *be a lattice.* $(L, *, \to) \in \pounds$ *if and only if the following conditions hold, for all* $a, b, c \in L$*:*

(i) $(R8)$ $(a \to b) * a \leq b$*;*

(ii) $(R9)$ $a \leq b \to a * b$*;*

(iii) $(R7)$ $1 * a = a$*.*

(iv) $(R20)$ $a \to (b \to c) = b \to (a \to c)$*;*

(v) $(R23)$ $(a \vee b) * c = (a * c) \vee (b * c)$*;*

(vi) $(R24)$ $a \to b \wedge c = (a \to b) \wedge (a \to c)$*.*

Proposition 1.8. *Let* $(L, \wedge, \vee, 0, 1)$ *be a lattice.* $(L, *, \to) \in \pounds$ *if and only if the following conditions hold, for all* $a, b, c \in L$*:*

(i) $(R4)$ $a * b \leq c \Leftrightarrow a \leq b \to c$

(ii) $(R'7)$ $1 \to a = a$*;*

(iii) $(R20)$ $a \to (b \to c) = b \to (a \to c)$*.*

Proof. It is easy to see that (i),(ii) and (iii) hold in any residuated lattice. Conversely, it suffices to show that (i),(ii) and (iii) imply that $(L, *, 1)$ is a commutative monoid. We have $x * 1 \leq t$ iff $x \leq (1 \to t)$ iff (by $(R'7)$) $x \leq t$, which implies $x * 1 = x$. Furthermore, $x * y \leq t$ iff $x \leq (y \to t)$ iff $1 \to x \leq y \to t$ iff (by

(R2))$x \to (1 \to x) \leq x \to (y \to t)$ iff (by (R20)) $(1 \to (x \to x) \leq y \to (x \to t)$ iff $1 \leq y \to (x \to t)$ iff (by (R4)) $1 * y \leq x \to t$ iff $y \leq 1 \to (x \to t)$ iff $y \leq x \to t$ iff $y * x \leq t$, i.e. $x * y = y * x$. Finally, $(x * y) * z \leq t$ iff ...iff $1 \to x \leq y \to (z \to t)$ iff $1 \leq x \to (y \to (z \to t))$ iff (by (R20)) $1 \leq z \to (y \to (x \to t))$ iff...iff $x * (y * z) \leq t$, i.e. $(x * y) * z = x * (y * z)$. Therefore $(L, *, 1)$ is a commutative monoid. $\square$

A residuated lattice satisfies the prelinearity axiom [3] if and only if $(x \to y) \vee (y \to x) = 1$ holds. A residuated lattice is *divisible* [3] if and only if $x \wedge y = x * (x \to y)$. It can be shown [3] that divisibility is equivalent to the following condition: for each $x \leq y$ there is z such that $x = y * z$. A residuated lattice satisfies the law of double negation(and is called integral, commutative Girard-monoid [3]) if and only if $x = (x \to 0) \to 0$ holds.

Several important algebras are special residuated lattices: Boolean algebras (algebraic counterpart of classical logic). Heyting algebras is a residuated lattice where $x * y = x \wedge y$. A BL-algebras [3] is a residuated lattice which is divisible and satisfies the prelinearity axiom. An MV-algebras [3, 17] is a residuated lattice in which $x \vee y = (x \to y) \to y$ holds. Equivalently [49], an MV-algebras is a residuated lattice which is divisible and satisfies the law of double negation. Thus, each BL-algebras satisfying the law of double negation is an MV-algebras(which is the way MV-algebras are defined in [46]). A Π-algebras(product algebras)[46] is a BL-algebras satisfying $(z \to 0) \to 0 \leq ((x * z) \to (y * z)) \to (x \to y)$ and $x \wedge (x \to 0) = 0$. A G-algebras (Gödel algebras) is a BL-algebras which satisfies $x * x = x$ (i.e. a Heyting algebras satisfying the prelinearity axiom). A Boolean algebras is a residuated lattice which is both a Heyting algebras and an MV-algebras(relation to the usual axiomatization is $x \to y = x' \vee y$).
More information about residuated and complete residuated lattices can be found in [5, 3, 8, 16, 24, 46, 50, 67, 61].

1.5. Fuzzy sets and fuzzy relations on residuated lattice

Analogously to the bivalent case, one can start developing a naive set theory with truth values in an (appropriately chosen) complete residuated lattice L (the classical bivalent case being a special case for $L = 2$). In this section we recall the basic notions of fuzzy logic. In the following, L will be a (complete) residuated lattice.

1.5.1. L-Fuzzy sets

An L-fuzzy subset(L-set, for short) in a universe set X is a mapping $A : X \to L$ assigning to every element $x \in X$ an element $A(x) \in L$ interpreted as the truth degree to which x belongs to A. Ordinary crisp subset of X are considered as fuzzy subset of X (or an L-subset) of X, taking membership values in the set $\{0, 1\} \subseteq L$. For later, L^X denotes the set of all L-subsets of X, i.e. the set of all mappings from X to L. Let A and B be two L-subsets of X.

The equality of A and B is defined as the usual equality of mappings, i.e. $A = B$ if and only if $A(x) = B(x)$, for every $x \in X$

The inclusion $A \leq B$ is also defined pointwise: $A \leq B$ if and only if $A(x) \leq B(x)$, for every $x \in X$.

Endowed with this partial order the set $\mathcal{L}(X)$ of all L-subsets of X forms a complete residuated lattice, in which the meet (intersection) $\bigwedge\limits_{i\in I} A_i$ and the join (union) $\bigvee\limits_{i\in I} A_i$ of an arbitrary family $\{A_i\}_{i\in I}$ of L-subsets of X, are mappings from X into L defined by

$$(\bigwedge_{i\in I} A_i)(x) = \bigwedge_{i\in I} A_i(x), \ (\bigvee_{i\in I} A_i)(x) = \bigvee_{i\in I} A_i(x)$$

The product $A \otimes B$ is an L-subset defined by $A \otimes B(x) = A(x) \otimes B(x)$, for every $x \in X$. The crisp part of an L-subset A of X is a crisp subset $\hat{A} = \{x \in X \mid A(x) = 1\}$ of X. We will also consider $\hat{A}$ as a mapping $\hat{A} : X \to L$ defined by $\hat{A}(x) = 1$, if $A(x) = 1$, and $\hat{A}(x) = 0$, if $A(x) \neq 1$.

For more information on L-set can be found in, for instance,[5, 44, 46, 76].

1.5.2. L-Fuzzy relations

Fuzzy relations were first introduced by Lotfi Zadeh [76] as a natural generalization of the usual crisp relations. Fuzzy relations play an important role in fuzzy modeling, fuzzy diagnosis, fuzzy control and relational databases. They also have applications in fields such as psychology, medicine, economics, and sociology. In many cases, fuzzy relations can handle real life problems better than the crisp ones. Some examples needed in this thesis are fuzzy tolerance, fuzzy equivalence, fuzzy equality, used to study the compatibility of a crisp relation with the fuzzy equivalence relations and the compatibility of fuzzy equivalence relations with a given order relation.

Basic definitions

A binary L-fuzzy relation (an L-relation, for short) on X is a mapping $R \in L^{X\times X}$, that is to say, any L-subsets of $X \times X$. For every $x, y \in X$, the value $R(x, y)$ is called the degree of membership of (x, y) in R, and the equality, inclusion, joins,

meets and ordering of fuzzy relations are defined as for fuzzy sets. The transpose R^t of R is the L-relation on X defined by $R^t(y,x) = R(x,y)$. For crisp relations, we use the usual infix notation, e.g. we write $a \leq b$ instead of $\leq (a,b)$.
For a t-norm T and L-relations R, S on X, the T-composition of R and S denoted by $R \circ S$, is a fuzzy relation on X defined by Note that if X is a finite set with n elements, then R and S can be treated as $n \times n$ fuzzy matrices over L and $R \circ S$ is the matrix product, whereas $A \circ R$ can be treated as the product of a $1 \times n$ matrix A and an $n \times n$ matrix R.
Consequently, an L-relation R is $*$-transitive if and only if $R \circ R \subseteq R$.
Moreover, for any L-relation R on a universe X, we will use the notation $R^{(i)} = R^{(i-1)} \circ R = R \circ R^{(i-1)},\ i \geq 2$ where $R^{(2)} = R \circ R$.

Main properties

Let $R \in L^{X \times X}$ be a binary L-relation on X. We are interested in the following properties (see, for example [4, 12, 13, 15, 25, 34, 39, 48, 77]):

- Reflexivity: $R(x,x) = 1$, for any $x \in X$,
- Irreflexivity: $R(x,x) = 0$, for any $x \in X$,
- Symmetry: $R(x,y) = R(y,x)$, for any $x, y \in X$,
- $*$-Asymmetric: $R(x,y) * R(y,x) = 0$, for any $x, y \in X$,
- $*$-Antisymmetric: $x \neq y$ implies $R(x,y) * R(y,x) = 0$, for any $x, y \in X$,
- $*$-Transitivity: $R(x,y) * R(y,z) \leq R(x,z)$, for any $x, y, z \in X$,
- Separability: $R(x,y) = 1$ implies that $x = y$, for any $x, y \in X$.

Note that R is called Strongly complete, if $\max(R(x,y), R(y,x)) = 1$, for any $x, y \in$.

Proposition 1.9. *[27] For any reflexive and $*$-transitive L-relation R on a universe X, it holds that $R^{(i)} = R$, for any $i \geq 2$.*

1.5.3. L-Fuzzy equivalence relation

Fuzzy equivalence relations were first introduced by Zadeh [77] as a generalization of the usual crisp equivalence relations. They were found to be extremely useful in such fields as Fuzzy Control, Approximate reasoning, Fuzzy Cluster Analysis etc..

Definition 1.7. *[2]*

(i) A binary L-relations E that are reflexive and symmetric are called L-fuzzy tolerances (L-tolerances relation,for short).

(ii) *A binary L-relations E that are L-tolerances and $*$-transitive are called L-equivalences or L-similarities.*

(iii) *A separable L-equivalences relations are called L-equalities.*

Note that 2-equality on X is precisely the usual equality (identity) I_X (i.e. $I_X(x, y) = 1$ for $x = y$ and $I_X(x, y) = 0$ for $x \neq y$). Therefore, the notion of L-equality is a natural generalization of the classical (bivalent) notion.

For an L-set A in X and an L-equality E on X we define the L-set $D_E(A)$ by $D_E(A)(x) = \bigvee_{x'} A(x') * E(x', x)$. It is easy to see that $D_E(A)$ is the smallest (w.r.t.$\subseteq$) L-set in X that is compatible with E and contains A.

Example 1.3. *The equality degree is an L-equality on L^X, for any X.*

Definition 1.8. *(Compatibility in sense of Bělohlávek)[2] A binary L-relation R between X and Y is compatibility w.r.t. L-equalities E_X on X and E_Y on Y if*

$$R(x_1, y_1) * (E_X(x_1, x_2) * E_Y(y_1, y_2) \leq R(x_1, y_2)$$

, for any $x_i \in X, y_i \in Y$ $(i = 1, 2)$.

By $L^{(X,E_X)} \times (y, E_Y)$ we denote the set of all L-relations between X and Y compatible w.r.t E_X and E_Y.

Definition 1.9. *An L-order relation (L-order, for short) on a set X with L-equality relation E is a binary L-relation R which is compatible w.r.t E and satisfies the following three axioms*

(i) $R(x, x) = 1$ *(reflexivity),*

(ii) $R(x, y) \wedge R(y, x) \leq E(x, y)$ *(antisymmetry),*

(iii) $R(x, y) * R(y, z) \leq R(x, z)$ *(transitivity).*

If R is an L-order on a set X with an L-equality E, we call the pair $X = ((X, E), R)$ an L-ordered set.

Remark 1.2. *(1) Clearly, if $L = 2$, the notion of L-order coincides with the usual notion of (partial) order.*

(2) For a similar approach to fuzzy order (however, with a different formulation of antisymmetry) see [10].

The concept of L-equivalence relation above mentioned has been introduced, named and studied in several different ways, it will be useful to give some other concepts related to compatibility that is one among our aims in this work.

Let $(L, *, \rightarrow)$ be a (complete) residuated lattice and E, F be L-equivalence relations on X and Y respectively. A fuzzy relation R on $L^{X\times Y}$ is called a perfect fuzzy function from X to Y w.r.t. E and F if and only if R satisfies the following four conditions:

(i) $R(x,y)*E(x,x') \leq R(x',y)$, for any $x,x' \in X$ and any $y \in Y$ (Extensionality w.r.t. E),

(ii) $R(x,y)*E(y,y') \leq R(x,y')$, for any $x \in X$ and any $y,y' \in Y$ (Extensionality w.r.t. F),

(iii) For each $x \in X$,$\exists y \in Y$ such that $R(x,y) = 1$,

(iv) $R(x,y)*R(x,y') \leq F(y,y')$, for any $x \in X$ and any $y,y' \in Y$.

A fuzzy relation R on $L^{(X\times X)\times X}$ satisfying the condition (iii) is said to be L-binary operation on X ([34]).

Theorem 1.3. *[9] Consider a reflexive and $*$-transitive binary fuzzy relation $R : X^2 \to [0,1]$ (often called fuzzy preordering). The relation R is a $*$-E-ordering for some $*$-equivalence E if and only if, for all $x,y \in X$,*

$$R(x,y)*R(y,x) \leq E(x,y) \leq \min(R(x,y),R(y,x))$$

Definition 1.10. *(Compatibility in sense Bodenhofer [9])Let $\preceq$ be a crisp ordering on X and let E be a fuzzy equivalence relation on X. E is called compatible with $\preceq$, if and only if the following implication holds for all $x,y,z \in X$:*

$$x \preceq y \preceq z \Rightarrow E(x,z) \leq \min(E(x,y),E(y,z))$$

.

2 The clone relation of a binary relation

In this chapter, we extend the notion of clone relation of a strict order relation introduced by De Baets et al.[32]to any binary relation. Although the definition of such extension is trivial, the corresponding properties significantly differ from those of the clone relation of a strict order relation. We analyse the most important ones among these properties, paying particular attention to a partition of the clone relation in terms of three different types of pairs of clones. Also in this chapter, we characterize the clone relation of the three different types of union of two relations defined on disjoint sets (the nondirectional disjoint union, the unidirectional disjoint union and the bidirectional disjoint). We have concluded this chapter by introducing the clone relation of order n.
The clone relation coincides with the covering relation, i.e., two elements are clones if and only if they are consecutive. This notion of consecutive elements in a totally ordered set was already independently considered in the field of social choice theory by Tideman under the same name: clones. Clones are important in the field of social choice theory. Several methods have been proposed in order to guarantee the independence of clones (see [68, 71, 73]).

2.1. The clone relation of a strict order relation

In this subsection, we recall the notion of clone relation of a strict order relation introduced by De Baets et al. [32]. The clone relation $\approx$ of a strict order relation $<$ is the binary relation on X defined by

$$x \approx y \quad \text{if} \quad \begin{cases} (\forall z \in X \setminus \{x,y\})(z < x \Leftrightarrow z < y) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(x < z \Leftrightarrow y < z)\,. \end{cases}$$

Note that the clone relation $\approx$ of a strict order relation $<$ is a tolerance relation on X. This clone relation can be partitioned[1] as follows:

$$\approx = \triangleleft \cup \triangleright \cup \diamond \cup \delta\,,$$

---

[1] Although the term ‘partition’ is used, any of the binary relations $\triangleleft$, $\triangleright$ and $\diamond$ might be empty.

where $\delta = \{(x,y) \in X^2 \mid x = y\}$ and the binary relations $\lhd$, $\rhd$ and $\diamond$ are pairwise disjoint relations given by:

$$\begin{aligned} \lhd &= \approx \cap \ll\,, \\ \rhd &= \approx \cap \gg\,, \\ \diamond &= \approx \cap \parallel\,, \end{aligned}$$

where $\ll = \{(a,b) \in X^2 \mid (a < b) \wedge (\nexists c \in X)(a < c < b)\}$ and $\gg = \ll^t$.

Note that, on the one hand, $\lhd$ and $\rhd$ are irreflexive, antisymmetric and antitransitive and it holds that $\lhd = \rhd^t$. On the other hand, $\diamond$ is irreflexive, symmetric and transitive. Hence, the clone relation of a poset can be partitioned in terms of two types of pairs of clones: pairs of comparable clones ($\lhd \cup \rhd$) and pairs of incomparable clones ($\diamond$).

2.2. The clone relation of a binary relation

In this section, we extend the notion of clone relation to an arbitrary binary relation. The study of the basic properties of this clone relation and its relation with set operations is also addressed.

2.2.1. Definition

The analysis of 'likeness' is a relevant matter of study in mathematics. Equivalence relations, which form a basic concept in mathematics, define a natural notion of 'likeness' grouping elements in equivalence classes. When we drop transitivity and allow an element to be 'alike' to two elements that are not 'alike' to each other, one does no longer talk about equivalence relations but about tolerance relations. Another natural way of defining such 'likeness' is based on how elements are related w.r.t. the other elements. In that way, two elements are said to be 'alike' (from now on clones) if they are related in the same way w.r.t. every other element.

Definition 2.1. *Let R be a relation on a set X. The clone relation $\approx_R$ of R is the binary relation on X defined by*

$$x \approx_R y \quad \text{if} \quad \begin{cases} (\forall z \in X \setminus \{x,y\})(zRx \Leftrightarrow zRy) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(xRz \Leftrightarrow yRz)\,. \end{cases} \tag{2.1}$$

If $x \approx_R y$, then we say that x and y are clones w.r.t. the relation R.

Remark 2.1. *Let R be a relation on a set X. Then the following statements hold:*

(i) For any $x, y \in X$, if $x \approx_R y$, then it holds that

$$(\forall z \in X \setminus \{x, y\})(z \parallel x \Leftrightarrow z \parallel y)\,.$$

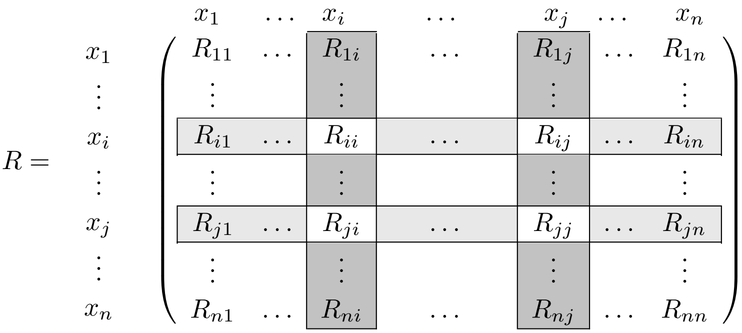
(ii) For any set X of two elements, it holds that $\approx_R = X^2$.

(iii) For any set X, it holds that $\approx_{X^2} = \approx_\emptyset = X^2$.

The matrix representation of a binary relation R can be used for illustrating the notion of clone relation and for facilitating the identification of clones in the finite case. Let R be a relation on a finite set $X = \{x_1, x_2, ..., x_n\}$ $(n \in \mathbb{N}^* = \{1, 2, 3, ...\})$. For any $x_i, x_j \in X$ with $1 \leqslant i, j \leqslant n$, it holds that[2]

$$R_{ij} = \begin{cases} 1 & \text{, if } x_i R x_j\,, \\ 0 & \text{, if } x_i R^c x_j\,. \end{cases}$$

By definition, it holds that $x_i \approx_R x_j$ if, and only if, for any $k \notin \{i, j\}$, it holds that $R_{ik} = R_{jk}$ and $R_{ki} = R_{kj}$. This means that x_i and x_j are clones if and only if the row and column corresponding to x_i coincide with the row and column corresponding to x_j, with the exception of the four elements contained in the intersection of these two rows with these two columns. This is illustrated in Figure 2.1.



$$R = \begin{array}{c} \\ x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{cccccccc} x_1 & \dots & x_i & \dots & x_j & \dots & x_n \end{array} \\ \begin{pmatrix} R_{11} & \dots & R_{1i} & \dots & R_{1j} & \dots & R_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ R_{i1} & \dots & R_{ii} & \dots & R_{ij} & \dots & R_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ R_{j1} & \dots & R_{ji} & \dots & R_{jj} & \dots & R_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ R_{n1} & \dots & R_{ni} & \dots & R_{nj} & \dots & R_{nn} \end{pmatrix} \end{array}$$

Figure 2.1: Natural interpretation of the clone relation by means of the matrix representation of R.

Example 2.1. *Let R be the relation on $X = \{a, b, c, d, e, f\}$ defined by the graph in Figure 2.2.*

---

[2] In this work, a relation R is identified with its characteristic mapping χ_R, i.e., $\chi_R(x, y) = 1$ means xRy and $\chi_R(x, y) = 0$ means $xR^c y$. In a finite setting, a relation can be conveniently represented as a matrix such that $R_{ij} = \chi_R(x_i, x_j)$.

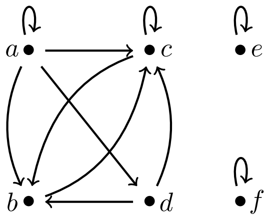


Figure 2.2: Graph of a relation R on the set $X = \{a, b, c, d, e, f\}$.

The matrix representation of the relation R is given by:

$$R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} .$$

Since the row and column corresponding to b coincide with the row and column corresponding to c (without taking the four elements in the intersection of rows and columns into account), it holds that $b \approx_R c$. In general, the clone relation of R is given by:

$$\approx_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} .$$

For any relation R, the clone relation of R obviously is reflexive and symmetric. Therefore, the following result is straightforward.

Proposition 2.1. *Let R be a relation on a set X. The clone relation $\approx_R$ of R is a tolerance relation.*

In general, the clone relation $\approx_R$ does not need to be an equivalence relation, as can be seen in Example 2.2.

Example 2.2. *Let $X = \{1, 2, 3\}$ and $<$ be the usual strict order relation. We can see that $\approx_<$ is not an equivalence relation. For instance, it holds that $1 \approx_< 2$ and $2 \approx_< 3$, while $1 \not\approx_< 3$.*

2.2.2. Basic properties

In this subsection, we discuss the most relevant properties of the clone relation. First, it is trivial to prove that the clone relation of a relation R always coincides with the clone relation of the complement, the transpose and the dual of R.

Proposition 2.2. *Let R be a relation on a set X. Then the following statements hold:*

(i) $\approx_{R^c} = \approx_R$.

(ii) $\approx_{R^t} = \approx_R$.

(iii) $\approx_{R^d} = \approx_R$.

Proof. (i) For any $x, y \in X$, it holds that

$$\begin{aligned} x \approx_{R^c} y &\Leftrightarrow \begin{cases} (\forall z \in X \setminus \{x,y\})(zR^cx \Leftrightarrow zR^cy) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(xR^cz \Leftrightarrow yR^cz) \end{cases} \\ &\Leftrightarrow \begin{cases} (\forall z \in X \setminus \{x,y\})(zRx \Leftrightarrow zRy) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(xRz \Leftrightarrow yRz) \end{cases} \\ &\Leftrightarrow x \approx_R y. \end{aligned}$$

(ii) For any $x, y \in X$, it holds that

$$\begin{aligned} x \approx_{R^t} y &\Leftrightarrow \begin{cases} (\forall z \in X \setminus \{x,y\})(zR^tx \Leftrightarrow zR^ty) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(xR^tz \Leftrightarrow yR^tz) \end{cases} \\ &\Leftrightarrow \begin{cases} (\forall z \in X \setminus \{x,y\})(xRz \Leftrightarrow yRz) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(zRx \Leftrightarrow zRy) \end{cases} \\ &\Leftrightarrow x \approx_R y. \end{aligned}$$

(iii) It is straightforward due to the two preceding statements.

□

Second, it can be proved easily that the reflexivity of R has no impact on the clone relation.

Proposition 2.3. *Let R and S be two relations on a set X. If for any $x, y \in X$ such that $x \neq y$ it holds that $xRy \Leftrightarrow xSy$, then the clone relation of R and the clone relation of S coincide, i.e., $\approx_R = \approx_S$.*

Note that, as a consequence of Proposition 2.3, we conclude that the clone relation does not take reflexivity or irreflexivity into account. Actually, the relation of an element with itself does not affect the clone relation.

Corollary 2.1. *Let R, R' and R'' be three relations on a set X. If $R' = R \cup \{(x,x) \in X^2\}$ and $R'' = R \setminus \{(x,x) \in X^2\}$, then it holds that $\approx_R = \approx_{R'} = \approx_{R''}$.*

This result is illustrated in the following example.

Example 2.3. *In Figure 2.3, the graphs of three relations R, R' and R'' on the set $X = \{a,b,c\}$ such that R is neither reflexive nor irreflexive, R' is reflexive and R'' is irreflexive are shown. Note that R, R' and R'' coincide for any two different elements. Hence, it holds that*

$$\approx_R = \approx_{R'} = \approx_{R''} = \{(a,a),(b,b),(c,c),(a,b),(b,a)\}\,.$$

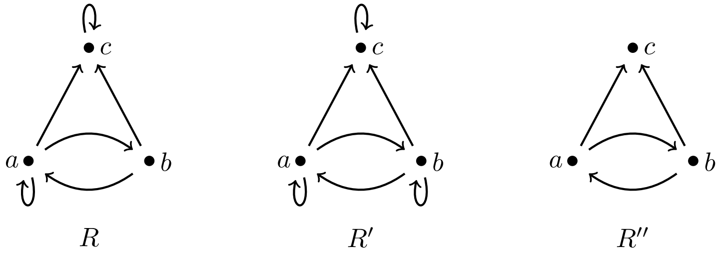


Figure 2.3: Graphs of three relations R, R' and R'' on the set $X = \{a,b,c\}$.

Remark 2.2. *If $(X, \leqslant)$ is a poset and $<$ is the strict order relation associated to the order relation $\leqslant$, then, from Corollary 2.1, it follows that $\approx_{\leqslant} = \approx_{<}$. Note that De Baets et al. [32] defined the clone relation of a poset $(X, \leqslant)$ by means of the strict order relation $<$ but, in fact, if they had defined this clone relation by means of the order relation $\leqslant$, then the result would have been the same.*

In the following proposition, we study when the clone relation $\approx_R$ is transitive, i.e., when it is an equivalence relation.

Proposition 2.4. *Let R be a relation on a set X. If there do not exist $x, y \in X$ such that $x \approx_R y$, xRy and $yR^c x$, then it holds that $\approx_R$ is an equivalence relation.*

Proof. Since $\approx_R$ is a tolerance relation (see Proposition 2.1), it suffices to prove that $\approx_R$ is transitive. Let $x, y, z \in X$ be such that $x \approx_R y$ and $y \approx_R z$. Suppose that $x \not\approx_R z$. It follows that there exists $t \in X \setminus \{x, z\}$ such that (tRx and $tR^c z$) or (xRt and $zR^c t$) or (tRz and $tR^c x$) or (zRt and $xR^c t$).

(i) Let us consider the case where tRx and $tR^c z$. We distinguish two cases: $t \neq y$ and $t = y$.

   (a) If $t \neq y$, then from $x \approx_R y$ and $y \approx_R z$, it follows that tRy and $tR^c y$, a contradiction.

   (b) If $t = y$, then it follows that yRx and $yR^c z$. Since $x \approx_R y$, $y \approx_R z$ and $x \not\approx_R z$, it follows that $x \neq y \neq z \neq x$. Moreover, as $y \approx_R z$, it follows that zRx and $yR^c z$ and, as $x \approx_R y$, this implies that zRy and $yR^c z$. At the same time it holds that $y \approx_R z$, a contradiction with the hypothesis. Therefore, $\approx_R$ is transitive.

(ii) The other cases where (xRt and $zR^c t$) or (tRz and $tR^c x$) or (zRt and $xR^c t$) are analogously proved.

□

In particular, the conditions of Proposition 2.4 are satisfied for any symmetric relation.

Corollary 2.2. *Let R be a relation on a set X. If R is symmetric, then it holds that $\approx_R$ is an equivalence relation.*

Corollary 2.3. *Let R be a relation on a set X. If $R = \approx_R$, then it holds that R is an equivalence relation.*

An equivalence relation is always included in its clone relation, as is expressed in the following proposition.

Proposition 2.5. *Let R be a relation on a set X. If R is an equivalence relation, then it holds that $R \subseteq \approx_R$.*

Proof. Let R be an equivalence relation and $x, y \in X$ be such that xRy. Let us suppose that $x \not\approx_R y$. Since R is an equivalence relation and $x \not\approx_R y$, it follows that there exists $z \in X \setminus \{x, y\}$ such that (zRx and $zR^c y$) or (zRy and $zR^c x$). Due to the symmetry and transitivity of R, it follows that (zRy and $zR^c y$) or (zRx and $zR^c x$), which leads to a contradiction. Hence, it holds that $x \approx_R y$ and, therefore, $R \subseteq \approx_R$. □

The necessary and sufficient conditions that an equivalence relation needs to satisfy in order to coincide with its clone relation are provided in the following proposition. In words, an equivalence relation coincides with its clone relation if and only if there is at most one singleton equivalence class.

Proposition 2.6. *Let R be a relation on a set X. If R is an equivalence relation, then it holds that $R = \approx_R$ if and only if there do not exist $x, y \in X$ such that $x \neq y$, $[x]_R = \{x\}$ and $[y]_R = \{y\}$.*

Proof. ($\Rightarrow$) Let R be an equivalence relation on X such that $R = \approx_R$ and suppose that there exist $x, y \in X$ such that $x \neq y$, $[x]_R = \{x\}$ and $[y]_R = \{y\}$. It follows that xR^cz and that yR^cz for any $z \in X \setminus \{x, y\}$, therefore it holds that $x \approx_R y$, a contradiction with $R = \approx_R$ and xR^cy. Hence, there do not exist $x, y \in X$ such that $x \neq y$, $[x]_R = \{x\}$ and $[y]_R = y$.

($\Leftarrow$) From Proposition 2.5, it follows that $R \subseteq \approx_R$. It remains to prove that $\approx_R \subseteq R$. Let R be an equivalence relation on X such that there do not exist $x, y \in X$ such that $x \neq y$, $[x]_R = \{x\}$ and $[y]_R = \{y\}$. Let us suppose that $\approx_R \not\subseteq R$. As R is reflexive, it holds that there exist $x, y \in X$ such that $x \neq y$, $x \approx_R y$ and xR^cy. Since ($[x]_R \neq \{x\}$ or $[y]_R \neq \{y\}$) and $x \neq y$, it follows that there exists $z \in X \setminus \{x, y\}$ such that xRz or yRz. As $x \approx_R y$, it implies that (xRz and yRz) or (yRz and xRz). Since R is an equivalence relation, it follows that xRy, a contradiction. Hence, $\approx_R \subseteq R$ and, therefore, $\approx_R = R$. ☐

In general, the fact that R is an equivalence relation does not necessarily lead to $\approx_R \subseteq R$, as can be seen from Example 2.4.

Example 2.4. *The relation R defined in Figure 2.4 is an equivalence relation on the set $X = \{a, b, c, d, e\}$.*

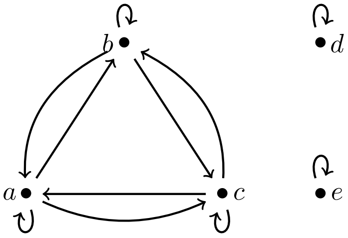


Figure 2.4: Graph of an equivalence relation R on the set $X = \{a, b, c, d, e\}$.

The matrix representations of R and $\approx_R$ are given by:

$$R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \qquad \approx_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

We can see that $d \approx_R e$ and dR^ce. Hence, it holds that $\approx_R \not\subseteq R$. Note that it *holds that* $\approx_R \not\subseteq R$*, due to the fact that there are two equivalence classes formed by singletons. Note that, as expected due to Proposition 2.5, it holds that* $R \subseteq \approx_R$*.*

The composition of any symmetric relation with its clone relation is always included in that relation. In addition, we will prove that the clone relation of any symmetric relation is the greatest symmetric relation that satisfies this inclusion.

Proposition 2.7. *Let* R *be a relation on a set* X*. If* R *is symmetric, then the following two statements hold:*

- (i) $\approx_R$ *is the greatest symmetric relation* S *such that* $R \circ S \subseteq R$*.*
- (ii) $\approx_R$ *is the greatest symmetric relation* S *such that* $S \circ R \subseteq R$*.*

Proof. Let R be a symmetric relation on X.

- (i) Let us suppose that there exists a symmetric relation S on X such that $R \circ S \subseteq R$ and $S \not\subseteq \approx_R$. It follows that there exist $x, y \in X$ such that xSy and $x \not\approx_R y$. As $x \not\approx_R y$ and R is symmetric, it follows that there exists $z \in X \setminus \{x, y\}$ such that (zRx and $zR^c y$) or (zRy and $zR^c x$). Let us consider, w.l.o.g, that zRx and $zR^c y$. Since zRx and xSy, it follows that $z(R \circ S)y$. As $R \circ S \subseteq R$, it follows that zRy, a contradiction. Hence, we conclude that $S \subseteq \approx_R$.
- (ii) As R and S are symmetric, it holds that $R \circ S = S \circ R$. Therefore, the result follows from statement (i).

□

2.2.3. Interaction of the clone relation with set operations

This subsection is devoted to discuss the interaction of the clone relation with the most common set operations.

Proposition 2.8. *Let* R *and* S *be two relations on a set* X*. If* $R \subseteq S$*, then the following statements hold:*

- *(i)* $\approx_R \cap \approx_{S \setminus R} \subseteq \approx_S$.
- *(ii)* $\approx_S \subseteq (\approx_R \cap \approx_{S \setminus R}) \cup ((\approx_R)^c \cap (\approx_{S \setminus R})^c)$.

Proof. (i) Suppose that $R \subseteq S$ and let $x, y \in X$ be such that $x(\approx_R \cap \approx_{S \setminus R})y$. It follows that $x \approx_R y$ and $x \approx_{S \setminus R} y$. Therefore, for any $z \in X \setminus \{x, y\}$, it holds that

$$\begin{aligned} xSz &\Leftrightarrow (xRz \vee x(S \setminus R)z) \\ &\Leftrightarrow (yRz \vee y(S \setminus R)z) \\ &\Leftrightarrow ySz\,. \end{aligned}$$

In a similar way, we prove that $zSx \Leftrightarrow zSy$. Hence, it holds that $x \approx_S y$. Therefore, it holds that $\approx_R \cap \approx_{S\setminus R} \subseteq \approx_S$.

(ii) Suppose that $R \subseteq S$ and let $x, y \in X$ be such that $x \approx_S y$. Since it trivially holds that $X^2 = (\approx_R \cup (\approx_R)^c) \cap (\approx_{S\setminus R} \cup (\approx_{S\setminus R})^c)$, it follows that one of the following statements holds: $x(\approx_R \cap \approx_{S\setminus R})y$ or $x(\approx_R \cap (\approx_{S\setminus R})^c)y$ or $x((\approx_R)^c \cap \approx_{S\setminus R})y$ or $x((\approx_R)^c \cap (\approx_{S\setminus R})^c)y$. We will prove that $x((\approx_R)^c \cap \approx_{S\setminus R})^c y$ and $x(\approx_R \cap (\approx_{S\setminus R})^c)^c y$.

  (a) Suppose that ($x(\approx_R)^c y$ and $x \approx_{S\setminus R} y$). Since $x(\approx_R)^c y$, it follows that there exists $z \in X \setminus \{x, y\}$ such that one of the following statements holds: (xRz and $yR^c z$) or (yRz and $xR^c z$) or (zRx and $zR^c y$) or (zRy and $zR^c x$). Any of these cases contradicts the fact that ($x \approx_S y$ and $x \approx_{S\setminus R} y$). For instance, if ($xRz$ and $yR^c z$), then, since $R \subseteq S$, it follows that xSz. Since $x \approx_S y$ and $z \in X \setminus \{x, y\}$, it follows that ySz. On the one hand, since ySz and $yR^c z$, it follows that $y(S \setminus R)z$. On the other hand, since xRz, it follows that $x(S \setminus R)^c z$. a contradiction with the fact that $x \approx_{S\setminus R} y$. The other cases where (yRz and $xR^c z$) or (zRx and $zR^c y$) or (zRy and $zR^c x$) are analogously proved.

  (b) Suppose that ($x \approx_R y$ and $x(\approx_{S\setminus R} y)^c$). Since $x(\approx_{S\setminus R})^c y$, it follows that there exists $z \in X \setminus \{x, y\}$ such that one of the following statements holds: ($x(S \setminus R)z$ and $y(S \setminus R)^c z$) or ($y(S \setminus R)z$ and $x(S \setminus R)^c z$) or ($z(S \setminus R)x$ and $z(S \setminus R)^c y$) or ($z(S \setminus R)y$ and $z(S \setminus R)^c x$). Any of these cases contradicts the fact that ($x \approx_S y$ and $x \approx_R y$). For instance, if ($x(S \setminus R)z$ and $y(S \setminus R)^c z$), then it follows that (xSz and $xR^c z$) and ($yS^c z$ or yRz). Therefore, it holds that (xSz and $yS^c z$) or ($xR^c z$ and yRz), a contradiction with the fact that ($x \approx_S y$ and $x \approx_R y$). The other cases where ($y(S \setminus R)z$ and $x(S \setminus R)^c z$) or ($z(S \setminus R)x$ and $z(S \setminus R)^c y$) or ($z(S \setminus R)y$ and $z(S \setminus R)^c x$) are analogously proved.

Hence, it holds that ($x \approx_R y$ and $x \approx_{S\setminus R} y$) or ($x(\approx_R)^c y$ and $x(\approx_{S\setminus R})^c y$). Therefore, it holds that $\approx_S \subseteq (\approx_R \cap \approx_{S\setminus R}) \cup ((\approx_R)^c \cap (\approx_{S\setminus R})^c)$.

□

The following corollary follows immediately from statement (ii) of Proposition 2.8.

Corollary 2.4. *Let R and S be two relations on a set X. Then it holds that*

$$\approx_{R\cup S}\subseteq(\approx_R\cap\approx_{S\setminus R})\cup((\approx_R)^c\cap(\approx_{S\setminus R})^c)\,.$$

Note that, in general, if R and S are two binary relations on a set X such that $S\subseteq R$ and $\approx_R$ and $\approx_S$ are their respective clone relations, then it does not necessarily hold that $\approx_S\subseteq\approx_R$, as can be seen in Example 2.5.

Example 2.5. *Let us consider two binary relations R and S on the set $X=\{a,b,c,d\}$ defined by $R=\{(a,a),(b,b),(a,b),(a,c)\}$ and $S=\{(a,a),(b,b)\}$. It holds that $S\subseteq R$, $a\approx_S b$, while $a(\approx_R)^c b$. Hence, $\approx_S\not\subseteq\approx_R$.*

The following corollary follows immediately from Proposition 2.2.

Corollary 2.5. *Let $(R_i)_{i\in I}$ be a finite family of relations on a set X. The following statements hold:*

(i) $\approx_{\bigcup\limits_{i\in I}R_i}=\approx_{\bigcap\limits_{i\in I}R_i^c}$.

(ii) $\approx_{\bigcap\limits_{i\in I}R_i}=\approx_{\bigcup\limits_{i\in I}R_i^c}$.

In the following, we discuss the interaction of the clone relation with the intersection and the union.

Proposition 2.9. *Let R and S be two relations on a set X. The following statements hold:*

(i) $\approx_R\cap\approx_S=\approx_{R\cap S}\cap\approx_{R\setminus S}\cap\approx_{S\setminus R}$.

(ii) $\approx_R\cap\approx_S=\approx_{R\cup S}\cap\approx_{R\setminus S}\cap\approx_{S\setminus R}$.

Proof. (i) We need to prove that $\approx_R\cap\approx_S\subseteq\approx_{R\cap S}\cap\approx_{R\setminus S}\cap\approx_{S\setminus R}$ and that $\approx_{R\cap S}\cap\approx_{R\setminus S}\cap\approx_{S\setminus R}\subseteq\approx_R\cap\approx_S$.

(a) First, we prove that $\approx_R\cap\approx_S\subseteq\approx_{R\cap S}\cap\approx_{R\setminus S}\cap\approx_{S\setminus R}$. Let $x,y\in X$ be such that $x(\approx_R\cap\approx_S)y$. It follows that $x\approx_R y$ and $x\approx_S y$. Therefore, for any $z\in X\setminus\{x,y\}$, it holds that

$$\begin{aligned}x(R\cap S)z&\Leftrightarrow xRz\wedge xSz\\&\Leftrightarrow yRz\wedge ySz\\&\Leftrightarrow y(R\cap S)z\,.\end{aligned}$$

In a similar way, we prove that $z(R\cap S)x\Leftrightarrow z(R\cap S)y$. Hence, it holds that $x(\approx_{R\cap S})y$ and, thus, that $\approx_R\cap\approx_S\subseteq\approx_{R\cap S}$. Moreover, for any

$z \in X \setminus \{x, y\}$, it holds that

$$\begin{aligned} x(R \setminus S)z &\Leftrightarrow xRz \wedge xS^c z \\ &\Leftrightarrow yRz \wedge yS^c z \\ &\Leftrightarrow y(R \setminus S)z\,. \end{aligned}$$

In a similar way, we prove that $z(R \setminus S)x \Leftrightarrow z(R \setminus S)y$. Hence, it holds that $x(\approx_{R\setminus S})y$ and, thus, that $\approx_R \cap \approx_S \subseteq \approx_{R\setminus S}$. The fact that $\approx_R \cap \approx_S \subseteq \approx_{S\setminus R}$ is proved in an analogous way.

(b) Second, we prove that $\approx_{R\cap S} \cap \approx_{R\setminus S} \cap \approx_{S\setminus R} \subseteq \approx_R \cap \approx_S$. We have that

$$\begin{aligned} \approx_{R\cap S} \cap \approx_{R\setminus S} \cap \approx_{S\setminus R} &= \approx_{R\cap S} \cap \approx_{R\setminus(R\cap S)} \cap \approx_{S\setminus(R\cap S)} \\ &= (\approx_{R\cap S} \cap \approx_{R\setminus(R\cap S)}) \cap (\approx_{R\cap S} \cap \approx_{S\setminus(R\cap S)}). \end{aligned}$$

From Proposition 2.8, it follows that $(\approx_{R\cap S} \cap \approx_{R\setminus(R\cap S)}) \subseteq \approx_R$ and $(\approx_{R\cap S} \cap \approx_{S\setminus(R\cap S)}) \subseteq \approx_S$.

(ii) From (i), it follows that $\approx_{R^c} \cap \approx_{S^c} = \approx_{R^c\cap S^c} \cap \approx_{R^c\setminus S^c} \cap \approx_{S^c\setminus R^c}$. Since $\approx_{R^c} = \approx_R, \approx_{S^c} = \approx_S, \approx_{R^c\cap S^c} = \approx_{(R\cup S)^c} = \approx_{R\cup S}$, $R^c \setminus S^c = S \setminus R$ and $S^c \setminus R^c = R \setminus S$, it follows that $\approx_R \cap \approx_S = \approx_{R\cup S} \cap \approx_{S\setminus R} \cap \approx_{R\setminus S}$.

□

The following corollary is a direct result of Proposition 2.9.

Corollary 2.6. *Let R and S be two relations on a set X. The following statements hold:*

(i) $\approx_R \cap \approx_S \subseteq \approx_{R\cap S}$.

(ii) $\approx_R \cap \approx_S \subseteq \approx_{R\cup S}$.

Note that the converse inclusions do not necessarily hold, as can be seen in Example 2.6.

Example 2.6. *Let R, S, $R \cup S$ and $R \cap S$ be the four relations defined on the set $X = \{a, b, c, d\}$ by the graphs in Figure 2.5*

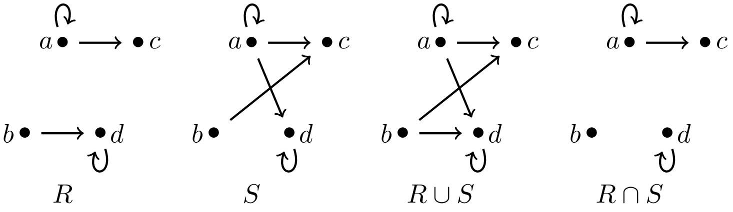


Figure 2.5: Graphs of four relations on the set $X = \{a, b, c, d\}$.

We can see that:

(a) $a \approx_{R\cap S} c$, *while* $a(\approx_S)^c c$. *Hence,* $\approx_{R\cap S} \not\subseteq \approx_R \cap \approx_S$.

(b) $c \approx_{R\cup S} d$, *while* $c(\approx_S)^c d$. *Hence,* $\approx_{R\cup S} \not\subseteq \approx_R \cap \approx_S$.

(c) $a \approx_{R\cup S} b$, *while* $a(\approx_R)^c b$ *and* $a(\approx_S)^c b$. *Hence,* $\approx_{R\cup S} \not\subseteq \approx_R \cup \approx_S$.

(d) $a \approx_R c$, *while* $a(\approx_{R\cup S})^c c$. *Hence,* $\approx_R \cup \approx_S \not\subseteq \approx_{R\cup S}$.

2.3. A partition of the clone relation

De Baets et al. [32] provided a partition of the clone relation for the special case of an order relation. Here, we extend this partition[3] to the case of an arbitrary binary relation.

Definition 2.2. *Let R be a relation on a set X. The following binary relations on X are defined:*

(i) $\lhd_R = \{(x,y) \in X^2 \mid x \approx_R y \wedge xRy \wedge yR^c x \wedge x \neq y\}$.

(ii) $\rhd_R = \{(x,y) \in X^2 \mid x \approx_R y \wedge yRx \wedge xR^c y \wedge x \neq y\}$.

(iii) $\circ_R = \{(x,y) \in X^2 \mid x \approx_R y \wedge xRy \wedge yRx \wedge x \neq y\}$.

(iv) $\diamond_R = \{(x,y) \in X^2 \mid x \approx_R y \wedge xR^c y \wedge yR^c x \wedge x \neq y\}$.

Remark 2.3. *Note that* $\lhd_R^t = \rhd_R$, $\circ_R^t = \circ_R$ *and* $\diamond_R^t = \diamond_R$.

Given Definition 2.2, it is immediately clear that the clone relation $\approx_R$ of any relation R can be written as follows:

$$\approx_R = \lhd_R \cup \rhd_R \cup \circ_R \cup \diamond_R \cup \delta\,,$$

where $\delta = \{(x,y) \in X^2 \mid x = y\}$.

Definition 2.3. *Let R be a relation on a set X. The triplet $(\lhd_R, \circ_R, \diamond_R)$ is called the (canonical) partition of the clone relation $\approx_R$.*

Note that in the canonical partition we do not explicitly mention $\rhd_R$ (as it equals $\lhd_R^t$) and δ (as it does not depend on the relation R).

Remark 2.4. *As discussed by Roubens and Vincke [65], any reflexive binary relation Q on a set X allows to partition X^2 into four disjoint parts: a strict preference relation $P_Q = Q \cap (Q^t)^c$ (which is irreflexive and asymmetric) and its transpose P_Q^t, an indifference relation $I_Q = Q \cap Q^t$ (which is reflexive and symmetric) and an incomparability relation $J_Q = Q^c \cap (Q^t)^c$ (which is irreflexive*

---

[3] Although the term ‘partition’ is used, any of the binary relations $\lhd_R$, $\rhd_R$, $\circ_R$ and $\diamond_R$ might be empty.

and symmetric)

$$X^2 = P_Q \cup P_Q^t \cup I_Q \cup J_Q \,.$$

We can see that the partition of the clone relation is closely related with this result. Indeed, extending the above definition to an arbitrary binary relation R, we can write

$$X^2 = P_R \cup P_R^t \cup (I_R \backslash \delta) \cup J_R \cup \delta \,,$$

and hence

$$\begin{aligned} \approx_R &= \approx_R \cap X^2 \\ &= (\approx_R \cap P_R) \cup (\approx_R \cap P_R^t) \cup (\approx_R \cap (I_R \backslash \delta)) \cup (\approx_R \cap J_R) \cup (\approx_R \cap \delta) \\ &= \lhd_R \cup \rhd_R \cup \circ_R \cup \diamond_R \cup \delta \,. \end{aligned}$$

Example 2.7. *Let R be the relation defined in Example 2.1. The matrix representations of the relations $\lhd_R$, $\rhd_R$, $\circ_R$ and $\diamond_R$ are given by:*

$$\lhd_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \rhd_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\circ_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \diamond_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \,.$$

Note that $\approx_R$ can be written as:

$$\approx_R = \lhd_R + \rhd_R + \circ_R + \diamond_R + \delta \,.$$

From the definition of the partition of the clone relation and from Proposition 2.2 and Corollary 2.1, the following results are straightforward

Corollary 2.7. *Let R be a relation on a set X. Then the following statements hold:*

(i) $(\lhd_{R^c}, \circ_{R^c}, \diamond_{R^c}) = (\rhd_R, \diamond_R, \circ_R)$.

(ii) $(\lhd_{R^t}, \circ_{R^t}, \diamond_{R^t}) = (\rhd_R, \circ_R, \diamond_R)$.

(iii) $(\lhd_{R^d}, \circ_{R^d}, \diamond_{R^d}) = (\lhd_R, \diamond_R, \circ_R)$.

Corollary 2.8. *Let R, R' and R'' be three relations on a set X. If $R' = R \cup \{(x,x) \in X^2\}$ and $R'' = R \setminus \{(x,x) \in X^2\}$, then it holds that*

(i) $\lhd_R = \lhd_{R'} = \lhd_{R''}$.

(ii) $\rhd_R = \rhd_{R'} = \rhd_{R''}$.

(iii) $\circ_R = \circ_{R'} = \circ_{R''}$.

(iv) $\diamond_R = \diamond_{R'} = \diamond_{R''}$.

Note that, depending on the properties of R, some of the relations $\lhd_R$, $\circ_R$ and $\diamond_R$ may be already determined.

Proposition 2.10. *Let R be a relation on a set X. The following statements hold:*

(i) If R is symmetric, then $\lhd_R = \rhd_R = \emptyset$.

(ii) If R is antisymmetric, then $\circ_R = \emptyset$.

(iii) If R is complete, then $\diamond_R = \emptyset$.

Proof. Let R be a relation on X.

(i) If R is symmetric, then, for any $x, y \in X$, it holds that xRy and yRx. Hence, it holds that $\lhd_R = \rhd_R = \emptyset$.

(ii) If R is antisymmetric, then, for any $x, y \in X$, it holds that xRy and yRx implies that $x = y$. Hence, it holds that $\circ_R = \emptyset$.

(iii) If R is complete, then, for any $x, y \in X$, it holds that xRy or yRx. Hence, it holds that $\diamond_R = \emptyset$.

□

Remark 2.5. *Note that $\circ_R$ was not considered in [32] because an order relation is always antisymmetric. In case the relation R is a total order relation (or, in general, antisymmetric and complete), the relations $\circ_R$ and $\diamond_R$ are no longer relevant as they are empty. In this case, the clone relation coincides with the usual covering relation for (total) order relations, as discussed in [32].*

The previous proposition serves to characterize the properties of the clone relation of particular types of binary relations, such as order relations or equivalence relations, in terms of the properties of its partition. For this purpose, we analyse some basic properties of the relations $\lhd_R$, $\rhd_R$, $\circ_R$ and $\diamond_R$.

Theorem 2.1. *Let R be a relation on a set X. The following statements hold:*

(i) If $x \lhd_R y$, then, for any $z \in X \setminus \{x, y\}$, $x \approx_R z$ implies that $x \rhd_R z$.

(ii) *If* $x \rhd_R y$*, then, for any* $z \in X \setminus \{x, y\}$*,* $x \approx_R z$ *implies that* $x \lhd_R z$*.*

(iii) *If* $x \circ_R y$*, then, for any* $z \in X \setminus \{x, y\}$*,* $x \approx_R z$ *implies that* $x \circ_R z$*.*

(iv) *If* $x \diamond_R y$*, then, for any* $z \in X \setminus \{x, y\}$*,* $x \approx_R z$ *implies that* $x \diamond_R z$*.*

Proof. (*i*) Let $x, y \in X$ and $z \in X \setminus \{x, y\}$ be such that $x \lhd_R y$ and $x \approx_R z$. Note that $x \neq y$. On the one hand, since xRy, yR^cx, $x \approx_R z$ and $y \in X \setminus \{x, z\}$, it follows that zRy and yR^cz. On the other hand, since zRy, yR^cz, $x \approx_R y$ and $z \in X \setminus \{x, y\}$, it follows that zRx and xR^cz. As $x \approx_R z$, it follows that $x \rhd_R z$.

(*ii*) The proof is analogous to that of (*i*).

(*iii*) Let $x, y \in X$ and $z \in X \setminus \{x, y\}$ be such that $x \circ_R y$ and $x \approx_R z$. Note that $x \neq y$. On the one hand, since xRy, yRx, $x \approx_R z$ and $y \in X \setminus \{x, z\}$, it follows that zRy and yRz. On the other hand, since zRy, yRz, $x \approx_R y$ and $z \in X \setminus \{x, y\}$, it follows that zRx and xRz. As $x \approx_R z$, it follows that $x \circ_R z$.

(*iv*) Let $x, y \in X$ and $z \in X \setminus \{x, y\}$ be such that $x \diamond_R y$ and $x \approx_R z$. Note that $x \neq y$. On the one hand, since $x \parallel y$, $x \approx_R z$ and $y \in X \setminus \{x, z\}$, it follows that $z \parallel y$. On the other hand, since $z \parallel y$, $x \approx_R y$ and $z \in X \setminus \{x, y\}$, it follows that $z \parallel x$. As $x \approx_R z$, it follows that $x \diamond_R z$.

□

Corollary 2.9. *Let* R *be a relation on a set* X*. Then there are no* $x, y, z \in X$ *such that* $x \lhd_R y$ *and* $y \lhd_R z$ *and* $z \lhd_R x$*.*

Proof. Suppose that there exist $x, y, z \in X$ such that $x \lhd_R y$, $y \lhd_R z$ and $z \lhd_R x$. Since xRy, $z \approx_R x$ and $y \in X \setminus \{x, z\}$, it follows that zRy, which contradicts zR^cy. □

The (ir)reflexivity and (anti)symmetry of the relations $\lhd_R$, $\rhd_R$, $\lhd_R \cup \rhd_R$, $\circ_R$ and $\diamond_R$ is discussed in the following proposition.

Proposition 2.11. *Let* R *be a relation on a set* X*. The following statements hold:*

(i) $\lhd_R$ *is irreflexive and antisymmetric.*

(ii) $\rhd_R$ *is irreflexive and antisymmetric.*

(iii) $\lhd_R \cup \rhd_R$ *is irreflexive and symmetric.*

(iv) $\circ_R$ *is irreflexive and symmetric.*

(v) $\diamond_R$ *is irreflexive and symmetric.*

Proof. By definition, the relations $\lhd_R$, $\rhd_R$, $\diamond_R$, $\circ_R$ and $\lhd_R \cup \rhd_R$ are irreflexive. Next, for any $x, y \in X$, it is immediate to see that both ($x \lhd_R y$ and $y \lhd_R x$) and ($x \rhd_R y$ and $y \rhd_R x$) are impossible; this implies that $\lhd_R$ and $\rhd_R$ are antisymmetric. Since $\lhd_R^t = \rhd_R$ and $\rhd_R^t = \lhd_R$, it follows that $\lhd_R \cup \rhd_R$ is symmetric. In addition, as $\circ_R^t = \circ_R$ and $\diamond_R^t = \diamond_R$, it follows that $\circ_R$ and $\diamond_R$ are symmetric. $\square$

In the following proposition, we discuss the (anti)transitivity of the relations $\lhd_R$, $\rhd_R$, $\lhd_R \cup \rhd_R$, $\circ_R$ and $\diamond_R$.

Proposition 2.12. *Let R be a relation on a set X. The following statements hold:*

(i) $\lhd_R$ *is antitransitive.*

(ii) $\rhd_R$ *is antitransitive.*

(iii) $\lhd_R \cup \rhd_R$ *is antitransitive.*

(iv) $\circ_R \cup \delta$ *is transitive.*

(v) $\diamond_R \cup \delta$ *is transitive.*

Proof. (i) Let $x, y, z \in X$ be such that $x \lhd_R y$ and $y \lhd_R z$. Suppose that $x \lhd_R z$. It follows that $x \lhd_R y$, $x \approx_R z$ and $z \in X \setminus \{x, y\}$. Therefore, from Theorem 2.1, it follows that $x \rhd_R z$, a contradiction. Hence, $\lhd_R$ is antitransitive.

(ii) The proof is analogous to that of (i) .

(iii) Let $x, y, z \in X$ be such that $x(\lhd_R \cup \rhd_R)y$ and $y(\lhd_R \cup \rhd_R)z$. From (i) and (ii) it follows that ($x \lhd_R y$ and $y \lhd_R z$) and ($x \rhd_R y$ and $y \rhd_R z$) lead to, respectively, $x(\lhd_R)^c z$ and $x(\rhd_R)^c z$. In addition, due to Corollary 2.9, we have that ($x \lhd_R y$ and $y \lhd_R z$ and $x \rhd_R z$) and ($x \rhd_R y$ and $y \rhd_R z$ and $x \lhd_R z$) are not possible. On the other hand, if $x \neq z$ then the cases ($x \lhd_R y$ and $y \rhd_R z$) or ($x \rhd_R y$ and $y \lhd_R z$) are not possible, due to Theorem 2.1; if $x = z$, then $x(\lhd_R \cup \rhd_R)^c x$, due to the irreflexivity of $\lhd_R \cup \rhd_R$. We conclude that $\lhd_R \cup \rhd_R$ is antitransitive.

(iv) Let $x, y, z \in X$ be such that $x(\circ_R \cup \delta)y$ and $y(\circ_R \cup \delta)z$.

  (a) If $x = z$ or $x = y$ or $y = z$, then it trivially holds that $x(\circ_R \cup \delta)z$.

  (b) The case $x \neq z$, $x \neq y$ and $y \neq z$. First, we prove that $x \approx_R z$. Suppose that $x \not\approx_R z$, then it follows that there exists $t \in X \setminus \{x, z\}$ such that (tRx and $tR^c z$) or (tRz and $tR^c x$) or (xRt and $zR^c t$) or (zRt and $xR^c t$). If, for instance, (tRx and $tR^c z$), then, since yRz, it follows that $t \neq y$. As tRx, $tR^c z$, $x \approx_R y$, $y \approx_R z$ and $t \in X \setminus \{x, y, z\}$, it follows that tRy and $tR^c y$, a contradiction. The other cases where (tRz and $tR^c x$) or (xRt and $zR^c t$) or (zRt and $xR^c t$) are analogously proved. We conclude that $x \approx_R z$. Second, as $z \circ_R y$, $z \approx_R x$ and $x \in X \setminus \{y, z\}$, it follows from Theorem 2.1 that $x \circ_R z$.

We conclude that $x(\circ_R \cup \delta)z$ and, therefore, $\circ_R \cup \delta$ is transitive.

(v) The proof is similar to that of (iv).

□

From Propositions 2.11 and 2.12, the following result follows.

Corollary 2.10. *Let R be a relation on a set X. Then it holds that $(\lhd_R \cup \rhd_R \cup \delta)$ is a tolerance relation and that $(\circ_R \cup \delta)$ and $(\diamond_R \cup \delta)$ are equivalence relations.*

For any $x, y \in X$, there exists at most one element z such that $x \approx_R z$ and $z \approx_R y$ and $x \not\approx_R y$.

Proposition 2.13. *Let R be a relation on a set X. For any two elements $x, y \in X$, if there exists $z \in X$ such that $x \approx_R z$, $z \approx_R y$ and $x \not\approx_R y$, then it holds that ($x \lhd_R z$ and $z \lhd_R y$) or that ($y \lhd_R z$ and $z \lhd_R x$), that z is unique and that $[z]_{\approx_R} = \{x, y, z\}$.*

Proof. Let $x, y \in X$ be such that there exists $z \in X$ such that $x \approx_R z$, $z \approx_R y$ and $x \not\approx_R y$. This implies that $x \neq z \neq y \neq x$. Hence, ($x \lhd_R z$ or $x \rhd_R z$ or $x \circ_R z$ or $x \diamond_R z$) and $z \approx_R y$. From Theorem 2.1 and Corollary 2.10, it follows that ($x \circ_R z$ or $x \diamond_R z$) and $z \approx_R y$ implies $y \approx_R x$, a contradiction. We only need to consider the cases ($x \lhd_R z$ or $x \rhd_R z$) and $z \approx_R y$. From Theorem 2.1, it follows that ($x \lhd_R z$ and $z \lhd_R y$) or ($y \lhd_R z$ and $z \lhd_R x$) are the only possible cases.

Suppose now that z is not unique, i.e., $\exists z' \in X \setminus \{x, y, z\}$ such that $x \approx_R z'$ and $y \approx_R z'$. Therefore, it holds that ($x \lhd_R z'$ and $z' \lhd_R y$) or ($y \lhd_R z'$ and $z' \lhd_R x$). From Theorem 2.1, as $x \lhd_R z$ and $x \approx_R z'$, it follows that $x \rhd_R z'$, a contradiction. Hence, $[z]_{\approx_R} = \{x, y, z\}$. □

Next, we provide an important result w.r.t. the structure of the intersection of two tolerance classes of the clone relation.

Proposition 2.14. *Let R be a relation on a set X. For any two elements $x, y \in X$, it holds that*

(i) If $x \approx_R y$, then it holds that

$$[x]_{\approx_R} \cap [y]_{\approx_R} = \begin{cases} [x]_{\approx_R} & \text{, if } x = y\,, \\ \{x, y\} & \text{, if } x \lhd_R y \vee x \rhd_R y\,, \\ \{x, y\} \cup \{z \in X \mid z \circ_R x \wedge z \circ_R y\} & \text{, if } x \circ_R y\,, \\ \{x, y\} \cup \{z \in X \mid z \diamond_R x \wedge z \diamond_R y\} & \text{, if } x \diamond_R y\,. \end{cases}$$

(ii) If $x \not\approx_R y$ and $[x]_{\approx_R} \cap [y]_{\approx_R} \neq \emptyset$, then it holds that

$$[x]_{\approx_R} \cap [y]_{\approx_R} = \{z\}\,,$$

where $z \in X$ is the unique element such that $x \rhd_R z$ and $z \rhd_R y$ or that $y \rhd_R z$ and $z \rhd_R x$.

Proof. (i) Let $x, y \in X$ be such that $x \approx_R y$.

(a) If $x = y$, then it trivially holds that

$$[x]_{\approx_R} \cap [y]_{\approx_R} = [x]_{\approx_R} = [y]_{\approx_R} \,.$$

(b) If $x \lhd_R y$, then we will prove that there does not exist any $z \in [x]_{\approx_R} \cap [y]_{\approx_R}$ such that $z \in X \setminus \{x, y\}$. Suppose that such z exists. It then follows from Theorem 2.1 that $z \lhd_R x$ and $y \lhd_R z$, a contradiction (Corollary 2.9). Hence, it holds that

$$[x]_{\approx_R} \cap [y]_{\approx_R} = \{x, y\} \,.$$

The proof is analogous for $x \rhd_R y$.

(c) If $x \circ_R y$, then, for any $z \in [x]_{\approx_R} \cap [y]_{\approx_R}$ such that $z \in X \setminus \{x, y\}$, it follows from Theorem 2.1 that $z \circ_R x$ and $z \circ_R y$. Hence, it holds that

$$[x]_{\approx_R} \cap [y]_{\approx_R} = \{x, y\} \cup \{z \in X \mid z \circ_R x \wedge z \circ_R y\} \,.$$

(d) If $x \diamond_R y$, then, for any $z \in [x]_{\approx_R} \cap [y]_{\approx_R}$ such that $z \in X \setminus \{x, y\}$, it follows from Theorem 2.1 that $x \diamond_R z$ and $y \diamond_R z$. Hence, it holds that

$$[x]_{\approx_R} \cap [y]_{\approx_R} = \{x, y\} \cup \{z \in X \mid z \diamond_R x \wedge z \diamond_R y\} \,.$$

(ii) Let $x, y \in X$ be such that $x \not\approx_R y$ and $[x]_{\approx_R} \cap [y]_{\approx_R} \neq \emptyset$ and let $z \in [x]_{\approx_R} \cap [y]_{\approx_R}$. It follows from Proposition 2.13 that z is the unique element such that $x \approx_R z$, $y \approx_R z$ and that $x \not\approx_R y$. Hence, it holds that

$$[x]_{\approx_R} \cap [y]_{\approx_R} = \{z\} \,.$$

□

Example 2.8. *Let* R *be the relation defined in Example 2.1. It holds that* $[a]_{\approx_R} = \{a, d\}, [b]_{\approx_R} = \{b, c\}, [c]_{\approx_R} = \{b, c\}, [d]_{\approx_R} = \{a, d\}, [e]_{\approx_R} = \{e, f\}$, $[f]_{\approx_R} = \{e, f\}$. *Therefore, it holds that:*

$$\begin{aligned} [a]_{\approx_R} \cap [d]_{\approx_R} &= \{a, d\} \,, \\ [b]_{\approx_R} \cap [c]_{\approx_R} &= \{b, c\} \,, \\ [e]_{\approx_R} \cap [f]_{\approx_R} &= \{e, f\} \,, \\ [e]_{\approx_R} \cap [d]_{\approx_R} &= \emptyset \,. \end{aligned}$$

Example 2.9. *Let R be the relation defined in Example 2.2. For any $n \in \mathbb{N}^*$ with $n \neq 1$, it holds that $[n]_{\approx_<} = \{n-1, n, n+1\}$ ($[1]_{\approx_<} = \{1,2\}$). As, for any $n_1, n_2 \in \mathbb{N}^*$, the fact that $[n_1]_{\approx_<} \cap [n_2]_{\approx_<} \neq \emptyset$ and that $n_1 \not\approx_R n_2$ implies that $n_1 = n_2 + 2$ or that $n_1 = n_2 - 2$, it follows that:*

$$[n_1]_{\approx_<} \cap [n_2]_{\approx_<} = \begin{cases} n_2 - 1 & , \textit{ if } n_2 > n_1 \, , \\ n_2 + 1 & , \textit{ if } n_1 > n_2 \, . \end{cases}$$

2.4. The clone relation and the different types of disjoint union

In this section, we characterize the clone relation of the three different types of union of two relations defined on disjoint sets.

For a relation R_P defined on a set P, we write $\mathbb{P} = (P, R_P)$ and we call $\mathbb{P}$ an equipped set.

Definition 2.4. *An equipped set $\mathbb{P} = (P, R_P)$ is called a reduction of another equipped set $\mathbb{Q} = (Q, R_Q)$ if the following two statements hold:*

(i) $P \subseteq Q$.

(ii) For any $x, y \in P$, it holds that xR_Py if and only if xR_Qy.

If an equipped set is a reduction of another equipped set, then the clone relation of the second one is included in that of the first, as can be seen in the following proposition.

Proposition 2.15. *Let $\mathbb{P} = (P, R_P)$ be a reduction of $\mathbb{Q} = (Q, R_Q)$. For any $x, y \in P$, it holds that $x \approx_{R_Q} y$ implies that $x \approx_{R_P} y$.*

Proof. Let $x, y \in P$ be such that $x \approx_{R_Q} y$. It holds that $(zR_Qx \Leftrightarrow zR_Qy)$ and $(xR_Qz \Leftrightarrow yR_Qz)$, for any $z \in Q \setminus \{x, y\}$. Since $\mathbb{P} = (P, R_P)$ is a reduction of $\mathbb{Q} = (Q, R_Q)$, it follows that, for any $z \in P \setminus \{x, y\}$, it holds that $(zR_Px \Leftrightarrow zR_Py)$ and $(xR_Pz \Leftrightarrow yR_Pz)$. Hence, it holds that $x \approx_P y$. ☐

Remark 2.6. *Note that, throughout this section, $\approx_{R_P}$ should be understood as the clone relation of R_P in P and not in $P \cup Q$. The same applies to $\approx_{R_Q}$.*

Note that the converse of the statement in Proposition 2.15 does not hold, as can be seen in Example 2.10.

Example 2.10. *Let us consider the sets $P = \mathbb{N}$ and $Q = \mathbb{R}$ equipped with the usual strict order relation $<$. It obviously holds that $\mathbb{P} = (\mathbb{N}, <_{\mathbb{N}})$ is a reduction of*

$\mathbb{Q} = (\mathbb{R}, <_{\mathbb{R}})$. *However, it holds that* $1 \approx_{<_{\mathbb{N}}} 2$, *while* $1 \not\approx_{<_{\mathbb{R}}} 2$. *Hence, if* $x \approx_P y$ *for some* $x, y \in P$, *then it does not necessarily hold that* $x \approx_Q y$.

For any two equipped sets $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$, we say that $\mathbb{P}$ and $\mathbb{Q}$ are disjoint if P and Q are disjoint. The union of two disjoint equipped sets is called a disjoint union. There are three different types of disjoint union: the nondirectional disjoint union, the unidirectional disjoint union and the bidirectional disjoint union.

The most common disjoint union is the nondirectional disjoint union, where the relations between elements in the same original set are kept and elements in different original sets are considered incomparable.

Definition 2.5. *Let* $\mathbb{P} = (P, R_P)$ *and* $\mathbb{Q} = (Q, R_Q)$ *be two disjoint equipped sets. The nondirectional disjoint union* $\mathbb{P} \cup \mathbb{Q}$ *of* $\mathbb{P}$ *and* $\mathbb{Q}$ *is the equipped set* $\mathbb{P} \cup \mathbb{Q} = (P \cup Q, R_P \cup R_Q)$.

The unidirectional disjoint union[4] is the disjoint union where the relations between elements in the same original set are kept and, for any element x in the first equipped set and any element y in the second equipped set, it holds that x is related with y but y is not related with x.

Definition 2.6. *Let* $\mathbb{P} = (P, R_P)$ *and* $\mathbb{Q} = (Q, R_Q)$ *be two disjoint equipped sets. The unidirectional disjoint union of* $\mathbb{P}$ *and* $\mathbb{Q}$ *is the equipped set* $\mathbb{P} \overrightarrow{\cup} \mathbb{Q} = (P \cup Q, R_P \overrightarrow{\cup} R_Q)$, *where*

$$R_P \overrightarrow{\cup} R_Q = R_P \cup R_Q \cup (P \times Q) .$$

The bidirectional disjoint union is the disjoint union where the relations between elements in the same original set are kept and, for any element x in the first equipped set and any element y in the second equipped set, it holds that x is related with y and y is related with x.

Definition 2.7. *Let* $\mathbb{P} = (P, R_P)$ *and* $\mathbb{Q} = (Q, R_Q)$ *be two disjoint equipped sets. The bidirectional disjoint union of* $\mathbb{P}$ *and* $\mathbb{Q}$ *is the equipped set* $\mathbb{P} \overleftrightarrow{\cup} \mathbb{Q} = (P \cup Q, R_P \overleftrightarrow{\cup} R_Q)$, *where*

$$R_P \overleftrightarrow{\cup} R_Q = R_P \cup R_Q \cup (P \times Q) \cup (Q \times P) .$$

Remark 2.7. *Both the nondirectional disjoint union and the bidirectional disjoint union are commutative, while the unidirectional disjoint union is not.*

The three types of disjoint union are illustrated in the following example.

Example 2.11. *Let* $P = \{a, b\}$, $Q = \{c, d\}$, $R_P = \{(b, a)\}$ *and* $R_Q = \{(c, d), (d, c)\}$. *The graphs of the three different types of disjoint union are shown in*

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[4] Note that, if $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ are two disjoint posets, then the unidirectional disjoint union of $\mathbb{P}$ and $\mathbb{Q}$ is known as the linear sum $\mathbb{P} \oplus \mathbb{Q}$ (see [24]).

Figure 2.6.

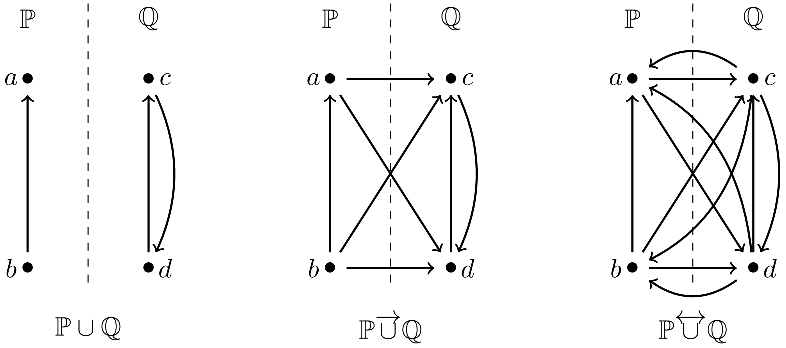

Figure 2.6: Graphs of the three different types of disjoint union of two disjoint equipped sets.

Now we characterize the clone relation of the three different types of disjoint union of two disjoint equipped sets. First, the clone relation of the nondirectional disjoint union is characterized.

Proposition 2.16. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The clone relation $\approx_R$ of the nondirectional disjoint union $R = R_P \cup R_Q$ is given by*

$$\approx_R = \approx_{R_P} \cup \approx_{R_Q} \cup (P_\| \times Q_\|) \cup (Q_\| \times P_\|)\,,$$

where $P_\| = \{x \in P \mid (\forall y \in P \setminus \{x\})(x \parallel_{R_P} y)\}$ and $Q_\| = \{x \in Q \mid (\forall y \in Q \setminus \{x\})(x \parallel_{R_Q} y)\}$.

Proof. (i) First, we prove that $\approx_{R_P} \cup \approx_{R_Q} \cup (P_\| \times Q_\|) \cup (Q_\| \times P_\|) \subseteq \approx_R$.

(a) Let $x, y \in P$ be such that $x \approx_{R_P} y$. By definition of the nondirectional disjoint union, it follows that, for any $z_Q \in Q$, $(z_Q R^c x \wedge z_Q R^c y)$ and $(x R^c z_Q \wedge y R^c z_Q)$. Therefore, it holds that $(z_Q R x \Leftrightarrow z_Q R y)$ and $(x R z_Q \Leftrightarrow y R z_Q)$. Since P and Q are disjoint sets and $x \approx_{R_P} y$, it follows that $x, y \notin Q$ and, for any $z_P \in P \setminus \{x, y\}$, $(z_P R_P x \Leftrightarrow z_P R_P y)$ and $(x R_P z_P \Leftrightarrow y R_P z_P)$. As $\mathbb{P}$ is a reduction of $\mathbb{P} \cup \mathbb{Q}$, it follows that, for any $z \in (P \cup Q) \setminus \{x, y\}$, $(z R x \Leftrightarrow z R y)$ and $(x R z \Leftrightarrow y R z)$. Hence, it holds that $x \approx_R y$, and, thus, $\approx_{R_P} \subseteq \approx_R$. In an analogous way, we can prove that $\approx_{R_Q} \subseteq \approx_R$.

(b) Let $x \in P$ and $y \in Q$ be such that $(x, y) \in (P_\| \times Q_\|)$. On the one hand, by definition of $P_\|$ and $Q_\|$, it holds that for any $z \in (P \cup Q) \setminus \{x, y\}$, $z \parallel_{R_P} x$ and $z \parallel_{R_Q} y$. On the other hand, by definition of the nondirectional disjoint union, it holds that $z \parallel_{R_P} y$ and $z \parallel_{R_Q} x$. Therefore, it follows that $z \parallel_R x$ and $z \parallel_R y$, for any $z \in (P \cup Q) \setminus \{x, y\}$. This implies that

$x \approx_R y$. Hence, it holds that $(P_{\|} \times Q_{\|}) \subseteq \approx_R$. In an analogous way, we can prove that $(Q_{\|} \times P_{\|}) \subseteq \approx_R$.

(ii) Second, we prove that $\approx_R \subseteq \approx_{R_P} \cup \approx_{R_Q} \cup (P_{\|} \times Q_{\|}) \cup (Q_{\|} \times P_{\|})$. Let $x, y \in P \cup Q$ be such that $x \approx_R y$. There are four cases to consider: ($x \in P$ and $y \in P$) or ($x \in Q$ and $y \in Q$) or ($x \in P$ and $y \in Q$) or ($x \in Q$ and $y \in P$).

(a) If $x, y \in P$, then, since $\mathbb{P}$ is a reduction of $\mathbb{P} \cup \mathbb{Q}$ and $x \approx_R y$, it follows from Proposition 2.15 that $x \approx_{R_P} y$.

(b) If $x, y \in Q$, then, again from Proposition 2.15, it follows that $x \approx_{R_Q} y$.

(c) If $x \in P$ and $y \in Q$, then one of the following cases holds: ($x \in P \setminus P_{\|}$ and $y \in Q$) or ($x \in P$ and $y \in Q \setminus Q_{\|}$) or ($x \in P_{\|}$ and $y \in Q_{\|}$). We will show that the two first cases lead to a contradiction.

(α) Suppose that $x \in P \setminus P_{\|}$ and $y \in Q$, then there exists $z \in P \setminus \{x\}$ such that zR_Px or xR_Pz. This implies that zRx or xRz. Since $y \in Q$, it follows that $z \parallel_R y$. From (zRx or xRz) and $z \parallel_R y$, it follows that $x \not\approx_R y$, a contradiction.

(β) Suppose that $x \in P$ and $y \in Q \setminus Q_{\|}$, then as in (α), it follows that $x \not\approx_R y$, a contradiction.

(d) If $x \in Q$ and $y \in P$, it follows analogously to (c) that $(x, y) \in Q_{\|} \times P_{\|}$.

□

Corollary 2.11. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The partition ($\lhd_R, \circ_R, \diamond_R$) of the clone relation $\approx_R$ of the nondirectional disjoint union $R = R_P \cup R_Q$ is given by:*

(i) $\lhd_R = \lhd_{R_P} \cup \lhd_{R_Q}$*;*

(ii) $\circ_R = \circ_{R_P} \cup \circ_{R_Q}$*;*

(iii) $\diamond_R = \diamond_{R_P} \cup \diamond_{R_Q} \cup (P_{\|} \times Q_{\|}) \cup (Q_{\|} \times P_{\|})$.

These results are illustrated in the following example.

Example 2.12. *Let $P = \{a, b, c\}$, $Q = \{d, e, f\}$, $R_P = \{(a, b)\}$ and $R_Q = \{(e, d), (d, e)\}$. The graphs of the relations R_P and R_Q are shown in Figure 2.7.*

The matrix representations of $\approx_{R_P}$ and $\approx_{R_Q}$ are given by:

$$\approx_{R_P} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \quad \approx_{R_Q} = \begin{array}{c} \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{ccc} d & e & f \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}.$$

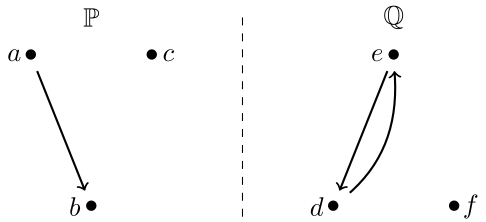

Figure 2.7: Graphs of the relations R_P and R_Q.

In addition, the matrix representation of the clone relation $\approx_{R_P \cup R_Q}$ *of the nondirectional disjoint union* $R_P \cup R_Q$ *is given by:*

$$\approx_{R_P \cup R_Q} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} .$$

Note that $P_\| = \{c\}$ *and* $Q_\| = \{f\}$ *and, therefore, it holds that* $(P_\| \times Q_\|) \cup (Q_\| \times P_\|) = \{(c,f),(f,c)\}$.

Next, the clone relation of the disjoint unidirectional union is characterized.

Proposition 2.17. *Let* $\mathbb{P} = (P, R_P)$ *and* $\mathbb{Q} = (Q, R_Q)$ *be two disjoint equipped sets. The clone relation* $\approx_R$ *of the unidirectional disjoint union* $R = R_P \overrightarrow{\cup} R_Q$ *is given by*

$$\approx_R = \approx_{R_P} \cup \approx_{R_Q} \cup (P_\rightarrow \times Q_\leftarrow) \cup (Q_\leftarrow \times P_\rightarrow)\,,$$

where $P_\rightarrow = \{x \in P \mid (\forall z_P \in P \setminus \{x\})(z_P R_P x \wedge x R_P^c z_P)\}$ *and* $Q_\leftarrow = \{y \in Q \mid (\forall z_Q \in Q \setminus \{y\})(y R_Q z_Q \wedge z_Q R_Q^c y)\}$[5].

Proof. (i) First, we prove that $\approx_{R_P} \cup \approx_{R_Q} \cup (P_\rightarrow \times Q_\leftarrow) \cup (Q_\leftarrow \times P_\rightarrow) \subseteq \approx_R$.

(a) Let $x, y \in P$ be such that $x \approx_{R_P} y$. By definition of the unidirectional disjoint union, it follows that, for any $z_Q \in Q$, $(z_Q R^c x \wedge z_Q R^c y)$ and $(x R z_Q \wedge y R z_Q)$. Therefore, it holds that $(z_Q R x \Leftrightarrow z_Q R y)$ and $(x R z_Q \Leftrightarrow y R z_Q)$. Since P and Q are disjoint sets and $x \approx_{R_P} y$, it follows that $x, y \notin Q$ and, for any $z_P \in P \setminus \{x, y\}$, $(z_P R_P x \Leftrightarrow z_P R_P y)$ and $(x R_P z_P \Leftrightarrow y R_P z_P)$. As $\mathbb{P}$ is a reduction of $\mathbb{P} \overrightarrow{\cup} \mathbb{Q}$, it follows that, for any $z \in (P \cup Q) \setminus \{x, y\}$, $(z R x \Leftrightarrow z R y)$ and $(x R z \Leftrightarrow y R z)$. Hence,

---

[5] Note that both $P_\rightarrow$ and $Q_\leftarrow$ are either the empty set or a singleton.

it holds that $x \approx_R y$, and, thus, $\approx_{R_P} \subseteq \approx_R$. In an analogous way, we can prove that $\approx_{R_Q} \subseteq \approx_R$.

(b) Let $x \in P$ and $y \in Q$ be such that $(x, y) \in P_{\rightarrow} \times Q_{\leftarrow}$. Let $z \in (P \cup Q) \setminus \{x, y\}$.

(α) If zRx, then, by definition of unidirectional disjoint union, it must hold that $z \in P$. It follows that $(z, y) \in P \times Q$ and, therefore, zRy.

(β) If zRy, then, since $y \in Q_{\leftarrow}$, it must hold that $z \in P$. Since $x \in P_{\rightarrow}$, it follows that zR_Px, and, therefore, zRx.

(γ) If xRz, then, since $x \in P_{\rightarrow}$, it must hold that $z \in Q$. Since $y \in Q_{\leftarrow}$, it follows that yR_Qz, and, therefore, yRz.

(δ) If yRz, then, by definition of unidirectional disjoint union, it must hold that $z \in Q$. It follows that $(x, z) \in P \times Q$ and, therefore, xRz.

Hence, it holds that $x \approx_R y$, and, thus, $P_{\rightarrow} \times Q_{\leftarrow} \subseteq \approx_R$. In an analogous way, we can prove that $Q_{\leftarrow} \times P_{\rightarrow} \subseteq \approx_R$.

(ii) Second, we prove that $\approx_R \subseteq \approx_{R_P} \cup \approx_{R_Q} \cup (P_{\rightarrow} \times Q_{\leftarrow}) \cup (Q_{\leftarrow} \times P_{\rightarrow})$. Let $x, y \in P \cup Q$ be such that $x \approx_R y$. There are four cases to consider: ($x \in P$ and $y \in P$), ($x \in Q$ and $y \in Q$), ($x \in P$ and $y \in Q$) or ($x \in Q$ and $y \in P$).

(a) If $x, y \in P$, then, since $\mathbb{P}$ is a reduction of $\mathbb{P} \overrightarrow{\cup} \mathbb{Q}$ and $x \approx_R y$, it follows from Proposition 2.15 that $x \approx_{R_P} y$.

(b) If $x, y \in Q$, then, again from Proposition 2.15, it follows that $x \approx_{R_Q} y$.

(c) If $x \in P$ and $y \in Q$, then, on the one hand, for any $z_Q \in Q$, it follows that xRz_Q and z_QR^cx. Since $x \approx_R y$, it holds that yRz_Q and z_QR^cy, for any $z_Q \in Q \setminus \{y\}$. Hence, it holds that $y \in Q_{\leftarrow}$. On the other hand, for any $z_P \in P$, it holds that z_PRy and yR^cz_P. Since $x \approx_R y$, it follows that z_PRx and xR^cz_P, for any $z_P \in P \setminus \{x\}$. Hence, it holds that $x \in P_{\rightarrow}$. Thus, it holds that $(x, y) \in P_{\rightarrow} \times Q_{\leftarrow}$.

(d) If $x \in Q$ and $y \in P$, it follows analogously to (c) that $(x, y) \in Q_{\leftarrow} \times P_{\rightarrow}$.

□

Corollary 2.12. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The partition $(\triangleleft_R, \circ_R, \diamond_R)$ of the clone relation $\approx_R$ of the unidirectional disjoint union $R = R_P \overrightarrow{\cup} R_Q$ is given by:*

(i) $\triangleleft_R = \triangleleft_{R_P} \cup \triangleleft_{R_Q} \cup (P_{\rightarrow} \times Q_{\leftarrow})$.

(ii) $\circ_R = \circ_{R_P} \cup \circ_{R_Q}$.

(iii) $\diamond_R = \diamond_{R_P} \cup \diamond_{R_Q}$.

These results are illustrated in the following example.

Example 2.13. *Let* $P = \{a, b, c\}$*,* $Q = \{d, e, f\}$*,* $R_P = \{(a, b), (c, b)\}$ *and* $R_Q = \{(e, d), (e, f)\}$*. The graphs of the relations* R_P *and* R_Q *are shown in Figure 2.8.*

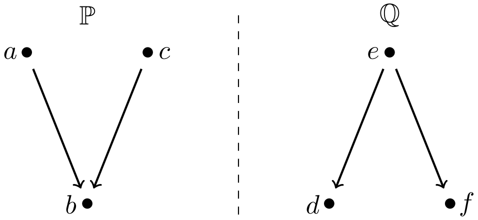

Figure 2.8: Graphs of the relations R_P and R_Q.

The matrix representations of $\approx_{R_P}$ *and* $\approx_{R_Q}$ *are given by:*

$$\approx_{R_P} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}, \qquad \approx_{R_Q} = \begin{array}{c} \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{ccc} d & e & f \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}.$$

In addition, the matrix representation of the clone relation $\approx_{R_P \overrightarrow{\cup} R_Q}$ *of the unidirectional disjoint union* $R_P \overrightarrow{\cup} R_Q$ *is given by:*

$$\approx_{R_P \overrightarrow{\cup} R_Q} = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 1 & 0 & 1 & 0 & 0 & 0 \\ b & 0 & 1 & 0 & 0 & 1 & 0 \\ c & 1 & 0 & 1 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 1 & 0 & 1 \\ e & 0 & 1 & 0 & 0 & 1 & 0 \\ f & 0 & 0 & 0 & 1 & 0 & 1 \end{array}.$$

Note that $P_{\rightarrow} = \{b\}$ *and* $Q_{\leftarrow} = \{e\}$ *and, therefore, it holds that* $(P_{\rightarrow} \times Q_{\leftarrow}) \cup (Q_{\leftarrow} \times P_{\rightarrow}) = \{(b, e), (e, b)\}$*.*

As the unidirectional disjoint union is not commutative, we also analyse the unidirectional disjoint union $R_Q \overrightarrow{\cup} R_P$*. The matrix representation of the clone relation*

$\approx_{R_Q \overrightarrow{\cup} R_P}$ *of the unidirectional disjoint union* $R_Q \overrightarrow{\cup} R_P$ *is given by:*

$$\approx_{R_Q \overrightarrow{\cup} R_P} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array} .$$

One may note that $\approx_{R_P \overrightarrow{\cup} R_Q}$ *and* $\approx_{R_Q \overrightarrow{\cup} R_P}$ *do not coincide. For instance, it holds that* $b \approx_{R_P \overrightarrow{\cup} R_Q} e$ *but* $b \not\approx_{R_Q \overrightarrow{\cup} R_P} e$.

We finish this subsection by characterizing the clone relation of the bidirectional disjoint union.

Proposition 2.18. *Let* $\mathbb{P} = (P, R_P)$ *and* $\mathbb{Q} = (Q, R_Q)$ *be two disjoint equipped sets. The clone relation* $\approx_R$ *of the bidirectional disjoint union* $R = R_P \overleftrightarrow{\cup} R_Q$ *is given by*

$$\approx_R = \approx_{R_P} \cup \approx_{R_Q} \cup (P_\leftrightarrow \times Q_\leftrightarrow) \cup (Q_\leftrightarrow \times P_\leftrightarrow) \, ,$$

where $P_\leftrightarrow = \{x \in P \mid (\forall z_P \in P \setminus \{x\})(x R_P z_P \wedge z_P R_P x)\}$ *and* $Q_\leftrightarrow = \{y \in Q \mid (\forall z_Q \in Q \setminus \{y\})(y R_Q z_Q \wedge z_Q R_Q y)\}$.

Proof. (i) First, we prove that $\approx_{R_P} \cup \approx_{R_Q} \cup (P_\leftrightarrow \times Q_\leftrightarrow) \cup (Q_\leftrightarrow \times P_\leftrightarrow) \subseteq \approx_R$.

(a) Let $x, y \in P$ be such that $x \approx_{R_P} y$. By definition of the bidirectional disjoint union, it follows that, for any $z_Q \in Q$, $(z_Q R x \wedge z_Q R y)$ and $(x R z_Q \wedge y R z_Q)$. Therefore, it holds that $(z_Q R x \Leftrightarrow z_Q R y)$ and $(x R z_Q \Leftrightarrow y R z_Q)$. Since P and Q are disjoint sets and $x \approx_{R_P} y$, it follows that $x, y \notin Q$ and, for any $z_P \in P \setminus \{x, y\}$, $(z_P R_P x \Leftrightarrow z_P R_P y)$ and $(x R_P z_P \Leftrightarrow y R_P z_P)$. As $\mathbb{P}$ is a reduction of $\mathbb{P} \overleftrightarrow{\cup} \mathbb{Q}$, it follows that, for any $z \in (P \cup Q) \setminus \{x, y\}$, $(z R x \Leftrightarrow z R y)$ and $(x R z \Leftrightarrow y R z)$. Hence, it holds that $x \approx_R y$, and, thus, $\approx_{R_P} \subseteq \approx_R$. In an analogous way, we prove that $\approx_{R_Q} \subseteq \approx_R$.

(b) Let $x \in P$ and $y \in Q$ be such that $(x, y) \in P_\leftrightarrow \times Q_\leftrightarrow$. Let $z \in (P \cup Q) \setminus \{x, y\}$.

(α) If $z R x$, then we distinguish two cases: $z \in Q$ or $z \in P$. If $z \in Q$, then, by definition of $Q_\leftrightarrow$, it follows that $z R_Q y$. Hence, it holds that $z R y$. If $z \in P$, then it holds that $(z, y) \in P \times Q$ and, therefore, $z R y$.

(β) If $z R y$, then we distinguish two cases: $z \in Q$ or $z \in P$. If $z \in Q$, then it holds that $(z, x) \in Q \times P$ and, hence, $z R x$. If $z \in P$, then by definition of $P_{\leftrightarrow}$, it follows that zR_Px. Hence, it holds that zRx.

(γ) If xRz, then we prove in an analogous way to (α) that yRz.

(δ) If yRz, then we prove in an analogous way to (β) that xRz.

Hence, it holds that $x \approx_R y$, and, thus, $P_{\leftrightarrow} \times Q_{\leftrightarrow} \subseteq \approx_R$. In an analogous way, we can prove that $Q_{\leftrightarrow} \times P_{\leftrightarrow} \subseteq \approx_R$.

(ii) Second, we prove that $\approx_R \subseteq \approx_{R_P} \cup \approx_{R_Q} \cup (P_{\leftrightarrow} \times Q_{\leftrightarrow}) \cup (Q_{\leftrightarrow} \times P_{\leftrightarrow})$. Let $x, y \in P \cup Q$ be such that $x \approx_R y$. There are four cases to consider: ($x \in P$ and $y \in P$) or ($x \in Q$ and $y \in Q$) or ($x \in P$ and $y \in Q$) or ($x \in Q$ and $y \in P$).

(a) If $x, y \in P$, then, since $\mathbb{P}$ is a reduction of $\mathbb{P} \overset{\leftrightarrow}{\cup} \mathbb{Q}$ and $x \approx_R y$, it follows from Proposition 2.15 that $x \approx_{R_P} y$.

(b) If $x, y \in Q$, then, again from Proposition 2.15, it follows that $x \approx_{R_Q} y$.

(c) If $x \in P$ and $y \in Q$, then, on the one hand, since $x \in P$, it follows, by definition of bidirectional disjoint union, that xRz_Q and z_QRx, for any $z_Q \in Q$. Since $x \approx_R y$, it follows that yRz_Q and z_QRy, for any $z_Q \in Q \setminus \{y\}$. Hence, it holds that $y \in Q_{\leftrightarrow}$. On the other hand, since $y \in Q$, it follows that yRz_P and z_PRy, for any $z_P \in P$. Since $x \approx_R y$, it follows that xRz_P and z_PRx, for any $z_P \in P \setminus \{x\}$. Hence, it holds that $x \in P_{\leftrightarrow}$. Thus, it holds that $(x, y) \in P_{\leftrightarrow} \times Q_{\leftrightarrow}$.

(d) If $x \in Q$ and $y \in P$, it follows analogously to (c) that $(x, y) \in Q_{\leftrightarrow} \times P_{\leftrightarrow}$.

□

Corollary 2.13. *Let* $\mathbb{P} = (P, R_P)$ *and* $\mathbb{Q} = (Q, R_Q)$ *be two disjoint equipped sets. The partition* $(\lhd_R, \circ_R, \diamond_R)$ *of the clone relation* $\approx_R$ *of the bidirectional disjoint union* $R = R_P \overset{\leftrightarrow}{\cup} R_Q$ *is given by:*

(i) $\lhd_R = \lhd_{R_P} \cup \lhd_{R_Q}$*;*

(ii) $\circ_R = \circ_{R_P} \cup \circ_{R_Q} \cup (P_{\leftrightarrow} \times Q_{\leftrightarrow}) \cup (Q_{\leftrightarrow} \times P_{\leftrightarrow})$*;*

(iii) $\diamond_R = \diamond_{R_P} \cup \diamond_{R_Q}$*.*

These results are illustrated in the following example.

Example 2.14. *Let* $P = \{a, b, c\}$*,* $Q = \{d, e, f\}$*,* $R_P = \{(a, c), (c, a), (c, b), (b, c)\}$ *and* $R_Q = \{(d, e), (e, d), (d, f), (f, d), (e, f), (f, e)\}$*. The graphs of the relations* R_P *and* R_Q *are shown in Figure 2.9.*

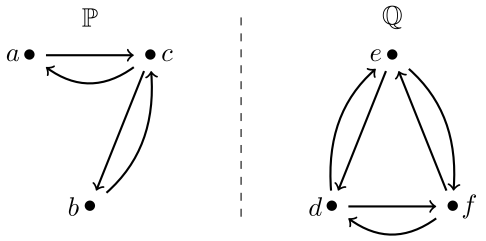

Figure 2.9: Graphs of the relations R_P and R_Q.

The matrix representations of $\approx_{R_P}$ *and* $\approx_{R_Q}$ *are given by:*

$$\approx_{R_P}= \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \quad \approx_{R_Q}= \begin{array}{c c} & \begin{array}{ccc} d & e & f \end{array} \\ \begin{array}{c} d \\ e \\ f \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}.$$

In addition, the matrix representation of the clone relation $\approx_{R_P \overset{\leftrightarrow}{\cup} R_Q}$ *of the bidirectional disjoint union* $R_P \overset{\leftrightarrow}{\cup} R_Q$ *is given by:*

$$\approx_{R_P \overset{\leftrightarrow}{\cup} R_Q}= \begin{array}{c c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

Note that $P_{\leftrightarrow} = \{c\}$ *and* $Q_{\leftrightarrow} = \{d, e, f\}$ *and, therefore, it holds that* $(P_{\leftrightarrow} \times Q_{\leftrightarrow}) \cup (Q_{\leftrightarrow} \times P_{\leftrightarrow}) = \{(c, d), (c, e), (c, f), (d, c), (e, c), (f, c)\}$*.*

We conclude this section by discussing when the clone relation of the different types of disjoint union of R_P and R_Q coincide with the union of the clone relations of R_P and R_Q.

Theorem 2.2. *Let* $\mathbb{P} = (P, R_P)$ *and* $\mathbb{Q} = (Q, R_Q)$ *be two disjoint equipped sets. The following statements hold:*

(i) $\approx_{R_P \cup R_Q} = \approx_R \cup \approx_Q$ *if and only if* $(\forall x \in P)(\exists y \in P \setminus \{x\})(x R_P y \vee y R_P x) \vee (\forall x \in Q)(\exists y \in Q \setminus \{x\})(x R_Q y \vee y R_Q x)$*.*

(ii) $\approx_{R_P \overset{\rightarrow}{\cup} R_Q} = \approx_R \cup \approx_Q$ *if and only if* $(\forall x \in P)(\exists y \in P \setminus \{x\})(x R_P y \vee y R_P^c x) \vee (\forall x \in Q)(\exists y \in Q \setminus \{x\})(x R_Q^c y \vee y R_Q x)$*.*

(iii) $\approx_{R_P \overset{\leftrightarrow}{\cup} R_Q} = \approx_R \cup \approx_Q$ *if and only if* $(\forall x \in P)(\exists y \in P \setminus \{x\})(x R_P^c y \vee y R_P^c x) \vee$

$(\forall x \in Q)(\exists y \in Q \setminus \{x\})(xR_Q^c y \vee yR_Q^c x)$.

Proof. (i) Note that, due to Proposition 2.16, $(\approx_{R_P \cup R_Q} = \approx_R \cup \approx_Q)$ is equivalent to $(P_\| = \emptyset) \vee (Q_\| = \emptyset)$. Furthermore, it trivially follows that, by definition of $P_\|$ and $Q_\|$, $(P_\| = \emptyset) \vee (Q_\| = \emptyset)$ is equivalent to $(\forall x \in P)(\exists y \in P \setminus \{x\})(xR_P y \vee yR_P x) \vee (\forall x \in Q)(\exists y \in Q \setminus \{x\})(xR_Q y \vee yR_Q x)$.

(ii) Note that, due to Proposition 2.17, $(\approx_{R_P \overrightarrow{\cup} R_Q} = \approx_R \cup \approx_Q)$ is equivalent to $(P_\rightarrow = \emptyset) \vee (Q_\leftarrow = \emptyset)$. Furthermore, it trivially follows that, by definition of $P_\rightarrow$ and $Q_\leftarrow$, $(P_\rightarrow = \emptyset) \vee (Q_\leftarrow = \emptyset)$ is equivalent to $(\forall x \in P)(\exists y \in P \setminus \{x\})(xR_P y \vee yR_P^c x) \vee (\forall x \in Q)(\exists y \in Q \setminus \{x\})(xR_Q^c y \vee yR_Q x)$.

(iii) Note that, due to Proposition 2.18, $(\approx_{R_P \overleftrightarrow{\cup} R_Q} = \approx_R \cup \approx_Q)$ is equivalent to $(P_\leftrightarrow = \emptyset) \vee (Q_\leftrightarrow = \emptyset)$. Furthermore, it trivially follows that, by definition of $P_\leftrightarrow$ and $Q_\leftrightarrow$, $(P_\leftrightarrow = \emptyset) \vee (Q_\leftrightarrow = \emptyset)$ is equivalent to $(\forall x \in P)(\exists y \in P \setminus \{x\})(xR_P^c y \vee yR_P^c x) \vee (\forall x \in Q)(\exists y \in Q \setminus \{x\})(xR_Q^c y \vee yR_Q^c x)$.

□

Corollary 2.14. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The following statements hold:*

(i) If either R_P or R_Q is complete, then $\approx_{R_P \cup R_Q} = \approx_R \cup \approx_Q$.

(ii) If either R_P or R_Q is symmetric, then $\approx_{R_P \overrightarrow{\cup} R_Q} = \approx_R \cup \approx_Q$.

(iii) If either R_P or R_Q is antisymmetric, then $\approx_{R_P \overleftrightarrow{\cup} R_Q} = \approx_R \cup \approx_Q$.

2.5. The clone relation of order n

In this section, we provide the definition of the clone relation of order n and we analyse its properties. The n-th power relation $(\approx_R)^n$ of the clone relation $\approx_R$ is addressed in this section.
The clone relation of order n of a relation is the clone relation of the n-th power of that relation.

Definition 2.8. *Let R be a relation on a set X and $n \in \mathbb{N}^*$. The clone relation of order n of R is the clone relation $\approx_{R^n}$ of R^n.*

If $x \approx_{R^n} y$, then we say that x and y are clones of order n.
The following corollary is a direct result from Proposition 1.2.

Corollary 2.15. *Let R be a relation on a set X, $n \in \mathbb{N}^*$ and $\approx_{R^n}$ be the clone relation of order n of R. The following statements hold:*

(i) If R is reflexive, then it holds that $(\forall n \in \mathbb{N}^*)(\approx_{\bigcup_{i=1}^{n} R^i} = \approx_{R^n})$.

(ii) If R is transitive, then it holds that $(\forall n \in \mathbb{N}^*)(\approx_{\bigcup_{i=1}^{n} R^i} = \approx_R)$.

Properties similar to that of (i) and (iii) in Proposition 1.2 are satisfied by the clone relation of order n, while a similar property to that of (ii) in the same proposition does not necessarily hold.

Proposition 2.19. *Let R be a relation on a set X, $n \in \mathbb{N}^*$ and $\approx_{R^n}$ be the clone relation of order n of R. The following statements hold:*

(i) If R is reflexive, then it holds that $(\forall n \in \mathbb{N}^*)(\approx_{R^n} \subseteq \approx_{R^{n+1}})$.

(ii) If R is transitive, then it does not necessarily hold that $(\forall n \in \mathbb{N}^*)(\approx_{R^{n+1}} \subseteq \approx_{R^n})$.

(iii) If R is reflexive and transitive, then it hold that $(\forall n \in \mathbb{N}^*)(\approx_{R^n} = \approx_R)$.

Proof. (i) Let R be a reflexive relation on X and $n \in \mathbb{N}^*$. For any $x, y \in X$, it holds that

$$\begin{aligned}
x \approx_{R^n} y &\Leftrightarrow \begin{cases} (\forall z \in X \setminus \{x,y\})(zR^n x \Leftrightarrow zR^n y) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(xR^n z \Leftrightarrow yR^n z) \end{cases} \\
&\Leftrightarrow \begin{cases} (\forall z \in X \setminus \{x,y\})(zR^n x \wedge xRx \Leftrightarrow zR^n y \wedge yRy) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(xRx \wedge xR^n z \Leftrightarrow yRy \wedge yR^n z) \end{cases} \\
&\Rightarrow \begin{cases} (\forall z \in X \setminus \{x,y\})(zR^{n+1} x \Leftrightarrow zR^{n+1} y) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(xR^{n+1} z \Leftrightarrow yR^{n+1} z) \end{cases} \\
&\Leftrightarrow x \approx_{R^{n+1}} y\,.
\end{aligned}$$

Thus, it holds that $\approx_{R^n} \subseteq \approx_{R^{n+1}}$.

(ii) Let us consider the relation R defined on the set $X = \{a,b,c\}$ by $R = \{(a,b),(a,c),(b,c)\}$. We can see that R is a transitive relation and that $R^2 = \{(a,c)\}$. It is clear that $a \approx_R b$, while $a \not\approx_{R^2} b$. Thus, it holds that $\approx_{R^2} \not\subseteq \approx_R$.

(iii) Let R be a reflexive and transitive relation on X. Due to Proposition 1.2, it holds that $(\forall n \in N^*)(R^n = R)$. For any $x, y \in X$, it holds that

$$\begin{aligned} x \approx_{R^n} y &\Leftrightarrow \begin{cases} (\forall z \in X \setminus \{x,y\})(zR^nx \Leftrightarrow zR^ny) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(xR^nz \Leftrightarrow yR^nz) \end{cases} \\ &\Leftrightarrow \begin{cases} (\forall z \in X \setminus \{x,y\})(zRx \Leftrightarrow zRy) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(xRz \Leftrightarrow yRz) \end{cases} \\ &\Leftrightarrow x \approx_R y\,. \end{aligned}$$

Thus, it holds that $\approx_{R^n} = \approx_R$.

□

Next we provide some properties of the n-th power relation $(\approx_R)^n$ of the clone relation $\approx_R$. It must be remarked that the n-nth power of the clone relation does not coincide with the clone relation of order n.

Proposition 2.20. *Let R be a relation on a set X, $n \in \mathbb{N}^*$ and $\approx_{R^n}$ be the clone relation of order n of R. The following statements hold:*

(i) $(\forall n \in \mathbb{N}^*)((\approx_R)^n \subseteq (\approx_R)^{n+1})$.

(ii) $(\forall n \in \mathbb{N}^*)(\bigcup_{i=1}^{n} (\approx_R)^i = (\approx_R)^n)$.

(iii) If R is symmetric, then it holds that $(\forall n \in \mathbb{N}^*)((\approx_R)^n = \approx_R)$.

Proof. Let R be a relation on a set X.

(i) Due to Proposition 2.1, we know that $\approx_R$ is reflexive. It follows from Proposition 1.2 that $(\forall n \in \mathbb{N}^*)((\approx_R)^n \subseteq (\approx_R)^{n+1})$.

(ii) It follows directly from (i).

(iii) From Corollary 2.2, it holds that, if R is symmetric, then $\approx_R$ is an equivalence relation. In particular, $\approx_R$ is a reflexive and transitive relation. It follows from Proposition 1.2 that $(\forall n \in \mathbb{N}^*)((\approx_R)^n = \approx_R)$. □

3 Clonal sets of a binary relation

In this chapter, we extend the notion of clone relation of tow elements to a set of elements and called the clonal set of a binary relation. In that way, the clonal set of a given relation is based on how any two elements of this set are related in same way w.r.t. to any other elements no in this set. We investigate the most important properties of the clonal sets of a given binary relation, paying particular attention to show that the set of all clonal sets of a binary relation is a complete lattice with the usual intersection and a clonal closure union.

3.1. Clonal sets of a binary relation

3.1.1. Definition and examples

The notion of clonal set is a natural extension of the clone relation to more than two elements. Informally, a clonal set is a set of which any two elements are related in the same way with any other element not belonging to this set.

Definition 3.1. *Let R be a relation on a set X. A subset A of X is called a clonal set of R if*

$$(\forall x, y \in A)(\forall z \in X \setminus A)((zRx \Leftrightarrow zRy) \wedge (xRz \Leftrightarrow yRz)) .$$

We denote by $\mathcal{C}_R$ the set of all clonal sets of R. Obviously, if $|X| \leqslant 2$, then it holds that $\mathcal{C}_R = \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X.

The matrix representation of a binary relation R on a set X can be used for illustrating the notion of a clonal set of R in the finite case. Let R be a relation on a finite set $X = \{x_1, x_2, ..., x_n\}$ and A be a subset of X. We denote by I_A the set of indices corresponding to A, i.e., $I_A = \{i \in \{1, 2, ..., n\} \mid x_i \in A\}$. By definition, A is a clonal set of R if and only if, for any $i, j \in I_A$ and any $k \notin I_A$, it holds that $R_{ik} = R_{jk}$ and $R_{ki} = R_{kj}$. This means that A is a clonal set of R if and only if the row and column corresponding to any element $x_i \in A$ coincide with the row and column corresponding to any other element $x_j \in A$ with the exception of the $|A|^2$ elements contained in the intersection of these $|A|$ rows with these $|A|$ columns. This is illustrated in Figure 3.1.

Note that, if A is a clonal set of a relation R on a set X, then A^c does not necessarily need to be a clonal set of R, as can be seen in Example 3.1.

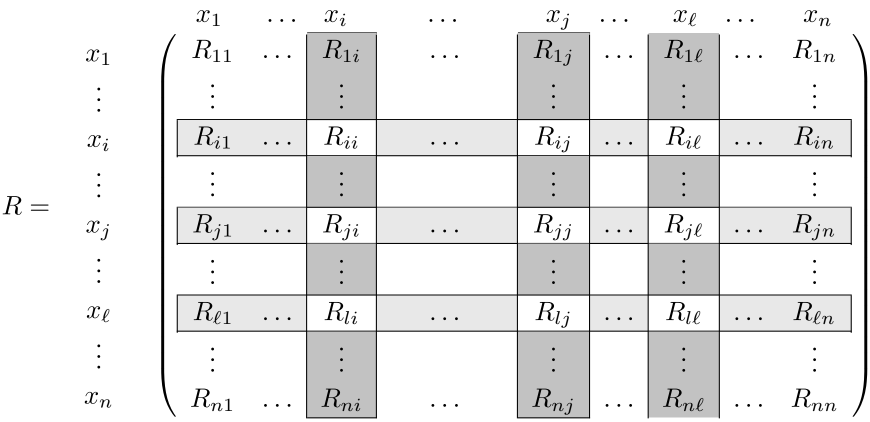

Figure 3.1: Natural interpretation of the clonal set $A = \{x_i, x_j, x_l\}$ by means of the matrix representation of R.

Example 3.1. *Let R be the relation on $X = \{a, b, c\}$ defined by $R = \{(a, c), (b, c)\}$. As $A = \{a\}$ is a singleton, it holds that $A \in \mathcal{C}_R$, while one could easily verify that $A^c = \{b, c\} \notin \mathcal{C}_R$.*

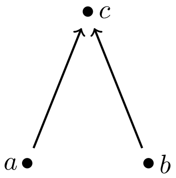

Figure 3.2: Graph of the relation R in example 3.1.

The following results easily follow from the definition of a clonal set.

Proposition 3.1. *Let R be a relation on a set X and A be a subset of X.*

(i) If $A = \emptyset$, then $A \in \mathcal{C}_R$. Therefore, $\emptyset$ is the smallest clonal set of R.

(ii) If $A = \{x\}$, then $A \in \mathcal{C}_R$.

(iii) If $A = \{x, y\}$, then $A \in \mathcal{C}_R$ if and only if $x \approx_R y$.

(iv) If $A = X$, then $A \in \mathcal{C}_R$. Therefore, X is the largest clonal set of R.

(v) For any element $a \in X$, it holds that the set $[a]_{\approx_R} = \{b \in X \mid b \approx_R a\}$ is a clonal set of R.

In case no element is related with any other element or all elements are related with all other elements, all subsets of X are assured to be clonal sets. We mention that throughout this work, $\mathcal{P}(X)$ always denotes the power set of the set X.

Proposition 3.2. *Let R be a relation on a set X and I_X be the identity relation on X. If $R \subseteq I_X$ or $X^2 \backslash I_X \subseteq R$, then $\mathcal{C}_R = \mathcal{P}(X)$.*

Any subset of elements that are unrelated to all other elements of a binary relation is a clonal set.

Proposition 3.3. *Let R be a relation on a set X. Any subset of the set $\{x \in X \mid (\forall y \in X \setminus \{x\})(x \parallel y)\}$ of incomparable elements of R is a clonal set of R.*

In particular, it holds that $\mathcal{C}_{X^2} = \mathcal{C}_{\emptyset} = \mathcal{P}(X)$.

Next, we denote by A_R the set of elements not belonging to A and to which an element of A is related, i.e. $A_R = \{y \in X \setminus A \mid (\exists x \in A)(xRy)\}$.

3.1.2. Properties of clonal sets of a binary relation

In this subsection, we discuss the most relevant properties of the clonal sets of a binary relation.
It can be easily seen that the relations between the elements of a subset A of X have no impact on this subset being a clonal set of R, as can be seen from the following proposition.

Proposition 3.4. *Let R_1 and R_2 be two binary relations on a set X and A be a subset of X. If $R_1 \backslash A^2 = R_2 \backslash A^2$, then it holds that $A \in \mathcal{C}_{R_1}$ if and only if $A \in \mathcal{C}_{R_2}$.*

Proof. Let A be a clonal set of R_1. Since $(\forall x, y \in A)(\forall z \in X \setminus A)(zR_1x \Leftrightarrow zR_1y \wedge xR_1z \Leftrightarrow yR_1z)$, it follows that $(\forall x, y \in A)(\forall z \in X \setminus A))(zR_2x \Leftrightarrow zR_2y \wedge xR_2z \Leftrightarrow yR_2z)$. Hence, A is a clonal set of R_2. ☐

The following proposition states that the set of clonal sets of a given relation always coincides with the set of clonal sets of its transpose, its complement and its dual.

Proposition 3.5. *Let R be a relation on a set X and A be a subset of X. Then it holds that $\mathcal{C}_R = \mathcal{C}_{R^t} = \mathcal{C}_{R^c} = \mathcal{C}_{R^d}$.*

Proof. First, we prove that $\mathcal{C}_R = \mathcal{C}_{R^t}$. For any $A \in \mathcal{C}_R$, it holds that

$$\begin{aligned} A \in \mathcal{C}_R &\Leftrightarrow (\forall x, y \in A)(\forall z \in X \setminus A)((zRx \Leftrightarrow zRy) \wedge (xRz \Leftrightarrow yRz)) \\ &\Leftrightarrow (\forall x, y \in A)(\forall z \in X \setminus A)((xR^tz \Leftrightarrow yR^tz) \wedge (zR^tx \Leftrightarrow zR^ty)) \\ &\Leftrightarrow A \in \mathcal{C}_{R^t}. \end{aligned}$$

Similarly, we show that $\mathcal{C}_R = \mathcal{C}_{R^c}$. For any $A \in \mathcal{C}_R$, it holds that

$$\begin{aligned} A \in \mathcal{C}_R &\Leftrightarrow (\forall x, y \in A)(\forall z \in X \setminus A)((zRx \Leftrightarrow zRy) \wedge (xRz \Leftrightarrow yRz)) \\ &\Leftrightarrow (\forall x, y \in A)(\forall z \in X \setminus A)((zR^c x \Leftrightarrow zR^c y) \wedge (xR^c z \Leftrightarrow yR^c z)) \\ &\Leftrightarrow A \in \mathcal{C}_{R^c}. \end{aligned}$$

Finally, the fact that $\mathcal{C}_R = \mathcal{C}_{R^d}$ follows from the two preceding results. □

The following proposition characterize the clonal sets of equivalence relation by mean of the equivalence classes of this relation.

Proposition 3.6. *Let R be an equivalence relation on a set X. A subset A of X it is a clone set of R if and only if is either a subset of an equivalence class of R or the union of two or more equivalence classes.*

Proof. ($\Rightarrow$): Let $A \in \mathcal{C}_R$. We need to prove that A is a subset of an equivalence class of R or the union of two or more equivalence classes of R. If $A = \emptyset$, then clearly it holds that $A \subset [x]_R$, for any $x \in X$. If $A \neq \emptyset$, then it follows that there exists a such that $a \in A$. We distinguish two cases

(i) For all $x \in A, xRa$, which implies that $A \subseteq [a]_R$.

(ii) There exists $b \in A$ such that $bR^c a$.

  (a) We prove that $[a]_R \subseteq A$ and $[b]_R \subseteq A$. Assume that $[a]_R \nsubseteq A$ or $[b]_R \nsubseteq A$. If for instance, $[a]_R \nsubseteq A$, then it follows that there exists c such that $cRa \wedge c \notin A$. Since $A \in \mathcal{C}_R$, $c \in X \setminus A$, $a, b \in A$ and cRa, it hold that cRb. Since R is equivalence, it follows that bRa, a contradiction. The other case is analogously proved. We conclude that $[a]_R \subseteq A\,, [b]_R \subseteq A$ and $[a]_R \neq [b]_R$.

  (b) We prove that for any $x \in A$, it holds that $[x]_R \subseteq A$. Let $x \in A$ and assume that there exists $x_0 \in X$ such that $x_0 \in [x]_R$ and $x_0 \notin A$. Since $A \in \mathcal{C}_R$, $x_0 \in (X \setminus A)$, $x, a \in A$ and $x_0 R x$, it follows that $x_0 R a$. Hence, $x_0 \in [a]_R$. Since $[a]_R \subseteq A$, it follows that $x_0 \in A$, a contradiction. We conclude that for any $x \in A$, $[x]_R \subseteq A$. Hence, $\cup_{x \in A}[x]_R \subseteq A$, and obviously, $A = \cup_{x \in A}[x]_R$.

  Since $a, b \in A$, $[a]_R \neq [b]_R$ and $A = \cup_{x \in A}[x]_R$, we conclude that A is the union of two or more equivalence classes of R.

($\Leftarrow$): Let $A \subseteq X$. We need to prove that if A is either a subset of an equivalence class of R or the union of two or more equivalence classes, then A is a clonal set of R.

(a) Suppose that there exists a such that $A \subseteq [a]_R$. For any $x, y \in A$ and $z \in X \setminus A$, it follows that $x, y \in [a]_R$, which implies that $zRx \Leftrightarrow zRy$ and

$xRz \Leftrightarrow yR$. Hence, $A \in \mathcal{C}_R$.

(b) Suppose that A is the union of two or more equivalence classes and let $x, y \in A$ and $z \in X \setminus A$. It follows that there exist a, b such that $x \in [a]_R \subseteq A$ and $y \in [b]_R \subseteq A$, this implies that $zR^c x$, $zR^c y$, $xR^c z$ and $yR^c z$ (other wise $z \in [a]_R \cup [b]_R$, and hence $z \in A$, a contradiction). Thus, $zR^c x \Leftrightarrow zR^c y$ and $xR^c z \Leftrightarrow yR^c z$. Therefore $A \in \mathcal{C}_R$.

□

Now, we discuss the intersection and union of clonal sets. First, we prove that the family of clonal sets is closed under intersection.

Proposition 3.7. *Let R be a relation on a set X and $(A_i)_{i\in I}$ a family of clonal sets of R. It holds that $\bigcap\limits_{i\in I} A_i \in \mathcal{C}_R$.*

Proof. Let $x, y \in \bigcap\limits_{i\in I} A_i$ and $z \in X \setminus \bigcap\limits_{i\in I} A_i$. It follows that there exists $i_0 \in I$ such that $z \in X \setminus A_{i_0}$, and $x, y \in A_{i_0}$. Since $A_{i_0} \in \mathcal{C}_R$, it follows that $zRx \Leftrightarrow zRy$ and $xRz \Leftrightarrow yRz$. Hence, it holds that $\bigcap\limits_{i\in I} A_i \in \mathcal{C}_R$. □

Together with Proposition 3.7, we obtain the following corollary.

Corollary 3.1. *Let R be a relation on a set X. It holds that $(\mathcal{C}_R, \subseteq, \cap, \emptyset, X)$ is a complete $\cap$-semi-lattice.*

In general, the union of clonal sets does not need to be a clonal set, as can be seen in Example 3.2.

Example 3.2. *Let R be the relation in Example 3.1. As every singleton is a clonal set, it holds that $\{a\}, \{c\} \in \mathcal{C}_R$, while $\{a, c\} \notin \mathcal{C}_R$ (it suffices to see that bRc while $bR^c a$).*

However, in case their intersection is not empty, the union of a family of clonal sets is a clonal set.

Proposition 3.8. *Let R be a relation on a set X and $(A_i)_{i\in I}$ a family of clonal sets of R. If $\bigcap\limits_{i\in I} A_i \neq \emptyset$, then it holds that $\bigcup\limits_{i\in I} A_i \in \mathcal{C}_R$.*

Proof. Let $x, y \in \bigcup\limits_{i\in I} A_i$ and $z \in X \setminus (\bigcup\limits_{i\in I} A_i)$. It follows that $z \in X \setminus A_i$, for any $i \in I$ and there exist $j, k \in I$ such that $x \in A_j$ and $y \in A_k$. Since $\bigcap\limits_{i\in I} A_i \neq \emptyset$, it follows that $A_j \cap A_k \neq \emptyset$, which implies that there exists t such that $t \in A_j$ and $t \in A_k$. As $A_j, A_k \in \mathcal{C}_R$, $x, t \in A_j$ and $y, t \in A_k$, it holds that

$$(zRx \Leftrightarrow zRt \Leftrightarrow zRy) \wedge (xRz \Leftrightarrow tRz \Leftrightarrow yRz)\,.$$

This implies that $(zRx \Leftrightarrow zRy) \wedge (xRz \Leftrightarrow yRz)$. Hence, $\bigcup\limits_{i\in I} A_i \in \mathcal{C}_R$. □

The following corollary follows immediately from the above propositions.

Corollary 3.2. *Let R be a relation on a set X. For any $x, y, z \in X$ such that $x \approx_R y$ and $y \approx_R z$, it holds that $\{x, y, z\} \in \mathcal{C}_R$.*

The following proposition states that any subset such that any two element on in it are clone related is a clonal set.

Proposition 3.9. *Let R be a relation on a set X and A be a subset of X. If $A^2 \subseteq \approx_R$, then any $A' \subseteq A$ is a clonal set of R.*

Proof. Let $A^2 \subseteq \approx_R$ and consider $A' \subseteq A$. For any $z \in X \setminus A'$ and $x, y \in A'$, it holds that $(x, y) \in (A')^2 \subseteq A^2 \subseteq \approx_R$. As $x \approx_R y$, it follows that $(zRx \Leftrightarrow zRy)$ and $(xRz \Leftrightarrow yRz)$. Hence, A' is a clonal set of R. □

For a relation R on a set X, let R^* denote its transitive closure, i.e., the smallest transitive relation on X that contains R. The transitive closure R^* of any relation R can be characterized as:

$$R^* = \bigcup_{k \geq 1} R^k \,,$$

where R^k is the k-th power of R. The transitive closure of a reflexive (resp. symmetric) relation is reflexive (resp. symmetric) as well.

Proposition 3.10. *Let R be a relation on a set X and R^* be its transitive closure. It holds that $\mathcal{C}_R \subseteq \mathcal{C}_{R^*}$.*

Proof. Let $A \in \mathcal{C}_R$. Consider $x, y \in A$ and $z \in X \setminus A$ such that zR^*x or xR^*z. For instance, if zR^*x, then it follows that there exists an integer $n \geqslant 1$ such that $zR^n x$, which implies that for any $i \in \{1, 2, ..., (n-1)\}$ there exists $t_i \in X$ such that

$$zR^{n-i}t_i \text{ and } t_i R^i x \,.$$

It holds that $\exists i \in \{1, 2, ..., (n-1)\}, t_i \notin A$ or $\forall i \in \{1, 2, ..., (n-1)\}, t_i \in A$.

(i) Suppose that $\exists i \in \{1, 2, ..., (n-1)\}, t_i \notin A$. Let $j = Max(1, 2, ..., (n-1))$ such that $t_j \in A$.

   (a) If $j = 1$, then it follows that $zR^{(n-1)}t_1$ and $t_1 R x$. Since $A \in \mathcal{C}_R$, $x, y \in A$, $t_1 \notin A$ and $t_1 R x$, it follows that $t_1 R y$. As $zR^{(n-1)}t_1$ and $t_1 R y$, it follows that $zR^n y$. Hence, $zR^* y$.

   (b) If $1 < j < n-1$, then it follows that $t_{j+1} \in A$, $zR^{(n-j)}t_j$, $t_j R t_{j+1}$ and $t_{j+1} R^{(j-1)} x$. Since $A \in \mathcal{C}_R$, $t_{j+1}, y \in A$, $t_j \notin A$ and $t_j R t_{j+1}$, it follows that $t_j R y$. As $zR^{(n-j)}t_j$ and $t_j R y$, it follows that $zR^{(n+1-j)}y$. Hence, $zR^* y$.

   (c) If $j = n-1$, then it follows that $t_{n-2} \in A$, $zR_2 t_{n-2}$, $t_{n-2} R t_{n-1}$ and $t_{n-1} R^{(n-3)} x$.

Since $A \in \mathcal{C}_R$, $t_{n-2}, y \in A$, $t_{n-1} \notin A$ and $t_{n-2}Rt_{n-1}$, it follows that $t_{n-2}Ry$. As $zR^{(n-2)}t_{n-2}$ and $t_{n-2}Ry$, it follows that $zR^{(n-1)}y$. Hence, zR^*y.

(ii) Suppose that $\forall i \in \{1, 2, ..., (n-1)\}, t_i \in A$. It follows that $t_1 \in A$, zRt_1 and $t_1R^{(n-1)}x$. Since $A \in \mathcal{C}_R$, $t_1, y \in A$, $z \notin A$ and zRt_1, it follows that zRy. Hence, zR^*y.

The case where xR^*z is analogously proved. We conclude that $\mathcal{C}_R \subseteq \mathcal{C}_{R^*}$. □

Note that the converse of the above proposition does not necessarily hold, as can be seen in the following example.

Example 3.3. *Let R be the relation R on $X = \{a, b, c, d\}$ defined as $R = \{(a, c), (a, d), (b, c), (c, d)\}$. One easily verifies that $R^* = \{(a, c), (a, d), (b, c), (b, d), (c, d)\}$. It holds that $\{a, b, c\} \in \mathcal{C}_{R^*}$, while $\{a, b, c\} \notin C_R$.*

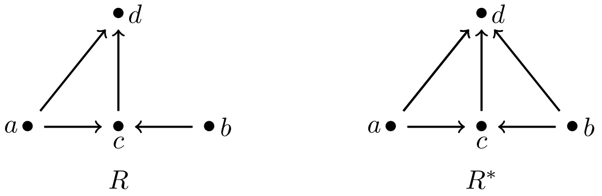

Figure 3.3: Graph of the relation R and its transitive closure R^* in Example 3.3.

Proposition 3.11. *Let R and S be two relations on a set X. The following statements hold:*

(i) $\mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}_{R \cap S} \cap \mathcal{C}_{R \setminus S} \cap \mathcal{C}_{S \setminus R}$;

(ii) $\mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}_{R \cup S} \cap \mathcal{C}_{R \setminus S} \cap \mathcal{C}_{S \setminus R}$.

Proof. (i) We need to prove that $\mathcal{C}_R \cap \mathcal{C}_S \subseteq \mathcal{C}_{R \cap S} \cap \mathcal{C}_{R \setminus S} \cap \mathcal{C}_{S \setminus R}$ and that $\mathcal{C}_{R \cap S} \cap \mathcal{C}_{R \setminus S} \cap \mathcal{C}_{S \setminus R} \subseteq \mathcal{C}_R \cap \mathcal{C}_S$.

(a) Let A be a subset of X such that $A \in \mathcal{C}_R \cap \mathcal{C}_S$. For any $x, y \in A$ and for any $z \in X \setminus A$, it holds that

$$\begin{aligned} z(R \cap S)x &\Leftrightarrow (zRx \wedge zSx) \\ &\Leftrightarrow (zRy \wedge zSy) \\ &\Leftrightarrow z(R \cap S)y\,. \end{aligned}$$

Similarly, it holds that

$$x(R \cap S)z \Leftrightarrow y(R \cap S)z\,.$$

Hence, $\mathcal{C}_R \cap \mathcal{C}_S \subseteq \mathcal{C}_{R\cap S}$.

Moreover, for any $z \in X \setminus A$and for any $x, y \in A$, it holds that

$$\begin{aligned} z(R \setminus S)x &\Leftrightarrow (zRx \wedge zS^c x) \\ &\Leftrightarrow (zRy \wedge zS^c y) \\ &\Leftrightarrow z(R \setminus S)y\,. \end{aligned}$$

Similarly, it holds that

$$x(R \setminus S)z \quad \Leftrightarrow \quad y(R \setminus S)z\,.$$

Hence, $\mathcal{C}_R \cap \mathcal{C}_S \subseteq \mathcal{C}_{R\setminus S}$. The fact that $\mathcal{C}_R \cap \mathcal{C}_S \subseteq \mathcal{C}_{S\setminus R}$ is analogously proved.

(b) Let A be a subset of X such that $A \in \mathcal{C}_{R\cap S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R}$. For any $x, y \in A$ and for any $z \in X \setminus A$, it holds that

$$\begin{aligned} zRx &\Leftrightarrow (zRx \wedge zSx) \vee (zRx \wedge zS^c x) \\ &\Leftrightarrow (zRy \wedge zSy) \vee (zRy \wedge zS^c y) \\ &\Leftrightarrow (z(R \cap S)y) \vee (z(R \setminus S)y) \\ &\Leftrightarrow zRy\,. \end{aligned}$$

Similarly, it holds that

$$xRz \quad \Leftrightarrow \quad yRz\,.$$

Hence, $\mathcal{C}_{R\cap S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R} \subseteq \mathcal{C}_R$.
Due to symmetry, it also holds that $\mathcal{C}_{R\cap S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R} \subseteq \mathcal{C}_S$.

Hence, $\mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}_{R\cap S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R}$.

(ii) From (i), it follows that $\mathcal{C}_{R^c} \cap \mathcal{C}_{S^c} = \mathcal{C}_{R^c\cap S^c} \cap \mathcal{C}_{R^c\setminus S^c} \cap \mathcal{C}_{S^c\setminus R^c}$. Since $\mathcal{C}_{R^c} = \mathcal{C}_R$, $\mathcal{C}_{S^c} = \mathcal{C}_S$, $\mathcal{C}_{R^c\cap S^c} = \mathcal{C}_{(R\cup S)^c} = \mathcal{C}_{R\cup S}$ (see Proposition 3.5), $R^c \setminus S^c = S \setminus R$ and $S^c \setminus R^c = R \setminus S$, it follows that $\mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}_{R\cup S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R}$.

□

Corollary 3.3. *Let R and S be two binary relations on a set X. The following statements hold:*

(i) $\mathcal{C}_R \cap \mathcal{C}_S \subseteq \mathcal{C}_{R\cap S}$*;*

(i) $\mathcal{C}_R \cap \mathcal{C}_S \subseteq \mathcal{C}_{R\cup S}$*.*

Corollary 3.4. *Let R and S be two relations on a set X. If $R \subseteq S$, then it holds that*

(i) $\mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}_R \cap \mathcal{C}_{S\setminus R}$*;*

(ii) $\mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}_S \cap \mathcal{C}_{S\setminus R}$;

(iii) $\mathcal{C}_R \cap \mathcal{C}_{S\setminus R} = \mathcal{C}_S \cap \mathcal{C}_{S\setminus R}$.

Proof. Suppose that $R \subseteq S$.

(i) Since $\mathcal{C}_\emptyset = \mathcal{P}(X)$, it follows from Proposition 3.11(i) that $\mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}_R \cap \mathcal{C}_{S\setminus R}$.

(ii) Since $\mathcal{C}_\emptyset = \mathcal{P}(X)$, it follows from Proposition 3.11(ii) that $\mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}_S \cap \mathcal{C}_{S\setminus R}$.

(iii) Follows from(i) and (ii).

□

3.1.3. Characterization of the set of clonal sets of the nondirectional disjoint union

In this subsection, we characterize the set of clonal sets of the unidirectional disjoint union of two relations defined on disjoint sets.

For a relation R_P defined on a set P, we write $\mathbb{P} = (P, R_P)$ and we call $\mathbb{P}$ an equipped set.

Proposition 3.12. *Let $\mathbb{P} = (P, R_P)$ be a reduction of $\mathbb{Q} = (Q, R_Q)$. It holds that $\mathcal{C}_{R_Q} \subseteq \mathcal{C}_{R_P}$.*

Proof. Let $A \in \mathcal{C}_{R_Q}$. It holds that $(zR_Qx \Leftrightarrow zR_Qy)$ and $(xR_Qz \Leftrightarrow yR_Qz)$, for any $z \in Q\setminus A$ and for any $x, y \in A$. Since $\mathbb{P} = (P, R_P)$ is a reduction of $\mathbb{Q} = (Q, R_Q)$, it follows that, for any $z \in P\setminus A$ and for any $x, y \in A$, it holds that $(zR_Px \Leftrightarrow zR_Py)$ and $(xR_Pz \Leftrightarrow yR_Pz)$. Hence, it holds that $A \in \mathcal{C}_{R_P}$. Thus, $\mathcal{C}_{R_Q} \subseteq \mathcal{C}_{R_P}$. □

Remark 3.1. *Note that, throughout this section, $\mathcal{C}_{R_P}$ should be understood as the set of clonal sets of R_P in P and not in $P \cup Q$. The same applies to $\mathcal{C}_{R_Q}$.*

Note that the converse of the statement in Proposition 3.12 does not hold, as can be seen in Example 3.4.

Example 3.4. *Let us consider the sets $P = \mathbb{N}$ and $Q = \mathbb{R}$ equipped with the usual strict order relation $<$. It obviously holds that $\mathbb{P} = (\mathbb{N}, <_\mathbb{N})$ is a reduction of $\mathbb{Q} = (\mathbb{R}, <_\mathbb{R})$. However, it holds that $\{0,1\} \in \mathcal{C}_{<_\mathbb{N}}$, while $\{0,1\} \notin \mathcal{C}_{<_\mathbb{R}}$. Hence, $\mathcal{C}_{<_\mathbb{N}} \subseteq \mathcal{C}_{<_\mathbb{R}}$.*

Proposition 3.13. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The set of clonal sets $\mathcal{C}_R$ of the nondirectional disjoint union $R = R_P \cup R_Q$ is given by*

$$\mathcal{C}_R = \mathcal{C}_{R_P} \cup \mathcal{C}_{R_Q} \cup \{A_{i\in I} \in \mathcal{P}(\underline{\underline{P}} \cup \underline{\underline{Q}}\},$$

where $A_i \cap \underline{\underline{P}} \neq \emptyset$ *and* $A_i \cap \underline{\underline{P}} \neq \emptyset$*, for any* $i \in I$ *such that* $\underline{\underline{P}} = \{S \subseteq P \mid (\forall x \in S)(\forall y \in P \setminus S)(x \parallel_{R_P} y)\}$, $\underline{\underline{Q}} = \{S \subseteq Q \mid (\forall x \in S)(\forall y \in Q \setminus S)(x \parallel_{R_Q} y)\}$ *and* $\mathcal{P}(\underline{\underline{P}} \cup \underline{\underline{Q}})$ *is the set of all subset of* $\underline{\underline{P}} \cup \underline{\underline{Q}}$ *.*

Proof. (i) First, we prove that $\mathcal{C}_{R_P} \cup \mathcal{C}_{R_Q} \cup \{A_{i\in I} \in \mathcal{P}(\underline{\underline{P}} \cup \underline{\underline{Q}})\} \subseteq \mathcal{C}_R$.

(a) Let $A \subseteq P$ be such that $A \in \mathcal{C}_{R_P}$. By definition of the nondirectional disjoint union, it follows that, for any $z_Q \in Q$ and for $x, y \in P$, $(z_Q R^c x \wedge z_Q R^c y)$ and $(x R^c z_Q \wedge y R^c z_Q)$. Therefore, it holds that $(z_Q R x \Leftrightarrow z_Q R y)$ and $(x R z_Q \Leftrightarrow y R z_Q)$. Since P and Q are disjoint sets and $A \in \mathcal{C}_{R_P}$, it follows that $A \nsubseteq Q$ and, for any $z_P \in P \setminus$ A, $(z_P R_P x \Leftrightarrow z_P R_P y)$ and $(x R_P z_P \Leftrightarrow y R_P z_P)$. As $\mathbb{P}$ is a reduction of $\mathbb{P} \cup \mathbb{Q}$, it follows that, for any $z \in (P \cup Q) \setminus A$, $(zRx \Leftrightarrow zRy)$ and $(xRz \Leftrightarrow yRz)$. Hence, it holds that $A \in \mathcal{C}_R$, and, thus, $\mathcal{C}_{R_P} \subseteq \mathcal{C}_R$. In an analogous way, we can prove that $\mathcal{C}_{R_Q} \subseteq \mathcal{C}_R$.

(b) Let $A \in \mathcal{P}(\underline{\underline{P}} \cup \underline{\underline{Q}})$ be such that $A \cap \underline{\underline{P}} \neq \emptyset$ and $A \cap \underline{\underline{Q}} \neq \emptyset$. By definitions of $\underline{\underline{P}}$, $\underline{\underline{Q}}$ and the nondirectional disjoint union, it holds that for any $z \in (P \cup Q) \setminus A$ and for $x, y \in A$, $z \parallel_{R_P} x$, $z \parallel_{R_Q} y$, $z \parallel_{R_P} y$ and $z \parallel_{R_Q} x$. Therefore, it follows that $z \parallel_R x$ and $z \parallel_R y$, for any $z \in (P \cup Q) \setminus A$ and for any $x, y \in A$. This implies that $A \in \mathcal{C}_R$. Hence, it holds that $\mathcal{P}(\underline{\underline{P}} \cup \underline{\underline{Q}}) \subseteq \mathcal{C}_R$.

(ii) Second, we prove that $\mathcal{C}_R \subseteq \mathcal{C}_{R_P} \cup \mathcal{C}_{R_Q} \cup \{A_{i\in I} \in \mathcal{P}(\underline{\underline{P}} \cup \underline{\underline{Q}})\}$. Let $A \subseteq P \cup Q$ be such that $A \in \mathcal{C}_R$. There are three cases to consider: $A \subseteq P$ or $A \subseteq Q$ or or $(A \cap P \neq \emptyset$ and $A \cap Q \neq \emptyset)$.

(a) If $A \subseteq P$, then, since $\mathbb{P}$ is a reduction of $\mathbb{P} \cup \mathbb{Q}$ and $A \in \mathcal{C}_R$, it follows from Proposition 3.12 that $A \in \mathcal{C}_{R_P}$.

(b) If $A \subseteq Q$, then, again from Proposition 3.12, it follows that $A \in \mathcal{C}_{R_Q}$.

(c) If $A \cap P \neq \emptyset$ and $A \cap Q \neq \emptyset$. It follows that there exist a, b such $a \in A \cap P$ and $b \in A \cap Q$. Assume that $A \notin \underline{\underline{P}} \cup \underline{\underline{Q}}$, it follows that there exists $z_0 \in (P \cup Q) \setminus A$ and there exists $x_0 \in A$ such that $z_0 R_p x_0$ or $x_0 R_p z_0$ or $z_0 R_Q x_0$ or $x_0 R_Q z_0$. If for instance, $z_0 R_p x_0$, then it follows that $z_0 R x_0$ and $z_0, x_0 \in P$. Since $A \in \mathcal{C}_R$, $z_0 \in (P \cup Q) \setminus A$, $x_0, b \in A$ and $z_0 R x_0$ it follows that $z_0 R b$, a contradiction with the fact that $z_0 \in P$ and $b \in Q$. The other cases where $x_0 R_p z_0$ or $z_0 R_Q x_0$ or $x_0 R_Q z_0$ are proved analogously. Hence, $A \in \underline{\underline{P}} \cup \underline{\underline{Q}}$.

Finally, we conclude that $\mathcal{C}_R = \mathcal{C}_{R_P} \cup \mathcal{C}_{R_Q} \cup \{A_{i\in I} \in \mathcal{P}(\underline{\underline{P}} \cup \underline{\underline{Q}}\}$.

$\square$

3.2. Lattice structure of the poset of clonal sets

In this section, we study the lattice structure of the poset of clonal sets of a given relation R. We show that the set of clonal sets of R ordered by the set inclusion is a complete lattice with set intersection as meet operation and clonal closure of set union as join operation. Also, we show that the principal filters of this complete lattice are complete sublattices with set intersection and union as meet and join operation, respectively.

3.2.1. The clonal closure operation and complete lattice structure of the poset of clonal sets

For a given relation R on a set X, we define the mapping $\widehat{\ }\colon \mathcal{P}(X) \to \mathcal{P}(X)$ for any subset A of X as follows:

$$\widehat{A} = \bigcap \{B \in \mathcal{C}_R \mid A \subseteq B\}\,.$$

The following proposition shows that the mapping $\widehat{\ }$ is a closure operator on X.

Proposition 3.14. *Let R be a relation on a set X. The mapping $\widehat{\ }\colon \mathcal{P}(X) \to \mathcal{C}_R$ defined by*

$$\widehat{A} = \bigcap \{B \in \mathcal{C}_R \mid A \subseteq B\}\,,$$

is a closure operator on X.

Proof. Let $A \in \mathcal{P}(X)$, from Proposition 3.7 it holds that $\widehat{A}$ is a clonal set.

(i) Let $A \in \mathcal{P}(X)$. Obviously, it holds that $A \subseteq \widehat{A}$.

(ii) Let $A, B \in \mathcal{P}(X)$ such that $A \subseteq B$. Since $B \subseteq \widehat{B}$, it follows that $A \subseteq \widehat{B}$. Since $\widehat{B}$ is a clonal set and $\widehat{A}$ is the smallest clonal set containing A, it follows that $\widehat{A} \subseteq \widehat{B}$.

(ii) Let $A \in \mathcal{P}(X)$. Since $\widehat{A}$ is a clonal set, it follows that $\widehat{A} \in \{B \in \mathcal{C}_R \mid \widehat{A} \subseteq B\}$. Hence, $\widehat{\widehat{A}} = \widehat{A}$.

We conclude that the mapping $\widehat{\ }$ is a closure operator on X. □

The operator $\widehat{\ }$ is called the *clonal closure operator* on X, and for any subset A of X, $\widehat{A}$ is called the *clonal closure* of A.

The following corollary is immediate.

Corollary 3.5. *Let R be a relation on a set X and A be a subset of X. A is a clonal set of R if and only if $\widehat{A} = A$.*

From Propositions 3.7 and 3.8, we obtain the following corollary.

Corollary 3.6. *Let R be a relation on a set X and $A, B \in \mathcal{C}_R$. Then it holds that*

(i) $\widehat{A \cap B} = A \cap B$;

(ii) If $A \cap B \neq \emptyset$, *then it holds that* $\widehat{A \cup B} = A \cup B$.

The following result discusses the complete lattice structure of the set of clonal sets. It follows from Proposition 3.14 and Theorem 1.2

Theorem 3.1. *Let R be a relation on a set X. Then it holds that* $(\mathcal{C}_R, \subseteq, \cap, \widehat{\cup}, \emptyset, X)$ *is a complete lattice in which*

$$\widehat{\bigcup_{i\in I}} A_i = \widehat{\bigcup_{i\in I} A_i} = \bigcap \{B \in \mathcal{C}_R \mid (\forall i \in I)(A_i \subseteq B)\} \,,$$

for any family $(A_i)_{i\in I}$ *in* $\mathcal{C}_R$.

Remark 3.2. *(i) It is important to mention that although* $(\mathcal{C}_R, \subseteq, \cap, \widehat{\cup})$ *is a complete lattice, it is not a complete sublattice of* $(\mathcal{P}(X), \subseteq, \cap, \cup)$.

(ii) In general, the complete lattice $(\mathcal{C}_R, \subseteq, \cap, \widehat{\cup})$ *is neither modular, nor complemented, as can be seen in the following example.*

Example 3.5. *Consider the set* $X = \{1, 2, 3\} \subseteq \mathbb{N}$ *equipped with the usual order relation* $\leqslant$. *It holds that* $\mathcal{C}_\leqslant = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, X\}$. *The Hasse diagram of the complete lattice* $(\mathcal{C}_\leqslant, \subseteq, \cap, \widehat{\cup})$ *is shown in Figure 3.4. Since* $\mathcal{N}_5 \subset \mathcal{C}_R$ *(consider, for example* $\mathcal{N}_5 = \{\emptyset, \{1\}, \{3\}, \{2, 3\}, X\}$*), it follows that* $\mathcal{C}_\leqslant$ *is neither distributive nor modular. Moreover, it is not complemented either. Indeed, there does not exist a clonal set* $A \in \mathcal{C}_\leqslant$ *such that* $\{2\} \cap A = \emptyset$ *and* $\{2\} \,\widehat{\cup}\, A = X$.

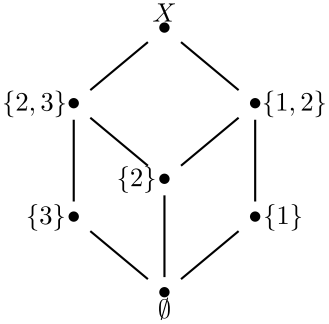

Figure 3.4: Hasse diagram of the complete lattice $(\mathcal{C}_\leqslant, \subseteq, \cap, \widehat{\cup})$

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3.2.2. Principal filters of the poset of clonal sets

In this subsection, we show that the principal filters of the poset of clonal sets $(\mathcal{C}_R, \subseteq)$ are complete sublattices of the complete lattice $(\mathcal{C}_R, \subseteq, \cap, \widehat{\cup}, \emptyset, X)$ with set intersection and union as meet and join operation, respectively.

A nonempty subset $\mathcal{F}$ of a poset $(\mathcal{P}, \subseteq)$ is called a filter if the following conditions hold:

(i) for any $A, B \in \mathcal{F}$, there exists element $C \in \mathcal{F}$ such that $C \leqslant A$ and $C \leqslant B$;

(ii) for any $A \in \mathcal{F}$ and $B \in \mathcal{P}$, $A \leqslant B$ implies that $B \in \mathcal{F}$, i.e., $\mathcal{F}$ is an upper set.

The principal filter generated by an element $A \in \mathcal{P}$ is the smallest filter that contains A, and is given by the set $\{B \in \mathcal{P} \mid A \subseteq B\}$.

The following example shows that principal filters of a complete lattice $(\mathcal{P}, \subseteq \cap, \widehat{\cup})$ does not necessarily constitute sublattices with set intersection and union as meet and join operation, respectively.

Example 3.6. *Consider the poset* $(\mathcal{P}, \subseteq)$ *of all filters of the lattice* $L = \{0, a, b, c, 1\}$ *given by the Hasse diagram in Figure 3.5.*

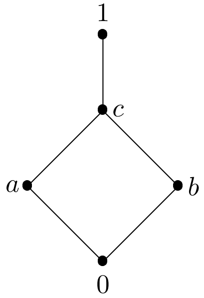

Figure 3.5: Hasse diagram of the lattice $L = \{0, a, b, c, 1\}$.

Let $(\mathcal{P}, \subseteq, \cap, \widehat{\cup})$ *be the complete lattice of filters of* $(\mathcal{P}, \subseteq)$ *in which* $\widehat{\cup}$ *is the filter closure of the union. Let* $\mathcal{F}_{\{1\}}$ *denote the principal filter of the complete lattice* $(\mathcal{P}, \subseteq, \cap, \widehat{\cup})$ *generated by the filter* $\{1\}$, *i.e.,* $\mathcal{F}_{\{1\}} = \{\{1\}, \{c, 1\}, \{a, c, 1\}, \{b, c, 1\}, L\}$. *Since* $\{a, c, 1\} \cup \{b, c, 1\} = \{a, b, c, 1\} \notin \mathcal{F}_{\{1\}}$, *it holds that* $\mathcal{F}_{\{1\}}$ *is not a sublattice with set intersection and union as meet and join operation, respectively.*

Let B be a clonal set of R and $\mathcal{F}_B$ be the principal filter of $(\mathcal{C}_R, \subseteq)$ generated by B, i.e.,

$$\mathcal{F}_B = \{C \in \mathcal{C}_R \mid B \subseteq C\}\,.$$

The following theorem shows that any principal filter of the complete lattice $(\mathcal{C}_R, \subseteq, \cap, \widehat{\cup}, \emptyset, X)$ is a complete sublattice with set intersection and union as meet and join operation, respectively. theorem

Theorem 3.2. *Let R be a relation on a set X, B be a noempty clonal set of R and $\mathcal{F}_B$ be the principal filter of $(\mathcal{C}_R, \subseteq)$ generated by B. Then it holds that $(\mathcal{F}_B, \subseteq, \cap, \cup, B, X)$ is a complete sublattice of $(\mathcal{C}_R, \subseteq, \cap, \widehat{\cup})$.*

Proof. Let $(A_i)_{i\in I}$ be a family in $\mathcal{F}_B$. From Proposition 3.7, it follows that $\underset{i\in I}{\cap} A_i$ is a clonal set of R. Since $B \subseteq \underset{i\in I}{\cap} A_i$, it follows that $\underset{i\in I}{\cap} A_i \in \mathcal{F}_B$. Also, from Proposition 3.8, it follows that $\underset{i\in I}{\cup} A_i$ is a clonal set of R. Hence, $\underset{i\in I}{\widehat{\cup}} A_i = \widehat{\underset{i\in I}{\cup} A_i} = \underset{i\in I}{\cup} A_i$. Since $B \subseteq \underset{i\in I}{\cup} A_i$, it follows that $\underset{i\in I}{\cup} A_i \in \mathcal{F}_B$. Hence, $(\mathcal{F}_B, \subseteq, \cap, \cup)$ is a complete sublattice of $(\mathcal{C}_R, \subseteq, \cap, \widehat{\cup})$. Moreover, it is clear that $0_{\mathcal{F}_B} = B$ and $1_{\mathcal{F}_B} = X$. □

Remark 3.3. *The fact that any principal filter $\mathcal{F}_B$ of the complete lattice $(\mathcal{C}_R, \subseteq, \cap, \widehat{\cup}, \emptyset, X)$ is a complete sublattice of $(\mathcal{P}(X), \subseteq, \cap, \cup)$ implies that it is a distributive and residuated sublattice, where $A \otimes B = A \cap B$ and $A \to B = \cup\{C \in \mathcal{F}_B \mid A \otimes C \subseteq B\}$, for any $A, B \in \mathcal{F}_B$.*

3.3. Clonal degrees

For any integer m, we can define a natural binary relation expressing that elements belong to a clonal set of size at most m.

Definition 3.2. *Let R be a relation on a finite set X. For any $m \in \mathbb{N}$, the binary relation φ_R^m on X is defined as*

$$\varphi_R^m = \{(x, y) \in X^2 \mid (\exists A \in \mathcal{C}_R)(x, y \in A \wedge |A| \leqslant m)\}\,.$$

It is straightforward to prove that the relations $(\varphi_R^m)_{m=1}^n$ constitute a nested family.

Proposition 3.15. *Let R be a relation on a finite set X. For any $m \in \mathbb{N}$, it holds that $\varphi_R^m \subseteq \varphi_R^{m+1}$.*

Some basic properties of this relation depend on the chosen integer.

Proposition 3.16. *Let R be a relation on a finite set X.*

(i) For any $x, y \in X$, it holds that $x = y$ if and only if $x\varphi_R^1 y$.

(ii) For any $x, y \in X$, it holds that $x \approx_R y$ if and only if $x\varphi_R^2 y$.

(iii) For any $x, y \in X$, it holds that $x\varphi_R^n y$.

Proof. Statement (i). For any $x, y \in X$ such that $x = y$, it holds that $\{x\} = A \in \mathcal{C}_R$, $x, y \in A$ and $|A| = 1 \leqslant 1$. Therefore, $x\varphi_R^1 y$. For any $x, y \in X$ such that $x\varphi_R^1 y$, it holds that there exists $A \in \mathcal{C}_R$ satisfying that $x, y \in A$ and $|A| = 1$. Therefore, $x = 1$.

Statement (ii). For any $x, y \in X$ such that $x \approx_R y$, we distinguish two cases $x = y$ and $x \neq y$. In case $x = y$, from(i), we know that $x\varphi^1_R y$, and, from Proposition 3.15, we conclude that $x\varphi^2_R y$. In case $x \neq y$, it holds that $\{x, y\} = A \in \mathcal{C}_R$ (and, additionally, $x, y \in A$ and $|A| = 2 \leqslant 2$). Therefore, $x\varphi^2_R y$. For any $x, y \in X$ such that $x\varphi^2_R y$, it holds that $\exists A \in \mathcal{C}_R$ satisfying that $x, y \in A$ and $|A| = 2$. Therefore, $x \approx_R y$.

Statement (iii). We recall that $X \in \mathcal{C}_R$. Therefore, for any $x, y \in X$, it holds that $x, y \in A = X$ and $|A| = n \leqslant n$). Therefore, $x\varphi^n_R y$. □

Obviously, the relations $(\varphi^m_R)^n_{m=1}$ are a tolerance relations.

Proposition 3.17. *Let R be a relation on a finite set X. For any $m \in \mathbb{N}$, φ^m_R is a tolerance relation.*

Proof. For any $x \in X$, due to the fact that $\{x\} \in \mathcal{C}_{\mathcal{R}}$ and $|\{x\}| \leqslant m$, for any $m \geqslant 1$, it follows that $x\varphi^m_R x$. Hence, φ^m_R is reflexive, for any $m \geqslant 1$. The symmetry property is evident. We conclude that, for any $m \geqslant 1$, φ^m_R is a tolerance relation. □

Obviously, as φ^1_R is the identity relation, it trivially is an equivalence relation. However, for any $m \geqslant 2$, the relation φ^m_R does not necessarily need to be an equivalence relation. For instance, let us consider the set of real numbers $\mathbb{R}$ equipped with the usual order relation $\leqslant$. It holds that $1\varphi^m_{\leqslant} m$ and $m\varphi^m_{\leqslant}(2m-1)$, for any $m \geqslant 2$. However, as it does not hold that $1\varphi^m_{\leqslant}(2m-1)$, we conclude that the transitivity property might not be fulfilled.

Note that the relations $(\varphi^m_R)^n_{m=1}$ can be characterized by means of the clonal closure operator of the set consisting of both elements.

Proposition 3.18. *Let R be a relation on a finite set X. For any $x, y \in X$ and any $m \in \mathbb{N}$, it holds that $x\varphi^m_R y$ if and only if $|\widehat{\{x, y\}}| \leqslant m$.*

Proof. Consider $m \in \mathbb{N}$ and $x, y \in X$ such that $x\varphi^m_R y$. It follows that there exists $A \in \mathcal{C}_R$ such that $x, y \in A$ and $|A| \leqslant m$. By definition of clonal closure of $\{x, y\}$, it is the smallest clonal set containing $\{x, y\}$. Therefore, $\widehat{\{x, y\}} \subseteq A$ and $|\widehat{\{x, y\}}| \leqslant m$.

Consider $m \in \mathbb{N}$ and $x, y \in X$ such that $|\widehat{\{x, y\}}| \leqslant m$. Note that for $A = \widehat{\{x, y\}}$ it holds that $A \in \mathcal{C}_R$, $x, y \in A$ and $|A| \leqslant m$. Therefore, $x\varphi^m_R y$. □

Finally, the preceding analysis allow us to introduce the notation of clonal degree of two elements as a tool to quantify how far two elements are from being clones.

The clonal degree is then introduced as a tool allowing to compare how far two elements are from being clones.

Definition 3.3. *Let R be a relation on a finite set X. For any $x, y \in X$, the clonal degree $c(x, y)$ of x and y is defined by*

$$c(x,y) = \min\{m \in \{1,\dots,n\} \mid x\varphi_R^m y\} - 1\,.$$

Remark 3.4. *As a consequence of Proposition 3.18, it holds that $c(x, y) = |\widehat{\{x, y\}}| - 1$, for any $x, y \in X$.*

An important observation concerns the fact that the clonal degree constitutes a metric on X.

Proposition 3.19. *Let R be a relation on a finite set X. The clonal degree function $c : X \times X \to \mathbb{R}$ defines a metric on X.*

Proof. Non-negativity. For any $x, y \in X$, it holds that $x\varphi_R^n y$. Therefore, it holds that $\min\{m \in \{1,\dots,n\} \mid x\varphi_R^m y\} \geq 1$ and, therefore, $c(x, y) \geq 0$.

Identity of indiscernibles. For any $x, y \in X$, it holds that

$$\begin{aligned} c(x,y) = 0 &\Leftrightarrow \min\{m \in \{1,\dots,n\} \mid x\varphi_R^m y\} = 1 \\ &\Leftrightarrow x\varphi_R^1 y \\ &\Leftrightarrow x = y\,. \end{aligned}$$

Symmetry. For any $x, y \in X$, it holds that

$$\begin{aligned} c(x,y) &= \min\{m \in \{1,\dots,n\} \mid x\varphi_R^m y\} - 1 \\ &= \min\{m \in \{1,\dots,n\} \mid y\varphi_R^m x\} - 1 \\ &= c(y,x)\,. \end{aligned}$$

Triangle inequality. For any $x, y, z \in X$, it holds that

$$\widehat{\{x,z\}} \subseteq \widehat{\{x,y\} \cup \{y,z\}} \subseteq \widehat{\{x,y\}} \cup \widehat{\{y,z\}}\,.$$

Removing $\{x\}$ on both sides, it follows that

$$\widehat{\{x,z\}} \backslash \{x\} \subseteq (\widehat{\{x,y\}} \backslash \{x\}) \cup (\widehat{\{y,z\}} \backslash \{x\})\,.$$

We conclude that

$$\begin{aligned} c(x,z) &= |\widehat{\{x,z\}}| - 1 \\ &= |\widehat{\{x,z\}} \backslash \{x\}| \\ &\leq |(\widehat{\{x,y\}} \backslash \{x\}) \cup (\widehat{\{y,z\}} \backslash \{x\})| \\ &\leq |\widehat{\{x,y\}} \backslash \{x\}| + |\widehat{\{y,z\}} \backslash \{x\}| \\ &= c(x,y) + c(y,z)\,. \end{aligned}$$

□

Example 3.7. *Let R be the relation on $X = \{a,b,c,d,e,f\}$ defined by the graph in Figure 3.6.*

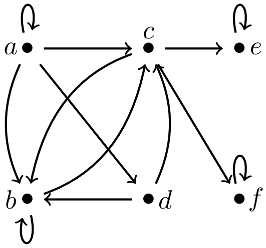

Figure 3.6: Graph of a relation R on the set $X = \{a,b,c,d,e,f\}$.

It holds that $\widehat{\{e,f\}} = \{e,f,c\}$, $\widehat{\{a,d\}} = \{a,d\}$, $\widehat{\{a,c\}} = \{a,c,b,d,e,f\} = X$. Hence, it holds that $c(e,f) = |\{e,f,c\}| - 1 = 2$, $c(a,d) = |\{a,d\}| - 1 = 1$, $c(a,c) = |X| - 1 = 5$.

Proposition 3.20. *Let R be a relation on a set X and R^* be its transitive closure. It $\mathcal{C}_{\mathcal{R}} \bigcup_{k=1} R_i$ holds that $\mathcal{C}_R \subseteq \mathcal{C}_{R^*}$.*

Proof. Let $A \in \mathcal{C}_R$. Consider $x, y \in A$ and $z \in X \setminus A$ such that zR^*x or xR^*z. For instance, if zR^*x, then it follows that there exists an integer $n \geqslant 1$ such that zR^nx, which implies that for any $i \in \{1,2,...,(n-1)\}$ there exists $t_i \in X$ such that

$$zR^{n-i}t_i \text{ and } t_iR^ix\,.$$

It holds that $\exists i \in \{1,2,...,(n-1)\}, t_i \notin A$ or $\forall i \in \{1,2,...,(n-1)\}, t_i \in A$.

(i) Suppose that $\exists i \in \{1,2,...,(n-1)\}, t_i \notin A$. Let $j = Max(1,2,...,(n-1))$ such that $t_j \in A$.

(a) If $j = 1$, then it follows that $zR^{(n-1)}t_1$ and t_1Rx. Since $A \in \mathcal{C}_R$, $x, y \in A$, $t_1 \notin A$ and t_1Rx, it follows that t_1Ry. As $zR^{(n-1)}t_1$ and t_1Ry, it follows that zR^ny. Hence, zR^*y.

(b) If $1 < j < n-1$, then it follows that $t_{j+1} \in A$, $zR^{(n-j)}t_j$, t_jRt_{j+1} and $t_{j+1}R^{(j-1)}x$. Since $A \in \mathcal{C}_R$, $t_{j+1}, y \in A$, $t_j \notin A$ and t_jRt_{j+1}, it follows that t_jRy. As $zR^{(n-j)}t_j$ and t_jRy, it follows that $zR^{(n+1-j)}y$. Hence, zR^*y.

(c) If $j = n-1$, then it follows that $t_{n-2} \in A$, zR_2t_{n-2}, $t_{n-2}Rt_{n-1}$ and $t_{n-1}R^{(n-3)}x$.

Since $A \in \mathcal{C}_R$, $t_{n-2}, y \in A$, $t_{n-1} \notin A$ and $t_{n-2}Rt_{n-1}$, it follows that $t_{n-2}Ry$. As $zR^{(n-2)}t_{n-2}$ and $t_{n-2}Ry$, it follows that $zR^{(n-1)}y$. Hence, zR^*y.

(ii) Suppose that $\forall i \in \{1, 2, ..., (n-1)\}, t_i \in A$. It follows that $t_1 \in A$, zRt_1 and $t_1R^{(n-1)}x$. Since $A \in \mathcal{C}_R$, $t_1, y \in A$, $z \notin A$ and zRt_1, it follows that zRy. Hence, zR^*y.

The case where xR^*z is analogously proved. We conclude that $\mathcal{C}_R \subseteq \mathcal{C}_{R^*}$. □

PART II

APPLICATIONS: COMPATIBILITY OF FUZZY EQUIVALENCE RELATIONS

4 Compatibility of a crisp relation with a fuzzy equivalence relation

The aim of this chapter is to generalize the characterization of the L-fuzzy tolerance/equivalence relations that a strict order relation is compatible with (see [32]), to any relation, i.e., the representation of the L-fuzzy tolerance/equivalence relations that a binary relation is compatible with.

The notion of compatibility is an important extension of the extensionality of a mapping between two universes with L-fuzzy equalities introduced by Höhle and Blanchard [47]. Also, this notion is similar to the compatibility of a fuzzy relation with respect to an L-fuzzy equality/equivalence rlation introduced by Bělohlávek [2]. The notion of compatibility has been used in various contexts. For example, it is used in the study of fuzzy lattices and fuzzy functions [5, 21, 34, 57], also it used to improve results on fuzzy partial orderings obtined by Zedeh [77].

After recalling some basic definitions and properties on compatibility of two L-fuzzy relations on a residuated lattice. In particular related to the clone relation of a crisp relation introduced in chapter 1 and the partition of this clone relation in terms of three different types of pairs of clones. More specifically, reflexive related clones and irreflexive unrelated clones turn out to play a key role in the characterization of the fuzzy tolerance and fuzzy equivalence relations that a given (crisp) relation is compatible with., we study two auxiliary relations associated with this clone relation. These auxiliary relations respectively gather the reflexive related clones and the irreflexive unrelated clones. Also we study the compatibility of a given crisp relation with the latter auxiliary relations. The results are exploited to characterize the fuzzy tolerance and fuzzy equivalence relations a given crisp relation is compatible with. These characterizations turn out to be pleasingly elegant and insightful.

4.1. Two auxiliary relations

In this section, we study a subrelation of $\circ_R$ and a subrelation of $\diamond_R$, associated with the clone relation $\approx_R$ of a given relation R. These subrelations will turn out to be useful technical tools in the following sections.

Definition 4.1. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ_R$ and $\diamond_R$. The following relations on X are defined:*

(i) $\circ_R^r = \{(x, y) \in \circ_R \mid xRx \wedge yRy\}$.

(ii) $\diamond^i_R = \{(x,y) \in \diamond_R \mid xR^cx \wedge yR^cy\}$.

Obviously, $\circ^r_R$ is a subrelation of $\circ_R$ and $\diamond^i_R$ is a subrelation of $\diamond_R$. Informally, $\circ^r_R$ consists of 'reflexive related clones', while $\diamond^i_R$ consists of 'irreflexive unrelated clones'.

Remark 4.1. *Definition 4.1 implies that, for a given relation R on a set X, the following two cases are impossible: ($x\circ^r_R y$ and $y\diamond^i_R z$), as well as ($x\diamond^i_R y$ and $y\circ^r_R z$).*

Proposition 4.1. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ^r_R$ and $\diamond^i_R$. Then the following statements hold:*

(i) If R is reflexive, then $\circ^r_R = \circ_R$ and $\diamond^i_R = \emptyset$.

(ii) If R is irreflexive, then $\diamond^i_R = \diamond_R$ and $\circ^r_R = \emptyset$.

Since $\circ_R$ and $\diamond_R$ are irreflexive and symmetric, the following proposition is immediate.

Proposition 4.2. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ^r_R$ and $\diamond^i_R$. Then the following statements hold:*

(i) The relation $\circ^r_R$ is irreflexive and symmetric.

(ii) The relation $\diamond^i_R$ is irreflexive and symmetric.

(iii) The relation $\circ^r_R \cup \diamond^i_R$ is irreflexive and symmetric.

Corollary 4.1. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ^r_R$ and $\diamond^i_R$. Then the following statements hold:*

(i) The relation $\circ^r_R \cup \delta$ is a tolerance relation.

(ii) The relation $\diamond^i_R \cup \delta$ is a tolerance relation.

(iii) The relation $\circ^r_R \cup \diamond^i_R \cup \delta$ is a tolerance relation.

The following proposition shows the relation between the subrelations associated with the clone relation $\approx_R$ and the subrelations associated with the clone relations $\approx_{R^c}, \approx_{R^t}$ and $\approx_{R^d}$. To that end, we need the following lemma.

Lemma 4.1. *Let R be a relation on a set X. Then the following statements hold:*

(i) $\approx_{R^c} = \approx_R$.

(ii) $\approx_{R^t} = \approx_R$.

(iii) $\approx_{R^d} = \approx_R$.

Proof. For any $x, y \in X$, it holds that

(i)

$$
\begin{aligned}
x \approx_{R^c} y &\Leftrightarrow \begin{cases} (\forall z \in X \setminus \{x,y\})(zR^c x \Leftrightarrow zR^c y) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(xR^c z \Leftrightarrow yR^c z) \end{cases} \\
&\Leftrightarrow \begin{cases} (\forall z \in X \setminus \{x,y\})(zRx \Leftrightarrow zRy) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(xRz \Leftrightarrow yRz) \end{cases} \\
&\Leftrightarrow x \approx_R y.
\end{aligned}
$$

(ii) Is proved analogously.

(iii) Follows immediately from (i) and (ii).

□

Proposition 4.3. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ^r_R$ and $\diamond^i_R$. Then the following statements hold:*

(i) $(\circ^r_{R^c}, \diamond^i_{R^c}) = (\diamond^i_R, \circ^r_R)$.

(ii) $(\circ^r_{R^t}, \diamond^i_{R^t}) = (\circ^r_R, \diamond^i_R)$.

(iii) $(\circ^r_{R^d}, \diamond^i_{R^d}) = (\diamond^i_R, \circ^r_R)$.

Proof. Let R be a relation on a set X.

(i) We need to prove that $\circ^r_{R^c} = \diamond^i_R$ and $\diamond^i_{R^c} = \circ^r_R$.

   (a) By definition, $\circ^r_{R^c} = \{(x,y) \in X^2 \mid (x \circ_{R^c} y) \wedge (xR^c x) \wedge (yR^c y)\}$. Since $\circ_{R^c} = \{(x,y) \in X^2 \mid x \approx_{R^c} y \wedge xR^c y \wedge yR^c x\}$ and $\approx_{R^c} = \approx_R$ (see Lemma 4.1), it follows that $\circ^r_{R^c} = \diamond^i_R$.

   (b) By definition, $\diamond^i_{R^c} = \{(x,y) \in X^2 \mid (x \diamond_{R^c} y) \wedge (xRx) \wedge (yRy)\}$. Since $\diamond_{R^c} = \{(x,y) \in X^2 \mid (x \approx_{R^c} y) \wedge (xRy) \wedge (yRx)\}$ and $\approx_{R^c} = \approx_R$, it follows that $\diamond^i_{R^c} = \circ^r_R$.

(ii) Is proved analogously to (i).

(iii) Follows immediately from (i) and (ii).

□

The following proposition identifies two important implications, which will be helpful in the proofs in Section 5.

Proposition 4.4. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ^r_R$ and $\diamond^i_R$. Then the following statements hold:*

(i) If $x \circ^r_R y$, then, for any $z \in X \setminus \{x, y\}$, $x \approx_R z$ and zRz imply that $x \circ^r_R z$ and $y \circ^r_R z$.

(ii) If $x \diamond^i_R y$, then, for any $z \in X \setminus \{x, y\}$, $x \approx_R z$ and zR^cz imply that $x \diamond^i_R z$ and $y \diamond^i_R z$.

Proof. Let $x, y \in X$

(i) Let $z \in X \setminus \{x, y\}$ be such that $x \circ^r_R y$, $x \approx_R z$ and zRz. Note that $x \neq y$. On the one hand, since xRy, yRx, $x \approx_R z$, and $y \in X \setminus \{x, z\}$, it follows that zRy and yRz. On the other hand, since zRy, yRz, $x \approx_R y$ and $z \in X \setminus \{x, y\}$, it follows that zRx and xRz. As $x \approx_R z$, xRx and zRz, it follows that $x \circ^r_R z$.

Next, we prove that $y \circ^r_R z$. As yRy and zRz, it remains to prove that $y \approx_R z$. Suppose that $y \not\approx_R z$, then there exists $t \in X \setminus \{y, z\}$ such that (yRt and zR^ct) or (zRt and yR^ct) or (tRy and tR^cz) or (tRz and tR^cy). Suppose, for instance, that (yRt and zR^ct). Since zRx, it follows that $t \in X \setminus \{x, y, z\}$. It then holds that $x \approx_R y$ and $x \approx_R z$ imply that (xRt and xR^ct), a contradiction. The three other cases lead to a similar contradiction. Hence, $y \approx_R z$, and, therefore, $y \circ^r_R z$.

(ii) Let $z \in X \setminus \{x, y\}$ be such that $x \diamond^i_R y$ and $x \approx_R z$. Note that $x \neq y$. On the one hand, since $x \parallel_R y$, $x \approx_R z$ and $y \in X \setminus \{x, z\}$, it follows that $z \parallel_R y$. On the other hand, since $z \parallel_R y$, $x \approx_R y$ and $z \in X \setminus \{x, y\}$, it follows that $z \parallel_R x$. As $x \approx_R z$, xR^cx and zR^cz, it follows that $x \diamond^i_R z$.

Next, we prove that $y \diamond^i_R z$. As $y \parallel_R z$, yR^cy and zR^cz, it remains to prove that $y \approx_R z$. Suppose that $y \not\approx_R z$, then it follows that there exists $t \in X \setminus \{y, z\}$ such that (yRt and zR^ct) or (zRt and yR^ct) or (tRy and tR^cz) or (tRz and tR^cy). Suppose, for instance, that(yRt and zR^ct). Since yR^ct, it follows that $t \in X \setminus \{x, y, z\}$. It then holds that $x \approx_R y$ and $x \approx_R z$ imply that (xRt and xR^ct), a contradiction. The three other cases lead to a similar contradiction. Hence, $y \approx_R z$, and, therefore, $y \diamond^i_R z$.

□

The following proposition discusses the transitivity of the relations $\circ^r_R \cup \delta$, $\diamond^i_R \cup \delta$ and $\circ^r_R \cup \diamond^i_R \cup \delta$.

Proposition 4.5. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ^r_R$ and $\diamond^i_R$. Then the following statements hold:*

(i) The relation $\circ^r_R \cup \delta$ is transitive.

(ii) The relation $\diamond^i_R \cup \delta$ is transitive.

(iii) The relation $\circ_R^r \cup \diamond_R^i \cup \delta$ *is transitive.*

Proof. Let R be a relation on a set X.

(i) Let $x, y, z \in X$ be such that $x(\circ_R^r \cup \delta)y$ and $y(\circ_R^r \cup \delta)z$.

  (a) If $x = z$ or $x = y$ or $y = z$, then it trivially holds that $x(\circ_R^r \cup \delta)z$.

  (b) If $x \neq z$, $x \neq y$ and $y \neq z$, then it holds that $(x \circ_R^r y)$ and $(y \circ_R^r z)$. Since $\circ_R$ is transitive, it holds that $x \circ_R z$. As xRx and zRz, it follows that $x \circ_R^r z$. Hence, $x(\circ_R^r \cup \delta)z$.

We conclude that $\circ_R^r \cup \delta$ is transitive.

(ii) Let $x, y, z \in X$ be such that $x(\diamond_R^i \cup \delta)y$ and $y(\diamond_R^i \cup \delta)z$.

  (a) If $x = z$ or $x = y$ or $y = z$, then it trivially holds that $x(\diamond_R^i \cup \delta)z$.

  (b) If $x \neq z$, $x \neq y$ and $y \neq z$, then it holds that $(x \diamond_R^i y)$ and $(y \diamond_R^i z)$. Since $\diamond_R$ is transitive, it holds that $x \diamond_R z$. As xR^cx and zR^cz, it follows that $x \diamond_R^i z$. Hence, $x(\diamond_R^i \cup \delta)z$.

We conclude that $\diamond_R^i \cup \delta$ is transitive.

(iii) Let $x, y, z \in X$ be such that $x(\circ_R^r \cup \diamond_R^i \cup \delta)y$ and $y(\circ_R^r \cup \diamond_R^i \cup \delta)z$.

  (a) If $x = z$ or $x = y$ or $y = z$, then it trivially holds that $x(\circ_R^r \cup \diamond_R^i \cup \delta)z$.

  (b) If $x \neq z$, $x \neq y$ and $y \neq z$, then since $(x \circ_R^r y \wedge y \diamond_R^i z)$ and $(x \diamond_R^i y \wedge y \circ_R^r z)$ are two impossible cases, it follows that $(x(\circ_R^r \cup \delta)y \wedge y(\circ_R^r \cup \delta)z)$ or $(x(\diamond_R^i \cup \delta)y \wedge y(\diamond_R^i \cup \delta)z)$. From (i) and (ii), it follows that $x(\circ_R^r \cup \delta)z$ or $x(\diamond_R^i \cup \delta)z$. Hence, it holds that $x(\circ_R^r \cup \diamond_R^i \cup \delta)z$.

We conclude that $\circ_R^r \cup \diamond_R^i \cup \delta$ is transitive.

□

From Corollary 4.1 and Proposition 4.5, the following result follows.

Corollary 4.2. *Let* R *be a relation on a set* X *and* $\approx_R$ *be the clone relation of* R *with corresponding* $\circ_R^r$ *and* $\diamond_R^i$*. Then it holds that* $\circ_R^r \cup \delta$, $\diamond_R^i \cup \delta$ *and* $\circ_R^r \cup \diamond_R^i \cup \delta$ *are equivalence relations.*

4.2. Compatibility of a relation with the two auxiliary relations associated with its clone relation

In this section, we study the compatibility of a relation with the two subrelations $\circ_R^r$ and $\diamond_R^i$ associated with its clone relation.

4.2.1. Compatibility of fuzzy relations

In this subsection, we recall some basic definitions and results on compatibility, right compatibility and left compatibility of two L-relations on a universe X. Further information can be found in [32, 53]. We pay particular attention to the case where the first L-relation considered is a crisp relation, in particular a pseudo-order relation.

Definition 4.2. [53] *Let R_1 and R_2 be two L-relations on a universe X.*

(i) *R_1 is called left compatible with R_2, denoted $R_1 \nabla_l R_2$, if it holds that $R_1(x,y) * R_2(x,z) \leqslant R_1(z,y)$, for any $x,y,z \in X$.*

(ii) *R_1 is called right compatible with R_2, denoted $R_1 \nabla_r R_2$, if it holds that $R_1(x,y) * R_2(y,t) \leqslant R_1(x,t)$, for any $x,y,t \in X$.*

(iii) *R_1 is called compatible with R_2, denoted $R_1 \nabla R_2$, if it holds that $R_1(x,y) * R_2(x,z) * R_2(y,t) \leqslant R_1(z,t)$, for any $x,y,z,t \in X$.*

Lemma 4.2. [53] *Let R_1 and R_2 be two L-relations on a universe X. Then it holds that*

(i) *If $R_1 \nabla_l R_2$ and $R_1 \nabla_r R_2$, then $R_1 \nabla R_2$.*

(ii) *If $R_1 \nabla R_2$ and R_2 is reflexive, then $R_1 \nabla_l R_2$ and $R_1 \nabla_r R_2$.*

Lemma 4.3. [53] *Let R be an L-relation on a universe X and $(S_i)_{i\in I}$ be a family of L-relations on X. Then it holds that*

(i) *$R \nabla_r S_i$, for any $i \in I$, if and only if $R \nabla_r \left(\bigcup_{i\in I} S_i\right)$.*

(ii) *$R \nabla_l S_i$, for any $i \in I$, if and only if $R \nabla_l \left(\bigcup_{i\in I} S_i\right)$.*

Let R be a relation on a set X and E be an L-relation on X, then compatibility of R with E states that

$$\tau(xRy) * E(x,z) * E(y,t) \leq \tau(zRt)\,, \tag{4.1}$$

for any $x,y,z,t \in X$. Note that throughout this work, we use the notation τ to refer to the characteristic mapping of a relation R on a set X, i.e., $\tau(xRy) = 1$ if xRy, while $\tau(xRy) = 0$ if xR^cy.

The following theorem shows that the only reflexive L-relation that a pseudo-order relation is compatible with, is the equality relation.

Theorem 4.1. *A pseudo-order relation R on a set X is compatible with a reflexive L-relation on X if and only if E is the crisp equality on X.*

Proof. Suppose that R is compatible with E. It follows that

$$\tau(xRx) * E(x,x) * E(x,y) \leq \tau(xRy)\,,$$

for any $x, y \in X$. Since R and E are reflexive, it follows that $E(x,y) \leq \tau(xRy)$, for any $x, y \in X$. Similarly, it holds that $E(x,y) \leq \tau(yRx)$, for any $x, y \in X$. Hence, $E(x,y) \leq \min(\tau(xRy), \tau(yRx))$, for any $x, y \in X$. On the one hand, since R is antisymmetric, it follows that $E(x,y) = 0$, for any $x, y \in X$ such that $x \neq y$. On the other hand, since E is reflexive, it holds that $E(x,x) = 1$, for any $x \in X$. Therefore, E is the crisp equality on X.

Conversely, it is obvious that R is compatible with the crisp equality. □

The above theorem implies the following corollary shown earlier in [53].

Corollary 4.3. *An order relation R on a set X is compatible with an L-tolerance or L-equivalence relation E if and only if E is the crisp equality on X.*

4.2.2. Compatibility of a relation R with the relations $\circ^r_R$ and $\diamond^i_R$

In this subsection, we prove some key results, which will be helpful in the proofs of our main theorems.

Proposition 4.6. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ^r_R$ and $\diamond^i_R$. Then the following statements hold:*

(i) R is compatible with $\circ^r_R$.

(ii) R is compatible with $\diamond^i_R$.

Proof. Let R be a relation on a set X.

(i) In view of Lemma 4.2, it suffices to prove that $R \,\nabla_l\, \circ^r_R$ and $R \,\nabla_r\, \circ^r_R$.

(a) Let $x, y, z \in X$, then we need to prove that

$$\tau(xRy) * \tau(x \circ^r_R z) \leq \tau(zRy)\,.$$

(1) If xR^cy or $x(\circ^r_R)^cz$, then $\tau(xRy) * \tau(x \circ^r_R z) = 0$ and the inequality is trivially fulfilled.

(2) If xRy and $x \circ^r_R z$, then we distinguish three cases: $x \neq y \neq z$, $x = y$ and $y = z$:

(α) If $x \neq y \neq z$, then since xRy and $x \approx_R z$, it follows that zRy. Hence, it holds that $\tau(xRy) * \tau(x \circ^r_R z) \leq \tau(zRy) = 1$.

(β) If $x = y$, then, by definition of $\circ^r_R$, it follows that $\tau(x \circ^r_R z) \leq \tau(zRx)$. Hence, it holds that $\tau(xRx) * \tau(x \circ^r_R z) \leq \tau(zRx)$.

(γ) If $y = z$, then, by definition of $\circ^r_R$, it follows that $\tau(x \circ^r_R y) \leq \tau(yRy)$. Hence, it holds that $\tau(xRy) * \tau(x \circ^r_R y) \leq \tau(yRy)$.

We conclude that $R \,\nabla_l \circ^r_R$.

(b) Let $x, y, t \in X$, then we need to prove that

$$\tau(xRy) * \tau(y \circ^r_R t) \leq \tau(xRt)\,.$$

(1) If $xR^c y$ or $y(\circ^r_R)^c t$, then $\tau(xRy) * \tau(y \circ^r_R t) = 0$ and the inequality is trivially fulfilled.

(2) If xRy and $y \circ^r_R t$, then we distinguish the following cases:

(α) If $y \neq x \neq t$, then since xRy and $y \approx_R t$, it follows that xRt. Hence, it holds that $\tau(xRy) * \tau(y \circ^r_R t) \leq \tau(xRt) = 1$.

(β) If $x = y$, then, by definition of $\circ^r_R$, it follows that $\tau(x \circ^r_R t) \leq \tau(xRt)$. Hence, it holds that $\tau(xRx) * \tau(x \circ^r_R t) \leq \tau(xRt)$.

(γ) If $x = t$, then, by definition of $\circ^r_R$, it follows that $\tau(y \circ^r_R x) \leq \tau(xRx)$. Hence, it holds that $\tau(xRy) * \tau(y \circ^r_R x) \leq \tau(xRx)$.

We conclude that $R \,\nabla_l \circ^r_R$.

(ii) Can be proved analogously.

$\square$

Combining Proposition 4.6, Lemma 4.3 and the fact R is compatible with δ, we obtain the following corollary.

Corollary 4.4. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ^r_R$ and $\diamond^i_R$. Then it holds that R is compatible with $(\circ^r_R \cup \diamond^i_R \cup \delta)$.*

From Propositions 4.1 and 4.6, we obtain the following corollary.

Corollary 4.5. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ^r_R$ and $\diamond^i_R$. Then the following statements hold:*

(i) If R is reflexive, then R is compatible with $\circ_R \cup \delta$.

(ii) If R is irreflexive, then R is compatible with $\diamond_R \cup \delta$.

4.3. Compatibility of a relation with an L-tolerance or L-equivalence relation

In this section, we study the compatibility of an arbitrary relation with an L-tolerance or L-equivalence relation.

4.3.1. Compatibility of a relation with an L-tolerance relation

In this subsection, we characterize the L-tolerance relations that a given relation is compatible with.

Theorem 4.2. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ_R^r$ and $\diamond_R^i$ and E be an L-tolerance relation on X. Then it holds that R is compatible with E if and only if $E \subseteq \circ_R^r \cup \diamond_R^i \cup \delta$.*

Proof. Suppose that R is compatible with E, i.e. $\tau(xRy) * E(x,z) * E(y,t) \leq \tau(zRt)$, for any $x, y, z, t \in X$. Let $a, b \in X$. If $a = b$ or $E(a,b) = 0$, then it trivially holds that $E(a,b) \leq \tau(a(\circ_R^r \cup \diamond_R^i \cup \delta)b)$. If $a \neq b$ and $E(a,b) > 0$, then we need to prove that $\tau(a(\circ_R^r \cup \diamond_R^i)b) = 1$, i.e. it holds that $(aRb \wedge bRa \wedge aRa \wedge bRb)$ or $(a \parallel b \wedge aR^ca \wedge bR^cb)$, as well as $a \approx_R b$.

(i) First, we prove that $(aRb \wedge bRa \wedge aRa \wedge bRb)$ or $(a \parallel b \wedge aR^ca \wedge bR^cb)$.
On the one hand, since E is symmetric and R is compatible with E, it follows that

$$\tau(aRb) * \underbrace{E(a,b)}_{>0} * \underbrace{E(b,a)}_{>0} \leq \tau(bRa)\,.$$

This implies that $\tau(aRb) \leq \tau(bRa)$. On the other hand, it follows that

$$\tau(bRa) * \underbrace{E(b,a)}_{>0} * \underbrace{E(a,b)}_{>0} \leq \tau(aRb)\,,$$

whence also $\tau(bRa) \leq \tau(aRb)$. Therefore, $\tau(aRb) = \tau(bRa)$, i.e. $(aRb \wedge bRa)$ or $a \parallel b$.

(a) If $(aRb \wedge bRa)$, then we need to prove that $(aRa \wedge bRb)$.
On the one hand, since E is reflexive and R is compatible with E, it follows that

$$\underbrace{\tau(aRb)}_{=1} * \underbrace{E(a,a)}_{=1} * \underbrace{E(b,a)}_{>0} \leq \tau(aRa)\,.$$

This implies that $\tau(aRa) = 1$, i.e. aRa. On the other hand, it follows that

$$\underbrace{\tau(bRa)}_{=1} * \underbrace{E(b,b)}_{=1} * \underbrace{E(a,b)}_{>0} \leq \tau(bRb)\,,$$

whence also bRb.

(b) If $a \parallel b$, then, in an analogous way, we can prove that $(aR^ca \wedge bR^cb)$.

We conclude that $(aRb \wedge bRa \wedge aRa \wedge bRb)$ or $(a \parallel b \wedge aR^ca \wedge bR^cb)$.

(ii) Second, it remains to prove that $a \approx_R b$. Let $c \in X \setminus \{a,b\}$. We distinguish four subcases.

(a) If aRc, then since E is reflexive and R is compatible with E, it follows that

$$0 < E(a,b) = \underbrace{\tau(aRc)}_{=1} * \underbrace{E(a,b)}_{>0} * \underbrace{E(c,c)}_{=1} \leq \tau(bRc)\,.$$

This implies that $\tau(bRc) > 0$, whence bRc.

(b) If bRc, then since E is reflexive, symmetric and R is compatible with E, it follows that

$$0 < E(a,b) = \underbrace{\tau(bRc)}_{=1} * \underbrace{E(b,a)}_{=E(a,b)>0} * \underbrace{E(c,c)}_{=1} \leq \tau(aRc)\,.$$

This implies that $\tau(aRc) > 0$, whence aRc.

(c) If cRa, then since E is reflexive and R is compatible with E, it follows that

$$0 < E(a,b) = \underbrace{\tau(cRa)}_{=1} * \underbrace{E(c,c)}_{=1} * \underbrace{E(a,b)}_{>0} \leq \tau(cRb)\,.$$

This implies that $\tau(cRb) > 0$, whence cRb.

(d) If cRb, then since E is reflexive, symmetric and R is compatible with E, it follows that

$$0 < E(a,b) = \underbrace{\tau(cRb)}_{=1} * \underbrace{E(c,c)}_{=1} * \underbrace{E(b,a)}_{=E(a,b)>0} \leq \tau(cRa)\,.$$

This implies that $\tau(cRa) > 0$, whence cRa.

We conclude that $a \approx_R b$

Finally, we conclude that $E \subseteq \circ_R^r \cup \diamond_R^i \cup \delta$.

Conversely, if $E \subseteq \circ_R^r \cup \diamond_R^i \cup \delta$, then due to the monotonicity of $*$, it holds that

$$\tau(xRy) * E(x,z) * E(y,t) \leq \tau(xRy) * \tau(x(\circ_R^r \cup \diamond_R^i \cup \delta)z) * \tau(y(\circ_R^r \cup \diamond_R^i \cup \delta)t)\,,$$

for any $x,y,z,t \in X$. From Corollary 4.4, it follows that

$$\tau(xRy) * E(x,z) * E(y,t) \leq \tau(zRt)\,,$$

for any $x,y,z,t \in X$. Hence, R is compatible with E. □

From Corollary 4.1 and Theorem 4.2, the following result is straightforward.

Corollary 4.6. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ_R^r$ and $\diamond_R^i$. Then it holds that $\circ_R^r \cup \diamond_R^i \cup \delta$ is the greatest L-tolerance relation on X that R is compatible with.*

Proposition 4.1 and Theorem 4.2 lead to the following corollary.

Corollary 4.7. *Let R be a relation on a set X, $\approx_R$ be the clone relation of R with corresponding $\circ_R^r$ and $\diamond_R^i$ and E be an L-tolerance relation on X. Then the following statements hold:*

(i) If R is reflexive, then R is compatible with E if and only if $E \subseteq \circ_R \cup \delta$.

(ii) If R is irreflexive, then R is compatible with E if and only if $E \subseteq \diamond_R \cup \delta$.

Combining Proposition 4.3 and Theorem 4.2, we obtain the following corollary.

Corollary 4.8. *Let R be a relation on a set X, $\approx_R$ be the clone relation of R with corresponding $\circ_R^r$ and $\diamond_R^i$ and E be an L-tolerance relation on X. If R is compatible with E, then it holds that the relations R^t, R^c and R^d are compatible with E.*

The following lemma will be useful in the proof of our representation theorem, i.e. the representation of the L-tolerance relations a given relation is compatible with. Its proof is straightforward.

Lemma 4.4. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ_R^r$ and $\diamond_R^i$. Let α and β be two L-tolerance relations on X such that $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$, then the union $E = \alpha \cup \beta$ is the L-tolerance relation on X given by*

$$E(x,y) = \begin{cases} 1 & \text{, if } x = y\,, \\ \alpha(x,y) & \text{, if } x \circ_R^r y\,, \\ \beta(x,y) & \text{, if } x \diamond_R^i y\,, \\ 0 & \text{, otherwise}\,. \end{cases}$$

Theorem 4.3. *Let R be a relation on a set X, $\approx_R$ be the clone relation of R with corresponding $\circ_R^r$ and $\diamond_R^i$ and E be an L-tolerance relation on X. Then it holds that R is compatible with E if and only if there exist two L-tolerance relations α and β on X with $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$ such that $E = \alpha \cup \beta$.*

Proof. Suppose that R is compatible with E. Consider the L-relations α and β on X defined by

$$\alpha(x,y) = \begin{cases} E(x,y) & , \text{ if } (x,y) \in \circ_R^r \cup \delta\,, \\ 0 & , \text{ otherwise}\,, \end{cases}$$

$$\beta(x,y) = \begin{cases} E(x,y) & , \text{ if } (x,y) \in \diamond_R^i \cup \delta\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

Note that if $(x,y) \in \circ_R^r \cup \delta$, then it also holds that $(y,x) \in \circ_R^r \cup \delta$. Similarly, if $(x,y) \in \diamond_R^i \cup \delta$, then also $(y,x) \in \diamond_R^i \cup \delta$. Hence, α and β are L-tolerance relations on X. Since R is compatible with E, it follows from Theorem 4.2 that $E \subseteq \circ_R^r \cup \diamond_R^i \cup \delta$, whence $E(x,y) = 0$ if $x(\circ_R^r \cup \diamond_R^i \cup \delta)^c\, y$. Hence, it follows that $E = \alpha \cup \beta$.

Conversely, let $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$ be two L-tolerance relations on X. Lemma 4.4 implies that $E = \alpha \cup \beta$ is an L-tolerance relation on X. Since $E \subseteq \circ_R^r \cup \diamond_R^i \cup \delta$, Theorem 4.2 guarantees that R is compatible with E. ☐

As a corollary, we obtain the following representation of the crisp tolerance relations a given relation is compatible with.

Corollary 4.9. *Let R be a relation on a set X, $\approx_R$ be the clone relation of R with corresponding $\circ_R^r$ and $\diamond_R^i$ and E be a tolerance relation on X. Then it holds that R is compatible with E if and only there exist two L-tolerance relations α and β on X with $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$ such that $E = \alpha \cup \beta$.*

Remark 4.2. *In the setting of Corollary 4.9, we have:*

(i) If $\alpha = \delta$ and $\beta = \delta$, then $E = \delta$.

(ii) If $\alpha = \circ_R^r \cup \delta$ and $\beta = \diamond_R^i \cup \delta$, then $E = \circ_R^r \cup \diamond_R^i \cup \delta$.

Combining Proposition 4.1 and Theorem 4.3, we obtain the following corollaries.

Corollary 4.10. *Let R be a reflexive relation on a set X, $\approx_R$ be the clone relation of R with corresponding $\circ_R^r$ and E be an L-tolerance relation on X. Then it holds that R is compatible with E if and only if there exists an L-tolerance relation α*

on X with $\alpha \subseteq \circ_R \cup \delta$ such that

$$E(x,y) = \begin{cases} \alpha(x,y) & \text{, if } (x,y) \in \circ_R \cup \delta\,, \\ 0 & \text{, otherwise}\,. \end{cases}$$

Corollary 4.11. *Let R be an irreflexive relation on a set X, $\approx_R$ be the clone relation of R with corresponding $\diamond_R^i$ and E be an L-tolerance relation on X. Then it holds that R is compatible with E if and only if there exists an L-tolerance relation β on X with $\beta \subseteq \diamond_R \cup \delta$ such that*

$$E(x,y) = \begin{cases} \beta(x,y) & \text{, if } (x,y) \in \diamond_R \cup \delta\,, \\ 0 & \text{, otherwise}\,. \end{cases}$$

Corollary 4.12. *Let $(X, \leqslant)$ be a poset, $\approx$ be the clone relation of the strict order relation corresponds the order relation $\leqslant$ and E be an L-tolerance relation on X. Then it holds that $\leqslant$ is compatible with E if and only if there exists an L-tolerance relation β on X with $\beta \subseteq \diamond \cup \delta$ such that*

$$E(x,y) = \begin{cases} \beta(x,y) & \text{, if } (x,y) \in \diamond \cup \delta\,, \\ 0 & \text{, otherwise}\,. \end{cases}$$

4.3.2. Compatibility of a relation with an L-equivalence relation

In this subsection, we characterize the L-equivalence relations a given relation is compatible with.

Proposition 4.7. *Let R be a relation on a set X and $\approx_R$ be the clone relation of R with corresponding $\circ_R^r$ and $\diamond_R^i$. Let α and β two L-equivalence relations on X such that $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$, then the union $E = \alpha \cup \beta$ is an L-equivalence relation on X.*

Proof. Due to Lemma 4.4, E is an L-tolerance relation. It remains to show that E is $*$-transitive. Let $x, y, z \in X$, then we need to show that

$$E(x,y) * E(y,z) \leq E(x,z)\,.$$

Let $x, y, x \in X$ such that $E(x,y) > 0$ and $E(y,z) > 0$. We consider the following cases.

(i) If $x = y$ or $y = z$ or $x = z$, then the inequality $E(x,y) * E(y,z) \leq E(x,z)$ trivially holds.

(ii) Suppose that $x \neq y$, $y \neq z$ and $x \neq z$. Since $E \subseteq \circ^r_R \cup \diamond^i_R$, it follows that $(x \circ^r_R y \wedge y \circ^r_R z)$ or $(x \circ^r_R y \wedge y \diamond^i_R z)$ or $(x \diamond^i_R y \wedge y \circ^r_R z)$ or $(x \diamond^i_R y \wedge \diamond^i_R z)$.

(a) From the definition of $\circ^r_R$ and $\diamond^i_R$, it follows that the cases $(x \circ^r_R y \wedge y \diamond^i_R z)$ and $(x \diamond^i_R y \wedge y \circ^r_R z)$ are impossible. Otherwise, it would follow that $(yRy \wedge yR^c y)$, a contradiction.

(b) If $(x \circ^r_R y \wedge y \circ^r_R z)$, then it follows that $x \circ^r_R y, y \approx_R z$, zRz and $z \in X \setminus \{x, y\}$. From Proposition 4.4, it follows that $x \circ^r_R z$. Hence, $E(x, y) = \alpha(x, y)$, $E(y, z) = \alpha(y, z)$ and $E(x, z) = \alpha(x, z)$. Since α is $*$-transitive, it holds that $\alpha(x, y) * \alpha(y, z) \leq \alpha(x, z)$, i.e., $E(x, y) * E(y, z) \leq E(x, z)$.

(c) If $(x \diamond^i_R y \wedge y \diamond^i_R z)$, then it follows that $x \diamond^i_R y$, $y \approx_R z$, $zR^c z$ and $z \in X \setminus \{x, y\}$. From Proposition 4.4, it follows that $x \diamond^i_R z$. Hence, $E(x, y) = \beta(x, y$, $E(y, z) = \beta(y, z)$ and $E(x, z) = \beta(x, z)$. Since β is $*$-transitive, it holds that $\beta(x, y) * \beta(y, z) \leq \beta(x, z)$, i.e., $E(x, y) * E(y, z) \leq E(x, z)$.

We conclude that $E = \alpha \cup \beta$ is an L-equivalence relation on X. □

Combining Theorem 4.3 and Proposition 4.7 easily leads to the following theorem.

Theorem 4.4. *Let R be a relation on a set X, $\approx_R$ be the clone relation of R with corresponding $\circ^r_R$ and $\diamond^i_R$ and E be an L-equivalence relation on X. Then it holds that R is compatible with E if and only if there exist two L-equivalence relations α and β on X with $\alpha \subseteq \circ^r_R \cup \delta$ and $\beta \subseteq \diamond^i_R \cup \delta$ such that $E = \alpha \cup \beta$.*

Proof. Theorem 4.3 states that R is compatible with E if and only if there exist two L-tolerance relations α and β on X with $\alpha \subseteq \circ^r_R \cup \delta$ and $\beta \subseteq \diamond^i_R \cup \delta$ such that $E = \alpha \cup \beta$, where

$$\alpha(x, y) = \begin{cases} E(x, y) & , \text{ if } (x, y) \in \circ^r_R \cup \delta\,, \\ 0 & , \text{ otherwise}\,, \end{cases}$$

$$\beta(x, y) = \begin{cases} E(x, y) & , \text{ if } (x, y) \in \diamond^i_R \cup \delta\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

Let $x, y, z \in X$, then we need to show that

$$\alpha(x, y) * \alpha(y, z) \leq \alpha(x, z)$$

and

$$\beta(x, y) * \beta(y, z) \leq \beta(x, z)\,.$$

(i) First, we prove that $\alpha(x, y) * \alpha(y, z) \leq \alpha(x, z)$.

   (a) If $x = y$ or $y = z$ or $x = z$, then this inequality trivially holds.

   (b) Suppose that $x \neq y$, $y \neq z$ and $x \neq z$.

      (1) If $x(\circ_R^r)^c y$ or $y(\circ_R^r)^c z$, then it follows that $\alpha(x, y) = 0$ or $\alpha(y, z) = 0$. Hence, it holds that $\alpha(x, y) * \alpha(y, z) \leq \alpha(x, z)$.

      (2) If $x \circ_R^r y$ and $y \circ_R^r z$, then it follows that $x \circ_R^r y$, $y \approx_R z$, zRz and $z \in X \setminus \{x, y\}$. From Proposition 4.4, it follows that $x \circ_R^r z$. As $x \circ_R^r y$, $y \circ_R^r z$ and $x \circ_R^r z$, it follows that $\alpha(x, y) = E(x, y)$, $\alpha(y, z) = E(y, z)$ and $\alpha(x, z) = E(x, z)$. Since E is $*$-transitive, it holds that $\alpha(x, y) * \alpha(y, z) \leq \alpha(x, z)$.

   Thus, α is $*$-transitive.

(ii) Second, we prove that $\beta(x, y) * \beta(y, z) \leq \beta(x, z)$.

   (a) If $x = y$ or $y = z$ or $x = z$, then this inequality trivially holds.

   (b) Suppose that $x \neq y$, $y \neq z$ and $x \neq z$.

      (1) If $x(\diamond_R^i)^c y$ or $y(\diamond_R^i)^c z$, then it follows that $\beta(x, y) = 0$ or $\beta(y, z) = 0$. Hence, it holds that $\beta(x, y) * \beta(y, z) \leq \beta(x, z)$.

      (2) If $x \diamond_R^i y$ and $y \diamond_R^i z$, then it follows that $x \diamond_R^r y$, $y \approx z$, $zR^c z$ and $z \in X \setminus \{x, y\}$. From Proposition 4.4, it follows that $x \diamond_R^i z$. As $x \diamond_R^i y$, $y \diamond_R^i z$ and $x \diamond_R^i z$, it follows that $\beta(x, y) = E(x, y)$, $\beta(y, z) = E(y, z)$ and $\beta(x, z) = E(x, z)$. Since E is $*$-transitive, it holds that $\beta(x, y) * \beta(y, z) \leq \beta(x, z)$.

   Thus, β is $*$-transitive.

We conclude that α and β are L-equivalence relations on X.

For the converse, Proposition 4.7 guarantees that $E = \alpha \cup \beta$ is an L-equivalence relation on X. ☐

As a corollary, we obtain the following representation of the L-equality or equivalence relations a relation is compatible with.

Corollary 4.13. *Let R be a relation on a set X, $\approx_R$ be the clone relation of R with corresponding $\circ_R^r$ and $\diamond_R^i$ and E be an L-equality relation on X. Then it holds that R is compatible with E if and only if there exist two L-equality relations α and β on X with $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$ such that $E = \alpha \cup \beta$.*

Corollary 4.14. *Let R be a relation on a set X, $\approx_R$ be the clone relation of R with corresponding $\circ_R^r$ and $\diamond_R^i$ and E be an equivalence relation on X. Then it holds that R is compatible with E if and only if there exist two equivalence relations α and β on X with $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$ such that $E = \alpha \cup \beta$.*

From Proposition 4.1 and Theorem 4.4, we obtain the following representation of the L-equivalence relations that a reflexive or irreflexive relation is compatible with.

Corollary 4.15. *Let R be a reflexive relation on a set X, $\approx_R$ be the clone relation of R with corresponding $\circ_R^r$ and E be an L-equivalence relation on X. Then it holds that R is compatible with E if and only if there exists an L-equivalence relation α on X with $\alpha \subseteq \circ_R \cup \delta$ such that*

$$E(x,y) = \begin{cases} \alpha(x,y) & , \text{ if } (x,y) \in \circ_R \cup \delta\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

Corollary 4.16. *Let R be an irreflexive relation on a set X, $\approx_R$ be the clone relation of R with corresponding $\diamond_R^i$ and E be an L-equivalence relation on X. Then it holds that R is compatible with E if and only if there exists an L-equivalence relation β on X with $\beta \subseteq \diamond_R \cup \delta$ such that*

$$E(x,y) = \begin{cases} \beta(x,y) & , \text{ if } (x,y) \in \diamond_R \cup \delta\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

Corollary 4.17. *Let $(X, \leqslant)$ be a poset, $\approx$ be the clone relation of the strict order relation corresponds the order relation $\leqslant$ and E be an L-equivalence relation on X. Then it holds that $\leqslant$ is compatible with E if and only if there exists an L-equivalence relation β on X with $\beta \subseteq \diamond \cup \delta$ such that*

$$E(x,y) = \begin{cases} \beta(x,y) & , \text{ if } (x,y) \in \diamond \cup \delta\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

5 Compatibility of a fuzzy equivalence relation with an order relation

In previous chapter, we studied the notion of compatibility of crisp relations with a fuzzy relation, and provided a representation of all fuzzy tolerance and fuzzy equivalence relations that a given crisp relation is compatible with. This representation generalized the characterization of the fuzzy tolerance/equivalence relations that a strict order relation is compatible with, introduced in [32].

In this chapter, we aim to highlight other points related to this notion of compatibility. We study the compatibility of a fuzzy equivalence relation with an order relation, and provided a representation of all fuzzy equivalence relations compatible with a given order relation. It shows that under mild conditions, the compatibility of a fuzzy equivalence with an order relation is a trivial notion.

After providing that the three types of compatibility of any fuzzy equivalence relation with an order relation are equivalent. We introduce three auxiliary relations associated with a poset $(X, \leqslant)$. These auxiliary relations are respectively, the set of couple elements where their set of lower bound no empty, the couple of elements where their set of upper bound no empty and the couple of elements where their set of lower or upper bound no empty.
We study the compatibility of the order relation with corresponding its latter auxiliary relations. The results turn out to play a key role in the characterization of the fuzzy equivalence relations compatible with an order relation.

5.1. Basic results

In what follows, we will use the following lemma.

Lemma 5.1. [53] *For any two L-relations R_1 and R_2 on X, the following equivalences hold:*

(i) R_1 is left-compatible with R_2 if and only if R_1^t is right-compatible with R_2;

(ii) R_1 is right-compatible with R_2 if and only if R_1^t is left-compatible with R_2;

(iii) R_1 is compatible with R_2 if and only if R_1^t is compatible with R_2.

Note that the relation $R = X^2$ is left-, right-compatible and compatible with any L-relation on X.

The following proposition provides that the three types of compatibility (see, Definition 4.2) of any L-tolerance (and, in particular, L-equivalence) relation with an order relation are equivalent.

Proposition 5.1. *Let $(X, \leqslant)$ be a poset and E be an L-tolerance relation on X. Then the following statements are equivalent:*

(i) E is left-compatible with $\leqslant$;

(ii) E is right-compatible with $\leqslant$;

(iii) E is compatible with $\leqslant$.

Proof. $(i) \Rightarrow (ii)$: Since E is left-compatible with $\leqslant$, it follows from Lemma 5.1(i) that E^t is right-compatible with $\leqslant$. As E is symmetric, it holds that E is right-compatible with $\leqslant$.
$(ii) \Rightarrow (iii)$: Since E is symmetric and E is right-compatible with $\leqslant$, it follows that

$$\begin{aligned} E(x,y) * \tau(x \leqslant z) * \tau(y \leqslant t) &= E(y,x) * \tau(x \leqslant z) * \tau(y \leqslant t) \\ &\leq E(y,z) * \tau(y \leqslant t) \\ &= E(z,y) * \tau(y \leqslant t) \\ &\leq E(z,t)\,, \end{aligned}$$

for any $x, y, z \in X$. Hence, E is compatible with $\leqslant$.
$(iii) \Rightarrow (i)$: Follows from Lemma 4.2(ii). □

Since the three types of compatibility of any L-tolerance (and, in particular, L-equivalence) relation with an order relation are equivalent, in the following two sections, we only consider the compatibility while using the inequality

$$E(x,y) * \tau(x \leqslant z) \leq E(z,y)\ , \tag{5.1}$$

for any $x, y, z \in X$.

Moreover, the left-compatibility (resp. right-compatibility) of any L-relation with a given strict order relation $<$ is equivalent to the left-compatibility (resp. right-compatibility) with the corresponding order relation $\leqslant$.

Proposition 5.2. *Let $(X, \leqslant)$ be a poset, $<$ be the corresponding strict order relation and E be an L-relation on X. Then it holds that*

(i) E is left-compatible with $\leqslant$ if and only if E is left-compatible with $<$;

(ii) E is right-compatible with $\leqslant$ if and only if E is right-compatible with $<$;

(iii) E is compatible with $\leqslant$ if and only if E is left- and right-compatible with $<$.

Proof. Let $x, y, z \in X$

(i) Suppose that E is left-compatible with $\leqslant$. Since $\tau(x < z) \leq \tau(x \leqslant z)$, it holds that $E(x,y) * \tau(x < z) \leq E(x,y) * \tau(x \leqslant z)$. Hence, $E(x,y) * \tau(x <$

$z) \leq E(z,y)$. Thus, E is left-compatible with $<$.

Conversely, the fact that E is left-compatible with $<$ and with the crisp equality then implies that

$$E(x,y) * \tau(x \leqslant z) = E(x,y) * (\tau(x < z) \vee \tau(x = z)) \leq E(z,y)\,.$$

Thus, E is left-compatible with $\leqslant$.

(ii) Follows from Lemma 5.1 and (i).

(iii) Follows from Lemma 4.2, (i) and (ii).

□

5.2. Compatibility of an L-tolerance relation with an order relation

In this section, we will provide a characterization of the L-tolerance relations that are compatible with a given order relation. First, we introduce the following binary relations $\triangledown$ and $\vartriangle$ on X associated with the poset $(X, \leqslant)$:

$$\begin{aligned} \triangledown &= \{(x,y) \in X^2 \mid \{x,y\}^l \neq \emptyset\}\,,\\ \vartriangle &= \{(x,y) \in X^2 \mid \{x,y\}^u \neq \emptyset\}\,. \end{aligned}$$

Also, we will use the following notation:

$$\boxtimes = \triangledown \cup \vartriangle = \{(x,y) \in X^2 \mid \{x,y\}^l \neq \emptyset \vee \{x,y\}^u \neq \emptyset\}\,.$$

Obviously, it holds that $(\leqslant \cup \leqslant^t) \subseteq (\triangledown \cap \vartriangle)$. Clearly, $\triangledown$, $\vartriangle$ and $\boxtimes$ are tolerance relations.

Example 5.1. *Let $(X, \leqslant)$ be the poset given by the Hasse diagram in Figure 5.1.*

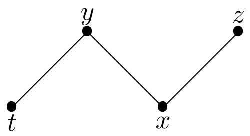

Figure 5.1: Hasse diagram of the poset $(X, \leqslant)$ with $X = \{x, y, z, t\}$.

It holds that

$$\begin{aligned}\triangledown &= \delta \cup \{(x,y),(y,x),(x,z),(z,x),(t,y),(y,t),(y,z),(z,y)\}\,,\\ \vartriangle &= \delta \cup \{(x,y),(y,x),(x,z),(z,x),(t,y),(y,t),(t,x),(x,t)\}\,.\end{aligned}$$

The following proposition is straightforward.

Proposition 5.3. *Let $(X, \leqslant)$ be a poset and consider the relations $\triangledown$, $\vartriangle$ and $\boxtimes$. Then it holds that*

(i) If $(X, \leqslant)$ is a $\wedge$-semi-lattice (resp. a $\vee$-semi-lattice), then $\triangledown = X^2$ (resp. $\vartriangle = X^2$).

(ii) If $(X, \leqslant)$ has a smallest (resp. a greatest) element, then $\triangledown = X^2$ (resp. $\vartriangle = X^2$).

(iii) If $(X, \leqslant)$ is a bounded poset or a lattice, then $\triangledown = \vartriangle = \boxtimes = X^2$.

Remark 5.1. *In general, the relations $\triangledown$, $\vartriangle$ and $\boxtimes$ are not transitive, as is the case for the poset of Example 5.1.*

The following proposition shows that the tolerance relation $\triangledown$ is compatible with $\leqslant$.

Proposition 5.4. *The relation $\triangledown$ associated with a poset $(X, \leqslant)$ is compatible with $\leqslant$.*

Proof. Let $x, y, z \in X$, then we need to prove that

$$\tau(x\triangledown y) * \tau(x \leqslant z) \leq \tau(z \triangledown y)\,.$$

If $x\triangledown^c y$, then this inequality trivially holds. If $x\triangledown y$, then there exists $c \in X$ such that $c \leqslant x$ and $c \leqslant y$. Hence,

$$\tau(x\triangledown y) * \tau(x \leqslant z) \leq (\tau(c \leqslant x) \wedge \tau(c \leqslant y)) * \tau(x \leqslant z)\,.$$

The fact that $* \leq \wedge$ implies that

$$(\tau(c \leqslant x) \wedge \tau(c \leqslant y)) * \tau(x \leqslant z) \leq \tau(c \leqslant z) \wedge \tau(c \leqslant y) \leq \tau(z\triangledown y)\,.$$

Hence,

$$\tau(x\triangledown y) * \tau(x \leqslant z) \leq \tau(z \triangledown y)\,.$$

Thus, $\triangledown$ is compatible with $\leqslant$. □

Remark 5.2. *The relations $\vartriangle$ and $\boxtimes$ associated with a poset $(X, \leqslant)$ are not necessarily compatible with $\leqslant$. Indeed, consider the poset of Example 5.1. It is* *clear that*

$$\underbrace{\tau(x \vartriangle t)}_{=1} * \underbrace{\tau(x \leqslant z)}_{=1} \nleq \underbrace{\tau(z \vartriangle t)}_{=0} .$$

This implies that $\vartriangle$ is not compatible with $\leqslant$. Similarly, it follows that $\boxtimes$ is not compatible with $\leqslant$.

The following proposition will be useful in the proof of our representation theorem of the L-tolerance relations that are compatible with a given order relation.

Proposition 5.5. *Let $(X, \leqslant)$ be a poset and E be an L-tolerance relation on X. If E is compatible with $\leqslant$, then $\triangledown \subseteq E$.*

Proof. Suppose that E is compatible with $\leqslant$ and consider $x, y \in X$ such that $x \triangledown y$. Then there exists $c \in X$ such that $c \leqslant x$ and $c \leqslant y$. Due to the compatibility of E with $\leqslant$ and the reflexivity of E, it holds that

$$\underbrace{E(c,c)}_{=1} * \underbrace{\tau(c \leqslant x)}_{=1} \leq E(c,x) .$$

Hence, $E(c,x) = 1$. Again, the symmetry of E and its compatibility with $\leqslant$ imply that

$$\underbrace{E(x,c)}_{=E(c,x)=1} * \underbrace{\tau(c \leqslant y)}_{=1} \leq E(x,y) .$$

Hence, $E(x,y) = 1$, and thus $\triangledown \subseteq E$. ☐

Since $\leqslant \subseteq \triangledown$, another necessary condition, but less demanding is given by $\leqslant \subseteq E$.

Proposition 5.5 states a necessary condition for the compatibility of an L-tolerance relation E with an order relation $\leqslant$. However, it is not a sufficient condition as can be seen in Example 5.2. In case E is an L-equivalence relation, we will show in the next section that this condition becomes necessary and sufficient.

Example 5.2. *Consider the poset of Example 5.1. Consider the tolerance relation E on X defined as: $E = X^2 \setminus \{(z,t),(t,z)\}$, then it holds that $\triangledown \subseteq E$. However, since*

$$\underbrace{\tau(E(x,t))}_{=1} * \underbrace{\tau(x \leqslant z)}_{=1} \nleq \underbrace{\tau(E(z,t))}_{=0} ,$$

it is clear that E is not compatible with $\leqslant$.

Also, we know that $\leqslant \cup \leqslant^t$ is a tolerance relation on X and $\leqslant \subseteq (\leqslant \cup \leqslant^t)$. However, since

$$\underbrace{\tau(x(\leqslant \cup \leqslant^t)y)}_{=1} * \underbrace{\tau(x \leqslant z)}_{=1} \nleq \underbrace{\tau(z(\leqslant \cup \leqslant^t)y)}_{=0} ,$$

it is clear that $\leqslant \cup \leqslant^t$ *is not compatible with* $\leqslant$.

Propositions 5.4 and 5.5 imply that the relation $\triangledown$ is the smallest L-tolerance relation on X that is compatible with $\leqslant$, while X^2 is the greatest one.

In view of Proposition 5.3, we obtain the following corollary. It shows that under mild conditions, the compatibility of an L-tolerance relation with an order relation is a trivial notion.

Corollary 5.1. *Let* $(X, \leqslant)$ *be a poset and* E *be an* L*-tolerance relation on* X. *If* $(X, \leqslant)$ *is a* $\wedge$*-semi-lattice or has a smallest element, then it holds that* E *is compatible with* $\leqslant$ *if and only if* $E = X^2$.

Definition 5.1. *Let* $(X, \leqslant)$ *be a poset,* Y *be a nonempty subset of* X^2 *and* E *be an* L*-relation on* X. E *is called increasing on* Y *if for any* (x, y), $(z, t) \in Y$ *such that* $x \leqslant z$ *and* $y \leqslant t$, *it holds that* $E(x, y) \leq E(z, t)$.

The following representation theorem characterizes the L-tolerance relations that are compatible with a given order relation.

Theorem 5.1. *Let* $(X, \leqslant)$ *be a poset and* E *be an* L*-tolerance relation on* X. *Then it holds that* E *is compatible with* $\leqslant$ *if and only if there exists an* L*-tolerance relation* α *on* X *with* $\alpha \subseteq \triangledown^c \cup \delta$ *such that* α *is increasing on* $\triangledown^c$ *and* $E = \triangledown \cup \alpha$.

Proof. Suppose that E is compatible with $\leqslant$, then it follows from Proposition 5.5(ii) that $\triangledown \subseteq E$. Consider the L-relation α on X defined by $\alpha = (E \setminus \triangledown) \cup \delta = (E \cap \triangledown^c) \cup \delta = E \cap (\triangledown^c \cup \delta)$, i.e.,

$$\alpha(x, y) = \begin{cases} E(x, y) & , \text{ if } (x, y) \in \triangledown^c \cup \delta\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

It is obvious that $\alpha \subseteq \triangledown^c \cup \delta$ and $E = \triangledown \cup \alpha$. It remains to show that α is an L-tolerance relation on X that is increasing on $\triangledown^c$. Note that if $(x, y) \in \triangledown^c \cup \delta$, then it also holds that $(y, x) \in \triangledown^c \cup \delta$. Hence, α is an L-tolerance relation on X. Since α is symmetric, in order to show that α is increasing on $\triangledown^c$, it suffices to show that $x \leqslant z$ implies that $\alpha(x, y) \leq \alpha(z, y)$, for any (x, y), $(z, y) \in \triangledown^c$. Let (x, y), $(z, y) \in \triangledown^c$ such that $x \leqslant z$, then it holds that $\alpha(x, y) = E(x, y)$ and $\alpha(z, y) = E(z, y)$. The compatibility of E with $\leqslant$ implies that

$$E(x, y) * \underbrace{\tau(x \leqslant z)}_{=1} \leq E(z, y)\,.$$

Hence, $\alpha(x, y) \leq \alpha(z, y)$.

Conversely, let α be an L-tolerance relation on X with $\alpha \subseteq \triangledown^c \cup \delta$ such that α is increasing on $\triangledown^c$ and $E = \triangledown \cup \alpha$. Since $\triangledown$ and α are L-tolerance relations on X, it holds that E is an L-tolerance relation on X. It remains to show that E is compatible with $\leqslant$, i.e.,

$$E(x,y) * \tau(x \leqslant z) \leq E(z,y)\,,$$

for any $x, y, z \in X$. We consider the following cases:

(i) The case $x \triangledown y$, i.e., $E(x,y) = \tau(x \triangledown y)$. It then follows from the compatibility of $\triangledown$ with $\leqslant$ (see Proposition 5.4) that

$$\begin{aligned} E(x,y) * \tau(x \leqslant z) &= \tau(x \triangledown y) * \tau(x \leqslant z) \\ &\leq \tau(z \triangledown y) \\ &\leq E(z,y)\,. \end{aligned}$$

(ii) The case $x \triangledown^c y$, i.e., $E(x,y) = \alpha(x,y)$. We consider two subcases:

  (a) The case $z \triangledown y$, i.e., $E(z,y) = \tau(z \triangledown y) = 1$. It then trivially holds that

$$E(x,y) * \tau(x \leqslant z) \leq E(z,y) = 1\,.$$

  (b) The case $z \triangledown^c y$, i.e., $E(z,y) = \alpha(z,y)$. If $\neg(x \leqslant z)$, then it trivially holds that

$$E(x,y) * \tau(x \leqslant z) = 0 \leq E(z,y)\,.$$

  If $x \leqslant z$, then the fact that α is increasing on $\triangledown^c$ implies that

$$\begin{aligned} E(x,y) * \tau(x \leqslant z) &= \alpha(x,y) * \tau(x \leqslant z) \\ &\leq \alpha(z,y) = E(z,y)\,. \end{aligned}$$

□

5.3. Compatibility of an L-equivalence relation with an order relation

In this section, we will provide a characterization of the L-equivalence relations that are compatible with a given order relation.

Theorem 5.2. *Let $(X, \leqslant)$ be a poset and E be an L-equivalence relation on X. Then the following statements are equivalent:*

(i) E is compatible with $\leqslant$;

(ii) $\leqslant \subseteq E$;

(iii) $\triangledown \subseteq E$;

(iv) $\vartriangle \subseteq E$*;*

(v) $\boxtimes \subseteq E$*.*

Proof. $(i) \Rightarrow (ii)$: Since E is compatible with $\leqslant$, it follows from Proposition 5.5 that $\triangledown \subseteq E$. Since $\leqslant \, \subseteq \triangledown$, it holds that $\leqslant \, \subseteq E$.
$(ii) \Rightarrow (iii)$: Let $x, y \in X$ such that $x \triangledown y$, then there exists $c \in X$ such that $c \leqslant x$ and $c \leqslant y$. Since $\leqslant \, \subseteq E$, it follows that $E(c,x) = E(c,y) = 1$. The symmetry and $*$-transitivity of E then imply that

$$\underbrace{E(c,x)}_{=1} * \underbrace{E(c,y)}_{=1} = E(x,c) * E(c,y) \leq E(x,y)\,.$$

Hence, $E(x,y) = 1$. Thus, $\triangledown \subseteq E$.
$(iii) \Rightarrow (iv)$: Let $x, y \in X$ such that $x \vartriangle y$, then there exists $c \in X$ such that $x \leqslant c$ and $y \leqslant c$. This implies that $x \triangledown c$ and $y \triangledown c$. Since $\triangledown \subseteq E$, it follows that $E(x,c) = E(y,c) = 1$. The symmetry and $*$-transitivity of E then imply that

$$\underbrace{E(x,c)}_{=1} * \underbrace{E(y,c)}_{=1} = E(x,c) * E(c,y) \leq E(x,y)\,.$$

Hence, $E(x,y) = 1$. Thus, $\vartriangle \subseteq E$.
$(iv) \Rightarrow (v)$: Let $x, y \in X$ such that $x \boxtimes y$, then it holds that $x \triangledown y$ or $x \vartriangle y$.

(a) If $x \vartriangle y$, then from the hypothesis $\vartriangle \subseteq E$ it follows that $E(x,y) = 1$.

(b) If $x \triangledown y$, then there exists $c \in X$ such that $c \leqslant x$ and $c \leqslant y$. This implies that $x \vartriangle c$ and $y \vartriangle c$. Since $\vartriangle \subseteq E$, it follows that $E(x,c) = E(y,c) = 1$. The symmetry and $*$-transitivity of E then imply that

$$\underbrace{E(x,c)}_{=1} * \underbrace{E(y,c)}_{=1} = E(x,c) * E(c,y) \leq E(x,y)\,.$$

Hence, $E(x,y) = 1$.

We conclude that $\boxtimes \subseteq E$.

$(v) \Rightarrow (i)$: Since $\leqslant \subseteq \boxtimes \subseteq E$, it follows that

$$E(x,y) * \tau(x \leqslant z) \leq E(x,y) * E(x,z)\,.$$

By the symmetry and $*$-transitivity of E, it follows that $E(x,y) * \tau(x \leqslant z) \leq E(z,y)$. Thus, E is compatible with $\leqslant$. □

In view of Proposition 5.3, we obtain the following corollary. In addition to the case of L-tolerance relations, it shows that there exist several other mild conditions under which the compatibility of an L-equivalence relation with an order relation is a trivial notion.

Corollary 5.2. *Let $(X,\leqslant)$ be a poset and E be an L-equivalence relation on X . If at least one of the following conditions is satisfied:*

(i) *$(X,\leqslant)$ is a $\wedge$-semi lattice;*

(ii) *$(X,\leqslant)$ is a $\vee$-semi lattice;*

(iii) *$(X,\leqslant)$ has a smallest element;*

(iv) *$(X,\leqslant)$ has a greatest element,*

then it holds that E is compatible with $\leqslant$ if and only if $E = X^2$.

For a relation R on a set X, let R^* denote its transitive closure, i.e., the smallest transitive relation on X that contains R. The transitive closure of a reflexive (resp. symmetric) relation is reflexive (resp. symmetric) as well.

As a consequence of Theorem 5.2, the following proposition shows that the transitive closures $\triangledown^*$, $\triangle^*$ and $\boxtimes^*$ coincide and are compatible with $\leqslant$.

Proposition 5.6. *Let $(X,\leqslant)$ be a poset, then it holds that the transitive closures $\triangledown^*$, $\triangle^*$ and $\boxtimes^*$ coincide and are compatible with $\leqslant$.*

Proof. First, note that the transitive closures $\triangledown^*$, $\triangle^*$ and $\boxtimes^*$ are equivalence relations on X. Next, we show that $\triangledown^* = \boxtimes^*$. Since $\triangledown \subseteq \triangledown^*$, it follows from Theorem 5.2 that $\triangledown^*$ is compatible with $\leqslant$. Again, from Theorem 5.2 it follows that $\boxtimes \subseteq \triangledown^*$. The fact that $\boxtimes^*$ is the smallest transitive relation containing $\boxtimes$ then implies that $\boxtimes^* \subseteq \triangledown^*$. Conversely, since $\triangledown \subseteq \boxtimes$, it holds that $\triangledown^* \subseteq \boxtimes^*$. Thus, $\triangledown^* = \boxtimes^*$. In similar way, we show that $\triangle^* = \boxtimes^*$. Finally, Theorem 5.2 guarantees that the relation $E = \triangledown^* = \triangle^* = \boxtimes^*$ is compatible with $\leqslant$. $\square$

General conclusions and future research

In this thesis, we have extended the notion of clone relation of a strict order relation to an arbitrary binary relation. Throughout this work, the basic properties of this clone relation have been analysed. We have also proposed a partition of the clone relation in terms of three different types of pairs of clones. One type of pairs of clones leads to an antitransitive relation, while both the two other types of pairs of clones lead to a transitive relation. This partition of the clone relation has not only been an important tool in the proofs of this work, but it also helps to gain a deeper understanding of the structure of the clone relation and it will be a key element in future work. The clone relation of the three different types of disjoint union has been characterized. We have investigated the most important properties of the clonal sets of a given binary relation, as well as we have provided that the set of all clonal sets of a binary relation is a complete lattice with the usual intersection and a clonal closure union.

In this work, we have tackled and solved the general problem of characterizing the L-tolerance and L-equivalence relations a given relation is compatible with. To that end, we have expanded our knowledge on the clone relation of a relation by studying two important subrelations of the partition of this clone relation, informally described as reflexive related clones and irreflexive unrelated clones. Also, we have studied the compatibility of a L-equivalence relations relation with an order relation, and we have provided a representation of all fuzzy equivalence relations compatible with a given order relation.

Future work is anticipated in multiple directions. First, we will extend the clone relation of a binary relation to fuzzy relations. In this context, connections with the field of fuzzy preference modelling, in particular the study of additive fuzzy preference structures [28, 72], will be explored. Second, we will extend the clonal set of binary relation to fuzzy clonal set. Third, the future work will be directed towards the characterization of the L-tolerance and L-equivalence relations a given L-relation is compatible with. This requires a lot of preparatory work, in particular the proper generalization of the notion of clone relation from crisp relations to L-relations.

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ملخص المذكـــــــــرة بالعربية.

الهدف الرئيسي من هذا العمل هو تحليل خصائص علاقة clone على مجموعة وتمديدها الى مجموعة Clonal كدراسة نظرية ، أما الجانب التطبيقي لهذه النتائج هو استغلالها من أجل حل المشكل العام لتمييز وتمثيل الـ L-علاقات التكافؤ الضبابية (المشوشة) (L- équivalence relations fuzzy) المعرفة على بنية جبرية تسمى residuated lattices و التي تكون علاقة ثنائية معطاة متوافقة معها. كما اننا قمنا بدراسة عكسية لتوافق هذه العلاقات الضبابية مع علاقة ترتيب كيفية و تمييز كل علاقات التكافؤ الضبابية التي تكون متوافقة مع علاقة ترتيب معطاة.

في هذه الأطروحة قمنا بتمديد مفهوم علاقة Clone من علاقة ترتيب صارم التي قدمها De Baets وآخرون إلى علاقة ثنائية كيفية. طوال هذا العمل تم تحليل الخصائص الأساسية لهذه العلاقة واقترحنا أيضا تجزئة لهذه العلاقة من حيث ثلاثة أنواع مختلفة من أزواجclones ، نوع واحد من أزواج clones يؤدي إلى علاقة ضد متعديه في حين أن كلا النوعين الآخرين من أزواج من clones تؤدي إلى علاقة متعدية. هذا التقسيم لعلاقة clone لم يكن فقط أداة مهمة في براهين نظرياتنا الأساسية في هذا العمل ولكنه يساعد أيضا على اكتساب فهم أعمق لهيكل علاقة clone وسيكون عنصرا أساسيا في العمل في المستقبل خاصة في دراسة توافق العلاقات الضبابية التي تعتبر تعميما مهما لـ لتمديد توافق الدوال (extensionality) المقدمة من قبل Hohle and Blanchard.

في هذا العمل قمنا بمعالجة وحل المشكلة العامة المتمثلة في تخصيص وتمييز الـ-Lعلاقات التكافؤ الضبابية (المشوشة) والتي تكون علاقة ثنائية معطاة متوافقة معها، تحقيقا لهذا الغاية قمنا بتوسيع معرفتنا على علاقة clone من خلال دراسة اثنين من العلاقات الفرعية الهامة لتقسيم هذه العلاقة هما على الترتيب علاقة clone ذات الصلة المرتبطة الانعكاسية وعلاقة clone غير مترابطة وغير انعكاسية. أيضا لقد درسنا توافق-L-علاقات التكافؤ الضبابية المتوافقة مع علاقة الترتيب. أخيرا قدمنا ملخص عام مع بعض الأفاق المستقبلية.

كلمات مفتاحيه: العلاقة ألثنائية علاقة ترتيب (صارمة) ، العلاقة Clone، مجموعةClonal ، الإتحاد المنفصل لعلاقتين، شبكية كاملة، residuated lattice ، t- معيار، العلاقات الضبابية، علاقة التكافؤ الضبابية، توافق العلاقات.

Résumé de la thèse en français.

Le but principal de ce travail est d'analyser les propriétés de la relation clone et l'ensemble clonal d'une relation binaire du côté théorique, afin de résoudre le problème général de caractérisation des relations L-équivalence floues définies sur une residuated lattice une relation binaire donnée est compatible avec (Le côté application). En outre, nous visons à fournir une représentation de toutes les relations L-tolerence floues et L-équivalence floues sont compatibles avec une relation d'ordre donnée.

Dans cette thèse, nous avons étendu la notion de relation clone d'une relation d'ordre stricte introduite par De Baets et al. a une relation binaire arbitraire. Les propriétés basiques de cette relation de clone ont été analysées. Nous avons également proposé une partition de la relation de clone en termes de trois types différents de paires de clones. Un type de paires de clones conduit à une relation anti transitive, tandis que les deux autres types de paires de clones conduisent à une relation transitive. Cette partition de la relation de clone a non seulement été un outil important dans les preuves de ce travail, mais elle permet également de mieux comprendre la structure de la relation de clone et ce sera un élément clé dans les travaux futurs particulier dans l'étude des compatible des relations floues. La relation clone des trois types différents d'union disjointe a été caractérisée. Dans ce travail, nous avons abordé et résolu le problème général de la caractérisation des relations L-équivalence, une relation donnée sont compatible avec. À cette fin, nous avons développé nos connaissances sur la relation clone d'une relation en étudiant deux auxiliaires importantes relations de la partition de cette relation de clone (clones liés réflexive et clones non reliés non réflexive). En outre, nous avons étudié la compatibilité d'une relation L-équivalence avec une relation d'ordre et nous avons fourni une représentation de toutes les relations L-équivalence floues compatibles avec une relation d'ordre donnée. Enfin, des conclusions générales et des recherches futures sont tirées.

Mots clés: Relation binaire, relation d'ordre (stricte), relation de clone, ensemble clonal, union disjointe, treillis complet, residuated lattice, T-norme, relation d'équivalence L-floue, relation de tolérance L-flou, Opération de fermeture, compatibilité.

Abstract of the dissertation in English.

The main aim of this work is to analyze the properties of the clone relation and clonal set of a binary relation on the theoretical side in order to solve the general problem of characterizing the L-fuzzy equivalence relations defined on resituated lattice a given relation is compatible with (◌ِthe application side). Also, we aim to provide a representation of all L-fuzzy equivalence relations compatible with a given order relation.

In this thesis, we have extended the notion of clone relation of a strict order relation introduced by De Baets et al. to an arbitrary binary relation. Throughout this work, the basic properties of this clone relation have been analysed. We have also proposed a partition of the clone relation in terms of three different types of pairs of clones. One type of pairs of clones leads to an antitransitive relation, while both the two other types of pairs of clones lead to a transitive relation. This partition of the clone relation has not only been an important tool in the proofs of this work, but it also helps to gain a deeper understanding of the structure of the clone relation and it will be a key element in future work especially in the study of the compatibility of fuzzy relations. . The clone relation of the three different types of disjoint union has been characterized. In this work, we have tackled and solved the general problem of characterizing the L-equivalence relations a given relation is compatible with. To that end, we have expanded our knowledge on the clone relation of a relation by studying two important subrelations of the partition of this clone relation, informally described as reflexive related clones and irreflexive unrelated clones. Also, we have studied the compatibility of a L-equivalence relations relation with an order relation, and we have provided a representation of all fuzzy equivalence relations compatible with a given order relation. Finally, general conclusions and future research are drawn.

Keywords: Crisp relation, order (strict) relation, Clone relation, Clonal set, Disjoint union, Complete lattice, residuated lattice, T-norm, L-fuzzy equivalence relation, L-fuzzy tolerance relation, Closure operation, Compatibility.