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Theme

On Some Elliptic Equations With $W_0^{1,1}(\Omega)$ *Solutions*

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Dedication

Dear Family,

As I stand on the threshold of graduation, my heart is filled with immense appreciation for each of you. Your love, encouragement, and constant presence have been the foundation of my strength throughout this journey.

This achievement is not mine aloneâ€”it belongs to all of us.

Your belief in me never wavered, even when I doubted myself. For that, I am forever grateful.

I proudly share this milestone with you, and I dedicate this special day to the family that made it all possible.

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Abstract

This work investigates the regularizing effects of lower-order terms in nonlinear Dirichlet problems of the form:

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + H(x,u,\nabla u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain, $1 < p \leq N$, and f has poor summability. We demonstrate how lower-order terms can enhance solution regularity, particularly when $f \in L^1(\Omega)$ or other Lebesgue spaces.

According to the work [8], this study focuses on four principal cases:

(A) For $H(x,u,\nabla u) = u|u|^{r-2}$, we establish existence of weak solutions in $W_0^{1,2}(\Omega)$ even when $f \in L^1(\Omega)$

(B) With polynomial nonlinearities, we prove existence of distributional solutions in $W_0^{1,1}(\Omega)$ for $f \in L^{r'/p}(\Omega)$ $(1 < p \leq r')$

(C) For gradient-dependent terms $H(x,u,\nabla u) = u|u|^{r-2}|\nabla u|$, we obtain solutions in $W_0^{1,1}(\Omega) \cap L^{r-1}(\Omega)$ when $f \in L^1(\Omega)$ and $1 < r \leq \frac{N-1}{N(p-1)}$

(D) We compare these results with the semilinear case $(p = 2)$, highlighting differences in regularization mechanisms

The analysis employs a unified three-step approach: (1) approximation by regular problems, (2) derivation of a priori estimates in $W_0^{1,1}(\Omega)$, and (3) passage to the limit. Our results significantly extend previous work by demonstrating existence in borderline cases where the unperturbed problem $(H = 0)$ admits no solutions. The findings have important implications for understanding nonlinear elliptic equations with non-regular data.

Key words: Nonlinear Dirichlet problem, Existence, Regularity, Regularizing effects, Non-regular data, $W_0^{1,1}(\Omega)$.

Résumé

Ce travail étudie les effets régularisants des termes d´ordre inférieur dans les problémes de Dirichlet non linéaires de la forme :

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + H(x,u,\nabla u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

oú $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ est un domaine borné, $1 < p \leq N$, et f le terme source a une sommabilité limitée. Nous montrons comment certains termes non linéaires d´ordre inférieur peuvent améliorer la régularité des solutions, en particulier lorsque $f \in L^1(\Omega)$ ou appartient á d´autres espaces de Lebesgue.

En nous appuyant sur le cadre développé dans [8], nous considérons quatre cas représentatifs :

(A) Lorsque $H(x,u,\nabla u) = u|u|^{r-2}$, nous établissons l´existence de solutions faibles dans $W_0^{1,2}(\Omega)$, méme pour $f \in L^1(\Omega)$

(B) Pour des non-linéarités polynomiales, nous prouvons l´existence de solutions distributionnelles dans $W_0^{1,1}(\Omega)$, sous la condition $f \in L^{r'/p}(\Omega)$, avec $1 < p \leq r'$.

(C) Lorsque le terme d´ordre inférieur dépend á la fois de la solution et de son gradient, c´est-á-dire $H(x,u,\nabla u) = u|u|^{r-2}|\nabla u|$, nous obtenons des solutions dans l´espace $W_0^{1,1}(\Omega) \cap L^{r-1}(\Omega)$, pour des données $f \in L^1(\Omega)$ et des exposants tels que $1 < r \leq \frac{N-1}{N(p-1)}$.

(D) Nous comparons ces résultats au cas semi-linéaire $(p = 2)$, en mettant en évidence les différences fondamentales dans les mécanismes de régularisation.

Notre analyse suit une méthodologie unifiée en trois étapes :

1. Approximation par des problémes régularisés,
2. Obtention d´estimations a priori dans $W_0^{1,1}(\Omega)$,
3. Passage á la limite á l´aide d´arguments de compacité et de semi-continuité inférieure.

Ces résultats étendent de maniére significative les travaux antérieurs en démontrant l´existence de solutions méme dans des cas limites, oúle probléme non perturbé $H = 0$ n´admet aucune solution. Ces travaux apportent un éclairage nouveau sur le róle des termes non linéaires d´ordre inférieur dans l´amélioration de la régularité des équations elliptiques avec des données singuliéres.

mots-clés: Probléme de Dirichlet non linéaire, Existence, Régularité, Effets régularisants, Données non réguliéres, $W_0^{1,1}(\Omega)$.

ملخص

تتناول هذه الدراسة التأثيرات المنتظمة (المرجّحة) للحدود ذات الرتبة الدنيا في مسائل *Dirichlet* غير الخطية من الشكل:

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)+H(x,u,\nabla u)=f(x),\Omega, \\ u=0,\partial\Omega, \end{cases} \tag{3}$$

حيث $\Omega\subset\mathbb{R}^N$ ($N\geq 2$)ميدان محدود، و $1<p\leq N$ ، و f تابع قابل للمكاملة. نُبيِّن كيف يمكن لبعض الشروط غير الخطية ذات الرتبة الدنيا أن تُحسِّن من انتظام الحلول، خاصة عندما يكون $f\in L^1(\Omega)$ أو ينتمي إلى فضاءات *Lebesgue* أخرى.

بإعتماد على العمل [٨] ، نركز على أربع حالات أساسية:

- عندما يكون $H(x,u,\nabla u)=u|u|^{r-2}$ ، نثبت وجود حلول ضعيفة في الفضاء $W_0^{1,2}(\Omega)$ ، حتى في حالة المعطيات $f\in L^1(\Omega)$
- في حالة H هو كثيرة الحدود، نبرهن على وجود حلول على شكل توزيعات في $W_0^{1,1}(\Omega)$ مع $f\in L^{r'/p}(\Omega)$ ، حيث $1<p\leq r'$.
- في حالة $H(x,u,\nabla u)=u|u|^{r-2}|\nabla u|$ ، نحصل على حلول تنتمي إلى الفضاء $W_0^{1,1}(\Omega)\cap L^{r-1}(\Omega)$ ، وذلك مع $f\in L^1(\Omega)$ و r يحقق $1<r\leq\frac{N-1}{N(p-1)}$.
- نقارن هذه النتائج مع الحالة شبه الخطية ($p=2$)، مبرزين الاختلافات الجوهرية في آليات التنظيم.

يعتمد دراستنا على ثلاث مراحل:

اولا إنشاء المسألة المتقاربة،ثانيا ايجاد تقديرات قبلية لحلول المسألة المتقاربة في الفضاء $W_0^{1,1}(\Omega)$ ، وأخيرا ا لمرور إلى النهاية في المسألة المتقاربة لإيجاد حلول للمسألة الأساسية، تمثل هذه النتائج امتدادًا كبيرًا للأعمال السابقة من خلال إثبات وجود حلول حتى في الحالات الحدية التي لا تقبل فيها المسألة ($H=0$) أي حل. وتُسهم هذه الدراسة في تعميق الفهم لدور الشروط غير الخطية من الرتبة الدنيا في تحسين انتظام الحلول لمعادلات ناقصية ذات معطيات غير منتظمة.

الكلمات الرئيسية: مسألة *Dirichlet* غير الخطية ، وجود ، انتظام، تأثيرات منتظمة، طرف ثاني غير منتظم، $W_0^{1,1}(\Omega)$.

List of Symbols

In what follows, we will use the following notations.

$\mathbb{R}^n$	Euclidean, n-dimensional space.
x	Vecteur de $\mathbb{R}^n$, $x=(x_1,x_2,...,x_n)$, $x_i \in \mathbb{R}$, $1 \leq i \leq n$.
Ω	Open set in $\mathbb{R}^n$.
$\partial\Omega$	The border of Ω.
$B(x,r)$	Open ball with center x and radius $r>0$.
$O(a,r)$	$=\Omega \cap B(a,r)=\{t \in \Omega : \lvert t-a\rvert < r\}$.
$S_\alpha(\mathbb{R}^n)$	$=\left\{f \in L^1_{loc}(\mathbb{R}^n) : \rho_\alpha f(r) \to 0 \text{ for } r \to 0\right\}$.
$\rho_\alpha f(r)$	$=\sup\limits_{x\in\Omega} \int_{B(x,r)} \frac{\lvert f(y)\rvert}{\lvert x-y\rvert^{n-\alpha}} dy < \infty, \forall r>0$.
$W^{k,p}(\Omega)$	$=\left\{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega)\ \forall \alpha \in \mathbb{N}^n \text{ such that } \lvert\alpha\rvert \leq j\right\}$.
$W_0^{k,p}(\Omega)$	Sobolev space with 0 on $\partial\Omega$.
$\mathscr{D}(\Omega)$	Space of unlimited differentiable functions on Ω with compact support.
p'	The conjugate exponent of p.
$p^\star$	$=\frac{Np}{N-p}$; Sobolev conjugate.
$C^\infty(\Omega)$	Is the set of functions in $C^k(\Omega)$ for all k.
∇v	The gradient of v.
$C^k(\Omega)$	Is the set of functions which have derivatives of $order \leq k$ that are continuous in Ω.

Introduction

In the study of nonlinear elliptic partial differential equations, understanding the analytic properties of solutions becomes particularly important when dealing with irregular or extreme data. This research highlights the crucial role of lower-order terms in enhancing solution regularity, as these terms play a pivotal role in compensating for weak data summability.

This work focuses on analyzing boundary value problems of the form:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + H(x,u,\nabla u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{4}$$

where Ω is a bounded ssubset in $\mathbb{R}^N$, $N \geq 2$, and f belongs to the class of weakly summable functions. We investigate how the additional terms $H(x,u,\nabla u)$ can improve the regularity of both the solution u and its gradient ∇u, compared to cases lacking such terms.

The problem (4) has been treated by [5, 6, 7, 8, 14], our aim to study the existence of solutions belonging to $W_0^{1,1}(\Omega)$ even for irregular data, demonstrating the regularizing effect of additional terms, we examine how polynomial and gradient-dependent terms enhance solution properties. In addition we give illustrative examples demonstrating the effectiveness of these results for irregular data cases.
- The proofs rely on careful a priori estimates, truncation techniques, and compactness arguments, leveraging tools from Sobolev spaces and nonlinear analysis. - The results highlight the interplay between the structure of lower-order terms and the summability of data, demonstrating how nonlinear terms can compensate for the lack of regularity in f.

This memory contains three chapters, The first chapter deals with some facts and difintions on Sobolev spaces, Hilbert spaces, some of their properties, and further details on these spaces because they also allows us to study the existence and regularity for our problem in Chapters $2-3$. We refer to [1] ,[12], [13], [16], [17], [18], [19],[22], and [3] for the theory of these spaces.

In the second chapter, we study Laplacian equation with $W_0^{1,2}(\Omega)$ solutions and p-Laplacian equation with $W_0^{1,p}(\Omega)$ solutions as follow:

- **(A)**: the Dirichlet problem:

$$\begin{cases} -\mathrm{div}(a(x)Du) + \gamma u|Du|^2 = f, & \text{in } \Omega \quad \gamma > 0; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

  with $f(x) \in L^1(\Omega)$, According to [20], this problem has at least weak solution $u \in W_0^{1,2}(\Omega)$.

- **(B)**: the Dirichlet problem:

$$\begin{cases} A(u) + g(x,u,Du) = f, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

  with $f(x) \in L^1(\Omega)$, According to [13] , this problem has at least weak solution $u \in W_0^{1,p}(\Omega)$.

The Chapter 3 is devoted to studying the general case:

- **(C)**: The case with polynomial lower order terms: We consider the problem:

$$\begin{cases} -\mathrm{div}\,(a(x,u,\nabla u)) + g(x,u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

  with

$$f \in L^{\frac{r'}{p}}(\Omega), \quad 1 < p \le r'.$$

  Then, there exists $u \in W_0^{1,1}(\Omega)$ such that $g(x,u) \in L^1(\Omega)$, which is a distributional solution as mentioned in [7].

- **(D)**: The case with gradient depending lower order terms: We take the following problem:

$$\begin{cases} -\mathrm{div}\,(a(x,u,\nabla u)) + Au|u|^{r-2}|\nabla u| = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

  where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies classical hypotheses (see [21]); namely, a is a Carathéodory function such that the following holds for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, for every $\xi \neq \eta \in \mathbb{R}^N$:

$$\begin{cases} a(x,s,\xi)\xi \ge \alpha|\xi|^p, \\ |a(x,s,\xi)| \le \beta|\xi|^{p-1}, \\ [a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0, \end{cases} \tag{5}$$

  where α, β are positive constants. Thus $\mathscr{A}$ is a pseudo-monotone (see [16]) and coercive differential operator.

  with $f \in L^1(\Omega)$ and

$$1 < r \le \frac{N-1}{N(p-1)}.$$

Then, there exists $u \in W_0^{1,1}(\Omega) \cap L^{r-1}(\Omega)$ such that

$$u|u|^{r-2}|\nabla u| \in L^1(\Omega),$$

which is a distributional solution as mentioned in [7] .

We highlight that all the results in Chapter 3 can be found in [7, 8]. Furthermore, some similar works that employ a variety of techniques can be accessed by looking at[3], [5], and[6].

Tanking everything into account, this study provides a comprehensive analysis of how auxiliary terms improve the properties of differential equation solutions, with special emphasis on critical cases requiring careful handling of data regularity issues. The work bridges theoretical analysis with practical applications, offering new insights into solving nonlinear elliptic equations under minimal regularity assumptions.

Chapter 1

Fundamental properties of Sobolev spaces

In this chapter, we recall some facts and definitions on Sobolev spaces and present some of their properties. For further details on Lebesgue and Sobolev spaces, we refer to [23], [24], [25], and [26]. We underline that Sobolev spaces constitute one of the most relevant functional settings for the treatment of boundary value problems.

1.1 Carathéodory's Function

The Carathéodory functions are used in solving problems in partial differential equations, so we have the following definition:

Definition 1.1.1. *[28] A function $f(t,x)$ defined on*

$$R : |t-\tau| \leq a, \quad |x-\xi| \leq b$$

is called a ***Carathéodory function*** *if:*

1. *$f(x,t)$ is continuous in t for almost every x*
2. *$f(x,t)$ is measurable in x for every t*
3. *$|f(t,x)| \leq m(t)$ for some integrable function $m(t)$*

Theorem 1.1.1 (Carathéodory's Existence Theorem)**.** *[28] Under the above conditions, there exists an absolutely continuous function $u(t)$ satisfying:*

$$u'(t) = f(t,u(t)) \text{ a.e.}, \; u(\tau) = \xi$$

Remarks 1.1.1. *A function $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is called a* ***Carathéodory function*** *if:*

1. *$a(x,s,\xi)$ is continuous in s,ξ for almost every x*
2. *$f(x,s,\xi)$ is measurable in x for every s,ξ*

1.2 The spaces $W^{j,p}(\Omega)$ and $W_0^{j,p}(\Omega)$

Definition 1.2.1.

let $1 \leq p < \infty$, such that
$L^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}; f$ measurable in addition $\int_\Omega |f|^p < \infty\}$
Note

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

$L^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}; f$ measurable in addition $\exists C$ such that $|f| < C a.e\}$
Note

$$\|f\|_{L^\infty(\Omega)} = \sup ess|f|.$$

Definition 1.2.2.

Suppose $1 \leq p < \infty$. Then

- $L^p_{loc}(\Omega) = \{u : u \in L^p(K)$ for every compact subset K of $\Omega\}$,
- u is locally integrable in Ω if $u \in L^1_{loc}(\Omega)$.
- Let u and v be locally integrable functions defined in Ω. We define $v$as the weak derivative of u with respect to α if, for every $\phi \in C_0^\infty(\Omega)$

  $$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v \phi dx.$$

  and we say that $D^\alpha u = v$ in the weak sense.
- Let u and v be in $L^p_{loc}(\Omega)$. We define v as the strong derivative of u with respect to α if, for every compact subset K of Ω, there exists a sequence $\{\phi_j\}$ in $C^{|\alpha|}(K)$ such that $\phi_j \to u$ in $L^p(K)$ and $D^\alpha \phi_j \to v$ in $L^p(K)$.

Theorem 1.2.1.

If $D^\alpha u = v$ and $D^\beta v = w$ in the weak sense then $D^{\alpha+\beta} u = w$
in the weak sense.

Proof. let $\psi \in C_0^\infty(\Omega)$ and $\phi = D^\beta \psi$.Then

$$\int_\Omega u D^{\alpha+\beta} \psi dx = (-1)^{|\alpha|} \int_\Omega \phi v dx = (-1)^{|\alpha|} \int_\Omega v D^\beta \psi dx = (-1)^{|\alpha|+|\beta|} \int_\Omega \psi w dx.$$

□

Definition 1.2.3.

Let $\mu \in C_0^\infty(\mathbb{R}^n)$ be such that

1. $supp \mu \subset B_1(0)$,(recall that "supp" denotes the support of a function, and $B_r(c)$ denotes an open ball of radius r and center c).
2. $\int \mu(x)dx = 1$.
3. $\mu(x) \geq 0$.

   if $\varepsilon > 0$ then we set (provided that the integral exists)

$$J_\varepsilon u(x) = \frac{1}{\varepsilon^n}\int_\Omega \mu(\frac{x-y}{\varepsilon})u(y)dy.$$

$J_\varepsilon u$ is called a mollifier of u. Note that if u is locally integrable in Ω and if K is a compact subset of Ω then $J_\varepsilon u \in C^\infty(K)$ provided that $\varepsilon < dist(K, \partial\Omega)$.Suppose now that $u \in L^p_{loc}(\Omega)$.

$$J_\varepsilon u(x) = \int_{B_1(0)} \mu(y)u(x-\varepsilon y)dy,$$

so for $p > 1$ we have (if $\frac{1}{p}+\frac{1}{q} = 1$)

$$\begin{aligned} |J_\varepsilon u(x)| &\leq \int_{B_1(0)} \{\mu(y)\}^{\frac{1}{q}}\{\mu(y)\}^{\frac{1}{p}}|u(x-\varepsilon y)|dy \\ &\leq (\int_{B_1(0)} (\{\mu(y)\}^{\frac{1}{q}})^q dx)^{\frac{1}{q}}(\int_{B_1(0)} (\{\mu(y)\}^{\frac{1}{p}}|u(x-\varepsilon y)|)^p dy)^{\frac{1}{p}}. \end{aligned}$$

Hence $|J_\varepsilon u(x)|^p \leq \int_{B_1(0)} \mu(y)|u(x-\varepsilon y)|^p dy$, and this trivially holds if $p = 1$ too. Integrating this, we see that

$$\begin{aligned} \int_K |J_\varepsilon u(x)|^p dx &\leq \int_{B_1(0)} \mu(y)\int_K |u(x-\varepsilon y)|^p dxdy \\ &\leq \int_{B_1(0)} \mu(y)\int_{K_0} |u(x)|^p dxdy \\ &= \int_{K_0} |u(x)|^p dx, \end{aligned}$$

where K_0 is a compact subset of Ω, $K \subset Interior(K_0)$ and $\varepsilon < dist(K, \partial K_0)$ i.e. we have

$$\|J_\varepsilon u\|_{L^p(K)} \leq \|u\|_{L^p(K_0)}. \tag{1.1}$$

Lemma 1.2.1.

If $u \in L^P_{loc}(\Omega)$ and K is a compct subset of Ω then $\|J_\varepsilon u - u\|_{L^p(K)} \to 0$ as $\varepsilon \to 0$

Proof. Let K_0 be a compact subset of Ω where $K \subset Interior(K_0)$ and let $\varepsilon < dist(K, \partial K_0)$. Let $\delta > 0$ and let $w \in C^\infty(K_0)$ be such that $\|u - w\|_{L^p(K_0)} < \delta$. Then applying (1.1) to $u - w$, we get

$$\|J_\varepsilon u - J_\varepsilon w\|_{L^p(K)} < \delta. \tag{1.2}$$

However $J_\varepsilon w(x) - w(x) = \int_{B_1(0)} \mu(y)\left\{w(x-\varepsilon y) - w(x)\right\} dy$, and this goes to zero uniformly on K as $\varepsilon \to 0$. Hence, if ε is sufficiently small, we have

$$\|J_\varepsilon w - w\|_{L^p(K)} < \delta. \tag{1.3}$$

Hence, by (1.2) and (1.3)

$$\|J_\varepsilon u - u\|_{L^p(K)} \leq \|w - u\|_{L^p(K)} + \|J_\varepsilon u - J_\varepsilon w\|_{L^p(K)} + \|J_\varepsilon w - w\|_{L^p(K)} < 3\delta. \tag{1.4}$$

Since δ is arbitrary, $\|J_\varepsilon u - u\|_{L^p(K)} \to 0$ as $\varepsilon \to 0$. □

Theorem 1.2.2.

Suppose that u and v are in $L^p_{loc}(\Omega)$. Then $D^\alpha u = v$ in the weak sense if and only if $D^\alpha u = v$ in the strong $L^P(\Omega)$ sense.

Proof. Suppose that $D^\alpha u = v$. Let $\phi \in C_0^\infty(\Omega)$ and let $K = supp\phi$. Let $\varepsilon > 0$ and take $\psi \in C^{|\alpha|}(K)$ so that $\left\|\psi - u\right\|_{L^p(K)} < \varepsilon$ and $\left\|D^\alpha \psi - v\right\|_{L^p(K)} < \varepsilon$. Then

$$\begin{aligned}
\left|\int_K u D^\alpha \phi dx - (-1)^{|\alpha|}\int_K v\phi dx\right| &\leq \left|\int_K \psi D^\alpha \phi dx - (-1)^{|\alpha|}\int_K \phi D^\alpha \psi dx\right| \\
&\quad + \left|\int_K (u-\psi) D^\alpha \phi dx\right| + \left|\int_K (v - D^\alpha\psi)\phi dx\right| \\
&\leq \left\|u-\psi\right\|_{L^p(K)} \left\|D^\alpha \phi\right\|_{L^q(K)} + \left\|v - D^\alpha \psi\right\|_{L^p(K)} \left\|\phi\right\|_{L^q(K)} \\
&\leq \varepsilon(\left\|D^\alpha\phi\right\|_{L^q(K)} + \left\|\phi\right\|_{L^q(K)}),
\end{aligned}$$

where q is the conjugate exponent of p (if $p = 1$ then $q = \infty$ and if $p > 1$ then $\frac{1}{p} + \frac{1}{q} = 1$). But ε is arbitrary, so the LHS must be zero. So $D^\alpha u = v$ in the weak sense.

Conversely, suppose that $D^\alpha u = v$ in the weak sense and let K be a compact subset of Ω. Then $J_\varepsilon u \in C^\infty(K)$ if $\varepsilon < dist(K, \partial\Omega)$ and we have for all x in K

$$\begin{aligned}
D^\alpha J_\varepsilon u(x) &= \varepsilon^{-n}\int_\Omega D_x^\alpha \mu(\frac{x-y}{\varepsilon}) u(y) dy \\
&= \varepsilon^{-n}(-1)^{|\alpha|}\int_\Omega D_y^\alpha \mu(\frac{x-y}{\varepsilon}) u(y) dy \\
&= \varepsilon^{-n}\int_\Omega \mu(\frac{x-y}{\varepsilon}) v(y) dy \\
&= J_\varepsilon v(x).
\end{aligned}$$

But by Lemma 1.2.1, $\|J_\varepsilon u - u\|_{L^p(K)} \to 0$ and $\|D^\alpha J_\varepsilon u - v\|_{L^p(K)} = \|J_\varepsilon v - v\|_{L^p(K)} \to 0$ as $\varepsilon \to 0$. Thus $D^\alpha u = v$ in the strong sense. □

Definition 1.2.4.

1. $\|u\|_{H^{j,p}(\Omega)} = \left(\sum_{|\alpha| \leq j} \int_\Omega |D^\alpha u(x)|^p dx\right)^{1/p}$.
2. $\hat{C}^{j,p}(\Omega) = \left\{u \in C^j(\Omega) : |u|_{H^{j,p}(\Omega)} < \infty\right\}$.

3. $H^{j,p}(\Omega)$ =completion of $\hat{C}^{j,p}(\Omega)$ with respect to the norm $|\ |_{H^{j,p}(\Omega)}$.

$H^{j,p}(\Omega)$ is called a Sobolev space. We will encounter other such spaces as well. Recall that for $1 \leq p < \infty$, $L^p(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the usual "p norm". This knowledge allows us to see what members of $H^{j,p}(\Omega)$. Suppose that u_m is a Cauchy sequence in $\hat{C}^{j,p}(\Omega)$ .Then for $|\alpha| \leq j$, $D^\alpha u_m$ is a Cauchy sequence in $L^p(\Omega)$. Hence, there are members u^α of $L^p(\Omega)$ such that $D^\alpha u_m \to u^\alpha$ in $L^p(\Omega)$. Hence, according to our definition of strong derivatives, u^0 is in $L^p(\Omega)$ and u^α is the α strong derivative of u^0. Hence we see that,

$$H^{j,p}(\Omega) = \Big\{u \in L^p(\Omega) : \ u \text{ has strong } L^p(\Omega) \text{ derivatives of order is less than or equals } j \text{ in } L^p(\Omega)$$
$$\text{and there exists a sequence } u_m \text{ in } \hat{C}^{j,p}(\Omega) \text{ such that } D^\alpha u_m \to D^\alpha u \text{ in } L^p(\Omega)\Big\} .$$

Definition 1.2.5.

$W^{j,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)\ \forall \alpha \in \mathbb{N}^n \text{ such that } |\alpha| \leq j\}$

Note

$$\|u\|_{W^{j,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{|\alpha| \leq j} \left\|D^\alpha u\right\|_{L^p(\Omega)} \tag{1.5}$$

Lemma 1.2.2.

Let $E \subset \mathbb{R}^n$ and let G be a collection of open sets U such that $E \subset \{\bigcup U : U \in G\}$. Then there exists a family F of non-negative functions $f \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq f(x) \leq 1$ and

(i) for each $f \in F$, there exists $U \in G$ such that $suppf \subset U$

(ii) if $K \subset E$ is compact then $suppf \cap K$ is non-empty for only finitely many $f \in F$,

(iii) $\sum_{f \in F} f(x) = 1$ for each $x \in E$,

(IV) if $G = \{\Omega_1, \Omega_2, \ldots\}$ where each Ω_i is bounded and $\bar{\Omega}_i \subset E$ then the family F of such functions can be constructed so that $F = \{f_1, f_2, \ldots\}$ and $suppf_j \subset \Omega_i$.

The family of functions F is called a partition of unity subordinate to the cover G.

Theorem 1.2.3. *(Meyers and Serrin, 1964)* $H^{j,p}(\Omega) = W^{j,p}(\Omega)$.

Proof. We already know that $H^{j,p}(\Omega) \subset W^{j,p}(\Omega)$.The opposite inclusion follows if we can show that for every $u \in W^{j,p}$ and for every $\varepsilon > 0$ we can find $w \in \hat{C}^{j,p}$ such that for $|\alpha| \leq j$,$\|D^\alpha w - D^\alpha u\|_{L^p(\Omega)} < \varepsilon$.

For $m \geq 1$ let

$$\Omega_m = \left\{x \in \Omega : \|x\| < m \qquad , dist(x, \partial\Omega) > \frac{1}{m}\right\}$$

and let $\Omega_0 = \Omega_{-1} = \emptyset$. let $\{\psi_m\}$ be the partition of unity of part (iv), Theorem (1.2.3) ,subordinate to the cover $\left\{\Omega_{m+2}-\overline{\Omega}_m\right\}$ .Each $u\psi_m$ is j times weakly differentiable and has support in $\Omega_{m+2}-\overline{\Omega}_m$ .As in the "conversely" part of the proof of Theorem (1.2.2), we can pick $\varepsilon_m>0$ so small that $w_m=J_{\varepsilon m}(u\psi_m)$ has support in $\Omega_{m+3}-\overline{\Omega}_{m-1}$ and $\left\|w_m-u\psi_m\right\|_{W^{j,p}(\Omega)}<\frac{\varepsilon}{2^m}$. Let $w=\sum\limits_{m=1}^{\infty}w_m$ .This is a C^∞ function because on each set $\Omega_{m+2}-\overline{\Omega}_m$ we have $w=w_{m-2}+w_{m-1}+w_m+w_{m+1}+w_{m+2}$. Further .

$$\begin{aligned}\left\|D^\alpha w-D^\alpha u\right\|_{L^p(\Omega)}&=\left\|\sum_{m=1}^{\infty}D^\alpha(w_m-u\psi_m)\right\|_{L^p(\Omega)}\\&\le\sum_{m=1}^{\infty}\left\|D^\alpha(w_m-u\psi_m)\right\|_{L^p(\Omega)}\\&\le\sum_{m=1}^{\infty}\frac{\varepsilon}{2^m}=\varepsilon.\end{aligned}$$

□

Remarks 1.2.1.

(i) The proof shows that in fact $C^\infty(\Omega)\cap\hat{C}^{j,p}(\Omega)$ is dense in $W^{j,p}(\Omega)$.

(ii) Clearly members of $C^\infty(\Omega)\cap\hat{C}^{j,p}(\Omega)$ are not necessarily continuous on $\partial\Omega$ or even bounded near $\partial\Omega$. It would be very useful to have the knowledge that $C^\infty(\bar{\Omega})\cup\hat{C}^{j,p}(\Omega)$ or $C^j(\bar{\Omega})\cup\hat{C}^{j,p}(\Omega)$ is also dense in $W^{j,p}(\Omega)$.

Theorem 1.2.4.

If Ω has the segment property then the set of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\Omega)$.

Theorem 1.2.5. ***Change of Variables and the Chain Rule.***

Let V,Ω be domains in $\mathbb{R}^n$ and let $T:V\to\Omega$ be invertible. Suppose that T and T^{-1} have continuous, bounded derivatives of order is j . Then if $u\in W^{j,p}(\Omega)$we have $v=u\circ T\in W^{j,p}(V)$ and the derivatives of v are given by the chain rule.

Proof. Let y denote coordinates in Ω and let x denote coordinates in V $(y=T(x))$. If $f\in L^p(\Omega)$ then $f\circ T\in L^p(V)$ because

$$\int_V|f\circ T|^p dx=\int_\Omega|f|^p J dy\le const.\int_\Omega|f|^p dy \tag{1.6}$$

(Here J is the Jacobian of T^{-1}).

If $u\in W^{j,p}(\Omega)$, let $\{u_m\}$ be a sequence in $\hat{C}^{j,p}(\Omega)$ converging to u in $W^{j,p}(\Omega)$ and set $v_m=u_m\circ T$. By the chain rule, if $|\alpha|\le j$

$$D_x^\alpha v_m=\sum_{\beta\le\alpha}(D_y^\beta u_m)\circ TR_{\alpha,\beta}$$

Where the $R_{\alpha,\beta}$ are bounded terms involving T and its derivatives. But for $|\beta| \le j \quad D_y^\beta u \in L^p(\Omega) \Rightarrow (D_y^\beta u) \circ T \in L^p(V) \Rightarrow (D_y^\beta u) \circ T R_{\alpha,\beta} \in L^p(V)$ since the $R_{\alpha,\beta}$ are bounded.

Further,

$$\begin{aligned}
\left\| D_x^\alpha v_m - \sum_{\beta\le\alpha} (D_y^\beta u) \circ T R_{\alpha,\beta} \right\|_{L^p(V)} &= \left\| \sum_{\beta\le\alpha} (D_y^\beta u_m - D_y^\beta u) \circ T R_{\alpha,\beta} \right\|_{L^p(V)} \\
&\le \sum_{\beta\le\alpha} \left\| (D_y^\beta u_m - D_y^\beta u) \circ T R_{\alpha,\beta} \right\|_{L^p(V)} \\
&\le const. \sum_{\beta\le\alpha} \left\| (D_y^\beta u_m - D_y^\beta u) \circ T \right\|_{L^p(V)} \\
&\le const. \sum_{\beta\le\alpha} \left\| (D_y^\beta u_m - D_y^\beta u) \right\|_{L^p(V)}
\end{aligned}$$

by (1.6). So ($\alpha = 0$ case), $v_m \to v = u \circ T$ in $L^p(V)$ and $D_x^\alpha v_m \to \sum_{\beta\le\alpha} (D_y^\beta u) \circ T R_{\alpha,\beta}$ in $L^p(V)$. This shows that $v \in W^{j,p}(V)$ and $D_x^\alpha v = \sum_{\beta\le\alpha} (D_y^\beta u) \circ T R_{\alpha,\beta}$. □

Definition 1.2.6.

$$W_0^{j,p}(\Omega) = \{ \textit{completion of } C_0^\infty(\Omega) \textit{ with respect to the norm } |\ |_{W^{j,p}(\Omega)} \}$$

Proposition 1.2.1. *[26] Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, the following statemente hold :*

(i) For each $1 \le p \le \infty$, $W^{1,p}(\Omega)$ is a Banach space.

(ii) For each $1 < p < \infty$, $W^{1,p}(\Omega)$ is refelexive.

(iii) For each $1 \le p < \infty$, $W^{1,p}(\Omega)$ is a separable.

Proof.

(i) Let $\{u_n\}_{n\in N}$ be a Cauchy sequence in $W^{1,p}(\Omega)$, with $1 \le p \le \infty$, Then, from (1.5) it follows that $\{u_n\}_{n\in N}$ and $\left\{(u_n)_{x_i}\right\}_{n\in N}$, with $1 \le i \le N$, are Cauchy sequences in $L^p(\Omega)$. Thus, since $L^p(\Omega)$ is a Banach space , it follows that $u_n \to n$ and $(u_n)_{x_i} \to g_i$ in $L^p(\Omega)$ with $u, g_i \in L^p(\Omega)$ Therefore, since

$$\int u_n \varphi_{x_i} = -\int_\Omega (u_n)_{x_i} \varphi \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Letting $n \to +\infty$

$$\int u \varphi_{x_i} = -\int_\Omega g_i \varphi \qquad \forall \varphi \in C_0^\infty(\Omega)$$

Therefore, we obtain that $u \in W^{1,p}(\Omega)$, $u_{xi} = g_i$ and thus

$$\|u_n - u\|_{W^{1,p}(\Omega)} = \|u_n - u\|_{L^p(\Omega)} + \sum_{i=1}^N \|U_n - g_i\|_{L^p(\Omega)} \to 0$$

as desired.

(ii) Consider the space $E = L^p(\Omega) \times L^p(\Omega)$ which is reflexive since it is the product of reflexive spaces. Set the operator $T : W^{1,p}(\Omega) \to E$ defined by $Tu = (u, \nabla u)$ Then, T is an isometry, and since $W^{1,p}(\Omega)$ is a Banach space, $M = T\left(W^{1,p}(\Omega)\right)$ is a closed subspace of E since E is reflexive, B_E is compact in the weak topology $\sigma\left(E, E^\star\right)$, and M is closed in the topology $\sigma\left(E, E^\star\right)$ Therefore, B_M is compact in $\sigma\left(E, E^\star\right)$, and Therefore $T\left(W^{1,p}(\Omega)\right)$ is reflexive .As a consequence, $W^{1,p}(\Omega)$ is also reflexive .

(iii) Under the notation of (ii), and taking into account that E is separable , it follows that $T\left(W^{1,p}(\Omega)\right)$ is separable and therefore $W^{1,p}(\Omega)$ is also separable . ☐

Remarks 1.2.2.

(i) Saying that $f \in W_0^{j,p}(\Omega)$ is a generalized way of saying that f and its derivatives of order is less than or equals $j-1$ vanish on $\partial\Omega$. e.g. $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ is a useful space for studying solutions of the Dirichlet problem for second order elliptic PDE's.

(ii) $C_0^j(\Omega) \subset W_0^{j,p}(\Omega)$ because if $f \in C_0^j(\Omega)$, we know that if ε is sufficiently small then $J_\varepsilon f \in C_0^\infty(\Omega)$ and $J_\varepsilon f \to f$ in $|\,|_{W^{j,p}(\Omega)}$ norm.

1.3 Extension Theorems

Most of the important Sobolev inequalities and embedding theorems that we will derive in the next section are most easily derived for the space $W_0^{j,p}(\Omega)$ which can be viewed as being a subspace of $W^{j,p}(\mathbb{R}^n)$.

Lemma 1.3.1.

Let $u \in \mathbb{R}^n$ and $f \in L^p(\mathbb{R}^n)$. Set $f_\delta(x) = f(x+\delta u)$. Then $\lim\limits_{\delta\to 0} f_\delta = f$ in $L^p(\mathbb{R}^n)$.

Proof. Given $\varepsilon > 0$, let $\phi \in C_0^\infty(\mathbb{R}^n)$ be such that $\|f-\phi\|_{L^p(\mathbb{R}^n)} < \varepsilon$. Since $\phi_\delta \to \phi$ uniformly on a sufficiently large ball containing the supports of all ϕ_δ (say, for $\delta \le 1$), we can pick δ so small that $\|\phi-\phi_\delta\|_{L^p(\mathbb{R}^n)} < \varepsilon$. Then

$$\|f-f_\delta\|_{L^p(\mathbb{R}^n)} \le \|f-\phi\|_{L^p(\mathbb{R}^n)} + \|\phi-\phi_\delta\|_{L^p(\mathbb{R}^n)} + \|\phi_\delta-f_\delta\|_{L^p(\mathbb{R}^n)} < 3\varepsilon.$$

$\square$

Lemma 1.3.2.

Let $\mathbb{R}^n_+ = \left\{x \in \mathbb{R}^n : x_i > 0\right\}$.$C^\infty(\bar{\mathbb{R}}^n_+) \cap \hat{C}^{j,p}(\mathbb{R}^n_+)$ is dense in $W^{j,p}(\mathbb{R}^n_+)$.

Proof. Suppose f is in $W^{j,p}(\mathbb{R}^n_+)$ let $\varepsilon > 0$ and pick $\phi \in C^\infty(\mathbb{R}^n_+) \cap \hat{C}^{j,p}(\mathbb{R}^n_+)$ so that $\|D^\alpha\phi - D^\alpha f\|_{L^p(\mathbb{R}^n_+)} < \varepsilon$ for all $|\alpha| \le j$.We take the vector of Lemma (1.3.1) to be $u = (0,0,0,\dots,1)$ and define functions $\psi^\alpha \in L^p(\mathbb{R}^n)$ as

$$\psi^\alpha(x) = \begin{cases} D^\alpha\phi(x) & ,x_i > 0 \\ 0 & ,x_i \le 0 \end{cases}$$

Observe that for each $\delta > 0$,$\phi_\delta \in C^\infty(\bar{\mathbb{R}}^n_+) \cap \hat{C}^{j,p}(\mathbb{R}^n_+)$. By Lemma (1.3.1), we can pick $\delta > 0$ so that, for all $|\alpha| \le j$,$\|\psi^\alpha_\delta - \psi^\alpha\|_{L^p(\mathbb{R}^n)} < \varepsilon$. But this implies that $\|D^\alpha\phi_\delta - D^\alpha\phi\|_{L^p(\mathbb{R}^n_+)} < \varepsilon$.
Hence

$$\|D^\alpha\phi_\delta - D^\alpha f\|_{L^p(\mathbb{R}^n_+)} \le \|D^\alpha\phi_\delta - D^\alpha\phi\|_{L^p(\mathbb{R}^n_+)} + \|D^\alpha\phi - D^\alpha f\|_{L^p(\mathbb{R}^n_+)} < 2\varepsilon.$$

$\square$

Lemma 1.3.3.

There exists a linear mapping $E_0 : W^{j,p}(\mathbb{R}^n_+) \to W^{j,p}(\mathbb{R}^n)$ such that $E_0 f = f$ in $\mathbb{R}^n_+$ and $\|E_0 f\|_{W^{j,p}(\mathbb{R}^n)} \le C\|f\|_{W^{j,p}(\mathbb{R}^n_+)}$, where C depends on only n and p.

Proof. If $f \in C^\infty(\bar{\mathbb{R}^n}_+)$, define

$$E_0 f(x) = \begin{cases} f(x) & ,x_n \ge 0 \\ \sum\limits_{k=1}^{j+1} c_k f(x_1, x_2, \dots, x_{n-1}, -kx_n) & ,x_n < 0 \end{cases}$$

Where the constants c_k are chosen so that $E_0 f(x) \in C^j(\mathbb{R}^n)$, i.e.

$$\sum_{k=1}^{j+1} (-k)^m c_k = 1,\ m = 0, 1, 2, \ldots, j.$$

It is easy to check that there is a constant C depending on only n and p such that

$$\left\| D^\alpha E_0 f \right\|_{L^p(\mathbb{R}^n)} \le C \left\| D^\alpha f \right\|_{L^p(\mathbb{R}^n_+)}. \tag{1.7}$$

If now $f \in W^{j,p}(\mathbb{R}^n_+)$, take a sequence $f_m \in C^\infty(\bar{\mathbb{R}}^n_+) \cap \hat{C}^{j,p}(\mathbb{R}^n_+)$ converging to f in $W^{j,p}(\mathbb{R}^n_+)$ (we can do this by Lemma 1.3.2). Then f_m is a Cauchy sequence and (1.7) implies that $E_0 f_m$ is a Cauchy sequence in $W^{j,p}(\mathbb{R}^n)$. We denote the limit by $E_0 f$. Since $\left\| D^\alpha E_0 f_m \right\|_{L^p(\mathbb{R}^n)} \le C \left\| D^\alpha f_m \right\|_{L^p(\mathbb{R}^n_+)}$ taking limits shows that f satisfies (1.7). ☐

Definition 1.3.1.

A domain Ω is of class C^m if $\partial\Omega$ can be covered by bounded open sets Ω_j such that there are mappings $\psi_j : \bar{\Omega}_j \to \bar{B}$,where B is the unit ball centered at the origin and

(i) $\psi_j(\Omega_j \cap \Omega) = B \cap R^n_+$

(ii) $\psi_j(\Omega_j \cap \partial\Omega) = B \cap \partial R^n_+$

(iii) $\psi_j \in C^m(\bar{\Omega}_j)$ and $\psi_j^{-1} \in C^m(\bar{B})$.

(Because of (iii), all derivatives of order is less than or equals m of ψ_j and its inverse are bounded).

Theorem 1.3.1.

If Ω is a bounded domain of class C^m then there exists a bounded linear extension operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n)$.

Definition 1.3.2.

A domain Ω is said to satisfy the cone property if there exist positive constants α , h such that for each $x \in \Omega$ there exists a right spherical cone $V_x \subset \Omega$ with height h and opening α.

1.4 Sobolev Inequalities and Embedding Theorems

Theorem 1.4.1.

If $\Omega \subset \mathbb{R}^n$ satisfies the cone condition (with height h and opening α) and if $P > 1$, $mp > n$ then $W^{m,p}(\Omega) \subset C_B(\Omega)$ and there is a constant C depending on only α , h, n and p such that for all $u \in W^{m,p}(\Omega)$, $\sup|u| \le C \|u\|_{W^{m,p}(\Omega)}$.

Proof. Initially, suppose that u is in $\hat{C}^{m,p}(\Omega)$ .Let $g \in C^{\infty}(\mathbb{R})$ be such that $g(t)=1$ if $t \leq \frac{1}{2}$ and $g(t)=0$ if $t \geq 1$.Let $x \in \Omega$ and let (r,θ) denote polar coordinates centered at x. Here, $\theta=(\theta_1,\theta_2,\dots,\theta_{n-1})$ denotes the angular coordinates and we can describe the cone with vertex x in polar coordinates as $V_x=\{(r,\theta):0\leq r\leq h,\theta\in A\}$ .Clearly, we have

$$\begin{aligned} u(x) &= -\int_0^h \frac{\partial}{\partial r}\left\{g(\frac{r}{h})u(r,\theta)\right\}dr,\\ &= \frac{(-1)^m}{(m-1)!}\int_0^h r^{m-1}\frac{\partial^m}{\partial r^m}\left\{g(\frac{r}{h})u(r,\theta)\right\}dr, \end{aligned}$$

After $m-1$ integrations by parts. Next, we integrate with respect to the angular measure dS_θ ,noting that the left-hand-side becomes a constant times $u(x)$.

$$\begin{aligned} u(x) &= c\int_A\int_0^h r^{m-1}\frac{\partial^m}{\partial r^m}\left\{g(\frac{r}{h})u(r,\theta)\right\}drdS_\theta\\ &= c\int_A\int_0^h r^{m-n}\frac{\partial^m}{\partial r^m}\left\{g(\frac{r}{h})u(r,\theta)\right\}r^{n-1}drdS_\theta\\ &= \int_{V_x} r^{m-n}\frac{\partial^m}{\partial r^m}\left\{g(\frac{r}{h})u(r,\theta)\right\}dV. \end{aligned}$$

Applying Hölder's inequality to this, we obtain

$$\begin{aligned} |u(x)| &\leq c\left\|r^{m-n}\right\|_{L^q(V_x)}\left\|\frac{\partial^m}{\partial r^m}\left\{g(\frac{r}{h})u(r,\theta)\right\}\right\|_{L^q(V_x)}\\ &\leq c\left\|r^{m-n}\right\|_{L^q(V_x)}\|u\|_{W^{m,p}(\Omega)}. \end{aligned}$$

But r^{m-n} is in $L^q(V_x)$ if $n-1+(m-n)q>-1$, which is the case because $q=\frac{p}{p-1}$ and $mp>n$ Thus, we obtain $\sup|u|\leq C\|u\|_{W^{m,p}(\Omega)}$. To extend this result to arbitrary $u\in W^{m,p}(\Omega)$,take a sequence $\{u_k\}$ of functions in $\hat{C}^{m,p}(\Omega)$ converging to u in the $|\ |_{W^{m,p}(\Omega)}$ norm.
Then $\sup|u_j-u_k|\leq C\left\|u_j-u_k\right\|_{W^{m,p}(\Omega)}$, showing that the sequence is a Cauchy sequence in $C_B(\Omega)$. Thus u is in $C_B(\Omega)$ and taking the limit of $\sup|u_j|\leq C\left\|u_j\right\|_{W^{m,p}(\Omega)}$shows that u satisfies the same inequality. ☐

Corollary 1.4.1.

If $\Omega\subset\mathbb{R}^n$ satisfies the cone condition (with height h and opening α) and if $p>1,(m-k)p>n$ then $W^{m,p}(\Omega)\subset C_B^k(\Omega)$and there is a constant C depending on only α,h,n,k and p such that for all $u\in W^{m,p}(\Omega)$ $\qquad \sup\limits_{|\alpha|\leq k}|D^\alpha u|\leq C\|u\|_{W^{m,p}(\Omega)}$.

Theorem 1.4.2.

If $\Omega\subset\mathbb{R}^n$ is any domain and $p>n$ then $W_0^{1,p}(\Omega)\subset C^{0,\alpha}(\bar{\Omega})$, where $\alpha=1-\frac{n}{p}$ and there exists α constant C depending on only p and n such that for all $u\in W_0^{1,p}(\Omega)$

$$\frac{|u(x)-u(y)|}{\left\|x-y\right\|^\alpha}\leq C\sum_{i=1}^n\|D_iu\|_{L^p(\Omega)}.$$

Theorem 1.4.3.

If $\Omega \subset \mathbb{R}^n$ is any domain and $p < n$ then $W_0^{1,p}(\Omega) \subset L^r(\Omega)$ where $r = \frac{np}{n-p}$ and there exists a constant C depending on only p and n such that for all $u \in W_0^{1,p}(\Omega)$

$$\|u\|_{L^r(\Omega)} \leq C \sum_{i=1}^{n} \|D_i u\|_{L^p(\Omega)}.$$

Remark 1.4.1.

Suppose that $a, b \geq 0$ and $1 < p, q < \infty$ in addition $\frac{1}{p} + \frac{1}{q} = 1$, the Young inequality is expressed by

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

which is more general than the previous one

$$ab \leq \frac{(a\varepsilon)^p}{p} + \frac{(\frac{b}{\varepsilon})^q}{q} = \delta a^p + C(\delta) b^q.$$

for all $\delta = \frac{\varepsilon^p}{p}$

Remark 1.4.2.

Suppose that $u_i \in L^{p_i}(\Omega)$, $i = (1, 2, 3, \ldots, m)$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \ldots + \frac{1}{p_m} = 1$.
The Hölder's inequality is expressed by

$$\int_\Omega |u_1 u_2 u_3 \ldots u_m| \, dx \leq \|u_1\|_{L^{p_1}(\Omega)} \|u_2\|_{L^{p_2}(\Omega)} \ldots \|u_m\|_{L^{p_m}(\Omega)} \tag{1.8}$$

*Proof. of Theorem*1.4.3 It suffices to prove the result for $u \in C_0^1(\mathbb{R}^n)$. First we prove the result for the case $p = 1$. For each i we have

$$|u(x)| \leq \int_{-\infty}^{x_i} |D_i u| dx_i \leq \int_{-\infty}^{\infty} |D_i u| dx_i.$$

Multiplying these n inequalities together and taking the $n-1$ the root gives

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |D_i u| dx_i \right)^{\frac{1}{n-1}} \tag{1.9}$$

Observe that $\int_{-\infty}^{\infty} |D_i x| dx_i$ does not depend on x_i , but it does depend on all $n-1$ of the remaining variables. We integrate each side of (1.9) with respect to x_1 and use the generalized Hölder inequality with $p_i = m = n-1$ to obtain

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 &\leq \left(\int_{-\infty}^{\infty} |D_1 u| dx_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |D_i u| dx_i \right)^{\frac{1}{n-1}} dx_1 \\ &\leq \left(\int_{-\infty}^{\infty} |D_1 u| dx_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_i u| dx_i dx_1 \right)^{\frac{1}{n-1}}. \end{aligned}$$

The RHS is still a product of $n-1$ functions of x_2 , so we integrate each side with respect to x_2, again applying (1.8) with $p_i = m = n-1$.Continuing in this manner, we finally obtain

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx \le \left(\prod_{i=1}^{n} \int_{\mathbb{R}^n} |D_i u| dx \right)^{\frac{1}{n-1}}$$

i.e.

$$\begin{aligned} \|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} &\le \left(\prod_{i=1}^{n} \int_{\mathbb{R}^n} |D_i u| dx \right)^{\frac{1}{n}} \\ &\le \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^n} |D_i u| dx \end{aligned}$$

Here we have used the fact that an arithmetic mean is no less than a geometric mean of the same numbers. This proves the result for the case $p = 1$.
For $p > 1$, let $\gamma = \frac{(n-1)p}{n-p} = 1 + \frac{n(p-1)}{n-p}$,Since $\gamma > 1$ and $u \in C_0^1(\mathbb{R}^n)$, it follows that $|u|^\gamma \in C_0^1(\mathbb{R}^n)$.

$$D_i |u|^\gamma = \frac{(n-1)p}{n-p} |u|^{\frac{n(p-1)}{n-p}} (\pm D_i u).$$

We apply the $p = 1$ case to $|u|^\gamma$ and obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}} dx \right)^{\frac{n-1}{n}} &\le \sum_{i=1}^{n} \frac{1}{n} \int_{\mathbb{R}^n} \frac{(n-1)p}{n-p} |u|^{\frac{n(p-1)}{n-p}} |D_i u| dx \\ &\le \frac{(n-1)p}{n(n-p)} \sum_{i=1}^{n} \left(\int_{\mathbb{R}^n} (|u|^{\frac{n(p-1)}{n-p}})^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \|D_i u\|_{L^p(\mathbb{R}^n)} \\ &= \frac{(n-1)p}{n(n-p)} \sum_{i=1}^{n} \left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}} dx \right)^{\frac{p-1}{p}} \|D_i u\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

Hence

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}} dx \right)^{\frac{n-p}{np}} \le \frac{(n-1)p}{n(n-p)} \sum_{i=1}^{n} \|D_i u\|_{L^p(\mathbb{R}^n)}$$

which is the desired result. As usual, to obtain the same result for a function $u \in W_0^{1,p}(\Omega)$,we just take a sequence of functions in $C_0^1(\mathbb{R}^n)$ converging to u. □

Remark 1.4.3.

$W_0^{1,p}(\Omega) \subset L^r(\Omega)$,where r is given above. But obviously $W_0^{1,p}(\Omega) \subset L^p(\Omega)$,so by the following interpolation lemma, $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ for all q satisfying $p \le q \le r$. If Ω is bounded then clearly this holds for all q satisfying $1 \le q \le r$.

Lemma 1.4.1. *If $s \le q \le r$ and $\phi \in L^s(\Omega) \cap L^r(\Omega)$, then $\phi \in L^q(\Omega)$ and*

$$\|\phi\|_{L^q(\Omega)} \le \|\phi\|_{L^s(\Omega)}^{\lambda} \|\phi\|_{L^r(\Omega)}^{1-\lambda},$$

where $\lambda = \frac{s(r-q)}{q(r-s)}$.

Corollary 1.4.2.

For every domain Ω in R^n there exists a constant C depending on only n and p such that

(i) if $kp < n$ then $W_0^{k,p}(\Omega) \subset L^{\frac{np}{n-kp}}(\Omega)$ and for each $u \in W_0^{k,p}(\Omega)$

$$\|u\|_{L^{\frac{np}{n-kp}}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}$$

(ii) if $kp > n$ then $W_0^{k,p}(\Omega) \subset C^{m,\alpha}(\bar{\Omega})$, where m is the integer satisfying $0 < k - m - \frac{n}{p} < 1$ and $\alpha = k - m - \frac{n}{p}$. Further, if $u \in W_0^{k,p}(\Omega)$ then

$$\|u\|_{C^{m,\alpha}(\bar{\Omega})} \leq C\|u\|_{W^{k,p}(\Omega)} .$$

Remarks 1.4.1.

(i) If $kp = n$ and $p > 1$ then $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ for all q satisfying $p \leq q < \infty$

(ii) If $kp > n, p > 1$ and $\frac{n}{p}$ is an integer then $W_0^{k,p}(\Omega) \subset W_0^{k-\frac{n}{p},q}(\Omega)$ for all q satisfying $p \leq q < \infty$.

(iii) If $kp > n$ and $p = 1$ (so $\frac{n}{p}$ is obviously an integer) then $W_0^{k,p}(\Omega) \subset C_\beta^{k-n}(\bar{\Omega})$.

Corollary 1.4.3.

If Ω is a bounded C^1 domain in $\mathbb{R}^n$ (or any other domain such that there exists a bounded extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$) then the statements concerning the spaces $W_0^{k,p}(\Omega)$ in Corollary (1.4.2) and in the remark following the corollary also apply to the spaces $W^{k,p}(\Omega)$. However, the constant C may also depend on Ω.

Proof. The cases for $k = 1$ dealt with in Theorems (1.4.2) and (1.4.3) are easily seen to have their counterparts here because of the extension operator. Inspection of the proof of Corollary (1.4.2) shows how the results for $k > 1$ may be derived from the results for $k = 1$ without any additional assumptions on the domain. ☐

Definition 1.4.1.

Let A and B be Banach spaces. If $A \subset B$, we say that A is continuously imbedded in B ((in symbols, this is written $A \hookrightarrow B$) if there is a constant C such that $\|x\|_B \leq C\|x\|_A$.

The theorems in this section provide examples of Embeddings and are called Sobolev Embedding Theorems.e.g. $W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{np}{n-p}}(\Omega)$ for $p > n$.

It is easy to see that $A \hookrightarrow B$ is equivalent to the identity mapping from A into B being continuous.

1.5 Compactness Theorems

Lemma 1.5.1.

Suppose that Ω is a bounded domain. If

1. $0 < \lambda \leq 1$ then $C^{m,\lambda}(\bar{\Omega})$ is compactly imbedded in $C^{m}(\bar{\Omega})$.
2. $0 < \nu < \lambda \leq 1$ then $C^{m,\lambda}(\bar{\Omega})$ is compactly embedded in $C^{m,\nu}(\bar{\Omega})$.

Proof. It suffices to prove the results for $m = 0$ because, once this is done, we can apply this case to the derivatives of the functions and deduce the result for general m. Let $\{f_j\}$ be a sequence in $C^{0,\lambda}(\bar{\Omega})$ such that $\|f_j\|_{C^{0,\lambda}(\bar{\Omega})} \leq M$. But this implies $|f_j(x) - f_j(y)| \leq M\|x-y\|^{\lambda}$, showing that the sequence is a bounded, equicontinuous set of functions. By the Arzela-Ascoli Theorem, there exists a subsequence $\{f_{jk}\}$ that converges in $C(\bar{\Omega})$. Thus $C^{0,\lambda}(\bar{\Omega})$ is compactly imbedded in $C(\bar{\Omega})$.

We show below that the same subsequence also converges in $C^{0,\nu}(\bar{\Omega})$. Suppose that $\psi \in C^{0,\lambda}(\bar{\Omega})$. Then

$$\begin{aligned}
[\psi]_{0,\nu} &= \sup \frac{|\psi(x) - \psi(y)|}{\|x-y\|^{\nu}} \\
&= \sup \left(\frac{|\psi(x) - \psi(y)|}{\|x-y\|^{\lambda}}\right)^{\frac{\nu}{\lambda}} |\psi(x) - \psi(y)|^{1-\frac{\nu}{\lambda}} \\
&\leq 2^{1-\frac{\nu}{\lambda}} \left([\psi]_{0,\lambda}\right)^{\frac{\nu}{\lambda}} \left(\max|\psi|\right)^{1-\frac{\nu}{\lambda}}
\end{aligned}$$

We apply this to $f_{jk} - f_{jr}$, noting that $[f_{jk} - f_{jr}]_{0,\lambda} \leq [f_{jk}]_{0,\lambda} + [f_{jr}]_{0,\lambda} \leq 2M$, and obtain

$$[f_{jk} - f_{jr}]_{0,\nu} \leq 2M^{\frac{\nu}{\lambda}}(\max|f_{jk} - f_{jr}|)^{1-\frac{\nu}{\lambda}},$$

showing that the subsequence is a Cauchy sequence in $C^{0,\nu}(\bar{\Omega})$ (because it converges in $C(\bar{\Omega})$). Thus the subsequence converges in $C^{0,\nu}(\bar{\Omega})$. ◻

Corollary 1.5.1.

If Ω is bounded, $kp > n$ and $0 < k - m - \frac{n}{p} < 1$ then $W_0^{k,p}(\Omega)$ is compactly embedded in $C^{m,\beta}(\bar{\Omega})$ if $\beta < k - m - \frac{n}{p}$.

Proof. Let $\alpha = k - m - \frac{n}{p}$ Then $W_0^{k,p}(\Omega) \hookrightarrow C^{m,\alpha}(\bar{\Omega}) \hookrightarrow C^{m,\beta}(\bar{\Omega})$,and the second, Embedding is compact. ◻

Corollary 1.5.2.

If Ω is a bounded C^1 domain (or any other domain for which there is a bounded extension operator $E : W^{1.p}(\Omega) \to W^{1.p}(R^n)$),$Kp > n$ and $0 < k - m - \frac{n}{p} < 1$ then $W^{k,p}(\Omega)$is compactly imbedded in $C^{m,\beta}\left(\overline{\Omega}\right) if \beta < k - m - \frac{n}{p}$.

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be such that supp ϕ is contained in some ball B containing Ω and $\phi = 1$ on Ω.Then we can define $\widetilde{E} : W^{1,p}(\Omega) \to W_0^{1,p}(B)$ by $\widetilde{E}\left(f\right) = \phi E\left(f\right)$.
By Corollary (1.5.1),$W_0^{1,p}(B)$ is compactly embedded in $C^{0,\beta}(\overline{B})$.Hence $W^{1,p}(\Omega)$is compactly embedded in $C^{0,\beta}(\overline{\Omega})$.The result for general k can be deduced from the $k = 1$ case by considering derivatives of the functions (as in the proof of Corollary (1.4.2) (b), deduce that if $u \in W^{k,p}(\Omega)$ and $|\beta| \leq m$ then $D^\beta u \in W^{1,\frac{n}{(1-\alpha)}}(\Omega)$,which is contained in $C^{0,\alpha}(\overline{\Omega})$). ☐

Definition 1.5.1.

A subset E of a metric space is said to be totally bounded if $\forall \varepsilon > 0$, E can be covered by finitely many balls of radius ε .

Theorem 1.5.1.

Let E be a subset of a complete metric space X. Then the following statements are equivalent.

(i) $\overline{E}$ is compact.

(ii) Every sequence in E has a convergent subsequence.

(iii) E is totally bounded.

Theorem 1.5.2.

if Ω is bounded and $p < n$, then $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q = \frac{np}{n-p}$.

Proof. Consider first the case $q = 1$. Let A be a bounded set in $W_0^{1,p}(\Omega)$.We may consider the members of A as members of $W^{1,p}(\mathbb{R}^n)$with supports contained in $\overline{\Omega}$. let $A_h = \{J_h u : u \in A\}$.Note that we have

$$|J_h u(x)| \leq h^{-n} \int_\Omega \rho\left(\frac{x-z}{h}\right) |u(z)|dz \leq h^{-n}(\max \rho) \|u\|_{L^1(\Omega)}$$

and

$$|D_i J_h u(x)| \leq h^{-n-1} \int_\Omega \left| D_i \rho\left(\frac{x-z}{h}\right)\right| |u(z)|\, dz \leq h^{-n-1}(\max |D_i \rho|) \|u\|_{L^1(\Omega)} .$$

Since Ω is bounded, $\|u\|_{L^1(\Omega)} \leq const. \|u\|_{L^p(\Omega)}$. The inequalities above show that A_h is a bounded equicontinuous set of functions in $C(\bar{\Omega})$. By the Arzela-Ascoli Theorem, every sequence in A_h has a subsequence that converges in $C(\bar{\Omega})$. Obviously, such subsequences also converge in $L^1(\Omega)$, so we see that A_h is totally bounded in $L^1(\Omega)$.

If $u \in A$ then

$$\begin{aligned} u(x) - J_h u(x) &= \int_{|z|\leq 1} \rho(z)(u(x) - u(x - hz))\, dz \\ &= \int_{|z|\leq 1} \rho(z) \int_0^{h\|z\|} -\frac{\partial}{\partial r} u\left(x - r\frac{z}{\|z\|}\right) dr\, dz. \end{aligned}$$

Thus

$$|u(x) - J_h u(x)| \le \int_{|z|\le 1} \rho(z) \int_0^{h\|z\|} \sum_{i=1}^{n} \left| D_i u\left(x - r\frac{z}{\|z\|}\right)\right| dr dz.$$

Integrating this with respect to x, we find

$$\begin{aligned}
\int_\Omega |u(x) - J_h u(x)|\, dx &\le \int_{|z|\le 1} \rho(z) \int_0^{h\|z\|} \sum_{i=1}^{n} \int_{\mathbb{R}^n} \left| D_i u\left(x - r\frac{z}{\|z\|}\right)\right| dx dr dz \\
&= \int_{|z|\le 1} \rho(z) \int_0^{h\|z\|} \sum_{i=1}^{n} \int_\Omega |D_i u(x)|\, dx dr dz \\
&= \int_{|z|\le 1} \rho(z) h \|z\| \sum_{i=1}^{n} \int_\Omega |D_i u(x)|\, dx dz \qquad (1.10) \\
&\le h \sum_{i=1}^{n} \int_\Omega |D_i u(x)|\, dx \\
&\le hB,
\end{aligned}$$

Where B is a constant depending on our bound of members of A in $W_0^{1,p}(\Omega)$.

Let $\varepsilon > 0$. Since A_h is totally bounded in $L^1(\Omega)$, we can cover A_h by a finite number of balls B_i of radius $\frac{\varepsilon}{2}$. Let $h = \frac{\varepsilon}{2B}$ .By (1.10), if $J_h u \in B_i$,then u is contained in a ball of radius ε centered at the center of B_i . Thus, A is covered by a finite number of balls of radius ε. i.e. A is totally bounded in $L^1(\Omega)$. Thus $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^1(\Omega)$.

Suppose $\phi \in W_0^{1,p}(\Omega)$. Then $\phi \in L^{\frac{np}{n-p}}(\Omega)$ by Theorem 1.4.3 and we get from Lemma 1.4.1 $\left(\text{with } s = 1 \text{ and } r = \frac{np}{n-p}\right)$ that

$$\|\phi\|_{L^q(\Omega)} \le \|\phi\|_{L^1(\Omega)}^{\lambda} \|\phi\|_{L^{\frac{np}{n-p}}(\Omega)}^{1-\lambda} \le C \|\phi\|_{L^1(\Omega)}^{\lambda} \left(\sum_{i=1}^{n} \|D_i \phi\|_{L^p(\Omega)}\right)^{1-\lambda}$$

Now let $\{u_m\}$ be a bounded sequence in $W_0^{1,p}(\Omega)$ and assume $\|u_m\|_{W_0^{1,p}(\Omega)} \le M$.Since $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^1(\Omega)$, we can extract a subsequence $\{u_m\}$ that converges in $L^1(\Omega)$. Applying the inequality above to $u_{m_f} - u_{m_k}$,noting that $\left\|u_{m_f} - u_{m_k}\right\|_{W_0^{1,p}(\Omega)} \le 2M$,
We obtain

$$\left\|u_{m_f} - u_{m_k}\right\|_{L^q(\Omega)} \le const. \left\|u_{m_f} - u_{m_k}\right\|_{L^1(\Omega)}^{\lambda},$$

showing that the subsequence is a Cauchy sequence in $L^q(\Omega)$. Hence the subsequence converges in $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$. ☐

Corollary 1.5.3.

If $kp < n$ and Ω is bounded then $W_0^{k,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$ for all $q < \frac{np}{n-kp}$.

Proof. $W_0^{k,p}(\Omega)$ is continuously imbedded in $W_0^{1,\frac{np}{n-(k-1)p}}(\Omega)$, which is compactly imbedded in $L^q(\Omega)$ if $q < \frac{np}{n-kp}$,by Theorem (1.5.2). ☐

Corollary 1.5.4.

The same compactness results hold for $W^{k,p}(\Omega)$ if Ω is a bounded, C^1 domain (or any other type of bounded domain for which there is an extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$).

1.6 Interpolation Results

The following results are very useful in PDE theory. We make use of Theorem 1.6.1 in our proof of Gårding's Inequality in our study of elliptic problems.

Theorem 1.6.1.

Let $u \in W_0^{k,p}(\Omega)$.Then for any $\varepsilon > 0$ and any $0 < |\beta| < k$

$$\left\| D^{\beta} u \right\|_{L^p(\Omega)} < \varepsilon \, \|u\|_{W^{k,p}(\Omega)} + C\varepsilon^{\frac{-|\beta|}{k-|\beta|}} \, \|u\|_{L^p(\Omega)}$$

Where C is a constant depending only on k.

Proof. We prove the result for $|\beta| = 1$, $k = 2$. The general result is easily obtained from this case by induction. In fact, we show that for each i

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{L^p} \leq \varepsilon \left\| \frac{\partial^2 u}{\partial x_i^2} \right\|_{L^p} + \frac{72}{\varepsilon} \, \|u\|_{L^p} \tag{1.11}$$

First suppose that $u \in C_0^2(R)$ and consider an interval (a,b) of length $b - a = \varepsilon$. If $y \in (a, \frac{a+\varepsilon}{3})$ and $z \in (\frac{b-\varepsilon}{3}, b)$, then by the Mean Value Theorem there is a $p \in (a,b)$ such that

$$|u'(p)| = \left| \frac{u(z) - u(y)}{z - y} \right| \leq \frac{3}{\varepsilon} \left(|u(z)| + |u(y)| \right)$$

Consequently, for every $x \in (a,b)$, we obtain

$$|u'(x)| = \left| u'(p) + \int_p^x u''(t)\,dt \right| \leq \frac{3}{\varepsilon} \left(|u(z)| + |u(y)| \right) + \int_a^b |u''(t)|\,dt.$$

Integrating with respect to y and z over the intervals $\left(a, \frac{a+\varepsilon}{3}\right)$ and $\left(\frac{b-\varepsilon}{3}, b\right)$ respectively, we obtain

$$\left| u'(x) \right| \leq \int_a^b \left| u''(t) \right| dt + \frac{18}{\varepsilon^2} \int_a^b |u(t)|\,dt,$$

so by Hölder's inequality and the inequality $(A+B)^p \leq 2^{p-1}(A^p + B^p)$,

$$\begin{aligned}
\left| u'(x) \right|^p &\leq 2^{p-1} \left(\left(\int_a^b \left| u''(t) \right| dt \right)^p + \frac{(18)^p}{\varepsilon^{2p}} \left(\int_a^b |u(t)|\,dt \right)^p \right) \\
&\leq 2^{p-1} \left(\left(\int_a^b \left| u''(t) \right|^p dt \right) \left(\int_a^b 1\,dt \right)^{p-1} + \frac{(18)^p}{\varepsilon^{2p}} \left(\int_a^b |u(t)|^p\,dt \right) \left(\int_a^b 1\,dt \right)^{p-1} \right) \\
&= 2^{p-1} \left(\varepsilon^{p-1} \int_a^b \left| u''(t) \right|^p dt + \frac{(18)^p}{\varepsilon^{p+1}} \int_a^b |u(t)|^p\,dt \right).
\end{aligned}$$

Integrating this with respect to x over the interval (a,b) gives

$$\int_a^b \left|u'(x)\right|^p dx = 2^{p-1}\left(\varepsilon^p \int_a^b \left|u''(t)\right|^p dt + \frac{(18)^p}{\varepsilon^p}\int_a^b |u(t)|^p dt\right).$$

We now subdivide $\mathbb{R}$ into intervals of length ϵ and obtain by adding all of these inequalities that

$$\int_{-\infty}^{\infty} \left|u'(x)\right|^p dx \leq 2^{p-1}\left(\varepsilon^p \int_{-\infty}^{\infty} \left|u''(t)\right|^p dt + \frac{(18)^p}{\varepsilon^p}\int_{-\infty}^{\infty} |u(t)|^p dt\right). \tag{1.12}$$

Suppose now that $u \in C_0^\infty(\mathbb{R}^n)$. Then we can apply (1.12) to u regarded as a function of x_i and integrate with respect to the remaining variables to obtain

$$\int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial x_i}\right|^p dx \leq 2^{p-1}\left(\varepsilon^p \int_{\mathbb{R}^n} \left|\frac{\partial^2 u}{\partial x_i^2}\right|^p dx + \frac{(18)^p}{\varepsilon^p}\int_{\mathbb{R}^n} |u|^p dx\right)$$

Taking the pth root of this and using $(A^p+B^p)^{1/p} \leq A+B$, we obtain (1.11). (Actually, we don't quite obtain (1.11). We actually obtain the inequality (1.11) for 2ε instead of ε . But since ε is an arbitrary positive constant, (1.11) holds). Finally, to obtain the result for $u \in W_0^\infty(\Omega)$, we take a sequence of functions in C_0^∞ converging to u. □

Corollary 1.6.1.

The interpolation inequality stated in Theorem 1.6.1 also applies to members of $W^{k,p}(\Omega)$, provided that Ω is a bounded C^2 domain (or any other domain for which there is a bounded extension operator $E: W^{2,p}(\Omega) \longrightarrow W^{2,p}(\mathbb{R}^n)$. Here the constant C may also depend on p and Ω.

Proof. Because of the extension operator, an inequality of the form (1.11) holds for functions in $W^{2,p}(\Omega)$. □

The Spaces $H^m(\Omega)$ and $H_0^m(\Omega)$

The following abstract theorem is a flexible tool for generating Sobolev Spaces. The ingredients of the construction are:

(i) The space $D'(\Omega;\mathbb{R}^n)$, in particular, for $n=1, D'(\Omega)$.

(ii) Two Hilbert spaces H and Z with $Z \hookrightarrow D'(\Omega;\mathbb{R}^n)$ for some $n \geq 1$. In particular

$$v_k \longrightarrow v \text{ in } Z \qquad \text{implies} \qquad v_k \longrightarrow v \text{ in } D'(\Omega;\mathbb{R}^n). \tag{1.13}$$

(iii) A linear continuous operator $L: H \longrightarrow D'(\Omega;\mathbb{R}^n)$ (such as a gradient or a divergence).

Theorem 1.6.2.

Define

$$W = \{v \in H : Lv \in Z\}$$

and

$$(u, v)_W = (u, v)_H + (Lu, Lv)_Z. \tag{1.14}$$

Then W is a Hilbert space with inner product given by (1.14). The embedding of W in H is continuous and the restriction of L to W is continuous from W into Z.

Proof. Thus W is an inner-product space. It remains to check its completeness. Let $\{v_k\}$ be a Cauchy sequence in W. We must show that there exists $v \in H$ such that

$$v_k \longrightarrow v \text{ in } H$$

and

$$Lv_k \longrightarrow Lv \text{ in } Z.$$

Observe that $\{v_k\}$ and $\{Lv_k\}$ are Cauchy sequences in H and Z, respectively. Thus, there exist $v \in H$ and $z \in Z$such that

$$v_k \longrightarrow v \text{ in } H \text{ and } Lv_k \longrightarrow z \text{ in } Z.$$

The continuity of L and (1.13) yield

$$Lv_k \longrightarrow Lv \text{ in } D'(\Omega;\mathbb{R}^n) \text{and} Lv_k \longrightarrow z \text{ in } D'(\Omega;\mathbb{R}^n).$$

Since the limit of a sequence in $D'(\Omega;\mathbb{R}^n)$ is unique, we infer that $Lv = z$. Therefore

$$Lv_k \longrightarrow Lv \text{ in } Z$$

and W is a Hilbert space.

The continuity of the embedding $W \subset H$ follows from

$$\|u\|_H \leq \|u\|_W$$

while the continuity of $L_{|W} : W \longrightarrow Z$ follows from

$$\|Lu\|_Z \leq \|u\|_W.$$

□

The space $H^1(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Choose in Theorem 1.6.2:

$$H = L^2(\Omega), Z = L^2\left(\Omega; \mathbb{R}^n\right) \hookrightarrow D'\left(\Omega; \mathbb{R}^n\right)$$

and $L : H \to D'(\Omega; \mathbb{R}^n)$given by

$$L = \nabla$$

where the gradient is considered in the sense of distributions. Then, W is the Sobolev space of the functions in $L^2(\Omega)$, whose first derivatives in the sense of distributions are functions in $L^2(\Omega)$. For this space we use the symbol $H^1(\Omega)$. Thus:

$$H^1(\Omega) = \left\{v \in L^2(\Omega) : \nabla v \in L^2(\Omega; \mathbb{R}^n)\right\}.$$

In other words, if $v \in H^1(\Omega)$, every partial derivative $\partial_{x_i} v$ is a function $v_i \in L^2(\Omega)$. This means that

$$\left\langle \theta_{x_i} v, \varphi \right\rangle = -\left(v, \theta_{x_i} \varphi\right)_{L^2}(\Omega) = \left(v_i, \varphi\right)_{L^2}(\Omega), \quad \forall \varphi \in D(\Omega)$$

Or, more explicitly,

$$\int_\Omega v(x) \theta_{x_i} \varphi(x) dx = -\int_\Omega v_i(x) \varphi(x) dx, \quad \forall \varphi \in D(\Omega).$$

In many applied situations, the Dirichlet integral

$$\int_\Omega |\nabla v|^2$$

represents an energy. The functions in $H^1(\Omega)$ are therefore associated with configurations having finite energy. From Theorem 1.6.2 and the separability of $L^2(\Omega)$, we have:

Proposition 1.6.1.

$H^1(\Omega)$ is a separable Hilbert space, continuously embedded in $L^2(\Omega)$. The gradient operator is continuous from $H^1(\Omega)$ into $L^2(\Omega; \mathbb{R}^n)$.

The inner product and the norm in $H^1(\Omega)$ are given, respectively, by

$$(u, v)_{H^1(\Omega)} = \int_\Omega uv dx + \int_\Omega \nabla u . \nabla v dx and \|u\|^2_{H^1(\Omega)} = \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx.$$

Exemple 1.6.1.

Let $\Omega = B_{1/2}(O) = \left\{x \in \mathbb{R}^2 : |x| < 1/2\right\}$ and $u(x) = (-log|x|)^a, x \neq O$. We have, using polar coordinates,

$$\int_{B_{1/2}(0)} u^2 dx = 2\pi \int_0^{1/2} (-\log r)^{2a} r dr < \infty, \text{ for every a } \in \mathbb{R},$$

so that $u \in L^2(B_{1/2}(0))$ for every $a \in \mathbb{R}$. Also:

$$u_{x_i} = -a x_i |x|^{-2}(-\log|x|)^{a-1}, i = 1,2$$

and therefore

$$|\nabla u| = \left|a(-\log|x|)^{a-1}\right| |x|^{-1}.$$

Using polar coordinates, we get

$$\int_{B_{1/2}(0)} |\nabla u|^2 dx = 2\pi a^2 \int_0^{1/2} |\log r|^{2a-2} r^{-1} dr.$$

This integral is finite only if $2-2a > 1$ or $a < 1/2$. In particular, ∇u represents the gradient of u in the sense of distribution as well. We conclude that $u \in H^1(B_1(0))$ only if $a < 1/2$. We point out that when $a > 0$, u is unbounded near 0.

Proposition 1.6.2.

Let $u \in L^2(a,b)$. Then $u \in H^1(a,b)$ if and only if u is continuous in $[a,b]$ and there exists $w \in L^2(a,b)$ such that

$$u(y) = u(x) + \int_x^y w(s)ds, \ \forall x, y \in [a,b]. \tag{1.15}$$

Also $u' = w$

Proof. Assume that u is continuous in $[a,b]$ and that (1.15) holds with $w \in L^2(a,b)$. Choose $x = a$. Replacing, if necessary, u by $u - u(a)$, we may assume $u(a) = 0$, so that

$$u(y) = \int_a^y w(s)ds, \ \forall x, y \in [a,b].$$

Let $\varphi \in D(a,b)$. We have:

$$\begin{aligned}\langle u', \varphi \rangle = -\langle u, \varphi' \rangle &= -\int_a^b u(s)\varphi'(s)ds \\ &= -\int_a^b \left[\int_a^b w(t)dt\right] \varphi'(s)ds\end{aligned}$$

(exchanging the order of integration)

$$\begin{aligned}&= -\int_a^b \left[\int_a^b \varphi'(s)ds\right] w(t)dt \\ &= \int_a^b \varphi(t)w(t)dt = \langle w, \varphi \rangle.\end{aligned}$$

Thus $u' = w$ in $D'(a,b)$ and therefore $u \in H^1(a,b)$. From the Lebesgue Differentiation Theorem we deduce that $u' = w$ a.e. as well. Viceversa, let $u \in H^1(a,b)$. Define

$$v(x) = \int_a^x u'(s)ds, \ x \in [a,b]. \tag{1.16}$$

The function v is continuous in $[a,b]$ and the above proof shows that $v' = u'$ in $D'(a,b)$.

$$u = v + C, \ C \in \mathbb{R},$$

and therefore u is continuous in $[a,b]$ as well. Moreover, (1.16) yields

$$u(y)-u(x)=v(y)-v(x)=\int_x^y u'(s)ds$$

which is (1.15). ☐

Since a function $u\in H^1(a,b)$ is continuous in $[a,b]$, the value $u(x_0)$ at every point $x_0\in[a,b]$ makes perfect sense. In particular the trace of u at the end points of the interval is given by the values $u(a)$ and $u(b)$.

The space $H_0^1(\Omega)$

Let $\Omega\subseteq\mathbb{R}^n$ be a domain. We study an important subspace of $H^1(\Omega)$.

Definition 1.6.1.

We denote by $H_0^1(\Omega)$ the closure of $D(\Omega)$ in $H^1(\Omega)$.

Thus $u\in H_0^1(\Omega)$ if and only if there exists a sequence $\{\varphi_k\}\subset D(\Omega)$ such that $\varphi_k\longrightarrow u$ in $H^1(\Omega)$, that is, such that both $\|\varphi_k-u\|_{L^2(\Omega)}\longrightarrow 0$ and $\|\nabla\varphi_k-\nabla u\|_{L^2(\Omega;\mathbb{R}^n)}\longrightarrow 0$ as $k\longrightarrow\infty$.

Since the test functions in $D(\Omega)$ have zero trace on $\partial\Omega$, every $u\in H_0^1(\Omega)$ inherits this property and it is reasonable to consider the elements $H_0^1(\Omega)$ as the functions in $H^1(\Omega)$ with zero trace on $\partial\Omega$. Clearly, $H_0^1(\Omega)$ is a Hilbert subspace of $H^1(\Omega)$.

An important property that holds in $H_0^1(\Omega)$, particularly useful in the solution of boundary value problems, is expressed by the following inequality of Poincaré. Recall that the diameter of a set Ω is given by

$$diam(\Omega)=\sup_{x,y\in\Omega}|x-y|.$$

Theorem 1.6.3.

Let $\Omega\subset\mathbb{R}^n$ be a bounded domain. There exists a positive constant C_P (a Poincaré's constant) depending only on n and $diam(\Omega)$, such that, for every $u\in H_0^1(\Omega)$,

$$\|u\|_{L^2(\Omega)}\le C_p\|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)}. \tag{1.17}$$

Proof. We use a strategy which is rather common for proving formulas in $H_0^1(\Omega)$. First, we prove the formula for $v\in D(\Omega)$; then, if $u\in H_0^1(\Omega)$, we select a sequence $vk\subset D(\Omega)$ converging to u in the $H^1(\Omega)$ norm as $k\to\infty$, that is

$$\|vk-u\|_{L^2(\Omega)}\to 0,\ \ \|\nabla vk-\nabla u\|_{L^2(\Omega;\mathbb{R}^n)}\to 0$$

In particular

$$\|v_k\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega)}, \quad \|\nabla v_k\|_{L^2(\Omega;\mathbb{R}^n)} \to \|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)}$$

Since (1.17) holds for every v_k, we have

$$\|v_k\|_{L^2(\Omega)} \leq Cp\|\nabla v_k\|_{L^2(\Omega;\mathbb{R}^n)}$$

Letting $k \to \infty$ we obtain (1.17) for u. Thus, it is enough to prove (1.17) for $v \in D(\Omega)$. Assume without loss of generality that $0 \in \Omega$, and set $\max_{x\in\Omega}|x| \leq M = diam(\Omega) < \infty$. Applying the Gauss Divergence Theorem, we can write

$$\int_\Omega div(v^2 x)dx = 0, \tag{1.18}$$

Since $v = 0$ on $\theta\Omega$. Now,

$$div(v^2 x) = 2v\nabla v \cdot x + nv^2$$

So that (1.18) yields

$$\int_\Omega^2 dx = -\frac{2}{n}\int_\Omega v\nabla v \cdot xdx$$

Since Ω is bounded, using Schwarz's inequality, we get

$$\int_\Omega v^2 dx = \frac{2}{n}|\int_\Omega v\nabla v \cdot xdx| \leq \frac{2M}{n}\left(\int_\Omega v^2 dx\right)^{1/2}\left(\int_\Omega |\nabla v|^2 dx\right)^{1/2}$$

Simplifying, it follows that

$$\|v\|_{L^2(\Omega)} \leq Cp\|\nabla v\|_{L^2(\Omega;\mathbb{R}^n)}$$

with $Cp = \frac{2M}{n}$. Inequality (1.17) implies that in $H_0^1(\Omega)$ the norm $\|u\|_{H^1(\Omega)}$ is equivalent to $\|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)}$.Indeed

$$\|u\|_{H^1(\Omega)} = \sqrt{\|u\|^2_{L^2}(\Omega) + \|\nabla u\|^2_{L^2}(\Omega;\mathbb{R}^n)}$$

and from (1.17),

$$\|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)} \leq \|u\|_{H^1(\Omega)} \leq \sqrt{C^2 p + 1}\,\|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)}$$

Unless explicitly stated, we will choose in $H_0^1(\Omega)$

$$(u,v)_{H_0^1(\Omega)} = (\nabla u, \nabla v)_{L^2(\Omega;\mathbb{R}^n)} \quad and \quad \|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)}$$

as inner product and norm, respectively.

□

The dual of $H_0^1(\Omega)$

In the applications of the Lax-Milgram theorem to boundary value problems, the dual of $H_0^1(\Omega)$ plays an important role. In fact it deserves a special symbol.

Definition 1.6.2.

We denote by $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$ with the norm

$$\|F\|_{H^{-1}(\Omega)} = \sup\left\{|Fv| : v \in H_0^1(\Omega), \|v\|_{H_0^1(\Omega)} \leq 1\right\}.$$

The first thing to observe is that, since $D(\Omega)$ is dense (by definition) and continuously embedded in $H_0^1(\Omega)$, $H^{-1}(\Omega)$ is a space of distributions. This means two things:

(a) If $F \in H^{-1}(\Omega)$, its restriction to $D(\Omega)$ is a distribution.

(b) If $F, G \in H^{-1}(\Omega)$ and $F\varphi = G\varphi$ for every $\varphi \in D(\Omega)$, then $F = G$.

To prove (a) it is enough to note that if $\varphi_k \longrightarrow \varphi$ in $D(\Omega)$, then $\varphi k \longrightarrow \varphi$in $H_0^1(\Omega)$ as well, and therefore $F\varphi k \longrightarrow F\varphi$ Thus $F \in D(\Omega)$. To prove (a) let $u \in H_0^1(\Omega)$ and $\varphi_k \longrightarrow u$ in $H_0^1(\Omega)$, with $\varphi k \in D(\Omega)$. Then, since $F\varphi_k = G\varphi_k$, we may write

$$Fu = \lim_{h\longrightarrow+\infty} F\varphi_k = \lim_{h\longrightarrow+\infty} G\varphi_k = G$$

whence $F = G$.

Thus, $H^{-1}(\Omega)$ is in one-to-one correspondence with a subspace of $D'(\Omega)$ and in this sense we will write $H^{-1}(\Omega) \subset D'(\Omega)$. Which distributions belong to $H^{-1}(\Omega)$? The following theorem gives a satisfactory answer.

Theorem 1.6.4.

$H^1(\Omega)$ is the set of distributions of the form

$$F = f_0 + div f \tag{1.19}$$

where $f_0 \in L^2(\Omega)$ and $f = (f_1, \cdots, f_n) \in L^2(\Omega; \mathbb{R}^n)$. Moreover:

$$\|F\|_{H^{-1}(\Omega)} \leq \left\{C_p \left\|f_0\right\|_{L^2(\Omega)} + \left\|f\right\|_{L^2(\Omega;\mathbb{R}^n)}\right\}. \tag{1.20}$$

Proof. Let $F \in H^{-1}(\Omega)$. From Riesz's Representation Theorem, there exists a unique $u \in H_0^1(\Omega)$ such that

$$(u, v)_{H_0^1(\Omega)} = Fv, \ \ \forall v \in H_0^1(\Omega)$$

Since

$$(u, v)_{H_0^1(\Omega)} = (\nabla u, \nabla v)_{L^2(\Omega;\mathbb{R}^n)} = -\langle div\nabla u, v\rangle$$

in $D'(\Omega)$, it follows that (1.19) holds with $f_0 = 0$ and $f = -\nabla u$. Moreover,

$$\|F\|_{H^{-1}(\Omega)} = \|u\|_{H_0^1(\Omega)} = \left\|f\right\|_{L^2(\Omega;\mathbb{R}^n)}$$

Viceversa, let $F = f_0 + divf$, with $f_0 \in L^2(\Omega)$ and $f = L^2(\Omega;\mathbb{R}^n)$. Then $F \in D'(\Omega)$ and, letting $\langle F, v\rangle = Fv$, we have,

$$Fv = \int_\Omega f_0 v dx + \int_\Omega f \cdot \nabla v dx, \quad \forall v \in D(\Omega)$$

From the Schwarz and Poincaré inequalities, we have

$$|Fv| \leq \left\{Cp\left\|f_0\right\|_{L^2(\Omega)} + \left\|f\right\|_{L^2(\Omega;\mathbb{R}^n)}\right\}\|v\|_{H_0^1(\Omega)}. \tag{1.21}$$

Thus, F is continuous in the H_0^1-norm. It remains to show that F has a unique continuous extension to all $H_0^1(\Omega)$. Take $u \in H_0^1(\Omega)$ and $\{v_k\} \subset D(\Omega)$ such that $\|v_k - u\|_{H_0^1(\Omega)}$. Then, (1.21) yields

$$|Fv_k - Fv_h| \leq \left\{Cp\left\|f_0\right\|_{L^2(\Omega)} + \left\|f\right\|_{L^2(\Omega;\mathbb{R}^n)}\right\}\|v_k - v_h\|_{H_0^1(\Omega)}$$

Therefore $\{Fv_k\}$ is a Cauchy sequence in $\mathbb{R}$ and converges to a limit we may denote by Fu, which is independent of the sequence approximating u, as it is not difficult to check. Finally, since

$$|Fu| = \lim_{k\to\infty}|Fv_k| \qquad and \qquad \|u\|_{H_0^1(\Omega)} = \lim_{k\to\infty}\|v_k\|_{H_0^1(\Omega)},$$

from (1.21) we get:

$$|Fu| \leq \left\{Cp\left\|f_0\right\|_{L^2(\Omega)} + \left\|f\right\|_{L^2(\Omega;\mathbb{R}^n)}\right\}\|u\|_{H_0^1(\Omega)}$$

showing that $F \in H^{-1}(\Omega)$. □

Theorem 1.6.5.

says that the elements of $H^{-1}(\Omega)$ are represented by a linear combination of functions in $L^2(\Omega)$ and their first derivatives (in the sense of distributions). In particular, $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$.

The spaces $H^m(\Omega), m > 1$

By involving higher order derivatives, we may construct new Sobolev spaces. Let $\mathbb{N}$ be the number of multi-indexes $\alpha = (\alpha_1, \cdots, \alpha_n)$ such that $|\alpha| = \sum_{i=1}^{n} \alpha_i \leq m$. Choose in Theorem (1.6.2)

$$H = L^2(\Omega), \ \ Z = L^2(\Omega;\mathbb{R}^N) \subset D'(\Omega;\mathbb{R}^N),$$

and $L : L^2(\Omega) \longrightarrow D'(\Omega;\mathbb{R}^N)$ given by

$$Lv = \{D^\alpha v\}_{|\alpha|\leq m}.$$

Then W is the Sobolev space of the functions in $L^2(\Omega)$, whose derivatives (in the sense of distributions) up to order m included, are functions in $L^2(\Omega)$. For this space we use the symbol $H^m(\Omega)$. Thus:

$$H^m(\Omega) = \left\{v \in L^2 : D^\alpha v \in L^2(\Omega), \ \forall \alpha \leq m\right\}.$$

From Theorem (1.6.2) and the separability of $L^2(\Omega)$, we deduce:

Proposition 1.6.3.

$H^m(\Omega)$ is a separable Hilbert space, continuously embedded in $L^2(\Omega)$. The operators $D^\alpha, |\alpha| \leq m$, are continuous from $H^m(\Omega)$ into $L^2(\Omega)$.
The inner product and the norm in $H^m(\Omega)$ are given, respectively, by

$$(u,v)_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u D^\alpha v dx.$$

and

$$\|u\|^2_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega \left|D^\alpha u\right|^2 dx.$$

If $u \in H^m(\Omega)$, any derivative of u of order $k \leq m$
belongs to $H^{m-k}(\Omega)$ and $H^m(\Omega) \hookrightarrow H^{m-k}(\Omega), k \geq 1$.
Trace Theorems In the following results, a vector x in $\mathbb{R}^n$ is denoted by $x = (x', x_n)$, where x' belongs to $\mathbb{R}^{n-1}$.

Lemma 1.6.1.

If $u \in W^{1,1}(\mathbb{R}^n)$, then for every $\zeta \in \mathbb{R}$, the function $v(x') = u(x',\zeta)$ is in $L^1(\mathbb{R}^{n-1})$, and

$$\|v\|_{L^1(\mathbb{R}^{n-1})} \leq \|u\|_{L^1(\mathbb{R}^n)} + \|D_n u\|_{L^1(\mathbb{R}^n)}$$

Proof. It suffices to prove the result for the case $\zeta = 0$ and $u \in C_0^\infty(\mathbb{R}^n)$. By the Mean Value Theorem for integrals

$$\int_0^1 \int_{\mathbb{R}^{n-1}} \left|u(x',x_n)\right| dx' dx_n = \int_{\mathbb{R}^{n-1}} \left|u(x',\sigma)\right| dx'$$

for some $\sigma \in [0,1]$. But

$$\begin{aligned} |u(x',0)| &= |u(x',\sigma) - \int_0^\sigma D_n u(x',t) dt| \\ &\leq |u(x',\sigma)| + \int_0^1 |D_n u(x',t)| dt. \end{aligned}$$

Integrating this over $\mathbb{R}^{n-1}$ gives

$$\begin{aligned} \|v\|_{L^1(\mathbb{R}^{n-1})} &\leq \int_{\mathbb{R}^{n-1}} |u(x',\sigma)| dx' + \int_{\mathbb{R}^{n-1}} \int_0^1 |D_n u(x',t)| dt dx' \\ &= \int_0^1 \int_{\mathbb{R}^{n-1}} |u(x',\sigma)| dx' dt + \int_{\mathbb{R}^{n-1}} \int_0^1 |D_n u(x',t)| dt dx'. \end{aligned}$$

□

Lemma 1.6.2.

If $u \in W^{1,p}(\mathbb{R}^n)$ where $p < n$, then for every $\zeta \in \mathbb{R}$, the function $v(x') = u(x',\zeta)$ is in $L^r(\mathbb{R}^{n-1})$, where

$$r = \frac{(n-1)p}{n-p} = 1 + \frac{n(p-1)}{n-p}$$

and there is a constant C depending on only n and p such that

$$\|v\|_{L^r(\mathbb{R}^{n-1})} \le C\,\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Proof. We can assume that $p > 1$ because the $p = 1$ case is dealt with in the previous lemma. We first show that if $u \in W^{1,p}(\mathbb{R}^n)$ then $w = |u|^r \in W^{1,1}(\mathbb{R}^n)$ and

$$\|w\|_{L^1(\mathbb{R}^n)} \le C\,\|Du\|^{r-1}_{L^p(\mathbb{R}^n)}\,\|u\|_{L^p(\mathbb{R}^n)}\,, \|D_i w\|_{L^1(\mathbb{R}^n)} \le C\,\|Du\|^r_{L^p(\mathbb{R}^n)}\,. \tag{1.22}$$

It suffices to prove this result for the case $u \in C_0^\infty(\mathbb{R}^n)$. Let $q = \frac{p}{(p-1)}$.Then $(r-1)\,q = \frac{np}{(n-p)}$, so by the Sobolev Embedding Theorem 1.4.3,

$$\left\||u|^{r-1}\right\|^q_{L^p(\mathbb{R}^n)} \le const.\|Du\|^{\frac{np}{(n-p)}}_{L^p(\mathbb{R}^n)}$$

and combining this with Hölder's Inequality, we get the first of (1.22):

$$\|w\|_{L^1(\mathbb{R}^n)} = \int |u|^r dx = \int |u|^{r-1}|u|dx \le \|u\|_{L^p(\mathbb{R}^n)}\left\||u|^{r-1}\right\|_{L^q(\mathbb{R}^n)} \le const.\|Du\|^{r-1}_{L^p(\mathbb{R}^n)}\|u\|_{L^p(\mathbb{R}^n)}.$$

Since $D_i w = \pm r|u|^{r-1}D_i u$, we obtain the second of (1.22):

$$\|D_i w\|_{L^1(\mathbb{R}^n)} = r\left\||u|^{r-1}\right\|_{L^p(\mathbb{R}^n)}\|D_i u\|_{L^p(\mathbb{R}^n)} \le const.\|Du\|^r_{L^p(\mathbb{R}^n)}.$$

We now apply Lemma 1.6.1 to w and immediately obtain the inequality

$$\begin{aligned}\|v\|_{L^r(\mathbb{R}^{n-1})} &\le const.\left(\|Du\|^{r-1}_{L^p(\mathbb{R}^{n-1})}\|u\|_{L^p(\mathbb{R}^{n-1})} + \|Du\|^r_{L^p(\mathbb{R}^{n-1})}\right)^{\frac{1}{r}}\\ &\le const.\left(\|Du\|^{1-\frac{1}{r}}_{L^p(\mathbb{R}^{n-1})}\|u\|^{\frac{1}{r}}_{L^p(\mathbb{R}^{n-1})} + \|Du\|_{L^p(\mathbb{R}^{n-1})}\right)\\ &\le const.\left(\|u\|_{L^p(\mathbb{R}^{n-1})} + \|Du\|_{L^p(\mathbb{R}^{n-1})}\right).\end{aligned}$$

□

Lemma 1.6.3.

If $u \in W^{k,p}(\mathbb{R}^n)$ where $kp < n$, then for every $\xi \in \mathbb{R}$,the function $v\left(x'\right) = u\left(x',\xi\right)$ is in $L^r\left(\mathbb{R}^{n-1}\right)$, where

$$r = \frac{(n-1)\,p}{n-kp}$$

and there is a constant C depending on only n, k and p such that

$$\|v\|_{L^r(\mathbb{R}^{n-1})} \le C\,\|u\|_{W^{k,p}(\mathbb{R}^n)}.$$

Penalization operators

Let E be a reflexive Banach space, we will always assume that the norm of E and that of its dual $E^{'}$ are strictly convex.

Definition 1.6.3.

We call penalization operator (attached to K) any operator β of $E \longrightarrow E^{'}$ having the following properties:

$$\beta \text{ is monotone bounded and semicontinuous of } E \longrightarrow E^{'}$$

$$S = \left\{\beta(v) = 0, v \in E\right\}.$$

Theorem 1.6.6.

We assume E defined as above and let F a duality operator of $E \longrightarrow E^{'}$ related to Φ we therefore have $(F(u), u) = \|F(u)\|_* \|u\|$, $\|F(u)\|_* = \Phi(\|u\|)$, where $\|\cdot\|_*$ is the norm in $E^{'}$,dual of $\|\cdot\|$ .So if P_S denotes the projection operator of $E \longrightarrow S$ such that $u \in E$,$P_S u$ is the only element of S such as

$$\|u - P_S u\| \leq \|u - s\| \ \ \forall s \in S.$$

the operator β given by

$$\beta(u) = F(u - P_S u),$$

is a penalty operator.

Penalty application

Theorem 1.6.7.

We assume E defined as before. Let A be an operator of $E \longrightarrow E^{'}$,pseudomonotonic and coercive in the sense :

$$\begin{cases} \exists v_0 \in S \text{ such as} \\ \frac{\langle A(v), v - v_0\rangle}{\|v\|} \longrightarrow +\infty, \text{ if } \|v\| \longrightarrow \infty. \end{cases}$$

So for everything $f \in E^{'}$,$\exists u \in S$ such as

$$\langle A(u), v - u\rangle \geq \left\langle f, v - u\right\rangle \ \ \forall v \in S.$$

Remark 1.6.1.

Let W be a Banach space of strictly convex norm as well as that of its dual and suppose that

1. $V \subset W$ with continuous injection , $V \subset W$ (so $W' \subset V'$)

2. K is a closed convex set in V and in W. We can then consider a penalization operator attached to K in the space W, such that:

$$\beta \text{ is monotone bounded and semicontinuous of } W \longrightarrow W'$$

$$k = \{\beta(w) = 0, w \in W\}.$$

Exemple 1.6.2.

We take $V = H_0^1(\Omega)$, A given by

$$\begin{cases} A(\varphi) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial \varphi}{\partial x_j}) + a_0\varphi, & a_0, a_{ij} \in L^{\infty}(\Omega), \\ \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \alpha(\xi_1^2 + \cdots + \xi_n^2), \alpha > 0, & \forall \xi_i \in \mathbb{R}, \text{ a.e. of } \Omega, \\ a_0(x) \geq \alpha_0, & \text{a.e. of } \Omega, \end{cases}$$

and either

$$k = \{\beta(w) = 0, w \in H_0^1\}.$$

We can still apply the remark 1.6.1.

we project in $L^2(\Omega)$, with $W = L^2(\Omega)$ and $k = \{\beta(w) = 0, w \in L^2(\Omega)\}$.

we choose

$$\beta(w) = F(w - P_k w)$$

with

$$F = identity$$

and $P_k w = w^+$ such as

$$\begin{cases} w(x) & \text{if } w(x) \geq 0 \\ 0 & \text{if } w(x) < 0 \end{cases}$$

The corresponding equation is therefore:

$$\begin{cases} Au_\varepsilon - \frac{1}{\varepsilon}u_\varepsilon^- = f \\ u_\varepsilon \in H_0^1(\Omega). \end{cases}$$

1.7 Some basic Formulas

Theorem 1.7.1. ***(Holder's inequality).***

Assume that $f \in L^P$ and $g \in L^{p'}$ with $1 \leqslant P \leqslant \infty$. then $fg \in L^1(\Omega)$ and

$$\int_\Omega |fg| \leqslant \|f\|_p \|g\|_{p'}$$

Proof. We first consider the case where $1 < p, q < \infty$ (the cases $p = 1, q = \infty$ or $p = \infty, q = 1$ are simpler and handled separately).

To begin with, the key ingredient is **Young's Inequality**, which states that for $a, b \geq 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

This follows from the concavity of the logarithm:

$$\ln(ab) = \ln a + \ln b = \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q) \leq \ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right).$$

In addition, assume $\|f\|_p = \|g\|_q = 1$ (otherwise, normalize f and g). Then, applying Young's Inequality pointwise:

$$|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}.$$

Integrating both sides over X gives:

$$\int_X |f(x)g(x)|\, d\mu(x) \leq \frac{1}{p}\int_X |f(x)|^p\, d\mu(x) + \frac{1}{q}\int_X |g(x)|^q\, d\mu(x) = \frac{1}{p} + \frac{1}{q} = 1.$$

Thus, $\|fg\|_1 \leq 1 = \|f\|_p \|g\|_q$.

Moreover, for arbitrary $f \neq 0$ and $g \neq 0$, define:

$$\tilde{f} = \frac{f}{\|f\|_p}, \quad \tilde{g} = \frac{g}{\|g\|_q}.$$

Then $\|\tilde{f}\|_p = \|\tilde{g}\|_q = 1$, and by Step 2:

$$\|\tilde{f}\tilde{g}\|_1 \leq 1.$$

Multiplying through by $\|f\|_p \|g\|_q$ gives the desired inequality.

Special Cases:

- If $p = 1$ and $q = \infty$, then $\|g\|_\infty = \operatorname{ess\,sup}|g|$, and the inequality follows from $|fg| \leq |f| \cdot \|g\|_\infty$ almost everywhere. - If $p = \infty$ and $q = 1$, the argument is symmetric. □

Theorem 1.7.2. ***(Poincaré's inequality)***

If $v \in W_0^{1,2}(\Omega)$, then there exists a positive constant $C = C(l)$ such that

$$\int_\Omega |v|^2 \leqslant C \int_\Omega |\nabla v|^2.$$

Proof. We proceed in several steps:

To start with, since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it suffices to prove the inequality for $u \in C_c^\infty(\Omega)$ and extend by density.

Additionally, fix $x \in \Omega$ and let $\omega \in \mathbb{S}^{n-1}$ be a unit vector. Since u has compact support, there exists $t_0 > 0$ such that $u(x + t\omega) = 0$ for $t \geq t_0$. By the fundamental theorem of calculus,

$$u(x) = -\int_0^\infty \frac{d}{dt} u(x + t\omega)\, dt.$$

Taking absolute values and using the chain rule gives

$$|u(x)| \le \int_0^\infty |\nabla u(x+t\omega)\cdot\omega|\,dt \le \int_0^\infty |\nabla u(x+t\omega)|\,dt.$$

Correspondingly, integrate over $\omega \in \mathbb{S}^{n-1}$:

$$|u(x)| \le \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\int_0^\infty |\nabla u(x+t\omega)|\,dt\,d\omega.$$

Besides, let $y = x + t\omega$, so $t^{n-1}dt\,d\omega = dy$. Then

$$|u(x)| \le \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{R}^n}\frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy.$$

Along with, square both sides and integrate over Ω:

$$\|u\|^2_{L^2(\Omega)} \le C\int_\Omega\left(\int_{\mathbb{R}^n}\frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy\right)^2 dx.$$

By the Hardy-Littlewood-Sobolev inequality for the Riesz potential, we obtain

$$\|u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}.$$

In conclusion, the inequality extends to all $u \in H^1_0(\Omega)$ by density of $C_c^\infty(\Omega)$. □

Theorem 1.7.3. *(**Sub Representation Formula**)*

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $u \in C^1(\Omega)\cap W^{1,p}(\Omega)$ for some $1 \le p < \infty$. Then, for any ball $B_R(x) \subset \Omega$, there exists a constant $C = C(n) > 0$ such that:

$$|u(x) - u_{B_R}| \le C\int_{B_R(x)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy,$$

where $u_{B_R} = \frac{1}{|B_R|}\int_{B_R(x)} u(y)\,dy$ is the average of u over $B_R(x)$.

Proof. We proceed in steps:
Firstly, let $x \in \Omega$ and $B_R(x) \subset \Omega$. For any $y \in B_R(x)$, we express y in polar coordinates centered at x:

$$y = x + r\omega, \quad \text{where } r = |x-y|, \quad \omega \in \mathbb{S}^{n-1}.$$

Then, the volume element becomes $dy = r^{n-1}dr\,d\omega$.
Secondly, for almost every $\omega \in \mathbb{S}^{n-1}$, we write $u(x)$ in terms of its radial derivative:

$$u(x) = u(x+r\omega) - \int_0^r \frac{\partial}{\partial s}u(x+s\omega)\,ds.$$

Averaging over $r \in [0,R]$ gives:

$$u(x) = \frac{1}{R^n}\int_0^R nr^{n-1}u(x+r\omega)\,dr - \frac{1}{R^n}\int_0^R nr^{n-1}\left(\int_0^r \frac{\partial}{\partial s}u(x+s\omega)\,ds\right)dr.$$

Thirdly, integrate over $\omega \in \mathbb{S}^{n-1}$ and divide by the surface area $|\mathbb{S}^{n-1}|$:

$$u(x) = \frac{n}{R^n|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\int_0^R r^{n-1}u(x+r\omega)\,drd\omega - \text{(remainder term)}.$$

The first term simplifies to the average u_{B_R}:

$$u_{B_R} = \frac{1}{|B_R|}\int_{B_R(x)} u(y)\,dy = \frac{n}{R^n|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\int_0^R r^{n-1}u(x+r\omega)\,drd\omega.$$

Fourthly,the remainder term is:

$$\left|\frac{n}{R^n|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\int_0^R r^{n-1}\left(\int_0^r \nabla u(x+s\omega)\cdot\omega\,ds\right)drd\omega\right|.$$

Since $|\omega| = 1$, we have:

$$|\nabla u(x+s\omega)\cdot\omega| \leq |\nabla u(x+s\omega)|.$$

Changing the order of integration and using Fubini's theorem, we get:

$$|u(x) - u_{B_R}| \leq C\int_0^R\int_{\mathbb{S}^{n-1}} |\nabla u(x+s\omega)|s^{n-1}\,d\omega ds.$$

Reverting back to Cartesian coordinates ($y = x + s\omega$, $dy = s^{n-1}dsd\omega$):

$$|u(x) - u_{B_R}| \leq C\int_{B_R(x)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy.$$

Finally, for $u \in W^{1,p}(\Omega)$, we use a density argument: approximate u by smooth functions $u_k \in C^\infty(\Omega)\cap W^{1,p}(\Omega)$ and pass to the limit. The constant C depends only on the dimension n.

□

Theorem 1.7.4. ***(Tonelli's Theorem)*** *Assume that $E \subset \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ are measurable sets and $f : E\times F \to \mathbb{R}$ is a nonnegative measurable function. Then, we have the followings:*

- *For a.e. $x \in E$, $f(x,.)$ is measurable in F.*
- *For a.e. $y \in F$, $f(.,y)$ is measurable in E.*
- *The function*

  $$g(x) = \int_F f(x,y)dy, \quad (resp. \quad h(y) = \int_E f(x,y)dx)$$

  is measurable in E (resp. F).
- *We have the following equality of the integrals(in $\overline{\mathbb{R}}$)*

$$\int_{E\times F} f(x,y)dxdy = \int_E\left(\int_F f(x,y)dy\right)dx = \int_F\left(\int_E f(x,y)dx\right)dy$$

Proof. We proceed in several steps.

First, assume $f = \chi_E$ where $E \in \mathscr{A} \otimes \mathscr{B}$ is a measurable set. By the definition of the product measure $\mu \times \nu$, we have:

$$\int_{X\times Y} \chi_E \, d(\mu \times \nu) = (\mu \times \nu)(E).$$

By the construction of the product measure, we know:

$$(\mu \times \nu)(E) = \int_X \nu(E_x) \, d\mu(x) = \int_Y \mu(E^y) \, d\nu(y),$$

where $E_x = \{y \in Y : (x, y) \in E\}$ and $E^y = \{x \in X : (x, y) \in E\}$. Thus, for indicator functions, the theorem holds:

$$\int_{X\times Y} \chi_E \, d(\mu \times \nu) = \int_X \left(\int_Y \chi_E(x, y) \, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X \chi_E(x, y) \, d\mu(x) \right) d\nu(y).$$

By linearity, the result extends to non-negative simple functions $f = \sum_{i=1}^n c_i \chi_{E_i}$.

Including, for a general non-negative measurable f, we approximate it by an increasing sequence of simple functions $f_n \uparrow f$. By the Monotone Convergence Theorem (MCT):

$$\int_{X\times Y} f \, d(\mu \times \nu) = \lim_{n\to\infty} \int_{X\times Y} f_n \, d(\mu \times \nu).$$

Applying the result for simple functions:

$$\int_{X\times Y} f_n \, d(\mu \times \nu) = \int_X \left(\int_Y f_n(x, y) \, d\nu(y) \right) d\mu(x).$$

For each x, $f_n(x, \cdot) \uparrow f(x, \cdot)$, so by MCT again:

$$\int_Y f_n(x, y) \, d\nu(y) \uparrow \int_Y f(x, y) \, d\nu(y).$$

Thus, taking the limit:

$$\int_X \left(\int_Y f_n(x, y) \, d\nu(y) \right) d\mu(x) \uparrow \int_X \left(\int_Y f(x, y) \, d\nu(y) \right) d\mu(x).$$

A symmetric argument applies to the other iterated integral, proving:

$$\int_{X\times Y} f \, d(\mu \times \nu) = \int_X \left(\int_Y f(x, y) \, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y) \, d\mu(x) \right) d\nu(y).$$

Furthermore, the measurability of $x \mapsto \int_Y f(x, y) \, d\nu(y)$ follows from the fact that it is the limit of the measurable functions $x \mapsto \int_Y f_n(x, y) \, d\nu(y)$ (which are measurable since f_n are simple). Similarly for $y \mapsto \int_X f(x, y) \, d\mu(x)$.

Since all steps hold for non-negative measurable functions on σ-finite spaces, Tonelli's theorem is proved. □

Consider the central hypothesis in the Lebesgue Dominated Convergence Theorem, namely that there is a function g integrable on E such that for all n, $|f_n| \leq g$ on E. This hypothesis implies two properties of $\{f_n\}$ that are important in their own right.

1.8 Vitali's Convergence Theorems

Proposition 1.8.1. *Let f be integrable over E. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that if $A \subseteq E$ and $m(A) < \delta$ then $\int_E |f| < \epsilon$.*

Definition 1.8.1. *A family $\mathscr{F}$ of measurable functions on E is uniformly integrable over E if given $\epsilon > 0$ there is a $\delta > 0$ such that if $A \subseteq E$ and $m(A) < \delta$ then $\int_E |f| < \epsilon$ for all $f \in \mathscr{F}$.*

Remark 1.8.1. *(1) Note that the only assumption about A in both the proposition and the definition is that $m(A) < \delta$. That is, the inequality holds for any subset A as long as $m(A) < \delta$.*
(2) Note that if the sequence f_n satisfies $|f_n| \leq g$ for some integrable function g on E, then the family $\{f_n\}$ is uniformly integrable over E.

Theorem 1.8.1 (Vitali)**.** *Let $m(E) < \infty$ and suppose that the sequence $\{f_n\}$ is uniformly integrable over E. If $f_n \to f$ pointwise a.e. on E, then f is integrable on E and $\int_E f_n \to \int_E f$ as $n \to \infty$.*

Proof. We proceed in several steps:
Since $\{f_n\}$ is uniformly integrable, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any measurable $A \subseteq E$ with $m(A) < \delta$, we have

$$\int_A |f_n| < \epsilon \quad \text{for all } n.$$

By Egorov's Theorem, since $f_n \to f$ pointwise a.e. on E and $m(E) < \infty$, for any $\delta > 0$, there exists a measurable subset $B \subseteq E$ such that $m(E \setminus B) < \delta$ and $f_n \to f$ uniformly on B.
The uniform convergence on B implies that f is measurable on B, and since $m(E \setminus B) < \delta$, the uniform integrability condition ensures that

$$\int_{E \setminus B} |f_n| < \epsilon \quad \text{for all } n.$$

By Fatou's Lemma, we have

$$\int_{E \setminus B} |f| \leq \liminf_{n \to \infty} \int_{E \setminus B} |f_n| \leq \epsilon.$$

Thus, f is integrable on $E \setminus B$. On B, f is the pointwise limit of integrable functions f_n, and hence integrable. Therefore, f is integrable on E.
We now show that $\int_E f_n \to \int_E f$. For any $\epsilon > 0$, choose $\delta > 0$ as in the uniform integrability condition. By Egorov's Theorem, select $B \subseteq E$ such that $m(E \setminus B) < \delta$ and $f_n \to f$ uniformly on B.
Then,

$$\left| \int_E f_n - \int_E f \right| \leq \int_E |f_n - f| = \int_B |f_n - f| + \int_{E \setminus B} |f_n - f|.$$

For the first term, since $f_n \to f$ uniformly on B, there exists N such that for all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$ for all $x \in B$. Thus,

$$\int_B |f_n - f| < \epsilon \cdot m(B) \leq \epsilon \cdot m(E).$$

For the second term, by the uniform integrability and the integrability of f, we have

$$\int_{E\setminus B} |f_n - f| \leq \int_{E\setminus B} |f_n| + \int_{E\setminus B} |f| < \epsilon + \epsilon = 2\epsilon.$$

Combining these, we get

$$\left|\int_E f_n - \int_E f\right| < \epsilon \cdot m(E) + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

□

Theorem 1.8.2 (Vitali converse). *Let $m(E) < \infty$ and suppose that $\{h_n\}$ is a sequence of non-negative integrable functions that converge pointwise a.e. on E to 0. Then $\lim_n \int_E h_n = 0$ if and only if $\{h_n\}$ is uniformly integrable over E.*

Proof. We prove both directions of the equivalence.

Part 1

Assume $\{h_n\}$ is uniformly integrable. Since $h_n \to 0$ pointwise a.e. and $m(E) < \infty$, by Vitali's Convergence Theorem (Theorem 0.1), we have:

$$\lim_{n\to\infty} \int_E h_n = \int_E 0 = 0.$$

Part 2

Assume $\lim_{n\to\infty} \int_E h_n = 0$. We must show that $\{h_n\}$ is uniformly integrable.
Fix $\epsilon > 0$. Since $\int_E h_n \to 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$\int_E h_n < \frac{\epsilon}{2}.$$

For each $n = 1, \ldots, N-1$, since h_n is integrable, by Proposition 0.1, there exists $\delta_n > 0$ such that for any measurable $A \subseteq E$ with $m(A) < \delta_n$:

$$\int_A h_n < \epsilon.$$

Let $\delta = \min(\delta_1, \ldots, \delta_{N-1}, \frac{\epsilon}{2C})$, where C is a bound for $\int_E h_n$ (which exists because $\int_E h_n \to 0$).
Now, for any measurable $A \subseteq E$ with $m(A) < \delta$: - If $n \geq N$, then:

$$\int_A h_n \leq \int_E h_n < \frac{\epsilon}{2} < \epsilon.$$

- If $n < N$, then by the choice of $\delta \leq \delta_n$:

$$\int_A h_n < \epsilon.$$

Thus, for all n, $\int_A h_n < \epsilon$ whenever $m(A) < \delta$, proving that $\{h_n\}$ is uniformly integrable.

□

Proposition 1.8.2. *Let* f *be integrable over* E*. Then for every* $\epsilon > 0$ *there is a subset* $E_0 \subseteq E$ *with* $m(E_0) < \infty$ *such that* $\int_{E-E_0} |f| < \epsilon$*.*

Definition 1.8.2. *A family* $\mathscr{F}$ *of measurable functions on* E *is tight over* E *if given* $\epsilon > 0$ *there is a subset* $E_0 \subseteq E$ *with* $m(E_0) < \infty$ *such that* $\int_{E-E_0} |f| < \epsilon$ *for all* $f \in \mathscr{F}$*.*

Remark 1.8.2. *Note that if the sequence* f_n *satisfies* $|f_n| \leq g$ *for some integrable function* g *on* E*, then the family* $\{f_n\}$ *is tight over* E*.*

Theorem 1.8.3 (Vitali)**.** *Let* $\{f_n\}$ *be a sequence that is uniformly integrable and tight over* E*. If* $f_n \to f$ *pointwise a.e. on* E*, then* f *is integrable on* E *and* $\int_E f_n \to \int_E f$ *as* $n \to \infty$*.*

Proof. The proof is based on the following steps:
Since $\{f_n\}$ is tight, for $\epsilon = 1$, there exists $E_0 \subseteq E$ with $m(E_0) < \infty$ such that:

$$\int_{E\setminus E_0} |f_n| < 1 \quad \text{for all } n.$$

On E_0 (which has finite measure), since $\{f_n\}$ is uniformly integrable and $f_n \to f$ a.e., by Theorem 0.1 (Vitali's Convergence Theorem for finite measure spaces), f is integrable on E_0 and:

$$\int_{E_0} f_n \to \int_{E_0} f.$$

For $E \setminus E_0$, by Fatou's Lemma:

$$\int_{E\setminus E_0} |f| \leq \liminf_{n\to\infty} \int_{E\setminus E_0} |f_n| \leq 1.$$

Thus f is integrable on all of E.
For any $\epsilon > 0$, by tightness there exists $E_1 \subseteq E$ with $m(E_1) < \infty$ such that:

$$\int_{E\setminus E_1} |f_n| < \epsilon \quad \text{for all } n.$$

Again by Fatou's Lemma:

$$\int_{E\setminus E_1} |f| \leq \liminf_{n\to\infty} \int_{E\setminus E_1} |f_n| \leq \epsilon.$$

Fix $\epsilon > 0$. Choose:

1. $E_2 \subseteq E$ with $m(E_2) < \infty$ such that $\int_{E\setminus E_2} |f_n| < \epsilon$ and $\int_{E\setminus E_2} |f| < \epsilon$ (by tightness)
2. $\delta > 0$ such that for any $A \subseteq E_2$ with $m(A) < \delta$, $\int_A |f_n| < \epsilon$ for all n (uniform integrability on E_2)
3. By Egorov's Theorem, find $B \subseteq E_2$ with $m(E_2 \setminus B) < \delta$ where $f_n \to f$ uniformly

Now decompose:

$$\left|\int_E f_n - \int_E f\right| \leq \int_{E\setminus E_2} |f_n| + \int_{E\setminus E_2} |f| + \int_{E_2\setminus B} |f_n| + \int_{E_2\setminus B} |f| + \left|\int_B (f_n - f)\right|$$

Each term is bounded:

- First two terms $< \epsilon$ by tightness
- Third term $< \epsilon$ by uniform integrability ($m(E_2 \setminus B) < \delta$)
- Fourth term $< \epsilon$ by Fatou's Lemma
- Last term $< \epsilon \cdot m(B)$ for large n by uniform convergence

Thus for sufficiently large n:

$$\left| \int_E f_n - \int_E f \right| < 5\epsilon$$

Since ϵ was arbitrary, the result follows. $\square$

Theorem 1.8.4 (Vitali converse)**.** *Let $\{h_n\}$ be a sequence of non-negative integrable functions on E that converge pointwise a.e. on E to 0. Then $\lim_n \int_E h_n = 0$ if and only if $\{h_n\}$ is uniformly integrable and tight over E.*

Proof. We prove both directions of the equivalence.

($\Rightarrow$) Assume $\lim_{n\to\infty} \int_E h_n = 0$:

1. For any $\epsilon > 0$, since $\int_E h_n \to 0$, there exists N such that for all $n \geq N$, $\int_E h_n < \epsilon/2$.

   For $1 \leq n < N$, since each h_n is integrable, by Proposition 0.1 there exists $\delta_n > 0$ such that for any $A \subseteq E$ with $m(A) < \delta_n$, we have $\int_A h_n < \epsilon$.

   Take $\delta = \min(\delta_1, \ldots, \delta_{N-1}, \epsilon/(2 \sup_{k\geq N} \|h_k\|_1))$. Then for any A with $m(A) < \delta$:

   - For $n < N$: $\int_A h_n < \epsilon$ by construction
   - For $n \geq N$: $\int_A h_n \leq \int_E h_n < \epsilon/2 < \epsilon$

2. For $\epsilon > 0$, since $\int_E h_n \to 0$, there exists N such that for $n \geq N$, $\int_E h_n < \epsilon$.

   For $1 \leq n < N$, since h_n is integrable, by Proposition 0.2 there exists $E_n \subseteq E$ with $m(E_n) < \infty$ such that $\int_{E \setminus E_n} h_n < \epsilon$.

   Take $E_0 = \bigcup_{n=1}^{N-1} E_n$, which has finite measure. Then:

   - For $n < N$: $\int_{E\setminus E_0} h_n \leq \int_{E\setminus E_n} h_n < \epsilon$
   - For $n \geq N$: $\int_{E\setminus E_0} h_n \leq \int_E h_n < \epsilon$

($\Leftarrow$) Assume $\{h_n\}$ is uniformly integrable and tight:

Given $\epsilon > 0$:

1. there exists $E_0 \subseteq E$ with $m(E_0) < \infty$ such that $\int_{E\setminus E_0} h_n < \epsilon/2$ for all n.

2. On E_0 (finite measure), since $\{h_n\}$ is uniformly integrable and $h_n \to 0$ a.e., by Theorem 0.1:

   $$\int_{E_0} h_n \to 0$$

   So there exists N such that for $n \geq N$, $\int_{E_0} h_n < \epsilon/2$.

3. Thus for $n \geq N$:

   $$\int_E h_n = \int_{E_0} h_n + \int_{E \setminus E_0} h_n < \epsilon/2 + \epsilon/2 = \epsilon$$

Hence $\lim_{n\to\infty} \int_E h_n = 0$.

□

Chapter 2

Elliptic equations with $W_0^{1,p}(\Omega)$ *solutions*

The compactness method is widely used in the study of Dirichlet problems involving the p-Laplacian and partial differential equations (PDEs) with growth conditions. In this chapter, we study the existence and uniqueness of solutions for both the Laplacian problem and the p-Laplacian problem using the compactness method as in [1]. Through this chapter we use $u(x) = u^+(x) - u^-(x)$ where u^+ and u^- denote the positive part and negative part of a function u, respectively. These are defined as follows: $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$, where $|u(x)| = u^+(x) + u^-(x)$.

2.1 Laplacian equations with $W_0^{1,2}(\Omega)$ solutions

In this part, we study the existence of solutions for nonlinear elliptic equations of the form:

$$u \in W_0^{1,2}(\Omega) : -\text{div}(a(x)Du) + \gamma u|Du|^2 = f, \quad \gamma > 0. \tag{2.1}$$

Where Ω is a bounded subset of $\mathbb{R}^N$ and $-\text{div}(a(x)Du)$ is a Leray-Lions operator with $a(x)Du$ is a Carathéodory function such that $a(\cdot) \geq C$.

$$f \in L^1(\Omega) \tag{2.2}$$

We give the definition of a weak solution to the problem (2.1).

Definition 2.1.1. *Let $f \in L^1(\Omega)$. A function $u \in W_0^{1,2}(\Omega)$ is called weak solution of problem (2.1) in the sense that*

$$\langle -div(a(x)Du), v\rangle + \int_\Omega \gamma u|Du|^2 v = \int_\Omega f v, \quad \forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega).$$

with $\gamma u|Du|^2 \in L^1(\Omega)$.

We have the following theorem

Theorem 2.1.1. *[13] Under the assumption (2.2), there exists a solution $u \in W_0^{1,2}(\Omega)$ to the problem (2.1) with $\gamma u|Du|^2 \in L^1(\Omega)$.*

The proof of theorem 2.1.1 is based on three steps:

Proof. **Step1: Approximation**

Let $\{f_\epsilon\}$ be a sequence in $L^{p'}(\Omega)$ such that $f_\epsilon \to f$ in $L^1(\Omega)$ with $\|f_\epsilon\|_{L^1} \le \|f\|_{L^1}$. Let u_ϵ be a solution of the approximate problem:

$$-\text{div}(a(x)Du_\epsilon) + \gamma u_\epsilon |Du_\epsilon|^2 = f_\epsilon \quad \text{in } \mathscr{D}'(\Omega), \tag{2.3}$$

where $u_\epsilon \in W_0^{1,2}(\Omega)$ and $\gamma u_\epsilon |Du_\epsilon|^2 \in L^1(\Omega)$.

Step 2: A Priori Estimates

Multiply the equation (2.3) by $T_k(u_\epsilon)$, where T_k is the truncation function at height k, we obtain:

$$\alpha \int_\Omega |DT_k(u_\epsilon)|^2 \le k\|f_\epsilon\|_{L^1}.$$

Next, we prove:

$$\int_{|u_\epsilon|>t} |\gamma u_\epsilon |Du_\epsilon|^2| \le \int_{|u_\epsilon|>t} |f_\epsilon|, \quad \text{for any } t \in \mathbb{R}^+.$$

This is achieved by testing the equation with $\psi_i(u_\epsilon)$, where $\{\psi_i\}$ is a sequence of smooth, increasing functions converging to the sign function:

$$\psi(s) = \begin{cases} 1 & \text{if } s \ge t, \\ 0 & \text{if } -t < s < t, \\ -1 & \text{if } s \le -t. \end{cases}$$

Passing to the limit, we derive:

$$\int_{|u_\epsilon|>t} |Du_\epsilon|^2 \le \frac{1}{\gamma} \int_{|u_\epsilon|>t} |f_\epsilon|, \quad \text{for } t \ge \sigma.$$

Combining these estimates, we get:

$$\int_\Omega |Du_\epsilon|^2 = \int_{|u_\epsilon|\le\sigma} |Du_\epsilon|^2 + \int_{|u_\epsilon|>\sigma} |Du_\epsilon|^2 \le \frac{\sigma}{\alpha}\|f_\epsilon\|_{L^1} + \frac{1}{\gamma}\int_{|u_\epsilon|>\sigma} |f_\epsilon| \le \left(\frac{\sigma}{\alpha} + \frac{1}{\gamma}\right)\|f\|_{L^1}.$$

Thus, $\{u_\epsilon\}$ is bounded in $W_0^{1,2}(\Omega)$..

Step3: Passage to the Limit

Since $\{u_\epsilon\}$ is bounded in $W_0^{1,2}(\Omega)$, we may extract a subsequence (still denoted u_ϵ) such that:

$$u_\epsilon \rightharpoonup u \quad \text{weakly in } W_0^{1,2}(\Omega), \quad \text{strongly in } L^2(\Omega), \quad \text{and a.e. in } \Omega.$$

We now prove:

$$u_\epsilon^+ \to u^+ \quad \text{strongly in } W_0^{1,2}(\Omega).$$

Let $k > \sigma$. Test the equation with $T_k(u_\epsilon^+ - u^+)$. On the set where $T_k(u_\epsilon^+ - u^+) > 0$, we have $u_\epsilon > 0$, and by the sign condition, $g(x, u_\epsilon, Du_\epsilon) \ge 0$. Thus:

$$\langle A(u_\epsilon), T_k(u_\epsilon^+ - u^+)\rangle \le \int_\Omega f_\epsilon T_k(u_\epsilon^+ - u^+).$$

Since $u_\epsilon = u_\epsilon^+$ on $\{u_\epsilon^+ > u^+\}$, we rewrite this as:

$$\int_\Omega a(x) Du_\epsilon^+ \cdot D(T_k(u_\epsilon^+ - u^+)) \leq \int_\Omega f_\epsilon T_k(u_\epsilon^+ - u^+).$$

Passing to the limit, we obtain:

$$\lim_{\epsilon \to 0} \int_\Omega \left[a(x) Du_\epsilon^+ - a(x) Du^+)\right] \cdot D(T_k(u_\epsilon^+ - u^+)) = 0.$$

For $k \to \infty$, the right-hand side vanishes uniformly in ϵ, leading to:

$$\lim_{\epsilon \to 0} \int_\Omega \left[a(x) Du_\epsilon^+ - a(x) Du^+)\right] \cdot D(u_\epsilon^+ - u^+) = 0.$$

By the Leray-Lions lemma, this implies the strong convergence of u_ϵ^+ in $W_0^{1,2}(\Omega)$.
A similar argument, using $T_k(u_\epsilon^- - u^-)^+$ and $\phi_\lambda((u_\epsilon^- - T_k(u^-))^-)$ as test functions, shows:

$$u_\epsilon^- \to u^- \quad \text{strongly in } W_0^{1,2}(\Omega).$$

Combining these results, we deduce:

$$Du_\epsilon \to Du \quad \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega.$$

The continuity of g in (s, ξ) and Vitali's theorem yield:

$$\gamma u_\epsilon |Du_\epsilon|^2 \to \gamma u |Du|^2 \quad \text{in } L^1(\Omega).$$

Finally, passing to the limit in the weak formulation:

$$\langle -\mathrm{div}(a(x) Du_\epsilon), v\rangle + \int_\Omega \gamma u_\epsilon |Du_\epsilon|^2 v = \int_\Omega f_\epsilon v,$$

we obtain the desired solution $u \in W_0^{1,2}(\Omega)$ satisfying:

$$\langle -\mathrm{div}(a(x) Du), v\rangle + \int_\Omega \gamma u |Du|^2 v = \int_\Omega f v, \quad \forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega).$$

□

Remark 2.1.1.
- *If $f \geq 0$, the solution u is non-negative.*
- *For $f \geq 0$, additional integrability properties may hold, but some terms could still be infinite.*
- *An example in $\mathbb{R}^2$ shows that certain nonlinear terms may not belong to L^1 even if the solution is in H_0^1.*

2.2 p-Laplacian equations with $W_0^{1,p}(\Omega)$ solutions

In this part, we study the existence of solutions of nonlinear elliptic equations of the type

$$u \in W_0^{1,p}(\Omega) : A(u) + g(x,u,Du) = f \in L^1(\Omega). \tag{2.4}$$

Where Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 2$, $1 < p < \infty$, and A is a nonlinear operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$, $1/p + 1/p' = 1$, defined by

$$A(u) = -\mathrm{div}(a(x,u,Du)),$$

with $a(x,s,\xi) : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that there exist $\beta > 0$, $k \in L^{p'}(\Omega)$, $\alpha > 0$ such that

$$|a(x,s,\xi)| \leq \beta(|s|^{p-1} + |\xi|^{p-1} + k(x)), \tag{2.5}$$

$$(a(x,s,\xi) - a(x,s,\eta)) \cdot (\xi - \eta) > 0, \quad \forall \xi \neq \eta, \tag{2.6}$$

$$a(x,s,\xi) \cdot \xi \geq \alpha |\xi|^p, \tag{2.7}$$

Assume that $g(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

$$g(x,s,\xi)s \geq 0, \tag{2.8}$$

and there exist $\sigma > 0$, $\gamma > 0$ such that $|g(x,s,\xi)| \geq \gamma |\xi|^p; |s| \geq \sigma$,

$$|g(x,s,\xi)| \leq b(|s|)(|\xi|^p + c(x)), \tag{2.9}$$

Where b is a continuous and increasing real function, $c(x) \in L^1(\Omega)$, $c(x) \geq 0$, and

$$f \in L^1(\Omega). \tag{2.10}$$

We are interesting in studying the following theorem.[20]

Theorem 2.2.1. *[8] Under the assumptions* (2.5)-(2.10)*, there exists a solution* $u \in W_0^{1,p}(\Omega)$ *to the problem* (2.4).

The proof of theorem 2.2.1 based on three steps:

Proof. **Step1: Approximation** We take a sequence $f_\epsilon (f_\epsilon \in L^p(\Omega), \forall \epsilon > 0)$ which converges to f in $L^1(\Omega)$ with $\|f_\epsilon\|_{L^1} \leq \|f\|_{L^1}$. Define u_ϵ to be a solution of the equation

$$\begin{aligned} A(u_\epsilon) + g(x,u_\epsilon,Du_\epsilon) &= f_\epsilon \quad \text{in } \mathscr{D}'(\Omega), \\ u_\epsilon \in W_0^{1,p}(\Omega), \quad g(x,u_\epsilon,Du_\epsilon) &\in L^1(\Omega). \end{aligned} \tag{2.11}$$

Step 2: A Priori Estimates Multiplying the equation in (2.11) by $T_k(u_\epsilon)$ and using (2.5)-(2.8), we get

$$\alpha \int_\Omega |DT_k(u_\epsilon)|^p \leq k \|f_\epsilon\|_{L^1}, \tag{2.12}$$

where $T_k(v), k \in \mathbb{R}^+$, is the usual truncation in $W_0^{1,p}(\Omega)$. Now we shall prove that

$$\int_{|u_\epsilon|>t} |g(x,u_\epsilon,Du_\epsilon)| \leq \int_{|u_\epsilon|>t} |f_\epsilon|, \quad \text{for any } t \in \mathbb{R}^+. \tag{2.13}$$

Let $\psi_i(s)$ be a sequence of real smooth increasing functions with $\psi_i' \in L^\infty(\mathbb{R})$ and $\psi_i(0) = 0$. The choice of $\psi_i(u_\epsilon)$ as test function in (2.11) yields

$$\int_\Omega g(x,u_\epsilon,Du_\epsilon)\psi_i(u_\epsilon) \leq \int_\Omega f_\epsilon \psi_i(u_\epsilon). \tag{2.14}$$

If $\psi_i(s)$ converges to the function $\psi(s)$ defined by

$$\psi(s) = \begin{cases} 1 & \text{if } s \geq t \\ 0 & \text{if } -t < s < t \\ -1 & \text{if } s \leq -t \end{cases}$$

We obtain the estimate (2.13) which implies

$$\int_{|u_\epsilon|>t} |Du_\epsilon|^p \leq \frac{1}{\gamma}\int_{|u_\epsilon|>t} |f_\epsilon|, \quad \text{for } t \geq \sigma. \tag{2.15}$$

Hence from (2.12) and (2.15) we get

$$\int_\Omega |Du_\epsilon|^p = \int_{|u_\epsilon|\leq\sigma} |Du_\epsilon|^p + \int_{|u_\epsilon|>\sigma} |Du_\epsilon|^p \leq \frac{\sigma}{\alpha}\|f_\epsilon\|_{L^1} + \frac{1}{\gamma}\int_{|u_\epsilon|>\sigma} |f_\epsilon| \leq \left(\frac{\sigma}{\alpha} + \frac{1}{\gamma}\right)\|f\|_{L^1}. \tag{2.16}$$

Step3: Passage to the Limit From the previous estimations, we can extract a subsequence, still denoted by u_ϵ, with

$$u_\epsilon \to u \quad \text{in } W_0^{1,p}(\Omega)\text{-weakly, } L^p(\Omega)\text{-strongly and a.e.} \tag{2.17}$$

First, we see that

$$u_\epsilon^+ \to u^+ \quad \text{in } W_0^{1,p}(\Omega)\text{-strongly.} \tag{2.18}$$

Let k be a positive constant greater than σ. We use in (2.11) $T_k(u_\epsilon^+ - u^+)$ as a test function (where T_k is the truncation at $\pm k$) and we have

$$\langle A(u_\epsilon), T_k(u_\epsilon^+ - u^+)\rangle + \int_\Omega g(x,u_\epsilon,Du_\epsilon)T_k(u_\epsilon^+ - u^+) + \int_\Omega f_\epsilon T_k(u_\epsilon^+ - u^+). \tag{2.19}$$

Note that where $T_k(u_\epsilon^+(\chi) - u^+(\chi))^+ > 0$, one has $u_\epsilon^+(\chi) > 0$, hence $u_\epsilon(\chi) > 0$ and from (2.8) $g(x,u_\epsilon(\chi),Du_\epsilon(\chi)) \geq 0$. Therefore from (2.19) we deduce

$$\langle A(u_\epsilon), T_k(u_\epsilon^+ - u^+)\rangle \leq \int_\Omega f_\epsilon T_k(u_\epsilon^+ - u^+).$$

Since $u_\epsilon(\chi) = u_\epsilon^+(\chi)$ on the set $\{x \in \Omega : u_\epsilon^+(\chi) > u^+(\chi)\}$, we can also write

$$\int_\Omega a(x,u_\epsilon,Du_\epsilon^+)DT_k(u_\epsilon^+ - u^+) \leq \int_\Omega f_\epsilon T_k(u_\epsilon^+ - u^+)$$

which implies

$$\lim_{\epsilon\to 0}\int_\Omega [a(x,u_\epsilon,Du_\epsilon^+) - a(x,u_\epsilon,Du^+)]DT_k(u_\epsilon^+ - u^+) = 0. \tag{2.20}$$

We recall again that, where $(u_\epsilon^+(\chi)-u^+(\chi))^+>0$, we have $u_\epsilon^+(\chi)=u_\epsilon(\chi)$. By (2.15), we deduce that

$$\begin{aligned}\int_{u_\epsilon^+-u^+>k}[a(x,u_\epsilon,Du_\epsilon^+)-a(x,u_\epsilon,Du^+)]D(u_\epsilon^+-u^+)&\le\int_{u_\epsilon>k}[a(x,u_\epsilon,Du_\epsilon)-a(x,u_\epsilon,Du^+)]D(u_\epsilon-u^+)\\ &\le c_1\left\{\int_{u_\epsilon>k}|Du_\epsilon|^p+\int_{u_\epsilon>k}|u_\epsilon|^p+\int_{u_\epsilon>k}k(\chi)^p+\int_{u_\epsilon>k}|Du^+|^p\right\}\\ &\le c_2\left\{\int_{u_\epsilon>k}|f_\epsilon|+\int_{u_\epsilon>k}|u_\epsilon|^p+\int_{u_\epsilon>k}k(x)^{p'}+\int_{u_\epsilon>k}|Du^+|^p\right\}:=R_\epsilon(k).\end{aligned} \tag{2.21}$$

If k tends to $+\infty$ the right-hand side of (2.21) tends to zero (uniformly with respect to ϵ). From this observation and (2.20) we deduce that

$$\lim_{\epsilon\to 0}\int_\Omega[a(x,u_\epsilon,Du_\epsilon^+)-a(x,u_\epsilon,Du^+)]D(u_\epsilon^+)-u^+)=0. \tag{2.22}$$

In the next step we study the behaviour of $z_\varepsilon^-:=(u_\varepsilon^+-T_k(u^+))^-$. We use as a test function in (2.5)

$$v_\varepsilon=\phi_\lambda((u_\varepsilon^+-T_k(u^+))^-) \tag{2.23}$$

where

$$\phi_\lambda(s)=se^{\lambda s^2},\quad \lambda=\frac{b(k)^2}{4\alpha^2} \tag{2.24}$$

Note that if $v_\varepsilon(x)\neq 0$ then $0\le u_\varepsilon^+(x)\le k$. Hence $v_\varepsilon\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and v_ε is an admissible test function in (2.5). We deduce

$$\int_\Omega a(x,u_\varepsilon,Du_\varepsilon)Dz_\varepsilon^-\phi_\lambda(z_\varepsilon^-)+\int_\Omega g(x,u_\varepsilon,Du_\varepsilon)\phi_\lambda(z_\varepsilon^-)=\int_\Omega f_\varepsilon\phi_\lambda(z_\varepsilon^-). \tag{2.25}$$

On the other hand since $\phi_\lambda(z_\varepsilon^-)\neq 0$, where $0\le u_\varepsilon^+(x)\le k$, we have $\phi_\lambda(z_\varepsilon^-)$ bounded in $L^\infty(\Omega)$, then

$$\int_\Omega f_\varepsilon\phi_\lambda(z_\varepsilon^-)\to\int_\Omega f\phi_\lambda((u^+-T_k(u^+))^-)\equiv 0.$$

Thus passing to the limit in ε, for k fixed, in (2.25) we have

$$\lim_{\varepsilon\to 0}\int_\Omega -[a(x,u_\varepsilon,Du_\varepsilon^+)-a(x,u_\varepsilon,DT_k(u^+))]D(u_\varepsilon^+)-T_k(u^+))^-\le 0. \tag{2.26}$$

We can write the following equalities

$$\begin{cases}\int_\Omega -[a(x,u_\varepsilon,Du_\varepsilon^+)-a(x,u_\varepsilon,Du^+)]D(u_\varepsilon^+)-u^+)\\ =\int_{T_k(u^+)<u_\varepsilon^+\le u^+}[a(x,u_\varepsilon,Du_\varepsilon^+)-a(x,u_\varepsilon,Du^+)]D(u_\varepsilon^+)-u^+)\\ \quad+\int_{u_\varepsilon^+\le T_k(u^+)}[a(x,u_\varepsilon,Du_\varepsilon^+)-a(x,u_\varepsilon,Du^+)]D(u_\varepsilon^+)-u^+))\\ =\int_{k<u_\varepsilon^+=u_\varepsilon\le u^+}[a(x,u_\varepsilon,Du_\varepsilon^+)-a(x,u_\varepsilon,Du^+)]D(u_\varepsilon^+-u^+)\\ \quad+\int_\Omega -[a(x,u_\varepsilon,Du_\varepsilon^+)-a(x,u_\varepsilon,DT_k(u^+))]D(u_\varepsilon^+-T_k(u^+))\\ \quad+\int_\Omega -[a(x,u_\varepsilon,DT_k(u^+))-a(x,u_\varepsilon,Du^+)]D(u_\varepsilon^+-T_k(u^+))^-\\ \quad+\int_{u_\varepsilon^+\le T_k(u^+)}[a(x,u_\varepsilon,Du_\varepsilon^+)-a(x,u_\varepsilon,Du^+)]D(T_k(u^+)-u^+),\end{cases} \tag{2.27}$$

because

$$\{x\in\Omega:T_k(u^+)<u_\varepsilon^+\le u^+\}=\{x\in\Omega:k<u_\varepsilon^+\le u^+\}\cup\{x\in\Omega:T_k(u^+)<u_\varepsilon^+\le u^+;u_\varepsilon^+\le k\}$$

and the last set is empty. Now we study the last four integrals. The first can be estimated as in (2.21). It goes to zero as $k \to \infty$, uniformly with respect to ϵ. For the second we have the limit (2.26). For fixed k, the third integral converges to zero (if $\epsilon \to 0$) and

$$\left| \int_{u_\varepsilon^+ \leq T_k(u^+)} [a(x,u_\varepsilon,Du_\varepsilon^+) - a(x,u_\varepsilon,Du^+)] D(T_k(u^+) - u^+) \right| \leq c_3 \left(\int_\Omega |D(T_k(u^+) - u^+)|^p \right)^{1/p}$$

which converges to zero, for $k \to +\infty$. Therefore (2.27) yields

$$\lim_{\varepsilon \to 0} \int_\Omega [a(x,u_\varepsilon,Du_\varepsilon^+) - a(x,u_\varepsilon,Du^+)] D(u_\varepsilon^+ - u^+) = 0. \tag{2.28}$$

From (2.22) and (2.28) we deduce that

$$\lim_{\varepsilon \to 0} \int_\Omega [a(x,u_\varepsilon,Du_\varepsilon^+) - a(x,u_\varepsilon,Du^+)] D(u_\varepsilon^+ - u^+) = 0. \tag{2.29}$$

By a variation of a result of Leray-Lions [21], (2.29) implies

$$u_\varepsilon^+ \to u^+ \quad \text{in} \quad W_0^{1,p}(\Omega)\text{-strongly}. \tag{2.30}$$

Now we want to prove that

$$u_\varepsilon^- \to u^- \quad \text{in} \quad W_0^{1,p}(\Omega)\text{-strongly}. \tag{2.31}$$

The proof of the convergence (2.31) is achieved using as test functions $T_k(u_\varepsilon^- - u^-)^+$ and $\phi_\lambda((u_\varepsilon^- - T_k(u^-))^-)$ and working as in the previous steps. From (2.30) and (2.31) we deduce that for some subsequence

$$Du_\varepsilon \to Du \quad \text{in} \quad L^p(\Omega)\text{-strongly} \tag{2.32}$$

and

$$Du_\varepsilon \to Du \quad \text{a.e. in} \quad \Omega. \tag{2.33}$$

Since $g(x,s,\xi)$ is continuous in (s,ξ) we have

$$g(x,u_\varepsilon(x),Du_\varepsilon(x)) \to g(x,u(x),Du(x)) \quad \text{a.e..} \tag{2.34}$$

Thus in order to prove that

$$g(x,u_e,Du_e) \to g(x,u,Du) \quad \text{in } L^1(\Omega). \tag{2.35}$$

It is sufficient to prove that, for any measurable subset E of Ω, we have

$$\lim_{|E| \to 0} \int_E |g(x,u_e,Du_e)| = 0, \quad \text{uniformly in } \varepsilon. \tag{2.36}$$

We can write

$$\int_E |g(x,u_e,Du_e)| = \int_{E \cap X_m^e} |g(x,u_e,Du_e)| + \int_{E \cap X_m^e} |g(x,u_e,Du_e)|,$$

where

$$X_m^e = \{x \in \Omega : |u_e(x)| \leq m\}$$

$$Y_m^e = \{x \in \Omega : |u_e(x)| > m\}.$$

So, using (2.7), we get

$$\int_E |g(x,u_e,Du_e)| \le b(m)\int_E (|Du_e|^p + c(x)) + \int_{|u_e|>m} |f_e|.$$

Using (2.17), (2.32) and (2.35) it is easy to pass to the limit in

$$\langle A(u_e), v\rangle + \int_\Omega g(x,u_e,Du_e)v = \int_\Omega f v$$

to obtain

$$\langle A(u), v\rangle + \int_\Omega g(x,u,Du)v = \int_\Omega f v, \tag{2.37}$$

for any $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Chapter 3

Elliptic equations with general lower-order terms

In this chapter, we discuss two cases: the case with a polynomial lower-order term, and the case with gradient-dependent lower-order terms employing the compactness method.
The results presented in this chapter have been published in [8, 11].

Assumptions on the principal part:

We study differential operators with a general principal part $\mathscr{A}$ acting on the Sobolev space $W_0^{1,p}(\Omega)$. The operator is defined as:

$$\mathscr{A}(v) = -\operatorname{div}\big(a(x, v, \nabla v)\big) \tag{3.1}$$

where the nonlinearity $a \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following standard conditions (cf. [21]):

- a is a Carathéodory function
- For almost every $x \in \Omega$, all $s \in \mathbb{R}$, and all $\xi \neq \eta \in \mathbb{R}^N$:

$$\begin{cases} a(x,s,\xi)\cdot\xi \geq \alpha|\xi|^p, & \text{(coercivity)} \\ |a(x,s,\xi)| \leq \beta|\xi|^{p-1}, & \text{(growth condition)} \\ \big[a(x,s,\xi) - a(x,s,\eta)\big]\cdot(\xi-\eta) > 0, & \text{(strict monotonicity)} \end{cases} \tag{3.2}$$

with $\alpha, \beta > 0$ being positive constants. Following [16], the operator $\mathscr{A}$ is pseudo-monotone and coercive.

Lower Order Terms

We consider two types of lower order terms H:

Polynomial Type: A Carathéodory function $g \colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying:

$$\begin{cases} g(x,s)\operatorname{sign}(s) \geq A|s|^{r-1}, \\ \forall t > 0 : \sup_{|s|\leq t} |g(x,s)| = h_t(x) \in L^1(\Omega), \end{cases} \tag{3.3}$$

where $A > 0$ and $r > 1$. A typical example is:

$$g(x,s) = b(x)s|s|^{r-2},$$

with $b \in L^1(\Omega)$ and $b(x) \geq A > 0$ almost everywhere in Ω.

Gradient-Dependent Type: A function of the form $g(x,u)|\nabla u|$, where $g(x,s) = As|s|^{r-2}$.

Now, let us distinguish the following cases:

3.1 Case with Polynomial lower order term

Theorem 3.1.1. *[8] We make the following assumptions:*

- *The structural conditions* (3.2) *and* (3.3)
- *The integrability condition:*

$$f \in L^{r'}(\Omega), \quad \text{where } 1 < p \leq r' \tag{3.4}$$

with r' denoting the conjugate exponent of r (i.e., $\frac{1}{r} + \frac{1}{r'} = 1$).

Under these hypotheses, there exists a function $u \in W_0^{1,1}(\Omega)$ with $g(x,u) \in L^1(\Omega)$ that serves as a distributional solution to the boundary value problem:

$$\begin{cases} -\operatorname{div}\big(a(x,u,\nabla u)\big) + g(x,u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Remark 3.1.1. *We shall investigate the regularizing effect induced by the lower-order term $g(x,u)$ on the solutions of the Dirichlet problem* (3.5)*. Specifically, we will examine how the presence of $g(x,u)$ influences.*

Remark 3.1.2. *Note that the summability condition* (3.4) *is weaker than the corresponding assumption in* (3.6)*, particularly in the case when*

$$r > \frac{pN}{N-1}.$$

This inequality implies the following relation between the exponents:

$$\frac{r'}{p} < \frac{N}{(p-1)N+1},$$

where r' denotes the conjugate exponent of r (i.e., $\frac{1}{r} + \frac{1}{r'} = 1$).

Remark 3.1.3.
- *A partial result for the case $r = 2$ was established in [4].*

Remark 3.1.4. *We recall that when $g \equiv 0$ and $f \in L^m(\Omega)$ with $m < (p^*)'$ (which implies that f does not belong to the dual of $W_0^{1,p}(\Omega)$ and consequently $u \notin W_0^{1,p}(\Omega)$), the Calderón-Zygmund theory for infinite energy solutions (see [11, 12, 14]) establishes the following regularity results:*

$$\begin{cases} \dfrac{N}{(p-1)N+1} < m < \dfrac{pN}{(p-1)N+p} \Rightarrow u \in W_0^{1,(p-1)m^*}(\Omega), \\ m = \dfrac{N}{(p-1)N+1} \text{ and } 1 < p < 2 - \dfrac{1}{N} \Rightarrow u \in W_0^{1,1}(\Omega). \end{cases} \tag{3.6}$$

Note that:

- *The condition* $m > \dfrac{N}{(p-1)N+1}$ *ensures that* $(p-1)m^* > 1$
- *The restriction* $1 < p < 2 - \dfrac{1}{N}$ *implies* $\dfrac{N}{(p-1)N+1} > 1$

Remark 3.1.5. *When considering boundary value problems with a lower order term* $g(x,u)$ *that maintains the same sign as* u*, the aforementioned existence results remain valid due to the coercive properties of* g*. Moreover, as demonstrated in [15, 20], the presence of a lower order term satisfying assumption* (3.3) *can enhance, in certain cases, the summability results for* ∇u *established in* (3.6)*. Specifically, [20] establishes the following regularity results:*

$$\begin{cases} \text{If } r' \le m < (p^*)' \text{ with } r > p^*, & \text{there exists a weak solution } u \in W_0^{1,p}(\Omega); \\ \text{If } \dfrac{r'}{p} < m < r', & \text{there exists a distributional solution } u \in W_0^{1,\frac{mp^*}{p}}(\Omega). \end{cases} \tag{3.7}$$

For the case $1 \le m < \dfrac{r'}{p}$*, the concept of distributional solution becomes inadequate, and the framework of entropy solutions becomes necessary (see [2]). In the present work, we extend this analysis to the borderline case* $m = \dfrac{r'}{p}$*, which yields distributional solutions in* $W_0^{1,1}(\Omega)$*.*

Proof. Consider the sequence of approximate Dirichlet problems:

$$\begin{cases} u_n \in W_0^{1,p}(\Omega), \\ \mathscr{A}(u_n) + g(x,u_n) = \dfrac{f(x)}{1+\frac{1}{n}|f(x)|}, \end{cases} \tag{3.8}$$

where:

- $\mathscr{A}$ is the operator defined in (3.1)
- $g(x,\cdot)$ satisfies assumption (3.3)
- $f \in L^m(\Omega)$ for some $m \ge 1$

Proposition 3.1.1. *For each* $n \in \mathbb{N}$*:*

1. *There exists a weak solution* u_n *to problem* (3.8) *(see [18])*
2. *Every solution* u_n *belongs to* $L^\infty(\Omega)$ *(see [22])*

To start with, for $s \in \mathbb{R}$ and $k \in \mathbb{R}^+$, we define the truncation operator T_k by

$$T_k(s) = \max\{-k, \min(k,s)\}.$$

Thanks to the positivity properties of g, the following estimates hold (using only $\|f\|_{L^1(\Omega)}$), as proved in [11, 2, 14]:

$$\int_\Omega |\nabla T_k(u_n)|^p \, dx \le \frac{\|f\|_{L^1(\Omega)}}{\alpha} k, \tag{3.9}$$

$$\alpha(p-1)\mathscr{S}\left[\int_\Omega \{\log(1+|u_n|)\}^{p^*} \, dx\right]^{\frac{p}{p^*}} \le \alpha(p-1)\int_\Omega |\nabla \log(1+|u_n|)|^p \, dx \le \int_\Omega |f| \, dx, \tag{3.10}$$

where $\mathscr{S}$ is the Sobolev constant.

Remark 3.1.6. *From estimate* (3.10), *we derive the following measure estimate:*

$$\operatorname{meas}\{x \in \Omega : |u_n(x)| \geq k\} \leq \left(\frac{\|f\|_{L^1(\Omega)}}{\alpha(p-1)\mathscr{S} \log(1+k)^p} \right)^{\frac{p^*}{p}}. \tag{3.11}$$

Proposition 3.1.2. *The sequence* $\{\log(1+|u_n|)\}$ *is bounded in* $W_0^{1,p}(\Omega)$. *Consequently, up to a subsequence (still denoted by* $\{u_n\}$*), there exists a measurable function u such that:*

$$u_n(x) \to u(x) \quad a.e.\ in\ \Omega. \tag{3.12}$$

Moreover, by (3.9), *we have:*

$$T_k(u_n) \rightharpoonup T_k(u) \quad weakly\ in\ W_0^{1,p}(\Omega). \tag{3.13}$$

In addition, we state a modified version of a classical result by H. Brezis and W. A. Strauss [19, 17, 11, 20]:

Theorem 3.1.2. *For* $f \in L^m(\Omega)$ *with* $m \geq 1$ *and* $k \geq 0$, *we have*

$$\int_{k \leq |u_n|} |g(x,u_n)||h(u_n)|^{m-1} dx \leq \frac{1}{A^{m/m'}} \int_{k \leq |u_n|} |f(x)|^m dx, \tag{3.14}$$

where $h(t) = t|t|^{r-2}$.

For $1 < m < \infty$, define the truncation function:

$$\psi_{k,\delta}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq k, \\ \frac{1}{\delta}(t-k) & \text{if } k < t < k+\delta, \\ 1 & \text{if } t \geq k+\delta, \\ -\psi_{k,\delta}(-t) & \text{if } t < 0. \end{cases}$$

Using $v = |h(u_n)|^{m-1}\psi_{k,\delta}(u_n) \in W_0^{1,p}(\Omega)$ as a test function in (3.8), we apply Hölder's inequality to obtain:

$$\begin{aligned} \int_{k+\delta \leq |u_n|} |g(x,u_n)||h(u_n)|^{m-1} dx \leq & \left(\int_{k \leq |u_n|} |f|^m dx \right)^{1/m} \\ & \times \left(\frac{1}{A} \int_{k \leq |u_n|} |g(x,u_n)||h(u_n)|^{m-1} dx \right)^{1/m'}. \end{aligned}$$

The result (3.14) follows by Fatou's lemma as $\delta \to 0$.

For the case $m = 1$, using $\psi_{k,\delta}(u_n)$ as test function yields:

$$\int_{k \leq |u_n|} |g(x,u_n)| dx \leq \int_{k \leq |u_n|} |f(x)| dx. \tag{3.15}$$

From (3.14) and assumption (3.3), we derive:

$$\int_{k \leq |u_n|} |u_n|^{m(r-1)} dx \leq \left(\frac{1}{A} \right)^{m/m'} \int_{k \leq |u_n|} |f|^m dx, \tag{3.16}$$

which implies the global bound:

$$\int_\Omega |u_n|^{m(r-1)} dx \le \left(\frac{\|f\|_m}{A^{1/m'}}\right)^m. \tag{3.17}$$

For $f \in L^m(\Omega)$, $m \ge 1$, we have:

$$g(x,u_n) \to g(x,u) \text{ in } L^1(\Omega). \tag{3.18}$$

The proof uses Vitali's theorem, relying on:

- Almost everywhere convergence from (3.12)
- Equi-integrability established via (3.11) and (3.15)

With $m = r'/p$, (3.16) becomes:

$$\int_{k\le|u_n|} |u_n|^{r/p} dx \le \left(\frac{1}{A}\right)^{m/m'} \int_{k\le|u_n|} |f|^{r'/p} dx. \tag{3.19}$$

Using careful test function selection and Hölder inequalities, we obtain:

$$\int_\Omega |\nabla u_n| dx \le C_{f,A}\left(\int_\Omega |f|^{r'/p} dx\right)^{1/p'}, \tag{3.20}$$

establishing $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$.

The sequence solves:

$$\mathscr{A}(u_n) = y_n \quad \text{with } \{y_n\} \text{ compact in } L^1(\Omega), \tag{3.21}$$

leading to the key result:

$$\nabla u_n(x) \to \nabla u(x) \text{ a.e. in } \Omega. \tag{3.22}$$

We need the following lemma:

Lemma 3.1.1. *Let $\{u_n\}$ be a sequence of solutions of the Dirichlet problems*

$$u_n \in W_0^{1,p}(\Omega) : -\operatorname{div}(a(x,u_n,\nabla u_n)) = y_n(x), \tag{3.23}$$

with $p > 1$, $\|y_n\|_{L^1(\Omega)} \le L$. Assume (3.2) and that

$$\|u_n\|_{W_0^{1,1}(\Omega)} \le M. \tag{3.24}$$

Then ∇u_n converges (up to a subsequence) a.e. to ∇u.

Proof. In this proof, many times we extract a subsequence $\{u_{n_k}\}$, but we still denote it by $\{u_n\}$ for simplicity. As a consequence of the bound (3.24), we have that

$$u_n(x) \quad \text{converges to } u(x) \text{ almost everywhere.}$$

Moreover, (3.9) gives

$$\int_\Omega |\nabla T_k(u_n)|^p \le \frac{L}{\alpha} k,$$

so that

$$\nabla T_k(u_n) \quad \text{converges weakly to } \nabla T_k(u) \text{ in } W_0^{1,p}(\Omega).$$

Let $0 < \theta < \frac{1}{p}$ and $k > 0$. Consider

$$I_{\Omega,n} = \int_\Omega \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\}^\theta.$$

We shall prove that the previous integral converges to zero. Indeed, it is equal to

$$\int_{C_k} \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\}^\theta + \int_{A_k} \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\}^\theta = I_{C_k,n} + I_{A_k,n},$$

where

$$C_k = \{x \in \Omega : |u(x)| \le k\}, \quad A_k = \{x \in \Omega : |u(x)| > k\}.$$

Here we repeat the proof of [3] (see also [10, 7]) and we use the same notations (we denote by $\omega_i(k)$ quantities such that $\lim_{k\to\infty} \omega_i(k) = 0$). The only difference is that we will only use the following inequality

$$\left| \int_\Omega y_n T_j[u_n - T_k(u)] \right| \le jL, \quad \forall j > 0.$$

so that we have the following estimate

$$\limsup_{n\to\infty} [I_{C_k,n} + I_{A_k,n}] \le \omega_1(k) + (jL)^\theta + \omega_2(k), \quad \forall j > 0.$$

Therefore

$$\int_\Omega \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\}^\theta \to 0,$$

that is

$$\|[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\|_{L^1(\Omega)}^\theta \to 0,$$

which implies

$$\{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\}^\theta \to 0 \quad \text{almost everywhere,}$$

and also (since θ is positive)

$$\{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\} \to 0 \quad \text{almost everywhere.}$$

In [21], it is proved that, under our assumptions on the function $a(x,s,\xi)$, the previous limit implies that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{almost everywhere.} \quad \square$$

Thus, we can use the last Lemma and we deduce that (up to a subsequence)

$$\nabla u_n(x) \quad \text{converges a.e. to } \nabla u(x). \tag{3.25}$$

Thanks to (3.20) and to (3.11), for every measurable subset $E \subset \Omega$, we have

$$\begin{aligned}\int_E |\nabla u_n| &\leq \int_E |\nabla T_k(u_n)| + \int_{\{k\leq |u_n|\}} |\nabla u_n| \\ &\leq \left[\int_\Omega |\nabla T_k(u_n)|^p\right]^{\frac{1}{p}} \operatorname{meas}(E)^{1-\frac{1}{p}} + (C_f)^{\frac{1}{p}} \left[\int_{\{k\leq |u_n|\}} |u_n|^{\frac{r}{p}}\right]^{\frac{1}{p'}} \\ &\leq \left[\frac{k\|f\|_{L^1(\Omega)}}{\alpha}\right]^{\frac{1}{p}} \operatorname{meas}(E)^{1-\frac{1}{p}} + C_{f,A} \left[\int_{\{k\leq |u_n|\}} |f|^{\frac{r'}{p}}\right]^{\frac{1}{p'}}.\end{aligned}$$

Thus, the sequence $\{\nabla u_n\}$ is equi-integrable. By Vitali's Theorem, we obtain

$$\nabla u_n \quad \text{converges to } \nabla u \text{ in } (L^1(\Omega))^N. \tag{3.26}$$

Moreover, (3.25) implies that $a(x, u_n(x), \nabla u_n(x))$ converges a.e. and, since (note that $p-1<1$)

$$\int_E |a(x, u_n(x), \nabla u_n(x))| \leq \beta \int_E |\nabla u_n(x)|^{p-1} \leq C_1 |E|^{\frac{2-p}{p-1}},$$

the Vitali Theorem yields

$$a(x, u_n, \nabla u_n) \quad \text{converges to } a(x, u, \nabla u) \text{ in } (L^1(\Omega))^N. \tag{3.27}$$

Finally, it is possible to pass to the limit in the weak formulation of (3.8) and we prove (recall (3.18), (3.26), (3.27)) that u is a distributional solution of (3.5):

$$\begin{cases} u \in W_0^{1,1}(\Omega), & g(x,u) \in L^1(\Omega): \\ \int_\Omega a(x,u,\nabla u)\nabla\varphi + \int_\Omega g(x,u)\varphi = \int_\Omega f(x)\varphi, & \\ \forall \varphi \in C_c^1(\Omega). & \end{cases}$$

The following example provides a finite energy solution of a boundary value problem with polynomial order term and non-smooth datum. ☐

Exemple 3.1.1. *Here, we consider the boundary value problem* (3.5) *in radial coordinates with*

$$\Omega = B(0,1), \quad p = 2, \quad g(x,s) = s|s|^{r-2} \quad \textit{and} \quad A = -\Delta.$$

Let $r > 2^*$*, so that the interval* $\left(-\frac{2}{r-2}, \frac{N-2}{2}\right)$ *is not empty. We choose* $\gamma \in \left(\frac{2}{r-2}, \frac{N-2}{2}\right)$*. The function* $w(|x|) = |x|^{-\gamma} - 1$ *is positive and belongs to* $W_0^{1,2}(\Omega)$*, since* $\gamma < \frac{N-2}{2}$*. Moreover,* w *is a solution of* (3.5) *with*

$$f(|x|) = (N-2-\gamma)\gamma |x|^{-(\gamma+2)} + (|x|^{-\gamma} - 1)^{r-1}.$$

If we write

$$f(|x|) = \gamma(N-2-\gamma)|x|^{-(\gamma+2)} + |x|^{-\gamma(r-1)} - \left[(|x|^{-\gamma})^{r-1} + (|x|^{-\gamma} - 1)^{r-1}\right],$$

we note that the second term is the most singular (since $r > 2 + \frac{2}{\gamma}$*) and it belongs to the Marcinkiewicz space* $M^{\frac{N}{2(r-1)}}(\Omega)$*. Then* $f \notin L^{\frac{2N}{N+2}}(\Omega)$ *if* $\frac{N}{2(r-1)} < \frac{2N}{N+2}$*, which is true since we have* $\frac{N+2}{2(r-1)} < \frac{N-2}{2}$ *(thanks to the assumption* $r > 2^*$*). Thus,* $w \in W_0^{1,2}(\Omega)$*, even if* $f \notin L^{\frac{2N}{N+2}}(\Omega)$*.*

Moreover, we note that in this example, the assumption $r' < m$ *means* $\frac{N}{2(r-1)} > r'$*, which is equivalent to* $\gamma < \frac{N}{r}$*; with our assumption we have*

$$\gamma < \frac{N-2}{2} < \frac{N}{r}.$$

3.2 Case with Gradient depending lower order terms

Theorem 3.2.1. *[7] We assume (3.2) and (3.30). Let $f \in L^1(\Omega)$ and*

$$1 < r \le \frac{N-1}{N(p-1)}. \tag{3.28}$$

Then, there exists $u \in W_0^{1,1}(\Omega) \cap L^{r-1}(\Omega)$ such that

$$u|u|^{r-2}|\nabla u| \in L^1(\Omega),$$

which is a distributional solution of the boundary value problem:

$$\begin{cases} -div(a(x,u,\nabla u)) + Au|u|^{r-2}|\nabla u| = f(x), & in\,\Omega, \\ u = 0, & on\,\partial\Omega. \end{cases} \tag{3.29}$$

Remark 3.2.1. *If $f \in L^1(\Omega)$, and the existence of $W_0^{1,p}$-solutions, if $f \in L^m(\Omega)$, with $m < (p^*)'$ suitably chosen.*
Concerning the assumption on p in the sequel we always assume that

$$1 < p \le 2 - \frac{1}{N}. \tag{3.30}$$

since otherwise both problems (3.5) and (3.29) have solutions in $W_0^{1,q}(\Omega)$, with $q > 1$, already in the case $f \in L^1(\Omega)$ (see [11, 12, 15, 20]).

Proof. To begin with, we consider the Dirichlet problems

$$u_n \in W_0^{1,p}(\Omega): \quad -\mathrm{div}(a(x,u_n,\nabla u_n)) + Au_n|u_n|^{r-2}|\nabla u_n| = \frac{f(x)}{1+\frac{1}{n}|f(x)|} \tag{3.31}$$

The existence of a weak solution u_n is proved in [18]. In addition, every function u_n belongs to $L^\infty(\Omega)$ (see [22]). We follow the approach already used in [13]: we take $T_k(u_n)$ as a test function, then we have

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p + Ak^r \int_{\{k \le |u_n|\}} |\nabla u_n| \le k\|f\|_{L^1(\Omega)}, \tag{3.32}$$

which implies

$$\begin{cases} \int_{\{|u_n| \le k\}} |\nabla u_n| \le |\Omega|^{1-\frac{1}{p}} \left(\frac{k\|f\|_{L^1(\Omega)}}{\alpha}\right)^{\frac{1}{p}}, \\ \int_{\{k \le |u_n|\}} |\nabla u_n| \le \frac{\|f\|_{L^1(\Omega)}}{Ak^{r-1}}, \end{cases}$$

and (with the simple choice $k = 1$)

$$\int_\Omega |\nabla u_n| \le |\Omega|^{1-\frac{1}{p}} \left(\frac{\|f\|_{L^1(\Omega)}}{\alpha}\right)^{\frac{1}{p}} + \frac{\|f\|_{L^1(\Omega)}}{A}. \tag{3.33}$$

Then the sequence $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$. Thus, up to a subsequence still denoted by $\{u_n\}$, the sequence $\{u_n\}$ converges to some function u strongly in $L^\rho(\Omega)$, $\rho < \frac{N}{N-1}$, and almost everywhere in Ω. Furthermore

$$\mathrm{meas}\{k < |u_n|\} \le \frac{C_1}{k!}. \tag{3.34}$$

Let $\psi_{k,\delta}(s)$ be the function already used in the previous section; choosing $\psi_{k,\delta}(u_n)$ as test function in (3.31) and dropping a positive term we have

$$A\int_{\{k+\delta<|u_n|\}} |\psi_{k,\delta}(u_n)||u_n|^{r-1}|\nabla u_n| = \int_{\{k\le|u_n|\}} |f(x)|.$$

Letting $\delta \to 0$ we deduce

$$A\int_{\{k\le|u_n|\}} |u_n|^{r-1}|\nabla u_n| \le \int_{\{k\le|u_n|\}} |f(x)|, \tag{3.35}$$

which also gives

$$A\int_\Omega |u_n|^{r-1}|\nabla u_n| \le \int_\Omega |f(x)|. \tag{3.36}$$

We notice that, the inequality (3.36) and the Sobolev inequality get

$$\frac{A}{r} S\left[\int_\Omega |u_n|^{1-r}\right]^{\frac{1}{r}} \le \int_\Omega |f(x)|. \tag{3.37}$$

which in turn gives $u \in L^{1-r}(\Omega)$.

Another factor is, since the lower order term is bounded in $L^1(\Omega)$,

$$-\operatorname{div}(a(x,u_n,\nabla u_n)) \text{ is bounded in } L^1(\Omega) \tag{3.38}$$

as in the proof of Theorem 3.2.1 we can again use Lemma 3.2.1 to deduce that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in } \Omega.$$

In order to pass to the limit in the approximate problems (3.31), first of all we have to prove the equi-integrability of the lower order term.

Let E be a measurable subset of Ω, by the Holder inequality, (3.32) and (3.35) we have

$$\int_E |u_n|^{r-1}|\nabla u_n| \le k^{r-1}\int_E |\nabla T_k(u_n)| + \int_{k<|u_n|} |u_n|^{r-1}|\nabla u_n| \le k^{r-1}\left(\|f\|_{L^1(\Omega)}\frac{k}{\alpha}\right)^{\frac{1}{p}} |E|^{\frac{1}{p'}} + \int_{\{k\le|u_n|\}} |f(x)|,$$

and working as in the proof of Theorem 3.2.1 we obtain

$$\lim_{|E|\to 0}\int_E |u_n|^{r-1}|\nabla u_n| \le \varepsilon, \quad \text{uniformly with respect to } n.$$

Now, we apply the Vitali theorem as follows:

First, we show that the sequence $\{\nabla u_n\}$ is equi-integrable. From(3.32), we have:

$$\int_{\{|u_n|\le k\}} |\nabla u_n| \le |\Omega|^{1-\frac{1}{p}}\left(\frac{k\|f\|_{L^1(\Omega)}}{\alpha}\right)^{\frac{1}{p}},$$

$$\int_{\{k\le|u_n|\}} |\nabla u_n| \le \frac{\|f\|_{L^1(\Omega)}}{Ak^{r-1}}.$$

For any measurable subset $E \subset \Omega$, the integral of $|\nabla u_n|$ over E is split into two parts:

$$\int_E |\nabla u_n| \le \int_{E\cap\{|u_n|\le k\}} |\nabla u_n| + \int_{\{k\le|u_n|\}} |\nabla u_n|.$$

Using the estimates above:

$$\int_E |\nabla u_n| \le \left(\frac{k\|f\|_{L^1(\Omega)}}{\alpha}\right)^{\frac{1}{p}} |E|^{1-\frac{1}{p}} + \frac{\|f\|_{L^1(\Omega)}}{Ak^{r-1}}.$$

For any $\varepsilon > 0$, choose k large enough so that the second term $\frac{\|f\|_{L^1(\Omega)}}{Ak^{r-1}} < \varepsilon/2$. Then, choose $|E|$ small enough so that the first term $\left(\frac{k\|f\|_{L^1(\Omega)}}{\alpha}\right)^{\frac{1}{p}} |E|^{1-\frac{1}{p}} < \varepsilon/2$. This ensures that for any E with $|E|$ sufficiently small, $\int_E |\nabla u_n| < \varepsilon$ uniformly in n.

Since $\{\nabla u_n\}$ is bounded in $L^1(\Omega)$ (from (3.33)) and equi-integrable, Vitali's theorem implies that ∇u_n converges strongly to ∇u in $L^1(\Omega)$ (as stated in (3.40)).

Second, we prove that the sequence $\{u_n|u_n|^{r-2}|\nabla u_n|\}$ is equi-integrable to pass to the limit in the weak formulation. From (3.35), we have:

$$\int_{\{k\le|u_n|\}} |u_n|^{r-1}|\nabla u_n| \le \int_{\{k\le|u_n|\}} |f(x)|.$$

For any measurable subset $E \subset \Omega$, split the integral:

$$\int_E |u_n|^{r-1}|\nabla u_n| \le k^{r-1}\int_E |\nabla T_k(u_n)| + \int_{\{k\le|u_n|\}} |u_n|^{r-1}|\nabla u_n|.$$

Using the estimates from (3.32) and (3.35):

$$\int_E |u_n|^{r-1}|\nabla u_n| \le k^{r-1}\left(\frac{k\|f\|_{L^1(\Omega)}}{\alpha}\right)^{\frac{1}{p}} |E|^{1-\frac{1}{p}} + \int_{\{k\le|u_n|\}} |f(x)|.$$

For any $\varepsilon > 0$, choose k large enough so that $\int_{\{k\le|u_n|\}} |f(x)| < \varepsilon/2$ (since $f \in L^1(\Omega)$). Then, choose $|E|$ small enough so that the first term $k^{r-1}\left(\frac{k\|f\|_{L^1(\Omega)}}{\alpha}\right)^{\frac{1}{p}} |E|^{1-\frac{1}{p}} < \varepsilon/2$. This ensures uniform equi-integrability of $\{u_n|u_n|^{r-2}|\nabla u_n|\}$.

The sequence $\{u_n|u_n|^{r-2}|\nabla u_n|\}$ is bounded in $L^1(\Omega)$ (from(3.36)) and equi-integrable. By Vitali's theorem, it converges strongly to $u|u|^{r-2}|\nabla u|$ in $L^1(\Omega)$.

The convergence of the principal part $a(x,u_n,\nabla u_n)$ in $L^1(\Omega)$ is also derived using Vitali's theorem. From the growth condition , $|a(x,u_n,\nabla u_n)| \le \beta|\nabla u_n|^{p-1}$. Since $p-1 < 1$, and $\{\nabla u_n\}$ is equi-integrable, $\{|\nabla u_n|^{p-1}\}$ is also equi-integrable. Vitali's theorem ensures that $a(x,u_n,\nabla u_n)$ converges to $a(x,u,\nabla u)$ in $L^1(\Omega)$.

Vitali's theorem bridges the gap between pointwise convergence (established earlier) and strong convergence in $L^1(\Omega)$, ensuring the existence of a distributional solution u to the boundary value problem (3.29). In the end, we conclude that

$$u_n|u_n|^{r-2}|\nabla u_n| \quad \text{converges in } L^1(\Omega) \text{ to } u|u|^{r-2}|\nabla u|. \tag{3.39}$$

Then, for every measurable subset $E \subset \Omega$, we have

$$\begin{aligned}
&\left[\int_E |\nabla u_n| \leq \int_E |\nabla T_k(u_n)| + \int_{\{k \leq |u_n|\}} |\nabla u_n|\right] \\
&\leq \left[\int_\Omega |\nabla T_k(u_n)|^p\right]^{\frac{1}{p}} \mathrm{meas}(E)^{1-\frac{1}{p}} + \frac{\|f\|_{L^1(\Omega)}}{Ak^{r-1}} \\
&\leq \left[\frac{k\|f\|_{L^1(\Omega)}}{\alpha}\right]^{\frac{1}{p}} \mathrm{meas}(E)^{1-\frac{1}{p}} + \frac{\|f\|_{L^1(\Omega)}}{Ak^{r-1}}.
\end{aligned}$$

Thus the sequence $\{\nabla u_n\}$ is equi-integrable and we get

$$\nabla u_n \quad \text{converges to } \nabla u \text{ in } (L^1(\Omega))^N. \tag{3.40}$$

Since $p-1<1$ we deduce that

$$a(x,u_n,\nabla u_n) \quad \text{converges to } a(x,u,\nabla u) \text{ in } (L^1(\Omega))^N.$$

Then it is possible to pass to the limit in the weak formulation and we prove (recall (3.40), (3.39)) that u is a distributional solution of (3.29), that is

$$\begin{aligned}
&u \in W_0^{1,1}(\Omega), \quad u|u|^{r-2}|\nabla u| \in L^1(\Omega): \\
&\int_\Omega a(x,u,\nabla u)\nabla\varphi + \int_\Omega u|u|^{r-2}|\nabla u|\varphi = \int_\Omega f(x)\varphi, \quad \square \\
&\forall \varphi \in C_c^1(\Omega).
\end{aligned}$$

Next, as in the previous section, we provide a Dirichlet problem with gradient depending lower order term, non smooth data and finite energy solutions. ☐

Exemple 3.2.1. *Let* $\Omega = B(0,1)$, $r > \frac{2^*}{2}$ *and* $\gamma = \frac{N}{2r}$. *Then the function*

$$u(x) = |x|^{-\gamma} - 1$$

is a positive weak solution of the problem

$$u \in W_0^{1,2}(\Omega): -\Delta u + u^{r-1}|\nabla u| = f(x),$$

with

$$f(x) = \gamma\left[(N-\gamma-2)|x|^{-(\gamma+2)} + |x|^{-(\gamma+1)}(|x|^{-\gamma}-1)^{r-1}\right]$$

and $f \notin L^{(2^*)'}(\Omega)$.

Conclusion and Further Prospects

In this work, we have studied the regularizing effects of lower-order terms in nonlinear Dirichlet problems of the form:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + H(x,u,\nabla u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ and f has poor summability. We discuss the following key findings: First, the presence of terms like $g(x,u) = u|u|^{r-2}$ can regularize solutions, allowing for distributional solutions in $W_0^{1,1}(\Omega)$ even when $f \in L^{r'/p}(\Omega)$, where r' is the conjugate exponent of r. This extends previous results by showing that weaker summability conditions on f can still yield solutions with minimal regularity. Second, terms like $H(x,u,\nabla u) = u|u|^{r-2}|\nabla u|$ provide stronger regularization, enabling solutions in $W_0^{1,1}(\Omega)$ even for $f \in L^1(\Omega)$, provided $1 < r \leq \frac{N-1}{N(p-1)}$. For $f \in L^m(\Omega)$ with m in a suitable range, solutions can achieve higher regularity, such as $W_0^{1,p}(\Omega)$. In addition, concrete examples illustrate the theory, showing how specific choices of f and lower-order terms lead to solutions with finite energy or minimal regularity. Moreover, the work bridges gaps in existing literature, particularly for borderline cases where traditional methods fail.

Overall, this work provides a comprehensive analysis of how lower-order terms can enhance the regularity of solutions to elliptic equations with non-smooth data, offering new insights into the solvability and properties of such problems. The results are significant for understanding the behavior of nonlinear PDEs in applied mathematics and engineering contexts.

This work opens several directions for future research. We propose the following problem: Find u satisfying:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + H(x,u,\nabla u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, and f is a measure on Ω.

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