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HPLTM for numerical simulation of multi-dimensional, time-fractional model of Navier–Stokes equation

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LIST OF SYMBOLS AND ABBREVIATIONS

$\Gamma(z)$	Gamma function
$\beta(z,w)$	Beta function
ERF	Error Function
$ERFC$	Complementary Error Function
I^{α}	Riemann-Liouville Fractional Integral
D^{α}	Riemann-Liouville Fractional Derivatives
${}^{H}I^{\alpha}$	Hadamard Fractional Integral
${}^{CH}D^{\alpha}$	Caputo-type hadamard Fractional Derivative
$\mathcal{J}^{\alpha}$	Katygampola Fractional Integral
${}^{K}D^{\alpha}$	Katygampola Fractional Derivatives
${}^{CK}D^{\alpha}$	Caputo-type Katygampola Fractional Derivative
NS	Navier-Stokes equations .
LT	Laplace transform
ET	Elzaki transform
MLF	Mittag-Leffler Function
RL	Riemann Liouville.
HPM	Homotopy perturbation method.
HPLTM	Homotopy perturbation Laplace transform method.
HPETM	Homotopy perturbation Elzaki transform method
ADM	Adomain decompositimethodon .
ADLTM	Adomain decomposition Laplace transform method
ADETM	Adomain decomposition Elzaki transform method

INTRODUCTION

Fractional calculus (Fc), an extension of traditional integer-order calculus, has gained significant attention across various scientific and engineering disciplines due to its capability to model complex phenomena with long-range memory and anomalous diffusion. This mathematical discipline dates back to the 17th century, with pioneers like Hospital and Leibniz exploring the idea of Fractional derivative (Fd). However, it was Liouville and Riemann in the late 19th century who laid the rigorous foundations of Fc [1, 2]. finds applications in various scientific and engineering fields, including physics, engineering, finance, biology, and signal processing. It allows the modeling of complex phenomena with long-range memory and anomalous diffusion, which cannot be adequately captured by traditional integer-order calculus [1, 3, 4, 5] Fc has emerged as a robust mathematical framework that extends the conventional integer-order calculus to accommodate derivatives and integrals of non-integer orders. This extension allows for the description and analysis of complex dynamical systems and processes that exhibit behaviors beyond the scope of traditional calculus. Over the past few decades, Fc has found applications in various scientific and engineering disciplines, such as physics, biology, economics, and engineering, enabling researchers to model intricate phenomena with long-range memory and anomalous diffusion [3, 6, 7, 8].

The study of FPDEs has gained significant attention in recent years. These equations are frequently encountered in various fields such as fluid mechanics, viscoelasticity, biology, engineering, and physics [9, 10]. However, most of these equations do not have exact analytical solutions, necessitating the use of approximation and numerical techniques. Several numerical methods have been developed to tackle these equations, including the Adomain decomposition method [11], Homotopy analysis method [12], Variational iteration method[13], and Homotopy perturbation method [14, 15, 16, 17]. The homotopy perturbation method [18] proposed by He, has proven to be a useful tool for obtaining exact and approximate solutions for both linear and nonlinear FPDEs.

One of the main advantages of the homotopy perturbation method is that it does not require a small parameter or linearization. The solution procedure is straightforward and only a few iterations are needed to achieve highly accurate solutions that are valid for the entire solution domain.

Moreover, the solution is expressed as the summation of an infinite series, which is expected to converge to the exact solution. The Laplace transform (LT), has emerged as a valuable mathematical tool for transforming a wide range of integral and DEs into algebraic equations and thus simplifying their solution process. This transform has proven particularly useful in the context of Fc, where it enables the conversion of FDEs into algebraic forms, facilitating the application of various solution methods. The LT has prominence due to its versatility and effectiveness in handling complex mathematical problems across different scientific and engineering disciplines. The LT, known for its scale and unit-preserving properties, eliminates the need for introducing a new frequency domain. It has been demonstrated that the LT maintains the units of the original problem, making it a valuable tool for solving problems without relying on the frequency domain.

In [18], Hamed, Yousif, and Arbab presented a novel approach for solving Schrödinger space-time fractional equations through the integration of the HPS with the ST. The researchers aimed to obtain both analytic and approximate solutions for these complex equations. The proposed method demonstrated its efficacy in providing accurate solutions, highlighting its potential in handling the challenges posed by Schrödinger space-time fractional equations. In his work, Khader explored the application of the HPSTS to solve nonlinear heat-like fractional equations. The study aimed to address the challenges posed by these intricate equations by combining the HPS with the ST technique. By doing so, the researcher sought to derive approximate solutions that capture the nonlinear dynamics of the equations [19]. By exploring the domain of mathematical challenges, recent studies have

showcased innovative approaches to overthrow the complexities of nonlinear FPDEs. Many researchers have solved different equations with the help of HPSTS, such as heat and wave-like equations [20], Black-Scholes European option pricing equations [21] and references therein.

Recently, El-Shahed and Salem generalized the traditional Navier-Stokes equations by introducing a fractional derivative of order α in place of the classical time derivative. They utilized Laplace transform, Fourier sine transform, and finite Hankel transforms to derive approximate solutions for three specific cases [22]. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one of the main reasons why fractional calculus has become more and more popular. However, due to the nonlinearity of the Navier-Stokes Time-Fractional Equation (NS-TFEs), there is currently no universally known method for analytically solving these equations. Obtaining an exact solution for these equations is rare and typically requires assumptions about the fluid's state and consideration of a simple flow pattern configuration. Subsequently, various numerical methods have been employed to study this equation, such as the Adomian decomposition method [23], Homotopy analysis method [24], and Modified Laplace decomposition method [25].

This work presents an approximate analytic solution of multi-dimensional, time fractional model of NS equation by adopting homotopy perturbation Laplace transform method (HPTM). The rest of the work is organized as follows: In chapter 1, some basic definitions and notations on fractional calculus are revisited. Chapter 2: we study some methods for solving fractional deferential equations by use Adomian decomposition method and homotopy perturbation method. In chapter 3, the approximate analytic solutions of two test problems of time-fractional order NS equation are obtained.

CHAPTER 1

PRELIMINARIES AND DEFINITIONS

Intrduction:

In this chapter, we recall some notions and definitions of specific useful functions. Next, we give the necessary definitions, property and lemmas of a fractional derivative and integral, which will be used through the whole of this work. Finelly, we will present the definition of the Laplace and Elzaki transforms with some properties and examples.

1.1 Special Functions

1.1.1 The Gamma Function

Definition 1.1 ([26, 27]) *The gamma function is most important in the fractional-order calculus, and it is written as:*

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx, \quad \Re(z) > 0, \tag{1.1}$$

where $\Re(z)$ is the real part of the complex number $z \in \mathbb{C}$.Equation (1.1) is convergent for all complex numbers z $(\Re(z) > 0)$. The gamma function is defined everywhere on the real axis except its singular points, viz. $0, -1, -2, \ldots$.

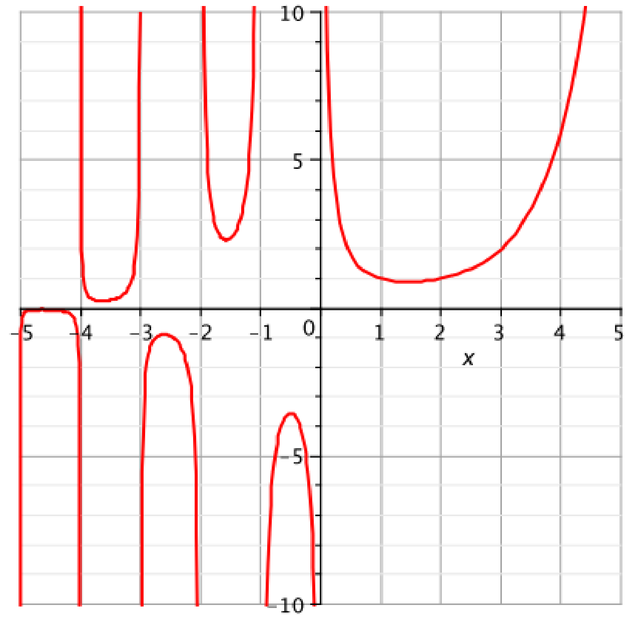

Figure 1.1 *The graph of gamma function in the real axis*

As a result, the domain of the gamma function is $\ldots \cup (-2,-1) \cup (-1,0) \cup (0,+\infty)$. The graph of the gamma function is depicted in the Figure 1.1.

Proposition 1.2 ([2, 26]) *Some properties of the gamma function are as follows*

i) $\Gamma(z+1) = z\Gamma(z)$ *for* $z \in \mathbb{R}^+$.

ii) $\Gamma(z+1) = z!, \Gamma(1) = 1$ *and* $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

iii) $\Gamma\left(\frac{1}{2} - z\right) = \frac{z!(-4)^z}{(2z)!}\sqrt{\pi}$.

iv) $\Gamma(z)\Gamma(-z) = \frac{-\pi}{n\sin(\pi z)}, \quad z \notin \mathbb{N}, \Re(z) < 1$.

Definition 1.3 (Euler psi Function [27]) *The Euler psi function is the logarithmic derivative of the gamma function, which is defined as:*

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \; z \in \mathbb{C},$$

with the following property:

$$\psi(z+n) = \psi(z) + \sum_{k=0}^{n-1}\frac{1}{z+k}, \quad z \in \mathbb{C}, n \in \mathbb{N}.$$

Definition 1.4 (Incomplete Gamma Function [27]) *The incomplete gamma function is derived from Eq. (1.1) by decomposing into an integral from* 0 *to* ω *and another from* ω *to* ∞ *as:*

$$\gamma(z,\omega) = \int_0^{\omega} e^{-x}x^{z-1}dx, z, \omega \in \mathbb{C}, \Re(z) > 0$$

$$\Gamma(z,\omega) = \int_{\omega}^{\infty} e^{-x}x^{z-1}dx, \quad |\arg(\omega)| < \pi, \Re(z) > 0.$$

Proposition 1.5 ([27]) *The incomplete gamma functions have the following properties:*

i) $\gamma(z,\infty) = \Gamma(z,0)\,\Gamma(z)\,,$

ii) $\gamma(z,\infty) + \Gamma(z,\omega) = \Gamma(z),\ \ \Re(z) > 0.$

1.1.2 The Beta Function

Definition 1.6 ([2, 27]) *The beta function is defined as:*

$$\beta(z,\omega) = \int_0^1 x^{z-1}(1-x)^{\omega-1}\,dx,\quad \Re(\omega)\,,\ \Re(z) > 0. \tag{1.2}$$

3D plot of the beta function Eq. (1.2) has been illustrated in Figure 1.2

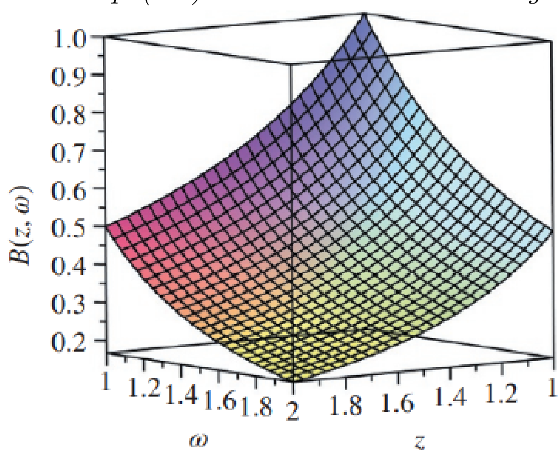

Figure 1.2 *The graph of buta function*

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Proposition 1.7 ([27]) *Some properties of the beta function are given as follows:*

i) $\beta(z,\omega) = \beta(\omega,z)\,,$

ii) $\beta(z,\omega) = 2\int\limits_0^{\frac{\pi}{2}} (\sin\theta)^{2z-1}(\cos\theta)^{2\omega-1}\,d\theta, \Re(\omega)\,, \Re(z) > 0,$

iii) $\beta(z,\omega) = \int\limits_0^{\infty} \frac{x^{z-1}}{(1+x)^{z+\omega}}dx,\quad \Re(\omega)\,, \Re(z) > 0,$

iv) $\beta(z,\omega) = \beta(z,\omega+1) + \beta(z+1,\omega)\,,$

v) $\beta(z,\omega+1) = \beta(z,\omega)\,\frac{\omega}{z+\omega},$

vi) $\beta(z+1,\omega) = \beta(z,\omega)\,\frac{z}{z+\omega},$

vii) $\beta(z,\omega)\,\beta(z+\omega,1-\omega) = \frac{\pi}{z\sin(\pi\omega)}.$

Notation 1.8 *The relationship between gamma and beta functions is written as:*

$$\beta(z,\omega) = \frac{\Gamma(z)\,\Gamma(\omega)}{\Gamma(z+\omega)}.$$

1. $\mathcal{L}_\rho \left\{ E_\alpha \left(-a \left(\frac{t^\rho}{\rho} \right)^\alpha \right) \right\} = \frac{s^\alpha}{s(s^\alpha + a)}$.

2. $\mathcal{L}_\rho \left\{ 1 - E_\alpha \left(-a \left(\frac{t^\rho}{\rho} \right)^\alpha \right) \right\} = \frac{a}{s(s^\alpha + a)}$.

3. $\mathcal{L}_\rho \left\{ (\frac{t^\rho}{\rho}^{\alpha-1}) E_{\alpha,\alpha} \left(-a \left(\frac{t^\rho}{\rho} \right)^\alpha \right) \right\} = \frac{1}{s(s^\alpha + a)}$.

4. $\mathcal{L}_\rho \left\{ \left(\frac{T^\rho}{\rho} \right) E_{\alpha,\beta} \left(a \left(\frac{T^\rho}{\rho} \right)^\alpha \right) \right\} = \frac{S^{\alpha-\beta}}{(S^\alpha - a)}$.

1.4 The amended Elzaki transform

Definition 1.46 *The amended elzaki transform of a given function $f(t)$*

$$H(s) = E[h(t)] = s \int_0^\infty e^{\frac{-t}{s}} f(t)\, dt, \ \ s \neq 0, t \geq 0$$

Lemma 1.47 **([26])** *.if $0 < \alpha \leq 1, s, t, p, \lambda$ are real numbers, $t \geq 0$. then*

1. $E\{1\} = s^2$, *where* $s > 0$,

2. $E\{t^\alpha\} = s^{\alpha+2}\Gamma(\alpha+1)$, *where* $s > 0$,

3. $E\{e^{at}\} = \frac{s^2}{1-as}$, *where* $s > 0$,

4. $E\{\sin(at)\} = \frac{as^3}{(1+a^2s^2)}$, *where* $s > 0$,

5. $E\{\cosh(at)\} = \frac{s}{1+s^2a^2}$, *where* $s > 0$

Theorem 1.48 *The ET of the Caputo fractional derivative is defined as [26]:*

$$E[D_t^{n\alpha} u(x,t)] = \frac{E[u(x,t)]}{s^{n\alpha}} - \sum_{k=0}^{n-1} s^{k-n\alpha+2} u^{(k)}(x,0), n-1 < n\alpha \leq n, n \in N.$$

CHAPTER 2

METHODS SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

Intrduction:

In this chapter, we present the Adomian decomposition method (ADM) and homotopy perturbation method (HPM) to solve linear and nonlinear fractional partial differential equations (PDEs).

2.1 Adomian Decomposition Method

The ADM was first introduced by Adomian in the early 1980s [33, 34]. It is a semi-analytical approach for solving linear and nonlinear ordinary/ partial/ fractional differential equations. It allows us to handle both nonlinear initial and boundary values problems. The method of solution of this method [35, 23] is based primarily on decomposing the nonlinear operator equation to a set of functions. Each series term is constructed from a polynomial generated by expanding an analytic function into a power series. The theoretical formulation of this technique is usually quite simple, but the actual difficulty arises when calculating the polynomials involved or when proving the convergence of the series of functions.

2.1.1 Basic Idea of ADM

Let us consider a general nonlinear nonhomogeneous PDE in the following form [37]

$$Lu\left(x,t\right)+Ru\left(x,t\right)+Nu\left(x,t\right)=f\left(x,t\right), \tag{2.1}$$

where L is the highest order differential operator and easily invertible, R is the linear differential operator of the order less than $L, f(x,t)$is the source term, and $Nu(x,t)$ represents the nonlinear term. The solution function $u(x,t)$ is assumed to be bounded, and the nonlinear term Nu satisfies the Lipschitz condition, that is $|Nu - Nv| \leq c|u - v|$, where c is a positive constant. Applying the inverse operator L^{-1} on both sides of Eq. (2.1), we obtain

$$u(x,t) = -L^{-1}Ru(x,t) - L^{-1}Nu(x,t) + L^{-1}f(x,t) + \phi \tag{2.2}$$

where ϕ satisfies $L\phi = 0$ and the initial conditions. If $L = D_t^{\alpha}$ in the Caputo sense, then $L^{-1} = J_t^{\alpha}$ hose expression is given in Chapter 1. Now, the solution may be defined by the method of decomposition provided in Eq.(2.2) by the following infinit series as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.3}$$

The nonlinear term Nu is then decomposed as:

$$Nu = \sum_{n=0}^{\infty} A_n. \tag{2.4}$$

where A_n's are the Adomian polynomials and An depends on $u_0, u_1, u_2, \ldots, u_n$. The Adomian polynomials can be defined as [33, 34]:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)\right]_{\lambda=0}, \quad n = 0,1,2,\ldots,$$

where λ is a grouping parameter of convenience.

For clarity, a few Adomian polynomials are listed as follows [27, 38]:

$$\begin{cases} A_0 = N(u_0), \\ A_1 = \frac{d}{d\lambda}N(u_0 + u_1\lambda)|_{\lambda=0} = u_1 N^{(1)}(u_0). \\ A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}N\left(u_0 + u_1\lambda + u_2\lambda^2\right)|_{\lambda=0} = u_2 N^{(1)}(u_0) + \frac{1}{2!}u_1^2 N^{(2)}(u_0) \\ A_3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}N\left(u_0 + u_1\lambda + u_2\lambda^2 + u_3\lambda^3\right)|_{\lambda=0} = u_3 N^{(1)}(u_0) + \left[\frac{1}{2!}u_2^2 + u_1u_3\right]N^{(2)}(u_0) + \frac{1}{3!}u_1^3 N^{(3)}(u_0). \\ A_4 = \frac{1}{4!}\frac{d^4}{d\lambda^4}N\left(u_0 + u_1\lambda + u_2\lambda^2 + u_3\lambda^3 + u_4\lambda^4\right)|_{\lambda=0} = u_4 N^{(1)}(u_0) + u_1u_2 N^{(2)}(u_0) + \\ \frac{1}{2!}u_1^2 u_2 N^{(3)}(u_0) + \frac{1}{4!}u_1^4 N^{(4)}(u_0). \end{cases}$$

Then, substituting Eqs. (2.3) and (2.4) into Eq. (2.2), we have

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = -L^{-1}R\left(\sum u_n(x,t)\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) + L^{-1}f(x,t) + \phi.$$

The following expressions can be obtained from Eq. (3.7)

$$\left.\begin{array}{l} u_0 = L^{-1}f + \phi \\ u_1 = -L^{-1}R(u_0) - L^{-1}A_0 \\ u_2 = -L^{-1}R(u_1) - L^{-1}A_1 \\ \vdots \\ u_{n+1} = -L^{-1}R(u_n) - L^{-1}A_n \end{array}\right\} \tag{2.5}$$

If we calculate all the terms $u_n's$, then we get the exact solution. But, in actual practice, this process may take a longer time. Toachieve an acceptable solution, it may be approximated by the truncated series $\sum_{n=0}^{N}$ (by using the convergence of the series).

2.2 Adomian Decomposition Transform Method

In this section, we will discuss the hybrid methods, which are the coupling of ADM with various transform methods, viz. Laplace transform (LT) and Elzaki transform (ET). With the combination of these transform methods, ADM is called the Adomian decomposition transform method (ADTM).

2.2.1 Transform Methods for the Caputo Sense Derivatives

Definition 2.1 *The LT of the Caputo fractional derivative is defined as [27]:*

$$L\left[D_t^{n\alpha}u\left(x,t\right)\right]=s^{n\alpha}L\left[u\left(x,t\right)\right]-\sum_{k=0}^{n-1}s^{(n\alpha-k-1)}u^{(k)}\left(x,0\right),n-1<n\alpha\leq 1,n\in\mathbb{N}. \tag{2.6}$$

Definition 2.2 *The ET of the Caputo fractional derivative is defined as [36]:*

$$E\left[D_t^{n\alpha}u\left(x,t\right)\right]=\frac{E\left[u\left(x,t\right)\right]}{s^{n\alpha}}-\sum_{k=0}^{\infty}s^{k-n\alpha+2}u^{(k)}\left(x,0\right),\ n-1<n\alpha\leq 1,n\in\mathbb{N}. \tag{2.7}$$

Following section deals with the systematic study of four hybrid methods, namely Adomian decomposition Laplace transform method (ADLTM) and Adomian decomposition Elzaki transform method (ADETM).

2.2.2 Adomian Decomposition Laplace Transform Method (ADLTM)

To clarify the fundamental idea of ADLTM, the fractional-order nonlinear nonhomogeneous partial differential equation(PDE) with initial conditions (ICs) are considered as:

$$D_t^{n\alpha}u\left(x,t\right)+Ru\left(x,t\right)+Nu\left(x,t\right)=f\left(x,t\right),\quad n-1<n\alpha\leq n. \tag{2.8}$$

subject to ICs

$$u^{(k)}\left(x,t\right)=g_k\left(x\right),\ldots,n-1.$$

where $D_t^{n\alpha}=\frac{\partial^{n\alpha}}{\partial t^{n\alpha}}=\frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$ is the fractional differential operator in the Caputo sense, R,N are respectively linear and nonlinear differential operators, respectively, and $f(x,t)$ is the source term. The ADLTM approach involves mainly two stages. In the firs stage, LT is taken on both sides of Eq. (2.8), and then in the second stage, ADM is applied where decomposition of the nonlinear term is done using Adomian polynomials. First, by operating LT on both sides of Eq. (2.8), we obtain

$$L\left[D_t^{n\alpha}u\left(x,t\right)\right]=L\left[f\left(x,t\right)\right]-L\left[Ru\left(x,t\right)\right]-L\left[Nu\left(x,t\right)\right].$$

Using differentiation property Eq. (2.6) of LT, we obtain

$$s^{n\alpha}Lu\left(x,t\right)-\sum_{k=0}^{n-1}s^{n\alpha-k-1}u^{(k)}\left(x,0\right)=L\left[f\left(x,t\right)\right]-L\left[Ru\left(x,t\right)\right]-L\left[Nu\left(x,t\right)\right].$$

$$L\left[u\left(x,t\right)\right]=\frac{1}{s^{n\alpha}}\sum_{k=0}^{n-1}s^{n\alpha-k-1}u^{(k)}\left(x,0\right)+\frac{1}{s^{n\alpha}}L\left[f\left(x,t\right)\right]-\frac{1}{s^{n\alpha}}L\left[Ru\left(x,t\right)\right]-\frac{1}{s^{n\alpha}}L\left[Nu\left(x,t\right)\right]. \tag{2.9}$$

Applying inverse LT on both sides of Eq. (2.9), we find

$$u(x,t) = F(x,t) - L^{-1}\left(\frac{1}{s^{n\alpha}} L\left[Ru(x,t)\right]\right) - L^{-1}\left(\frac{1}{s^{n\alpha}} L\left[Nu(x,t)\right]\right). \tag{2.10}$$

Here $F(x,t)$ repesents the term coming from the IC and source term (first two terms on the right-hand side of Eq. (2.9)).Next, in order to implement ADM, first we need to consider the solution in series form as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.11}$$

and the nonlinear term may be decomposed by using Adomian polynomials ([38]) as

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n. \tag{2.12}$$

where A_n denotes the Adomian polynomials and which is defined as follows:

$$A_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[N\left(\sum_{t=0}^{\infty} \lambda^i u_i(x,t) \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots \tag{2.13}$$

Substituting Eqs. (2.11) and (2.12) into Eq. (2.10), one may get the following expression:

$$\sum_{n=0}^{\infty} u_n(x,t) = F(x,t) - L^{-1}\left(\frac{1}{s^{n\alpha}} L\left[R\sum_{n=0}^{\infty} u_n(x,t)\right]\right) - L^{-1}\left(\frac{1}{s^{n\alpha}} L\left[\sum_{n=0}^{\infty} A_n\right]\right). \tag{2.14}$$

An iterative algorithm may be obtained by matching both sides of Eq. (2.14) as follows:

$$\begin{aligned}
u_0(x,t) &= F(x,t).\\
u_1(x,t) &= -L^{-1}\left(\frac{1}{s^{n\alpha}} L\left[Ru_0(x,t)\right]\right) - L^{-1}\left(\frac{1}{s^{n\alpha}} L\left[A_0\right]\right),\\
u_2(x,t) &= -L^{-1}\left(\frac{1}{s^{n\alpha}} L\left[Ru_1(x,t)\right]\right) - L^{-1}\left(\frac{1}{s^{n\alpha}} L\left[A_1\right]\right),\\
u_3(x,t) &= -L^{-1}\left(\frac{1}{s^{n\alpha}} L\left[Ru_2(x,t)\right]\right) - L^{-1}\left(\frac{1}{s^{n\alpha}} L\left[A_2\right]\right),\\
&\vdots\\
u_n(x,t) &= -L^{-1}\left(\frac{1}{s^{n\alpha}} L\left[Ru_{n-1}(x,t)\right]\right) - L^{-1}\left(\frac{1}{s^{n\alpha}} L\left[A_{n-1}\right]\right)\\
&\vdots
\end{aligned}$$

So, the solution of Eq. (2.8) may be obtained as:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots.$$

2.2.3 Adomian Decomposition Elzaki Transform Method (ADETM)

Applying ET on both sides of the $Eq.$(4.5), we obtain

$$E\left[D_t^{n\alpha}u\left(x,t\right)\right]=E\left[f\left(x,t\right)\right]-E\left[Ru(x,t)\right]-E\left[Nu(x,t)\right]$$

Using differentiation property Eq. (2.7) of ET, we have

$$S^{-n\alpha}E\left[u\left(x,t\right)\right]-\sum_{k=0}^{n-1}S^{k-n\alpha+2}u^{(k)}\left(x,0\right)=E\left[f\left(x,t\right)\right]-E\left[Ru\left(x,t\right)\right]-E\left[Nu\left(x,t\right)\right]. \quad (2.15)$$

$$E\left[u\left(x,t\right)\right]=S^{n\alpha}\sum_{k=0}^{n-1}S^{k-n\alpha+2}u^{(k)}\left(x,0\right)+S^{n\alpha}E\left[f\left(x,t\right)\right]-S^{n\alpha}E\left[Ru\left(x,t\right)\right]-S^{n\alpha}E\left[Nu\left(x,t\right)\right]. \quad (2.16)$$

.Inverse ET on both sides of Eq. (2.16) reduces to the following equation:

$$u\left(x,t\right)=F\left(x,t\right)-E^{-1}\left(s^{n\alpha}E\left[Ru\left(x,t\right)\right]\right)-E^{-1}\left(s^{n\alpha}E\left[Nu\left(x,t\right)\right]\right). \quad (2.17)$$

By plugging Eqs. (2.11) and (2.12) into Eq. (2.17), we have the expression as follows:

$$\sum_{n=0}^{\infty}u_n\left(\left(x,t\right)\right)=F\left(x,t\right)-E^{-1}\left(S^{n\alpha}E\left[R\sum_{n=0}^{\infty}p^nu_n\left(x,t\right)\right]\right)+E^{-1}\left(S^{n\alpha}E\left[\sum_{n=0}^{\infty}p^nH_n\left(u\right)\right]\right). \quad (2.18)$$

comparing both sides of Eq. (2.18), we have the following approximations successively:

$$\begin{aligned}
u_0\left(x,t\right) &= F\left(x,t\right),\\
u_1\left(x,t\right) &= -E^{-1}\left(s^{n\alpha}E\left[Ru_0\left(x,t\right)\right]\right)-E^{-1}\left(s^{n\alpha}E\left[H_0\left(u\right)\right]\right),\\
u_2\left(x,t\right) &= -E^{-1}\left(s^{n\alpha}E\left[Ru_1\right]\right)-E^{-1}\left(s^{n\alpha}E\left[H_1\left(u\right)\right]\right),\\
u_3\left(x,t\right) &= -E^{-1}\left(s^{n\alpha}E\left[Ru_2\right]\right)-E^{-1}\left(s^{n\alpha}E\left[H_2\left(u\right)\right]\right),\\
&\vdots\\
u_n\left(x,t\right) &= -E^{-1}\left(s^{n\alpha}E\left[Ru_{n-1}\right]\right)-E^{-1}\left(s^{n\alpha}E\left[H_{n-1}\left(u\right)\right]\right),\\
&\vdots
\end{aligned}$$

So, the solution of Eq. (2.8) may be written as:

$$u\left(x,t\right)=\lim_{n\to\infty}u_n\left(x,t\right)=u_0\left(x,t\right)+u_1\left(x,t\right)+u_2\left(x,t\right)+\cdots.$$

2.3 Homotopy Perturbation Method

In this section, we will discuss about homotopy perturbation method (HPM). HPM was first proposed by He ([39]). This approach has been established using artificial parameters ([40]). Interested readers may visit references ([41, 42]) for more information.

2.3.1 Procedure for HPM

In order to illustrate the fundamental idea of HPM, the fractional-order nonlinear non-homogeneous partial differential equation with initial conditions (ICs) is considered as follows:

$$D_t^{\alpha} u(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), \quad n-1 < \alpha \le n, \tag{2.19}$$

subject to ICs:

$$u^{(k)}(x,0) = g_k(x), \quad k = 0,1,\ldots,n-1, \tag{2.20}$$

where $D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the differential operator, $D_t^{\alpha} u(x,t)$ is the derivative of $u(x,t)$ in the Caputo sense, R, N are the linear and nonlinear differential operators, and $f(x,t)$ is the source term. We shall next present the solution approach based on the standard HPM. Let us now construct the following homotopy of Eq. (2.19) as ([39]):

$$(1-P) D_t^{\alpha} u(x,t) + P\left(D_t^{\alpha} u(x,t) + Ru(x,t) + Nu(x,t) - f(x,t)\right) = 0, \tag{2.21}$$

or

$$D_t^{\alpha} u(x,t) + P\left(Ru(x,t) + Nu(x,t) - f(x,t)\right) = 0, \tag{2.22}$$

where $p[0,1]$ is an embedding parameter. If $p=0$, then Eqs. (2.21) and (2.22) become

$$D_t^{\alpha} u(x,t) = 0,$$

and when $p=1$ Eqs. (2.21) and (2.22) turn out to be the original Eq. (2.19)

First, we need to consider the solution in series form containing the embedding parameter $p \in [0,1]$ as:

$$u(x,t) = \sum_{n=0}^{\infty} P^n u_n(x,t), \tag{2.23}$$

and the nonlinear term may be decomposed by using He's polynomials as:

$$Nu(x,t) = \sum_{n=0}^{\infty} P^n H_n(u). \tag{2.24}$$

where $H_n(u)$ denotes the He's polynomials and which is defined as follows:

$$H_n(u_0,u_1,\ldots,u_n) = \frac{1}{n!}\frac{\partial}{\partial p^n}\left[N\left(\sum_{n=0}^{\infty} P^n u_n(x,t)\right)\right]_{p=0}, n = 0,1,2,\ldots \tag{2.25}$$

Substituting Eqs. (2.23) and (2.24)into Eq. (2.22), we get the following expression

$$D_t^{\alpha}\left\{\sum_{n=0}^{\infty} P^n u_n(x,t)\right\} = P\left(f(x,t) - R\sum_{n=0}^{\infty} P^n u_n(x,t) - N\sum_{n=0}^{\infty} P^n H_n(u)\right). \tag{2.26}$$

By comparing the coefficients of the same powers of "p" on both sides of Eq. (2.26), we may have the following approximations successively:

$$\begin{aligned}
p^0 &: D_t^\alpha \{u_0(x,t)\} = 0,\\
P^1 &: D_t^\alpha \{u_1(x,t)\} = f(x,t) - Ru_0(x,t) - NH_0(u),\\
P^2 &: D_t^\alpha \{u_2(x,t)\} = -Ru_1(x,t) - NH_1(u),\\
P^3 &: D_t^\alpha \{u_3(x,t)\} = -Ru_2(x,t) - NH_2(u),\\
&\vdots\\
P^n &: D_t^\alpha \{u_n(x,t)\} = -Ru_{n-1}(x,t) - NH_{n-1}(u),\\
&\vdots
\end{aligned}$$

Applying the operator J_t^α , the inverse operator of D_t^α and using the ICs in Eq. (2.20), the first few terms of the HPM solution may be written as:

$$\begin{aligned}
u_0(x,t) &= \sum_{k=0}^{n-1} u^k(x,0)\frac{t^k}{k!} = \sum_{k=0}^{n-1} g_k(x)\frac{t^k}{k!},\\
u_1(x,t) &= J_t^\alpha[f(x,t)] - J_t^\alpha[Ru_0(x,t)] - J_t^\alpha[NH_0(u)],\\
u_2(x,t) &= -J_t^\alpha[Ru_1(x,t)] - J_t^\alpha[NH_1(u)],
\end{aligned}$$

and so on. So, the solution of Eq. (2.19) may be obtained as:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

2.4 Homotopy Perturbation Transform Method

In this section, we will discuss about thehybrid methods, which are the coupling of HPM with various transform methods, viz. Laplace transform (LT) and Elzaki transform (ET). As said earlier, HPM with the combination of these transform methods is called as homotopy perturbation transform method (HPTM) [36, 27, 43]. Now a days, these methods, namely homotopy perturbation Laplace transform method (HPLTM) and homotopy perturbation Elzaki transform method (HPETM) are getting popular recently. Although these four transform methods are effective methods for solving fractional differential equations, but these methods sometimes fail to address nonlinear terms arising from the fractional differential equations. These difficulties may be overcome by coupling these transforms with that of HPM.

2.4.1 Transform Methods for the Caputo Sense Derivatives

Definition 2.3 *The LT of the Caputo fractional derivative is defined as [43]:*

$$L[D_t^{n\alpha}u(x,t)] = s^{n\alpha}L[u(x,t)] - \sum_{k=0}^{n-1} s^{(n\alpha-k-1)}u^{(k)}(x,t), \quad n-1 < n\alpha \le 1, n\in\mathbb{N}. \tag{2.27}$$

Definition 2.4 *The ET of the Caputo fractional derivative is defined as [36]:*

$$E\left[D_t^{n\alpha}u\left(x,t\right)\right]=\frac{E\left[u\left(x,t\right)\right]}{s^{n\alpha}}-\sum_{K=0}^{n-1}s^{k-n\alpha+2}u^{(k)}\left(x,0\right),\quad n-1<n\alpha\leq 1,n\in\mathbb{N}. \tag{2.28}$$

Following section deals with the systematic study of four hybrid methods, namely HPLTM and HPETM, one after another.

2.4.2 Homotopy Perturbation Laplace Transform Method (HPLTM)

In order to clarify the basic idea of HPLTM, the fractional-order nonlinear nonhomogeneous partial differential equationwith initial conditions (ICs) is considered as follows:

$$D_t^{n\alpha}u\left(x,t\right)+Ru\left(x,t\right)+Nu\left(x,t\right)=f\left(x,t\right),\quad n-1<n\alpha\leq n. \tag{2.29}$$

subject to ICs:

$$u^{(k)}\left(x,0\right)=g_k\left(x\right),\quad k=0,1,\ldots,n-1. \tag{2.30}$$

where $D_t^{n\alpha}=\frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$ is the differential operator, $D_t^{n\alpha}u\left(x,t\right)$ is the derivative of $u(x,t)$ in the Caputo sense, R, N are the linearand nonlinear differential operators, and $f(x,t)$ is the source term. The HPLTM approach involves mainly two stages. Inthe first stage, LT is taken on both sides of Eq. (2.29), and then in the second stage, HPM is applied where decompositio of the nonlinear term is done using He's polynomials. First, by operating LT on both sides of Eq. (2.29), we obtain

$$L\left[D_t^{n\alpha}u\left(x,t\right)\right]=L\left[f\left(x,t\right)\right]-L\left(Ru\left(x,t\right)\right)-L\left[Nu\left(x,t\right)\right]. \tag{2.31}$$

Using differentiation property Eq. (2.27) of LT, we obtain

$$s^{n\alpha}L\left[u\left(x,t\right)\right]-\frac{1}{s^{n\alpha}}\sum_{k=0}^{n-1}s^{n\alpha-k-1}u^{(k)}\left(x,0\right)=L\left[f\left(x,t\right)\right]-L\left[Ru\left(x,t\right)\right]-L\left[Nu\left(x,t\right)\right]. \tag{2.32}$$

$$L\left[u\left(x,t\right)\right]=\frac{1}{s^{n\alpha}}\sum_{k=0}^{n-1}s^{n\alpha-k-1}u^{(k)}\left(x,0\right)+\frac{1}{s^{n\alpha}}L\left[f\left(x,t\right)\right]-\frac{1}{s^{n\alpha}}L\left[Ru\left(x,t\right)\right]-\frac{1}{s^{n\alpha}}L\left[Ru\left(x,t\right)\right]. \tag{2.33}$$

Applying inverse LT on both sides of Eq. (2.33), we find

$$u\left(x,t\right)=F\left(x,t\right)-L^{-1}\left(\frac{1}{s^{n\alpha}}L\left[Ru\left(x,t\right)\right]\right)-L^{-1}\left(\frac{1}{s^{n\alpha}}L\left[Nu\left(x,t\right)\right]\right). \tag{2.34}$$

where F(x, t) represents the term coming from the IC and source term (first two terms on the right-hand side of Eq. (2.33)).Next, to implement HPM, first, we need to consider the solution as in series form containing the embedding parameters $p\in[0,1]$ as:

$$u\left(x,t\right)=\sum_{n=0}^{\infty}p^n u_n\left(x,t\right). \tag{2.35}$$

and the nonlinear term may be decomposed by using He's polynomials as

$$Nu\left(x,t\right)=\sum_{n=0}^{\infty}p^{n}H_{n}\left(u\right). \tag{2.36}$$

where $H_n(u)$ denotes the He's polynomials and which is defined as follows:

$$H_{n}\left(u_{0},u_{1},\ldots,u_{n}\right)=\frac{1}{n!}\frac{\partial^{n}}{\partial p^{n}}\left[N\left(\sum_{n=0}^{\infty}p^{n}u_{n}\left(x,t\right)\right)\right]_{p=0}, \quad n=0,1,2,\ldots \tag{2.37}$$

Substituting Eqs. (2.35) and(2.36) into Eq. (2.34) and applying LT with HPM, one may get the following expression:

$$\sum_{n=0}^{\infty}p^{n}u_{n}\left(x,t\right)=F\left(x,t\right)-p\left(L^{-1}\left(\frac{1}{s^{n\alpha}}L\left[R\sum_{n=0}^{\infty}p^{n}u_{n}\left(x,t\right)\right]\right)+L^{-1}\left(\frac{1}{s^{n\alpha}}L\left[\sum_{n=0}^{\infty}p^{n}H_{n}\left(u\right)\right]\right)\right). \tag{2.38}$$

By comparing the coefficients of the same powers of "p" on both sides of Eq. (2.38), we may have the following approximations successively:

$$\begin{aligned}
p^{0} &: \ u_{0}\left(x,t\right)=F\left(x,t\right),\\
p^{1} &: \ u_{1}\left(x,t\right)=-L^{-1}\left(\frac{1}{s^{n\alpha}}L\left[Ru_{0}\left(x,t\right)\right]\right)-L^{-1}\left(\frac{1}{s^{n\alpha}}L\left[H_{0}\left(u\right)\right]\right),\\
p^{2} &: \ u_{2}\left(x,t\right)=-L^{-1}\left(\frac{1}{s^{n\alpha}}L\left[Ru_{1}\left(x,t\right)\right]\right)-L^{-1}\left(\frac{1}{s^{n\alpha}}L\left[H_{1}\left(u\right)\right]\right),\\
p^{3} &: \ u_{3}\left(x,t\right)=-L^{-1}\left(\frac{1}{s^{n\alpha}}L\left[Ru_{2}\left(x,t\right)\right]\right)-L^{-1}\left(\frac{1}{s^{n\alpha}}L\left[H_{2}\left(u\right)\right]\right),\\
p^{n} &: \ u_{n}\left(x,t\right)=-L^{-1}\left(\frac{1}{s^{n\alpha}}L\left[Ru_{n-1}\left(x,t\right)\right]\right)-L^{-1}\left(\frac{1}{s^{n\alpha}}L\left[H_{n-1}\left(u\right)\right]\right)
\end{aligned}$$

So, the solution of Eq. (2.29) may be obtained as:

$$u\left(x,t\right)=\lim_{n\to\infty}u_{n}\left(x,t\right)=u_{0}\left(x,t\right)+u_{1}\left(x,t\right)+u_{2}\left(x,t\right)+\cdots. \tag{2.39}$$

2.4.3 Homotopy Perturbation Elzaki Transform Method (HPETM)

Applying ET on both sides of the Eq. (2.29), we obtai

$$E\left[D_{t}^{n\alpha}u\left(x,t\right)\right]=E\left[f\left(x,t\right)\right]-E\left[Ru\left(x,t\right)\right]-E\left[Nu\left(x,t\right)\right]. \tag{2.40}$$

Using differentiation property Eq. (2.28) of ET, we have

$$s^{-n\alpha}E\left[u\left(x,t\right)\right]-\sum_{k=0}^{n-1}s^{k-n\alpha+2}u^{(k)}\left(x,0\right)=E\left[f\left(x,t\right)\right]-E\left[Ru\left(x,t\right)\right]-E\left[Nu\left(x,t\right)\right]. \tag{2.41}$$

$$E\left[u\left(x,t\right)\right]=s^{n\alpha}\sum_{k=0}^{n-1}s^{k-n\alpha+2}u^{(k)}\left(x,0\right)+s^{n\alpha}E\left[f\left(x,t\right)\right]-s^{n\alpha}E\left[Ru\left(x,t\right)\right]-s^{n\alpha}E\left[Nu\left(x,t\right)\right]. \tag{2.42}$$

Conclusion

In this work, homotopy perturbation Laplace transform method (HPLTM) is adopte d for the numerical simulation of time-fractional model of Navier-Stokes equations with initial conditions. Thefrac- tional derivative is considered in the Caputo sense. The analytical results have been given interms of a power series. tow test problems are carriedout in order to validate and illustrate the e¢ciency of the method. The proposed solutions agree excellently with HPM[45] and ADM [44], andare approximated without any discretization, transformation, perturbation, or restrictive conditions.However, the performed calculations show that the described method needs very small size of com-putation in comparison with HPM [45] and ADM [44]. Small size of computation contrary to the otherschemes, is the strength of the scheme.

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ملخص

تم الحصول في هذه الدراسة على حل تقريبي لمعادلة نافييه-ستوكس متعددة الأبعاد بالترتيب الزمني الكسري باعتماد نهج شبه تحليلي: " طريقة تحويل لابلاس بالإضطراب التماثلي *(HPLTM)*"، تم اختبار مشكلتين للتحقق من فعالية الطريقة وتوضيحها. وقد وجد أن النهج يعتبر تقنية قوية وفعالة لحل مجموعة واسعة من المشاكل المتعلقة بالهندسة والعلوم.

الكلمات الرئيسية: معادلة نافييه-ستوكس، مشتق كابوتو الزمني الكسري، *HPLTM*، دالة ميتاغ-ليفلر.

Abstract

In this work, approximate solution of time-fractional order multi-dimensional Navier-Stokes equation is obtained by adopting a semi-analytical scheme: "Homotopy perturbation Laplace transform Method (HPLTM)". Tow test problems are carried out in order to validate and illustrate the efficiency of the method. The scheme is found to be very reliable, effective and efficient powerful technique to solve wide range of problems arising in engineering and sciences. The small size of computation contrary to the other schemes.

KEYWORDS: *Navier--Stokes equation, Caputo time-fractional derivative, HPLTM, Mittag--Leffler function.*

Résume

Dans ce mémoire, une solution approchée de l'équation de Navier-Stokes multidimensionnelle d'ordre fractionnaire dans le temps est obtenue en adoptant un schéma semi-analytique : "Méthode de transformation de Laplace par perturbation homotopique (HPLTM)". Deux problèmes de test sont réalisés afin de valider et d'illustrer l'efficacité de la méthode. Le schéma s'avère être une technique très fiable, efficace et puissante pour résoudre une large gamme de problèmes rencontrés en ingénierie et en sciences. Sa petite taille de calcul, contrairement aux autres schémas.

MOTS CLÉS: *équation de Navier-Stokes, dérivée temporelle de Caputo fractionnaire, HPLTM, fonction de Mittag-Leffler*