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Faculty of Mathematics And Informatics
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# Master of Mathematics

Mathematics And Informatics

**Specialty** : Mathematics

**Option** : Mathematical and Numerical Analysis

## Theme

# Spline Collocation For Fredholm Integral Equations

**Persented By :**

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**in front of the jury :**

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# Dedication

To my Father and Mother, to whom I owe everything and who supported me to the end.

To my Brothers and Sisters, to whom I wish much success in their lives.

To all my friends who have a special place in my life and to whom I wish a lot of happiness and success.

To all those who have contributed far or closely to this work.

# ملخص

*في هذه الأطروحة، قمنا بدراسة المعادلات التكاملية والتي تكتب من* الشكل :

$$u(x)=f(x)+\lambda\int_{\beta(x)}^{\alpha(x)} k(x,t)u(t)dt$$

بحيث$\alpha(x)$و$\beta(x)$حدودالتكامل و $u(x)$ مجهول ، $k(x,t)$ النواة لهذه المعادلة والدالة *f(x)* و $\lambda$ معامل ثابت

تناولنا معادلة فريدهولم التكاملية بشكل خاص

1. تعريف معادلة فريدهولم التكاملية
2. برهان الوجود والوحدانية
3. الحل التقريبي لمعادلة فريدهولم. وذلك بإستعمال طريقة سبلاين كولكشن (التجميع في سبلاين) بالمقارنة مع لاغرونج كولكشن.

# Abstract

In this thesis we study the integral equations, whose the general form is

$$u(x) = f(x) + \lambda \int_{\beta(x)}^{\alpha(x)} k(x,t)\, u(t)\, dt$$

where $\alpha(x)$ and $\beta(x)$ the limits of integration and $u(x)$ unknown function, $K(x,t)$the kernel and $f(x)$ function in equation, and $\lambda$ is a constant parameter.

particularly, we study the Fredholm Integral equation, we

1. Outline these equation using basic definitions.
2. Discuss the existence and uniqueness of solutions.
3. Approximate the solution of Fredholm integral equation, using spline colloction method compared to colloction basis on Lagrange method.

# Résumé

Dans cette thése nous étudions les équations intégrqles, dont la form générale est :

$$u(x) = f(x) + \lambda \int_{\beta(x)}^{\alpha(x)} k(x,t)\, u(t)\, dt$$

où $\alpha(x)$ et $\beta(x)$ les limites d'intégration $u(x)$ fonction inconnue, $K(x,t)$ le noyau et la fonction $f(x)$ dans l'équation, et $\lambda$ est un paramétre constant.

En particulier, nous étudions l'équation intégrale de type Fredholm, Nous

1. décrivons ces équations en utilisant des définitions de base.
2. Discutons léexistence et d'unicitédes solutions.
3. Approchons la solution des équation intégrale de Fredholm en utilisant méthode de colloction spline comparée à collection basée sur Lagrange méthode

# Notation

| | |
|---|---|
| $\mathbb{R}^n$ | Set of n-tuples $x = (x_1, x_2, ..., x_n)$ the integral. |
| $X, E, F$ | Metris spaces, Banach or Hilbert spaces. |
| $A$ | Integral operator. |
| $\varphi$ | Unknown function in the integral equation. |
| $k(x, t, \varphi(t))$ | Kernel of the integral equation. |
| $C([a, b], R^n)$ | Set continuously differentiable functions $\varphi : [a, b] \to R^n, ([a, b] \subset R^n \text{ open})$. |
| S.L | spline colloction. |
| C.L | colloction basis on lagrange. |

# Table of Contents

# Introduction

The subject of integral equations is one of the most useful tools in both pure and applied mathematics. It has enormous operations in numerous physical problems. Numerous original and boundary value problems associated with ordinary discriminational equation (ODE) and partial discriminational equation (PDE) can be converted into problems of working some approximate integral equation.[20]

In this work ,we will study Fredholm integral equation, this kind of equation arises in many scientific application. It was also shown that Fredholm integral equation can be derived from boundary value problems. Erik Ivar Fredholm (1866–1927) is best remembered for his work on integral equations and spectral theory. Fredholm was a Swedish mathematician who established the theory of integral equations and his paper in Acta Mathematica (...) played a major role in the establishment of operator theory. [22][20]

Many authors give numerical solutions for different types of Fredholm integral equations, in this dissertation, we show the efficiency of a numerical method which are the collocation methods based on cubic B-spline, can be used to solve the linear Fredholm integral equation of the second kind and compared them with the collocation methods based on Lagrange basis.

Our work is presented as follows/

It the first chapter come a preface to the Integral equations and their classifications, we focus on Fredholm integral equations. Riez thoeory, Fredholm Alternative and the fixed point theory are mentioned in order to used it to discuss the existence and uniqueness of solutions of integral equations.

In the second chapter, we talked about some important definitions that we need to make thesis asuccess. and we defined colloction method based on Lagrange polynomials and spline method , The Regularization method which talks about how to change an equation of the first kind that is difficult to solve into an equation of the second kind . And defined the Newton-Cotes method (simpson method and Tapezoid method) , which searches for an approximate solution to an equation to find its approximate solu .

As for the third chapter , provides a solution to the Fredholm integral equations using colloction method based on lagrange .The basis of this work was dealt with , which is the spline colloction method .Which was applied to the Fredholm integral equation.It was shown that these spline colloction method can provide us accurate approximate solution . We show a convergence result and we give the error estimate for these method.

Finalement chapter , numerical examples are included to demonstrate the validity and applicability of these approache .

# Chapter 1

# Preliminaries

## 1.1 Compact linear operators

A linear operator $A$ defined from a normed space $E$ into a normed space $F$ is called a compact linear operator or completely continuous linear operator if for every bounded subset $\Omega$ of $E$, the image $A(\Omega)$ is relatively compact in $F$. In other words, the closure $\overline{A(\Omega)}$ is compact. [17]

## 1.2 Collocation method

A collocation method is based on the idea of approximating the exact solution of a given integral equation with a suitable function belonging to a chosen finite dimensional space such that the approximated solution satisfies the integral equation on a certain subset of the interval on which the equation has to be solved (called the set of collocation points).

( section (2.3.1) (2.3.2) ,[1] [23])

Pick distinct node points $x_1, ...., x_m \in D$, and require

$$r_m(x_i) = 0, \qquad i = 1, ..., m_n \tag{1.2.1}$$

This leads to determine $\{c_1, ..., c_k\}$ as the solution of the linear system

$$\sum_{j=1}^{m} c_j \left\{ u_j(x_i) - \lambda \int_D K(x_j, t)\, u_j(t_i)\, ty \right\} = k(x_i), \qquad i = 1, ..., m. \tag{1.2.2}$$

We should have written the node points as u$\{x_{1,n}, ..., x_{m,n}\}$ , but for notational simplicity, the explicit dependence on $n$ has been suppressed, to be understood only implicitly.

The function space framework for collocation methods is often $C(D)$ ,which is what we use here.

As a part of writing (1.2.2) in a more abstract form, we introduce a projection operator $P_n$ that maps $X = C(D)$ onto $X_n$ . Given $f \in C(D)$ define $P_n f$ to be that element of $X_n$ that interpolates $f$ at the nodes $\{x_1, ..., x_m\}$ . This means writing

$$P_n f(x) = \sum_{j=1}^{m_n} \alpha_j u_j(x) \tag{1.2.3}$$

with the coefficients $\{\alpha_j\}$ determined by solving the linear system

$$\sum_{j=1}^{m_n} \alpha_j u_j(x_i) = f(x_i), \qquad i = 1, ..., m_n \tag{1.2.4}$$

This linear system has a unique solution if

$$\det[u_j(x_i)] \neq 0. \tag{1.2.5}$$

we assume this is true whenever the collocation method is being discussed.By a simple argument, this condition also implies that the functions $\{u_1, ..., u_m\}$ are a linearly independent set over $D$.

In the case of polynomial interpolation for functions of one variable and monomials $\{1, x, ..., x^n\}$ as the basis functions, the determinant in (1.2.5) is referred to as the Vandermonde determinant .To see more clearly that $P_n$ is linear. and to give a more explicit formula.

for more details see [1].

## 1.3 Lagrange polynomial interpolation

Let $f$ be a continuous function defined on a finite closed interval $[a, b]$ Let

$$\Delta : a \leq x_0 < x_1 < ... < x_n \leq b$$

be a partition of the interval $[a,b]$. Choose $X = C[a,b]$ , the space of continuous functions $f:[a,b]\to F$ ; (where $F$ is real or complex) and choose $X_{n+1}$ to be $p_n$, the space of the polynomials of degree less than or equal to $n$. Then the Lagrange interpolant of degree $n$ of $f$ is defined by the conditions[23]

$$p_n(x_i) = f(x_i), \qquad 0 \le i \le n, \qquad p_n \in P_n \tag{1.3.1}$$

Here the interpolation linear functionals are

$$L_i f = f(x_i), \qquad 0 \le i \le n \tag{1.3.2}$$

If we choose the regular basis $v_j(x) = x_j \,(0 \le j \le n)$ for $P_n$, then it can be shown that Furthermore, we have the representation formula

$$\det (L_i v_j)_{(n+1)\times(n+1)} = \prod_{j>i} (x_j - x_i) \neq 0 \tag{1.3.3}$$

Thus there exists a unique Lagrange interpolation polynomial.

Furthermore, we have the representation formula

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\,\varphi_i(x), \qquad \varphi_i(x) = \prod_{j\neq i} \frac{x - x_j}{x_i - x_j}, \tag{1.3.4}$$

called Lagrange's formula for the interpolation polynomial. The functions $\varphi_i$satisfy the special interpolation conditions

$$\varphi_i(x) = \delta_{ij} = \begin{cases} 0, i \neq j, \\ 1, i = j, \end{cases} \tag{1.3.5}$$

The functions $\{\varphi_i\}_{i=0}^{n}$ form a basis for $P_n$, and they are often called Lagrange basis functions.

**Theorem 1.3.1** *The following statements are equivalent:*

1. *The interpolation problem has a unique solution.*
2. *The functionals $L_1, ..., L_n$ are linearly independent over $X_n$.*

*3. The only element $f_n \in X_n$ satisfying*

$$L_i f_n = 0, \qquad 1 \leq i \leq n,$$

*is $f_n = 0$*

*4. For any data $\{b_i\}_{i=1}^{n}$ , there exists one $f_n \in X_n$ such that*

$$L_i f_n = b_i, \qquad 1 \leq i \leq n$$

*Outside of the framework of thise Theorem,the formula (**??**) shows that there is a solution to the Lagrange interpolation problem (1.3.1). The singularity result can be demonstrated by showing that the corresponding interpolation for homogeneous data is zero.*

*let $p_n \in P_n$ with $p_n(x_i) = 0, 1 \leq i \leq n$ Then the polynomial $p_n$ must contain the factors $(x - x_i), 1 \leq i \leq n$. Since deg $(p_n) \leq n$ and*

$$\deg \prod_{i=1}^{n} (x - x_i) = n$$

we have

$$p_n(x) = c \prod_{i=1}^{n} (x - x_i) \qquad (1.3.6)$$

for some constant $c$. Using the condition $p_n(x_0) = 0$ we see that $c = 0$ and there fore,$(p_n \equiv 0)$ . We note that by Theorem Lagrenge, this result on the uniqueness of the solvability of the homogeneous problem also implies the existence of a solution.

In the above, we have indicated three methods for showing the existence and uniqueness of a solution to the interpolation problem (1.3.1). The method based on showing the determinant of the coefficient is nonzero ,as in (1.3.3) , this can be done easily only in simple situations such as Lagrange polynomial interpolation. Usually it is simpler to show that the interpolant corresponding to the homogeneous data is zero, even for complicated interpolation conditions. For practical calculations, it is also useful to have a representation formula that is the analogue of (1.3.4) ,but such a formula is sometimes difficult to find.

For more details see [4].

## 1.4 Spline method

In this section, we discuss how to approximate a sufficiently smooth functin in the spline space spanned a given set of *B-spline* . Using the properties of the *B-spline* basis, we explicitly construct a spline that achives for the function and its derivatives,amd determines an apprpriate error estimate. The construction methods we will introduce are local and linear.

The weakness of the Lagrange interpolation is that the interpolation error increases with the number of interpolation points $n$. This results experimentally in large oscillations of the interpolating polynomial, even if f is very simple. For example(Runge1901), when we interpolate the function $x \to 1/\left(1+25x^2\right)$ at uniformly distributed points on the interval $[a,b]$the Lagrange polynomials do not converge to $f$. Hence the idea of interpolating by piecewise polynomial functions, the degree of which does not increase with the number of interpolation points.

In the mathematical field of numerical analysis, a spline is a function defined by pieces, by polynomials.

Given $(n+1)$ nodes $a = x_0 < x_1 < ... < x_n = b$ and the corresponding valuer $f_i, i = 0, 1, ..., n$, the spline $S(x)$ is defined by

$$S(x) = \begin{cases} s_0(x) & si, x \in [x_0, x_1] \\ s_1(x) & si, x \in [x_1, x_2] \\ & . \\ & . \\ s_{n-1}(x) & si, x \in [x_{n-1}, x_n] \end{cases}$$

When the polynomials $S_i(x)$are of degree 1, we speak of linear spline, when they are of degree 2, we speak of quadratic spline,are of degree3, we by the cubic spline:

### 1.4.1 Cubic spline

**Definition 1.4.1** *A cubic spline is a noted function $S$ , which verifies the following properties:*

*1-* $S \in C^2([a,b])$ .

*2- $S$ coincided on each interval$[x_i, x_{i+1}]$ with a polynomial of degree less than or equal to* 3.

*3- $S(x_i) = f_i$, for $i = 0, 1, ...n$.*

### 1.4.2 Cubic spline calculation

Interpolation by cubic splines consists in replacing, on each sub-interval, the function $f$ by a polynomial of the third degree, so that the interpolating function is continuous as well as its first and second derivatives over the entire interval $[x_0, x_n]$.

Our goal is to define an efficient method to build the cubic spline interpolating these values.

Introducing the following notations

$$f_i = S(x_i), m_i = S'(x_i) \text{ and } M_i = S''(x_i), i = 0, 1, ...n.$$

As $S_{i-1}(x)$ is a polynomial of degree 3, $S''_{i-1}(x)$ is linear, let's set $h_i = x_i - x_{i-1}$ for $i = 1, ..., n$.

$$S''_{i-1}(x) = M_{i-1}\frac{(x_i - x)}{h_i} + M_i\frac{(x - x_{i-1})}{h_i}, \text{for } x \in [x_{i-1}, x_i]. \qquad (1.4.1)$$

Integrating twice (1.4.1) we get.

$$S_{i-1}(x) = M_{i-1}\frac{(x_i - x)^3}{6h_i} + M_i\frac{(x - x_{i-1})^3}{6h_i} + C_{i-1}(x - x_{i-1}) + D_{i-1},$$

the constants $C_{i-1}$ and $D_{i-1}$are determined by imposing the values at the ends $S(x_{i-1}) = f_{i-1}$and $S(x_i) = f_i$

This gives, for $i = 1, ..., n-1$

$$C_{i-1} = \frac{f_i - f_{i-1}}{h_i} - \frac{h_i}{6}(M_i - M_{i-1}), \quad D_{i-1} = f_{i-1} - M_{i-1}\frac{h_i^2}{6}$$

Let us now impose the continuity of the first derivative in xi, we obtain

$$\begin{aligned} S'\left(x_i^-\right) &= \frac{h_i}{6}M_{i-1} + \frac{h_i}{3}M_i + \frac{f_i - f_{i-1}}{h_i} \\ &= -\frac{h_{i+1}}{3}M_i - \frac{h_{i+1}}{6}M_{i+1} + \frac{f_{i+1} - f_i}{h_{i+1}} \\ &= S'\left(x_i^+\right) \end{aligned}$$

$S'\left(x_i^{\pm}\right) = \lim_{t\to 0} S'\left(x_i \pm t\right)$ .This leads to the following linear system

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_{i,} \qquad i = 1, ..., n-1. \tag{1.4.2}$$

where we posed

$$\begin{aligned} \mu_i &= \frac{h_i}{h_i + h_{i+1}}, \lambda = \frac{h_{i+1}}{h_i + h_{i+1}} \\ d_i &= \frac{6}{h_i + h_{i+1}}\left(\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right), i = 1, ..., n-1. \end{aligned}$$

The system (1.4.2) has $(n+1)$ unknowns and $(n-1)$equations, two conditions there fore remain to be fixed. In general, these conditions are of the form

$$\begin{aligned} 2M_0 + \lambda_0 M_1 &= d_0 \\ \mu_n M_{n-1} + 2M_n &= d_n \end{aligned} \tag{1.4.3}$$

as $0 < \lambda_0, \mu_n < 1$ and $d_0, d_n$ are given values.

To get the natural splines:

$$S''(a) = S''(b) = 0$$

We must cancel the above coefficients.

A frequent choice is to set $\lambda_0 = \mu_n = 1$ and $d_0 = d_1, d_{n-1} = d_n$,which amounts to extending the spline beyond the extreme points of the interval$[a, b]$and treat $a$ and $b$ as internal points. This strategy gives a spline with "**regular**” behavior

In general, the linear system obtained is tridiagonal of the form:i-1

$$\begin{bmatrix} 2 & \lambda_0 & 0 & . & . & 0 \\ \mu_1 & 2 & \lambda_1 & . & & . \\ 0 & . & . & & & . \\ . & . & . & . & & . \\ . & & . & \mu_{n-1} & 2 & \lambda_{n-1} \\ 0 & . & . & 0 & \mu_n & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ . \\ . \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ . \\ . \\ d_{n-1} \\ d_n \end{bmatrix}$$

### 1.4.3 Approximation of an integral by splines

Let $P_i(x)$ be the polynomial of degree 3 of the spline in the interval $[x_{i-1}, x_i]$. The expression of this polynomial is:

$$P_i(x) = -f''_{i-1}\frac{(x-x_i)^3}{6h_i} + f''_i\frac{(x-x_{i-1})^3}{6h_i} - \left(\frac{f(x_{i-1})}{h_i} - \frac{h_i f''_{i-1}}{6}\right)(x-x_i) + \left(\frac{f(x_i)}{h_i} - \frac{h_i f''_{i-1}}{6}\right)(x-x_{i-1})$$

By integrating this polynomial we get

$$\begin{aligned} \int_a^b f(x)\,dx &\simeq \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} P_i(x)\,dx \\ &= \sum_{i=1}^{n}\left[f''_{i-1}\frac{(x_{i-1}-x_i)^4}{24h_i} + f''_i\frac{(x_i-x_{i-1})^4}{24h_i} + \left(\frac{f(x_{i-1})}{h_i} - \frac{h_i f''_{i-1}}{6}\right)\frac{(x_{i-1}-x_i)^2}{2} + \left(\frac{f(x_i)}{h_i} - \right.\right. \end{aligned}$$

since $h_i = x_i - x_{i-1}$ the last expression is equivalent to

$$\begin{aligned} &\sum_{i=1}^{n}\left[\left(f''_{i-1} + f''_i\right)\frac{h_i^3}{24} + \left(f(x_{i-1}) + f(x_i)\right)\frac{h_i}{2} - \left(f''_{i-1} + f''_i\right)\frac{h_i^3}{12}\right] \\ = &\sum_{i=1}^{n}\left[\left(f(x_{i-1}) + f(x_i)\right)\frac{h_i}{2} - \left(f''_{i-1} + f''_i\right)\frac{h_i^3}{24}\right] \end{aligned}$$

We thus obtain the following approximation of the integral of $f(x)$ in the interval $[a, b] = [x_0, x_n]$

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n}[\left(f(x_{i-1}) + f(x_i)\right)\frac{h_i}{2} - \left(f''_{i-1} + f''_i\right)\frac{h_i^3}{24}] \tag{1.4.4}$$

In the case where the abscissas x$_i$ are equidistant $(h_i = h)$ we can further simplify the previous expression to get :

$$\int_a^b f(x)\,dx \simeq \frac{h}{2}\left[f(x_0) + 2(f(x_1) + f(x_2) + ... + f(x_{n-1})) + f(x_n)\right] - \frac{h^3}{24}\left[f_0^{"} + 2\left(f_1^{"} + f_2^{"} + ... + f_{n-1}^{"}\right) + \right.$$

**since** $f_0^{"} = f_n^{"} = 0$ in the case of the natural spline , we have

$$\int_a^b f(x)\,dx \simeq \frac{h}{2}\left[f(x_0) + 2(f(x_1) + f(x_2) + ... + f(x_{n-1})) + f(x_n)\right] - \frac{h^3}{12}\left[f_1^{"} + f_2^{"} + ... + f_{n-1}^{"}\right]$$

**Remark 1.4.1** *In the general case we use the expression* (1.4.4) *to approximate the integral using the spline.If the abscissas are equidistant, the expression*(**??**) *is preferably used.*

## 1.5 Newton-Cotes method

### 1.5.1 Tapezoid method

**Principle**

We replace the curve representative of $f$ , on each segment of the subdivision, by the segment which joins $(x_i, f(x_i))$ to $(x_{i+1}, f(x_{i+1}))$ . This therefore amounts to interpolating the function $f$ on the segment $[x_i, x_{i+1}]$ by the Lagrange polynomial of degree 1 at the points $x_i$ and $x_{i+1}$.

**Proposition 1.5.1** *The approximate value of the integral of $f$ over $[a, b]$ by the trapezoid method is then given by*

$$I_n(f) = \frac{b-a}{n}\left(\frac{f(a) + f(b)}{2} + \sum_{i=1}^{n-1} f(x_i)\right)$$

**Evidence** :

The area of the base trapezoid $[x_i, x_{i+1}]$ is

$$\frac{(x_{i+1} - x_i)(f(x_i) + f(x_{i+1}))}{2}$$

It is deduced that

$$I_n(f) = \sum \frac{(x_{i+1} - x_i)}{2}(f(x_i) + f(x_{i+1})) = \frac{b-a}{n}\left(\frac{f(a) + f(b)}{2} + \sum_{i=1}^{n-1} f(x_i)\right)$$

**Error evaluation:**

**Proposition 1.5.2** *If $f$ is of class $C^2$ on $[a,b]$, so we have , for any non-zero integer $n$*

$$\|E_n(f)\|_\infty \leq (b-a)^3 \frac{\left\|f^{(2)}\right\|_\infty}{12n}$$

We deduce that In $I_n(f)$ converges to $I(f)$.

**Evidence :**

This method consists in replacing $f$ on the segment $[x_i, x_{i+1}]$by its Lagrange interpolation polynomial$P_i$of degree 1 having the same values as $f$ at the limits of the interval , and as $f$is of class $C^2$, we has

$$\forall x \in [x_i, x_{i+1}], \qquad f(x) - P_i(x) = (x_{i+1} - x)(x - x_i)\frac{f^{(2)}(x)}{2}$$

It is deduced that

$$\begin{aligned}
\left|\int_{x_i}^{x_{i+1}} (f(x) - P_i(x))\,dx\right| &\leq \int_{x_i}^{x_{i+1}} |(x_{i+1}-x)(x-x_i)| \frac{\left\|f^{(2)}\right\|_\infty}{2} dx \\
&\leq \int_{x_i}^{x_{i+1}} (|(x_{i+1}-x)|\,|(x-x_i)|\,|(x_{i+1}-x)|\,|(x_{i+1}+x)|) \frac{\left\|f^{(2)}\right\|_\infty}{2} dx \\
&\leq |(x_{i+1}-x_i)|^3 \frac{\left\|f^{(2)}\right\|_\infty}{12}
\end{aligned}$$

To conclude, it suffices to notice that

$$\begin{aligned}
|E_n(f)| &= \left|\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (f(x) - P_i(x))\,dx\right| \\
&\leq \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |(f(x) - P_i(x))|\,dx \\
&\leq \sum_{i=0}^{n-1} (x_{i+1}-x_i)^3 \frac{\left\|f^{(2)}\right\|_\infty}{12} = n\frac{\left(\frac{b-a}{n}\right)^3 \left\|f^{(2)}\right\|_\infty}{12} = (b-a)^3 \frac{\left\|f^{(2)}\right\|_\infty}{12},
\end{aligned}$$

So

$$\|E_n(f)\|_\infty \leq (b-a)^3 \frac{\left\|f^{(2)}\right\|_\infty}{12}$$

### 1.5.2 Simpson method

We replace $f$, on each segment $[x_i, x_{i+1}]$ by its Lagrange interpolation polynomial $P_i$ of degree 2 having the same values as $f$ at the limits of the interval and in its middle.

The approximate value of the integral of $f$over$[a, b]$ by Simpson's method is given by

$$I_n(f) = \frac{b-a}{6n}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i) + 4\sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)\right)$$

If $f$ is of class $C^3$ on $[a, b]$ , then ,for any non-zero natural number $n$ , we have

$$\|E_n(f)\|_\infty \leq \frac{(b-a)^4}{192n^3}\left\|f^{(3)}\right\|_\infty$$

# Chapter 2

# Classifications and theory of integral equations

In this chapter we define Integral Equations and give its classification.

Next we discuss the existence and uniqueness of solutions of Fredholm integral equation by using two basic theorems ,which is the Riez thoeory and Fredholm alternative, and the fixed point theory by using Banach's fixed point theorem.

## 2.1 Definition of Integral Equations

An integral equation is defined as an equation in which the unknown function $u(x)$ to be determined appear under the integral sign [20][18][6].

A typical form of an integral equation in $u(x)$ is of the form

$$u(x) = f(x) + \lambda \int_{\beta(x)}^{\alpha(x)} k(x,t)\, u(t)\, dt \tag{2.1.1}$$

where $K(x,t)$ is called the kernel of the integral equation (2.1.1), and $\alpha(x)$ and $\beta(x)$ are the limits of integration. It can be fluently observed that the unknown function $u(x)$ appears under the integral sign. It's to be noted then that both the kernel $K(x,t)$ and the function $f(x)$ in equation (2.1.1) are given functions; and $\lambda$ is a constant parameter. The high ideal of this project is to determine the unknown function $u(x)$that will satisfy equation(2.1.1) using a number of result ways. We shall devote considerable sweats in exploring these styles to find results of the unknown function..

## 2.2 Classification Integral Equations

An integral equation can be classified as a linear or nonlinear integral equation as we have noticed that the differential equation can be equivalently represented by the integral equation. Therefore, there is a good relationship between these two equations. The most frequently used integral equations fall under two major classes, namely Volterra and Fredholm integral equations. Of course, we have to classify them as homogeneous or nonhomogeneous; and also linear or nonlinear. In some practical problems, we come across singular equations also. In this text, we shall distinguish four major types of integral equations , the two main classes and two related types of integral equations. In particular, the five types are given below[18]:

- Fredholm integral equations
- Volterra integral equations
- Volterra-Fredholm integral equations
- Integro-differential equations
- Singular integral equations

We shall outline these equations linear using basic definitions properties and each type.

## 2.3 Fredholm Integral Equations

For Fredholm integral equations, the limits of integration are fixed. Moreover, the unknown function $u(x)$ may appear only inside integral equation in the form:

$$f(x) = \int_a^b k(x,t)u(t)dt \qquad a \leq x \leq b \tag{2.3.1}$$

This is called Fredholm integral equation of the first kind. However, for Fredholm integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral sign. as the non homogeneous Fredholm linear integral equation second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)dt \qquad a \leq x \leq b \tag{2.3.2}$$

this equation

$$u(x) = f(x) + \lambda \int_a^b k(x,t) \{u(t)\}^2 dt, \qquad a \leq x \leq b \tag{2.3.3}$$

is the Fredholm nonlinear integral equation of the second kind. $K(x,t)$ and$f(x)$ are known functions. $k(x,t)$ is called the kernel of the integral equation defined in the rectangle $R$, for which $a \leq x \leq b$ and $a \leq t \leq b$ and $f(x)$ is called the forcing term defined in$a \leq x \leq b$. If $f(x) = 0$, then the equations are homogeneous. The functions $u, f, k$may be complex-valued functions. The linear and nonlinear integral equations are defined when the unknown function appears linearly or nonlinearly under the integral sign. The parameter $\lambda$ is a known quantity.

But in this thesis we study on the linear Fredholm integral equation.

## 2.4 Existence Solutions of Linear Fredholm Integral Equations

In this party,We will talk about the Existence and uniqueness Solutions of Integral Equations ,By applying two theorems :

### 2.4.1 Riez Thoeory and Fredholm Alternative

In this dissertation , we denote by $A : X \to X$ the compact linear operator in a space normed in itself.

We present the basic theory for an equation

$$\varphi - A\varphi = f$$

We define the operator $T$, by

$$T = I - A$$

and $I$ denotes the identity operator[11].

**Theorem 2.4.1** $(Riesz's\ First\ Theorem)$

*The null space of the operator $T$, i.e. the kernel of the operator $L$*

$$Ker(T) = \{\varphi \in X : T\varphi = 0\}$$

*is a finite dimensional subspace.*

**proof.** The kernel of the bounded linear operator $T$ is a closed subspace of $X$ . Since for each sequence $\varphi_n \to \varphi, n \to \infty$ and $T\varphi_n = 0$ , then we have $T\varphi = 0$ , so

$$\varphi \in \ker(T) is equivalent to A\varphi = \varphi.$$

And so the restriction of $A$ on $\ker(T)$ coincides with the identity operator on $\ker(T)$,the operator $A$ is compact in $X$ and thus to make compact from $\ker(T)$ onto $\ker(T)$, since $\ker(T)$ is closed. Therefore $\ker(T)$ is finite dimensional. ■

**Theorem 2.4.2** $(Riesz\prime s\ Second\ Theorem)$

*The image of operator $T$, i.e.*

$$\mathrm{Im}\,(T) = \{T\varphi : \varphi \in X\}$$

*is a closed linear subspace of finite co-dimension*

**proof.** The image of the operator $T$ is a subspace. Let $f$ be an element of $\overline{T(X)}$,then there exists a sequence $(\varphi_n)$ of $X$ such that $T\varphi_n \to f, n \to \infty$, we choose the best approximation $x_n$ , ie.

$$\|\varphi_n - x_n\| = \inf_{x_n \in \mathrm{Im}(T)} \|\varphi_n - x_n\|$$

we define the sequence

$$\widetilde{\varphi}_n = \varphi_n - x_n . n \in N$$

which is bounded

Assuming that the sequence $(\widetilde{\varphi}_n)$ is unbounded, then we can extract a subsequence $\left(\widetilde{\varphi}_{n(K)}\right)$, such that $\left\|\widetilde{\varphi}_{n(K)}\right\| \geq K$, for all $k \in N$, now we set

$$\psi_K = \frac{\widetilde{\varphi}_{n(K)}}{\left\|\widetilde{\varphi}_{n(K)}\right\|}, K \in N,$$

with $\|\psi_K\| = 1$ and $A$ is compact, then there exists a subsequence $\psi_{K(j)}$ such that $A\psi_{K(j)} \to \psi, j \to \infty$ en on the other hand

$$T\psi_K = \frac{\left\|T\widetilde{\varphi}_{n(K)}\right\|}{\left\|\widetilde{\varphi}_{n(K)}\right\|} \leq \frac{\left\|T\widetilde{\varphi}_{n(K)}\right\|}{K} \to 0, K \to 0$$

since the sequence $(T\varphi_n)$ is convergent and therefore bounded. Therefore

$$T\psi_{K(j)} \to 0, j \to \infty$$

then, we get

$$\psi_{K(j)} = T\psi_{K(j)} + A\psi_{K(j)} \to \psi, j \to \infty$$

and since $T$ is bounded, and by the two previous equations we conclude that $T\varphi = 0$. But since

$$x_{n(K)} + \left\|\widetilde{\varphi}_{n(K)}\right\| \psi \in \mathrm{Im}\,(T)\,, \forall K \in N,$$

we find

$$\begin{aligned}
\|\psi_K - \psi\| &= \frac{1}{\left\|\widetilde{\varphi}_{n(K)}\right\|} \left\|\varphi_{n(K)} - \{x_{n(k)} + \left\|\widetilde{\varphi}_{n(K)}\right\|\}\psi\right\| \\
&\geq \frac{1}{\left\|\widetilde{\varphi}_{n(K)}\right\|} \inf_{x \in \mathrm{Im}(T)} \left\|\varphi_{n(K)} - x_n\right\| \\
&= \frac{1}{\left\|\widetilde{\varphi}_{n(K)}\right\|} \left\|\varphi_{n(K)} - x_{n(K)}\right\| = 1.
\end{aligned}$$

This contradicts the fact that $\psi_{K(j)} \to \psi, j \to \infty$ . Therefore $(\widetilde{\varphi}_n)$ is bounded, and since $A$ is compact, we can extract a subsequence $\left(\widetilde{\varphi}_{n(K)}\right)$ such that $\left(A\widetilde{\varphi}_{n(K)}\right)$ converges for $K \to \infty$. Because $T\widetilde{\varphi}_{n(K)} \to f, K \to \infty$, and by

$$\widetilde{\varphi}_{n(K)} = T\widetilde{\varphi}_{n(K)} + A\widetilde{\varphi}_{n(K)}$$

We observe that $\widetilde{\varphi}_{n(K)} \to \varphi \in X, K \to \infty$ , but $.T\widetilde{\varphi}_{n(K)} \to T\varphi \in X, K \to \infty$. ■

**Theorem 2.4.3** (*Third Riesz Theorem*)

*There exists a unique $r \in N$ called the Riesz number of the operator $T$ such that:*

$\{0\} = \ker(T^0) \subset \ker(T^1) \subset ... \subset \ker(T^r) \subset \ker(T^{r+1})$

$E = \operatorname{Im}(T^0) \supset \operatorname{Im}(T^1) \supset ........ \supset \operatorname{Im}(T^r) \supset \operatorname{Im}(T^{r+1})$.

*And we have the direct sum*

$$E = \ker(T^r) \oplus \operatorname{Im}(T^r)$$

**proof.** For the proof see [15] ■

To demonstrate the existence of linear integral equations,we apply Riesz's theory and Fredholm's alternative,the first two corollaries being direct results of linear equations

**Corollary 2.4.1** *Let $A$ a compact operator of a normed space $X$ in itself ,for $\lambda \neq 0$ the nonhomogeneous equation*

$$T\varphi = \varphi - \lambda A\varphi = f$$

*has a unique solution $\varphi \in X$, for all $f \in X$,if and only if the homogeneous equation*

$$T\varphi = \varphi - \lambda A\varphi = 0$$

*has the trivial solution $\varphi = 0$.*

**proof.** In fact ,if the first equation has a solution for all $f \in X$,then $T$ is surjective ,the result $T$ is injective ,proving that the second equation has a unique solution $\varphi = 0$ ,We do the other direction in the some way .

Also,if $\varphi = 0$ is not a solutions that are linearly independent ,in this case is either that the nonhomogeneous is unsolvable ,or there is a solution given in the form

$$\varphi = \tilde{\varphi} + \sum_{i=1}^{m} \alpha_i \varphi_i.$$

where $\alpha_1, \alpha_2, ..., \alpha_m$ are arbitarary complex numbers and $\tilde{\varphi}$ a particular solution of the inhomogeneous equation . ■

**Corollary 2.4.2** *Let $\Omega \subset R^m$, and let $K(x,t)$a continuous function .then either homogeneous integral equations*

$$\begin{aligned} u(x) - \int_\Omega K(x,t)\, u(t)\, dt &= f(x), \quad x \in \Omega \\ v(x) - \int_\Omega K(t,x)\, v(t)\, dy &= g(x), \quad x \in \Omega \end{aligned}$$

*have only the trivial solution $u = 0$ and $v = 0$ and in this case the nonhomogeneous equations*

$$\begin{aligned} u(x) - \int_\Omega K(x,t)\, u(t)\, dt &= f(x), \quad x \in \Omega \\ v(x) - \int_\Omega K(t,x)\, v(t)\, dt &= g(x), \quad x \in \Omega \end{aligned}$$

*have a unique solution $u \in C(\Omega)$and $v \in C(\Omega)$ respectively for any $f \in C(\Omega)$ and $g \in C(\Omega)$,*

*Or the homogeneous integral equations have the some finite number $m \in N$ of solutions linearly independent, and in thise case the nonhomogeneous integral equations are solvable if and only if*

$$\int_\Omega f(x)\, v(x)\, dx = \int_\Omega g(x)\, u(x)\, dx = 0$$

*for all $v$ solution of the adjoint homogeneous equation and for all $u$ solution of the homogeneous equation.*

### 2.4.2 The fixed point theory

In analysis,The fixed point theoremis a result that allows us to affirm that a function $F$ admits under certain conditions a fixed point. these theorem turn out to be very useful tools in mathematics,mainly in the field of solving differential equations

The purpose of this party is to recall some theorem of the fixed point .We will start by the simplest and best knows of them: Banach's fixed point theorem for contracting maps.

#### Banach Fixed point:

Banach's fixed point theorem, it is the basis of the theory of the fixed point.this principle guarantees the existence of a unique fixed point for any contracting application of a complete metric space in itself.

**Definition 2.4.1** *Let $T$ be an operator of a Banach space $E$ in itself ,$T$ is a contraction (or contracting map ) , if there exists a constant $0 \leq k < 1$ such that ,for all $x, y \in E$ we have*

$$\|T(x) - T(y)\| \leq k \|x - y\|$$

**Theorem 2.4.4** *Let $T$be a contraction in a Banch space $X$.then $T$ admits a unique fixed point*

**Remark 2.4.1** *$T$ note to the operator which obtained by composing $T$withe itself $n$ times*

$$T^n = \underbrace{T \circ T \circ ... \circ T}_{n \ \ element}$$

**proof.** Fix an arbitrary element $z \in X$ and consider the sequence

$$(T^n(z))_{n=1}^{\infty}$$

We denote by $z_n$ the element $T^n(z)$ for $n = 1, 2, ...$ we have

$$\begin{aligned} \|z_n - z_m\| &\leq \|z_n - z_{n-1}\| + ... + \|z_{m+1} - z_m\| \\ &= \|T(z_{n-1}) - T(z_{n-2})\| + ... + \|T(z_m) - T(z_{m-1})\| \\ &\leq c\|z_{n-1} - z_{n-2}\| + ... + c\|z_m - z_{m-1}\| \leq ... \leq \\ &\leq \left(c^{n-1} + c^{n-2} + ... + c^{m-1}\right)\|z_1 - z\| \leq \frac{c^{m-1}}{1-c}\|z_1 - z\| \end{aligned}$$

Where we assumed $n > m \geq 1$: $\|z_n - z_m\| \to 0$ when $n, m \to \infty$ so $(z_n)_{n=1}^{\infty}$ is a Cauchy sequence .as $X$ is a Banach space the sequence converges.there is $x_0 \in X$ such that $z_n \to x_0$ when $n \to \infty$ .Here $x_0$ is a fixed point for $T$ as

$$\|T(x_0) - x_0\| \leq \|T(x_0) - T(z_n)\| + \|z_{n+1} - x_0\| \leq c\|x_0 - z_n\| + \|z_{n+1} - x_0\|$$

wher the first member of the equation is independent of $n$ and the other member tends to 0 when $n \to \infty$

Uniqueness results from property of contraction for $T$ .if $x_0 \neq y_0$ two fixed points of $T$ then we obtain

$$\|x_0 - y_0\| = \|T(x_0) - T(y_0)\| \leq c\|x_0 - y_0\| < \|x_0 - y_0\|$$

it is a contradiction. ■

**From the proof we have:**

1. The sequence $(T^n(z))_{n=1}^{\infty}$ converges to a single fixed point independent of the choice of $z$.

2.for an arbitrary element $x \in X$ we have

$$\|x - x_0\| \leq \frac{1}{1-c}\|x - T(x)\|,$$

where $x_0$ denotes the fixed point of $T$, as

$$\|x - x_0\| \leq \|x - T(x)\| + \|T(x) - T(x_0)\| \leq \|x - T(x)\| + c\|x - x_0\|$$

**Banach 's fixed point theorem generalize:**

Let $T$ be an application in a Banach space $X$ such that $T^N$ is a contraction in $X$ for some positive integer $N$ .then $T$ admits a unique fixed point .

It is not necessery to assume that $T$ is a cotinuous map.

**proof.** Banach's fixed point theorem implies that there is a unique fixed point for $T^N$.this elements named $x_0$

Now we note that

$$\|T(x_0) - x_0\| = \left\|T^N(T(x_0)) - T^N(x_0)\right\| \leq c\|T(x_0) - x_0\|$$

Implies that $T(x_0) = x_0$ as fixed point $0 < c < 1$.uniqueness is clear as fixed point for $T$ is also a fixed point for $T^N$. ■

**Theorem 2.4.5** *Let $F$ be a closed subset in a Banach space and let $T : F \to F$ a contracting apllication,then*

*a) The equation $Tx = x$ ,has only one unique solution.*

*b) The unique solution $x$ can be obtained by the limit of the sequence $(x_n)$ of $F$ defined by $x_n = Tx_{n-1}, n = 1, 2, ...,$ where $x_0$ is an arbitrary $F$.*

$$x = \lim_{n\to\infty} T^n x_0.$$

## 2.5 Approximate the Solutions of Linear Fredholm Integral Equations

The collocation method has been developed in many fields .In this thesis, we'll see two ways of solving a Linear Fredholm integral equation.by using a colloction method

- The collocation basis on Lagrange method.

Let $[a, b]$ a finite closed interval and Let $\pi : \{a = x_0 < x_1 < ... < x_n = b\}$ $n \in N$, where $h = \frac{b-a}{n}, x_i = a + ih$, $i = 0, 1, ..., m_n$. We introduce a new set of basis functions. For each $i, 1 \leq i \leq m_n$ , let $L_i \in X_n$ be that element that satisfies the interpolation conditions

$$L_i(x_j) = \delta_{i,j}, \qquad j = 1, ..., m_n \tag{2.5.1}$$

$$L_i(x_j) = \prod_{j\neq i} \frac{x - x_j}{x_i - x_j}, \qquad j = 0, 1, ..., m_n,$$

By (1.2.5), there is a unique such $L_i$ , and the set $\{L_1, ..., L_m\}$ is a new basis for d . With polynomial interpolation, such functions $L_i$ are called Lagrange basis functions, and we use this name with all types of approximating subspaces $X_n$. With this new basis, we can write

$$P_n u(x) = \sum_{j=1}^{m_n} u(x_j) L_j(x), \qquad x \in D \tag{2.5.2}$$

In the view of Lagrange polynomial interpolation (which is illustrated above) Clearly, $P_n$ is linear and finite rank. In addition, as an operator on $C(D)$ onto $C(D)$

$$\|P_n\| = \max_{x\in D} \sum_{j=1}^{m_n} |l_j(x)|. \tag{2.5.3}$$

For more details see [2] [19].

- Cubic B-spline collocation method, Adetailed explanation is provided in chapter three.

# Chapter 3

# Application of Spline Collocation for Fredholm Integral Equations

In this chapter we will approximate the solution of Fredholm integral equations of the second using the collocation methods based on cubic B-spline. First we collocate the solution by B-spline and the Newton-Cotes formula is used to approximate integral. Convergence analysis has been investigated and proved that the quadratue rule is fourth order convergent, then we give some numerical examples.

## 3.1 Solving Fredholm Integral Equations using Cubic B-spline collocation method

B-splines are a set of special spline functions that be used to construct the piece wise polynomial by computing the appropriate linear combination. These functions have their computational advantage from the fact that any B-spline basis function of order m is nonzero over at al most m adjacent intervals and zero otherwise,and since they have compact support, numerical cal methods in which B-spline functions are used

as a basic function led matrix systems, including [21]

band matrices. Through smoothness and capability to handle local phenomena, B-spline basis functions offer distinct advantages in comparison to other basis functions. Such systems can be handled and solved with low computational cost. Cubic B-spline function (CBS) has already been used as the basis functions to solve many physical models.

To develop the colloction method based on cubic B-splin for the solution of fredholm integral equation second kind.

**Definition 3.1.1** *let* $\pi$ *be a uniform partition of the inverval* $[a, b]$*such as*

$\pi = a = t_0 < t_1 < ... < t_{n+2} = b$ where$h = \frac{b-a}{n+2}$ ,$t_i = a + ih, i = 0, 1, 2..., n + 2$
We introduce the spline space$S_3(\pi) = v \in C^2[a, b]; v|_{[t_i,t_{i+1}]} \in P_3, i = 0, 1, ..., n + 2$
Where $P_3$ is the class polynomials .By introducing adjacent knots
$t_{-2} < t_{-1} < t_0 < ... < t_{n+2} < t_{n+3} < t_{n+4}$.

**Definition 3.1.2**

***Table 1:*** *The values of B-spline*

| $t$ | $t_{i-2}$ | $t_{i-1}$ | $t_i$ | $t_{i+1}$ | $t_{i+2}$ |
|---|---|---|---|---|---|
| $B_i(t)$ | 0 | 1 | 4 | 1 | 0 |

and the functions $B_i(t), S(t)$ which are defined in the following form:

$$B_i(t) = \begin{cases} (t - t_{i-2})^3 / h^{3,} & if\ t \in [t_{i-2}, t_{i-1}] \\ \left(h^3 + 3h^2(t - t_{i-1}) + 3h(t - t_{i-1})^2 - 3(t - t_{i-1})^3\right)/h^3, & if\ t \in [t_{i-1}, t_i] \\ \left(h^3 + 3h^2(t_{i+1} - t) + 3h(t_{i+1} - t)^3\right)/h^3, & if\ \ t \in [t_i, t_{i+1}] \\ (t_{i+2} - t)^3/h^3, & if\ \ t \in [t_i, t_{i+1}] \\ 0, & otherwise. \end{cases} \tag{3.1.1}$$

$$S(t) = \sum_{i=-2}^{n+2} c_i B_i(t). \tag{3.1.2}$$

In the case of integral equations of the second kind (2.3.2)using cubic B-splines (3.1.1)we can approximation the solution as well as the integrand Newton-Cotes type methods.if $n$ is even,you can use Simpson's rule if $n$ is odd ,we have to apply the three-eigths rule ,

$$\sum_{i=-2}^{n+2} c_i B_i\left(x_j\right) = f\left(x_j\right) + h \sum_{i=-1}^{n+1} \left[ w_{i,j} k\left( x_j, t_i, \sum_{k=-2}^{n+2} c_k B_k\left(t_i\right) \right) \right], \quad j = -1, ..., n+1. \quad (3.1.3)$$

where$h = \frac{b-a}{n+2}$ $\quad j = 0, ..., n+2, t_j = x_j, \; k = -1, .., n+1.$

By solving system (3.1.3)we obtain the vector $c_i$ and assume that $c_{-2} = c_{n+2} = 0$ hase a cubic B-spline relationship,then substitute in $c_i$ exist (3.1.2) we can obtian an approximate solution to $(2.3.2)$.

## 3.2 Error analysis: convergence of the approximate solution

To study the convergence analysis, first we need to recall the folloing basic theorem in (1.4.3)

**Remark 3.2.1** *the most immediate error analysis for spline approximates $S$ to a given function $u$ defined on an interval $[a, b]$ follows from the first and second integral relations .Throughout our discussion $\pi : \{a = t_0 < t_1 < ... < t_{n+2} = b\}$ is partition in $[a, b]$ and $h = \frac{b-a}{n+2}$ is the mesh of our partition*

if $u \in C^4[a, b]$ ,then

$$\left\| D^j (u - S) \right\|_\infty \leq \gamma_j h^{4-j}, \quad j = 0, 1, 2, 3, 4$$

Where $\|u\|_\infty = \max_{0 \leq i \leq n+2} \sup_{t_{i-1} \leq t \leq t_i} |u(t)|$ and$D^j$the $j - th$ derivative

The numerical method is said to be convergent if the solution of the approximating set of converges to the solution of the step length $h$ tends to zero ;that is,if $\lim_{h \to 0} |x_{i+1} - x_i| = 0$. consider the equation

$$u(x) = \int_a^b k(x, t)\, u(t)\, dt + f(x), \; a \leq x \leq b \quad (3.2.1)$$

and suppose that at $x = x_i$, where $z_i = a + ih, i = 0, ..., n+2, x_i = t_i = z_{i+1}, i = -1, ..., n+1$. the quadrature formula

$$\int_{a}^{a+ih} k\left(x_{i},t\right)u\left(t\right)dt = h\sum_{j=-1}^{n+1} w_{ij}k\left(x_{i},t_{j}\right)u\left(t_{j}\right) + E_{i,t}\left(k\left(x_{i},t\right)u\left(t\right)\right) \tag{3.2.2}$$

At $x = x_i, i = -1, ..., n+1$, by substituting 3.2.2in3.2.1 we have

$$u\left(x_{i}\right) = f\left(x_{i}\right) + h\sum_{j=-1}^{n+1} w_{ij}k\left(x_{i},t_{j}\right)u\left(t_{j}\right) + E_{i,t}\left(k\left(x_{i},t\right)u\left(t\right)\right) \qquad i = -1, ..., n+1. \tag{3.2.3}$$

and corresponding approximating function equations are

$$S\left(x_{i}\right) = f\left(x_{i}\right) + h\sum_{j=-1}^{n+1} w_{ij}k\left(x_{i},t_{j}\right)S\left(t_{j}\right) \qquad i = -1, ..., n+1. \tag{3.2.4}$$

Thus we have

$$u\left(x_{i}\right)-S\left(x_{i}\right) = +h\sum_{j=-1}^{n+1} w_{ij}k\left(x_{i},t_{j}\right)\left(u\left(t_{i}\right)-S\left(t_{i}\right)\right)+E_{i,t}\left(k\left(x_{i},t\right)u\left(t\right)\right), \qquad i = -1, ..., n+1. \tag{3.2.5}$$

We set $e_i = u(x_i) - S(x_i)$, it follows that

$$|e_{i}| \leq h\sum_{j=-1}^{n+1} |w_{ij}|\left|k\left(x_{i},t_{j}\right)\right|\|e_{j}\| + E_{i,t}\left(k\left(x_{i},t\right)u\left(t\right)\right), \qquad i = -1, ..., n+1 \tag{3.2.6}$$

Let $w = \max_{i,j}|w_{i,j}|$ and $e = \max_{-1\leq i\leq n+1}|e_i|$, $|e_i| \leq \{|E_i, t(k(x_i,t)u(t))| + hkw(n+3)e\}$, hence $|ei| \to 0$ as $h \to 0$, we may write equivalently $\|e_i\| = O\left(h^{p+1}\right) + O\left(h^{q+1}\right)$, where the error in the quadrature rule is $O\left(h^5\right)$ and the error in the function approximate is $O\left(h^5\right)$. If we set $r = \min(p,q) = \min(4,4)$ then we say the quadrature rule is convergen of order 4

## 3.3 Numerical examples

To compare results and justify the accuracy and efficiency of our presented method, we consider the following examples which are considered . The solution of the given examples is obtained for different values of $n$. errors solutions $E = \left(\frac{1}{n}\sum_{i=0}^{n}\left[u\left(x_{i}\right)-S\left(x_{i}\right)\right]^{2}\right)^{\frac{1}{2}}$ where

$u(x)$is the exact solution and $S(x)$ is the approximated solution of integral equation which are given by the suggested methods [21].

**Example 3.3.1**

$$u(x) = e^{2x+\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2x-\frac{5t}{3}} u(t)dt, 0 \leq x \leq 1,$$

*with the exact solution* $u(x) = e^{2x}$.

*Table 1* : $(C.L)$ *and* $(S.C)$ ,$n = 30$ .

| $x$ | *sol exact* | *error* $(C.L)$ | *error* $(S.C)$ |
|---|---|---|---|
| *0* | *1.0000* | *9.4490*$(-3)$ | *1.85*$(-11)$ |
| *0.1* | *1.2214* | *1.1541*$(-2)$ | *2.38*$(-11)$ |
| *0.2* | *1.4918* | *1.4096*$(-2)$ | *2.87*$(-11)$ |
| *0.3* | *1.8221* | *1.7217*$(-2)$ | *3.46*$(-11)$ |
| *0.4* | *2.2255* | *2.1029*$(-2)$ | *4.17*$(-11)$ |
| *0.5* | *2.7182* | *2.5685*$(-2)$ | *5.04*$(-11)$ |
| *0.6* | *3.3201* | *3.1372*$(-2)$ | *6.47*$(-11)$ |
| *0.7* | *4.0551* | *3.8318*$(-2)$ | *7.80*$(-11)$ |
| *0.8* | *4.9530* | *4.6801*$(-2)$ | *9.41*$(-11)$ |
| *0.9* | *6.0496* | *5.7163*$(-2)$ | *1.13*$(-10)$ |
| *1* | *7.3890* | *6.9819*$(-2)$ | *1.37*$(-10)$ |

**Example 3.3.2**

$$u(x) = -\frac{e^x\left(N\pi - eN\pi\cos(N\pi) + eN\pi\sin(N\pi)\right)}{1+N^2\pi^2} + \sin(Nx\pi) + \int_0^1 e^{x+t}u(t)dt \quad 0 \le x \le 1,$$

*with the exact solution* $u(x) = \sin(N\pi x)$.

*Table 2 : The RMS errors* $(C.L)$ *and* $(S.C)$ $, h = \left(\frac{1}{3}, \frac{1}{4}, ..., \frac{1}{10}\right)$.

| $h$ | *sol exact* | *error* $(C.L)$ | *error* $(S.C)$ |
|---|---|---|---|
| $\frac{1}{3}$ | 3.6545(−2) | 1.10(−1) | 1.797(−04) |
| $\frac{1}{4}$ | 2.7412(−2) | 7.68(−2) | 1.914(−04) |
| $\frac{1}{5}$ | 2.1930(−2) | 5.74(−2) | 1.296(−03) |
| $\frac{1}{6}$ | 1.8276(−2) | 4.50(−2) | 1.293(−03) |
| $\frac{1}{7}$ | 1.5665(−2) | 3.65(−2) | 1.122(−03) |
| $\frac{1}{8}$ | 1.3707(−2) | 3.04(−2) | 2.916(−03) |
| $\frac{1}{9}$ | 1.1218(−2) | 2.58(−2) | 8.348(−03) |
| $\frac{1}{10}$ | 1.0966(−2) | 2.23(−2) | 4.690(−03) |

# Conclusion.

There are many ways to solve linear Fredholm integral equation of the second kind. Among them Colloction methods, in this work we solve the linear Fredholm integral equation of the second kind using Spline Colloction method then compared it with the colloction method *basis on Lagrange polynomials.*

Through the solving the examples, we find that the spline collocation for fredholm integral equation in more accurate than to collocation *basis on lagrange method.*

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