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Analytical and numerical study of nonlinear integral equations

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Dedication

To my parents,

my husband and my children, Mayar and Mohamed-Amine,

I dedicate this thesis

Acknowledgements

*I thank first and foremost Allah for giving me the courage, the patience, and the will to achieve this thesis.
I wish to express my dearest gratitude to my supervisor Prof. Azedine RAHMOUNE for his continuous support, ideas, and patience in guiding me during the preparation of my thesis, I also acknowledge the opportunity he provided me to grow in this field.
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Finally, I would like to thank my family, my friends , and all the people whose support and assistance was instrumental in the completion of this work.*

ملخص

الهدف الرئيسي من هذه الأطروحة هو تقديم دراسة نظرية وعددية حول المعادلات التكاملية غير الخطية. استخدمنا مختلف نظريا ت النقطة الثابتة ، كما قدمنا مبدأ ليراي - شودر لتوفير نتائج حول وجود حلول للمعادلات التكاملية غير الخطية في مجالات محدودة وغير محدودة ، كما قدمنا أساليب فعالة لحل هذه المعادلات مع دراسة متعمقة حول التقارب. بالإضافة إلى ذلك، طبقنا بعض هذه الطرق، منها طريقة التجميع الطيفي و طريقة سينك-نيستروم من أجل إيجاد حلول تقريبية لبعض المعادلات التكاملية غير الخطية، هذه الطرق تحول المعادلة التكاملية غير الخطية إلى نظام من المعادلات الجبرية غير الخطية وهذا النظام الجبري تم حله بطريقة نيوتن. واستخلصنا تحليلا للخطأ فيما يتعلق بالأساليب الحالية التي تثبت أن لها ترتيبا أسيا للتقارب. وأخيرا، قدمنا عدة أمثلة رقمية لإثبات فعالية مناهجنا التقريبية.

كلمات مفتاحية : معادلات تكاملية غير خطية، نظريات النقطة الثابتة، معادلة أوريسون التكاملية، معادلة تكاملية من نوع هامرشتاين، نصف المستقيم العددي، طريقة الإسقاط، طريقة سينك-نستروم، تحليل التقارب.

ABSTRACT

The main objective of this thesis is to offer a theoretical and numerical study on nonlinear integral equations. We have used different fixed point theorems, and Leray-Schauder principle to provide existence results for nonlinear integral equations on bounded and unbounded domains, we have also presented efficient methods for solving such equations with a thorough study on the convergence analysis. Furthermore, we have applied some of these methods, specially, spectral collocation methods and Sinc-Nyström methods in order to find numerical solutions of certain nonlinear integral equations, these methods reduce the nonlinear integral equation to a system of nonlinear algebraic equations and that algebraic system has been solved by Newton's method. We have derived an error analysis for the current methods, which prove that they have exponential convergence order. Finally, several numerical examples are given to show the effectiveness of our approaches.

Keywords: Nonlinear integral equations, fixed-point theorems, Urysohn integral equation, Hammerstein integral equation half-line, projection method, Sinc-Nyström method, convergence analysis.

RÉSUMÉ

L'objectif principal de cette thèse est de proposer une étude théorique et numérique sur les équations intégrales non linéaires. nous avons utilisé différents théorèmes du point fixe, et le principe de Leray-Schauder pour fournir des résultats d'existence pour les équations intégrales non linéaires sur des domaines bornés et non bornés, nous avons également présenté des méthodes efficaces pour résoudre de telles équations avec une étude approfondie sur la convergence. En outre, nous avons appliqué certaines de ces méthodes, notamment, la méthode spectrale de collocation, et la méthode de Sinc-Nyström pour trouver des solutions numériques de quelques équations intégrales non lineaires, ces méthodes transforment l'équation intégrale non linéaire en un système d'équations algébriques non linéaires ce système algébrique a été résolu par la méthode de Newton. Nous avons dérivé une analyse d'erreur pour les méthodes actuelles qui prouvent qu'elles ont un ordre de convergence exponentielle. Enfin, plusieurs exemples numériques sont donnés pour montrer l'efficacité de nos approches.

Mots clés: Equations intégrales non lineaires, théorèmes du point fixe, équation intégrale d'Urysohn, équation intégrale de type Hammerstien, demi-droite, méthode de projection, méthode de Sinc-Nyström, analyse de la convergence.

CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

$B_R(u,X)$:	Open ball $\{v \in X: \ |u-v| < R\}$.
$\overline{B}_R(u,X)$:	Closed ball $\{v \in X: \ |u-v| \leq R\}$.
$C(\Omega)$:	Space of continuous functions $u : \Omega \to \mathbb{R}^n$.
$\|u\|_\infty$:	$\sup\limits_{s\in\Omega} |u(s)|$ ($\Omega \in \mathbb{R}^n$ bounded open, $u \in C(\Omega)$).
C_l:	Space of continuous functions on $[0,\infty)$ having a limit at infinity.
$\|u\|_0$:	$\sup\limits_{s\in[0,\infty)} |u(s)|$, norm of the space C_l.
$C^{r,k}(\Omega)$:	Space of functions whose r-th derivatives are Hölder continuous with exponent k.
$\|u\|_{r,k}$:	$\max\limits_{0\leq k\leq r} \max\limits_{x\in\Omega} \left|\partial_x^k u(x)\right| + \sup\limits_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\left|\partial_x^r u(x) - \partial_x^k u(y)\right|}{\left|x-y\right|^k}, \quad u \in C^{r,k}(\Omega)$.
$E,\ X,\ Y$:	Vectorial space, Banach or Hilbert space.
$J_{N+1}^{\alpha,\beta}$:	Roots of Jacobi polynomials.
$H_w^m(\Omega)$:	Sobolev space.
$I_N^{\alpha,\beta}$:	Lagrange interpolation operator associated with the Gauss-Jacobi points.
$\mathcal{K}$:	Integral operator.
$K(s,t),\ k(s,t)$:	Kernel of integral operator.
$\{L_j^{\alpha,\beta}\}_{j=0}^N$:	Lagrange interpolation basic function associated with Gauss-Jacobi points.
$L^p(\Omega)$:	Space of measurable functions $u : \Omega \to \mathbb{R}$ such that $\|u\|_p < \infty$.
$L_w^2(\Omega)$:	Space of measurable functions $u : \Omega \to \mathbb{R}$ such that $\|u\|_{w^{\alpha,\beta}} < \infty$.
$L^\infty(\Omega)$:	Space of measurable functions $u : \Omega \to \mathbb{R}$ such that are bounded outside a set of measure zero.
N.I.Es:	Nonlinear integral equations.

$\|u\|_p$:	Norm in $L^p(\Omega)$, $\|u\|_{\,p} = \left(\int_\Omega \|u(s)\|^P ds\right)^{\frac{1}{p}}$.
$\|u\|_{w^{\alpha,\beta}}$:	Norm in $L^2_w(\Omega)$, $\|u\|_{w^{\alpha,\beta}} = \left(\int_\Omega \|u(s)\|^2 w^{\alpha,\beta}(s) ds\right)^{\frac{1}{2}}$.
$\|u\|_{L_\infty}$:	Norm in $L^\infty(\Omega)$ defined by $ess \sup_{s\in\Omega} \|u(s)\|$.
$P,\ P_N$:	Projection operators.
$\mathcal{P}_N$:	Space of all algebraic polynomials of degree up to N.
$\{\sigma^{\alpha,\beta}_{N,j}\}^N_{j=0}$:	Points of the Gauss-Jacobi quadrature formula.
$\mathbb{R}[x]$:	Set of all polynomials (of all degrees) in one variable x.
$\mu(\Omega)$:	Lebesgue measure of Ω.
$\partial(G)$:	Boundary of G.
$\mathcal{O}(.)$:	Order of convergence.

INTRODUCTION

The subject of integral equations has held an eminent place in the attention of mathematicians, such equations arise naturally in applications in diverse areas of applied mathematics and physical sciences, engineering, biology and in many other fields, they also provide an effective technique for solving a wide range of practical problems.

Abel is the initiator of the mention of integral equations, in 1823 he proposed a generalization of the Tautochrone problem whose solution referred the solution of an integral equation, recently has been dubbed ”an integral equation of the first kind”, and in 1837 Liouville proved that determining a particular solution of a linear differential equation of the second order might be obtained by solving an integral equation, known as the integral equation of second kind, however, the term integral equation was first proposed by Du Bois-Reymond in 1888, who defined an integral equation as an equation in which the unknown function appears under one or more signs of definite integration. Afterwards, in 1896, Vito Volterra built up a theory of integral equations, viewing their solutions as a problem of finding the inverses of certain integral operators, without forgetting to mention the famous paper of Fredholm which he published in 1903 and it presented the fundamentals of the Fredholm integral equation theory. Poincaré, Fréchet, Hilbert, Schmidt, Hardy and Riesz have also participated in the development of this area of research.

In this thesis we are focused our study on the nonlinear integral equations. The most general form of a nonlinear integral equation is:

$$F(u)(s) = \lambda \int_{\Omega} K(s,t,u(t))dt, \quad s,t \in \Omega,$$

where F a measurable function given on $\mathbb{R}$, λ a non zero real or complex scalar, $K(s,t,u(t))$ a measurable function on $\Omega^2 \times \mathbb{R}$ called the kernel of the integral equation and $u(s)$ is the unknown function.

In our work we will consider the theory of this class of integral equations, and we will provide a comprehensive existence theory for them which have been given in various literature on this regard [1, 4, 11, 18, 39, 41, 42, 59, 60].

In practice the nonlinear integral equations yield a considerable amount of difficulties in solving them analytically, for this, there's a great interest to solve them numerically, in this thesis we will explain the most numerical methods which can be used to solve these equations, namely, Nyström method [2, 6, 5, 7, 26, 57], Projection methods, (see the references [5, 7, 8, 26, 49, 58]) and Sinc approximation methods, (for more details, see [51, 53, 54, 52]), we shall also conduct a thorough consideration to explicate the convergence analysis for these methods.

Although the resolution of nonlinear integral equations is received a significant place in the attention of researchers, their solving on the unbounded intervals is still a challenge where there is a scarcity of research that's interested to treating them in infinite intervals (see [36, 48]), that's why one of our main objectives of the present work is to solve and study the convergence analysis for solutions of one of nonlinear integral equations, namely, Hammerstein type integral equation on half-line (see [50, 3, 39, 2, 36]).

The layout of this thesis is as follows:

The first chapter introduces some basic results and fundamental theorems, such as, compactness, compact operators and integral operators in Banach spaces, then, we recall some definitions and techniques that are very important in the next chapters as Fréchet derivative, Hölder space and Gronwall inequalities.

In chapter two we present an introduction to the terminology and classification of integral equations, and some theorems of fixed point to offer a comprehensive existence theory for nonlinear integral equations.

In chapter three we will describe a various numerical methods with their convergence properties, such as Nyström method, Projection methods and Sinc approximation method.

The subject of the last chapter is the application of certain numerical methods for solving some nonlinear integral equations, such as nonlinear Volterra-Fredholm integral equations and integral equation of Hammerstein type on half-line, in which we will discuss our convergence analysis, and illustrate the efficiency of the present methods by instructive examples.

CHAPTER 1

PRELIMINARY CONCEPTS

In this chapter we recall briefly some basic concepts and fundamental theorems concerning the compactness, compact operators and integral operators in Banach spaces, and we mention some examples of nonlinear integral operators with certain important results that will be used through the thesis. Some definitions and techniques will also be discussed in this chapter are very important in the next chapters as Fréchet derivative, Hölder space and Gronwall inequalities.

1.1 Compactness

Definition 1.1. (Compact set) A subset G of a normed space X is called compact if every open cover $H = \{\Omega_j\}$ of G contains a finite subcover of G. In other words, for every family $H = \{\Omega_j\}, j \in J$ of an open sets with the property

$$G \subset \bigcup_{j \in J} \Omega_j,$$

there exists a finite subfamily $\{\Omega_{j(k)}\}, j(k) \in J, k = 1, 2, \cdots, n$, such that

$$G \subset \bigcup_{k=1}^{n} \Omega_{j(k)}.$$

Definition 1.2. (Relatively compact set) A subset G of a normed space X is called relatively compact if its closure $\overline{G}$ is compact .

Theorem 1.1. A bounded and finite dimensional subset G of a normed space X is relatively compact.

Definition 1.3. (Collectively compact set) Let X and Y be normed spaces, and let $\mathcal{B}$ denote the closed unit ball in X. Then a set Λ of bounded operators from X into Y is collectively compact if and only if the set $\Lambda(\mathcal{B}) = \{Ku, \ K \in \Lambda, \ u \in \mathcal{B}\}$ is a relatively compact subset of Y.

Compactness in the space $C(\Omega)$

The space of continuous functions consist of all continuous maps of the closed interval Ω into $\mathbb{R}^n$ denoted $C(\Omega)$ is equipped with the norm

$$\|u\|_\infty = \sup_{s\in\Omega} |u(s)|.$$

We recall that if Ω bounded and closed, $C(\Omega)$ is a Banach space.

Definition 1.4. (Equicontinuity) Let $G : \Omega \to \mathbb{R}^n$ a set of continuous functions, we say that G is equicontinuous at a point $s_0 \in \Omega$ if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$\forall u \in G, \forall s \in \Omega, \|s - s_0\| < \delta \text{ then } \|u(s) - u(s_0)\|_\infty < \varepsilon.$$

Theorem 1.2. (Arzela-Ascoli theorem) A subset G of $C(\Omega)$ is relatively compact if and only if it is bounded and equicontinuous.

Proof. (See [5]) □

Compactness in the space C_l

Let C_l denote the space of continuous functions on $[0,\infty)$ having a limit as $s \to \infty$. If $u \in C_l$, then $u(\infty)$ denotes $\lim_{s\to\infty} u(s)$. The space C_l is equipped with the norm

$$\|u\|_0 = \sup_{s\in[0,\infty)} |u(s)|.$$

With this norm it is a Banach space, since it is a closed subspace of Banach space of all bounded continuous functions on $[0,\infty)$. The notation C_l is that of Corduneanu [15].

Theorem 1.3. [15] Let $\Omega \subset C_l$. Then Ω is compact in C_l if the following conditions hold:

i) Ω is bounded in C_l,

ii) the functions belonging to Ω are equicontinuous on any compact interval of $[0,\infty)$,

iii) the functions from Ω are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $S(\varepsilon) > 0$ such that $|u(s) - u(\infty)| < \varepsilon$ for any $s \geq S(\varepsilon)$ and $u \in \Omega$.

1.2 Compact operators

Definition 1.5. Let X and Y be two normed linear spaces and $A : X \to Y$ a linear map between X and Y.
A is called compact operator if and only if one of the following three equivalent proposition is verified

i) For all bounded sets $G \subseteq X$, $A(G)$ is relatively compact in Y.

ii) The image of the open unit ball under A is relatively compact in Y.

iii) For any bounded sequence $\{u_n\}$ in X there exists a subsequence $\{Au_{n_k}\}$ of $\{Au_n\}$ that converge in Y.

Definition 1.6. A compact operator A is called completely continuous if it is continuous.

Theorem 1.4. The sequence A_n of compact operators defined from a normed space X into a normed space Y converge uniformly to an operator A, say,

$$\lim_{n \to \infty} \|A_n - A\| = 0.$$

Then the operator A is compact.

Proof. See [34] □

1.2.1 Compact integral operators

Definition 1.7. An integral operator $\mathcal{K}$ with continuous kernel $K(s,t)$ is a linear operator defined from $C(\Omega)$ into $C(\Omega)$ by

$$\mathcal{K}(u)(s) = \int_\Omega K(s,t)u(t)dt, \quad s,t \in \Omega, \tag{1.1}$$

with Ω is a compact set of $\mathbb{R}^n$ and K is continuous function from $\Omega \times \Omega$ to $\mathbb{R}^n$.
This operator is bounded with

$$\|\mathcal{K}\| = \max_{s \in \Omega} \int_\Omega |K(s,t)|dt, \tag{1.2}$$

Theorem 1.5. Let $K(s,t)$ is Riemann-integrable as a function of t, for all $s \in \Omega$, and further we assume the following

i. $\lim_{h\to 0} \max_{s,\tau\in\Omega} \max_{|s-\tau|\le h} \int_\Omega |K(s,t)-K(\tau,t)|dt = 0.$

ii. $\max_{s\in\Omega} \int_\Omega |K(s,t)|dt < \infty.$

Then the integral operator defined by (1.1) is compact on $C(\Omega)$ to $C(\Omega)$.

Proof. See [5] □

Remark 1.1. The assumptions *i-ii* are satisfied if $K(s,t)$ is a continuous function of $(s,t) \in \Omega$.

1.3 Nonlinear integral operators

1.3.1 The Urysohn integral operator

Let $\Omega \subset \mathbb{R}^n$ a bounded set, the nonlinear integral operator $\mathcal{K}$ defined by

$$\mathcal{K}(u)(s) = \int_\Omega K(s,t,u(t))dt, \quad \forall s \in \Omega, \;\; u \in C(\Omega), \tag{1.3}$$

is called Urysohn integral operator with kernel K, this type of operators is also called nonlinear Fredholm integral operators.
In the case that $\Omega \subset \mathbb{R}$ and has the form $\Omega = [a,b]$ or $\Omega = [a,\infty)$, and if the kernel $K : \Omega^2 \times \mathbb{R}^n \to \mathbb{R}^n$ has the property that $K(s,t,x) = 0$ whenever $t > s$ and $x \in \mathbb{R}^n$, then the Urysohn operator (1.3) has the form:

$$\mathcal{K}(u)(s) = \int_a^s K(s,t,u(t))dt, \quad \forall s \in \Omega, \tag{1.4}$$

and $\mathcal{K}$ is called nonlinear Volterra integral operator in the Urysohn form.

Theorem 1.6. [42] Let $K : \overline{\Omega}^2 \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping. Then the Fredholm operator $\mathcal{K} : C(\overline{\Omega}) \to C(\overline{\Omega})$ associated to K and given by (1.3) is completely continuous.

Proof. Firstly, we need to prove that $\mathcal{K}$ is compact, for this it must be shown that $\mathcal{K}(G)$ where G is a bounded subset of $C(\overline{\Omega})$ is relatively compact in $C(\overline{\Omega})$.
The boundedness of G implies that

$$\|u\|_\infty < c, \quad \forall u \in G,$$

where c is positive constant, then

$$\|\mathcal{K}(u)\|_\infty < M\mu(G),$$

with

$$M = \max_{\overline{\Omega}^2 \times \overline{B}_c(0,\mathbb{R}^n)} |K(s,t,z)|,$$

hence, the set $\mathcal{K}(G)$ is bounded in $C(\overline{\Omega})$.
On the other hand, using the uniform continuity of K on the compact $\overline{\Omega}^2 \times \overline{B}_c(0,\mathbb{R}^n)$, for each $\varepsilon > 0$ there exists a δ_ε such that

$$|K(s,t,u(t)) - K(s',t,u(t))| < \varepsilon, \text{ for all } s,s',t \in \overline{\Omega} \text{ with } |s-s'| < \delta_\varepsilon \text{ and } u \in G.$$

This immediately yields

$$|\mathcal{K}(u)(s) - \mathcal{K}(u)(s')| < \varepsilon\mu(\Omega), \quad \text{for all } s,s' \in \overline{\Omega} \text{ satisfying } |s-s'| < \delta_\varepsilon \text{ and } u \in G.$$

Thus $\mathcal{K}(G)$ is equicontinuous, and from the Arzela-Ascoli theorem we obtain that $\mathcal{K}$ is compact.
Next, we need to prove that $\mathcal{K}$ is continuous. Let $u_0 \in C(\overline{\Omega})$ and choose any number $R > \|u_0\|_\infty$. Let $\varepsilon > 0$, since K is uniformly continuous on the compact set $\overline{\Omega}^2 \times \overline{B}_R(0;\mathbb{R}^n)$, there exists a $\delta_\varepsilon > 0$ such that for every $u \in C(\overline{\Omega})$ satisfying $\|u - u_0\|_\infty < \delta_\varepsilon$ one has $u(t) \in \overline{B}_R(0;\mathbb{R}^n)$ and

$$|K(s,t,u(t)) - K(s,t,u_0(t))| < \varepsilon, \quad \text{for all } s,t \in \overline{\Omega}.$$

Then

$$\begin{aligned} |\mathcal{K}(u)(s) - \mathcal{K}(u_0)(s)| &\leq \int_\Omega |K(s,t,u(t)) - K(s,t,u_0(t))|dt \\ &< \varepsilon\mu(\Omega), \end{aligned}$$

for every $s \in \overline{\Omega}$. Hence

$$\|\mathcal{K}(u) - \mathcal{K}(u_0)\|_\infty < \varepsilon\mu(\Omega),$$

whenever $\|u - u_0\|_\infty < \delta_\varepsilon$. Therefore $\mathcal{K}$ is continuous at u_0. □

Corollary 1.1. Let $R > 0$ and $K : \overline{\Omega}^2 \times \overline{B}_R(0;\mathbb{R}^n) \to \mathbb{R}^n$ be a continuous mapping. Then the operator $\mathcal{K} : \overline{B}_R(0;C(\overline{\Omega})) \to C(\overline{\Omega})$ given by (1.3) is completely continuous.

We can followed the same reasoning as in the proof of the previous theorem to establish the following result.

Theorem 1.7. [42] Let $K : [a,b]^2 \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping. Then the Volterra operator $\mathcal{K} : C[a,b] \to C[a,b]$ associated to K and given by (1.4) is completely continuous.

Corollary 1.2. Let $R > 0$ and $K : [a,b]^2 \times \overline{B}_R(0;\mathbb{R}^n) \to \mathbb{R}^n$ be a continuous mapping. Then the operator $\mathcal{K} : \overline{B}_R(0;C[a,b]) \to C[a,b]$ given by (1.4) is completely continuous.

1.3.2 The Nemytskii operator

Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a given function, for each u define the member $N_f u$ by

$$[N_f(u)](s) = f(s,u(s)), \qquad \text{for all } s \in \Omega. \tag{1.5}$$

The operator N_f is said to be substitution or Nemytskii operator generated by f, this operator occur frequently in connection with integral equations.

Definition 1.8. **(L^q-Carathéodory function)** Let Ω be a closed domain in $\mathbb{R}^n$, a function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a L^q-Carathéodory function if the following conditions hold:

i) $f(.,t) : s \to f(s,t)$ is measurable on Ω for each $t \in \mathbb{R}^n$.

ii) $f(s,.) : t \to f(s,t)$ is continuous on $\mathbb{R}^n$ for almost all $s \in \Omega$.

ii) For any $r > 0$, there exists $\mu_r \in L^q$ such that $|u| \le r$ implies that $|f(s,u)| \le \mu_r(s)$ for almost all $s \in \Omega$.

1.3.3 The Hammerstein integral operator

Let A be a linear Fredholm operator given by

$$A(u)(s) = \int_\Omega K(s,t)u(t)dt, \qquad s \in \Omega,$$

where K is the kernel, a Hammerstein operator is a nonlinear integral operator $\mathcal{H}$ obtained from the composition of a Fredholm integral operator A with a kernel K and a Nemytskii operator N_f associated to f as follows

$$\mathcal{H}(u)(s) = (AN_f u)(s) = \int_\Omega K(s,t)f(t,u(t))dt. \tag{1.6}$$

Let X and Y a Banach spaces.
We assume that the following assumptions are fulfilled:

B_1. A is compact operator.

B_2. N_f is continuous from its domain $D(N_f) \subset X$ into Y and N_f is bounded.

Lemma 1.1. [32] if B_1 and B_2 are fulfilled, then the Hammerstein operator $\mathcal{H}$ defined by (1.6) is completely continuous on $D(N_f)$.

1.4 The Fréchet derivative

The notion of differentiability can be generalized to Banach space in various ways, such as, Fréchet differentiability, named after Maurice Fréchet (1878-1973), this technique of differential calculus allows further investigation about nonlinear operators by establishing a connection between them and linear operators, more precisely by the technique of local approximation to a nonlinear operator by a linear one as we will see in chapter three. Suppose we have an operator A mapping an open set Ω of a Banach space X into a set D of a Banach space Y, we take a fixed point $x_0 \in \Omega$ and assume that there exists a continuous linear operator $F \in \mathcal{L}(X;Y)$ such that, for every $x \in X$

$$\lim_{r \to 0} \frac{A(x_0 + rx) - A(x_0)}{r} = F(x), \tag{1.7}$$

if the limit in equation (1.7) is uniform for all $x \in X$ with $\|x\| = 1$, then A is said to be Fréchet differentiable at x_0 and in this case $F = A'(x_0)$ is called the Fréchet derivative (which is unique) of A at x_0.
In other words, the Fréchet differentiability of an operator A at point x_0 in the direction of $h \in X$ means that

$$\lim_{h \to 0} \frac{\|A(x_0 + h) - A(x_0) - F(h)\|_Y}{\|h\|_X} = 0.$$

Example 1.1. For $X = C[0,1]$, $Y = \mathbb{R}$ and a fixed polynomial $A(x) \in \mathbb{R}[x]$, the nonlinear operator $A : X \to Y$ defined by

$$A(u) = \int_0^1 A(t)[u(t)]^3 dt,$$

is Fréchet differentiable on X, because, for fixed $u,\ h \in X$, and $r > 0$ we have

$$
\begin{aligned}
A'(u)(x) &= \lim_{r\to 0} \frac{A(u+rx)-A(u)}{r} \\
&= \lim_{r\to 0} \frac{1}{r}\left\{\int_0^1 A(t)[u(t)+rx(t)]^3 dt - \int_0^1 A(t)[u(t)]^3 dt\right\} \\
&= \lim_{r\to 0} \int_0^1 A(t)[3u^2(t)x(t) + 3ru(t)x^2(t) + r^2x^3(t)]dt \\
&= \int_0^1 3A(t)u^2(t)x(t)dt,
\end{aligned}
$$

hence the guess for the Fréchet derivatives of A at u is the linear operator,

$$
F(x) = \int_0^1 3A(t)[u(t)]^2 x(t)dt,
$$

since

$$
\|F\| = \sup_{x\neq 0} \frac{|F(x)|}{\|x\|_\infty} \le 3\|A\|_\infty \|u\|_\infty^2,
$$

thus, F is a bounded linear operator. Now we show that our guess F is the Fréchet derivative of A at u:

$$
\begin{aligned}
\lim_{h\to 0} \frac{|A(u+h)-A(u)-F(h)|}{\|h\|_\infty} &= \lim_{h\to 0} \frac{1}{\|h\|_\infty}\Bigg|\int_0^1 A(t)[u(t)+h(t)]^3 dt - \int_0^1 A(t)[u(t)]^3 dt \\
&\qquad - \int_0^1 3A(t)[u(t)]^3 h(t)dt\Bigg| \\
&= \lim_{h\to 0} \frac{1}{\|h\|_\infty}\left|\int_0^1 A(t)[3u(t)h^2(t) + h^3(t)]dt\right| \\
&\le \lim_{h\to 0} \frac{\|h\|_\infty^2}{\|h\|_\infty} \int_0^1 \Big|A(t)([u(t)]^2 + h(t))\Big| dt = 0,
\end{aligned}
$$

because the integral is bounded for small h.
Therefore, A is Fréchet differentiable on X where for each $u \in X$, and we have

$$
A'(u)(x) = \int_0^1 A(t)[u(t)]^2 x(t)dt, \quad x \in X.
$$

1.4.1 Differentiation of nonlinear integral operators

Consider the integral operator $\mathcal{K}$ defined by

$$\mathcal{K}(u)(s) = \int_0^1 K(s,t,u(t))dt. \tag{1.8}$$

Theorem 1.8. [23] Assume that the function $K(s,t,u)$ is defined and continuous, and has continuous derivatives $K'_u(s,t,u)$ and $K''_{u^2}(s,t,u)$, throughout the region Ω, then the operator $\mathcal{K}$ defined by (1.8) maps Ω into $C[0,1]$ and is twice differentiable at each interior point $u_0 \in \Omega$, and

$$\mathcal{K}'(u_0)(x(s)) = \int_0^1 K'_u(s,t,u_0(t))x(t)dt. \tag{1.9}$$

$$\mathcal{K}''(u_0)((x,y)(s)) = \int_0^1 K''_{u^2}(s,t,u_0(t))x(t)y(t)dt. \tag{1.10}$$

1.5 Some useful spaces

In this section we recall definitions and properties of some useful spaces to the next chapters, these spaces are required for the analysis of various numerical approaches for solving integral equations.

Definition 1.9. Let X and Y be normed spaces, we say that a function u is Lipschitz continuous if there exists a positive constant L such that

$$\|u(x) - u(y)\| \le L\|x-y\|, \quad \forall x,y \in X.$$

Definition 1.10. We say that a function u is Hölder continuous with exponent $k \in (0,1]$ if for some positive constant L,

$$\|u(x) - u(y)\| \le L\|x-y\|^k, \quad \text{for } x,y \in \Omega.$$

Sobolev space

We define

$$H_w^m(\Omega) = \{v \in L_w^2(\Omega), \|v\|_{m,w} < \infty\},$$

the Sobolev space of all functions $v(x)$ on Ω such that $v(x)$ and all its weak derivatives up to order m are in $L^2_w(\Omega)$, with the semi-norm

$$|v|_{H^{m,N}_w(\Omega)} = \left(\sum_{k=\min(m,N+1)}^{m} |v|^2_{k,w} \right)^{1/2}, \tag{1.11}$$

where

$$\|v\|_{m,w} = \left(\sum_{k=0}^{m} |v|^2_{k,w} \right)^{1/2}, \quad |v|_{k,w} = \left\| \partial^k_x v \right\|_{w^{\alpha,\beta}}.$$

It follows from [12] this useful lemma.

Lemma 1.2. Let $v \in H^m_w(\Omega)$ for $m > \frac{1}{2}$, there exists a positive constant C independent of N such that

$$|\langle v, \phi\rangle - \langle v, \phi\rangle_N| \le CN^{-m} |v|_{H^{m,N}_w(\Omega)} \|\phi\|_{w^{\alpha,\beta}}, \qquad \forall \phi \in \mathcal{P}_N, \tag{1.12}$$

where

$$\langle v, \phi\rangle = \int_{-1}^{1} v(x)\phi(x)dx, \text{ and } \langle v, \phi\rangle_N = \sum_{j=0}^{N} v(x_j)\phi(x_j)w_j.$$

Hölder space

For $k \in (0,1]$, and $r \ge 0$, the Hölder space $\mathcal{C}^{r,k}(\Omega)$ will denote the space of functions whose r-th derivatives are Hölder continuous with exponent k, this is a Banach space equipped with the semi-norm

$$\|v\|_{r,k} = \max_{0\le k\le r} \max_{x\in\Omega} \left|\partial^k_x v(x)\right| + \sup_{\substack{x,y\in\Omega \\ x\ne y}} \frac{\left|\partial^r_x v(x) - \partial^r_y v(y)\right|}{\left|x-y\right|^k},$$

particularly, the Hölder space $\mathcal{C}^{0,k}(\Omega)$ is a subspace of $C(\Omega)$ that involves functions that are Hölder continuous with the exponent k, and the Hölder space $\mathcal{C}^{0,1}(\Omega)$ involves all the Lipschitz continuous functions.

The following lemmas are necessary for determining our main results.

Lemma 1.3. [43] For a nonnegative integer r and $k \in (0,1)$, there exists a constant $C_{r,k} > 0$, such that for any function $v \in \mathcal{C}^{r,k}(\Omega)$, there exists a polynomial function $T_N v \in \mathcal{P}_N$ such that

$$\|v - T_N v\|_{L^\infty} \le C_{r,k} N^{-(r+k)} \|v\|_{r,k}. \tag{1.13}$$

Lemma 1.4. [14] Let $\mathcal{A}_v$ be the operator defined by

$$\mathcal{A}_v(s) = \int_{-1}^{x} K(s,t)v(t)dt,$$

then for any function $v \in C(\Omega)$, there exists a positive constant C such that

$$\|\mathcal{A}_v\|_{0,k} \leq C\|v\|_{L^\infty}, \quad 0 < k < 1. \tag{1.14}$$

1.6 The Gronwall inequalities

In the following part we will present a generalisation of the Gronwall inequalities which are used to prove convergence of a numerical method that it will be proposed to approximate Volterra operators.
The following lemma of generalized Gronwall inequality can be found in [21].

Lemma 1.5. [21] Suppose that $H \geq 0$, let $E(x)$ be a non-negative integrable function satisfying

$$E(x) \leq H \int_{-1}^{x} E(y)dy + G(x), \qquad x \in \Omega,$$

where $G(x)$ is also an integrable function, then there exists a positive constant C such that

$$\|E\|_{w^{\alpha,\beta}} \leq C\,\|G\|_{w^{\alpha,\beta}}. \tag{1.15}$$

$$\|E\|_{L^\infty} \leq C\,\|G\|_{L^\infty}. \tag{1.16}$$

CHAPTER 2

GENERALITIES ON THE THEORY OF NONLINEAR INTEGRAL EQUATIONS

In this chapter we present an introduction to the terminology and classification of integral equations through certain characteristics and criterions. We recall some theorems of fixed point such as Banach's fixed point theorem, Brouwer and Schauder's theorems, and nonlinear alternative of Leray-Schauder, then we applied them to a class of nonlinear integral equations (which is the most general form of integral equations) to establish some existence and uniqueness results of solution for these equations.

2.1 Classification of integral equations

An integral equation is an equation in which the unknown function of one or more variables appears under an integral sign, one of the most inclusive and most standard categories is that of equations assuming the form

$$u(s) = f(s) + \lambda \int_{g(s)}^{h(s)} K(s,t,u(t))dt, \tag{2.1}$$

where $g(s)$ and $h(s)$ are the limits of integration, $f(s)$ and $K(s,t,u)$ are known functions, λ is a nonzero real or complex scalar, the function $u(s)$ appears in the inside and outside of the integral sign is the unknown function and the function $K(s,t,u)$ is called the kernel.

The integral equations can be classified according to the following basic characteristics.

- The first one depends on the location of the unknown function, if it only appears in the inside of the integral sign, we say integral equation of first kind, however, in the equations of second kind, the unknown function also appear in the outside of the integral sign.

- The second characteristic depends on the limits of integration,
  - an integral equation in which the limits of integration are fixed is called a Fredholm integral equation given in the form:

$$u(s) = f(s) + \lambda \int_a^b K(s,t,u(t))dt, \tag{2.2}$$

  - an integral equation in which one limit of integration is a variable is called a Volterra integral equation given in the form:

$$u(s) = f(s) + \lambda \int_a^s K(s,t,u(t))dt,$$

  - it should also be noted that the Volterra integral equation is a special case of Fredholm integral equation, it is enough to take the kernel $K(s,t,u(t)) = 0$ if $a \leq s < t \leq b$.

- Moreover, we say that the equation (2.1) is homogeneous if the function $f(x) = 0$ in $[a,b]$.

- Another description is related to the linearity of the kernel $K(s,t,u)$ in respect of third variable,
  - if $K(s,t,u(t))$ is linear in respect of the the third variable .e.i.,

$$K(s,t,u(t)) = K(s,t)u(t),$$

  the integral equation is called linear equation, otherwise, the equation is nonlinear, in this case we have the following two most frequently forms given by

$$u(s) = f(s) + \lambda \int_\Omega K(s,t,u(t))dt, \tag{2.3}$$

$$u(s) = f(s) + \lambda \int_\Omega K(s,t)G(t,u(t))dt, \tag{2.4}$$

are called Urysohn integral equation and Hammerstein integral equation respectively.

- The last criterion in this classification is connected to improper integrals, an equation is called singular if at least one limit of integration is infinite, or if the kernel is unbounded in given interval, such as:
  - The generalized Abel's integral equation:

$$f(s) = \int_0^s \frac{u(t)}{(s-t)^\alpha} dt, \quad 0 < \alpha < 1.$$

  - The Cauchy type integral equation:

$$u(s) = f(s) + \lambda \int_b^a \frac{u(t)}{(t-s)} dt, \text{ where } \lambda \text{ is a parameter.}$$

  - The Wiener-Hopf integral equation:

$$f(s) = u(s) + \lambda \int_0^\infty k(s-t)u(t)dt, \text{ where } \lambda \text{ is a parameter.}$$

2.2 Fixed-point theorems and their applications to N.I.Es

In various branches of mathematical analysis we can solved many problems defined by nonlinear functional equations by converting them to an equivalent problem of a fixed point problems, in fact, an operator equation $Gu = u$ maybe expressed as a fixed point equation $Au = u$, where A is an operator. Here we discuss some different types of fixed point theorems that prove the existence of solutions for nonlinear integral equations.
Starting from the basics of Banach's contraction theorem, one of the most main results and techniques that have been developed (see [11, 1, 60, 4, 45]), this theorem establishes a general criterion guarantees that the iteration procedure of a function results a fixed point. Then, we discuss the Brouwer fixed-point theorem which the finite dimensional version of Schauder's theorem, this latter stated that if Ω is a convex and compact subset of a Banach space X and A is a self-continuous mapping, then A has at least one fixed point, nevertheless, one of the defects of Schauder's fixed point theorem is the invariance condition $A(\Omega) \subset \Omega$ this must be insured for a bounded closed convex subset Ω in a Banach space. The principle of Leray-Schauder allows for the avoidance of such

a condition by requiring the fulfilment of a boundary condition.

2.2.1 Banach's fixed-point theorem

Definition 2.1. A bounded operator A on a Banach space X is a contraction, if there exists a constant L with $0 < L < 1$, such that

$$\|A(u_1) - A(u_2)\| \leq L\|u_1 - u_2\|, \qquad \forall u_1, u_2 \in X.$$

Theorem 2.1. (Banach's fixed-point theorem (1922)) Let A be a contraction on a Banach space X.
Then the equation

$$A(u) = u, \tag{2.5}$$

has a unique solution on X, this solution is the fixed point of operator A.

Proof. Let $u_0 \in X$ be arbitrary, make a sequence in X by establishing $u_1 = A(u_0)$ and $u_{n+1} = A(u_n)$ for $n > 0$.
Let we first demonstrate that $\{u_n\}$ is a Cauchy sequence. From the contractivity of A, we have

$$\|u_{n+1} - u_n\| \leq L\|u_n - u_{n-1}\| \leq \cdots \leq L^n\|u_1 - u_0\|,$$

for $m \geq n \geq 1$,

$$\begin{aligned}
\|u_m - u_n\| &\leq \sum_{j=0}^{m-n-1} \|u_{n+j+1} - u_{n+j}\| \\
&\leq \sum_{j=0}^{m-n-1} L^{n+j}\|u_1 - u_0\| \\
&\leq \frac{L^n}{1-L}\|u_1 - u_0\|.
\end{aligned}$$

Since $L \in (0,1), \|u_m - u_n\| \to 0$ as $m, n \to \infty$. Thus $\{u_n\}$ is a Cauchy sequence in the Banach space X, therefore $\{u_n\}$ converge to a limit $u \in X$, by the continuity of A we have

$$A(u) = A(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} A(u_n) = \lim_{n\to\infty} u_{n+1} = u,$$

then u is a fixed point of A.
Now by contradiction we prove that u is unique.

Assume that A possesses two or more distinct fixed points, say u_1 and u_2, Then

$$\|u_1 - u_2\| = \|A(u_1) - A(u_2)\| \leq L\|u_1 - u_2\|,$$

this implies that $\|u_1 - u_2\| = 0$ since $0 < L < 1$, so a contractive mapping's fixed point is unique. □

Theorem 2.2. Let A be an operator on a Banach space X with A^n is a contraction. Then the equation

$$A(u) = u,$$

has a unique solution on X.

Proof. As a result of the theorem, A^n has a fixed point noted by u_0, then

$$\begin{aligned}\|A(u_0) - u_0\| &= \|A^n(A(u_0)) - A^n(u_0)\| \\ &\leq L\|A(u_0) - u_0\|,\end{aligned}$$

implies that $Au_0 = u_0$ since $0 < L < 1$.
The uniqueness is clear since a fixed point for A is also a fixed point for A^n. □

2.2.2 Existence and uniqueness of solution for N.I.E via Banach's fixed point theorem

2.2.2.1 For nonlinear Volterra integral equation

Theorem 2.3. Assume that $K(s,t,u)$ is defined and continuous on the square $a \leq s,t \leq b$ and that it satisfies a Lipschitz condition of the form

$$\|K(s,t,u_1) - K(s,t,u_2)\| \leq L\|u_1 - u_2\|.$$

Assume further that $f \in C[a,b]$. Then the nonlinear Volterra integral equation

$$u(s) = f(s) + \lambda \int_a^s K(s,t,u(t))dt, \tag{2.6}$$

has a unique solution on the interval $[a,b]$ for every value of λ, where $a \leq s \leq b$.

Proof. If it can be proved that an adequate power of the continuous operator

$$\mathcal{K} : C[a,b] \to C[a,b],$$

defined by

$$\mathcal{K}(u)(s) = f(s) + \lambda \int_a^s K(s,t,u(t))dt, \tag{2.7}$$

is a contraction for every valued of λ, then it will be obvious as an application of theorem (2.1) that $\mathcal{K}$ has a unique fixed point, implying that the equation (2.6) has a unique solution.

Let $u_1, u_2 \in C[a,b]$ and $s \in [a,b]$, we will show that for any $n \geq 1$

$$\|\mathcal{K}^n(u_1) - \mathcal{K}^n(u_2)\| \leq \frac{\lambda^n L^n (b-a)^n}{n!} \|u_1 - u_2\|_\infty, \tag{2.8}$$

For $n = 1$, we have

$$\begin{aligned}
|\mathcal{K}(u_1)(s) - \mathcal{K}(u_2)(s)| &= |\lambda \int_a^s [K(s,t,u_1(t)) - K(s,t,u_2(t))]dt| \\
&\leq |\lambda L| \int_a^s |u_1(t) - u_2(t)|dt \\
&\leq |\lambda L|(b-a)\|u_1 - u_2\|_\infty,
\end{aligned}$$

this implies that $\|\mathcal{K}(u_1) - \mathcal{K}(u_2)\| \leq |\lambda| L(b-a)\|u_1 - u_2\|_\infty$.

Assume that the property (2.8) is verified for $n = m$, and we will show that (2.8) is verified for $n = m+1$.

$$\begin{aligned}
|\mathcal{K}^{m+1}(u_1)(s) - \mathcal{K}^{m+1}(u_2)(s)| &= |\mathcal{K}(\mathcal{K}^m(u_1))(s) - \mathcal{K}(\mathcal{K}^m(u_2))(s)| \\
&= |\lambda \int_a^s [K(s,t,\mathcal{K}^m u_1(t)) - K(s,t,\mathcal{K}^m u_2(t))]dt| \\
&\leq |\lambda| \int_a^s L|\mathcal{K}^m(u_1)(t) - \mathcal{K}^m(u_2)(t)|dt \\
&\leq |\lambda| \int_a^s L \frac{|\lambda|^m L^m (b-a)^m}{m!} \|u_1 - u_2\|_\infty dt \\
&\leq \frac{|\lambda|^{m+1} L^{m+1} (b-a)^{m+1}}{(m+1)!} \|u_1 - u_2\|_\infty,
\end{aligned}$$

hence

$$\|\mathcal{K}^{m+1}(u_1) - \mathcal{K}^{m+1}(u_2)\| \leq \frac{|\lambda|^{m+1} L^{m+1} (b-a)^{m+1}}{(m+1)!} \|u_1 - u_2\|_\infty,$$

then the property (2.8) is valid for all $n > 0$.

Since the sequence $\frac{|\lambda|^n L^n (b-a)^n}{n!} \to 0$ there exist a power n_0 such that $\frac{|\lambda|^{n_0} L^{n_0} (b-a)^{n_0}}{n_0!} < 1$, this implies that $\mathcal{K}^{n_0}$ is a contraction. □

2.2.2.2 For nonlinear Fredholm integral equation

Theorem 2.4. Assume that $K(s,t,u)$ is defined and continuous on the square $a \leq s,t \leq b$ and that it satisfies a Lipschitz condition of the form

$$|K(s,t,u_1)-K(s,t,u_2)| < L|u_1-u_2|.$$

Assume further that $f \in C[a,b]$. Then the nonlinear Fredholm integral equation

$$u(s) = f(s) + \lambda \int_a^b K(s,t,u(t))dt, \tag{2.9}$$

has a unique solution on the interval $[a,b]$ whenever $\lambda < 1/(L(b-a))$.

Proof. If it can be proved that the operator

$$\mathcal{K} : C[a,b] \to C[a,b]$$

defined by

$$\mathcal{K}(u)(s) = f(s) + \lambda \int_a^b K(s,t,u(t))dt,$$

is a contraction for the constrained values of λ specified in the statements of the theorem, then it will be obvious as an application of Banach's fixed point theorem (theorem 2.1) that $\mathcal{K}$ has an unique fixed point, implying that the integral equation (2.9) has a unique solution.

Let $u_1, u_2 \in C[a,b]$ and $s \in [a,b]$, we have

$$\begin{aligned} |\mathcal{K}(u_1)(s)-\mathcal{K}(u_2)(s)| &= |\lambda \int_a^b (K(s,t,u_1(t))-K(s,t,u_2(t))dt| \\ &\leq |\lambda| L(b-a) \|u_1-u_2\|_\infty, \end{aligned}$$

this implies that

$$|\lambda| L(b-a) < 1,$$

(we can select $\lambda < \frac{1}{L(b-a)}$), then $\mathcal{K}$ is a contraction operator. □

2.2.3 Brouwer's fixed-point theorem

Theorem 2.5. (Brouwer's fixed-point theorem (1912)) Let $\Omega \subset \mathbb{R}^n$ be a nonempty, convex and compact set and let $A : \Omega \to \Omega$ be a continuous mapping. Then A has at least one fixed point.

Proof. (See [59]) □

2.2.4 Schauder's fixed-point theorem

Theorem 2.6. (Schauder's fixed-point theorem (1930)) Let X be a Banach space, $\Omega \subset X$ a nonempty convex compact set and let $A : \Omega \to \Omega$ be a continuous operator. Then A has at least one fixed point.

Proof. Since A is a continuous mapping and Ω is a compact, A is uniformly continuous, hence, for all $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for every $x, y \in \Omega$ satisfying $\|x - y\| \leq \delta$, we have that $\|A(x) - A(y)\| \leq \varepsilon$, furthermore, there exists a finite set of points $\{x_1, \cdots, x_p\} \subset \Omega$ such that the open balls with radius δ centred at x_j cover Ω, i.e. $\Omega \subset \bigcup\limits_{1 \leq j \leq p} B(x_j, \delta)$.
Let $G = \text{ vect } (A(x_j))_{1 \leq j \leq p}$, then G is of finite dimension, and $\Omega^* = \Omega \cap G$ is a compact convex and of finite dimension.
For $1 \leq j \leq p$, we define the continuous mappings $\Psi_j : X \to \mathbb{R}$ by

$$\Psi_j = \begin{cases} 0, & \text{if } \|x - x_j\| \geq \delta, \\ 1 - \frac{\|x - x_j\|}{\delta}, & \text{if otherwise} . \end{cases}$$

Obviously Ψ_j is strictly positive on $B(x_j, \delta)$ and is equal to zero outside, hence

$$\forall x \in \Omega, \sum_{j=1}^{p} \Psi_j(x) > 0,$$

this implies that the continuous mapping φ_j defined as follows

$$\varphi_j(x) = \frac{\Psi_j(x)}{\sum_{k=1}^{p} \Psi_k(x)},$$

satisfying

$$\sum_{j=1}^{p} \varphi_j(x) = 1,$$

for every $x \in \Omega$.
Let us now define for all $x \in \Omega$, the mapping g by

$$g(x) = \sum_{j=1}^{p} \varphi_j(x) A(x_j),$$

g is continuous mapping defined on Ω into Ω^*, and if we take the restriction $g/\Omega^* \to \Omega^*$, the Brouwer fixed-point theorem implies that g has a fixed point $y \in \Omega^*$, furthermore,

$$\begin{aligned} A(y) - y &= A(y) - g(y) \\ &= \sum_{j=1}^{p} \varphi_j(y) A(y) - \sum_{j=1}^{p} \varphi_j(y) A(x_j) \\ &= \sum_{j=1}^{p} \varphi_j(y)(A(y) - A(x_j)), \end{aligned}$$

if $\varphi_j(y) \neq 0$ then $\|y - x_j)\| \leq \delta$, this yields, $\|A(y) - A(x_j)\| \leq \varepsilon$, and for all j

$$\|\varphi_j(y)(A(y) - A(x_j))\| \leq \varepsilon \varphi_j(y),$$

then

$$\begin{aligned} \|A(y) - y\| &\leq \sum_{j=1}^{p} \|\varphi_j(y)(A(y) - A(x_j))\| \\ &\leq \sum_{j=1}^{p} \varepsilon \varphi_j(y) = \varepsilon, \end{aligned}$$

this means that, for each $\varepsilon > 0$, there exists $y = y(\varepsilon) \in \Omega$ such that $\|A(y) - y\| < \varepsilon$, hence

$$\forall m \in \mathbb{N} \quad \exists y_m \in \Omega : \|A(y_m) - y_m\| < 2^{-m},$$

Since Ω is a compact, $\{y_m\}_{m \in \mathbb{Z}}$ has a subsequence $\{y_{mk}\}_{k \in \mathbb{Z}}$ convergent to some element $y^* \in \Omega$, and since A is continuous, the sequence $A(y_{mk})$ converge to $A(y^*)$, thus $A(y^*) = y^*$, i.e. y^* is at least one fixed point of A on Ω. □

The following variation of Schauder's theorem is more commonly utilised in applications.

Theorem 2.7. Let X be a Banach space, $\Omega \subset X$ a non empty convex bounded closed set and let $A : \Omega \to \Omega$ be a completely continuous operator. Then A has a fixed point

2.2.5 Existence of continuous solutions for N.I.E via Schauder's fixed point theorem

2.2.5.1 For nonlinear Fredholm integral equation

We consider the nonlinear Fredholm integral equation of second kind defined by

$$u(s) = \int_a^b K(s,t,u(t))dt, \qquad s \in [a,b]. \tag{2.10}$$

Theorem 2.8. [42] Let $R > 0$ and $K : [a,b]\times[a,b]\times\overline{B}_R(0;\mathbb{R}) \to \mathbb{R}$ be a continuous mapping. Assume

$$M|b-a| \leq R, \tag{2.11}$$

where

$$M = \max_{[a,b]^2\times\overline{B}_R(0;\mathbb{R})} |K(s,t,u)|,$$

Then (2.10) has at least one solution $u \in C[a,b]$ with $\|u\|_\infty < R$.

Proof. In accordance with the corollary (1.1), the operator

$$\mathcal{K} : \overline{B}_R(0;C[a,b]) \to C[a,b]$$

is completely continuous, at the same time, (2.11) implies that

$$\mathcal{K}(\overline{B}_R(0;C[a,b])) \subset \overline{B}_R(0;C[a,b]).$$

Thus, the conclusion follows from the variant of Schauder's theorem (theorem 2.7). □

2.2.5.2 For nonlinear Volterra integral equation

We consider the nonlinear Volterra integral equation of second kind defined by

$$u(s) = \int_a^s K(s,t,u(t))dt, \qquad a \leq s \leq b. \tag{2.12}$$

Theorem 2.9. [42] Let $R > 0$ and let $K : [a,b] \times [a,b] \times \overline{B}_R(0,\mathbb{R}) \to \mathbb{R}$ be continuous. Assume that there exist constants $\alpha, \beta \in \mathbb{R}_+$ such that

$$|K(s,t,y)| \leq \alpha|y| + \beta,$$

for all $s,t \in [a,b], y \in \mathbb{R}$. Then (2.12) has at least one solution $u \in C[a,b]$.

Proof. We can followed the same reasoning as in the proof of theorem 2.8 with the use of corollary 1.2 and the variation of Schauder's theorem 2.7 to establish the proof of this theorem. For more details see [42]. □

2.2.6 Nonlinear alternative of Leray-Schauder

Theorem 2.10. (Leray-Schauder alternative) Let X be a Banach space, Ω a closed, convex subset of X, and let $G \subset \Omega$ an open subset of Ω, with $u_0 \in G$. Suppose that $A : \overline{G} \to \Omega$ is a continuous, compact. Then either

(i) A has a fixed point in $\overline{G}$, or

(ii) there is a $u \in \partial G$, with $u = (1-\lambda)u_0 + \lambda A(u)$ for some $\lambda \in (0,1)$.

Proof. A proof of this classical result and of other similar results of Leray-Schauder principle is given in [18, 22, 41, 42] and it is based on transversality theory. □

2.2.7 Existence of continuous solutions for Hammerstien integral equation on half-line via nonlinear alternative

We consider the Hammerstien integral equation defined by

$$u(s) = g(s) + \int_0^\infty K(s,t) f(t,u(t))dt, \quad s \in [0,\infty). \tag{2.13}$$

In this section we apply a nonlinear Alternative of Leray-Schauder to establish conditions under which equation (2.13), will have solutions $u \in C_l$ such that $\lim\limits_{s\to\infty} u(s)$ exists.

Theorem 2.11. [39] Suppose that $1 \leq p \leq \infty$ and let q be such that $(1/p) + (1/q) = 1$. Assume that:

- $g \in C_l.$ (2.14)
- f in L^q-Carathéodory. (2.15)
- $K_s = K(s,t) \in L^p[0,\infty)$ for each $s \in [0,\infty)$. (2.16)
- The map $s \to K_s$ is continuous from $[0,\infty)$ to $L^p[0,\infty)$, (2.17)

and

- there exists $\tilde{K} \in L^p[0,\infty)$ such that $K_s \to \tilde{K}$ in $L^p[0,\infty)$ as $s \to \infty$ (2.18)

hold.
In addition, suppose there exists a constant $M > 0$, independent of λ, with

$$\|u\|_0 = \sup_{s\in[0,\infty)} |u(s)| \neq M,$$

for any solution $u \in C_l$ to

$$u(s) = \lambda\left(g(s) + \int_0^\infty K(s,t)f(t,u(t))dt\right), \quad s \in [0,\infty), \tag{2.19}$$

for each $\lambda \in (0,1)$. Then (2.13) has at least one solution $u \in C_l$.

Proof. The authors in this proof showed that the well defined operator $\mathcal{H} : C_l \to C_l$ defined by:

$$\mathcal{H}(u)(s) = g(s) + \int_0^\infty K(s,t)f(t,u(t))dt, \quad s \in [0,\infty),$$

has a fixed point, since this latter is a solution of equation (2.13).
To justify the above statement, firstly, they have showed that $\mathcal{H} : C_l \to C_l$ is a continuous on $[S_0,\infty)$, with $S_0 > 0$, through the assumptions 2.14-2.18 and the Lebesgue dominated convergence theorem. Then, by means of the assumptions 2.14-2.16 and the pointwise convergence of $\mathcal{H}$ on the interval $[0,S_0]$, together with the Lebesgue Dominated Convergence Theorem and the compactness of $[0,S_0]$, it have been proved that $\mathcal{H}$ is continuous also on $[0,S_0]$ and thereby they obtain the continuity of $\mathcal{H}$ on $[0,\infty)$.
Then, for proving the compactness of $\mathcal{H} : C_l \to C_l$ it enough to select a bounded set Λ in C_l and it have been showed that $\mathcal{H}(\Lambda)$ satisfies the three conditions of theorem 1.3.
Therefore, by applying the Nonlinear Alternative with $\Omega = X = C_l$ and $G = \{u \in C_l : |u|_0 < M\}$ it can be concluded that $\mathcal{H}$ has a fixed point (the possibility (ii) of the theorem 2.10 cannot occur), that is, equation (2.13) has a solution $u \in C_l$. □

CHAPTER 3

SOME NUMERICAL METHODS FOR SOLVING NONLINEAR INTEGRAL EQUATIONS

In this chapter, various numerical methods are presented and analysed for the solution of nonlinear integral equations. The main aspect of these methods is the discretization of the equation $u = \mathcal{K}u$ by substituting it with a series of finite-dimensional approximation problems $u_n = \mathcal{K}_n u_n$, with $n \to \infty$, the types of this discretization are: (1) Projection methods, with the most common ones being collocation and Galerkin methods, and (2) Nyström method. We also presented in this chapter the Sinc approximation methods which are an optimal basis for the approximation in spaces of functions that are analytic.

3.1 Projection methods

Projection methods for solving the nonlinear integral equation

$$u(s) = \int_{\Omega} K(s,t,u(t))dt, \qquad s \in \Omega, \tag{3.1}$$

are mainly based to the choice of a finite-dimensional set of functions thought to include a function $u_n(s)$ approach to the exact solution $u(s)$, this required numerical solution $u_n(s)$ is chosen by having it satisfy (3.1) approximately. There are various senses in which $u_n(s)$ can be said to "satisfy (3.1) approximately," and these lead to different types of methods. The most popular of these are collocation methods and Galerkin methods, and they are defined below. General theoretical frameworks for projection methods have been provided in number of references, such as [5, 7, 8, 16, 17, 25, 49, 58].

3.1.1 Projection operators

Definition 3.1. Let X be a normed space and $Y \subset X$ a non trivial subspace, a projection operator from X into Y is a bounded linear operator $P : X \to Y$ such that

$$P(u) = u \qquad \forall u \in X. \tag{3.2}$$

from (3.2) we have

$$P^2(u) = P(P(u)) = P(u),$$

thus

$$P^2 = P, \tag{3.3}$$

furthermore

$$\|P\| = \|P^2\| \le \|P\|^2,$$

this mean that

$$\|P\| \ge 1. \tag{3.4}$$

Orthogonal projection operators

Definition 3.2. Let X be a Hilbert space, and let $\langle \cdot, \cdot \rangle$ denote the inner product for X, a projection operator P is orthogonal if and only if

$$\langle Pu, (I - P)v \rangle = 0, \qquad \forall u, v \in X.$$

Example 3.1. Let $\{\varphi_n\}_{n\ge1}$ be an orthonormal basis of the space X and $X_n =$ span $\{\varphi_1, \cdots, \varphi_n\}$, for any $u \in X$, the formula

$$P_n(u) = \sum_{i=1}^{n} \langle u, \varphi_i \rangle \varphi_i,$$

defines an orthogonal projection from X onto X_n.

Interpolatory projection operators

Let X be a normed vector space, and X_n is an n-dimensional subspace of X with a basis $\{\varphi_1, \cdots, \varphi_n\}$, let $x_1, \cdots, x_n$ be interpolation nodes in the interpolation region Ω, the interpolation function $u_n \in X_n$ is given to approximate a given function $u \in X$ where the interpolation problem is as follows: Given data $y_1, \cdots, y_n$ find $u_n \in X_n$ given by

$$u_n(x) = \sum_{j=1}^{n} \alpha_j \varphi_j(x),$$

such that the interpolation conditions

$$u_n(x_i) = y_i, \qquad i = 1, \cdots, n,$$

are satisfied.

Thus, we find the coefficient $\alpha_1, \cdots, \alpha_n$ by solving the system

$$\sum_{j=1}^{n} \alpha_j \varphi_j(x_i) = y_i, \qquad i = 1, \cdots, n. \tag{3.5}$$

The necessary and sufficient condition for the system (3.5) to have a unique solution is

$$\det[\varphi_j(x_i)] \neq 0.$$

Thus, the solution of the interpolation problem is

$$P_n(u)(x) = \sum_{i=1}^{n} u(x_i)\varphi_i(x),$$

where P_n is the interpolatory projection operator from X into X_n.

Example 3.2. (Lagrange interpolation) Let X is $C[a,b]$ and $\mathcal{P}_N$ the space of all polynomials of degree not more than N. For any $u \in C[a,b]$ we can define the Lagrange interpolating polynomials $I_N^{\alpha,\beta} u \in \mathcal{P}_N$ as

$$I_N^{\alpha,\beta} u(x) = \sum_{j=0}^{N} u(x_j^{\alpha,\beta}) L_j^{\alpha,\beta}(x), \qquad L_j^{\alpha,\beta}(x) = \prod_{i \neq j} \frac{x - x_i^{\alpha,\beta}}{x_j^{\alpha,\beta} - x_i^{\alpha,\beta}}, \tag{3.6}$$

where $\{L_j^{\alpha,\beta}\}_{j=0}^{N}$ is the Lagrange interpolation basic functions associated with Gauss-Jacobi quadrature points $\{x_j^{\alpha,\beta}\}_{j=0}^{N}$, which are the roots of the Jacobi polynomials $J_{N+1}^{\alpha,\beta}$.

The functions $L_j^{\alpha,\beta}$ satisfy the special interpolation conditions

$$L_j^{\alpha,\beta}(x_i) = \delta_{ji} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{3.7}$$

and they form a basis for $\mathcal{P}_N$.

The following lemmas are necessary for determining our main results.

Lemma 3.1. [37] Let $I_N^{\alpha,\beta} v$ be the interpolation operator associated with the Gauss-Jacobi points, then for any function $v \in H_w^m(I)$, there exists a positive constant C independent of N such that

$$\|v - I_N^{\alpha,\beta} v\|_{L^\infty} \leq CN^{\frac{1}{2}-m} |v|_{H_w^m(I)}, \tag{3.8}$$

$$\|v - I_N^{\alpha,\beta} v\|_{w^{\alpha,\beta}} \leq CN^{-m} |v|_{H_w^m(I)}. \tag{3.9}$$

Lemma 3.2. [33] Let $\{L_j\}_{j=0}^N$ be the N-th Lagrange interpolation polynomials associated with the Gauss-Jacobi points. Then

$$\|I_N^{\alpha,\beta}\|_{L^\infty} = \max_{x\in I} \sum_{j=0}^{N} |L_j(x)| = \begin{cases} \mathcal{O}(\log N), & if\ -1 < \alpha,\beta \leq -\frac{1}{2}, \\ \mathcal{O}(N^{\frac{1}{2}+\max\{\alpha,\beta\}}), & if \text{ otherwise.} \end{cases} \tag{3.10}$$

Lemma 3.3. [37] For every bounded function $v(x)$, there exists a positive constant C independent of v such that

$$\|I_N^{\alpha,\beta} v\|_{w^{\alpha,\beta}} \leq C\|v\|_{L^\infty}. \tag{3.11}$$

3.1.2 Principle of projection methods

Let X be a Banach space, the most popular choices are $C(\Omega)$ and $L^2(\Omega)$, consider the Urysohn integral operator

$$\mathcal{K}(u)(s) = \int_\Omega K(s,t,u(t))dt, \qquad s \in \Omega, \tag{3.12}$$

we are interested in a solution of the operator integral equation

$$u = \mathcal{K}u, \tag{3.13}$$

let $X_n, n \geq 1$, be a sequence of finite dimensional subspaces, with X_n having dimension N for notational simplicity, and let P_n be a projection of X onto X_n, it is usually assumed that

$$P_n u \to u, \text{ as } n \to \infty, \quad \forall u \in X, \tag{3.14}$$

the projection method for solving (3.13) is to seek an approximate solution $u_n \in X_n$, such that u_n satisfies the operator equation

$$u_n = P_n \mathcal{K}(u_n), \tag{3.15}$$

thus, we find an approximate fixed point problem which can be written in the following equivalent form

$$P_n(I - \mathcal{K})(u_n) = 0, \qquad u_n \in X_n. \tag{3.16}$$

Lemma 3.4. [5] Let X be a Banach space and let $\{P_n\}$ be a family of bounded projections on X with

$$P_n u \to u \text{ as } n \to \infty, \quad \forall u \in X,$$

if $\mathcal{K} : X \to X$ is a compact operator, then

$$\|\mathcal{K} - P_n \mathcal{K}\| \to 0 \text{ as } n \to \infty. \tag{3.17}$$

Theorem 3.1. [5] Assume $\mathcal{K} : X \to X$ is bounded, with X a Banach space, and assume that

$$\lambda - \mathcal{K} : X \longrightarrow X.$$

Further assume

$$\|\mathcal{K} - P_n \mathcal{K}\| \to 0 \text{ as } n \to \infty. \tag{3.18}$$

Then for all sufficiently large n, say $n \geq N$, the operator $(\lambda - P_n \mathcal{K})^{-1}$ exists as a bounded operator from X to X, moreover, it is uniformly bounded:

$$\sup_{n \geq N} \|(\lambda - P_n \mathcal{K})^{-1}\| < \infty.$$

3.1.3 Convergence of projection methods

After we approximated the equation (3.13) by the system (3.15), we want to prove that for all sufficiently large n, (3.15) has a unique solution, we also trying to obtain bounds on the rate of convergence of u_n to u. For this we apply the procedure of linearisation (see [5]), in this approach we linearise the problem and apply the Banach Fixed-point

theorem.
Firstly, we substitute the nonlinear function by an approximating linear Taylor series in a point u_0

$$\mathcal{K}(u) = \mathcal{K}(u_0) + \mathcal{K}'(u_0)(u - u_0),$$

where $\mathcal{K}'(u_0)$ denote the Fréchet derivative of $\mathcal{K}(u)$ at u_0.
Then, we discussed the error in linearisation of $\mathcal{K}(u)$ about a point (u_0):

$$R(u;u_0) = \mathcal{K}(u) - [\mathcal{K}(u_0) + \mathcal{K}'(u_0)(u - u_0)],$$

the error $R(u;u_0)$ has properties that are useful in the study of convergence.

Lemma 3.5. [5] Let X be a Banach space, and let E be an open subset of X, let $\mathcal{K} : E \subseteq X \to X$ be twice continuously differentiable with $\mathcal{K}''(u)$ bounded over any bounded subset of E.
Let $B \subset E$ be a closed, bounded, and convex set with a non-empty interior. Let u_0 belong to the interior of B, and define $R(u;u_0)$, as above. Then for all $u_1, u_2 \subset B$,

$$\|R(u_1;u_2)\| \leq \frac{1}{2}M\|u_1 - u_2\|^2, \tag{3.19}$$

with $M = \sup_{u\in B} \|\mathcal{K}''(u)\|$. Moreover

$$\|\mathcal{K}'(u_2) - \mathcal{K}'(u_1)\| \leq M\|u_2 - u_1\|,$$

implying $\mathcal{K}'(u)$ is Lipschitz continuous, and

$$\|R(u_1;u_0) - R(u_2;u_0)\| \leq M\left[\|u_1 - u_0\| - \frac{1}{2}\|u_1 - u_2\|\right]\|u_1 - u_2\|. \tag{3.20}$$

After that, we make on $\mathcal{K}$ and $\{P_n\}$ the following assumptions.

A_1. $\mathcal{K} : E \subseteq X \to X$ is a completely continuous nonlinear operator.

A_2. The equation (3.13) has a unique isolated solution $u^* \in X$ within the ball

$$B(u^*, \varepsilon) = \{u \in E \;/\; \|u - u^*\| \leq \varepsilon\}.$$

for some $\varepsilon > 0$,

A_3. $\mathcal{K}$ is twice continuously differentiable over E with

$$M = \sup_{u \in B(u^*, \varepsilon)} \|\mathcal{K}''(u)\| < \infty. \tag{3.21}$$

A_4. The projection operators $\{P_n\}$ are pointwice convergent to the identity on X

$$P_n(u) \to u \text{ as } n \to \infty, \quad \forall u \in X.$$

The assumption A_1 implies that $L = \mathcal{K}'(u^*)$ is a compact linear operator, also from the assumption A_4 we can apply the lemma 3.4 to obtain

$$\|(I - P_n)L\| \to 0 \text{ as } n \to \infty, \tag{3.22}$$

hence for all sufficiently large n, $n \geq N$, the operator $(I - P_nL)^{-1}$ exists and

$$\sup_{n \geq N} \|(I - P_nL)^{-1}\| < \infty,$$

that is a result from the theorem 3.1.
By this result, we seek to show that for all sufficiently large n, (3.15) has a unique solution within $B(u^*, \varepsilon_1)$, for some $0 \leq \varepsilon_1 \leq \varepsilon$, by expending $\mathcal{K}(u_n)$ about u^*, we obtain

$$\mathcal{K}(u_n) = \mathcal{K}(u^*) + L(u_n - u^*) + R(u_n; u^*),$$

hence, we can rewrite the equation (3.15) as the equivalent equation

$$(I - P_nL)(u_n - u^*) = P_nu^* - u^* + P_nR(u_n; u^*), \tag{3.23}$$

let $\gamma_n = u_n - u^*$, then

$$\begin{aligned} \gamma_n &= (I - P_nL)^{-1}(P_nu^* - u^*) + (I - P_nL)^{-1}R(\gamma_n + u^*; u^*) \\ &\equiv F_n(\gamma_n). \end{aligned} \tag{3.24}$$

From the various results and the Banach contractive mapping theorem, we can show that the fixed-point equation (3.24) has a unique solution γ_n, this prove that the approximating equation (3.15) has a unique solution u_n in some ball of fixed radius about u^*.

The rate of convergence

Theorem 3.2. [8] If 1 is not eigenvalue of $L = \mathcal{K}'(u^*)$, then there are nonnegative sequence $\{w_n\}$ convergent to zero, such that

$$\|u_n - u^*\| \le C(1 + w_n)\|P_n u^* - u^*\|. \tag{3.25}$$

Proof. We can rewrite (3.23) as the equivalent identity

$$(I - L)(u_n - u^*) = (P_n - I)L(u_n - u^*) + (P_n - I)u^* + P_n R(u_n; u^*),$$

and then bounding the right side, we obtain (3.25). The constants are given by

$$w_n = \frac{C(a_n + Pr_n)}{1 - C(a_n + Pr_n)}, \qquad a_n = \|(I - P_n)L\|, \qquad r_n = \frac{\|R(u_n; u^*)\|}{\|u_n - u^*\|},$$
$$C = \|(I - L)^{-1}\|, \qquad \sup_n \|P_n\| \le P < \infty.$$

□

3.1.4 Collocation method

We consider the Urysohn integral equation

$$u(s) = f(s) + \int_\Omega K(s, t, u(t))dt, \qquad s \in \Omega. \tag{3.26}$$

let X be $C(\Omega)$, and let X_n be a finite dimensional subsequence of X, define P_n to be the interpolatory projection operator of X onto X_n.
let

$$u_n(s) = \sum_{i=1}^{N} \alpha_j \varphi_j(s),$$

where $\{\varphi_j\}_{j=1}^N$is a basis of X_n and pick distinct node points $s_1, \cdots, s_N \in \Omega$ and require

$$R_n(s_i) \equiv \sum_{i=1}^{N} \alpha_j \varphi_j(s_i) - f(s_i) - \int_\Omega K(s_i, t, \sum_{i=1}^{N} \alpha_j \varphi_j(t))dt = 0, \tag{3.27}$$

where R_n is called the residual in the approximation of the equation when using $u \simeq u_n$. By forcing R_n be approximately 0 at the collocation points $s_1, s_2, \cdots, s_N$ we can determine the coefficients $\{\alpha_1, \cdots, \alpha_N\}$. The hope is for the resulting function $u_n(s)$ to have a good approximation of the exact solution $u(s)$.

3.1.5 Galerkin method

Let X be $L^2(\Omega)$ or some other Hilbert space, and let X_n be a finite dimensional subspace of X. Define P_n to be the orthogonal projection operator of X onto X_n, based on using the inner product of $L^2(\Omega)$, thus

$$\langle P_n u, v\rangle = \langle u, v\rangle, \qquad \text{all } v \in X_n,$$

let $\{\varphi_j\}_{j=1}^N$ be a basis of X_n, the Galerkin method for solving the Urysohn integral equation (3.26) is to seek an approximate solution $u_n \in X_n$ where,

$$u_n(s) = \sum_{j=1}^{N} \alpha_j \varphi_j(s),$$

such that

$$\langle R_n, \varphi_j\rangle = 0, \qquad j = 1, \cdots, N$$

This yields to the non-trivial system

$$\sum_{j=1}^{N} \alpha_j \langle \varphi_j, \varphi_i\rangle = \langle f, \varphi_i\rangle + \langle \int_\Omega K(\cdot, t, \sum_{j=1}^{N} \alpha_j \varphi_j(t))dt, \varphi_i\rangle, \qquad i = 1, \cdots, N. \tag{3.28}$$

Solving this system leads to determine the coefficients $\{\alpha_j\}_{j=1}^N$, therefore we find the resulting function $u_n(s)$ which we hope will be very close to the exact solution $u(s)$.

3.2 The Nyström methods

In this section, we will approximate a class of nonlinear integral equations of the second kind by the so-called quadrature or Nyström method, in which the integral terms is approximated by an ordinary quadrature rule.
Let $Q : X = C[a,b] \to \mathbb{R}$ be an integral operator defined by

$$Q(g) = \int_a^b g(t)dt,$$

and let $Q_n : X \to \mathbb{R}$ be a discrete operator defined by the quadrature rule

$$Q_n(g) = \sum_{j=1}^{n} \omega_j^{(n)} g(x_j^{(n)}), \tag{3.29}$$

the values $\{x_j^{(n)}\}_{j=1}^n$ are called the quadrature nodes and $\{\omega_j^{(n)}\}_{j=1}^n$ are called weights.

Definition 3.3. A sequence of quadrature rules $Q_n(g)$ is called convergent if

$$Q_n(g) \to Q(g) \quad \text{as} \quad n \to \infty, \text{ for all } g \in X,$$

i.e., if the sequence of linear functionals $Q_n(g)$ converges pointwise to the integral $Q(g)$.

Theorem 3.3. [26] The quadrature rules (Q_n) converge if and only if

$$Q_n(g) \to Q(g) \text{ as } n \to \infty, \text{ for all } g \text{ in some dense subset } G \subset C[a,b],$$

and

$$\sup_{n\in\mathbb{N}} \sum_{j=1}^{n} |\omega_j^{(n)}| < \infty.$$

3.2.1 Principle of Nyström methods

Consider the Urysohn integral equation

$$u(s) = f(s) + \int_a^b K(s,t,u(t))dt, \qquad s \in \Omega = [a,b], \tag{3.30}$$

where $f \in X = C(\Omega)$ and $K(s,t,u(t))$ given with an appropriate smoothness assumption, the right-hand side of (3.30) defines a completely continuous operator from some open domain $D \subset X$ into X, explicitly

$$\mathcal{K}(u)(s) = \int_a^b K(s,t,u(t))dt, \qquad s \in \Omega.$$

Thus, solving (3.30) is equivalent to solve the operator equation

$$u = f + \mathcal{K}(u), \tag{3.31}$$

using the quadrature formula (3.29) to approximate the integral in (3.30) and the Nyström method to (3.30) is : Find $u_n(s)$ such that

$$u_n(s) = f(s) + \sum_{j=1}^{n} \omega_j^{(n)} K(s, t_j^{(n)}, u_n(t_j^{(n)})), \qquad s \in \Omega, \tag{3.32}$$

where $u_n(s)$ is an approximation to $u(s)$, we rewrite the approximating numerical integral equation (3.32) in operator notation as

$$u_n = f + \mathcal{K}_n(u_n), \tag{3.33}$$

where the discrete integral operators $\mathcal{K}_n$, $n \geq 1$, is defined by

$$\mathcal{K}_n(u)(s) = \sum_{j=1}^{n} \omega_j^{(n)} K(s, t_j^{(n)}, u(t_j^{(n)})), \qquad s \in \Omega. \tag{3.34}$$

A solution to the functional equation (3.32) may be obtained by determining $\{u_n(s_i^{(n)})\}$, thus (3.32) is reduced to the finite nonlinear system

$$z_i = f(s_i^{(n)}) + \sum_{j=1}^{n} \omega_j^{(n)} K(s_i^{(n)}, t_j^{(n)}, z_j^{(n)}), \qquad i = 1, \cdots, n, \tag{3.35}$$

where the interpolatory function

$$z(s) = f(s) + \sum_{j=1}^{n} \omega_j^{(n)} K(s, t_j^{(n)}, z_j^{(n)}), \qquad s \in \Omega,$$

satisfies (3.32).

The formulas (3.32) and (3.35) are completely equivalent in their resolvability (for more details see[28, 5]), practically, we solve (3.35), however we use (3.32) for the theoretical convergence analysis.

3.2.2 Convergence analysis

In the present section we'll discuss the existence and the convergence of an approximate solution of (3.33) in neighbourhood of an isolated solution of (3.31), where the basic notions for deriving the order of convergence is based on the theory of collectively compact operators (see[13, 2]) for this we shall assume that (3.31) has an isolated solution $u_0 \in X$, e.i., there is some ball

$$B(u_0,\delta) = \{u \in X : \|u-u_0\| \leq \delta,\ \delta > 0\}$$

that contains no solution of (3.31) other than u_0, and the compact operator $\mathcal{K}$ is possesses a continuous first and a bounded second derivative on $B(u_0,\delta)$.
For ease of reference, the following required assumptions are mentioned from [6, 57].

A_1. $\{\mathcal{K}_n : n \geq 1\}$ is collectively compact family on X.

A_2. $\mathcal{K}_n$ is pointwise convergent to $\mathcal{K}$ on X.

A_3. For $n \geq 1$, $\mathcal{K}_n$ possesses a continuous first and a bounded second Fréchet derivatives on $B(u_0,\delta)$. Moreover,

$$\|\mathcal{K}_n''\| \leq \lambda < \infty.$$

Lemma 3.6. (**Weiss** [57]) Assume that $[I - \mathcal{K}'(u_0)]$ is nonsingular and that the hypotheses $A_1 - A_3$ hold. Then the linear operator $[I - \mathcal{K}_n'(u_0)]$ are nonsingular for sufficiently large n, say $n \geq n_1$, and

$$\|[I - \mathcal{K}_n'(u_0)]^{-1}\| \leq \beta < \infty. \tag{3.36}$$

Theorem 3.4. Assume that the assumptions of lemma 3.6 hold. Then there exists a positive integer n_1 such that, for all $n \geq n_1$, (3.33) has a unique solution $u_n \in B(u_0,\delta)$. Furthermore, there exists a constant C independent of n such that

$$\|u_0 - u_n\| \leq C\|\mathcal{K}(u_0) - \mathcal{K}_n(u_0)\|.$$

Proof. By subtracting (3.31) from (3.33) we obtain

$$u_0 - u_n = \mathcal{K}(u_0) - \mathcal{K}_n(u_n),$$

by adding the term $\mathcal{K}_n'(u_0)(u_0 - u_n)$ on both sides we have

$$[I - \mathcal{K}_n'(u_0)](u_0 - u_n) = \mathcal{K}(u_0) - \mathcal{K}_n(u_0) - [\mathcal{K}_n(u_n) - \mathcal{K}_n(u_0) - \mathcal{K}_n'(u_0)(u_n - u_0)].$$

The term $\mathcal{K}_n(u_n) - \mathcal{K}_n(u_0) - \mathcal{K}_n'(u_0)(u_n - u_0)$ has been bounded by the term $\frac{1}{2}\lambda\|u_0 - u_n\|^2$ (see [6]), then from lemma 3.6 we have

$$\|u_0 - u_n\| \le \beta[\|\mathcal{K}(u_0) - \mathcal{K}_n(u_0)\| + \frac{1}{2}\lambda\|u_0 - u_n\|^2],$$

hence

$$\begin{aligned}\|u_0 - u_n\| &\le \frac{\beta\|\mathcal{K}(u_0) - \mathcal{K}_n(u_0)\|}{1 - \frac{\beta\lambda\delta}{2}} \\ &\le \frac{\beta}{1 - \frac{\beta\lambda\delta}{2}}\|\mathcal{K}(u_0) - \mathcal{K}_n(u_0)\|.\end{aligned}$$

This completes the proof. □

Remark 3.1. From this theorem we can conclude that the rate of convergence of u_n to u_0 is that of numerical integration approach applied to $\mathcal{K}(u_0)$ and this is usually simple to get it.

3.3 Sinc approximation methods

The Sinc approximation methods are based primarily on the Cardinal function which plays the same role as polynomials in the classical numerical methods. These techniques of approximation was first considered in [24, 29], nevertheless, Stenger in [51] was the first who explained a detailed and a thorough consideration about the Sinc approximation methods. This Cardinal function which was mentioned earlier is noted by Sinc (x), and it is defined in the whole real line by:

$$\text{Sinc }(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0, \end{cases} \tag{3.37}$$

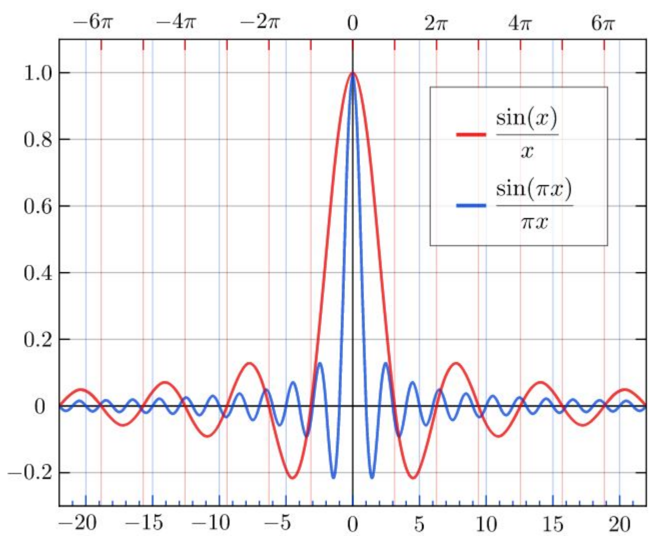

Figure 3.1: The Sinc function.

although, there is more powerful notation for the Sinc function. Namely, if k an element of $\mathbb{Z}$, and $h > 0$, we define the translated Sinc functions with evenly spaced nodes by:

$$S(k,h)(x) = \text{ Sinc }(\frac{x}{h} - k),$$

where $S(k,h)(x)$ will be referred as the K'th Sinc function, with step size h evaluated at x.

we note that

$$S(k,h)(jh) = \text{ Sinc } (\frac{jh}{h} - k) = \text{ Sinc } (j-k)$$
$$= \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

The following properties are easily shown.

(i) Sinc (.) is an even function.

(ii) $\{x \in \mathbb{R} : \text{ Sinc } (x) = 0\} = \mathbb{Z}^*$.

(iii) $\int_{-\infty}^{+\infty} \text{ Sinc } (x)dx = \int_{-\infty}^{+\infty} (\text{ Sinc } (x))^2 dx = 1$.

3.3.1 Exact interpolation and quadrature via Sinc method

Definition 3.4. Let $h > 0$, and let $W(\frac{\pi}{h})$ denote the family of all analytic functions $f \in \mathbb{C}$, such that

$$f \in L^2(\mathbb{R}),$$

and such that for all $z \in \mathbb{C}$

$$|f(z)| \leq Ce^{\pi|z|/h},$$

with C is a positive constant.

Theorem 3.5. (Paley-Wiener theorem) Assume that $f \in W(\frac{\pi}{h})$, Then $\hat{f} \in L^2(\frac{-\pi}{h}, \frac{\pi}{h})$ and

$$f(z) = \frac{1}{2\pi} \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} \hat{f}(x) e^{-ixz} dx.$$

Theorem 3.6. If $f \in W(\frac{\pi}{h})$, Then for all $z \in \mathbb{C}$,

$$f(z) = C(f,h)(z) = \sum_{k=-\infty}^{+\infty} f(kh)S(k,h)(z), \tag{3.38}$$

and

$$f(kh) = \frac{1}{h} \int_{-\infty}^{+\infty} f(t)S(k,h)(t)dt, \tag{3.39}$$

furthermore, if $f \in L^1(\mathbb{R})$,

$$\int_{-\infty}^{+\infty} f(t)dt = h \sum_{k=-\infty}^{+\infty} f(kh). \tag{3.40}$$

Proof. Since $f \in W(\frac{\pi}{h})$, The Theorem (3.5) gives

$$f(z) = \frac{1}{2\pi} \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} \hat{f}(x) e^{-ixz} dx,$$

and we have

$$e^{-ixz} = \sum_{k=-\infty}^{+\infty} e^{ikhx} S(k,h)(z), \quad \frac{-\pi}{h} < x < \frac{\pi}{h},$$

yields

$$\begin{aligned}
f(z) &= \frac{1}{2\pi} \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} \hat{f}(x) \sum_{k=-\infty}^{+\infty} e^{ikhx} S(k,h)(z) dx \\
&= \sum_{k=-\infty}^{+\infty} S(k,h)(z) \left[\frac{1}{2\pi} \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} \hat{f}(x) e^{ikhx} dx \right] \\
&= \sum_{k=-\infty}^{+\infty} f(-kh) S(k,h)(z) \\
&= \sum_{k=-\infty}^{+\infty} f(kh) S(k,h)(z).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi} \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} \hat{f}(x) e^{-ixz} dx \\
&= \frac{1}{2\pi} \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} \left[\int_{-\infty}^{+\infty} f(t) e^{ixt} dt \right] e^{-ixz} dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) \left[\int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} e^{ix(t-z)} dx \right] dt \\
&= \frac{1}{h} \int_{-\infty}^{+\infty} f(t) \left[\frac{-ih}{2\pi(t-z)} \right] \left[e^{(i\pi(t-z)/h)} - e^{(-i\pi(t-z)/h)} \right] dt \\
&= \frac{1}{h} \int_{-\infty}^{\infty} f(t) \text{ sinc } (\frac{t-z}{h}) dt,
\end{aligned}$$

then

$$f(kh) = \frac{1}{h}\int_{-\infty}^{\infty} f(t)S(k,h)(t)dt.$$

Now, for proving the quadrature formula (3.40) we replace z by $t \in \mathbb{R}$ in (3.38) and integrate the result over $\mathbb{R}$,

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)dt &= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{+\infty} f(kh)S(k,h)(t)dt \\ &= \sum_{k=-\infty}^{+\infty} f(kh)\int_{-\infty}^{\infty} S(k,h)(t)dt,\end{aligned}$$

from the equality

$$\begin{aligned}\int_{-\infty}^{\infty} S(k,h)(t)dt &= h\int_{-\infty}^{\infty} \text{Sinc }(s)ds \\ &= h,\end{aligned}$$

we obtain

$$\int_{-\infty}^{\infty} f(t)dt = h\sum_{k=-\infty}^{\infty} f(kh).$$

□

3.3.2 Sinc approximation methods on real-line

Whereas the interpolation and quadrature formulas in the previous section are exact, however, there are some functions don't belongs to $W(\frac{\pi}{h})$ when we approximated it by its Cardinal series has errors that decrease exponentially.
So we need to substitute the class $W(\frac{\pi}{h})$ by class of functions with less constraints where the formulas of theorem (3.5) are fulfilled, nevertheless, because the approximation of f will be on $\mathbb{R}$, it must be assumed that f is analytic in a domain containing $\mathbb{R}$, let D_d be the infinite strip domain of width $2d,\ \ d > 0$, defined by

$$D_d = \{z \notin \mathbb{C} : z = x + iy, |y| < d\}. \tag{3.41}$$

3.3.2.1 Infinite term of Sinc approximation on real-line

Definition 3.5. Let $H^1(\mathcal{D}_d)$ denote the family of all functions f that are analytic in $\mathcal{D}_d$, such that

$$N(f,\mathcal{D}_d) = \left(\int_{\partial \mathcal{D}_d} |f(z)||dz| \right) < \infty.$$

Let us use the notation $\|f\|$ to denote $\sup_{x\in\mathbb{R}} |f(x)|$.

Theorem 3.7. [54] Let $f \in H^1(\mathcal{D}_d)$, Then

$$\|f - C(f,h)\| \leq Ce^{-\pi d/h} = \mathcal{O}(e^{-\pi d/h}). \tag{3.42}$$

Moreover

$$\left| \int_{-\infty}^{\infty} f(x)dx - h \sum_{k=-\infty}^{\infty} f(kh) \right| \leq C^* e^{-2\pi d/h} = \mathcal{O}(e^{-2\pi d/h}), \tag{3.43}$$

where C and C^* are constants.

3.3.2.2 Finite term of Sinc approximation on real-line

although the infinite Sinc approximation may be very accurate, as stated in theorem 3.7, there might still be a problem from the points of view of numerical computation. So, these functions must be identified as analytical functions in $\mathcal{D}_d$, for which we can get a close approximation of f via

$$C_N(f,h)(x) = \sum_{k=-N}^{N} f(kh)S(k,h)(x), \qquad \text{for relatively small } N.$$

Definition 3.6. Let $\alpha > 0$ and let $\mathcal{L}_\alpha(\mathcal{D}_d)$ denote the set of all functions f analytic in $\mathcal{D}_d$, such that for some constant $C > 0$, and all $z \in \mathcal{D}_d$, we have

$$|f(z)| \leq C \frac{|e^{\alpha z}|}{(1+|e^z|)^{2\alpha}},$$

and for all $x \in \mathbb{R}$

$$|f(x)| \leq Ce^{-\alpha|x|}.$$

Theorem 3.8. [53] Let $f \in \mathcal{L}_\alpha(\mathcal{D}_d)$, then taking

$$h = \left(\frac{\pi d}{\alpha N}\right)^{1/2}, h^* = \left(\frac{2\pi d}{\alpha N}\right)^{1/2},$$

there exist a positive constants C and C^* depending only on f, d and α, such that

a)
$$\|f - C_N(f,h)\| \le C\sqrt{N}e^{-\sqrt{\pi d \alpha N}}, \tag{3.44}$$

b)
$$\left|\int_{-\infty}^{\infty} f(x)dx - h^* \sum_{k=-N}^{N} f(kh^*)\right| \le C^* e^{-\sqrt{2\pi d \alpha N}}. \tag{3.45}$$

3.3.3 Sinc approximation methods on arcs Γ

In this section we'll expand the results for approximation on $\mathbb{R}$ to approximate functions over infinite, semi infinite, and finite intervals, in fact, over arcs Γ.
For this purpose a conformal maps ϕ that transform Γ to $\mathbb{R}$ are desirable.

Definition 3.7. Let $\mathcal{D}$ be a domain in $\mathbb{C}$ with boundary points $a \neq b$. Let ϕ denote a conformal map of $\mathcal{D}$ into $\mathcal{D}_d$ such that, $\phi(a) = -\infty$ and $\phi(b) = \infty$.
Denote by $w = \psi(z)$ the inverse of mapping ϕ and let

$$\Gamma = \{w \in \mathbb{C} : w = \psi(x), x \in \mathbb{R}\} = \psi(\mathbb{R}).$$

Let $H^1(\mathcal{D})$ denote the set of all functions f that are analytic in $\mathcal{D}$, such that

$$N(f,\mathcal{D}) \equiv \int_{\partial\mathcal{D}} |f(z)||dz| < \infty,$$

and let $L_\alpha(\mathcal{D})$ denote the set of all functions f analytic in $\mathcal{D}$, such that for some constant $C > 0$, and all $z \in \mathcal{D}$, we have

$$|f(z)| \le C \frac{|e^{\phi(z)}|^\alpha}{\left(1 + |e^{\phi(z)}|\right)^{2\alpha}}.$$

As we have see earlier the assumption that $f \in H^1(\mathcal{D}_d)$ guarantees us the rapid convergence of Sinc approximation on $\mathbb{R}$, the same convergence rate, is reachable under more general conditions, to accomplish an accurate interpolation (respectively, quadrature) on the arcs Γ, all that is required is that $\psi' f$ (respectively, f) is belongs to $H^1(\mathcal{D})$, because if $f \in H^1(\mathcal{D}_d)$, then

$$\int_{\partial \mathcal{D}_d} |f(u)||du| < \infty,$$

this means that

$$\int_{\partial \mathcal{D}} |f(\psi(z))||\psi'(z)dz| < \infty,$$

that it is necessary to have $f\psi' \in H^1(\mathcal{D})$.

Theorem 3.9. [53] Assume that $f\psi' \in L_\alpha(\psi(\mathcal{D}_d))$ for d with $0 < d < \frac{\pi}{2}$, let N be a positive integer, and

$$h = \left(\frac{\pi d}{\alpha N}\right)^{1/2}, h^* = \left(\frac{2\pi d}{\alpha n}\right)^{1/2},$$

then there exist constants C and C^*, independent of N, such that

a)

$$\|f - \sum_{k=-N}^{N} f(x_k) S(k,h) \circ \phi\| \le C\sqrt{N} e^{-\sqrt{d\pi\alpha N}}.$$

b)

$$\left| \int_\Gamma f(x)dx - h^* \sum_{k=-N}^{N} f(x_k)\psi'(kh) \right| \le C^* e^{-\sqrt{2\pi\alpha d N}}.$$

Example 3.3. The case of $\Gamma = (a,b)$, where $-\infty < a < b < \infty$.
In this case we take

$$w = \phi(z) = \ln \frac{z-a}{b-z},$$

hence

$$\psi(w) = \frac{a + be^w}{1 + e^w}.$$

The conformal map ϕ transforms the complex domain

$$\mathcal{D} = \left\{ z = x + iy : \left| \arg \frac{z-a}{b-z} \right| < d < \frac{\pi}{2} \right\},$$

into the infinite strip

$$\mathcal{D}_d = \left\{ w = \alpha + i\beta : |\beta| < d < \frac{\pi}{2} \right\}.$$

the basic functions in the interval (a, b) defined by

$$S(k,h)\circ\phi(x) = \begin{cases} \frac{\sin[\pi(\phi(x)-kh)/h]}{\pi(\phi(x)-kh)/h}, & \phi(x)\neq kh, \\ 1, & \phi(x)=kh, \end{cases}$$
$$= \text{Sinc}\,[(\phi(x)-kh)/h]$$

Since

$$S(k,h)(jh) = \delta_{kj} = \begin{cases} 1, & k=j, \\ 0, & k\neq j. \end{cases}$$

Then, the nodes of the function $S(k,h)\circ\phi(x)$ are:

$$x_k = \phi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}},$$

and the finite approximation of interpolation and quadrature for a function $f(x)$ on $[a, b]$ are defined by

$$f(x) \approx \sum_{k=-N}^{N} f(x_k)S(k,h)\circ \ln\frac{x-a}{b-x},$$

$$\int_a^b f(x)dx \approx h(b-a)\sum_{k=-N}^{N}\frac{f(x_k)e^{kh}}{(1+e^{kh})^2}.$$

Numerical test

We wish to approximate the function g on the interval $(0, 1)$ where

$$g(x) = \frac{1}{\sqrt{x^2+1}},$$

the numerical results of the approximation of quadrature and interpolation for the function g on $(0, 1)$ are represented in Table 3.1 and Figure 3.2.

Table 3.1: Absolute error between the exact and the Sinc Approximate integration of g

N	h	errors
10	9.935e-01	4.8473e-05
20	7.025e-01	9.3278e-07
30	5.736e-01	4.2550e-08
40	4.967e-01	3.0967e-09
50	4.443e-01	3.0548e-10
60	4.056e-01	3.0548e-10
70	3.755e-01	5.2651e-12
80	3.512e-01	9.9476e-13
90	3.312e-01	3.979e-13
100	3.142e-01	5.6843e-14

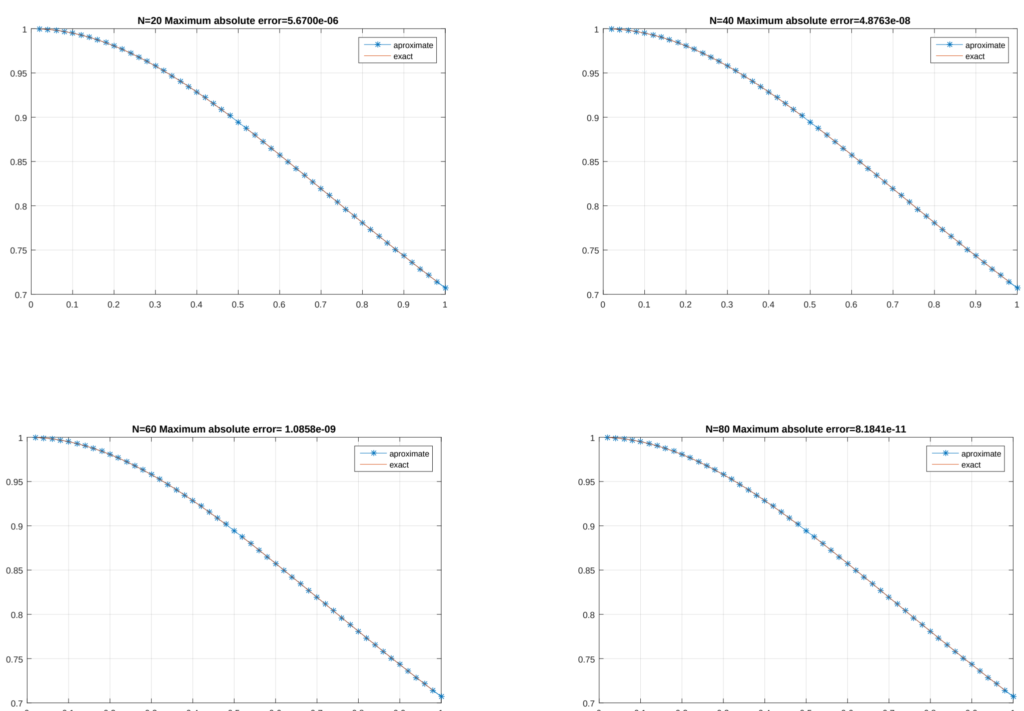

Figure 3.2: Sinc approximate interpolation of g.

Example 3.4. The case $\Gamma = [0,\infty)$
We take

$$w = \phi(z) = \log(z),$$

hence

$$\psi(w) = e^{w}.$$

The conformal map ϕ transforms the complex domain

$$\mathcal{D} = \{z \in \mathbb{C} : |\arg(z)| < d\},$$

into the infinite strip

$$\mathcal{D}_d = \left\{ w = \alpha + i\beta : |\beta| < d < \frac{\pi}{2} \right\}.$$

The nodes of the function $S(k,h) \circ \phi(x)$ are:

$$x_k = \phi^{-1}(kh) = e^{kh},$$

and the finite approximation of interpolation and quadrature for a function $f(x)$ on $[0,\infty)$ are defined by

$$f(x) \approx \sum_{k=-N}^{N} f(x_k) S(k,h) \circ \log(x),$$

$$\int_0^{\infty} f(x)dx \approx h \sum_{k=-N}^{N} f(x_k) e^{kh}.$$

Numerical test

We wish to approximate the function f on the interval $[0,\infty)$ where

$$f(x) = \frac{1}{x^3+1},$$

the numerical results of the approximation of quadrature and interpolation for the function f on $(0,1)$ are represented in Table 3.2 and Figure 3.3.

Table 3.2: Absolute error between the exact and the Sinc Approximate integration of f

N	h	errors
10	9.935e-01	4.8005e-03
20	7.025e-01	3.0982e-04
30	5.736e-01	3.7787e-05
40	4.967e-01	6.4089e-06
50	4.4443e-01	1.3422e-06
60	4.065e-01	3.2657e-07
70	3.755e-01	8.9018e-08
80	3.512e-01	2.6549e-08
90	3.312e-01	8.5220e-09
100	3.114e-01	2.9092e-09

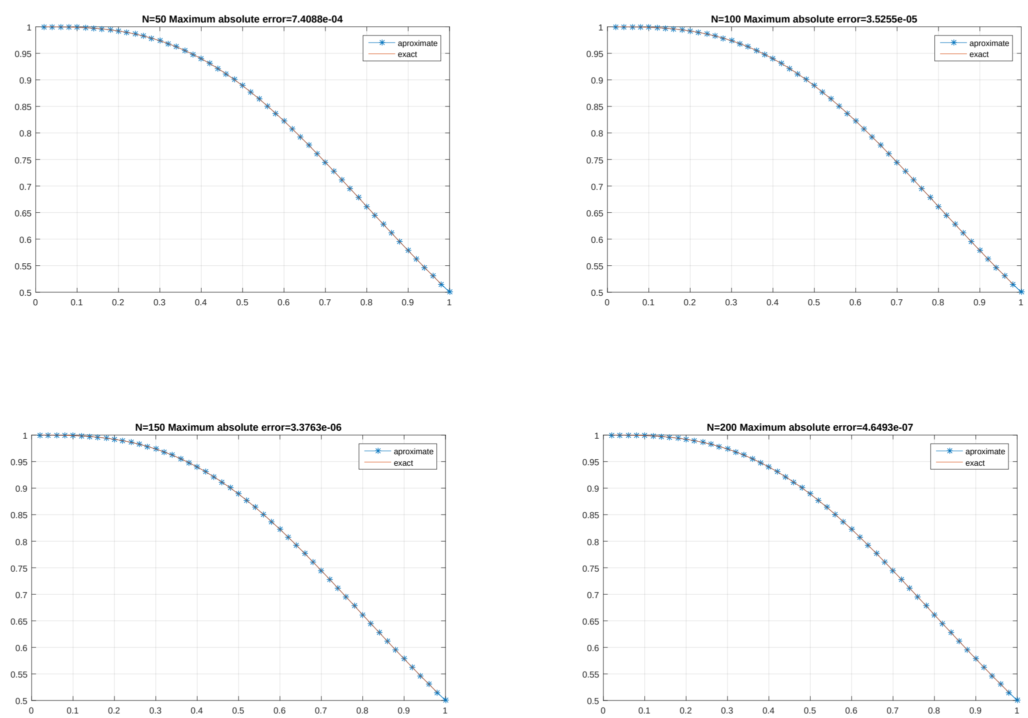

Figure 3.3: Sinc approximate interpolation of f.

In our work and as we will see in the next chapter, we're particularly interested to explain Sinc-approximations in the semi infinite interval $[0,\infty)$.
In this interval, we can adapt the Sinc approximation with the aid of the following useful " Single-Exponential transformations " given by Stenger [53] (denoted briefly by SE):

$$\psi_{SE_1}(t) = e^t, \quad \psi_{SE_2}(t) = \operatorname{arcsinh}(e^t),$$

or by ones of the following transformations

$$\psi_{DE_1}(t) = \psi_{SE_1}(\frac{\pi}{2}\sinh t), \quad \psi_{DE_2}(t) = \psi_{SE_2}(\frac{\pi}{2}\sinh t), \quad \psi_{DE_3}(t) = e^{t-exp(-t)}.$$

These are known as " Double-Exponential transformations " (DE), where Takahasi and Mori was the first to introduce them [55, 56]. The following theorems show the exponential convergence of the SE-Sinc approximations and the DE-Sinc approximations.

Theorem 3.10. [53] The following is true for $i = 1, 2$. Assume that $f\psi' \in L_\alpha(\psi_{SE_i}(\mathcal{D}_d))$ for d with $0 < d < \pi/2$. Then, there exists a constant C, independent of N, such that

$$\left|\int_0^\infty f(t)dt - h\sum_{k=-N}^{N} f(\psi_{SE_i}(kh))\psi'_{SE_i}(kh)\right| \leq Ce^{-\sqrt{2\pi\alpha d N}}, \tag{3.46}$$

where $h = \sqrt{\frac{2\pi d}{\alpha N}}$.

Theorem 3.11. [56] The following is true for $i = 1, 2, 3$. Assume that $f\psi' \in L_\alpha(\psi_{DE_i}(\mathcal{D}_d))$ for d with $0 < d < \pi/2$. Then, there exists a constant C, independent of N, such that

$$\left|\int_0^\infty f(t)dt - h\sum_{k=-N}^{N} f(\psi_{DE_i}(kh))\psi'_{DE_i}(kh)\right| \leq C\exp\left(\frac{-2\pi d N}{\log(8dN/\alpha)}\right), \tag{3.47}$$

where $h = \frac{\log(8dN/\alpha)}{N}$.

CHAPTER 4

RESOLUTION OF NONLINEAR INTEGRAL EQUATIONS

In this chapter we try to apply some of methods derived from those that we saw earlier. In the first section, we presented a Jacobi spectral collocation method to solve nonlinear Volterra-Fredholm integral equations with smooth kernels, the main idea in this approach is to convert the original problem into an equivalent one through an appropriate variable transformations, so that the resulting equation can be accurately solved by using spectral collocation at the Jacobi-Gauss points. The convergence and errors analysis are discussed for both L^∞ and weighted L^2 norms, and we confirm the theoretical prediction of the exponential rate of convergence by the numerical results which are compared with well known methods. In the second section, we proposed two numerical methods to solve the nonlinear integral equation of Hammerstein type on the half-line, by using a Sinc-Nyström method based on Single-Exponential (SE) and Double-Exponential (DE) transformations, where the problem is converted into a nonlinear system of equations. We provided an error analysis of the proposed schemes and showed that these methods have exponential convergence rates. Finally, several numerical examples are given to show the effectiveness of the present methods.

4.1 Solving nonlinear Volterra-Fredholm integral equation by using a Jacobi spectral collocation method

We consider the nonlinear Volterra-Fredholm integral equations given by the general form:

$$u(s) = g(s) + \int_0^s V(s,t)\psi_1(u(t))dt + \int_0^1 F(s,t)\psi_2(u(t))dt, \qquad s \in [0,1], \tag{4.1}$$

where the kernels V, F, and g, ψ_1, ψ_2, are given smooth functions about their variables, and $u(s)$ is the unknown function to be determined.
the present section is the subject of our research which was published in [19] where we developed an accurate spectral method based on useful generating set of orthogonal polynomials to solve equation (4.1). Evidently, at first we must transform the problem set in the given interval to $[-1,1]$ by means of an appropriate variable transformation. Then, we apply the standard spectral Jacobi-collocation method to the resulting equation in which Jacobi-Gauss points are utilized together with Legendre-Gauss quadrature to reduce it to the solution of nonlinear equations.

4.1.1 Description of the method

Before proceeding to discuss the Jacobi spectral collocation method for Eq. (4.1), we need to introduce the following notations, let $I = [-1,1]$, and $w^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ be a weight function, for $\alpha, \beta > -1$. For a given positive integer N, we denote by $\{\sigma_{N,j}^{\alpha,\beta}\}_{j=0}^N$ the points of Gauss-Jacobi quadrature formula, which are the roots of Jacobi polynomials $J_{N+1}^{\alpha,\beta}$.
For convenience, we consider the following variable transformations

$$s = \frac{x+1}{2}, \quad t = \frac{y+1}{2}, \ x, y \in I,$$

so that Eq. (4.1) becomes

$$U(x) = G(x) + \int_{-1}^x \overline{V}(x,y)\psi_1(U(y))dy + \int_{-1}^1 \overline{F}(x,y)\psi_2(U(y))dy, \tag{4.2}$$

where

$$U(x) = u\left(\frac{x+1}{2}\right),$$
$$G(x) = g\left(\frac{x+1}{2}\right),$$
$$\overline{V}(x,y) = \frac{1}{2}V\left(\frac{x+1}{2}, \frac{y+1}{2}\right),$$
$$\overline{F}(x,y) = \frac{1}{2}F\left(\frac{x+1}{2}, \frac{y+1}{2}\right).$$

Then we introduce the linear transformation

$$y = y(x,\theta) = \frac{x+1}{2}\theta + \frac{x-1}{2}, \quad x,\theta \in \Omega,$$

which transfers the first integral term in (4.2) to the form

$$\int_{-1}^{x} \overline{V}(x,y)\psi_1(U(y))dy = \frac{x+1}{2}\int_{-1}^{1} \overline{V}(x,y(x,\theta))\psi_1\left(U(y(x,\theta))\right)d\theta.$$

Hence, Eq. (4.2) becomes

$$U(x) = G(x) + \int_{-1}^{1} \widetilde{V}(x,\theta)\psi_1(U(y(x,\theta)))d\theta + \int_{-1}^{1} \widetilde{F}(x,\theta)\psi_2(U(\theta))d\theta, \tag{4.3}$$

where

$$\widetilde{V}(x,\theta) = \frac{x+1}{2}\overline{V}(x,y(x,\theta)), \quad \widetilde{F}(x,\theta) = \overline{F}(x,\theta).$$

The spectral Jacobi-collocation method for solving Eq. (4.1) is to seek an approximate solution $U_N(x) \in \mathcal{P}_N$, such that $U_N(x)$ satisfies Eq. (4.3) at the collocation points $\sigma_{N,j}^{\alpha,\beta}$, i.e.,

$$U_N(\sigma_{N,j}^{\alpha,\beta}) = G(\sigma_{N,j}^{\alpha,\beta}) + \int_{-1}^{1} \widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\theta)\psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta)))d\theta + \int_{-1}^{1} \widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\theta)\psi_2(U_N(\theta))d\theta, \tag{4.4}$$

where $y_{N,j}^{\alpha,\beta}(\theta) := y(\sigma_{N,j}^{\alpha,\beta},\theta)$. The integral terms in the above equation can be accurately approximated by using Legendre-Gauss quadrature formula. Let $\{\theta_k,\omega_k\}_{k=0}^{N}$ be the

Legendre-Gauss nodes and weights, and there holds

$$\int_{-1}^{1} \widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\theta)\psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta))d\theta \sim \sum_{k=0}^{N}\omega_k\widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\theta_k)\psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta_k))), \tag{4.5}$$

$$\int_{-1}^{1} \widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\theta)\psi_2(U_N(\theta))d\theta \sim \sum_{k=0}^{N}\omega_k\widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\theta_k)\psi_2(U_N(\theta_k)). \tag{4.6}$$

Next, by using the Lagrange interpolation to approximate the nonlinear parts in (4.5) and (4.6), namely,

$$\psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta))) \sim I_N^{\alpha,\beta}(\psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta)))) = \sum_{j=0}^{N}\psi_1(\hat{U}_{N,j}^{\alpha,\beta})L_j(y_{N,j}^{\alpha,\beta}(\theta)),$$

$$\psi_2(U_N(\theta)) \sim I_N^{\alpha,\beta}(\psi_2(U_N(\theta))) = \sum_{j=0}^{N}\psi_2(\hat{U}_{N,j}^{\alpha,\beta})L_j(\theta),$$

where $\hat{U}_{N,j}^{\alpha,\beta} := U_N(\sigma_{N,j}^{\alpha,\beta})$, we get the full collocation scheme

$$\hat{U}_{N,j}^{\alpha,\beta} = G(\sigma_{N,j}^{\alpha,\beta}) + \sum_{k=0}^{N}\omega_k\widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\theta_k)I_N^{\alpha,\beta}(\psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta_k)))) + \sum_{k=0}^{N}\omega_k\widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\theta_k)I_N^{\alpha,\beta}(\psi_2(U_N(\theta_k))). \tag{4.7}$$

We can get the values of $\hat{U}_{N,j}^{\alpha,\beta}, j = 0,1,\ldots,N$, by solving (4.7).
An approximate solution of Eq. (4.2) will be given by:

$$U(x) \sim U_N(x) = \sum_{j=0}^{N}\hat{U}_{N,j}^{\alpha,\beta}L_j(x),$$

since the exact solution of the original equation (4.1) can be written as

$$u(s) = U(x), \text{ where } x = 2s-1, \quad s,x \in I.$$

4.1.2 Error estimates

From all the following, we assume that the nonlinear functions ψ_1 and ψ_2 and all its derivatives up to order m satisfy a local Lipschitz condition, such that for every v_1, $v_2 \in C(I)$

$$\left|\frac{\partial^k}{\partial y^k}\psi_i(v_1) - \frac{\partial^k}{\partial y^k}\psi_i(v_2)\right| \le L_{ik}|v_1 - v_2|, \quad i = 1, 2. \quad k = 1, 2, \cdots, m,$$

we also consider that

$$L = \max_{1\le k\le m} L_{ik}.$$

Error estimate in L^∞-norm

Theorem 4.1. Let U be the exact solution of the nonlinear Volterra-Fredholm integral equation (4.2), which is assumed to be sufficiently smooth. Let the approximated solution U_N be obtained by using the spectral scheme (4.7), assume that in Eq. (4.1) the nonlinear functions ψ_1 and ψ_2 and all its derivatives up to order m satisfy a local Lipschitz condition. Then there is a positive constant C such that the errors satisfy for $m \ge 1$,

$$\|U - U_N\|_{L^\infty} \le CN^{\frac{1}{2}-m}(|U|_{H_w^m(I)} + \chi\|I_N^{\alpha,\beta}\|_{L^\infty}), \tag{4.8}$$

where

$$\chi = \|\widetilde{V}\|_{L^\infty}|\psi_1(U)|_{H_w^{m,N}(I)} + \|\widetilde{F}\|_{L^\infty}|\psi_2(U)|_{H_w^{m,N}(I)} + N^{-\frac{1}{2}}(\vartheta_{m,N}^{\alpha,\beta}\|\psi_1(U)\|_{L^\infty} + \tau_{m,N}^{\alpha,\beta}\|\psi_2(U)\|_{L^\infty}),$$

$$\vartheta_{m,N}^{\alpha,\beta} = \max_{0\le j\le N}\left|\widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\cdot)\right|_{H_w^{m,N}(I)}, \quad \tau_{m,N}^{\alpha,\beta} = \max_{0\le j\le N}\left|\widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\theta)\right|_{H_w^{m,N}(I)}.$$

Proof. At the collocation points $x = \sigma_{N,j}^{\alpha,\beta}$, we have $U_N(\sigma_{N,j}^{\alpha,\beta}) = \hat{U}_{N,j}^{\alpha,\beta}$. We subtract (4.7) from (4.3) we obtain

$$\begin{aligned} U(\sigma_{N,j}^{\alpha,\beta}) - U_N(\sigma_{N,j}^{\alpha,\beta}) &= \langle \widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\theta), \psi_1(U(y_{N,j}^{\alpha,\beta}(\theta)))\rangle \\ &\quad + \langle \widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\theta), \psi_2(U(\theta))\rangle \\ &\quad - \langle \widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\cdot), I_N^{\alpha,\beta}\psi_1(U_N(y_{N,j}^{\alpha,\beta}(\cdot)))\rangle_N \\ &\quad - \langle \widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\cdot), I_N^{\alpha,\beta}\psi_2(U_N(\cdot))\rangle_N \\ &= \langle \widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\theta), \psi_1(U(y_{N,j}^{\alpha,\beta}(\theta))) - \psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta)))\rangle \\ &\quad + \langle \widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\theta), \psi_2(U(\theta)) - \psi_2(U_N(\theta))\rangle + J(\sigma_{N,j}^{\alpha,\beta}). \end{aligned}$$

where

$$\begin{aligned} J(\sigma_{N,j}^{\alpha,\beta}) &= \langle \widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\theta), \psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta)))\rangle \\ &\quad - \langle \widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\cdot), I_N^{\alpha,\beta}\psi_1(U_N(y_{N,j}^{\alpha,\beta}(\cdot)))\rangle_N \\ &\quad + \langle \widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\theta), \psi_2(U_N(\theta))\rangle - \langle \widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\cdot), I_N^{\alpha,\beta}\psi_2(U_N(\cdot))\rangle_N. \end{aligned} \tag{4.9}$$

Let the error function be written as

$$E(x) = U(x) - U_N(x),$$

then

$$\begin{aligned} E(\sigma_{N,j}^{\alpha,\beta}) &= \langle \widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\theta), \psi_1(U(y_{N,j}^{\alpha,\beta}(\theta))) - \psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta)))\rangle \\ &\quad + \langle \widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\theta), \psi_2(U(\theta)) - \psi_2(U_N(\theta))\rangle + J(\sigma_{N,j}^{\alpha,\beta}), \end{aligned} \tag{4.10}$$

multiplying $L_j(x)$ on both sides of (4.10) and summing up from $j = 0$ to N, yield

$$I_N^{\alpha,\beta}(U(x) - U_N(x)) = I_N^{\alpha,\beta} Q_1(x) + I_N^{\alpha,\beta} Q_2(x) + \sum_{j=0}^{N} J(\sigma_{N,j}^{\alpha,\beta}) L_j(x), \tag{4.11}$$

where

$$Q_1(x) = \int_{-1}^{x} \widetilde{V}(x,y)[\psi_1(U(y)) - \psi_1(U_N(y))]\,dy,$$

and

$$Q_2(x) = \int_{-1}^{1} \widetilde{F}(x,y)[\psi_2(U(y)) - \psi_2(U_N(y))]\,dy.$$

By adding and subtracting $Q_1(x)$ and $U(x)$ into the right-hand side of (4.11) we get

$$\begin{aligned}
|E(x)| &\le \int_{-1}^{x} |\widetilde{V}(x,y)[\psi_1(U(y)) - \psi_1(U_N(y))]|dy + |U(x) - I_N^{\alpha,\beta}(U(x))| \\
&\quad + |Q_1(x) - I_N^{\alpha,\beta} Q_1(x)| + |I_N^{\alpha,\beta} Q_2(x)| + |\sum_{j=0}^{N} J(\sigma_{N,j}^{\alpha,\beta}) L_j(x)| \\
&\le L \max_{(x,y)\in I\times I} |\widetilde{V}(x,y)| \int_{-1}^{x} |E(y)|dy + \sum_{j=1}^{4} I_j(x), \qquad (4.12)
\end{aligned}$$

where

$$I_1(x) = |U(x) - I_N^{\alpha,\beta}(U(x))|, \qquad (4.13)$$

$$I_2(x) = \sum_{j=0}^{N} |J(\sigma_{N,j}^{\alpha,\beta}) L_j(x)|, \qquad (4.14)$$

$$I_3(x) = |Q_1(x) - I_N^{\alpha,\beta} Q_1(x)|, \qquad (4.15)$$

$$I_4(x) = |I_N^{\alpha,\beta} Q_2(x)|. \qquad (4.16)$$

Then the Gronwall inequality (1.16) gives

$$\|E\|_{L^\infty} \le C(\|I_1\|_{L^\infty} + \|I_2\|_{L^\infty} + \|I_3\|_{L^\infty} + \|I_4\|_{L^\infty}). \qquad (4.17)$$

Due to (3.8) we have

$$\|I_1\|_{L^\infty} = \|U - I_N^{\alpha,\beta} U\|_{L^\infty} \le C N^{\frac{1}{2}-m} |U|_{H_w^{m,N}(I)}. \qquad (4.18)$$

From (4.9) we have

$$\begin{aligned}
|J(\sigma_{N,j}^{\alpha,\beta})| &\le \\
&|\langle \widetilde{V}(\sigma_{N,j}^{\alpha,\beta}, \theta), \psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta)))\rangle - \langle \widetilde{V}(\sigma_{N,j}^{\alpha,\beta}, \cdot), I_N^{\alpha,\beta} \psi_1(U_N(y_{N,j}^{\alpha,\beta}(\cdot))\rangle_N| \\
&\quad + |\langle \widetilde{F}(\sigma_{N,j}^{\alpha,\beta}, \theta), \psi_2(U_N(\theta))\rangle - \langle \widetilde{F}(\sigma_{N,j}^{\alpha,\beta}, \cdot), I_N^{\alpha,\beta} \psi_2(U_N(\cdot))\rangle_N|.
\end{aligned}$$

Thus, the right hand side of the above inequality is less or equal to

$$
\begin{aligned}
&|\langle \widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\theta), \psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta))) - I_N^{\alpha,\beta}\psi_1(U_N(y_{N,j}^{\alpha,\beta}(\cdot)))\rangle| \\
&+ |\langle \widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\theta), I_N^{\alpha,\beta}\psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta)))\rangle - \langle \widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\cdot), I_N^{\alpha,\beta}\psi_1(U_N(y_{N,j}^{\alpha,\beta}(\cdot)))\rangle_N| \\
&+ |\langle \widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\theta), \psi_2(U_N(\theta)) - I_N^{\alpha,\beta}\psi_2(U_N(\theta))\rangle| \\
&+ |\langle \widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\theta), I_N^{\alpha,\beta}\psi_2(U_N(\theta))\rangle - \langle \widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\cdot), I_N^{\alpha,\beta}\psi_2(U_N(\cdot))\rangle_N|,
\end{aligned}
$$

hence by using (1.12) we have

$$
\begin{aligned}
\max_{0\le j\le N} |J(\sigma_{N,j}^{\alpha,\beta})| \le{}& 2\|\widetilde{V}\|_{L^\infty}\|\psi_1(U_N) - I_N^{\alpha,\beta}\psi_1(U_N)\|_{L^\infty} \\
&+ CN^{-m}\max_{0\le j\le N}|\widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\cdot)|_{H_w^m(I)}\max_{0\le j\le N}\|I_N^{\alpha,\beta}\psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta)))\|_{w^{\alpha,\beta}} \\
&+ 2\|\widetilde{F}\|_{L^\infty}\|\psi_2(U_N) - I_N^{\alpha,\beta}\psi_2(U_N)\|_{L^\infty} \\
&+ CN^{-m}\max_{0\le j\le N}|\widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\cdot)|_{H_w^m(I)}\|I_N^{\alpha,\beta}\psi_2(U_N)\|_{w^{\alpha,\beta}}.
\end{aligned}
$$

From (3.8) we obtain

$$
\begin{aligned}
\max_{0\le j\le N} |J(\sigma_{N,j}^{\alpha,\beta})| \le{}& CN^{\frac{1}{2}-m}\|\widetilde{V}\|_{L^\infty}|\psi_1(U_N)|_{H_w^{m,N}(I)} \\
&+ CN^{-m}\max_{0\le j\le N}|\widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\cdot)|_{H_w^{m,N}(I)}\max_{0\le j\le N}\|\psi_1(U_N(y_{N,j}^{\alpha,\beta}(\theta)))\|_{L^\infty} \\
&+ CN^{\frac{1}{2}-m}\|\widetilde{F}\|_{L^\infty}|\psi_2(U_N)|_{H_w^{m,N}(I)} \\
&+ CN^{-m}\max_{0\le j\le N}|\widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\cdot)|_{H_w^{m,N}(I)}\|\psi_2(U_N)\|_{L^\infty}.
\end{aligned} \tag{4.19}
$$

From the definition of semi-norm (1.11) we have

$$
\begin{aligned}
|\psi_1(U_N)|_{H_w^m(I)} &= |\psi_1(U_N) - \psi_1(U) + \psi_1(U)|_{H_w^{m,N}(I)} \\
&\le |\psi_1(U) - \psi_1(U_N)|_{H_w^{m,N}(I)} + |\psi_1(U)|_{H_w^{m,N}(I)} \\
&\le \left(\sum_{k=\min(m,N+1)}^{m}\left\|\frac{\partial^k\psi_1}{\partial y^k}(U-U_N)\right\|_{w^{\alpha,\beta}}^2\right)^{1/2} + |\psi_1(U)|_{H_w^{m,N}(I)}.
\end{aligned}
$$

Since the nonlinear function ψ_1 and its derivatives of orders $1,\dots,m$ satisfy the Lipschitz condition, we have

$$\left\| \frac{\partial^k \psi_1}{\partial y^k}(U - U_N) \right\|^2_{w^{\alpha,\beta}} \leq L \|(U - U_N)\|^2_{w^{\alpha,\beta}}, \quad \min(m, N+1) \leq k \leq m,$$

hence

$$\begin{aligned} |\psi_1(U_N)|_{H^{m,N}_w(I)} &\leq L' \|E\|_{w^{\alpha,\beta}} + |\psi_1(U)|_{H^m_w(I)} \\ &\leq L'' \|E\|_{L^\infty} + |\psi_1(U)|_{H^{m,N}_w(I)}, \end{aligned} \tag{4.20}$$

where L' and L'' are positive constants.
Similarly, we can obtain

$$\begin{aligned} |\psi_2(U_N)|_{H^m_w(I)} &\leq L' \|E\|_{w^{\alpha,\beta}} + |\psi_2(U)|_{H^{m,N}_w(I)} \\ &\leq L'' \|E\|_{L^\infty} + |\psi_2(U)|_{H^{m,N}_w(I)}. \end{aligned} \tag{4.21}$$

Therefore, combining (4.19), (4.20), (4.21), with (3.10) yields

$$\begin{aligned} \|I_2\|_{L^\infty} = \| \sum_{j=0}^{N} |J(\sigma^{\alpha,\beta}_{N,j})| L_j(x) \|_{L^\infty} &\leq \max_{0 \leq j \leq N} |J(\sigma^{\alpha,\beta}_{N,j})| \| I^{\alpha,\beta}_N \|_{L^\infty} \\ &\leq C N^{\frac{1}{2}-m} \|\widetilde{V}\|_{L^\infty} (L'' \|E\|_{L^\infty} + |\psi_1(U)|_{H^{m,N}_w(I)}) \| I^{\alpha,\beta}_N \|_{L^\infty} \\ &+ C N^{-m} \max_{0 \leq j \leq N} |\widetilde{V}(\sigma^{\alpha,\beta}_{N,j}, \cdot)|_{H^{m,N}_w(I)} (L \|E\|_{L^\infty} + \|\psi_1(U)\|_{L^\infty}) \| I^{\alpha,\beta}_N \|_{L^\infty} \\ &+ C N^{\frac{1}{2}-m} \|\widetilde{F}\|_{L^\infty} (L'' \|E\|_{L^\infty} + |\psi_2(U)|_{H^{m,N}_w(I)}) \| I^{\alpha,\beta}_N \|_{L^\infty} \\ &+ C N^{-m} \max_{0 \leq j \leq N} |\widetilde{F}(\sigma^{\alpha,\beta}_{N,j}, \cdot)|_{H^{m,N}_w(I)} (L \|E\|_{L^\infty} + \|\psi_2(U)\|_{L^\infty}) \| I^{\alpha,\beta}_N \|_{L^\infty}. \end{aligned} \tag{4.22}$$

Let $\mathcal{A}_v$ be the operator be defined in Lemma 1.4 and we consider $v(x) = \psi_1(U(x)) -$

$\psi_1(U_N(x))$, from (1.13) and (1.14) we obtain

$$\begin{aligned}
\|I_3\|_{L^\infty} &= \|\mathcal{A}_v(I - I_N^{\alpha,\beta})\|_{L^\infty} \\
&= \|(\mathcal{A}_v - T_N\mathcal{A}_v)(I - I_N^{\alpha,\beta})\|_{L^\infty} \\
&\le \|\mathcal{A}_v - T_N\mathcal{A}_v\|_{L^\infty}\|I - I_N^{\alpha,\beta}\|_{L^\infty} \\
&\le C_{0,k}N^{-k}\|\mathcal{A}_v\|_{0,k}(1 + \|I_N^{\alpha,\beta}\|_{L^\infty}) \\
&\le C_{0,k}N^{-k}\|v\|_{L^\infty}(1 + \|I_N^{\alpha,\beta}\|_{L^\infty}) \\
&\le C_{0,k}N^{-k}\|\psi_1(U(x)) - \psi_1(U_N(x))\|_{L^\infty}(1 + \|I_N^{\alpha,\beta}\|_{L^\infty}) \\
&\le C_{0,k}LN^{-k}\|E\|_{L^\infty}(1 + \|I_N^{\alpha,\beta}\|_{L^\infty}). \qquad (4.23)
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\|I_4\|_{L^\infty} &= \|I_N^{\alpha,\beta}Q_2\|_{L^\infty} \\
&\le \max_{0\le j\le N} |Q_2(\sigma_{N,j}^{\alpha,\beta})|\|I_N^{\alpha,\beta}\|_{L^\infty} \\
&\le \max_{0\le j\le N} \left|\int_{-1}^{1} \widetilde{F}(\sigma_{N,j}^{\alpha,\beta}, y)[\psi_2(U(y)) - \psi_2(U_N(y))]dy\right| \|I_N^{\alpha,\beta}\|_{L^\infty} \\
&\le 2L\|\widetilde{F}\|_{L^\infty}\|E\|_{L^\infty}\|I_N^{\alpha,\beta}\|_{L^\infty}. \qquad (4.24)
\end{aligned}$$

Combining (4.18), (4.22), (4.23), (4.24) with (4.17) gives the desired estimate (4.8). □

Error estimate in $L^2_{w^{\alpha,\beta}}$–norm

Theorem 4.2. Assume that the hypotheses in Theorem 4.1 hold, and $k \in (0,1)$. Then

$$\|U - U_N\|_{w^{\alpha,\beta}} \le CN^{-m}(|U|_{H_w^{m,N}(I)} + N^{-\frac{1}{2}}\chi + \rho(|U|_{H_w^{m,N}(I)} + \chi\|I_N^{\alpha,\beta}\|_{L^\infty})), \qquad (4.25)$$

where $\vartheta_{m,N}^{\alpha,\beta}, \tau_{m,N}^{\alpha,\beta}$ and χ are given as in Theorem 4.1 and

$$\rho = N^{\frac{1}{2}}\|\widetilde{F}\|_{L^\infty} + N^{\frac{1}{2}-k} + N^{\frac{1}{2}-m}(\vartheta_{m,N}^{\alpha,\beta} + \tau_{m,N}^{\alpha,\beta}).$$

Proof. By using (4.12) and the Gronwall inequality (1.15) we have

$$\|E\|_{w^{\alpha,\beta}} \le C(\|I_1\|_{w^{\alpha,\beta}} + \|I_2\|_{w^{\alpha,\beta}} + \|I_3\|_{w^{\alpha,\beta}} + \|I_4\|_{w^{\alpha,\beta}}). \qquad (4.26)$$

From (3.9) we have

$$\|I_1\|_{w^{\alpha,\beta}} = \|U - I_N^{\alpha,\beta} U\|_{w^{\alpha,\beta}} \leq CN^{-m}|U|_{H_w^{m,N}(I)}. \tag{4.27}$$

By Lemma 3.3 we have

$$\|I_2\|_{w^{\alpha,\beta}} = \|\sum_{j=0}^{N} J(\sigma_{N,j}^{\alpha,\beta})L_j(x)\|_{w^{\alpha,\beta}} \leq C\|J\|_{L^\infty} \tag{4.28}$$

Combining (4.19), (4.20), (4.21) leads

$$\begin{aligned}\|I_2\|_{w^{\alpha,\beta}} \leq\; & CN^{\frac{1}{2}-m}\|\widetilde{V}\|_{L_\infty}(L''\|E\|_{w^{\alpha,\beta}} + |\psi_1(U)|_{H_w^{m,N}(I)}) \\ & + CN^{-m}\max_{0\leq j\leq N}|\widetilde{V}(\sigma_{N,j}^{\alpha,\beta},\cdot)|_{H_w^{m,N}(I)}(L\|E\|_{L^\infty} + \|\psi_1(U)\|_{L^\infty}) \\ & + CN^{\frac{1}{2}-m}\|\widetilde{F}\|_{L^\infty}(L''\|E\|_{w^{\alpha,\beta}} + |\psi_2(U)|_{H_w^{m,N}(I)}) \\ & + CN^{-m}\max_{0\leq j\leq N}|\widetilde{F}(\sigma_{N,j}^{\alpha,\beta},\cdot)|_{H_w^{m,N}(I)}(L\|E\|_{L^\infty} + \|\psi_2(U)\|_{L^\infty}).\end{aligned} \tag{4.29}$$

Let consider $v(x) = \psi_1(U(x)) - \psi_1(U_N(x))$, similarly to the estimate of $\|I_3\|_{L_\infty}$ and by using (1.13) and (1.14) we have

$$\begin{aligned}\|I_3\|_{w^{\alpha,\beta}} &= \|(I_N^{\alpha,\beta} - I)\mathcal{A}_v\|_{w^{\alpha,\beta}} \\ &= \|(I - I_N^{\alpha,\beta})(\mathcal{A}_v - T_N\mathcal{A}_v)\|_{w^{\alpha,\beta}} \\ &\leq \|\mathcal{A}_v - T_N\mathcal{A}_v\|_{w^{\alpha,\beta}} + \|I_N^{\alpha,\beta}(\mathcal{A}_v - T_N\mathcal{A}_v)\|_{w^{\alpha,\beta}} \\ &\leq C\|\mathcal{A}_v - T_N\mathcal{A}_v\|_{L^\infty} \\ &\leq CN^{-k}\|\mathcal{A}_v\|_{0,k} \\ &\leq CN^{-k}\|v\|_{L^\infty} \\ &\leq CN^{-k}\|\psi_1(U) - \psi_1(U_N)\|_{L^\infty} \\ &\leq CLN^{-k}\|E\|_{L^\infty}.\end{aligned} \tag{4.30}$$

From (3.11) we obtain

$$\begin{aligned}
\|I_4\|_{w^{\alpha,\beta}} &= \|I_N^{\alpha,\beta} Q_2\|_{w^{\alpha,\beta}} \\
&\le C\|Q_2\|_{L^\infty} \\
&\le C \max_{0\le j\le N} |Q_2(\sigma_{N,j}^{\alpha,\beta})| \\
&\le C \max_{0\le j\le N} \left| \int_{-1}^{1} \widetilde{F}(\sigma_{N,j}^{\alpha,\beta}, y)[\psi_2(U(y)) - \psi_2(U_N(y))]dy \right| \\
&\le CL\|\widetilde{F}\|_{L^\infty}\|E\|_{L^\infty}.
\end{aligned} \tag{4.31}$$

The desired estimate (4.25) is obtained by combining (4.27)-(4.31), (4.26) and taking into account the convergence result in Theorem **??**. □

4.1.3 Numerical examples

In this section, some illustrative examples are provided to demonstrate the applicability of the designed method. The calculations performed in the examples are calculated by Matlab software, and a Core i5-2520M CPU 2.5 GHZ and 4 GB RAM are used to run the programs.

Example 4.1. Consider the nonlinear Volterra-Fredholm integral equation,

$$u(s) = g(s) + \int_0^s (s-t)u^2(t)dt + \int_0^1 (s+t)u(t)dt, \quad s \in [0,1],$$

with

$$g(s) = -\frac{1}{30}s^6 + \frac{1}{3}s^4 - s^2 + \frac{5}{3}s - \frac{5}{4},$$

the exact solution of which is

$$u(s) = s^2 - 2.$$

Table 4.1 shows the numerical errors obtained by using the spectral method described above with $\alpha = \beta = -3/4$. Table 4.2 shows the comparison of the absolute errors of this method for $N = 5$ at some equally-spaced points on $[0,1]$ with those previously obtained by using three other collocation methods: The first is based on Chebyshev approximation [10], the second is based on rationalized Haar functions (RH) [38], while the third is based on multiquadrics radial basis functions (MQ-RBFs)[40] in which the zeros of the shifted Legendre polynomial are chosen as collocation points. We observe that the numerical results are in good accordance with the theoretical analysis and from comparison, we can see that the proposed method is better than the considered literature methods.

Table 4.1: The L^∞ and L^2_w errors for Example 4.1

N	2	4	6	8
L^∞ Error	6.0526e-02	4.5375e-04	1.3323e-15	1.5543e-15
L^2_w Error	9.7168e-02	2.7076e-04	9.3726e-16	8.2961e-16

Table 4.2: Absolute errors for Example 4.1

s	Present method for $N = 5$	Method of [10] for $N = 5$	Method of [38] for $N = 16$	Method of [40] for $N = 10$
0	8.88e-16	0.20e-9	8.00e-6	1.92e-8
0.2	4.44e-16	7.35e-9	4.00e-6	4.16e-9
0.4	2.22e-16	7.93e-9	1.10e-5	2.53e-9
0.6	0.00e+00	2.55e-9	1.30e-5	1.92e-9
0.8	0.00e+00	3.98e-9	1.40e-5	1.83e-9
1	6.66e-16	2.64e-9	1.40e-5	5.07e-9

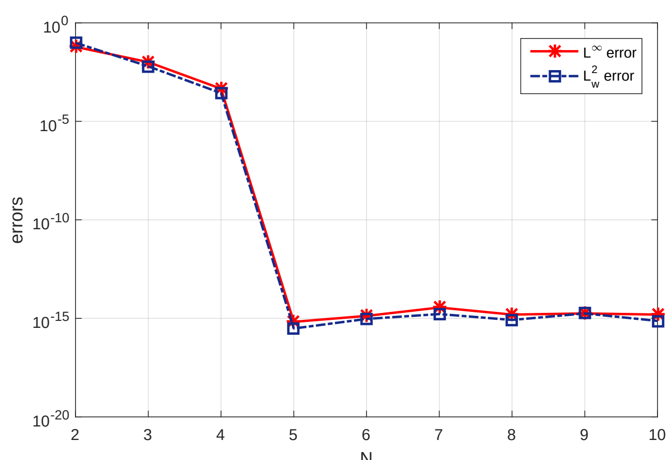

Figure 4.1: The errors in L^∞ and L^2_w norms versus the number of collocation points for Example 4.1.

Example 4.2. Consider the Volterra integral equation

$$u(s) = s + \cos s - 1 + \int_0^s \sin(u(t))dt, \quad s \in [0,1],$$

where $u(s) = s$ is the exact solution. Table 4.3 shows the numerical errors for $\alpha = 1/4$ and $\beta = 1/3$. Table 4.4 shows a comparison of the absolute errors at some points with those obtained using fixed point technique and cubic B-spline wavelets [31].

Table 4.3: The L^∞ and L^2_w errors for Example 4.2

N	2	4	6	8	10
L^∞ error	5.1387e-03	1.8428e-05	2.7832e-08	2.3683e-11	1.2546e-14
L^2_w error	2.7245e-03	9.1760e-06	1.3662e-08	1.1807e-11	6.5577e-15

Table 4.4: Absolute errors for Example 4.2

s	Present method for $N = 10$	Method of [31] for $N = 10$
0	5.38e-14	0.00e+00
0.2	4.69e-15	4.22e-08
0.4	1.08e-14	1.09e-08
0.6	5.55e-16	2.35e-08
0.8	6.66e-15	1.42e-08
1	3.56e-14	2.63e-08

Example 4.3. Consider the Fredholm integral equation

$$u(s) = e^{s+1} - \int_0^1 e^{s-2t} u^3(t)dt, \quad s \in [0,1],$$

where $u(s) = e^s$ is the exact solution. Table 4.5 shows the numerical errors for $\alpha = \beta = 1/2$. Table 4.6 shows the absolute errors at some points compared with those obtained Haar wavelets [9].

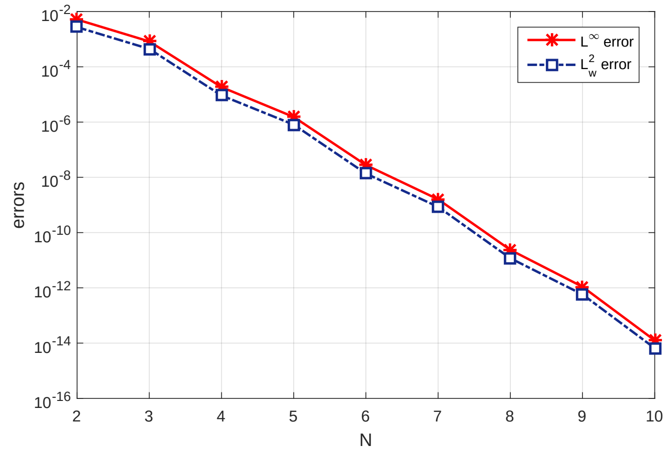

Figure 4.2: The errors in L^∞ and L^2_w norms versus the number of collocation points for Example 4.2.

Table 4.5: The L^∞ and L^2_w errors for Example 4.3.

N	2	4	6	8	10
L^∞ error	6.3352e-02	2.4390e-03	3.5130e-05	2.8389e-07	1.4804e-09
L^2_w error	5.0454e-02	1.2316e-03	1.7110e-05	1.3598e-07	7.0234e-10

Table 4.6: Absolute errors for Example 4.3

s	Present method for $N = 10$	Method of [9] for $N = 32$
0.1	6.14e-10	2.05e-03
0.3	7.50e-10	8.69e-03
0.5	9.17e-10	1.87e-02
0.7	1.12e-09	2.93e-03
0.9	1.37e-09	2.16e-02

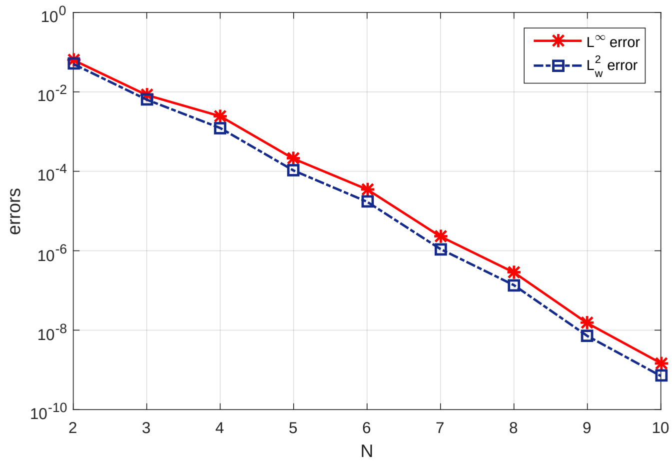

Figure 4.3: The errors in L^∞ and L^2_w norms versus the number of collocation points for Example 4.3.

4.2 Solving nonlinear integral equation of Hammerstein type on half-line by using a Sinc-Nyström method

We consider the nonlinear integral equation of Hammerstein type on the half-line given by the general form

$$u(s) - \int_0^\infty k(s,t)f(t,u(t))dt = g(s), \qquad s \in I = [0,\infty), \tag{4.32}$$

where k, g, and f are known functions and u is a solution to be determined.
Recently, there are a various of researches that was interested about solving integral equations on bounded intervals, such as [27, 30, 35, 44, 46, 47]. However, the numerical solving of N.I.Es on unbounded domains are still a challenge where there is a scarcity of researches that's interested in this regard.

This section is the subject of our work which have been published [20], where we presented two numerical schemes for solving Eq. (4.32). The first one is based on the SE-Sinc transformations, that have indicated the order of convergence $\mathcal{O}(\exp(-C\sqrt{N}))$ and the second one is based on the DE-Sinc transformations, which improve the rate of convergence to $\mathcal{O}(\exp(-C(N/\log N)))$.

We can write Eq.(4.32) in operator notation as

$$(\mathcal{I} - \mathcal{K}f)u = g, \tag{4.33}$$

where

$$(\mathcal{K}f)(u)(s) = \int_I k(s,t)f(t,u(t))dt, \quad s \in I. \tag{4.34}$$

This operator is defined on the Banach space $X = Hol(\mathcal{D}) \cap C(\overline{\mathcal{D}})$, where $Hol(\mathcal{D})$ denotes the family of analytic functions, and $\mathcal{D}$ is a simply connected domain in $\mathbb{C}$ which satisfies $I \subset \mathcal{D}$.

4.2.1 Sinc-Nyström method

4.2.2 SE-Sinc scheme

Let $k(s,.)f(.,u(.))\psi'_{SE_i}(.) \in L_\alpha\left(\psi_{SE_i}(\mathcal{D}_d)\right)$ for all $s \in I$. Then the discrete SE-Sinc operator can be defined by

$$\left(\mathcal{K}_N^{SE_i} f\right)(u(s)) = h \sum_{j=-N}^{N} k(s,t_j^{SE_i}) f\left(t_j^{SE_i}, u(t_j^{SE_i})\right) \psi'_{SE_i}(jh). \tag{4.35}$$

The Nyström method applied to (4.32) is exploited to find $u_N^{SE_i}$ such that

$$u_N^{SE_i}(s) - h \sum_{j=-N}^{N} k(s,t_j^{SE_i}) f\left(t_j^{SE_i}, u(t_j^{SE_i})\right) \psi'_{SE_i}(jh) = g(s), \tag{4.36}$$

where the quadrature points are defined by

$$t_j^{SE_i} = \psi_{SE_i}(jh), \quad j = -N, \cdots, N.$$

Solving (4.36) reduces to solving a finite dimensional nonlinear system. For any solution of (4.36) the values $u_N^{SEi}(t_j^{SE_i})$ at the quadrature points satisfy the nonlinear system

$$u_N^{SE_i}(t_l^{SE_i}) - h \sum_{j=-N}^{N} k(t_l^{SE_i}, t_j^{SE_i}) f\left(t_j^{SE_i}, u(t_j^{SE_i})\right) \psi'_{SE_i}(jh) = g(t_l^{SE_i}), \quad l = -N, \cdots, N.$$

Then the approximate solution $u_N^{SE_i}(s)$ at an arbitrary point s, can be expressed as

$$u_N^{SE_i}(s) = g(s) + h\sum_{j=-N}^{N} k(s, t_j^{SE_i}) f(t_j^{SE_i}, u(t_j^{SE_i}))\psi'_{SE_i}(jh). \tag{4.37}$$

Equation (4.36) can be written in the following discrete SE-sinc operator equation

$$\left(\mathcal{I} - \mathcal{K}_N^{SE_i} f\right) u_N^{SE_i} = g. \tag{4.38}$$

4.2.3 DE-Sinc scheme

Let $k(s,.)f(.,u(.))\psi'_{DE_i}(.) \in L_\alpha(\psi_{DE_i}(\mathcal{D}_d))$ for all $s \in I$. Then, the discrete DE-Sinc operator can be defined by

$$\left(\mathcal{K}_N^{DE_i} f\right)(u(s)) = h\sum_{j=-N}^{N} k(s, t_j^{DE_i}) f\left(t_j^{DE_i}, u(t_j^{DE_i})\right)\psi'_{DE_i}(jh). \tag{4.39}$$

The Nyström method applied to (4.32) is exploited to find u_N^{DEi} such that

$$u_N^{DE_i}(s) - h\sum_{j=-N}^{N} k(s, t_j^{DE_i}) f\left(t_j^{DE_i}, u(t_j^{DE_i})\right)\psi'_{DE_i}(jh) = g(s), \tag{4.40}$$

where the quadrature points are defined by

$$t_j^{DE_i} = \psi_{DE_i}(jh), \quad j = -N \cdots N.$$

solving (4.40) reduce to solving a finite dimensional nonlinear system. For any solution of (4.40) the value $u_N^{DE_i}(t_j^{DE_i})$ at the quadrature points satisfy the nonlinear system

$$u_N^{DE_i}(t_l^{DE_i}) - h\sum_{j=-N}^{N} k(t_l^{DE_i}, t_j^{DE_i}) f\left(t_j^{DE_i}, u(t_j^{DE_i})\right)\psi'_{DE_i}(jh) = g(t_l^{DE_i}), \quad l = -N, \cdots, N.$$

Then the approximate solution $u_N^{DE_i}(s)$ at an arbitrary point s can be expressed as

$$u_N^{DE_i}(s) = g(s) + h\sum_{j=-N}^{N} k(s, t_j^{DE_i}) f(t_j^{DE_i}, u(t_j^{DE_i}))\psi'_{DE_i}(jh). \tag{4.41}$$

Equation (4.40) can be written in the following discrete DE-Sinc operator equation

$$\left(\mathcal{I}-\mathcal{K}_N^{DE_i} f\right) u_N^{DE_i} = g. \tag{4.42}$$

4.2.4 Convergence analysis

Throughout this section, we discuss the convergence of the SE and DE Sinc-Nyström methods on the semi-infinite interval $I=[0,\infty)$, we first consider the SE-case. Assume that u and g belong to the space C_l, the space of all continuous functions on $[0,\infty)$ having a limit at infinity.
Also, we suppose that (4.33) has an isolated solution $u_0 \in C_l$ and the compact operator $\mathcal{K}f$ possesses a continuous first and a bounded second derivative on $B(u_0,\delta)$ where

$$B(u_0,\delta)=\{u\in C_l : \|u-u_0\|_0 \leqq \delta,\ \delta>0\}.$$

For prove the following theorem we need to mentioned the following required conditions, let the kernel $k(.,.)$ satisfy

A_1. $k(s,t)$ is bounded and continuous on $I\times I$.

A_2. $k(s,t)$ is continuous uniformly with respect to s for all $s,t\in I$.

A_3. For each $t\in I$, $k(s,t)\to 0$ as $s\to\infty$.

We also assume that the following conditions are met on the nonlinear function $f(.,u(.))$

B_1. $f(s,u)$ is defined and continuous on $I\times\mathbb{R}$.

B_2. $f(s,u)$ is bounded for $s\in I$ uniformly for u in any bounded set.

B_3. the partial derivative $f_u(s,u)=\frac{\partial}{\partial u} f(s,u)$ exists and is continuous on $I\times\mathbb{R}$.

B_4. the second partial derivative $f_{uu}(s,u)=\frac{\partial^2}{\partial u^2} f(s,u)$ exists, continuous on $I\times\mathbb{R}$, and bounded for $s\in I$ uniformly for u in any bounded set.

Theorem 4.3. Let A_1-A_3 and B_1-B_4 hold, assume that $k(s,.)f(.,u(.))\psi'(.)\in L_\alpha(\psi_{SE_i}(\mathcal{D}_d))$ with $0<d<\pi/2$ and $u\in B(u_0,\delta)$, then

C_1. $\{\mathcal{K}_N^{SE_i} f : N\geq 1\}$ is a collectively compact family on C_l.

C_2. $\mathcal{K}_N^{SE_i} f$ is pointwise convergent to $\mathcal{K}f$ on C_l.

C_3. For $N \geq 1$, $\mathcal{K}_N^{SE_i} f$ possesses a continuous first and a bounded second Fréchet derivatives on $B(u_0, \delta)$. Moreover,

$$\|(\mathcal{K}_N^{SE_i} f)''\| \leq \gamma < \infty,$$

where γ is a constant independent of N.

Proof We recall that the set $\{\mathcal{K}_N^{SE_i} f : N \geq 1\}$ is a collectively compact family on the Banach space C_l if the set $\Lambda = \{(\mathcal{K}_N^{SE_i} f)u : N \geq 1, u \in \mathcal{B}\}$, (where $\mathcal{B}$ is the unit ball in C_l) is a relatively compact subset of C_l, we deduce that the set $\{\mathcal{K}_N^{SE_i} f : N \geq 1\}$ is collectively compact if Λ is equicontinuous at each point $s \in I$, equiconvergent at infinity and bounded.
From (4.35) we have

$$\left|\left(\mathcal{K}_N^{SE_i} f\right)(u(s')) - \left(\mathcal{K}_N^{SE_i} f\right)(u(s))\right| \leq h \sum_{j=-N}^{N} \left|k(s', t_j^{SE_i}) - k(s, t_j^{SE_i})\right| \left|f(t_j^{SE_i}, u(t_j^{SE_i}))\psi'_{SE_i}(jh)\right|,$$

due to the uniform continuity of the kernel $k(s,t)$ with respect to s, we can obtain

$$\left|\left(\mathcal{K}_N^{SE_i} f\right)(u(s')) - \left(\mathcal{K}_N^{SE_i} f\right)(u(s))\right| \to 0 \text{ as } s' \to s, \forall s \in I, \text{ uniformly for } N \geq 1, \qquad (4.43)$$

hence, we conclude that Λ is equicontinuous at each point of I.
Also from (4.35)

$$\left|\left(\mathcal{K}_N^{SE_i} f\right)(u(s))\right| \leq h \sum_{j=-N}^{N} \left|k(s, t_j^{SE_i})\right| \left|f\left(t_j^{SE_i}, u(t_j^{SE_i})\right)\psi'_{SE_i}(jh)\right|, \qquad (4.44)$$

hence by the condition $A3$, the set $\{(\mathcal{K}_N^{SE_i} f)u \mid N \geq 1,\ u \in B\}$ is equiconvergent to zero at infinity.
Next, we seek to show that Λ is bounded. It follows from the assumption of theorem 3.10 that

$$(\mathcal{K}_N^{SE_i} f)u(s) \to (\mathcal{K} f)u(s), \text{ for all } s \in I.$$

It is known that pointwise convergence on the interval $[0,\infty)$ of a family that is equicontinuous at each point of $[0,\infty)$ and equiconvergent at infinity is sufficient to guarantee uniform convergence, hence

$$\lim_{N\to\infty} \|(\mathcal{K}_N^{SE_i} f)u - (\mathcal{K} f)u\|_0 = 0,$$

for all $u \in C_I$, then

$$\sup_N \|(\mathcal{K}_N^{SE_i} f)u\|_0 < \infty,$$

since $\{\mathcal{K}_N^{SE_i} f\}$ is a sequence of bounded operators on the Banach space C_I, it follows from the uniform-boundedness (Banach-Steinhaus) theorem that

$$\sup_N \|\mathcal{K}_N^{SE_i} f\| < \infty,$$

thus, Λ is bounded.
So from the above-mentioned discussions, C_1 holds.
Due to the Theorem 3.9 the assumption C_2 holds immediately.
The condition B_3 implies that $\mathcal{K}f$ is Fréchet differentiable with

$$(\mathcal{K}f)'(u)x(s) = \int_I k(s,t) f_u(t, u(t)) x(t) dt, \qquad s \in I, \;\; x \in B(u_0, \delta),$$

and the condition B_4 leading to the existence and the boundedness of the second Fréchet derivative with

$$(\mathcal{K}f)''(u)(x,y)(s) = \int_I k(s,t) f_{uu}(t, u(t)) x(t) y(t) dt, \qquad s \in I, \;\; x, y \in B(u_0, \delta),$$

similar to $\left(\mathcal{K}_N^{SE_i} f\right)$, $(\mathcal{K}_N^{SE_i} f)'$ and $(\mathcal{K}_N^{SE_i} f)''$ can be defined by the SE-Sinc quadrature formula as follows

$$\left(\mathcal{K}_N^{SE_i} f\right)'(u)x(s) = h \sum_{j=-N}^{N} k(s, t_j^{SE_i}) f_u\left(t_j^{SE_i}, u(t_j^{SE_i})\right) \psi'_{SE_i}(jh) x(t_j^{SE_i}), \tag{4.45}$$

$$\left(\mathcal{K}_N^{SE_i} f\right)''(u)(x,y)(s) = h \sum_{j=-N}^{N} k(s, t_j^{SE_i}) f_{uu}\left(t_j^{SE_i}, u(t_j^{SE_i})\right) \psi'_{SE_i}(jh) x(t_j^{SE_i}) y(t_j^{SE_i}), \tag{4.46}$$

by considering 4.45 and 4.46 in $B(u_0, \delta)$ and the boundedness of $f_{uu}(s,u)$ it is easily to concluded C_3.

Theorem 4.4. Assume that the assumptions of Lemma 3.6 hold. Then there exists a positive integer N_1 such that, for all $N \geq N_1$, Eq. (4.38) has a unique solution $u_N^{SE_i} \in B(u_0, \delta)$. Furthermore, there exists a constant C independent of N such that

$$\left\| u_0 - u_N^{SE_i} \right\|_0 \leq C \exp(-\sqrt{2\pi d \alpha N}). \tag{4.47}$$

Proof By subtracting (4.33) from (4.38) we obtain

$$u_0 - u_N^{SE_i} = (\mathcal{K}f)(u_0) - (\mathcal{K}_N^{SEi} f)(u_N^{SEi}),$$

by adding the term $(\mathcal{K}_N^{SE_i} f)'(u_0)(u_0 - u_N^{SE_i})$ on both sides we have

$$\left[\mathcal{I} - (\mathcal{K}_N^{SE_i} f)'(u_0)\right](u_0 - u_N^{SE_i}) = (\mathcal{K}f)(u_0) - (\mathcal{K}_N^{SE_i} f)(u_0) - \Big[(\mathcal{K}_N^{SE_i} f)(u_N^{SE_i}) - (\mathcal{K}_N^{SE_i} f)(u_0) - (\mathcal{K}_N^{SE_i} f)'(u_0)(u_N^{SE_i} - u_0)\Big].$$

By condition C_3, the term $(\mathcal{K}_N^{SE_i} f)(u_N^{SE_i}) - (\mathcal{K}_N^{SE_i} f)(u_0) - (\mathcal{K}_N^{SE_i} f)'(u_0)(u_N^{SE_i} - u_0)$ has been bounded by the term $\frac{1}{2}\gamma \|u_0 - u_N^{SE_i}\|_0^2$, then from Lemma 3.6 we have

$$\left\| u_0 - u_N^{SE_i} \right\|_0 \leq \beta \left[\|(\mathcal{K}f)(u_0) - (\mathcal{K}_N^{SE_i} f)(u_0)\|_0 + \frac{1}{2}\gamma \|u_0 - u_N^{SE_i}\|_0^2 \right],$$

hence

$$\begin{aligned} \left\| u_0 - u_N^{SE_i} \right\|_0 &\leq \frac{\beta \|(\mathcal{K}f)(u_0) - (\mathcal{K}_N^{SE_i} f)(u_0)\|_0}{(1 - \frac{\beta\gamma\delta}{2})} \\ &\leq \frac{\beta}{1 - \frac{\beta\gamma\delta}{2}} \|(\mathcal{K}f)(u_0) - (\mathcal{K}_N^{SE_i} f)(u_0)\|_0. \end{aligned}$$

Then by using Theorem 3.10 we obtain the desired result.
Concerning the convergence of the DE-Sinc Nyström method, we can define the assumptions $C_1 - C_3$ for the DE-case by replacing the SE-transformation ψ_{SE_i} with DE-transformation ψ_{DE_i}. Then we can formulate and prove the following Theorem in the same way as in the SE-case.

Theorem 4.5. Assume that the same assumptions of Lemma 3.6 are satisfied for the DE-case. Then there exists a positive integer N_1 such that, for all $N \geq N_1$, Eq.(4.42) has a unique solution $u_N^{DE_i} \in B(u_0, \delta)$. Furthermore, there exist a constant C independent of N such that

$$\|u_0 - u_N^{DE_i}\|_0 \leq C \exp\left(\frac{-2\pi d N}{\log(8dN/\alpha)} \right). \tag{4.48}$$

4.2.5 Illustrating examples

In this section, we show numerical results that illustrate the theoretical results obtained previously. As we mentioned in the preceding section, the convergence of the SE-Sinc and DE-Sinc methods depends to the parameters α and d, the important parameter d value is 1.57 for both methods, and the parameter α changes by each Example.

Example 4.4. Consider the following nonlinear integral equation

$$u(s) + \int_0^\infty e^{-(s+t)} u^2(t)dt = 6e^{-s},$$

where the exact solution is given by $u(s) = 3e^{-s}$, Table 4.7 shows a comparison of the maximum absolute errors obtained using SE-Sinc and DE-Sinc methods with $\alpha = 1$, respectively $\alpha = 3$, and those obtained from [36].

Table 4.7: Maximum absolute errors for Example 4.4.

N	SE1	SE2	DE1	DE2	DE3	[36]
8	2.17e-04	9.98e-04	6.10e-05	8.84e-09	6.36e-08	
16	3.31e-05	2.38e-05	4.79e-08	3.55e-15	1.33e-15	6.54e-03
32	2.54e-07	1.19e-07	5.03e-11	1.78e-15	1.33e-15	8.39e-04
64	1.02e-10	6.94e-11	2.22e-15	2.66e-15	2.22e-15	1.05e-04

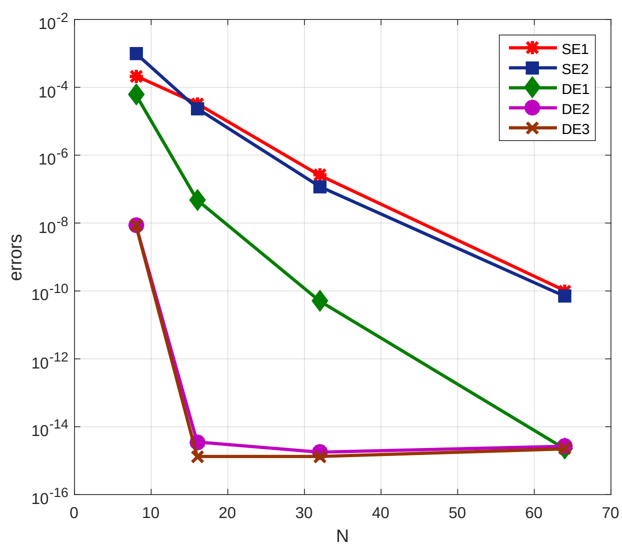

Figure 4.4: The SE and DE-sinc Nyström results for Example 4.4.

Example 4.5. Consider the following nonlinear integral equation

$$u(s) + \int_0^\infty \frac{e^{-(s+t)}}{1 + u(t) + u^2(t)} dt = (1 - \frac{\pi}{3\sqrt{3}})e^{-s},$$

whose exact solution is $u(s) = e^{-s}$, we choose $\alpha = 1$, for the SE-Sinc method and $\alpha = 2$, for the DE-Sinc method, the obtained numerical results are given in Table 4.8.

Table 4.8: Maximum absolute errors for Example 4.5.

N	SE1	SE2	DE1	DE2
4	1.83e-02	2.00e-03	1.46e-02	1.46e-05
8	2.33e-04	1.16e-04	1.50e-03	1.09e-08
16	1.67e-05	2.12e-06	3.15e-05	6.88e-15
32	1.18e-08	7.65e-09	1.08e-08	2.22e-16
64	9.69e-11	2.62e-12	5.15e-13	4.44e-16

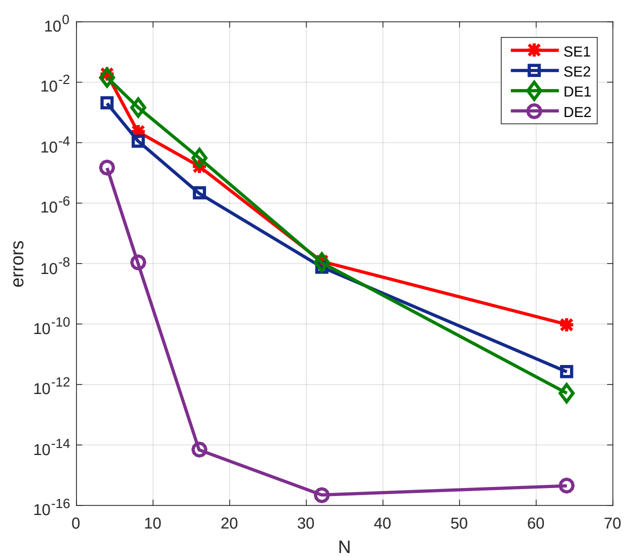

Figure 4.5: The SE and DE-sinc Nyström results for Example 4.5.

Example 4.6. Consider the following nonlinear integral equation

$$u(s) + \int_0^\infty e^{-t(s+1)} u^2(t) dt = \sin(s) - \frac{2}{s^3 + 3s^2 + 7s + 5},$$

whose exact solution is $u(s) = \sin(s)$, we choose $\alpha = 5$, for the SE-Sinc method and $\alpha = 9$, for the DE-Sinc method, the numerical results for this Example are given in Table 4.9.

Table 4.9: Maximum absolute errors for Example 4.6.

N	SE1	SE2	DE1	DE2	DE3
4	2.45e-02	1.32e-02	4.72e-02	7.59e-04	1.30e-03
8	4.70e-03	3.70e-03	6.50e-03	6.86e-05	2.05e-04
16	6.01e-04	7.75e-04	2.80e-03	4.40e-07	1.75e-06
32	8.86e-06	4.51e-05	1.93e-04	4.73e-11	2.36e-10
64	2.03e-07	1.70e-06	3.03e-06	3.05e-16	2.16e-16

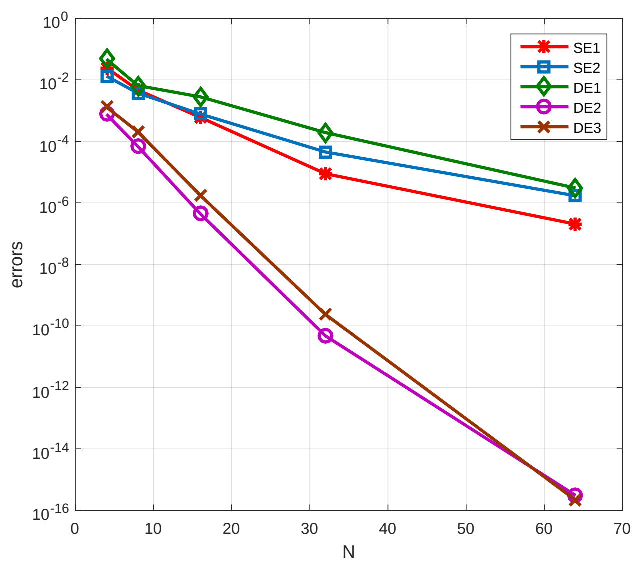

Figure 4.6: The SE and DE-sinc Nyström results for Example 4.6.

Example 4.7. Consider the following nonlinear integral equation on the half-line

$$u(s) + \int_0^\infty e^{-(s+t)} \cos(u(t))dt = e^{-s}(1 - \sin(1)),$$

where the exact solution is given by $u(s) = e^{-s}$, we choose $\alpha = 1$, for SE-Sinc method and $\alpha = 2$, for DE-Sinc method, the numerical results for this Example are given in Table 4.10.

Table 4.10: Maximum absolute errors for Example 4.7.

N	SE1	SE1	DE1	DE2
4	9.00e-03	3.60e-03	1.35e-02	1.39e-05
8	1.65e-04	2.15e-04	1.40e-03	1.92e-08
16	4.43e-05	2.15e-04	3.23e-05	9.77e-15
32	8.32e-08	1.54e-08	1.10e-08	3.33e-16
64	1.41e-10	6.12e-12	5.08e-13	6.66e-16

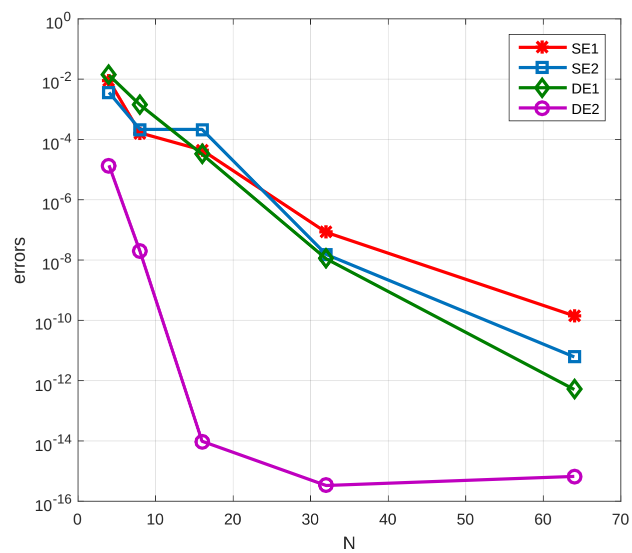

Figure 4.7: The SE and DE-sinc Nyström results for Example 4.7.

Example 4.8. Consider the linear integral equation

$$u(s) + \int_0^\infty e^{-t^2-s} u(t)dt = g(s),$$

where $g(s)$ is selected so that the exact solution is $u(s) = \frac{1}{s^4+2s^2+1}$, Table 4.11 shows the numerical results using DE1-Sinc and DE2-Sinc methods with $\alpha = 7$.

Table 4.11: Maximum absolute errors for Example 4.8.

N	DE1	DE2
4	1.63e-03	9.91e-04
9	7.81e-04	1.97e-07
12	1.50e-05	4.96e-10
15	2.65e-06	1.90e-11
18	9.94e-08	5.28e-13
21	9.96e-08	1.65e-14

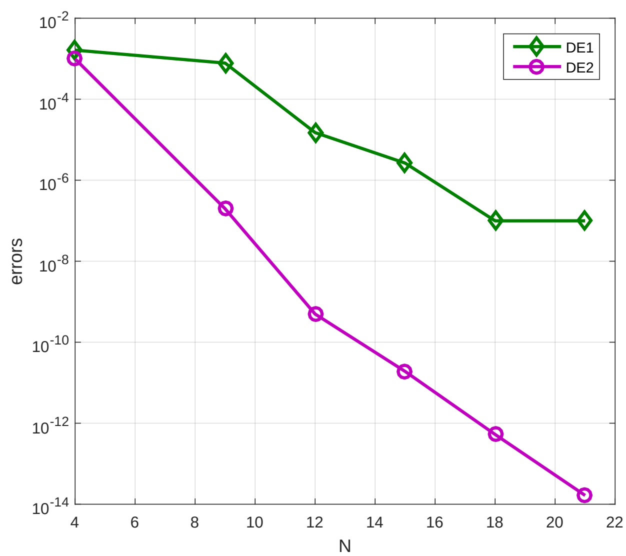

Figure 4.8: The SE and DE-sinc Nyström results for Example 4.8.

CONCLUSION AND PROSPECTS

In this work we've been concerned with the study and the resolution of nonlinear integral equations, where we have provided a comprehensive existence theory for these equations, specially, for Urysohn integral equation which is the most general form of a nonlinear integral equations, and also for the integral equation of Hammerstein type in half-line which will later be included on subject of our work. We have also presented the main types of numerical approximation methods for solving nonlinear integral equations, such as Projection methods, Nyström methods, and Sinc methods.
The projection methods is based on the projection of our equation in a subspace of finite dimension, where we select a finite dimensional set of functions that is hoped to include a function $u_n(s)$ close to the exact solution $u^*(s)$, this required numerical solution $u_n(s)$ is chosen by having it satisfy our integral equation approximately. In the convergence analysis we have used the procedure of linearisation and the standard contractive mapping theorem, to show that the speed of convergence of u_n to u^* is exactly the same as that of $P_n u^*$ to u^*, thus it does not depend explicitly on the operator $\mathcal{K}$, but only on the approximation properties of $P_n u^*$.
The Nyström methods or quadrature methods is depend on the approximation of the definite integral that appear in our integral equation over the interval $\Omega = [a, b]$ by a finite sum, to seek an approximate solution in a finite number of points. The study of convergence analysis of these methods is based on the collectively compact operators theory, and it leads to consider that the speed of this convergence is the same of the numerical integration technique which applied.
As regards the Sinc methods, through our work we have explained that these methods are suited and very accurate for solving nonlinear integral equations, for such problems,

the resulting system of algebraic equations can always be written down explicitly, and is relatively small in size, furthermore, we have proved that these methods have an exponential convergence order.

Finally, we have tried to apply the present methods for solving some nonlinear integral equations, specifically, the nonlinear Volterra-Fredholm integral equations which have been solved by a Jacobi spectral collocation method, where we have stated the theorems on the convergence and error estimates of the method, and we have proved them for both L^∞ and weighted L^2 norms. Furthermore, we have approximated the solution of the integral equation of Hammerstein type on half-line by using a combination of the Sinc quadrature rule with the Nyström method, such method have been developed by means of the Single Exponential (SE) and Double Exponential (DE) transformations, and their convergence has been discussed where the numerical results have confirmed the theoretical prediction of the exponential rate of convergence.

This work could be extended to solve nonlinear quadratic integral equations on unbounded intervals by using the Sinc-Nyström method where we anticipate a good results.

We look forward to consider a suitable numerical methods for solving quadratic integral equations defined on bounded domains, specifically, the Chandrasekhar's equation, that is given by the the following expression:

$$u(s) = 1 + u(s)\int_0^1 \frac{s}{s+t}\psi(t)u(t)dt,$$

where $u(s)$ is a continuous function in $[0,1]$, and the kernel $\frac{s}{s+t}\psi(t)$ is continuous and nonnegative in $[0,1]\times[0,1]$.

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ملخص :

الهدف الرئيسي من هذه الأطروحة هو تقديم دراسة نظرية وعددية حول المعادلات التكاملية غير الخطية. استخدمنا مختلف نظريا ت النقطة الثابتة، كما قدمنا مبدأ ليراي ـ شودر لتوفير نتائج حول وجود حلول للمعادلات التكاملية غير الخطية في مجالات محدودة وغير محدودة ، كما قدمنا أساليب فعالة لحل هذه المعادلات مع دراسة متعمقة حول التقارب.
بالإضافة إلى ذلك، طبقنا بعض هذه الطرق، منها طريقة التجميع الطيفي و طريقة سينك-نيستروم من أجل إيجاد حلول عددية لبعض المعادلات التكاملية غير الخطية، هذه الطرق تحول المعادلة التكاملية غير الخطية إلى نظام من المعادلات الجبرية غير الخطية وهذا النظام الجبري تم حله بطريقة نيوتن. واستخلصنا تحليلا للخطأ فيما يتعلق بالأساليب الحالية التي تثبت أن لها ترتيبا أسيا للتقارب. وأخيرا، قدمنا عدة أمثلة رقمية لإثبات فعالية مناهجنا التقريبية.

كلمات مفتاحية : معادلات تكاملية غير خطية، نظريات النقطة الثابتة، معادلة أوريسون التكاملية، معادلة تكاملية من نوع هامرشتاين، نصف المستقيم العددي، طريقة الإسقاط، طريقة سينك-نستروم، تحليل التقارب.

Abstract :

The main objective of this thesis is to offer a theoretical and numerical study on nonlinear integral equations. We have used different fixed point theorems, and Leray-Schauder principle to provide existence results for nonlinear integral equations on bounded and unbounded domains, we have also presented efficient methods for solving such equations with a thorough study on the convergence analysis. Furthermore, we have applied some of these methods, specially, spectral collocation method and Sinc-Nyström method in order to find numerical solutions of certain nonlinear integral equations, these methods reduce the nonlinear integral equation to a system of nonlinear algebraic equations and that algebraic system has been solved by Newton's method. We have derived an error analysis for the current methods, which prove that they have exponential convergence order. Finally, several numerical examples are given to show the effectiveness of our approaches.

Keywords: Nonlinear integral equations, fixed-point theorems, Urysohn integral equation, Hammerstein integral equation, half-line, projection method, Sinc-Nyström method, convergence analysis.

Résumé :

L'objectif principal de cette thèse est de proposer une étude théorique et numérique sur les équations intégrales non linéaires. Nous avons utilisé différents théorèmes du point fixe, et le principe de Leray-Schauder pour fournir des résultats d'existence pour les équations intégrales non linéaires sur des domaines bornés et non bornés, nous avons également présenté des méthodes efficaces pour résoudre de telles équations avec une étude approfondie sur la convergence. En outre, nous avons appliqué certaines de ces méthodes, notamment, la méthode spectrale de collocation, et la méthode de Sinc-Nyström pour trouver des solutions numériques de quelques équations intégrales non linéaires, ces méthodes transforment l'équation intégrale non linéaire en un système d'équations algébriques non linéaires ce système algébrique a été résolu par la méthode de Newton. Nous avons dérivé une analyse d'erreur pour les méthodes actuelles qui prouvent qu'elles ont un ordre de convergence exponentielle. Enfin, plusieurs exemples numériques sont donnés pour montrer l'efficacité de nos approches.

Mots clés: Equations intégrales non linéaires, théorèmes du point fixe, équation intégrale d'Urysohn, équation intégrale de type Hammerstien, demi-droite, méthode de projection, méthode de Sinc-Nyström, analyse de la convergence.