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Theme

Types of fuzzy ideals on a ring

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ملخص

في هذه المذكرة، قمنا بدراسة مفهوم المثالي الضبابي داخل حلقة كتعميم لمفهوم المثالي الكلاسيكي داخل حلقة. كما تطرقنا الى نوعين من هذه المثاليات وهي المثاليات الاولية الضبابية والمثاليات الاعظمية الضبابية وكذا العلاقة بينهما.

كلمات مفتاحية

مجموعة ضبابية، حلقة، مثالي ضبابي، مثالي اولي ضبابي، مثالي اعظمي ضبابي.

Abstract

In this memory, we have studied the concept of a fuzzy ideal on a ring as a generalization of crisp ideal on a ring. Moreover, we studied two types of these ideals which are fuzzy prime ideals and fuzzy maximal ideals, and relationship between them.

Key words

Fuzzy set, Ring, Fuzzy ideal, Prime fuzzy ideal, Maximal fuzzy ideal.

Résumé

Dans cette mémoire, nous avons étudié le concept d'idéal flou sur un anneau comme une généralisation d'idéal classique sur un anneau. De plus, nous avons également abordé deux types de ces idéaux qui sont les idéaux premier flous et les idéaux maximaux flous et la relation entre eux.

Mot-clés

Ensemble flou, Anneau, Idéal flou, Idéal premier flou, Idéal maximal flou.

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Contents

Introduction

The concept of the fuzzy set was introduced by professor Zadeh in Berkeley University, California in 1965 in his article "Fuzzy sets" as an extension of the classical notion of set, then this theory has been developed by many authors.
There are many application fields of fuzzy set theory found in automatic [4], in robotics [23], in image processing [8]... etc.
The notions of fuzzy rings and fuzzy ideals on a ring, which are fuzzy subsets are just a part of fuzzy logic and they were introduced by Liu [14] and they were developed by Mukherjee and Sen [20]. Azam and his colleagues [6] introduced the notion of an anti-ideal fuzzy in a ring. Recently, Alam [3] Studied and introduced some operators on anti-rings and fuzzy ideal. For the types of fuzzy ideals on a ring, Malik and Mordeson [15] introduced the notion of the prime fuzzy ideal on a ring and Huang [26] introduced the notion of the maximal fuzzy ideal on a ring. Later, Swamy and his colleague[24] studied some properties of the prime fuzzy ideal on a ring and Raj et al. [22] studied some properties of the maximal fuzzy ideal on a ring.
The aim of this memory is to investigate fuzzy ideals concepts on rings and their fundamental properties. We present interesting characterizations of these notions in terms of their α-level sets. Moreover, we study some types of fuzzy ideals on a ring; the prime fuzzy ideal on a ring and the maximal fuzzy ideal on a ring and the relationship between them.
The memory is divided into three chapters:
In the first chapter, we recall the fundamental concepts of groups, rings and types of ideals on a ring. Also, we recall some properties of crisp ideals on a ring.
In the second chapter, we give fundamental concepts of fuzzy sets, operations of fuzzy sets, characteristics of fuzzy set, triangular norm and conorm, cartesian product and projection of fuzzy sets.
In the third chapter, we treat the notions of the fuzzy ring and fuzzy ideal on a ring, we characterize these notions in terms of their level sets and support, and we study some types of fuzzy ideals on a ring; the fuzzy prime ideal and the maximal fuzzy

ideal. Finally, we study their properties and the relationship between them.

Chapter 1

Generalities on crisp rings and ideals

The notion of a ring is one of the fundamental algebraic structures used in abstract algebra. In this chapter, we recall some definitions of group, ring, ideal, morphism of a ring and some types of ideals on a ring. Also, we show several properties of ideals on a ring, for more details please refer to [11].

1.1 Definitions

This section contains the basic definitions and properties of a group, ring, ideal on a ring and some related notions that will be needed throughout the next chapters.

1.1.1 Group structure

In this subsection, we recall definitions and examples of a group, abelian group and sub-group.

Definition 1.1 (Group)**.** *A group $(G, \cdot)$ is a set with a composition low "$\cdot$" satisfying the following axioms.*

(i) G is closed under "$\cdot$", i.e., $x \cdot y \in G$ for all $x, y \in G$;

(ii) *the operation* $" \cdot "$ *is associative, i.e.,* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *for all* $x, y, z \in G$ *;*

(iii) *there is an identity element* $e \in G$ *such that* $x \cdot e = e \cdot x = x$ *for all* $x \in G$ *;*

(iv) *each element* $x \in G$ *has an inverse element* $x^{-1} \in G$ *such that* $x \cdot x^{-1} = x^{-1} \cdot x = e$*.*

Definition 1.2 (Abelian group)**.** *Let* G *be a group with respect to* $" \cdot "$*. Then* G *is called an abelian group, or commutative group, if* $x \cdot y = y \cdot x$ *for all* $x, y \in G$*.*

Example 1.1. $(\mathbb{Z}, +), (\mathbb{R}^*, \times)$ *are abelian groups.*

Definition 1.3 (Subgroup)**.** *A nonempty subset* H *of a group* G *is called a sub group of* G *if* H *is itself a group with respect to the operatin on* G*.*

In the following theorem, we present an equivalent definition of the subgroup.

Theorem 1.1. *A subset* H *of the group* G *is a subgroup of* G *if and only if these conditions are satisfied :*

(i) H *is nonempty ;*

(ii) $x \in H$ *and* $y \in H$ *imply* $x - y \in H$*.*

Example 1.2. $(\mathbb{Z}, +)$ *is a subgroup of* $(\mathbb{R}, +)$

1.1.2 Ring structure

In this subsection, we recall definitions and examples of a ring, abelian ring and subring.

Definition 1.4 (Ring)**.** *A ring* $(R, +, \cdot)$ *is a set* R *with two binary operations* $" + "$ *and* $" \cdot "$ *on* R *satisfying the following conditions:*

(i) $(R, +)$ *is an abelian group;*

(ii) *associativity of multiplication i.e.,* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *for all* $x, y, z \in R$ *;*

(iii) distributive laws hold in R such that $x.(y+z) = x.y + x.z$ for all $x, y, z \in R$.

Definition 1.5 (Abelian ring)**.** *Let $(R, +, \cdot)$ be a ring. Then $(R, +, \cdot)$ is called a abelian ring, or commutative ring, if $x \cdot y = y \cdot x$ for all $x, y \in R$.*

Example 1.3. $(\mathbb{Z}, +, \times)$, $(\mathbb{Q}, +, \times)$, $(\mathbb{R}, +, \times)$ *and* $(\mathbb{C}, +, \times)$ *are commutative rings.*

Definition 1.6 (Subring)**.** *If $(R, +, \cdot)$ is a ring. a nonempty subset S of R is called a subring of R if S itself a ring with respect to the operations on R.*

Next, we provide an equivalent definition of the subring.

Theorem 1.2. *A subset S of the ring R is a subring of R if and only if these conditions are satisfied :*

(i) S is nonempty ;

(ii) $(S, +)$ is a subgroup de R ;

(iii) for all $x, y \in S$, $x \cdot y \in S$.

Example 1.4. $(\mathbb{Z}, +, \times)$ *is a subring of* $(\mathbb{R}, +, \times)$.

1.1.3 Morphism of rings

A morphism (or homomorphism) between two rings is a function their underlying sets that preserves the two operations of addition and multiplication and also the element identity e.

Definition 1.7. *Let $(R, +, \cdot)$ and $(S, \circ, *)$ are two rings. The function $f : R \longrightarrow S$ is called a ring morphism if for all $x, y \in R$ we have :*

(i) $f(x+y) = f(x) \circ f(y)$;

(ii) $f(x \cdot y) = f(x) * f(y)$;

(iii) $f(1_R) = 1_S$.

A ring isomorphism is a bijective ring morphism, and we say that R and S are isomorphic rings and we write $R \cong S$.

Definition 1.8 (Field). *A field is a nonempty set F with two operations "$+$" (called addition) and "$\cdot$" (called multiplication) satisfying the following condition :*

(i) $(F, +, \cdot)$ *is a commutative ring;*

(ii) If $a \in F$, and $a \neq 0$, there exists $b \in F$ such that $a \cdot b = b \cdot a = 1$ (The element b is called the multiplicative inverse of a).

Example 1.5. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ *are fields but* $\mathbb{Z}$ *is not a field.*

1.1.4 Ideals on a ring

The notion of ideal is an important subset of the ring structure.

Definition 1.9. *A nonempty subset I of a ring R is called an ideal on R if*

(i) $(I, +)$ *is a subgroup of the group* $(R, +)$*;*

(ii) $\forall a \in I, \forall x \in R,\ a \cdot x \in I, x \cdot a \in I$.

Example 1.6. *Consider the ring* $(\mathbb{Z}, +, \times)$*. Let $n \in \mathbb{N}$. Then $I = \{qn \mid q \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z}$.*

Definition 1.10. *Let I be an ideal on a ring R then the set $R/I = a + I : a \in R$ is called cosets of I and R is called quotient ring where the addition and the multiplication are defined as:*

(i) $(a + I) + (b + I) = (a + b) + I \ a, b \in R$*;*

(ii) $(a + I) \cdot (b + I) = a \cdot b + I \ a, b \in R$.

Remark 1.1. *(i) If R is a commutative ring with unity then R/I is also a commutative ring with unity;*

(ii) $1+I$ *is the multiplicative identity of* R/I *and* $0+I=I$ *is the additive identity of* R/I.

Properties of ideals on a ring

In this part, we show some properties of ideals on a ring; intersection, union, sum, product and complement.

Proposition 1.1. *Let* R *be a commutative ring with identity and let* $\{I_j \mid j \in J\}$ *be a nonempty collection of ideals of* R*. Then* $\cap_{j\in J} I_j$ *is an ideal of* R.

Remark 1.2. *The union of two ideals on a ring* R *is not an ideal in general on* R*. Indeed, let* $2\mathbb{Z}$ *and* $3\mathbb{Z}$ *are an ideals on the ring* $(\mathbb{Z},+,\times)$, $3,2 \in 2\mathbb{Z}\cup 3\mathbb{Z}$ *but* $3-2=1 \notin 2\mathbb{Z}\cup 3\mathbb{Z}$*. Hence,* $2\mathbb{Z}\cup 3\mathbb{Z}$ *is not an ideal on* $(\mathbb{Z},+,\times)$.

Proposition 1.2. *The sum of two ideals on a ring* R *is an ideal of* R.
i.e., if I *and* J *are two ideals on a ring* R *we have. Then*

$$I+J=\{a+b : a\in I, b\in J\}$$

is an ideal on R

Proposition 1.3. *If* I *and* J *are an ideals on* R*. The product of* I *and* J*, written* $I\cdot J$ *is the set*

$$I\cdot J=\left\{\sum_{k=1}^{n} i_k j_k \mid i_k\in I, j_k\in J, k=1,...,n; n\in\mathbb{N}\right\}$$

$I\cdot J$ *is an ideal on* R.

Remark 1.3. *The complement of an ideal on a ring* R *is not an ideal on* R*. Indeed, let* $2\mathbb{Z}$ *is a an ideal on the ring* $(Z,+,\times)$*. The set* $C(2\mathbb{Z})$ *set of the odd numbers. We have* $3,1\in C(2\mathbb{Z})$ *but* $3-1=2\in 2\mathbb{Z}$ *this implies that* $2\notin C(2\mathbb{Z})$*. Hence,* $C(2\mathbb{Z})$ *is not an ideal.*

1.2 Types of ideals on a ring

In this subsection, we give definitions and examples of some types of ideals on a ring.

1.2.1 Principal ideal

Definition 1.11. *Principal ideal is an ideal I on a ring R that is generated by a single element a of R through multiplication by every element of R.*

Definition 1.12. *Let R be a commutative ring with identity and let X be a subset of R. Define $\langle X\rangle$ to be the intersection of all ideals of R which contain X. Then $\langle X\rangle$ is called the ideal of R generated by X.*

In Definition 1.12, $\langle X\rangle$ is the smallest ideal of R which contains X.
The next theorem states that $\langle X\rangle$ is precisely the set of all such sums. This follows by showing that $\{\sum_{i=1}^{n} r_i x_i \mid r_i \in R, x_i \in X, i = 1, 2, ..., n; n \in \mathbb{N}\}$ is an ideal of R containing X and then using the fact that $\langle X\rangle$ is the smallest ideal of R containing X.

Theorem 1.3. *Let R be a commutative ring with identity and let X be a nonempty subset of R. Then $\langle X\rangle = \{\sum_{i=1}^{n} r_i x_i \mid r_i \in R, x_i \in X, i = 1, 2, ..., n; n \in \mathbb{N}\}$.*

Remark 1.4. *Let R be a commutative ring with identity and let $x \in R$.*
Then $\langle\{x\}\rangle = \{rx \mid r \in R\}$.
If R is a commutative ring with identity and $x \in R$, we often write $\langle x\rangle$ for $\langle\{x\}\rangle$.
Theorem 1.3 gives us a method to construct examples of ideals. This can be seen from the following example.

Example 1.7. *(i) Consider any commutative ring R with identity. Let x be any element (fixed) of R. Then $\{rx \mid r \in R\}$ is an ideal of R by remarque 1.4 ;*

(ii) Let $R[x]$ be a polynomial ring in the indeterminate x, where R is a commutative ring with identity. Then $\langle x\rangle = \{r(x)x \mid r(x) \in R[x]\}$ is an ideal of $R[x]$. $\langle x\rangle$ is the set of all polynomial with zero constant term.

Definition 1.13. *Let R be a commutative ring with identity. Then R is satisfy the ascending chain condition for ideals or to be Noetherian if for every ascending chain of ideals $I_1 \subseteq I_2 \subseteq ... \subseteq I_n \subseteq ...$ there exists a positive integer m such that $\forall n \geq m$, $I_n = I_m$.*

Example 1.8. *The ring $\mathbb{Z}$ of integers is Noetherian. We can see this from the following argument. It can be shown that for every ideal I of $\mathbb{Z}$, there exists $n \in \mathbb{N}$ such that $I = \langle n \rangle$. Let $\langle K_1 \rangle \subseteq ... \subseteq \langle K_n \rangle \subseteq ...$ be an ascending chain of ideals of $\mathbb{Z}$ for $K_n \in \mathbb{N}$,$n = 1, 2, ...$ Then, $K_n \in \langle K_{n+1} \rangle$ and $\exists r_n \in \mathbb{Z}$ such that $K_n = r_n K_{n+1}$. Thus, $K_n \geqslant K_{n+1} \geqslant 0$ for $n = 1, 2, ...$ Hence, $\exists m \in \mathbb{N}$ such that $\forall n \geqslant m$, $K_n = K_m$.*

1.2.2 Prime ideal

Definition 1.14. *Let R be a commutative ring with identity and let P be an ideal of R. Then P is said to be a prime ideal of R if $\forall a, b \in R, ab \in P$ and $a \notin P$ implies $b \in P$.*

Example 1.9. *Consider the ring of integers $\mathbb{Z}$ and let p be a prime element of $\mathbb{Z}$. Then $\langle p \rangle$ is a prime ideal of $\mathbb{Z}$. We can see this from the following reasoning. Let $ab \in \langle p \rangle$. By remarque 1.4, $\exists r \in Z$ such that $ab = rp$. Hence either a or b is a multible of p since p is prime and so either $a \in \langle p \rangle$ or $b \in \langle p \rangle$, respectively.*

In the following proposition, we present relationship between prime ideal and quotient ring.

Proposition 1.4. *Let R be a commutative ring with identity and let I is ideal on R. Then I is prime ideal if and only if R/I integral domain.*

Proof. ($\Rightarrow$)
Let I be a prime ideal on a ring R, let $(x + I), (y + I) \in R/I$.
Since $(x + I) \cdot (y + I) \in R/I$ it follows that $x \cdot y + I = I$ then $x \cdot y \in I$. Since I is a prime ideal, it holds that $x \in I$ or $y \in I$. This implies that $x + I = I$ or $y + I = I$. Hence, R/I is integral domain.
($\Leftarrow$)
Let R/I integral domain, and let $x, y \in I \mid\ x \cdot y \in I$ then it follows that $x \cdot y + I = I$ then $(x + I) \cdot (y + I) = I$. Since R/I integral domain, it holds that $(x + I) = I$ or $(y + I) = I$. Thus, $x \in I$ or $y \in I$. Hence, I is prime ideal. □

1.2.3 Maximal ideal

Definition 1.15. *An ideal I of R is called maximal if $I \subsetneq R$ and there does not exist an ideal J of R such that $I \subsetneq J \subsetneq R$, i.e., for any ideal J with $I \subseteq J$, either $J = I$ or $J = R$.*

Example 1.10. *In the ring $\mathbb{Z}$ of integers, the maximal ideals are the principal ideals generated by a prime numbre.*

In the following theorem, we present a relationship between a maximal ideal and quotient ring.

Theorem 1.4. *Let R be a commutative ring with identity and let I is an ideal on R. Then I is a maximal ideal if and only if R/I is a field.*

Proof. $(\Rightarrow)$
Let I is a maximal ideal and let $\bar{J}$ is an ideal of R/I, then $\bar{J} = J/I$ and J are ideals of R and $I \subset J$, this implies that $J = I$ or $J = R$. So, $\bar{J} = I/I = \{\bar{0}\}$ or $\bar{J} = R/I$. Hence, R/I is a field.
$(\Leftarrow)$
Let R/I is a field and let J is an ideal on R and $I \subset J$, this implies that $\bar{J} = J/I$ is an ideal on R/I. So, R/I is a field. Therefore, $\bar{J} = J/I = \{\bar{0}\}$ or $\bar{J} = J/I = R/I$. Thus, $J = I$ or $J = R$. Hence, I is a maximal ideal. □

In the following theorem, we provide the relationship between the maximal ideal and the prime ideal on the same ring.

Theorem 1.5. *Every maximal ideal on a ring is a prime ideal on this ring.*

Proof. Suppose that I is a maximal ideal on a ring R. We show that I is a prime ideal on R.
I is a maximal ideal $\Longleftrightarrow$ R/I is a field $\Rightarrow$ R/I is an integral domain $\Longleftrightarrow$ I is a prime ideal. □

Chapter 2

Generalities on fuzzy sets

Fuzzy sets were introduced by L. Zadeh in [27] as a generalization of crisp sets. The purpose of this chapter is to provide a basic introduction to the notion of fuzzy sets, operations of fuzzy sets, characteristics of fuzzy sets, triangular norms, triangular conorms, projection and Cartesian product. Many of the properties of these concepts will be used in the next chapter. For more details please refer to [13], [27].

2.1 Definitions

Crisp set is an unodered collection of different elements. We represent a set by

(i) Enumerating its elements $a_1, a_2, ..., a_n$ are the element of the set A, it is represented as follows $A = \{a_1, a_2, ..., a_n\}$;

(ii) specifying the conditions of elements i.e., $A = \{x \mid P(x)\}$;

(iii) Characteristic function of A is a function on X.

$$\chi_A : X \longrightarrow \{0, 1\}$$

$$x \longmapsto \begin{cases} 0 \;\; if \;\; x \;\notin\; A; \\ 1 \;\; if \;\; x \;\in\; A. \end{cases}$$

2.1.1 Fuzzy sets

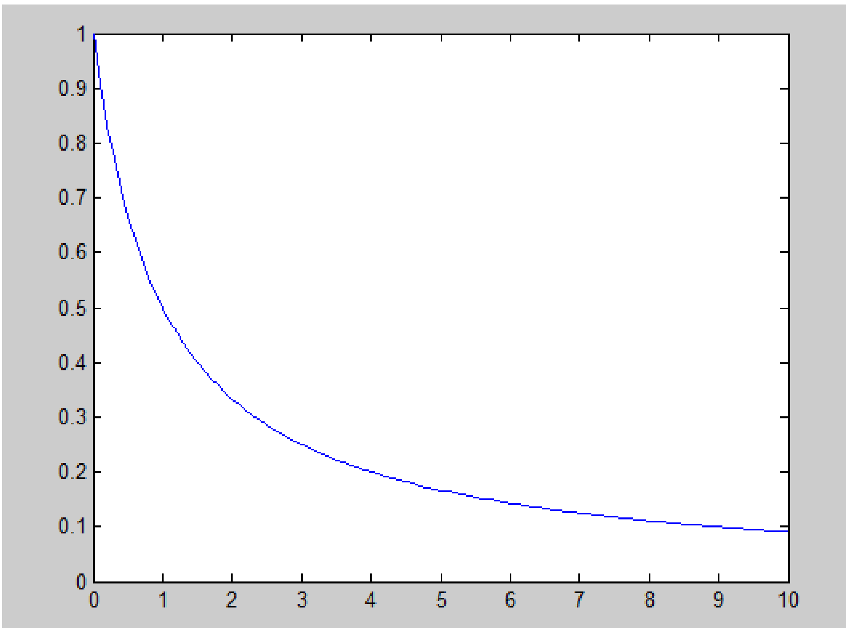
The notion of fuzzy sets was first introduced by Zadeh [27].

Definition 2.1. [27] *Let X be a nonempty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for each $x \in X$.*

Example 2.1. *(1) Let $X = \{a, b, c\}$ be an universal set. Then $A = \{(a, 0.2), (b, 0.8), (c, 1)\}$ is a fuzzy subset in X ;*

(2) Let $X = [0, 10]$, and A is a fuzzy subset in X, defined by :

$$\mu_A(x) = \frac{1}{1 + x}$$



graph of μ_A

2.1.2 Operations of fuzzy sets

In this section, we give definitions for some operations of fuzzy sets; equality, inclusion, intersection, union, sum and product of two fuzzy subsets and complement of a fuzzy set with an example.

Definition 2.2 (Equality). *Let X be a nonempty set and let A and B two fuzzy subsets, we say that $A = B$, if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$.*

Definition 2.3 (Inclusion). *Let X be a nonempty set and let A and B are two fuzzy subsets, we say that $A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$ for all x in X.*

Definition 2.4 (Intersection). *Let X be a nonempty set and let A and B are two fuzzy subsets, the intersection defined by for all $x \in X$*

$$\mu_{A\cap B}(x) = min\left\{\mu_A(x), \mu_B(x)\right\} = \mu_A(x) \wedge \mu_B(x)$$

Definition 2.5 (Union). *Let X be a nonempty set and let A and B are two fuzzy subsets, the union defined by for all $x \in X$*

$$\mu_{A\cup B}(x) = max\left\{\mu_A(x), \mu_B(x)\right\} = \mu_A(x) \vee \mu_B(x)$$

Definition 2.6 (Complement). *The complement of a fuzzy set A denoted by $C(A)$ and is defined by : for all $x \in X$*

$$\mu_{C(A)}(x) = 1 - \mu_A(x)$$

Definition 2.7 (Sum). *Let X be a nonempty set and let A and B are two fuzzy subsets, the sum defined by for all $x \in X$*

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$$

Definition 2.8 (Product). *Let X be a nonempty set and let A and B are two fuzzy subsets, the product defined by for all $x \in X$*

$$\mu_{A\times B}(x) = \mu_A(x)\mu_B(x)$$

Example 2.2. *Let $X = \{a, b, c\}$, and let $A = \{(a, 0.2), (b, 0.6), (c, 0.5)\}$, and $B = \{(a, 0.7), (b, 0.1), (c, 1)\}$ we have :*

1. $A \cap B = \{(a, 0.2), (b, 0.1), (c, 0.5)\}$

2. $A \cup B = \{(a, 0.7), (b, 0.6), (c, 1)\}$

3. $A \times B = \{(a, 0.14), (b, 0.06), (c, 0.5)\}$

4. $A + B = \{(a, 0.76), (b, 0.74), (c, 1)\}$

5. $C(A) = \{(a, 0.8), (b, 0.4), (c, 0.5)\}$

2.1.3 Characteristics of fuzzy set

In this section, we give definitions for characteristics of fuzzy set : support, kernel, height and cardinality of a fuzzy set with an example.

Definition 2.9 (α-cuts)**.** [27] Let A be a fuzzy set in X and let $\alpha \in]0, 1]$. The α-cut of A, denoted by A_α. We mean all elements of X that belong to A to a degree of at least α. That is A_α is a classical set defined by

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

Definition 2.10 (Support)**.** *[27] The support of a fuzzy set A, denoted by $Supp(A)$, we mean all elements of X that belong to a nonzero degree. That is $S(A)$is a clssical set defined by*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

Definition 2.11 (Kernel)**.** *[27] The kernel of a fuzzy set A, denoted by $ker(A)$, is defined as the set of elements whose membership degree is esual to* 1*. That is $ker(A)$is a calssical set as formalized in*

$$ker(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

Definition 2.12 (Height)**.** *[27] The height of a fuzzy set A is the largest membership grade of any element in A.*

$$H(A) = Max \mu_A(x)$$

Definition 2.13 (Cardinality). *[27] The cardinality of a finite fuzzy set A, denoted $\mid A \mid$ is defined as*

$$\mid A \mid = \sum_{x \in X} \mu_A(x).$$

Example 2.3. *Let $X = \{a, b, c, d\}$, and $A = \{(a, 0.6), (b, 1), (c, 0.3), (d, 0)\}$*

$A_{0.5} = \{a, b\}$

$Supp(A) = \{a, b, c\}$

$ker(A) = \{b\}$

$H(A) = 1$

$|A| = 1.9$

2.2 T-norm and T-conorm

The history of triangular-norms (*t-norms*) started with Menger [12]. His main idea was to construct metric spaces where probability distributions are used to describe the distance between two elements.

2.2.1 Triangular norm

Definition 2.14. [12] Triangular norm is a binary operation T on the unit interval $[0, 1]$, i.e., it is a function $T : [0, 1]^2 \longrightarrow [0, 1]$ with the following conditions

(T1) Commutativity i.e., $T(x, y) = T(y, x)$;

(T2) Associativity i.e., $T(x, T(y, z)) = T(T(x, y), z)$;

(T3) Monotonicity i.e., $T(x, y) \leq T(x, z)$ whenever $y \leq z$;

(T4) Boundary condition i.e., $T(x, 1) = x$.

Example 2.4. *The following four operations are the most common t-norms:*

1. *Minimum:* $T_M(x, y) = \min\{x, y\}$

2. *Product:* $T_P(x, y) = x.y$

3. *Lukasiewicz:* $T_L(x, y) = \max\{x + y - 1, 0\}$

4. *Drastic product:*

$$T_D(x,y) = \begin{cases} x & \text{if } \; y = 1 \\ y & \text{if } \; x = 1 \\ 0 & \text{if } \; x, y < 1. \end{cases}$$

2.2.2 Triangular conorm

Definition 2.15. *[12] A triangular conorm is a binary operation* S *on the unit interval* $[0, 1]$*, i.e., it is a function* $S : [0, 1]^2 \longrightarrow [0, 1]$ *with the following conditions*

(S1) Commutativity : $S(x, y) = S(y, x)$ *;*

(S2) Associativity : $S(x, S(y, z)) = S(S(x, y), z)$ *;*

(S3) Monotonicity : $S(x, y) \leq S(x, z)$ *whenever* $y \leq z$ *;*

(S4) Boundary condition : $S(x, 0) = x$.

Example 2.5. 1. *Maximum:* $S_M(x, y) = \max\{x, y\}$

2. *Probabilistic sum:* $S_P(x, y) = x + y - s.y$

3. *Lukasiewicz:* $S_L(x, y) = \min\{x + y, 1\}$

4. *Drastic sum:*

$$S_D(x,y) = \begin{cases} 1, & \text{if } \; (x, y) \in [0, 1)^2 \\ \max\{x, y\}, & \text{otherwise} \end{cases}$$

2.3 Cartesian product and projection on fuzzy sets

2.3.1 Cartesian product on fuzzy sets

The cartesian product of the fuzzy subsets is the minimum of these degrees of belonging.

Definition 2.16. *The cartesian product applied to n fuzzy sets can be defined as follows. Let* $\mu_{A_1}, \mu_{A_2}, ...\mu_{A_n}$, *be the membership functions of* $A_1, A_2, ..., A_n$. *Then, the membership degree of* $(x_1, x_2, ...x_n) \in X_1 \times X_2 \times ... \times X_n$ *on the fuzzy set* $A_1 \times A_2 \times ... \times A_n$ *is given by*

$$\mu_{A_1 \times A_{2 \times ... \times A_n}}(x_1, x_2, ..., x_n) = min\left\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\right\}.$$

Example 2.6. *Lets* $X_1 = \{a, b, c, \}$, $X_2 = \{\alpha, \beta\}$ *and let* A_1, A_2 *be two fuzzy subset, respectively defined on* X_1 *and* X_2 *by:*
$A_1 = \{(a, 0.1), (b, 0.4), (c, 0.8)\}$;
$A_2 = \{(\alpha, 0.2), (\beta, 0.6)\}$.
Therefore,

$$A_1 \times A_2 = \{((a, \alpha), 0.1), ((a, \beta), 0.1), ((b, \alpha), 0.2), ((b, \beta), 0.4), ((c, \alpha), 0.2), ((c, \beta), 0.6)\}.$$

2.3.2 Projection on fuzzy sets

Definition 2.17. *The projection on* X_1 *of the fuzzy set* A *of* $X_1 \times X_2 \times ... \times X_n$ *is the fuzzy set* $Proj_{X_1}(A)$ *of* X_1, *whose membership function is defined for any* $x_1 \in X_1$ *by:*

$$\mu_{ProjX_{1(A)}}(x_1) = sup_{x_2 \in X_2, x_3 \in X_3, ..., x_n \in X_n}(\mu_A(x_1, x_2, ..., x_n)).$$

Example 2.7. *Let* $X = X_1 \times X_2$ *the set of reference such that* X_1 *and* X_2 *are two subsets of* X, *we consider* $A_1 \times A_2 = A$ *defined by:*
$A = \{((a, \alpha), 0.1), ((a, \beta), 0.1), ((b, \alpha), 0.2), ((b, \beta), 0.4), ((c, \alpha), 0.2), ((c, \beta), 0.6)\}$

Therefore,

$$
\begin{aligned}
Proj_{X_1}(A) &= \{(a, max(0.1, 0.1)), (b, max(0.2, 0.4)), (c, max(0.2, 0.6))\}; \\
&= \{(a, 0.1), (b, 0.4), (c, 0.6)\}.
\end{aligned}
$$

Chapter 3

Types of fuzzy ideals on a ring

In this chapter, we provide interesting characterizations of fuzzy ideals on a ring in terms of their α-level sets and support. Moreover, we study the notion of fuzzy prime ideal and fuzzy maximal ideal on a ring, their properties and the relationship between them.

3.1 Fuzzy ideals on a ring

In this section, we recall the definition of fuzzy ring and fuzzy ideal on a ring with some examples.

Definition 3.1 (Fuzzy ring on a ring)**.** [20] *Let R be a ring and A be a fuzzy set on R. Then A is called a fuzzy ring if for any $x, y \in R$, the following conditions are satisfied :*

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(x \cdot y) \geq \mu_A(x) \wedge \mu_A(y)$.

Example 3.1. *Let* $(\mathbb{Z}, +, \cdot)$ *be a ring. The fuzzy set* A *on* $(\mathbb{Z}, +, \cdot)$ *defined by:*

$$\mu_A : \mathbb{Z} \longrightarrow [0,1]$$
$$x \longmapsto \begin{cases} 0.6 & if \quad x = 0; \\ 0.2 & if \quad x \neq 0. \end{cases}$$

is a fuzzy ring.

Definition 3.2 (Fuzzy ideal on a ring)**.** [20] *Let* R *be a ring and* I *be a fuzzy subset on* R*. Then* I *is called a fuzzy ideal if, for any* $x, y \in R$*, the following conditions are satisfied :*

(i) $\mu_I(x - y) \geq \mu_I(x) \wedge \mu_I(y)$ *;*

(ii) $\mu_I(x \cdot y) \geq \mu_I(x) \vee \mu_I(y)$*.*

Example 3.2. *Let* $(\mathbb{Z}_8, +, \cdot)$ *be a ring. The fuzzy subset* I *on the ring* $\mathbb{Z}_8$ *defined by :*

$$\mu_I : \mathbb{Z}_8 \longrightarrow [0,1]$$
$$x \longmapsto = \begin{cases} \textit{0.6} & \textit{if} \quad \textit{x=0;} \\ \textit{0.3} & \textit{if} \quad \textit{x=2 or x=4;} \\ \textit{0.1} & \textit{else.} \end{cases}$$

is a fuzzy ideal on $(\mathbb{Z}, +, \cdot)$*.*

Remark 3.1. *Every fuzzy ideal is a fuzzy ring, but the converse is not true in general. Indeed, let* $(\mathbb{Z}, +, \cdot)$ *be a ring. The fuzzy subset* A *on the ring defined by:*

$$\mu_A : \mathbb{Z} \longrightarrow [0,1]$$
$$x \longmapsto \begin{cases} 0.6 & if \quad x = 0; \\ 0.5 & if \quad x = 1; \\ 0.2 & if \quad x \notin \{0,1\} \end{cases}$$

is a fuzzy ring, but is not a fuzzy ideal, because $\mu_A(1 \cdot 3) = \mu_A(3) = 0.2 \not\geq 0.5 = \mu_A(1) \vee \mu_A(3)$*.*

In the following proposition, we study the intersection of two fuzzy ideals on a ring.

Proposition 3.1. *Let R be a ring. If I and J are two fuzzy ideals on R, then $I \cap J$ is a fuzzy ideal on R.*

Proof. Let I, J be two fuzzy ideals on R. We will show that $I \cap J$ is a fuzzy ideal on R. Since I is an ideal, it follows that

$$\begin{cases} \mu_I(x-y) \geq \mu_I(x) \wedge \mu_I(y); \\ \mu_I(xy) \geq \mu_I(x) \vee \mu_I(y). \end{cases}$$

and J is an ideal, then

$$\begin{cases} \mu_J(x-y) \geq \mu_J(x) \wedge \mu_J(y); \\ \mu_J(xy) \geq \mu_J(x) \vee \mu_J(y). \end{cases}$$

Then,

$$\begin{aligned} \mu_{I\cap J}(x-y) &= \mu_I(x-y) \wedge \mu_J(x-y); \\ &\geq [\mu_I(x) \wedge \mu_I(y)] \wedge [\mu_J(x) \wedge \mu_J(y)]; \\ &\geq [\mu_I(x) \wedge \mu_J(x)] \wedge [\mu_I(y) \wedge \mu_J(y)]; \\ &\geq \mu_{I\cap J}(x) \wedge \mu_{I\cap J}(y). \end{aligned}$$

And

$$\begin{aligned} \mu_{I\cap J}(xy) &= \mu_I(xy) \wedge \mu_J(xy); \\ &\geq [\mu_I(x) \vee \mu_I(y)] \wedge [\mu_J(x) \vee \mu_J(y)]; \\ &\geq [\mu_I(x) \wedge \mu_J(x)] \vee [\mu_I(y) \wedge \mu_J(y)]; \\ &\geq \mu_{I\cap J}(x) \vee \mu_{I\cap J}(y). \end{aligned}$$

Then, $I \cap J$ is a fuzzy ideal on a ring R.

□

Remark 3.2. *The union of two fuzzy ideals does not necessarily be a fuzzy ideal.*

Theorem 3.1. *Let I be a fuzzy ideal on R and $X_I = \{r \in R | \mu_I(r) = \mu_I(0)\}$, where 0 is the unit for the sum operation of R. Then the classical subset X_I of R is an ideal on R.*

Proof. Let I be a fuzzy ideal on R and suppose that $r, s \in X_I$ and $x \in R$.

1. First, we prove that $r - s \in X_I$. Since $\mu_I(r) = \mu_I(0) = \mu_I(s)$ and by the condition $(I1)$, it follows that $\mu_I(r - s) \geq \mu_I(r) \wedge \mu_I(s) = \mu_I(0)$.
   Since the inequality $\mu_I(0) \geq \mu_I(r - s)$ is always satisfied, we obtain that $\mu_I(r - s) = \mu_I(0)$. Hence, $r - s \in X_I$.

2. Second, we prove that $r \cdot x \in X_I$, i.e., $\mu_I(r \cdot x) = \mu_I(0)$.
   Since $\mu_I(r) = \mu_I(0)$ and from $(I2)$, it holds that $\mu_I(r \cdot x) \geq \mu_I(r) \vee \mu_I(x) = \mu_I(0) \vee \mu_I(0) = \mu_I(0)$.
   Since always $\mu_I(0) \geq \mu_I(r \cdot x)$, then $\mu_I(r \cdot x) = \mu_I(0)$. Hence, $r.s \in X_I$. We conclude that X_I is an ideal on R.

□

In the following theorem, we study the interaction of fuzzy ideals with a rings-homomorphism.

Theorem 3.2. *Let R_1, R_2 be two rings and $f : R_1 \to R_2$ be a homomorphism of rings. If I is a fuzzy ideal on R_1, then $f(I)$ is a fuzzy ideal on R_2.*

Proof. Let I be a fuzzy ideal on R_1. We show that $f(I)$ is a fuzzy ideal on R_2 i.e.,

$$f(I)(x - y) \geq f(I)(x) \wedge f(I)(y);$$
$$f(I)(x.y) \geq f(I)(x) \vee f(I)(y).$$

We have

$$
\begin{aligned}
f(I)(x-y) &= \mu_I(f(x-y)) \\
&= \mu_I(f(x)-f(y)) \\
&\geq \mu_I(f(x)) \wedge \mu_I(f(y)) \quad (\textit{by the condition } (I1)) \\
&= f(I)(x) \wedge f(I)(y).
\end{aligned}
$$

And

$$
\begin{aligned}
f(I)(x.y) &= \mu_I(f(x \cdot y)) \\
&= \mu_I(f(x) \cdot f(y)) \\
&\geq \mu_I(f(x)) \vee \mu_I(f(y)) \quad (\textit{by the condition } (I2)) \\
&= f(I)(x) \vee f(I)(y).
\end{aligned}
$$

Then, $f(I)$ is a fuzzy ideal on R_2. □

In the following theorem, we will show that the inverse image of a fuzzy ideal is a fuzzy ideal.

Theorem 3.3. *Let R_1, R_2 be two rings and $f : R_1 \to R_2$ be a homomorphism of rings. If I is a fuzzy ideal on R_2, then $f^{-1}(I)$ is a fuzzy ideal on R_2.*

Proof. Let I be a fuzzy ideal on R_2. We show that $f^{-1}(I)$ is a fuzzy ideal on R_1 i.e.,

$$
f^{-1}(I)(x-y) \geq f^{-1}(I)(x) \wedge f^{-}(I)(y);
$$
$$
f^{-1}(I)(x.y) \geq f^{-1}(I)(x) \vee f^{-1}(I)(y).
$$

We have

$$
\begin{aligned}
f^{-1}(I)(x-y) &= \mu_I(f^{-1}(x-y)) \\
&= \mu_I(f^{-1}(x) - f^{-1}(y)) \\
&\geq \mu_I(f^{-1}(x)) \wedge \mu_I(f^{-1}(y)) \quad (\textit{by the condition } (I1)) \\
&= f^{-1}(I)(x) \wedge f^{-1}(I)(y).
\end{aligned}
$$

And

$$\begin{aligned} f^{-1}(I)(x.y) &= \mu_I(f^{-1}(x \cdot y)) \\ &= \mu_I(f^{-1}(x) \cdot f^{-1}(y)) \\ &\geq \mu_I(f^{-1}(x)) \vee \mu_I(f^{-1}(y)) \quad \textit{(by the condition } (I2)) \\ &= f^{-1}(I)(x) \vee f^{-1}(I)(y). \end{aligned}$$

Then, $f^{-1}(I)$ is a fuzzy ideal on R_1. □

3.2 Characterization of fuzzy ideals on a ring

In this section, we provide interesting characterizations of fuzzy ideals on a ring in terms of their α-level sets and support.

Proposition 3.2. *Let R be a ring. If I is a fuzzy ideal on R, then its support $Supp(I)$ is an ideal on R.*

Proof. Let I is a fuzzy ideal on a ring R.

(1) Let $x, y \in Supp(I)$. Since $x \in Supp(I)$, it follows that $\mu_I(x) > 0$. Since, $y \in Supp(I)$, it follows that $\mu_I(y) > 0$, we have that I is a fuzzy ideal on a ring R, then $\mu_I(x - y) \geq \mu_I(x) \wedge \mu_I(y) > 0$. Hence, $x - y \in Supp(I)$.

(2) Let $x \in Supp(I)$ and $a \in A$. We prove that $x.a \in Supp(I)$ and $a.x \in Supp(I)$. Since $x \in Supp(I)$ it holds that $\mu_I(x) > 0$, we have that I is a fuzzy ideal on R, then $\mu_I(x.a) \geq \mu_I(x) \vee \mu_I(a) > 0$. Hence, $x.a \in Supp(I)$. By using the same method as above we get that $a.x \in Supp(I)$.

Thus, $Supp(I)$ is an ideal on R. □

Remark 3.3. *The converse of the above implication does not necessarily hold. Indeed, let us consider the ring $(\mathbb{Z}, +, \cdot)$, and the fuzzy set I on the ring $\mathbb{Z}$ defined by*

:

$$\mu_I : \mathbb{Z} \longrightarrow [0,1]$$
$$x \longmapsto \begin{cases} 0.8 & if \quad x = 1; \\ 0.3 & if \quad x \neq 1. \end{cases}$$

Supp(I) $= \mathbb{Z}$ *is an ideal on* $\mathbb{Z}$*, but* I *is not a fuzzy ideal on* $\mathbb{Z}$*, because* $\mu_I(1-1) = \mu_I(0) = 0.3 \ngeq 0.8 = \mu_I(1) \wedge \mu_I(1)$.

In this theorem, we characterize the notion of fuzzy ideals on a ring in terms of their level sets.

Theorem 3.4. *Let* R *be a ring and* I *be a subset of* R*. Then,* I *is a fuzzy ideal on* R *if and only if their level sets are ideals on* R*.*

Proof.
$\Longrightarrow$) Suppose that I is a fuzzy ideal. We show that I_α is an ideal on R for $\alpha \in [0,1]$:

(i) Let $x, y \in I_\alpha$, then $\mu_I(x) \geq \alpha$ and $\mu_I(y) \geq \alpha$.
Since I is a fuzzy ideal on R, hence $\mu_I(x-y) \geq \mu_I(x) \wedge \mu_I(y) \geq \alpha$.
Then, $x - y \in I_\alpha$.

(ii) Let $x \in I_\alpha$ and $a \in A$, hence $\mu_I(x) \geq \alpha$ and $a \in A$. Since I is a fuzzy ideal on R, hence $\mu_I(x.a) \geq \mu_I(x) \vee \mu_I(a) \geq \alpha$.
then, $x.a \in I_\alpha$. By using the same method as above we get that $a.x \in I_\alpha$.

Hence, I_α are ideals on R.
$\Longleftarrow$) Suppose that I_α are ideals on R for all $\alpha \in [0,1]$. We show that I is a fuzzy ideal on R.

(i) Let $x, y \in R$. Setting $\alpha = \mu_I(x) \wedge \mu_I(y)$, then it follows that $\mu_I(x) \geq \alpha$ and $\mu_I(y) \geq \alpha$.
The case $\alpha = 0$ is obvious. Let $\alpha \in]0,1]$ and $x, y \in I_\alpha$.
Since I_α are ideals on R, then $x - y \in I_\alpha$ for all $\alpha \in]0,1]$ this implies that $\mu_I(x-y) \geq \alpha$. Then, $\mu_I(x-y) \geq \mu_I(x) \wedge \mu_I(y)$.

(ii) Let $x, y \in I_\alpha$ then $x \cdot y \in I_\alpha$ because I_α is an ideal. Then $\mu_I(x) \geq \alpha$ and it follows that $\mu_I(x \cdot y) \geq \alpha$. Hence $\mu_I(x \cdot y) \geq \mu_I(x)$ and $\mu_I(x \cdot y) \geq \mu_I(y)$. Thus, $\mu_I(x \cdot y) \geq \mu_I(x) \vee \mu_I(y)$.

We conclude that I is a fuzzy ideal on R. □

3.3 Prime fuzzy ideals on a ring

In this section, we treat the concept of prime fuzzy ideal and its properties. First we recall the following definition of prime fuzzy ideal on a ring.

Definition 3.3. [24] *Let R be a ring, and let I be a fuzzy subset on R, I is called a prime fuzzy ideal of R if the following two conditions are satisfied :*

(i) $\mu_I(x - y) \geq \mu_I(x) \wedge \mu_I(y)$ *;*

(ii) $\mu_I(x \cdot y) = \mu_I(x) \vee \mu_I(y)$.

Example 3.3. *Let $(\mathbb{Z}, +, \cdot)$ be a ring. The fuzzy set I on the ring defined by :*

$$\mu_I : \mathbb{Z} \longrightarrow [0,1]$$
$$x \longmapsto \begin{cases} 0.8 & if \quad 5 \mid x; \\ 0.3 & else. \end{cases}$$

is a prime fuzzy ideal on $(\mathbb{Z}, +, \cdot)$.

In the following proposition, we study the intersection of two prime fuzzy ideals on a ring.

Proposition 3.3. *Let R be a ring. If I and J are two prime fuzzy ideals on R, then $I \cap J$ is a prime fuzzy ideal on R.*

Proof. Suppose that I, J are prime fuzzy ideals on R. We show that $I \cap J$ is a prime fuzzy ideal on R.

Since I and J are prime fuzzy ideals, then

$$\begin{cases} \mu_I(x-y) \geq \mu_I(x) \wedge \mu_I(y); \\ \mu_I(xy) = \mu_I(x) \vee \mu_I(y). \end{cases}$$

and

$$\begin{cases} \mu_J(x-y) \geq \mu_J(x) \wedge \mu_J(y); \\ \mu_J(xy) = \mu_J(x) \vee \mu_J(y). \end{cases}$$

This implies that,

$$\begin{aligned} \mu_{I\cap J}(x-y) &= \mu_I(x-y) \wedge \mu_J(x-y); \\ &\geq [\mu_I(x) \wedge \mu_I(y)] \wedge [\mu_J(x) \wedge \mu_J(y)]; \\ &\geq [\mu_I(x) \wedge \mu_J(x)] \wedge [\mu_I(y) \wedge \mu_J(y)]; \\ &\geq \mu_{I\cap J}(x) \wedge \mu_{I\cap J}(y). \end{aligned}$$

and

$$\begin{aligned} \mu_{I\cap J}(xy) &= \mu_I(xy) \wedge \mu_J(xy); \\ &= [\mu_I(x) \vee \mu_I(y)] \wedge [\mu_J(x) \vee \mu_J(y)]; \\ &= [\mu_I(x) \wedge \mu_J(x)] \vee [\mu_I(y) \wedge \mu_J(y)]; \\ &= \mu_{I\cap J}(x) \vee \mu_{I\cap J}(y). \end{aligned}$$

Hence, $I \cap J$ is a prime fuzzy ideal on a ring R. □

The following proposition shows that the support of a prime fuzzy ideal on a ring is a prime ideal in that ring.

Proposition 3.4. *Let R be a ring. If I is a prime fuzzy ideal on R, then its support $Supp(I)$ is a prime ideal on R.*

Proof. Suppose that I is a prime fuzzy ideal on a ring R. We show that $Supp(I)$ is a prime ideal on R. Let $a \cdot b \in Supp(I)$, and $a \notin Supp(I)$, we show that $b \in Supp(I)$. We have $\mu_I(a \cdot b) > 0$ and $\mu_I(a) = 0$. Since I is a prime fuzzy ideal on R it follows that $\mu_I(a \cdot b) = \mu_I(a) \vee \mu_I(b) = 0 \vee \mu_I(b) > 0$. This implies that $\mu_I(b) > 0$. Hence, $b \in Supp(I)$. Thus, $Supp(I)$ is a prime ideal on R. □

In this theorem, we characterize the notion of prime fuzzy ideals on a ring in terms of their level sets.

Theorem 3.5. *Let R be a ring and I be a subset on a ring R. Then, I is a prime fuzzy ideal on R if and only if their level sets are prime ideals on R.*

Proof.
$\Longrightarrow$) Suppose that I is a prime fuzzy ideal on a ring R. We show that I_α is a prime ideal on R, for all $\alpha \in [0,1]$. Let $a \cdot b \in I_\alpha$, and $a \notin I_\alpha$, we show that $b \in I_\alpha$. We have $\mu_I(a \cdot b) \geq \alpha$, and $\mu_I(a) < \alpha$. Since I is a prime fuzzy ideal on R it follows that $\mu_I(a \cdot b) = \mu_I(a) \vee \mu_I(b) \geq \alpha$. Hence, $\mu_I(b) \geq \alpha$ (because $\mu_I(a) < \alpha$). Thus, $b \in I_\alpha$. Therefore, I_α is a prime ideal on R.
$\Longleftarrow$) Suppose that I_α are prime ideals on R for all $\alpha \in [0,1]$. We show that I is a fuzzy prime ideal on R.

(i) I_α are prime ideals. From Definition 3.2, it holds that $\mu_I(x-y) \geq \mu_I(x) \wedge \mu_I(y)$.

(ii) Let $x \cdot y \in I_\alpha$. Since I_α are prime ideals, it follows that $x \in I_\alpha$ or $y \in I_\alpha$. This implies that $\mu_I(x \cdot y) \geq \alpha$. Hence, $\mu_I(x) \vee \mu_I(y) \geq \alpha$. Thus, $\mu_I(y) \vee \mu_I(y) \geq \mu_I(x \cdot y)$. and $\mu_I(x \cdot y) \geq \mu_I(y) \vee \mu_I(y)$. Therefore, $\mu_I(x \cdot y) = \mu_I(x) \vee \mu_I(y)$.

We conclude that I is a fuzzy prime ideal on R. □

3.4 Maximal fuzzy ideals on a ring

In this section, we introduce and characterize the notion of a maximal fuzzy ideal on a ring.

Definition 3.4. *A maximal fuzzy ideal of a ring R is a fuzzy ideal I, not equal to R, such that there are no ideals in between I and R.*

Example 3.4. *Let $R = \mathbb{Z}/4\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ be a ring and $I = \{\bar{0}, \bar{2}\}$ such that $\mu_I(\bar{0}) = \mu_I(\bar{2}) = 1$. Then, I is a maximal fuzzy ideal on $R = \mathbb{Z}/4\mathbb{Z}$.*

The following proposition shows that the support of a maximal fuzzy ideal on a ring is a prime ideal on that ring.

Proposition 3.5. *Let R be a ring. If I is a maximal fuzzy ideal on R, then its support $Supp(I)$ is a maximal ideal on R.*

Proof. Suppose that I is a maximal fuzzy ideal on a ring R and we will show that $Supp(I)$ is a maximal ideal on R. We suppose that $Supp(I)$ is not a maximal i.e., there exists a subset K such that $Supp(I) \subset K$, this means that there exists a fuzzy ideal J on R such that $Supp(J) = K$ satisfies $\mu_J(x) = \mu_I(0)$, then it holds that $I \subset J$. That is a contradiction with the fact that I is a fuzzy maximal ideal. Hence, $Supp(I)$ is a maximal ideal on R. □

In this theorem, we characterize the notion of maximal fuzzy ideals on a ring in terms of their level sets.

Theorem 3.6. *Let R be a ring and I be a fuzzy ideal on R. Then, I is a maximal fuzzy ideal on R if and only if their level sets are maximal ideals on R.*

Proof. ($\Rightarrow$)
Suppose that I is a maximal fuzzy ideal on a ring R and we will show that I_α is a maximal ideal on R, for $\alpha \in [0,1]$. Suppose that I_α is not a maximal ideal (for some alpha) i.e., there exists an ideal J on R such that $I_\alpha \subset J$. We can consider the set J as one of the level sets of another fuzzy set J', for example, J'_{β_0} such that $\beta_0 \geq \alpha$ and $\beta_0 \in [0,1]$. From the definition of level sets, it follows that $\mu_I(x) \leq \mu_{J'}(x)$, for any $x \in R$. That is a contradiction with the fact that I is a fuzzy maximal ideal on R. Hence, I_α is a maximal ideal. Because for a given alpha I is single on R, for all $\alpha \in [0,1]$.
($\Leftarrow$)
Suppose that I_α are maximal ideals on R for all $\alpha \in [0,1]$ and we will show that I is a fuzzy maximal ideal on R. Suppose that I is not a maximal fuzzy ideal i.e., there exists a fuzzy ideal J on R such that $I \subset J$. From the properties of $\alpha-$cuts, we get

that $I_\alpha \subset J_\alpha$. That is a contradiction with the fact that I_α is a maximal ideal on R. Hence, I is a fuzzy maximal ideal. □

In the following theorem, we show the relationship between maximal fuzzy ideals and prime fuzzy ideals on a ring.

Theorem 3.7. *Let R be a ring. Then, every maximal fuzzy ideal on R is a prime fuzzy ideal on R.*

Proof. Suppose that I is a maximal fuzzy ideal on a ring R. Theorem 3.6 guarantees that I_α is a maximal ideal on R, for all $\alpha \in [0, 1]$. From Theorem 1.5, it holds that I_α is a prime ideal on R, for all $\alpha \in [0, 1]$. Theorem 3.5 guarantees that I is a prime fuzzy ideal on a ring R. □

Remark 3.4. *The converse of the above implication does not necessarily hold. Indeed, let us consider the ring $R = \mathbb{Z}/4\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ and let $I = \{\bar{0}\}$ such that $\mu_I(\bar{0}) = 1$. It is easy to verify that I is a prime fuzzy ideal but I is not a maximal fuzzy ideal, because there exists another ideal $J = \{\bar{0}, \bar{2}\}$ such that $\mu_I(\bar{0}) = \mu_J(\bar{2}) = 1$ and $I \subset J$. Hence, I is not a maximal fuzzy ideal.*

Conclusion

In this memory, we characterized the notion of a fuzzy ideal on a ring in terms of level sets. Also, we treat some types of fuzzy ideals on a ring such as the prime fuzzy ideal and the maximal fuzzy ideal. Moreover, we studied the relationship between them.

Future work is anticipated in multiple directions. We think it makes sense to study the notion of fuzzy ideal for other structures such as semigroups or fields. Moreover, we intend to extend this work to other types of fuzzy ideals such as a principal ideal and radical ideal.

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