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Theme

Solution for a Fractional Boundary Value Problems on the Infinite Interval

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ملخص

الهدف الرئسي من هذا العمل هو دراسة وجود ووحدانية حل معادلة تفاضلية كسرية غير خطية على فترة لانهائية بواسطة نظريتين للنقطة الثابتة ، نظرية بناخ ولاري شودر .

الكلمات الرئيسية: معادلة تفاضلية كسرية، وجود الحل، نظرية شودر للنقطة الصامدة ، نظرية بناخ للنقطة الصامدة، المشتقة الكسرية من نوع كاتوغامبولا.

Abstract

The principal objective of this work is the study the existence and uniqueness of solution for a fractional differential equation boundary value problem on an infinite interval by two fixed point theorems, Banach theorem and Leray-Schauder Alternative.

Keywords: Fractional differential equation,Boundary problem, Existence of solution,Leray-Schauder fixed point theorem,Banach fixed point theorems, Katugampola fractional derivative.

Résumé

Le but de ce travail est d'étudier l'existence et l'uniquence de la solution de problème aux limites d'équation différentielle fractionnaire non linéaire sur un intervalle infini par deux théorèmes du point fixe , theorème de Banach et Leray-Schauder Alternative.

Mots-Clés : Equation différentielle fractionnaire, Problème aux limites Existence de la solution, Théorème du point fixe de Leray-Schauder Alternative,Théorème du point fixe de Banach , Dérivée fractionnaire de Katugampola.

Introduction

Fractional calculus generalizes the order of derivative and integral from positive integers to real numbers, or even to complex numbers. In the last few decades, it is found that a series of natural phenomena can be modelled robustly in terms of fractional calculus, see [2, 10, 15]. As a result, fractional calculus gained a rapid development recently, both in the aspect of mathematics and many disciplines of applied sciences, being nowadays recognized as an excellent tool for describing complex systems and practical matters, especially involving long range memory effects and non-locality, such as viscoelastic theory, fluid dynamics, biology, image processing, one may refer [4].

Boundary value problem of fractional differential equations is recently approached by various researchers ([4], [13], [14]).

This thesis is devoted to the study of some nonlinear fractional boundary value problems on the infinite interval by using fixed point theorems. Since Riemann-Liouville and Caputo fractional derivatives are the most used in differential equations, we investigate, in the second chapter, the existence of solutions for differential equations involving Riemann-Liouville type derivative. As it is natural to look for generalizations of fractional derivatives and integrals, for which the known ones are particular cases, Katugampola introduced in [8] a new type of fractional derivative that generalizes both the Riemann–Liouville and Hadamard fractional derivatives.

In the first chapter, we introduce some functions that are of fundamental importance in the theory of fractional differential equations, Gamma function and Beta function. We provide some basic knowledge about fractional integrals and derivatives, such Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative, Caputo fractional derivative, Katugampola derivative and Caputo–Katugampola derivative, and some fixed point theorems .

In the second chapter, we detail the work submitted by Guotao Wang [14] , we prove the existence and uniquense of solutions for a fractional higher order boundary value problem on the infinite interval by tow fixed point theorems, Banach fixed point theorem and Leray-Schauder fixed point theorem

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t)) = 0, \quad n < \alpha \leq n+1, \\ u(0) = u'(0) = ... = u^{(n-1)} = 0, \qquad D^{\alpha-1}u(\infty) = \xi I^{\beta}u(\eta), \quad \beta > 0, \end{cases}$$

where $t \in J = [0, +\infty)$, $f \in C[J\text{X}\mathbb{R}, \mathbb{R}]$, $\xi \in \mathbb{R}$, $\eta \in J$, D^{α} is the Riemann-Liouville fractional derivatives of order α, and I^{β} is the Riemann-Liouville fractional integral of order β.

In the third chapter, we study the existence of solution for the following fractional initial value problem

$$\begin{cases} D^{\alpha,\rho}u(t) + f(t, u(t)) = 0, \quad n < \alpha \leq n+1, \\ \delta^{k}u(0) = 0, \quad k = 0, 1, ..., n-1, \qquad D^{\alpha-1,\rho}u(\infty) = \xi^{\rho} I^{\beta}u(\eta), \quad \beta > 0, \end{cases}$$

where $t \in J = [0, +\infty)$, $f \in C[J\text{X}\mathbb{R}, \mathbb{R}]$, $\xi \in \mathbb{R}$, $\eta \in J$, $D^{\alpha,\rho}$ is the Katugampola fractional derivatives of order α, and ${}^{\rho}I^{\beta}$ is the Katugampola fractional integral of order β.

Chapter 1

Preliminaries

In this chapter, we present the elementary definitions and basic notion of the theory of fractional calculus (Gamma function, Beta function), then we define the integral and the derivative of fractional order in the sense of Riemann-Liouville, Caputo, Katugampola and Caputo-Katugampola and their properties and some fixed point theorems.

1.1 Special Functions

1.1.1 The Gamma Function

Definition 1.1.1 *The Gamma function* $\Gamma(.)$ *is defined by the integral*

$$\Gamma(z)=\int_0^{\infty} \mathrm{e}^{-t}t^{z-1}\ \mathrm{d}t, \tag{1.1}$$

which converges in the right half of the complex plane, that is, $Re(z)>0$.

1. The Gamma function satisfies

$$\Gamma(z+1)=z\Gamma(z), \mathrm{Re}(z)>0,$$

2. For any integer$n\geq 0,$ we have

   a. $\Gamma(n+1)=n!,$

   b. $\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n!)\sqrt{\pi}}{4^n n!}.$

Proof.

1. We use the integration by parts

$$\begin{aligned}\Gamma(z+1)&=\int_0^{+\infty} t^z e^{-t}dt=\left[-t^z e^{-t}\right]_0^{+\infty}+z\int_0^{+\infty} t^{z-1}e^{-t}dt\\&=z\Gamma(z).\end{aligned}$$

2. a. For $z = n$

$$\begin{aligned}\Gamma(n+1) &= (n)\Gamma(n)\\ &= (n)(n-1)\Gamma(n-1)\\ &= n(n-1)(n-2)(n-3)\ldots\Gamma(1)\\ &= n!.\end{aligned}$$

b. By induction, $n \in \mathbb{N}$

- For $n = 0$, we have $\Gamma\left(0+\frac{1}{2}\right) = \frac{(0!)\sqrt{\pi}}{400!} = \sqrt{\pi}$.

- Assume that the formula is satisfying for $(n-1)$

i.e $\Gamma\left((n-1)+\frac{1}{2}\right) = \frac{(2(n-1))!\sqrt{\pi}}{4(n-1)(n-1)!}$,

We have

$$\begin{aligned}\Gamma\left(n+\frac{1}{2}\right) &= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\frac{((2(n-1))!\sqrt{\pi}}{4^{(n-1)}(n-1)!}\\ &= \left(\frac{2n-1}{2}\right)\frac{(2n-1)\sqrt{\pi}}{4^{(n-1)}(n-1)!} = \frac{2n(2n-1)(2n-2)!}{2n(2)(4)^{(n-1)}(n-1)!}\\ &= \frac{(2n)!\sqrt{\pi}}{4^n n!}.\end{aligned}$$

Hence $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n!)\sqrt{\pi}}{4^n n!}$.

Remark 1.1.1 *The Gamma function is not difined in* Z^-

For $z \in \mathbb{Q}^-$ *we use the formula*

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}.$$

Example 1.1.1 *For* $z = \frac{-1}{2}$

1. $\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi}$.

2. $\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{4\sqrt{\pi}}{3}$.

Graph of the Gamma function

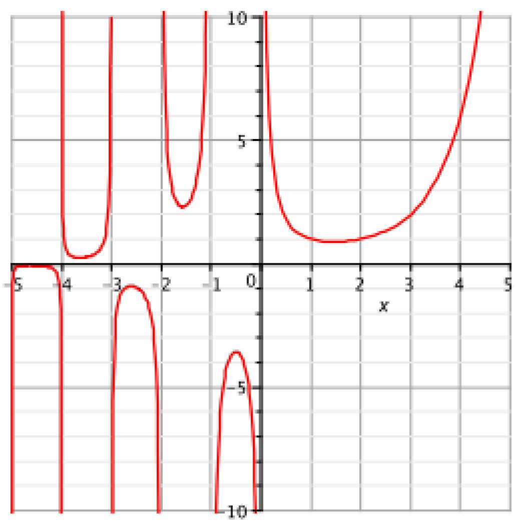

Figure 1.1: Graph of the Gamma function $\Gamma(x)$ in a real domain

Definition 1.1.2 *The reciprocal Gamma function is defined by*

$$\frac{1}{\Gamma(z)} = \frac{z}{\Gamma(z+1)}. \tag{1.2}$$

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \frac{z(z+1)\dots(z+n)}{n!n^z}. \tag{1.3}$$

Graph of the reciprocal Gamma function

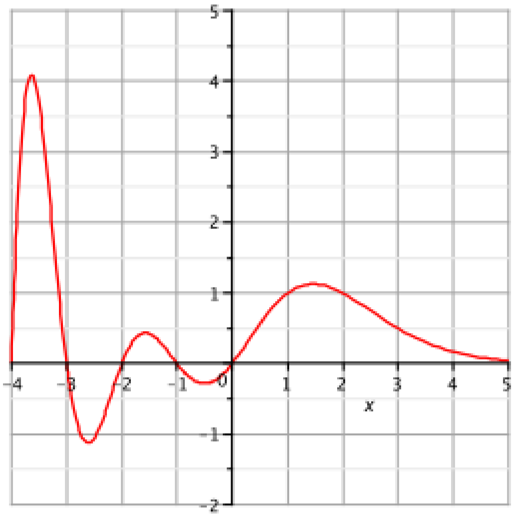

Figure 1.2: Graph of the reciprocal Gamma function $\frac{1}{\Gamma(x)}$ in a real domain.

1.1.2 The Beta Function

For every z, w satisfying $\Re\mathfrak{e}(z) > 0$ and $\Re\mathfrak{e}(w) > 0$.the Beta function is defined by:

$$B(z,w) = \int_0^1 \tau^{z-1}(1-\tau)^{w-1}\, \mathrm{d}\tau. \tag{1.4}$$

An interesting formula relating the Gamma and Beta functions is

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{1.5}$$

1.2 Fractional integrals and fractional derivatives

In this section, we focus on the Riemann-Liouville integrals and derivatives and the Caputo derivative since they are the most used ones in applications.

Definition 1.2.1 *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (a, +\infty) \to \mathbb{R}$ is giving by*

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds.$$

Definition 1.2.2 *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (a, +\infty) \to \mathbb{R}$ is given by*

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{ds}\right)^n \int_a^x \frac{f(s)}{(t-s)^{\alpha-n+1}} ds = \left(\frac{d}{ds}\right)^n I_{a^+}^{n-\alpha} f(s),$$

provided that the right side is pointwise defined on $(a, +\infty)$, where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of α.

Lemma 1.2.1 *Let $\alpha \geq \beta > 0$, then for $f \in L^P[a, b](1 \leq p \leq \infty)$ the relation*

$$\left(D_{a^+}^{\beta} I_{a+}^{\alpha} f\right)(t) = I_{a^+}^{\alpha-\beta} f(t),$$

holds almost everywhere on $[a, b]$. In particular if $\alpha = \beta$ we get

$$\left(D_{a+}^{\alpha} I_{a+}^{a} f\right)(t) = f(t).$$

Lemma 1.2.2 *The fractional integral operator I_{a+}^{α} is bounded from $L^P(a, b)(1 \leq p \leq \infty)$ into itself*

$$\left\|I_{a+}^{a} f\right\|_{L^p} \leq k\|f\|_{L^p}, \quad k = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

Definition 1.2.3 *Let $\alpha > 0$ and $n = [\alpha] + 1$, for a function $f \in AC^n([a, b], \mathbb{R})$ the Caputo fractional derivative of order α of f is defined by*

$$\begin{aligned}\left({}^C D_{a+}^{\alpha} f\right)(t) &= I^{n-\alpha} D^{(n)} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (t-s)^{n-\alpha-1} f^{(n)}(s) ds,\end{aligned}$$

where $D = \frac{d}{dt}$ denotes the classical derivative and $AC^n[a, b] = \{f \in C^{n-1}[a, b], f^{(n-1)}$ absolutely continuous function .

Properties. Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$, then the following relations hold:

$$\begin{aligned}I_{a^+}^{\alpha}(x-a)^{\beta-1}(t) &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1}, \\ D_{a^+}^{\alpha}(x-a)^{\beta-1}(t) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \\ {}^C D_{a^+}^{\alpha}(x-a)^{\beta-1}(t) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \beta > n.\end{aligned}$$

On the other hand, for $k = 1, 2, \ldots, n$, we have

$$D_{a^+}^{\alpha}(x-a)^{\alpha-k}(t) = 0, \quad k = 0, 1, \ldots, n-1.$$

and

$${}^C D_{a+}^{\alpha}(x-a)^{k}(t) = 0. \quad k = 0, 1, \ldots, n-1.$$

In particular,

$${}^{C}D^{\alpha}_{a+}(1) = 0.$$

The Riemann-Liouville fractional derivative of a constant is in general not equal to zero, in fact

$$D^{\alpha}_{a^+}(1) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0 < \alpha < 1.$$

Lemma 1.2.3 *Let* $\alpha > 0, \quad n = [\alpha] + 1$ *and* $f : [a, b] \to \mathbb{R}$ *be a given fonction. Assume that* $D^{\alpha}_{a^+} f$ *and* ${}^{C}D^{\alpha}_{a+} f$ *exist. Then*

$${}^{C}D^{\alpha}_{a+} f(t) = D^{\alpha}_{a+} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}.$$

Lemma 1.2.4 *Let* $\alpha > 0$*, then the solution of the fractional differential equation*

$$D^{\alpha}_{0^+} f(t) = 0,$$

is giving by $f(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \ldots + c_n t^{\alpha-n}, \quad c_i \in \mathbb{R}, i = 1, 2, \ldots, n.$

Lemma 1.2.5 *Let* $\alpha > 0, \quad n = [\alpha] + 1$*. If* $f \in L^1[a, b]$ *and* $f_{n-\alpha} \in AC^n[a, b]$*, then the equality*

$$\left(I^{\alpha}_{a^+} D^{\alpha}_{a^+} f\right)(t) = f(t) - \sum_{j=1}^{n} \frac{f^{(n-j)}_{n-a}(a)}{\Gamma(\alpha-j+1)} (t-a)^{\alpha-j},$$

holds almost everywhere on $[a, b]$*. In particular, if* $0 < \alpha < 1$*, then*

$$\left(I^{\alpha}_{a+} D^{\alpha}_{a^+} f\right)(t) = f(t) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1},$$

where $f_{n-\alpha} = I^{n-\alpha}_{a^+} f$ *and* $f_{1-\alpha} = I^{1-\alpha}_{a^+} f$*.*

Theorem 1.2.1 *Let* $\beta > \alpha > 0$*, then we have*

$$\left(I^{\alpha}_{a^+}\, {}^{C}D^{\alpha}_{a+} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k.$$

$$\left(D^{\alpha}_{a+} I^{\beta}_{a+} f\right)(t) = I^{\beta-\alpha}_{a+} f(t).$$

$$D^m D^{\alpha}_{a+} f(t) = D^{\alpha+m}_{a^+} f(t), m \in N.$$

Definition 1.2.4 *The Hadamard fractional integral of order* $\alpha > 0$ *of a function* f *is defined by*

$$\boldsymbol{I}^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds, \quad a < t < b.$$

A more general fractional integral referred as Hadamard fractional integral of order α *is given by*

$$\boldsymbol{I}^{\alpha,\mu}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{s}{t}\right)^{\mu} \left(\log \frac{t}{s}\right)^{a-1} \frac{f(s)}{s} ds, \quad a < t < b, \quad \mu \in \mathbb{R}.$$

Definition 1.2.5 *The Hadamard fractional derivative of order $\alpha > 0$ of a function f is defined by*

$$\boldsymbol{D}^{\alpha}_{a^+} f(t) = \left(t\frac{d}{dt}\right)^n \boldsymbol{I}^{n-\alpha}_{a^+} f(t), \quad a < t < b, \quad n = [\alpha] + 1.$$

A more general fractional derivative referred as Hadamard fractional derivative of order α is given by

$$\boldsymbol{D}^{\alpha,\mu}_{a^+} f(t) = t^{-\mu} \left(t\frac{d}{dt}\right)^n t^{\mu} \boldsymbol{I}^{n-\alpha,\mu}_{a^+} f(t), \quad a < t < b, \quad n = [\alpha] + 1.$$

1.3 Generalized fractional integrals and fractional derivatives

Katugampola in [8] introduced a new type of fractional derivative generalizing Riemann-Liouville and Hadamard fractional derivatives.

Definition 1.3.1 *(Katugampola fractional integrals) Let a, b be two real and $f : [a, b] \to \mathbb{R}$ be an integrable function. The Katugampola fractional integrals of order $\alpha > 0$, parameter $\rho > 0$, of f is defined as*

$${}^{\rho}I^{\alpha}_{a^+} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} \left(t^{\rho} - s^{\rho}\right)^{\alpha-1} f(s) ds.$$

Definition 1.3.2 *(Katugampola fractional derivative) Let $0 < a < b < \infty$ be two real $f : [a, b] \to \mathbb{R}$ be an integrable function. The Katugampola fractional derivative of order $\alpha > 0$, and panameter $\rho > 0$, is defined as*

$$\begin{aligned} D^{\alpha,\rho}_{a+} f(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^n {}^{\rho}I^{n-\alpha}_{a+} f(t) \\ &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t s^{\rho-1} \left(t^{\rho} - s^{\rho}\right)^{n-\alpha-1} f(s) ds. \end{aligned}$$

Proposition 1.3.1 *We have the following properties for Katugompola fractional integral and derivative.*

1. $D^{\alpha,\rho}_{a+} \left({}^{\rho}I^{\alpha}_{a+}\right) f(t) = f(t)$.
2. ${}^{\rho}I^{\alpha}_{a^+} \left({}^{\rho}I^{\beta}_{a+}\right) f(t) = {}^{\rho}I^{\alpha+\beta}_{a+} f(t)$.
3. $\lim\limits_{\rho \to 1} {}^{\rho}I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} f(s) ds$.
4. $\lim\limits_{\rho \to 0^+} {}^{\rho}I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}$.
5. $\lim\limits_{\rho \to 1} D^{\alpha,\rho}_{a+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \displaystyle\int_a^t (t-s)^{n-\alpha-1} f(s) ds$.

1.4 Caputo-Katugampola fractional derivative

Definition 1.4.1 *(Caputo-Katugampola fractional derivative) Let $0 < a < b < \infty$ be two real, $\rho > 0$ be a positive real number and $f \in AC^n([a,b],\mathbb{R})$. The Caputo-Katugampola fractional derivative of order $\alpha > 0$ of the function f is defined by*

$$\begin{aligned} {}^C D_{a^+}^{\alpha,\rho} f(t) &= {}^\rho I_{a^+}^{n-\alpha} \left(t^{1-\rho} \frac{d}{dt} \right)^n f(t) \\ &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \int_a^t s^{\rho-1} \left(t^\rho - s^\rho\right)^{n-\alpha-1} \left(t^{1-\rho} \frac{d}{dt} \right)^n f(s) ds \end{aligned}$$

where n is the smallest integer greater than α.

Properties.

1- When $\rho = 1$, the Caputo-Katugampola derivative coincides with Caputo derivative.

2- In the case $0 < \alpha < 1$ and $\rho > 0$, then

$${}^C D_{a^+}^{\alpha,\rho} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} t^{1-\rho} \frac{d}{dt} \int_a^t \frac{s^{(\rho-1)}(f(s) - f(a))}{\left(t^\rho - s^\rho\right)^\alpha} ds.$$

3- If $f \in C[a,b]$ then

$${}^C D_{a^+}^{\alpha,\rho} I_{a^+}^{\alpha,\rho} f(t) = f(t),$$

and if $f \in C^1[a,b]$ then

$$I_{a^+}^{\alpha,\rho} \; {}^C D_{a^+}^{\alpha,\rho} f(t) = f(t) - f(a).$$

4- If $f(a) = 0$, then the Caputo Katugampola and the Katugampola fractional derivatives coincide. Moreover if both types of derivatives exist then

$${}^C D_{a^+}^{\alpha,p} f(t) = D_{a^+}^{\alpha\rho\rho} f(t) - \frac{f(a)\rho^\alpha \left(t^\rho - s^\rho\right)^{-\alpha}}{\Gamma(1-\alpha)}.$$

1.5 Fixed point theorems

Fixed point theory is an important topic with a big number of applications in various fields of mathematics. The fixed point theorems concern a function f satisfying some conditions and admits a fixed point, that is $f(x) = x$.

Theorem 1.5.1 ***(Banach contraction principle)*** *Let T be a contraction on a Banach space X. Then T has a unique fixed point.*

Theorem 1.5.2 ***(Leray-SchauderAlternative)****Let C be a convex subset of a Banach space, U be an open subset of C with $0 \in U$. Then every completely continuous map $N : \bar{U} \to C$ has at least one of the following two properties: (A1) N has a fixed point in $\bar{U}$; or (A2) There is an $x \in \partial U$ and $\lambda \in (0,1)$ with $x = \lambda N x$.*

Theorem 1.5.3 ***(Schauder fuxed point theorem)*** *Let Ω be a nonempty closed bounded and convex subset of a normed space. Let N be a continuous mapping from Ω into a compact subset of Ω, then N has a fixed point in Ω.*

Theorem 1.5.4 ***(Arzela-Ascoli theorem)*** *A set $\Omega \subset C([a,b]\ )$ is relatively compact in $C([a,b])$ iff the functions in Ω are uniformly bounded and equicontinuous on $[a,b]$.*
We recall that a family Ω of continuous functions is uniformly bounded if there exists $M > 0$ such that

$$\|f\| = \max_{x\in[a,b]} |f(x)| \leq M, f \in \Omega.$$

The family Ω is equicontinuous on $[a,b]$, if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall t_1, t_2 \in [a,b]$ and $\forall f \in \Omega$, we have

$$|t_1 - t_2| < \delta \Rightarrow |f(t_1) - f(t_2)| < \varepsilon.$$

The criteria for compactness for sets in the space of integrable functions $L^p(0,1)$ is the following.

Chapter 2

Fractionel Boundary Value Problems on the infinit interval

In this chapter, we will study the existence and the uniqueness of solution of a fractional integral boundary value problem for nonlinear fractional differential equation on an infinite interval indicate in (2.1) (we will detail the work submitted by Guotao Wang (see [14])) by some fixed point theorems.

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t))=0, \quad n<\alpha\leq n+1, \\ u(0)=u'(0)=...=u^{(n-1)}=0, \qquad D^{\alpha-1}u(\infty)=\xi I^{\beta}u(\eta), \quad \beta>0, \end{cases} \tag{2.1}$$

where $t\in J=[0,+\infty)$, $f\in C[J\text{X}\mathbb{R},\mathbb{R}]$, $\xi\in\mathbb{R}$, $\eta\in J$. D^{α} is the Riemann-Liouville fractional derivatives of order α, and I^{β} is the Riemann-Liouville fractional integral of order β.

2.1 Lemmas

In this section, we will introduce notations, definitionas and some useful lemmas, which will play an important role in the proof of our main results([13]). Denote the space

$$FC\,(J,\mathbb{R})=\left\{u\in C(J,\mathbb{R}):\sup_{t\in J}\frac{|u(t)|}{1+t^{\alpha-1}}<\infty\right\},$$

with norm

$$\|u\|_F=\sup_{t\in J}\frac{|u(t)|}{1+t^{\alpha-1}}.$$

Obviously, $FC(J,\mathbb{R})$ is a Banach space.
A map $u(t)\in C(J,\mathbb{R})$ with its Riemann-Liouville derivative of order α existing on J is called a solution of (1.1) if it satisfies (1.1).

For convenience, we list the following assumptions:
(H_1) $\xi\geq 0,\quad \Gamma(\alpha+\beta)>\xi\eta^{a+\beta-1}$.

(H$_2$) there exist a positive function $p(t)$ with $p^* = \int_0^\infty (1+t^{\alpha-1})\, p(t)dt < \infty$ such that

$$|f(t,u) - f(t,v)| \leq p(t)|u-v|, \quad t \in J, u, v \in \mathbb{R},$$

and

$$\lambda = \int_0^\infty |f(t,0)|dt < \infty.$$

(H$_3$) Let $F(t,u) = f\left(t, (1+t^{\alpha-1})\, u\right), |F(t,u)| \leq \varphi(t)w(|u|)$ on $[0,\infty)X\mathbb{R}$ with $w \in C([0,\infty),[0,\infty))$ nondecreasing and $\varphi \in L^1[0,+\infty)$.

Lemma 2.1.1 *Let $h \in C([0,+\infty))$ with $\int_0^\infty h(s)ds < \infty$, if $\Gamma(\alpha+\beta) \neq \xi\eta^{a+\beta-1}$, then the fractional integral boundary value problem*

$$\begin{cases} D^\alpha u(t) + h(t) = 0, \\ u(0) = u'(0) = ... = u^{(n-1)} = 0. \quad D^{\alpha-1}u(\infty) = \xi I^{\dot{\beta}} u(\eta), \quad \beta > 0, \end{cases} \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^{+\infty} G(t,s)h(s)ds.$$

where

$$G(t,s) = \frac{1}{\Delta}\begin{cases} \left[\Gamma(\alpha+\beta) - \xi(\eta-s)^{\alpha+\beta-1}\right] t^{\alpha-1} - \left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right)(t-s)^{\alpha-1}, & s \leq t, s \leq \eta. \\ \left[\Gamma(\alpha+\beta) - \xi(\eta-s)^{\alpha+\beta-1}\right] t^{\alpha-1}, & 0 \leq t \leq s \leq \eta, \\ \Gamma(\alpha+\beta)\left[t^{\alpha-1} - (t-s)^{\alpha-1}\right] + \xi\eta^{\alpha+\beta-1}(t-s)^{a-1}, & 0 \leq \eta \leq s \leq t, \\ \Gamma(\alpha+\beta)t^{\alpha-1}, & s \geq t, s \geq \eta, \end{cases} \tag{2.3}$$

and

$$\Delta = \Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right].$$

Proof. Let $h \in C([0,+\infty))$ with $\int_0^\infty h(s)ds < \infty$, if $\Gamma(\alpha+\beta) \neq \xi\ \eta^{\alpha+\beta-1}$ we have the probleme

$$\begin{cases} D^\alpha u(t) + h(t) = 0, & n < \alpha < n+1, \\ u(0) = u'(0) = ... = u^{(n-1)} = 0; & D^{\alpha-1}u(\infty) = \xi I^\beta u(\eta), \beta > 0. \end{cases}$$

By applying I_{0+}^α to equation we get

$$I^\alpha D^\alpha u(t) + I^\alpha h(t) = 0,$$

we have

$$I^\alpha D^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_{n+1}t^{\alpha-n-1}, \qquad c_1, c_2, ..., c_{n+1} \in \mathbb{R},$$

then

$$u(t) = -c_1 t^{\alpha-1} - c_2 t^{\alpha-2} - ... - c_{n+1}t^{\alpha-n-1} - I^\alpha h(t).$$

The conditions $u(0) = u'(0) = ... = u^{(n-1)} = 0$ implies that

$$u(0) = 0 \Leftrightarrow c_{n+1} \lim_{t \to 0} t^{\alpha-n-1} = 0 \Rightarrow c_{n+1} = 0.$$

$$u'(t) = -c_1(\alpha - 1)t^{\alpha-2} - c_2(\alpha - 2)t^{\alpha-3} - ..... - c_n(\alpha - n)t^{\alpha-n-1} - I^{\alpha-1}h(t),$$

$$u'(0) = 0 \Leftrightarrow -c_n(\alpha - n) \lim_{t \to 0} t^{\alpha-n-1} = 0,$$

$$\Leftrightarrow c_n = 0.$$

.

.

$$u^{(n-1)} = 0 \Leftrightarrow -c_2(\alpha - 2)(\alpha - 3)....(\alpha - n) \lim_{t \to 0} t^{\alpha-n-1} = 0,$$

$$\Leftrightarrow c_2 = 0.$$

So

$$u(t) = -c_1 t^{\alpha-1} - I^\alpha h(t).$$

and the conditions $D^{\alpha-1}u(\infty) = \xi I^\beta u(\eta)$ implies that

$$\begin{aligned} D^{\alpha-1}u(t) &= -c_1 D^{\alpha-1}t^{\alpha-1} - D^{\alpha-1}I^\alpha h(t) \\ &= -c_1\Gamma(\alpha) - I_1 h(t) \\ &= -c_1\Gamma(\alpha) - \int_0^t h(s)ds. \end{aligned}$$

Then

$$D^{\alpha-1}u(\infty) = -c_1\Gamma(\alpha) - \int_0^\infty h(s)ds = \xi I^\beta u(\eta).$$

So

$$-c_1\Gamma(\alpha) - \int_0^{+\infty} h(s)ds = -c_1\xi I^\beta \eta^{\alpha-1} - \xi I^{\alpha+\beta}h(\eta)$$

$$c_1\left(\xi I^\beta \eta^{\alpha-1} - \Gamma(\alpha)\right) = \int_0^{+\infty} h(s)ds - \xi I^{\alpha+\beta}h(\eta).$$

$$\begin{aligned} c_1 &= \frac{1}{\xi I^\beta \eta^{\alpha-1} - \Gamma(\alpha)}\left[\int_0^{+\infty} h(s)ds - \xi I^{\alpha+\beta}h(n)\right] \\ &= \frac{1}{\xi\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\eta^{\alpha+\beta-1}\right) - \Gamma(\alpha)}\left[\int_0^{+\infty} h(s)ds - \xi\left(\frac{1}{\Gamma(\alpha+\beta)}\int_0^\eta (\eta - s)^{\alpha+\beta-1}h(s)ds\right)\right] \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left(\xi\eta^{\alpha+\beta-1} - \Gamma(\alpha+\beta)\right)}\int_0^{+\infty} h(s)ds \\ &- \frac{\xi}{\Gamma(\alpha)\left(\xi\eta^{\alpha+\beta-1} - \Gamma(\alpha+\beta)\right)}\int_0^\eta (\eta - s)^{\alpha+\beta-1}h(s)ds. \end{aligned}$$

So

$$-c_1 = \frac{1}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta) - \{\eta^{\alpha+\beta-1}\right)} \left[\Gamma(\alpha+\beta)\int_0^{+\infty} h(s)ds - \xi\int_0^{\eta}(\eta-s)^{\alpha+\beta-1}h(s)ds\right]$$
$$= \frac{1}{\Delta}\left[\int_0^{\eta}\left(\Gamma(\alpha+\beta) - \xi(\eta-s)^{\alpha+\beta-1}\right)h(s)ds + \int_{\eta}^{+\infty}\Gamma(\alpha+\beta)h(s)ds\right].$$

Hence

$$u(t) = \frac{1}{\Delta}\left[\int_0^{\eta}\left(\Gamma(\alpha+\beta) - \xi(\eta-s)^{\alpha+\beta-1}\right)h(s)ds + \int_{\eta}^{+\infty}\Gamma(\alpha+\beta)h(s)ds\right]t^{\alpha-1}$$
$$- \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds$$
$$= \int_0^{+\infty} G(t,s)h(s)ds,$$

where

$$G(t,s) = \frac{1}{\Delta}\begin{cases} \left[\Gamma(\alpha+\beta) - \xi(\eta-s)^{\alpha+\beta-1}\right]t^{\alpha-1} - \left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right)(t-s)^{\alpha-1}, & s \le t, s \le \eta. \\ \left[\Gamma(\alpha+\beta) - \xi(\eta-s)^{\alpha+\beta-1}\right]t^{\alpha-1}, & 0 \le t \le s \le \eta, \\ \Gamma(\alpha+\beta)\left[t^{\alpha-1} - (t-s)^{\alpha-1}\right] + \xi\eta^{\alpha+\beta-1}(t-s)^{a-1}, & 0 \le \eta \le s \le t, \\ \Gamma(\alpha+\beta)t^{\alpha-1}, & s \ge t, s \ge \eta. \end{cases}$$

Lemma 2.1.2 *If (H1) holds, then Green's function $G(t,s)$ satisfies*

1. $G(t,s) \ge 0, \quad \forall t, s \in (0,\infty)$

2. $\dfrac{G(t,s)}{1+t^{\alpha-1}} \le \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right]}, \quad \forall t, s \in (0,\infty).$

Proof.

1. We have four cases

Case 1: For $s \le t$ and $\xi \le \eta$,

we have $\quad t \ge t-s \Leftrightarrow t^{\alpha-1} \ge (t-s)^{\alpha-1},$

$$\text{and } \eta - s \leqslant \eta \Leftrightarrow (\eta-s)^{\alpha+\beta-1} \leqslant \eta^{\alpha+\beta-1}$$
$$\Leftrightarrow -\xi(\eta-s)^{\alpha+\beta-1} \geqslant -\xi\eta^{\alpha+\beta-1}$$
$$\Leftrightarrow \Gamma(\alpha+\beta) - \xi(\eta-s)^{\alpha+\beta-1} \geqslant \Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}$$
$$\Leftrightarrow \left[\Gamma(\alpha+\beta) - \xi(\eta-s)^{\alpha+\beta-1}\right]t^{\alpha-1} \geqslant \left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right](t-s)^{\alpha-1}$$
$$\Leftrightarrow \left[\Gamma(\alpha+\beta) - \xi(\eta-s)^{\alpha+\beta-1}\right]t^{\alpha-1} - \left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right](t-s)^{\alpha-1} \geqslant 0,$$

an other hand by (H1), $\Gamma(\alpha+\beta) > \xi\eta^{\alpha+\beta-1}$ so $\frac{1}{\Delta} > 0$,

Hence $G(t,s) > 0$.

Case 2: For $0 \le t \le s \le \eta$, We have also by (H1)

$$\begin{aligned}\Gamma(\alpha+\beta) > \xi\eta^{\alpha+\beta-1} &\geq \xi(\eta-s)^{\alpha+\beta-1}\\ &\Rightarrow \Gamma(\alpha+\beta)-\xi(\eta-s)^{\alpha+\beta-1} \geq 0\\ &\Rightarrow \left[\Gamma(\alpha+\beta)-\xi(\eta-s)^{\alpha+\beta-1}\right] t^{\alpha-1} \geq 0.\end{aligned}$$

Case 3: For $0 \leq \eta \leq s \leq t$.
We have

$$\begin{aligned}t \geq t-s &\Leftrightarrow t^{\alpha-1} \geq (t-s)^{\alpha-1}\\ &\Leftrightarrow t^{\alpha-1}-(t-s)^{\alpha-1} \geq 0\\ &\Leftrightarrow \Gamma(\alpha+\beta)\left[t^{\alpha-1}-(t-s)^{\alpha-1}\right] \geq 0.\end{aligned}$$

We also have

$$\begin{aligned}t \geq s &\Leftrightarrow t-s \geq 0\\ &\Leftrightarrow (t-s)^{\alpha-1} \geq 0\\ &\Leftrightarrow \xi\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} \geq 0.\end{aligned}$$

So

$$\Gamma(\alpha+\beta)\left[t^{\alpha-1}-(t-s)^{\alpha-1}\right]+\xi\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} \geq 0.$$

Hence

$$G(t,s) > 0.$$

Case 4: For $s \geq t$ and $s \geq \eta$,
we have

$$\Gamma(\alpha+\beta)t^{\alpha-1} \geq 0.$$

According the four cases precedding if $(H1)$ holds, then $G(t,s) > 0$.
2. Now we will prove that:

$$\frac{G(t,s)}{1+t^{\alpha-1}} \leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right]} = \frac{\Gamma(\alpha+\beta)}{\Delta}.$$

Case 1: For $s \leqslant t$ and $s \leq \eta$,
we have :

$$\begin{aligned}\left[\Gamma(\alpha+\beta)-\xi(\eta-s)^{\alpha+\beta-1}\right] t^{\alpha-1}-\left[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right](t-s)^{\alpha-1} &\leqslant \Gamma(\alpha+\beta)t^{\alpha-1}\\ &\leqslant \Gamma(\alpha+\beta)(1+t^{\alpha-1}),\end{aligned}$$

So

$$\Delta \cdot G(t,s) \leqslant \Gamma(\alpha+\beta)\left(1+t^{\alpha-1}\right).$$

Then

$$\frac{G(t,s)}{1+t^{\alpha-1}} \leqslant \frac{\Gamma(\alpha+\beta)}{\Delta}.$$

Case 2: For $0 \leqslant t \leqslant s \leqslant \eta$,
we have:

$$\left[\Gamma(\alpha+\beta) - \xi(\eta - s)^{\alpha+\beta-1}\right] t^{\alpha-1} \leqslant \Gamma(\alpha+\beta)t^{\alpha-1} \leqslant \Gamma(\alpha+\beta)\left(1 + t^{\alpha-1}\right),$$

So

$$\Delta G(t,s) \leq \Gamma(\alpha+\beta)(1+t^{\alpha-1}) \Rightarrow \frac{G(t,s)}{1+t^{\alpha-1}} \leqslant \frac{\Gamma(\alpha+\beta)}{\Delta}.$$

Case 3: For $0 \leq \eta \leq s \leq t$,
we have

$$\Gamma(\alpha+\beta) > \xi\eta^{\alpha+\beta-1},$$

so

$$\Gamma(\alpha+\beta)(t-s)^{\alpha-1} > \xi\eta^{\alpha+\beta-1}(t-s)^{\alpha-1},$$

then

$$\xi\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} - \Gamma(\alpha+\beta)(t-s)^{\alpha-1} \leq 0.$$

Hence

$$\Gamma(\alpha+\beta)t^{\alpha-1} - \Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \xi\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} \leqslant \Gamma(\alpha+\beta)t^{\alpha-1},$$

then

$$\Gamma(\alpha+\beta)\left[t^{\alpha-1} - (t-s)^{\alpha-1}\right] + \xi\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} \leqslant \Gamma(\alpha+\beta)t^{\alpha-1} \leqslant \Gamma(\alpha+\beta)(1+t^{\alpha-1}).$$

So

$$\Delta G(t.s) \leq \Gamma(\alpha+\beta)(1+t^{\alpha-1}),$$

Hence

$$\frac{G(t.s)}{1+t^{\alpha-1}} \leq \frac{\Gamma(\alpha+\beta)}{\Delta}.$$

Case 4: For $s \geq t$ and $s \geq \eta$,
we have

$$\Gamma(\alpha+\beta)t^{\alpha-1} \leq \Gamma(\alpha+\beta)(1+t^{\alpha-1}),$$

so

$$\Delta G(t.s) \leq \Gamma(\alpha+\beta)(1+t^{\alpha-1}),$$

hence

$$\frac{G(t.s)}{1+t^{\alpha-1}} \leq \frac{\Gamma(\alpha+\beta)}{\Delta}.$$

By according to the precedding cases we have

$$\frac{G(t,s)}{1+t^{\alpha-1}} \leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right]}.$$

Lemma 2.1.3 *If $(H2)$ holds, then for any $u \in FC(J,\mathbb{R})$*

$$\int_0^{\infty} |f(s,u(s))|ds \leq p^* \|u\|_F + \lambda. \tag{2.4}$$

Proof . For any $u \in FC(J,\mathbb{R})$, by taking $v = 0$ in (H_1), we have

$$\begin{aligned} |f(t,u(t))| &\leq p(t)|u(t)| + |f(t,0)| \\ &\leq p(t)\left(1+t^{\alpha-1}\right)\frac{|u(t)|}{1+t^{\alpha-1}} + |f(t,0)| \\ &\leq p(t)\left(1+t^{\alpha-1}\right)\|u\|_F + |f(t,0)|, \end{aligned} \tag{2.5}$$

we integre

$$\int_0^{+\infty} |f(s,u(s))|ds \leq \int_0^{+\infty} p(s)\left(1+s^{\alpha-1}\right)ds\|u\|_F + |f(t,0)|$$

$$\int_0^{+\infty} |f(s,u(s))|ds \leq p^*||u||_F + \lambda. \tag{2.6}$$

2.2 Existence and uniqueness of solution by Banach fixed point theorem

In this section, we study the existence and uniqueness of solution by Banach fixed point theorem, we assume that $(H1)$ and $(H2)$ hold

Theorem 2.2.1 *Assume that $(H1)$ and $(H2)$ hold. If*

$$m = \frac{p^*\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right]} < 1, \tag{2.7}$$

then the fractional integral boundary value problem (2.1) has a unique solution $\overline{u}(t)$ in $FC(J,\mathbb{R})$. Moreover, there exists a monotone iterative sequence $un(t)$such that $u_n(t) \longrightarrow \overline{u}(t)$ as $n \to \infty$ uniformly on any finite sub-interval of J, where

$$u_n(t) = \int_0^{\infty} G(t,s) f\left(s, u_{n-1}(s)\right) ds, \tag{2.8}$$

In addition, there exists an error estimate for the approximation sequence

$$\|u_n - \bar{u}\|_F \leq \frac{m^n}{1-m}\|u_1 - u_0\|_F, \quad (n = 1,2,\ldots). \tag{2.9}$$

Proof. Define the operator Q by

$$(Qu)(t) = \int_0^{\infty} G(t,s)f(s,u(s))ds. \tag{2.10}$$

By Lemma 2.1.1, fractional integral boundary value problem (2.1) has a solution u if and only if u solves the operator equation $u = Qu$, *i.e.* The solution of (2.1) coincides with the fixed point of the operator Q. First, for any $u \in FC(J, \mathbb{R})$, by Lemmas 2.1.2 and 2.1.3, we have

$$\begin{aligned} \frac{|(Qu)(t)|}{1+t^{\alpha-1}} &\leq \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}}|f(s,u(s))|ds \\ &\leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right]} \\ &= m\|u\|_F + k. \end{aligned} \tag{2.11}$$

where m is defined in (2.17) and $k = \frac{\lambda\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right]}$. In addition, for any $u, v \in FC(J, \mathbb{R})$, we have

$$\begin{aligned} \frac{|(Qu)(t)-(Qv)(t)|}{1+t^{\alpha-1}} &\leq \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}}|f(s,u(s))-f(s,v(s))|ds \\ &\leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right]}\int_0^\infty p(s)\left(1+s^{\alpha-1}\right)\|u-v\|_F ds \\ &\leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right]}p^*\|u-v\|_F ds \\ &= m\|u-v\|_F. \end{aligned} \tag{2.12}$$

Thus,

$$\|Qu-Qv\|_F \leq m\|u-v\|_F, \quad \forall u,v \in FC(J,\mathbb{R}) \tag{2.13}$$

As $m < 1$, then the Banach fixed point theorem ensures that Q has a unique fixed point $\bar{u}$ in $FC(J, \mathbb{R})$, i.e. The integral boundary value problem (2.1) has a unique solution $\bar{u} \in FC(J, \mathbb{R})$ ndoreover, for any $u_0 \in FC(J, \mathbb{R})$, $\|u_n - \bar{u}\|_F \to 0$ as $n \to \infty$, where $u_n = Qu_{n-1}(n = 1, 2, \ldots)$

From (2.14), we have

$$\|u_n - u_{n-1}\|_F \leq m^{n-1}\|u_1 - u_0\|_F,$$

and

$$\begin{aligned} \|u_n - u_j\|_F &\leq \|u_n - u_{n-1}\|_F + \|u_{n-1} - u_{n-2}\|_F + \cdots + \|u_{j+1} - u_j\|_F \\ &\leq \frac{m^n\left(1-m^{n-j}\right)}{1-m}\|u_1 - u_0\|_F. \end{aligned} \tag{2.14}$$

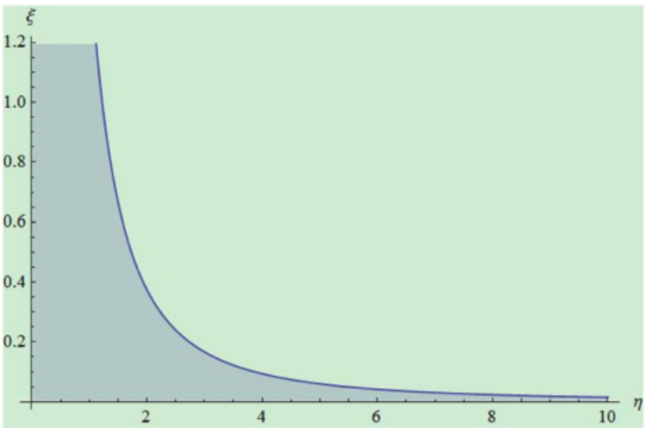

Figure 2.1: set of $(\eta, \xi) \in \mathbb{R}^+\mathbb{R}^+$ such that $\xi\eta^2 \leqslant 2 - \frac{8}{\sqrt{\pi}}$

Letting $n \to \infty$ in both sides of (2.15), we can obtain

$$\|u_n - \bar{u}\|_F \leq \frac{m^n}{1-m} \|u_1 - u_0\|_F ,$$

Hence Eq. (2.10) holds, and the theorem is proved.

2.2.1 Example

Consider the problem

$$\begin{cases} D^{\frac{5}{2}}u(t) + \frac{e^{-3t}}{(1+t^3)^3} \sin\left(3t^2 + u(t)\right) = 0, \quad t \in J = [0, \infty), \\ u(0) = u'(0) = 0, \quad D^{\frac{3}{2}}u(\infty) = \xi I^{\beta} u(\eta), \end{cases} \tag{2.15}$$

where $\alpha = \frac{5}{2}, \beta = \frac{1}{2}, f(t,u) = \frac{e^{-3t}}{(1+t^3)^3} \sin\left(3t^2 + u\right), \xi, \eta$ satisfy $0 \leq \xi\eta^2 \leq 2 - \frac{3}{9\sqrt{*}}$ (see Fig, 2.1). For example, we can take $\xi = 4, \eta = \frac{1}{2}$. Next, we will verify the conditions of Theorem (2.2.1): First, it is clear that $\Gamma(\alpha + \beta) = \Gamma(3) = 2 > \xi\eta^2 = \xi\eta^{\alpha+\beta-1}$, (H_1) holds. Second,

$$\begin{aligned} |f(t,-u) - f(t,v)| &\leq \frac{e^{-3t}}{\left(1+t^{\frac{3}{2}}\right)^3} \left|\sin\left(3t^2 + u\right) - \sin\left(3t^2 + v\right)\right| \\ &\leq \frac{e^{-3t}}{\left(1+t^{\frac{3}{2}}\right)^3} |u - v|. \end{aligned} \tag{2.16}$$

Noting that $p(t) = \frac{e^{-3t}}{\left(1+t^{\frac{3}{2}}\right)^3}$, thus, we have

$$p^* = \int_0^\infty \left(1 + t^{\frac{3}{2}}\right) p(t)dt \leq \int_0^\infty e^{-3t} \, \mathrm{d}t = \frac{1}{3} < \infty,$$

and

$$\lambda = \int_0^\infty |f(t,0)|dt \leq \int_0^\infty e^{-3t} dt = \frac{1}{3} < \infty.$$

Then (H_2) holds. At last, by a simple computation, we have

$$m = \frac{p^*\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right]} \leq \frac{8}{9\sqrt{\pi}\,(2 - \xi\eta^2)} \leq \frac{8}{9\sqrt{\pi}} < 1. \tag{2.17}$$

Hence, all conditions of Theorem 2.2.1 are satisfied. Therefore, Theorem 2.2.1 implies that (2.16) has a unique solution, which can be obtained by taking limits from some iterative sequences.

2.3 Existence of solution by Leray-schauder Nonlinear Alternative theorem

In this section, we study the existence of solution by Leray-schauder Nonlinear Alternative theorem.

For $u \in FC(J, \mathbb{R})$, define the operator Q by

$$(Qu)(t) = \int_0^\infty G(t,s) f(s, u(s)) ds.$$

Boundary value problem (2.1) has a solution u if and only if u solves the operator equation $u = Qu$. Thus we set out to verify that the operator Q satisfies Theorem 1.5.2, which will prove the existence of the fixed points of Q. Since the Arzela-Ascoli theorem fails to work in the space $FC(J, \mathbb{R})$, we need a modified compactness criterion to prove T is compact.

Lemma 2.3.1 *Let* $V = \{u \in FC(J, \mathbb{R}) \mid \|u\|_F < l\}\ (l > 0), V_1 = \left\{\frac{u(t)}{1+t^{\alpha-1}} \mid u \in V\right\}$. *If* V_1 *is equicontinuous on any compact intervals of* $[0, +\infty)$ *and equiconvergent at infinity, then* V *is relatively compact on* $FC(J, \mathbb{R})$.

Remark 2.3.1 V_1 *is called equiconvergent at infinity if and only if for all* $\epsilon > 0$, *there exists* $v = v(\epsilon) > 0$ *such that for all* $u \in V_1, t_1, t_2 \geq v$, *it holds,*

$$\left| \frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}} \right| < \epsilon.$$

Lemma 2.3.2 *If* (H1) – (H3) *hold, then* $Q : FC(J, \mathbb{R}) \to FC(J, \mathbb{R})$ *is completely continuous.*

Proof We divided the proof into three steps.
Step 1: We show that $Q : FC(J, \mathbb{R}) \to FC(J, \mathbb{R})$ is continuous.
Let $u_n \to u$ as $n \to +\infty$ in $FC(J, \mathbb{R})$,
We have

$$\begin{aligned} \frac{|(Qu_n)(t) - (Qu)(t)|}{1+t^{\alpha-1}} &\leq \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} |f(s, u_n(s)) - f(s, u(s))| ds \\ &\leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right]} \int_0^\infty p(s)\left(1+s^{\alpha-1}\right) \|u_n - u\|_F ds \\ &\leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right]} p^* \|u_n - u\|_F ds \\ &= m\|u_n - u\|_F. \end{aligned}$$

Thus,

$$\|Qu_n - Qu\|_F \leq m\|u_n - u\|_F, \quad \to 0, \quad \text{as} \quad n \to +\infty$$

So, Q is continuous.
Step 2: We show that $Q : FC(J, \mathbb{R}) \to FC(J, \mathbb{R})$ is relatively compact.
Let Ω be any bounded subset of $FC(J, \mathbb{R})$, then there exists $K > 0$ such that $\|u\|_F \leq K$.
We have

$$\begin{aligned} \|Qu\|_F = \sup_{t \in J} \frac{|(Qu)(t)|}{1+t^{\alpha-1}} &\leq \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} |f(s, u(s))| ds \\ &\leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right]} \left[p^*\|u\|_F + \lambda\right] \\ &= m\|u\|_F + k < \infty. \end{aligned}$$

where m is defined in (3.4) and $k = \dfrac{\lambda\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right]}$.
Hence $Q\Omega$ is uniformly bounded.

Now we show that $Q\Omega$ is equicontinuous on any compact interval of $[0,\infty)$. For any $T > 0, t_1, t_2 \in [0,T]$, and $u \in \Omega$, without loss of generality, we may assume that $t_2 > t_1$. In fact,

$$\begin{aligned}
\left|\frac{(Qu)(t_2)}{1+t_2^{\alpha-1}} - \frac{(Qu)(t_1)}{1+t_1^{\alpha-1}}\right| &= \left|\int_0^\infty \frac{G(t_2,s)}{1+t_2^{\alpha-1}} f(s,u(s))ds - \int_0^\infty \frac{G(t_1,s)}{1+t_1^{\alpha-1}} f(s,u(s))ds\right| \\
&= \int_0^\infty \left[\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_2^{\alpha-1}} + \frac{G(t_1,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right] |f(s,u(s))|ds \\
&= \int_0^\infty \left[\frac{G(t_2,s) - G(t_1,s)}{1+t_2^{\alpha-1}} + G(t_1,s)\left(\frac{1}{1+t_2^{\alpha-1}} - \frac{1}{1+t_1^{\alpha-1}}\right)\right] |f(s,u(s))|ds \\
&\leq \int_0^\infty \left[\frac{2\Gamma(\alpha+\beta)\left({t_2}^{\alpha-1} - {t_1}^{\alpha-1}\right)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right]} + G(t_1,s)\left({t_1}^{\alpha-1} - {t_2}^{\alpha-1}\right)\right] |f(s,u(s))|ds \\
&\leq (p^*||u||_F + \lambda)\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right]}\left({t_2}^{\alpha-1} - {t_1}^{\alpha-1}\right)\right],
\end{aligned}$$

Hence $Q\Omega$ is locally equicontinuous on $[0,\infty)$.

Step 3: We show that $Q : FC(J,\mathbb{R}) \to FC(J,\mathbb{R})$ is equiconvergent at ∞.

For any $u \in \Omega$,

$$\int_0^{+\infty} |f(s,u(s))|ds \leq p^*||u||_F + \lambda < \infty,$$

$$\lim_{t\to+\infty}\left|\frac{(Qu)(t)}{1+t^{\alpha-1}}\right| = \frac{\xi\eta^{\alpha+\beta-1}}{\left(\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right)\Gamma(\alpha)} \int_0^\infty f(s,u(s))ds,$$

Hence $Q\Omega$ is equiconvergent at infinity.

By using Lemma 3.5, we obtain that $T : FC(J,\mathbb{R}) \to FC(J,\mathbb{R})$ is completely continuous.

Theorem 2.3.1 *Assume that (H1)-(H3) hold, Let ,φ satisfies the following condition: (B) $\exists\varrho > 0$ such that*

$$\frac{\varrho\Gamma(\alpha)\left(\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right)}{(\varrho)\int_0^\infty \varphi(s)ds} > 1, \tag{2.18}$$

Then boundary value problem (1.1) has an unbounded solution $u = u(t)$ such that

$$0 \leq \frac{u(t)}{1+t^{\alpha-1}} \leq \varrho, \quad \text{for } t \in [0,\infty).$$

Proof: We consider the fractional boundary value problem

$$\begin{cases} D^\alpha u(t) + \lambda f(t,u(t)) = 0, & t \in (0,\infty), \alpha \in (n,n+1) \\ u(0) = u'(0) = ...u^{(n-1)} = 0. \quad D^{\alpha-1}u(\infty) = \xi I^{\dot{\beta}} u(\eta), & \beta > 0 \end{cases} \tag{2.19}$$

for $0 < \lambda < 1$. Solving (4.2) is equivalent to solving the fixed point problem $u = \lambda Qu$.

Let

$$U = \left\{u \in FC(J,\mathbb{R}) \mid \|u\|_F < \varrho\right\}.$$

We claim that $u \neq \lambda Qu$ for $u \in \partial U$ and $\lambda \in (0,1)$. The claim is immediate, since if there exists $u \in \partial U$ with $u = \lambda Qu$, then for $\lambda \in (0,1)$ we have

$$\begin{aligned}
\|u\|_F &= \sup_{t\in[0,\infty)} \left| \frac{(\lambda Qu)(t)}{1+t^{\alpha-1}} \right| \\
&\leq \sup_{t\in[0,\infty)} \left| \frac{(Qu)(t)}{1+t^{\alpha-1}} \right| \\
&= \sup_{t\in[0,\infty)} \left| \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f\left(s, \frac{u(s)\left(1+s^{\alpha-1}\right)}{1+s^{\alpha-1}}\right) ds \right| \\
&= \sup_{t\in[0,\infty)} \left| \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right) ds \right| \\
&\leq \int_0^\infty \frac{1}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right)} \varphi(s) \left(\frac{|u(s)|}{1+s^{\alpha-1}}\right) ds \\
&\leq \frac{1}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right)} (\varrho) \int_0^\infty \varphi(s) ds,
\end{aligned}$$

So

$$\varrho \leq \frac{1}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right)} (\varrho) \int_0^\infty \varphi(s) ds.$$

Hence

$$\frac{\varrho\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right)}{(\varrho)\int_0^\infty \varphi(s)ds} \leq 1,$$

which contradicts with (4.1). By Theorem 1.1 and Lemma 3.4, boundary value problem (1.1) has an unbounded solution $u = u(t)$ such that

$$0 \leq \frac{u(t)}{1+t^{\alpha-1}} \leq \varrho, \qquad \text{for } t \in [0,\infty).$$

2.3.1 Example

Let $\alpha = \frac{5}{2}, \xi = 4, \eta = \frac{1}{2}, \beta = \frac{1}{2}, f(t,u) = \sqrt{\left|\frac{u}{1+t^{\frac{3}{2}}}\right|} e^{-t}, F(t,u) = \sqrt{|u|} e^{-t}$ in problem (2.1). Now we consider the following fractional boundary value problem

$$\begin{cases} D^{\frac{5}{2}} u(t) + f(t,u) = 0, & t \in (0,\infty) \\ u(0) = u' u(0) = 0, & D^{\frac{3}{2}} u(\infty) = \xi I^{\frac{1}{2}} u(\eta), \end{cases} \tag{2.20}$$

Choose $w(u) = \sqrt{u}, \varphi(t) = e^{-t}, \varrho > \left(\frac{1}{\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right)}\right)^2$,

We have $\Gamma(\alpha+\beta) = \Gamma(3) = 2 > \xi\eta^{\alpha+\beta-1}$,

Then (H_1) holds.

Second,

$$|f(t,u)-f(t,v)| \leq e^{-t}\left|\sqrt{\left|\frac{u}{1+t^{\frac{3}{2}}}\right|}-\sqrt{\left|\frac{v}{1+t^{\frac{3}{2}}}\right|}\right|$$
$$\leq \frac{e^{-t}}{\left(1+t^{\frac{3}{2}}\right)^{\frac{1}{2}}}|u-v|$$
$$\leq p(t)|u-v|.$$

Noting that $p(t)=\frac{e^{-t}}{\left(1+t^{\frac{3}{2}}\right)^{\frac{1}{2}}}$, thus, we have

$$p^{*}=\int_{0}^{\infty}\left(1+t^{\frac{3}{2}}\right)p(t)dt \leq \int_{0}^{\infty} t^{\frac{3}{4}}e^{-t}dt=\frac{3\Gamma(\frac{3}{4})}{4}<\infty,$$

and

$$\lambda=\int_{0}^{\infty}|f(t,0)|dt=\int_{0}^{\infty}0dt<\infty.$$

Then (H_2) holds. At last, by a simple computation, we have

$$m=\frac{p^{*}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right]} \leq \frac{\Gamma(\frac{3}{4})}{2\sqrt{\pi}\left(2-\xi\eta^{4}\right)} \leq \frac{\Gamma(\frac{3}{4})}{2\sqrt{\pi}}<1.$$

$f:[0,\infty)R \to [0,\infty)$ is continuous;

$|F(t,u)|=\varphi(t)(|u|)$ on $[0,\infty)R$ with $\in C([0,\infty),[0,\infty))$ nondecreasing and $\varphi \in L^{1}[0,+\infty)$;

$$\frac{\varrho\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right]}{(\varrho)\int_{0}^{\infty}\varphi(s)ds}=\sqrt{\varrho}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}\right]>1.$$

Hence all conditions of Theorem 4.1 hold. Thus with Theorem 4.1, the problem (4.3) has at least a positive solution u such that

$$0 \leq \frac{u(t)}{1+t^{\frac{3}{2}}} \leq \varrho, \qquad \text{for } t \in [0,\infty),$$

Hence all conditions of Theorem 4.1 hold. Thus with Theorem 4.1, the problem (4.3) has at least a positive solution u such that

$$0 \leq \frac{u(t)}{1+t^{\frac{3}{2}}} \leq \varrho, \qquad \text{for } t \in [0,\infty).$$

Chapter 3

Fractionel boundary value problem on the infinit interval in the sens of generalized fractional derivatives

We investigate in this chapter the existence of unbounded solution of a fractional integral boundqryary value problem for nonlinear fractional differential equation on an infinīte interval:

$$\begin{cases} D^{\alpha,\rho}u(t)+f(t,u(t))=0, \quad n<\alpha\leq n+1, \\ \delta^k u(0)=0, k=0,1...,n-1, \qquad D^{\alpha-1,\rho}u(\infty)=\xi^{\rho}I^{\beta}u(\eta), \quad \beta>0, \end{cases} \tag{3.1}$$

where $t \in J=[0,+\infty)$, $f \in C\left[J X \mathbb{R}, \mathbb{R}\right]$, $\xi \in \mathbb{R}$, $\eta \in J$. $D^{\alpha,\rho}$ is the katugampola fractional derivatives of order α, and ${}^{\rho}I^{\beta}$ is the katugampola fractional integral of order β.

By means of fixed point theorems, sufficient conditions are obtained that guarantee the existence of solutions to the above boundary value problem [12].

3.1 Lemmas

In this section, we will introduce notations, definitionas and some useful lemmas, which will play an important role in the proof of our main results[12]. Denote the space

$$FC\left(J,\mathbb{R}\right)=\left\{u\in C(J,\mathbb{R}):\sup_{t\in J}\frac{|u(t)|}{1+t^{\rho(\alpha-1)}}<\infty\right\},$$

with norm

$$\|u\|_F=\sup_{t\in J}\frac{|u(t)|}{1+t^{\rho(\alpha-1)}}.$$

Obviously, $FC(J,\mathbb{R})$ is a Banach space.

A map $u(t)\in C(J,\mathbb{R})$ with its katugampola derivative of order α existing on J is called a solution of (3.1) if it satisfies (3.1).

For convenience, we list the following assumptions:

(H_1) $\xi \geq 0, \quad \Gamma(\alpha+\beta) > \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}$.

(H_2) there exist a positive function $p(t)$ with $p^* = \int_0^\infty \left(1+t^{\rho(\alpha-1)}\right)p(t)dt < \infty$ such that

$$|f(t,u)-f(t,v)| \leq \frac{p(t)}{t^{\rho-1}}|u-v|, \quad t \in J \qquad u,v \in \mathbb{R}$$

and

$$\lambda = \int_0^\infty t^{\rho-1}|f(t,0)|dt < \infty.$$

(H_3) Let $F(t,u) = t^{\rho-1}f\left(t,\left(1+t^{\rho(\alpha-1)}\right)u\right), |F(t,u)| \leq \varphi(t)w(|u|)$ on $[0,\infty)X\mathbb{R}$ with $w \in C([0,\infty),[0,\infty))$ nondecreasing and $\varphi \in L^1[0,+\infty)$.

Lemma 3.1.1 *Let $h \in C([0,+\infty))$ with $\int_0^\infty h(s)ds < \infty$, if $\Gamma(\alpha+\beta) \neq \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}$ then the fractional integral boundary value problem*

$$\begin{cases} D^{\alpha,\rho}u(t)+h(t)=0, \\ \delta^k u(0)=0, \quad k=0,1...,n-1, \qquad D^{\alpha-1,\rho}u(\infty)=\xi^\rho I^\beta u(\eta), \quad \beta>0, \end{cases} \tag{3.2}$$

has a unique solution

$$u(t) = \int_0^{+\infty} s^{\rho-1}G(t,s)h(s)ds.$$

where

$$G(t,s) = \frac{1}{\Delta}\begin{cases} \left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}(\eta^\rho-s^\rho)^{\alpha+\beta-1}\right]t^{\rho(\alpha-1)} \\ -\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right](t^\rho-s^\rho)^{\alpha-1}, & s\leq t, s\leq\eta. \\ \left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}(\eta^\rho-s^\rho)^{\alpha+\beta-1}\right]t^{\rho(\alpha-1)}, & 0\leq t\leq s\leq\eta, \\ \Gamma(\alpha+\beta)\left[t^{\rho(\alpha-1)}-(t^\rho-s^\rho)^{\alpha-1}\right] \\ +\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}(t^\rho-s^\rho)^{\alpha-1}, & 0\leq\eta\leq s\leq t, \\ \Gamma(\alpha+\beta)t^{\rho(\alpha-1)}, & s\geq t, s\geq\eta, \end{cases}$$

and

$$\Delta = \rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right].$$

Proof. Let $h \in C([0,+\infty))$ with $\int_0^\infty h(s)ds < \infty$, if $\Gamma(\alpha+\beta) \neq \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}$ we have the probleme

$$\begin{cases} D^{\alpha,\rho}u(t)+h(t)=0, \\ \delta^k u(0)=0, k=0,1...,n-1, \qquad D^{\alpha-1,\rho}u(\infty)=\xi\ {}^\rho I^\beta u(\eta), \quad \beta>0. \end{cases}$$

By applying ${}^\rho I^\alpha_{0+}$ to equation we get

$${}^\rho I^\alpha D^{\alpha,\rho}u(t) + {}^\rho I^\alpha h(t) = 0.$$

by lemma (1.2.5) we have

$$^{\rho}I^{\alpha}D^{\alpha,\rho}u(t) = u(t) - \sum_{k=1}^{n+1} C_k(\frac{t^{\rho}}{\rho})^{\alpha-k}, \qquad c_1, c_2, ..., c_{n+1} \in \mathbb{R}.$$

So

$$u(t) - \sum_{k=1}^{n+1} C_k(\frac{t^{\rho}}{\rho})^{\alpha-k} + {}^{\rho}I^{\alpha}h(t) = 0.$$

Hence

$$u(t) = c_1(\frac{t^{\rho}}{\rho})^{\alpha-1} + c_2(\frac{t^{\rho}}{\rho})^{\alpha-2} + ... + c_{n+1}(\frac{t^{\rho}}{\rho})^{\alpha-n-1} - {}^{\rho}I^{\alpha}h(t).$$

The conditions $\delta^k u(0) = 0$ implies that

$$u(0) = 0 \Leftrightarrow \frac{c_{n+1}}{\rho^{\alpha-n-1}} \lim_{t\to 0} t^{\rho(\alpha-n-1)} = 0 \Rightarrow c_{n+1} = 0,$$

$$(t^{1-\rho}\frac{d}{dt})u(t) = t^{1-\rho}\left[\frac{c_1}{\rho^{\alpha-1}}\rho(\alpha-1)t^{\rho(\alpha-1)-1} + \frac{c_2}{\rho^{\alpha-2}}\rho(\alpha-2)t^{\rho(\alpha-2)-1} + ... \right.$$
$$\left. + \frac{c_n}{\rho^{\alpha-n}}\rho(\alpha-n)t^{\rho(\alpha-n)-1} - {}^{\rho}I^{\alpha-1}h(t)\right],$$

$$(t^{1-\rho}\frac{d}{dt})u(0) = 0 \Leftrightarrow \frac{c_n}{\rho^{\alpha-n}}\rho(\alpha-n)\lim_{t\to 0} t^{\rho(\alpha-n-1)} = 0 \Leftrightarrow c_n = 0,$$

.

.

$$(t^{1-\rho}\frac{d}{dt})^{n-1}u(0) = 0 \Leftrightarrow \frac{c_2}{\rho^{\alpha-2}}\rho(\alpha-2)(\alpha-3)....(\alpha-n)\lim_{t\to 0} t^{\rho(\alpha-n-1)} = 0 \Leftrightarrow c_2 = 0,$$

So

$$\begin{aligned} u(t) &= c_1(\frac{t^{\rho}}{\rho})^{\alpha-1} - {}^{\rho}I^{\alpha}h(t) \\ &= c_1(\frac{t^{\rho}}{\rho})^{\alpha-1} - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t^{\rho} - s^{\rho})^{\alpha-1}s^{\rho-1}h(s)ds. \end{aligned}$$

and the conditions $D^{\alpha-1,\rho}u(\infty) = \xi^{\rho}I^{\beta}u(\eta)$ implies that

$$\begin{aligned} D^{\alpha-1,\rho}u(t) &= D^{\alpha-1,\rho}\left[c_1(\frac{t^{\rho}}{\rho})^{\alpha-1}\right] - {}^{\rho}D^{\alpha-1\rho}I^{\alpha}h(t) \\ &= c_1\Gamma(\alpha) - {}^{\rho}Ih(t) \\ &= c_1\Gamma(\alpha) - \int_0^t s^{\rho-1}h(s)ds, \end{aligned}$$

$$D^{\alpha-1,\rho}u(\infty) = c_1\Gamma(\alpha) - \Gamma(\alpha)h(s)ds = \xi^{\rho}I^{\beta}u(\eta).$$

So

$$c_1\Gamma(\alpha) - c_1\frac{\xi}{\rho^{\alpha-1}}{}^{\rho}I^{\beta}\eta^{\rho(\alpha-1)} = \int_0^{\infty} s^{\rho-1}h(s)ds - \xi^{\rho}I^{\alpha+\beta}h(\eta),$$

$$c_1\left[\Gamma(\alpha) - \frac{\xi}{\rho^{\alpha-1}}{}^{\rho}I^{\beta}\eta^{\rho(\alpha-1)}\right] = \int_0^{\infty} s^{\rho-1}h(s)ds - \xi^{\rho}I^{\alpha+\beta}h(\eta),$$

Hence

$$c_1 = \frac{\int_0^\infty s^{\rho-1}h(s)ds - \xi^\rho I^{\alpha+\beta}h(\eta)}{\Gamma(\alpha) - \frac{\xi}{\rho^{\alpha-1}}\rho I^\beta \eta^{\rho(\alpha-1)}}$$
$$= \frac{1}{\Gamma(\alpha) - \xi\rho^{1-\alpha-\beta}\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\eta^{\rho(\alpha+\beta-1)}}\left[\int_0^{+\infty} s^{\rho-1}h(s)ds\right.$$
$$\left.-\xi\left(\frac{\rho^{1-\alpha-\beta}}{\Gamma(\alpha+\beta)}\int_0^\eta(\eta^\rho - s^\rho)^{\alpha+\beta-1}s^{\rho-1}h(s)ds\right)\right]$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)(\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)})}\int_0^{+\infty} s^{\rho-1}h(s)ds - \frac{\xi\rho^{1-\alpha-\beta}}{\Gamma(\alpha)\left[(\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)})\right]}$$
$$\text{X}\int_0^\eta(\eta^\rho - s^\rho)^{\alpha+\beta-1}s^{\rho-1}h(s)ds$$
$$= \frac{1}{\Gamma(\alpha)(\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)})}\left[\Gamma(\alpha+\beta)\int_0^{+\infty} s^{\rho-1}h(s)ds\right.$$
$$\left.-\xi\rho^{1-\alpha-\beta}\int_0^\eta(\eta^\rho - s^\rho)^{\alpha+\beta-1}s^{\rho-1}h(s)ds\right].$$

$$\frac{c_1}{\rho^{\alpha-1}} = \frac{1}{\rho^{\alpha-1}\Gamma(\alpha)(\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)})}\left[\Gamma(\alpha+\beta)\int_0^{+\infty} s^{\rho-1}h(s)ds\right.$$
$$\left.-\xi\rho^{1-\alpha-\beta}\int_0^\eta(\eta^\rho - s^\rho)^{\alpha+\beta-1}s^{\rho-1}h(s)ds\right]$$
$$= \frac{1}{\Delta}\left[\int_0^\eta\left(\Gamma(\alpha+\beta)s^{\rho-1} - \xi\rho^{1-\alpha-\beta}(\eta^\rho - s^\rho)^{\alpha+\beta-1}s^{\rho-1}\right)h(s)ds\right.$$
$$\left.+\int_\eta^{+\infty}\Gamma(\alpha+\beta)s^{\rho-1}h(s)ds\right],$$

with

$$\Delta = \rho^{\alpha-1}\Gamma(\alpha)(\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}).$$

So

$$u(t) = \frac{1}{\Delta}\left[\int_0^\eta\left(\Gamma(\alpha+\beta)s^{\rho-1} - \xi\rho^{1-\alpha-\beta}(\eta^\rho - s^\rho)^{\alpha+\beta-1}s^{\rho-1}\right)h(s)ds\right.$$
$$\left.+\int_\eta^{+\infty}\Gamma(\alpha+\beta)s^{\rho-1}h(s)ds\right]t^{\rho(\alpha-1)} - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t^\rho - s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds$$
$$= \int_0^{+\infty} s^{\rho-1}G(t,s)h(s)ds,$$

$$G(t,s) = \frac{1}{\Delta}\begin{cases} \left[\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}(\eta^\rho - s^\rho)^{\alpha+\beta-1}\right]t^{\rho(\alpha-1)} & \\ -\left[\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right](t^\rho - s^\rho)^{\alpha-1}, & s \le t, s \le \eta \\ \left[\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}(\eta^\rho - s^\rho)^{\alpha+\beta-1}\right]t^{\rho(\alpha-1)}, & 0 \le t \le s \le \eta, (2.2) \\ \Gamma(\alpha+\beta)\left[t^{\rho(\alpha-1)} - (t^\rho - s^\rho)^{\alpha-1}\right] & \\ +\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}(t^\rho - s^\rho)^{\alpha-1}, & 0 \le \eta \le s \le t, \\ \Gamma(\alpha+\beta)t^{\rho(\alpha-1)}, & s \ge t, s \ge \eta, \end{cases}$$

Lemma 3.1.2 *If $(H1)$ holds, then Green's function $G(t,s)$ satisfies*

1. $G(t,s) \geq 0$,

2. $\dfrac{G(t,s)}{1+t^{\rho(\alpha-1)}} \leq \dfrac{\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]}.$

Proof.

1. We have four cases

Case 1: For $s \leq t$ and $\xi \leq \eta$,

Then

$$t^\rho \geq t^\rho - s^\rho \Leftrightarrow t^{\rho(\alpha-1)} \geq (t^\rho - s^\rho)^{\alpha-1},$$

We have

$$\begin{aligned}
&\eta^\rho - s^\rho \leqslant \eta^\rho \\
&\quad \Leftrightarrow (\eta^\rho - s^\rho)^{\alpha+\beta-1} \leqslant \eta^{\rho(\alpha+\beta-1)} \\
&\quad \Leftrightarrow -\xi\rho^{1-\alpha-\beta}(\eta^\rho - s^\rho)^{\alpha+\beta-1} \geqslant -\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)} \\
&\quad \Leftrightarrow \Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}(\eta^\rho - s^\rho)^{\alpha+\beta-1} \geqslant \Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)} \\
&\quad \Leftrightarrow \left[\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}(\eta^\rho - s^\rho)^{\alpha+\beta-1}\right] t^{\rho(\alpha-1)} \geqslant \left[\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right] (t^\rho - s^\rho)^{\alpha-1} \\
&\quad \Leftrightarrow \left[\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}(\eta^\rho - s^\rho)^{\alpha+\beta-1}\right] t^{\rho(\alpha-1)} - \left[\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right] (t^\rho - s^\rho)^{\alpha-1} \geqslant 0,
\end{aligned}$$

we have $\frac{1}{\Delta} > 0$ so $G(t,s) > 0$.

Case 2: For $0 \leq t \leq s \leq \eta$,

We have

$$\Gamma(\alpha+\beta) > \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)} \geq \xi\rho^{1-\alpha-\beta}(\eta^\rho - s^\rho)^{\alpha+\beta-1},$$

Then

$$\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}(\eta^\rho - s^\rho)^{\alpha+\beta-1} \geq 0,$$

Hence

$$\left[\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}(\eta^\rho - s^\rho)^{\alpha+\beta-1}\right] t^{\rho(\alpha-1)} \geq 0,$$

Then

$$G(t,s) > 0.$$

Case 3: For $0 \leq \eta \leq s \leq t$,

We have

$$\begin{aligned}
t^\rho \geq t^\rho - s^\rho &\Leftrightarrow t^{\rho(\alpha-1)} \geq (t^\rho - s^\rho)^{\alpha-1} \\
&\Leftrightarrow t^{\rho(\alpha-1)} - (t^\rho - s^\rho)^{\alpha-1} \geq 0 \\
&\Leftrightarrow \Gamma(\alpha+\beta)\left[t^{\rho(\alpha-1)} - (t^\rho - s^\rho)^{\alpha-1}\right] \geq 0.
\end{aligned}$$

We also have

$$\begin{aligned}
t^\rho \geq s^\rho &\Leftrightarrow t^\rho - s^\rho \geq 0 \\
&\Leftrightarrow (t^\rho - s^\rho)^{\alpha-1} \geq 0 \\
&\Leftrightarrow \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}(t^\rho - s^\rho)^{\alpha-1} \geq 0,
\end{aligned}$$

So

$$\Gamma(\alpha+\beta)\left[t^{\rho(\alpha-1)}-(t^\rho-s^\rho)^{\alpha-1}\right]+\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}(t^\rho-s^\rho)^{\alpha-1}\geq 0,$$

Hence

$$G(t,s)>0.$$

Case 4: For $s\geq t$ and $s\geq\eta$,

We know that

$$\Gamma(\alpha+)t^{\rho(\alpha-1)}\geq 0,$$

So

$$G(t,s)\geq 0.$$

According the four cases precedding

if $(H1)$ holds $\quad G(t,s)\geq 0.$

2. Now we will prove that:

$$\frac{G(t,s)}{1+t^{\rho(\alpha-1)}}\leq\frac{\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\;\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}]}.$$

Case 1: For $s\leq t$ and $s\leq\eta$,

We have :

$\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}(\eta^\rho-s^\rho)^{\alpha+\beta-1}\right]t^{\rho(\alpha-1)}-\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right](t^\rho-s^\rho)^{\alpha-1}$
$\leqslant\Gamma(\alpha+\beta)t^{\rho(\alpha-1)}\leqslant\Gamma(\alpha+\beta)(1+t^{\rho(\alpha-1)}),$

So

$$\Delta.G(t,s)\leqslant\Gamma(\alpha+\beta)(1+t^{\rho(\alpha-1)}),$$

Hence

$$\frac{G(t,s)}{1+t^{\rho(\alpha-1)}}\leqslant\frac{\Gamma(\alpha+\beta)}{\Delta}.$$

Case 2: For $0\leqslant t\leqslant s\leqslant\eta$,

We have

$$\begin{aligned}\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}(\eta^\rho-s^\rho)^{\alpha+\beta-1}\right]t^{\rho(\alpha-1)}&\leqslant\Gamma(\alpha+\beta)t^{\rho(\alpha-1)}\\&\leqslant\Gamma(\alpha+\beta)(1+t^{\rho(\alpha-1)}),\end{aligned}$$

So

$$\Delta.G(t,s)\leqslant\Gamma(\alpha+\beta)(1+t^{\rho(\alpha-1)}),$$

Then

$$\frac{G(t,s)}{1+t^{\rho(\alpha-1)}}\leqslant\frac{\Gamma(\alpha+\beta)}{\Delta}.$$

Case 3: For $0\leq\eta\leq s\leq t$,

We have

$$\Gamma(\alpha+\beta)>\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)},$$

So

$$\Gamma(\alpha+\beta)(t^\rho-s^\rho)^{\alpha-1}>\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}(t^\rho-s^\rho)^{\alpha-1},$$

Then

$$\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}(t^\rho - s^\rho)^{\alpha-1} - \Gamma(\alpha+\beta)(t^\rho - s^\rho)^{\alpha-1} \leq 0,$$

Hence

$$\Gamma(\alpha+\beta)t^{\rho(\alpha-1)} - \Gamma(\alpha+\beta)(t^\rho - s^\rho)^{\alpha-1} + \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}(t^\rho - s^\rho)^{\alpha-1} \leqslant \Gamma(\alpha+\beta)t^{\rho(\alpha-1)},$$

Then

$$\Gamma(\alpha+\beta)\left[t^{\rho(\alpha-1)} - (t^\rho - s^\rho)^{\alpha-1}\right] + \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}(t^\rho - s^\rho)^{\alpha-1} \leqslant \Gamma(\alpha+\beta)t^{\rho(\alpha-1)},$$

So

$$\Delta.G(t,s) \leqslant \Gamma(\alpha+\beta)t^{\rho(\alpha-1)} \leq \Gamma(\alpha+\beta)(1+t^{\rho(\alpha-1)}),$$

Hence

$$\frac{G(t,s)}{1+t^{\rho(\alpha-1)}} \leqslant \frac{\Gamma(\alpha+\beta)}{\Delta}.$$

Case 4: For $s \geq t$ and $s \geq \eta$,
We have

$$\Gamma(\alpha+\beta)t^{\rho(\alpha-1)} \leq \Gamma(\alpha+\beta)(1+t^{\rho(\alpha-1)}),$$

So

$$\Delta.G(t.s) \leq \Gamma(\alpha+\beta)(1+t^{\rho(\alpha-1)}),$$

Then

$$\frac{G(t,s)}{1+t^{\rho(\alpha-1)}} \leqslant \frac{\Gamma(\alpha+\beta)}{\Delta}.$$

By according to the precedding cases we have

$$\frac{G(t,s)}{1+t^{\rho(\alpha-1)}} \leq \frac{\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]}.$$

Lemma 3.1.3 *If $(H2)$ holds, then for any $u \in FC(J,\mathbb{R})$*

$$\int_0^{+\infty} s^{\rho-1}|f(s,u(s))|ds \leq p^*||u||_F + \lambda.$$

Proof . For any $u \in FC(J,\mathbb{R})$, by taking $v = 0$ in (H_1), we have

$$\begin{aligned} t^{\rho-1}|f(t,u(t))| &= t^{\rho-1}|f(t,u(t)) - f(t,0) + f(t,0)| \\ &\leq t^{\rho-1}|f(t,u(t)) - f(t,0)| + t^{\rho-1}|f(t,0)| \\ &\leq t^{\rho-1}\frac{p(t)}{t^{\rho-1}}|u(t)| + t^{\rho-1}|f(t,0)| \\ &\leq p(t)\left(1+t^{\rho(\alpha-1)}\right)\frac{|u(t)|}{1+t^{\rho(\alpha-1)}} + t^{\rho-1}|f(t,0)| \\ &\leq p(t)\left(1+t^{\rho(\alpha-1)}\right)\|u\|_F + t^{\rho-1}|f(t,0)|. \end{aligned}$$

we integre

$$\int_0^{+\infty} s^{\rho-1}|f(s,u(s))|ds \leq \int_0^{+\infty} p(s)\left(1+s^{\rho(\alpha-1)}\right)ds\|u\|_F + \int_0^{+\infty} s^{\rho-1}|f(s,0)|,$$

so

$$\int_0^{+\infty} s^{\rho-1}|f(s,u(s))|ds \leq p^*||u||_F + \lambda.$$

3.2 Existence and uniqueness of solution by Banach fixed point theorem

In this section, we study the existence and uniqueness of solution by Banach fixed point theorem, we assume that $(H1)$ and $(H2)$ hold

Theorem 3.2.1 *Assume that $(H1)$ and $(H2)$ hold. If*

$$m = \frac{p^*\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)(\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)})} < 1, \tag{3.3}$$

then the fractional integral boundary value problem (3.1) has a unique solution $\overline{u}(t)$ in $FC(J,\mathbb{R})$. Moreover, there exists a monotone iterative sequence $u_n(t)$such that $u_n(t) \longrightarrow \overline{u}(t)$ as $n \to \infty$ uniformly on any finite sub-interval of J, where

$$u_n(t) = \int_0^{\infty} s^{\rho-1}G(t,s)f\left(s,u_{n-1}(s)\right)ds.$$

In addition, there exists an error estimate for the approximation sequence

$$\|u_n - \bar{u}\|_F \leq \frac{m^n}{1-m}\|u_1-u_0\|_F, \quad (n=1,2,\ldots)$$

Proof. Define the operator Q by

$$(Qu)(t) = \int_0^{\infty} s^{\rho-1}G(t,s)f(s,u(s))ds?$$

By Lemma 3.1, fractional integral boundary value problem (1.1) has a solution u if and only if u solves the operator equation $u = Qu$, *i.e.* The solution of (1.1) coincides with the fixed point of the operator Q. First, for any $u \in FC(J,\mathbb{R})$, by Lemmas 3.2 and 3.3, we have

$$\begin{aligned}
\frac{|(Qu)(t)|}{1+t^{\rho(\alpha-1)}} &\leq \int_0^{\infty} \frac{s^{\rho-1}G(t,s)}{1+t^{\rho(\alpha-1)}}|f(s,u(s))|ds \\
&\leq \frac{\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]}\left[p^*\|u\|_F+\lambda\right] \\
&= m\|u\|_F + k.
\end{aligned}$$

where m is defined in (4.3) and $k = \dfrac{\lambda\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]}$. In addition, for any $u, v \in FC(J,\mathbb{R})$, we have

$$\begin{aligned}
\frac{|(Qu)(t)-(Qv)(t)|}{1+t^{\rho(\alpha-1)}} &\le \int_0^\infty \frac{s^{\rho-1}G(t,s)}{1+t^{\rho(\alpha-1)}}|f(s,u(s))-f(s,v(s))|ds \\
&\le \frac{\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]}\int_0^\infty p(s)\left(1+s^{\rho(\alpha-1)}\right)\|u-v\|_F ds \\
&\le \frac{\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]}p^*\|u-v\|_F \\
&= m\|u-v\|_F.
\end{aligned}$$

Thus,

$$\|Qu-Qv\|_F \le m\|u-v\|_F, \quad \forall u,v \in FC(J,\mathbb{R})$$

As $m<1$, then the Banach fixed point theorem ensures that Q has a unique fixed point $\bar{u}$ in $FC(J,\mathbb{R})$, i.e. The integral boundary value problem (1.1) has a unique solution $\bar{u} \in FC(J,\mathbb{R})$ ndoreover, for any $u_0 \in FC(J,\mathbb{R})$, $\|u_n-\bar{u}\|_F \to 0$ as $n\to\infty$, where $u_n = Qu_{n-1}(n=1,2,\ldots)$.

3.2.1 Example

Consider the problem

$$\begin{cases} D^{\frac{5}{2},\rho}u(t) + \frac{e^{-3t}}{(1+t^3)^2}\cos\left(3t^2+u(t)\right) = 0, \quad t \in J = [0,\infty), \\ u(0) = (t^{1-\rho}\frac{d}{dt})u(0) = 0, \quad D^{\frac{3}{2},\rho}u(\infty) = \xi\ {}^{\rho}I^{\frac{1}{2}}u(\eta), \end{cases}$$

where $\alpha=\frac{5}{2}, \beta=\frac{1}{2}, f(t,u) = \frac{e^{-3t}}{(1+t^3)^2}\cos\left(3t^2+u\right)$.
For example, we can take $\rho=2, \xi=4, \eta=\frac{1}{2}$.
Next, we will verify the conditions of Theorem 3.1: First, it is clear that $\Gamma(\alpha+\beta)=\Gamma(3)= 2 > \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}$, ($H_1$) holds. Second,

$$\begin{aligned}
|f(t,u)-f(t,v)| &\le \frac{e^{-3t}}{(1+t^3)^2}\left|\cos\left(3t^2+u\right)-\cos\left(3t^2+v\right)\right| \\
&\le \frac{e^{-3t}}{(1+t^3)^2}|u-v| \\
&\le \frac{p(t)}{t}|u-v|.
\end{aligned}$$

Noting that $p(t) = \frac{te^{-3t}}{(1+t^3)^2}$, thus, we have

$$p^* = \int_0^\infty \left(1+t^3\right)p(t)dt \le \int_0^\infty te^{-3t}\ \mathrm{dt} = \frac{1}{9} < \infty,$$

and

$$\lambda = \int_0^\infty t|f(t,0)|dt \le \int_0^\infty te^{-3t}dt = \frac{1}{9} < \infty.$$

Then (H_2) holds. At last, by a simple computation, we have

$$m = \frac{p^*\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]} \leq \frac{4}{27\sqrt{2\pi}\left(2-\xi\rho^{-2}\eta^4\right)} \leq \frac{4}{27\sqrt{2\pi}} < 1.$$

Hence, all conditions of Theorem 3.1 are satisfied. Therefore, Theorem 3.1 implies that (4.1) has a unique solution, which can be obtained by taking limits from some iterative sequences.

3.3 Existence of solution by Leray-schauder Nonlinear Alternative theorem

In this section, we study the existence of solution by Leray-schauder Nonlinear Alternative theorem.
For $u \in FC(J, \mathbb{R})$, define the operator Q by

$$(Qu)(t) = \int_0^\infty s^{\rho-1} G(t,s) f(s, u(s)) ds.$$

Boundary value problem (3.1) has a solution u if and only if u solves the operator equation $u = Qu$. Thus we set out to verify that the operator Q satisfies Theorem 2.1, which will prove the existence of the fixed points of Q. Since the Arzela-Ascoli theorem fails to work in the space $FC(J, \mathbb{R})$, we need a modified compactness criterion to prove T is compact.

Lemma 3.3.1 *Let* $V = \{u \in FC(J, \mathbb{R}) \mid \|u\|_F < l\}\,(l > 0), V_1 = \left\{\frac{u(t)}{1+t^{\rho(\alpha-1)}} \mid u \in V\right\}$. *If* V_1 *is equicontinuous on any compact intervals of* $[0, +\infty)$ *and equiconvergent at infinity, then* V *is relatively compact on* $FC(J, \mathbb{R})$.

Remark 3.3.1 V_1 *is called equiconvergent at infinity if and only if for all* $\epsilon > 0$, *there exists* $v = v(\epsilon) > 0$ *such that for all* $u \in V_1, t_1, t_2 \geq v$, *it holds,*

$$\left| \frac{u(t_1)}{1+t_1^{\rho(\alpha-1)}} - \frac{u(t_2)}{1+t_2^{\rho(\alpha-1)}} \right| < \epsilon.$$

Lemma 3.3.2 *If* (H1) − (H3) *hold, then* $Q : FC(J, \mathbb{R}) \to FC(J, \mathbb{R})$ *is completely continuous.*

Proof We divided the proof into three steps.
Step 1: We show that $Q : FC(J, \mathbb{R}) \to FC(J, \mathbb{R})$ is continuous.
Let $u_n \to u$ as $n \to +\infty$ in $FC(J, \mathbb{R})$,

We have

$$\begin{aligned}
\frac{|(Qu_n)(t)-(Qu)(t)|}{1+t^{\rho(\alpha-1)}} &\leq \int_0^\infty \frac{s^{\rho-1}G(t,s)}{1+t^{\rho(\alpha-1)}}|f(s,u_n(s))-f(s,u(s))|ds \\
&\leq \frac{\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]}\int_0^\infty p(s)\left(1+s^{\rho(\alpha-1)}\right)\|u_n-u\|_F ds \\
&\leq \frac{\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]}p^*\|u_n-u\|_F ds \\
&= m\|u_n-u\|_F.
\end{aligned}$$

Thus,

$$\|Qu_n-Qu\|_F \leq m\|u_n-u\|_F, \quad \to 0, \quad \text{as} \quad n\to+\infty$$

So, Q is continuous.

Step 2: We show that $Q: FC(J,\mathbb{R}) \to FC(J,\mathbb{R})$ is relatively compact.
Let Ω be any bounded subset of $FC(J,\mathbb{R})$, then there exists $K>0$ such that $\|u\|_F \leq K$.
We have

$$\begin{aligned}
\|Qu\|_F = \sup_{t\in J}\frac{|(Qu)(t)|}{1+t^{\rho(\alpha-1)}} &\leq \int_0^\infty \frac{s^{\rho-1}G(t,s)}{1+t^{\rho(\alpha-1)}}|f(s,u(s))|ds \\
&\leq \frac{\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]}\left[p^*\|u\|_F+\lambda\right] \\
&= m\|u\|_F + k < \infty,
\end{aligned}$$

where m is defined in (4.3) and $k = \dfrac{\lambda\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]}$.

Hence $Q\Omega$ is uniformly bounded.
Now we show that $Q\Omega$ is equicontinuous on any compact interval of $[0,\infty)$. For any $T>0, t_1, t_2 \in [0,T]$, and $u\in\Omega$, without loss of generality, we may assume that $t_2>t_1$. In fact,

$$\begin{aligned}
&\left|\frac{(Qu)(t_2)}{1+t_2^{\rho(\alpha-1)}}-\frac{(Qu)(t_1)}{1+t_1^{\rho(\alpha-1)}}\right| \\
&= \left|\int_0^\infty \frac{s^{\rho-1}G(t_2,s)}{1+t_2^{\rho(\alpha-1)}}f(s,u(s))ds - \int_0^\infty \frac{s^{\rho-1}G(t_1,s)}{1+t_1^{\rho(\alpha-1)}}f(s,u(s))ds\right| \\
&= \int_0^\infty s^{\rho-1}\left[\frac{G(t_2,s)}{1+t_2^{\rho(\alpha-1)}}-\frac{G(t_1,s)}{1+t_2^{\rho(\alpha-1)}}+\frac{G(t_1,s)}{1+t_2^{\rho(\alpha-1)}}-\frac{G(t_1,s)}{1+t_1^{\rho(\alpha-1)}}\right]|f(s,u(s))|ds \\
&= \int_0^\infty s^{\rho-1}\left[\frac{G(t_2,s)-G(t_1,s)}{1+t_2^{\rho(\alpha-1)}}+G(t_1,s)\left(\frac{1}{1+t_2^{\rho(\alpha-1)}}-\frac{1}{1+t_1^{\rho(\alpha-1)}}\right)\right]|f(s,u(s))|ds \\
&\leq \int_0^\infty s^{\rho-1}\left[\frac{2\Gamma(\alpha+\beta)\left({t_2}^{\rho(\alpha-1)}-{t_1}^{\rho(\alpha-1)}\right)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]}+G(t_1,s)\left({t_1}^{\rho(\alpha-1)}-{t_2}^{\rho(\alpha-1)}\right)\right]|f(s,u(s))|ds \\
&\leq (p^*||u||_F+\lambda)\left[\frac{\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]}\left({t_2}^{\rho(\alpha-1)}-{t_1}^{\rho(\alpha-1)}\right)\right],
\end{aligned}$$

Hence $Q\Omega$ is locally equicontinuous on $[0,\infty)$.

Step 3: We show that $Q : FC(J,\mathbb{R}) \to FC(J,\mathbb{R})$ is equiconvergent at ∞.

For any $u \in \Omega$,

$$\int_0^{+\infty} s^{\rho-1}|f(s,u(s))|ds \leq p^*||u||_F + \lambda < \infty,$$

$$\lim_{t\to+\infty}\left|\frac{(Qu)(t)}{1+t^{\rho(\alpha-1)}}\right| = \frac{\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}}{\rho^{\alpha-1}\left(\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right)\Gamma(\alpha)}\int_0^{\infty} s^{\rho-1}f(s,u(s))ds,$$

Hence $Q\Omega$ is equiconvergent at infinity.

By using Lemma 3.5, we obtain that $T : FC(J,\mathbb{R}) \to FC(J,\mathbb{R})$ is completely continuous.

Theorem 3.3.1 *Assume that (H1)-(H3) hold, Let* , φ *satisfies the following condition:* $\exists\varrho > 0$ *such that*

$$\frac{\varrho\rho^{\alpha-1}\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right)}{(\varrho)\int_0^{\infty}\varphi(s)ds} > 1, \tag{3.4}$$

Then boundary value problem (3.1) has an unbounded solution $u = u(t)$ *such that*

$$0 \leq \frac{u(t)}{1+t^{\rho(\alpha-1)}} \leq \varrho, \quad \text{for } t \in [0,\infty).$$

Proof: We consider the fractional boundary value problem

$$\begin{cases} D^{\alpha,\rho}u(t) + \lambda f(t,u(t)) = 0, & t \in (0,\infty), \alpha \in (n,n+1) \\ \delta^k u(0) = 0, k = 0,1...,n-1, & D^{\alpha-1,\rho}u(\infty) = \xi^\rho I^\beta u(\eta), \quad \beta > 0 \end{cases} \tag{3.5}$$

for $0 < \lambda < 1$. Solving (4.2) is equivalent to solving the fixed point problem $u = \lambda Qu$.

Let

$$U = \{u \in FC(J,\mathbb{R}) \mid \|u\|_F < \varrho\}.$$

We claim that $u \neq \lambda Qu$ for $u \in \partial U$ and $\lambda \in (0,1)$. The claim is immediate, since if there exists $u \in \partial U$ with $u = \lambda Qu$, then for $\lambda \in (0,1)$ we have

$$\begin{aligned}
\|u\|_F &= \sup_{t\in[0,\infty)}\left|\frac{(\lambda Qu)(t)}{1+t^{\rho(\alpha-1)}}\right| \\
&\leq \sup_{t\in[0,\infty)}\left|\frac{(Qu)(t)}{1+t^{\rho(\alpha-1)}}\right| \\
&= \sup_{t\in[0,\infty)}\left|\int_0^{\infty}\frac{s^{\rho-1}G(t,s)}{1+t^{\rho(\alpha-1)}}f\left(s,\frac{u(s)\left(1+s^{\rho(\alpha-1)}\right)}{1+s^{\rho(\alpha-1)}}\right)ds\right| \\
&= \sup_{t\in[0,\infty)}\left|\int_0^{\infty}\frac{s^{\rho-1}G(t,s)}{1+t^{\rho(\alpha-1)}}F\left(s,\frac{u(s)}{1+s^{\rho(\alpha-1)}}\right)ds\right| \\
&\leq \int_0^{\infty}\frac{1}{\rho^{\alpha-1}\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right)}\varphi(s)\left(\frac{|u(s)|}{1+s^{\alpha-1}}\right)ds \\
&\leq \frac{1}{\rho^{\alpha-1}\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right)}(\varrho)\int_0^{\infty}\varphi(s)ds,
\end{aligned}$$

So

$$\varrho \leq \frac{1}{\rho^{\alpha-1}\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right)}(\varrho)\int_0^{\infty}\varphi(s)ds.$$

Hence

$$\frac{\varrho\rho^{\alpha-1}\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right)}{(\varrho)\int_0^\infty \varphi(s)ds} \leq 1,$$

which contradicts with (3.1). By Theorem 2.1 and Lemma 3.4, boundary value problem (3.1) has an unbounded solution $u = u(t)$ such that

$$0 \leq \frac{u(t)}{1+t^{\rho(\alpha-1)}} \leq \varrho, \qquad \text{for } t \in [0,\infty).$$

3.3.1 Example

Let $\alpha = \frac{5}{2}, \xi = 4, \eta = \frac{1}{2}, \beta = \frac{1}{2}, \rho = 2, f(t,u) = \sqrt{\left|\frac{u}{1+t^3}\right|}e^{-t}, F(t,u) = t\sqrt{|u|}e^{-t}$ in problem (1.1). Now we consider the following fractional boundary value problem

$$\begin{cases} D^{\frac{5}{2},\rho}u(t) + f(t,u) = 0, \quad t \in (0,\infty) \\ u(0) = (t^{1-\rho}\frac{d}{dt})u(0) = 0, \quad D^{\frac{3}{2},\rho}u(\infty) = \xi^{\rho} I^{\frac{1}{2}}u(\eta), \end{cases} \tag{3.6}$$

Choose $(u) = \sqrt{u}, \varphi(t) = te^{-t}, \varrho > \left(\frac{1}{\rho^{\alpha-1}\Gamma(\alpha)\left(\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right)}\right)^2$, we have $\Gamma(\alpha+\beta) = \Gamma(3) = 2 > \xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}$, ($H_1$) holds. Second,

$$\begin{aligned} |f(t,u) - f(t,v)| &\leq e^{-t}\left|\sqrt{\left|\frac{u}{1+t^3}\right|} - \sqrt{\left|\frac{v}{1+t^3}\right|}\right| \\ &\leq \frac{e^{-t}}{(1+t^3)^{\frac{1}{2}}}|u-v| \\ &\leq \frac{p(t)}{t}|u-v|. \end{aligned}$$

Noting that $p(t) = \frac{te^{-t}}{(1+t^3)^{\frac{1}{2}}}$, thus, we have

$$p^* = \int_0^\infty \left(1+t^3\right)p(t)dt \leq \int_0^\infty t^{\frac{5}{2}}e^{-t}dt = \frac{15\sqrt{\pi}}{8} < \infty,$$

and

$$\lambda = \int_0^\infty t|f(t,0)|dt^{\frac{-2}{3}} = \int_0^\infty 0dt < \infty.$$

Then (H_2) holds. At last, by a simple computation, we have

$$m = \frac{p^*\Gamma(\alpha+\beta)}{\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]} \leq \frac{15\sqrt{\pi}}{12\sqrt{2\pi}\,(2-\xi\rho^{-2}\eta^4)} \leq \frac{15\sqrt{\pi}}{12\sqrt{2\pi}} < 1,$$

$f : [0,\infty)R \to [0,\infty)$ is continuous.

$|F(t,u)| = \varphi(t)(|u|)$ on $[0,\infty)R$ with $\in C([0,\infty),[0,\infty))$ nondecreasing and $\varphi \in L^1[0,+\infty)$; $\frac{\varrho\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right]}{(\varrho)\int_0^\infty \varphi(s)ds} = \sqrt{\varrho}\rho^{\alpha-1}\Gamma(\alpha)\left[\Gamma(\alpha+\beta)-\xi\rho^{1-\alpha-\beta}\eta^{\rho(\alpha+\beta-1)}\right] > 1.$

Hence all conditions of Theorem 4.1 hold. Thus with Theorem 4.1, the problem (4.3) has at least a positive solution u such that

$$0 \leq \frac{u(t)}{1+t^3} \leq \varrho, \qquad \text{for } t \in [0, \infty),$$

Hence all conditions of Theorem 3.1 hold. Thus with Theorem 3.1, the problem (3.3) has at least a positive solution u such that

$$0 \leq \frac{u(t)}{1+t^3} \leq \varrho, \qquad \text{for } t \in [0, \infty).$$

Conclusion

In this thesis, we have generalized the solution of the fractional boundary value problem on the infinite interval.

This work can be divided in two main parts:
The first parts is devoted to the study of the existence and uniqueness of solutions for Riemann-Liouville fractional boundary value problem on the Infinite Interval:

$$\begin{cases} D^{\alpha}u(t) + f(t,u(t)) = 0, \quad n < \alpha \leq n+1, \\ u(0) = u'(0) = ... = u^{(n-1)} = 0, \qquad D^{\alpha-1}u(\infty) = \xi I^{\beta}u(\eta), \quad \beta > 0 \end{cases}$$

where $t \in J = [0, +\infty),\ f \in C[J\text{X}\mathbb{R}, \mathbb{R}], \xi \in \mathbb{R}, \eta \in J.D^{\alpha}$ is the Riemann-Liouville fractional derivatives of order α, and I^{β} is the Riemann-Liouville fractional integral of order β.

In the second part, we studied the existence and uniqueness of solutions for a Katugampola fractional nonlinear differential equation of order $n < \alpha < n+1$

$$\begin{cases} D^{\alpha,\rho}u(t) + f(t,u(t)) = 0, \quad n < \alpha \leq n+1 \\ \delta^{k}u(0) = 0, k = 0, 1..., n-1, \qquad D^{\alpha-1,\rho}u(\infty) = \xi^{\rho} I^{\beta}u(\eta), \quad \beta > 0 \end{cases}$$

where $t \in J = [0, +\infty),\ f \in C[J\text{X}\mathbb{R}, \mathbb{R}], \quad \xi \in \mathbb{R}, \quad \eta \in J, \quad D^{\alpha,\rho}$ is the katugampola fractional derivatives of order α, and ${}^{\rho}I^{\beta}$ is the katugampola fractional integral of order β.

Bibliography

[1] A.A. Kilbas, Hadamard type fractional calculus, J. Korean Math. Soc, 38(2001), 1191-1204.

[2] A.A. Kilbas, H.H. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[3] Butzer, P. L., Kilbas, A. A., and Trujillo, J.J., Compositions of Hadamard-type frac- tional integration operators and the semigroup property, Journal of Mathematical Anal- ysis and Applications, 269, (2002),387-400.

[4] B. Basti, Y. Arioua , N. Benhamidouche, Existence and Uniqueness of Solutions for Nonlinear Katugampola Fractional Differential Equations , J. Math. Appl. 42 (2019), 35-61.

[5] B. Basti, Y. Arioua, N. Benhamidouche, Existence results for nonlinear Katugampola fractionnal differentia equations with an integral condition, J. Acta Mathematica Universitatis Comenianae. 2(2020), pp. 243-260.

[6] P. L. Butzer, A. A. Kilbas, and J. J. Trujillo, Fractional calculus in the Mellin set- ting and Hadamard-type fractional integrals, Journal of Mathematical Analysis and Applications, 269, (2002), 1-27.

[7] K. Diethelm, The Analysis of fractional differential equations. An applicationoriented exposition using differential operators of Caputo type. Springer, Heidelberg, 2010.

[8] U.N. Katugampola, New approach to a generalized fractional integral, Applied Mathematics and Computation 218 (3) (2011) 860–865.

[9] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer New York Dordrecht Heidelberg London (2010).

[10] I. Podlubny, Fractional Differential Equations, San Diego, Academic Press, New York 1999.

[11] R. Almeida, A. B. Malinowska, T. Odzijewicz, Fractional differential equations with dependence on the Caputo Katugampola derivative, J. Comput. Nonlinear Dynam, Vol. 11, 061017-1, (2016).

[12] D. Baleanu, G. C. Wu, S. D. Zeng, Chaos analysis and asymptotic stability of generalized Caputo fractional differential equations, Chaos, Solitons and Fractals, (2017), 1–7.

[13] X. Zhao ,G. Weigao, Unbounded Solutions for a Fractional Boundary Value Problems on the Infinite Interval, Acta Appl Math 109(2010), 495–505.

[14] G. Wang, Explicit iteration and unbounded solutions for fractional integral boundary value problem on an infinite interval, Applied Mathematics Letters 47 (2015) 1–7.

[15] R. Hilfer, Applications of Fractional Calculus in Physics, Singapore, World Scientific, 2000.