J. Math. Anal. Appl. 491 (2020) 124346

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Two-Lipschitz operator ideals

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A R T I C L E   I N F O

Article history:
Received 22 April 2020
Available online 7 July 2020
Submitted by R.M. Aron

Keywords:
Lipschitz operator ideal
Non-linear compact mapping
Two-Lipschitz mapping
Two-Lipschitz operator ideal

A B S T R A C T

We introduce and investigate the concept of two-Lipschitz operator ideal between pointed metric spaces and Banach spaces. We show the basics of this new theory and we give a procedure to create a two-Lipschitz operator ideal from a linear operator ideal. We apply our result to the ideals of strongly p-summing and compact linear operator to obtain their corresponding two-Lipschitz operator ideal. Also, we establish a natural relation between two-Lipschitz and bilinear maps and show that the two-Lipschitz factorable p-dominated operators are those which are associated to the well-known p-semi-integral bilinear operators.

0. Introduction

The theory of operator ideals has proved to be a strong tool for the investigation and classification of linear operators between Banach spaces. Nowadays, it has become the starting point to understand and solve new problems related to non-linear operators. The linear theory has spread to Lipschitz operators, leading to the notions of Lipschitz operator ideals. A first outline of such a Lipschitz theory was given by Farmer and Johnson in 2009 (see [18]). In 2016 an axiomatic theory of Lipschitz operator ideals for Banach spaces-valued Lipschitz mappings was given by Achour et al. in [3] (see also [35]) and in [7] for Lipschitz operator ideals between pointed metric spaces. These new Lipschitz operator ideals could also be considered to be a point for the study of some specific properties of non-linear operators, that can be considered as a new area in non-linear functional analysis.

An ideal of Lipschitz mappings $\mathcal{I}_{Lip}$ is a subclass of the class of all Lipschitz mappings between pointed metric spaces and Banach spaces such that for a metric space X and Banach space E, the component

$$\mathcal{I}_{Lip}(X,E) := Lip_0(X,E) \cap \mathcal{I}_{Lip},$$

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https://doi.org/10.1016/j.jmaa.2020.124346
0022-247X/

is a vector subspace of $Lip_0(X, E)$ that is invariant by the composition of a linear operator on the right and Lipschitz operator on the left and which contains the Lipschitz finite rank mappings type.

A number of operator ideals have been fruitfully generalized to the Lipschitz setting in recent years by several authors (see [36], [4], [5], [6], [24], [12] and the references therein).

In 2009, Dubei et al. introduced in [17] the definition of two-Lipschitz maps which is defined on the cartesian product of two pointed metric spaces with values in a Banach space, that is Lipschitz separately in each variable. Under some difficult adequate requirements it is shown that every two-Lipschitz mapping is associated with continuous bilinear mapping from product of two suitable free Banach spaces to another Banach space (see [17]). This idea worked successfully without any conditions in [33]. There, Sánchez-Pérez present a suitable definition of real-valued two-Lipschitz mappings (under the name of Lipschitz bi-forms) that admits a good continuous bi-linearization between Banach spaces.

Our purpose is to study the new concept of two-Lipschitz operator ideals between pointed metric spaces and Banach spaces. We extend to the two-Lipschitz mappings setting a linear procedure for creating ideals of two-Lipschitz operators from a given linear operator ideal. We apply our results to the well known operator ideal of strongly p-summing operators ideals. We present and characterize the notion of compactness for the two-Lipschitz mappings.

The article is divided as follows. In Section 1, after fixing some notation, we recall the most important results on the theory of Lipschitz operators and projective tensor product theory that we will use throughout the manuscript. In Section 2, we discuss some properties of the Banach space of two-Lipschitz operators, we state and prove the bi-linearization theorem. The third section deals with the basics of the theory of two-Lipschitz operator ideals, that are introduced in a natural way. We present the composition method to produce an ideal of two-Lipschitz operators from a given operator ideal $\mathcal{I}$. We show that a two-Lipschitz operator T belongs to the resulting ideal if it can be written as $T = u \circ S$ with u belonging to $\mathcal{I}$ and S is a two-Lipschitz operator. Finally, in the last section, we introduce the ideal of two-Lipschitz compact operators from the product of two pointed metric spaces X, Y into a Banach space E and we study a two-Lipschitz version of strongly p-summing linear operators in order to apply the technique of composition previously developed. Also we extend to the two-Lipschitz case the concept of absolutely $(p; p_1, p_2)$-summing bilinear operators. Finally, the notion of two-Lipschitz factorable p-dominated operators is introduced, showing that the bi-linearization of these mappings are exactly the well-known p-semi-integral bilinear operators.

1. Notation and preliminaries

As usual, X and Y will be pointed metric spaces with a base point denoted by 0 and a metric that will be denoted by d. We denote by B_X the closure of the ball centered at 0 with radius 1. Also, E, F and G will stand for Banach spaces over the same field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$) with dual spaces E^* and F^*. A Banach space E will be considered as a pointed metric space with distinguished point 0 and distance $d(x, x') = \|x - x'\|$. Given a Banach space E, B_E is its closed unit ball (which is coherent with the above notation) and S_E is the unit sphere of E. With $Lip_0(X, Y)$ we denote the set of all Lipschitz mappings from X to Y that map 0 to 0. In particular, $Lip_0(X, E)$ is the Banach space of all Lipschitz mappings T from X to E that vanish at 0, under the Lipschitz norm

$$Lip\,(T) = \inf\{C > 0 : \|T(x) - T(x')\| \leq Cd(x, x');\ \forall x, x' \in X\}.$$

When $E = \mathbb{K}$, $Lip_0(X, \mathbb{K})$ is denoted by $X^{\#}$ and it is called the Lipschitz dual of X. The space of all linear operators from E to F is indicated by $\mathcal{L}(E, F)$ and it is a Banach space with the usual supremum norm. It is clear that $\mathcal{L}(E, F)$ is a subspace of $Lip_0(E, F)$ and, in particular, E^* is a subspace of $E^{\#}$. Along the paper we consider $B_{X^{\#}}$ endowed with the pointwise topology. One of the main tools that we will use is the Arens–Ells space Æ(X) [9] (also known as the Lipschitz-free Banach space of a metric space X, $\mathcal{F}(X)$).

A molecule on X is a scalar valued function m on X with finite support that satisfies $\sum_{x\in X} m(x) = 0$. We denote by $\mathcal{M}(X)$ the linear space of all molecules on X. For $x, x' \in X$ the molecule $m_{xx'}$ is defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$, where χ_A is the characteristic function of the set A. For $m \in \mathcal{M}(X)$ we can write $m = \sum_{j=1}^{n} \lambda_j m_{x_j x'_j}$ for some suitable scalars λ_j and we write

$$\|m\|_{\mathcal{M}(X)} = \inf \left\{ \sum_{j=1}^{n} |\lambda_j|\, d(x_j, x'_j),\ m = \sum_{j=1}^{n} \lambda_j m_{x_j x'_j} \right\},$$

where the infimum is taken over all representations of the molecule m. Denote by Æ(X) the completion of the normed space $(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)})$. The canonical Lipschitz injection map $\delta_X : X \to$Æ(X) defined by $\delta_X(x) = m_{x0}$ isometrically embeds X in Æ(X). For any Lipschitz mapping $T \in Lip_0(X, Y)$, there exists a unique linear operator $\widehat{T} \in \mathcal{L}($Æ$(X)$, Æ$(Y))$ such that

$$\widehat{T} \circ \delta_X = \delta_Y \circ T. \tag{1}$$

Furthermore, $\left\|\widehat{T}\right\| = Lip(T)$ (see [21, Lemma 3.1]).

Given $T \in Lip_0(X, E)$, there exists a unique linear map $T_L :$ Æ$(X) \longrightarrow E$ such that $T = T_L \circ \delta_X$ and $\|T_L\| = Lip(T)$. The operator T_L is referred to as the linearization of T (see [34, Theorem 2.2.4 (b)]). The correspondence $T \longleftrightarrow T_L$ establishes an isomorphism between the vector spaces $Lip_0(X, E)$ and $\mathcal{L}($Æ$(X), E)$. In particular, the spaces $X^{\#}$ and Æ$(X)^*$ are isometrically isomorphic via the linearization $R(f) := f_L$, where $f_L(m) = \sum_{x\in X} f(x)m(x)$ (see [34, Theorem 2.2.2]).

By $\mathcal{L}(E, F; G)$ we denote the Banach space of all continuous bilinear mappings $T : E \times F \longrightarrow G$ with the usual sup norm

$$\|T\| = \sup_{x\in B_E, y\in B_F} \|T(x, y)\|.$$

By $E\widehat{\otimes}_\pi F$ we denote the completed projective tensor product of E and F. Given $T \in \mathcal{L}(E, F; G)$ consider $T_L \in \mathcal{L}(E\widehat{\otimes}_\pi F, G)$, the unique linearization of T defined by $T_L(x \otimes y) = T(x, y)$, for all $x \in E$, $y \in F$. Moreover, $\|T\| = \|T_L\|$. In other words $T = T_L \circ \sigma_2$, where σ_2 is the canonical continuous bilinear mapping $\sigma_2 : E \times F \longrightarrow E\widehat{\otimes}_\pi F$, $\sigma_2(x, y) = x \otimes y$. For the theory of topological tensor products we refer to [31].

Let $1 \le p < \infty$ we write p^* for the extended real number that satisfies $1/p + 1/p^* = 1$, as usual, when $p = 1$, $p^* = \infty$. We denote by $\ell_p^n(E)$ the space of all sequences $(x_i)_{i=1}^n$ in the Banach space E with the norm $\left\|(x_i)_{i=1}^n\right\|_p = \left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}}$, and by $\ell_{p,\omega}^n(E)$ the space of all sequences $(x_i)_{i=1}^n$ in E with the norm $\left\|(x_i)_{i=1}^n\right\|_{p,\omega} = \sup_{\|x^*\|_{E^*}\le 1} \left\|(\langle x_i, x^*\rangle)_{i=1}^n\right\|_p$.

2. Bi-linearization of two-Lipschitz operators

Definition 2.1. [33, Section 3.4] Let (X, d_X) and (Y, d_Y) be pointed metric spaces and let E be a Banach space, we say that a map $T : X \times Y \longrightarrow E$ is a two-Lipschitz operator if there is a constant $C > 0$ such that for each $x, x' \in X$ and $y, y' \in Y$,

$$\|T(x, y) - T(x, y') - T(x', y) + T(x', y')\| \le C.d_X(x, x')\, d_Y(y, y'). \tag{2}$$

By $BLip_0(X, Y; E)$ we denote the set of all two-Lipschitz operators from $X \times Y$ to E such that

$$T(x, 0) = T(0, y) = 0, \tag{3}$$

for all $x \in X$ and $y \in Y$. For $T \in BLip_0(X, Y; E)$ we set

$$BLip(T) = \inf C = \sup_{x \neq x', y \neq y'} \frac{\|T(x, y) - T(x, y') - T(x', y) + T(x', y')\|}{d_X(x, x') d_Y(y, y')}. \tag{4}$$

For the mapping $T : X \times Y \longrightarrow E$, consider $A_y : X \longrightarrow E$ and $A_x : Y \longrightarrow E$ such that $A_y(x) = T(x, y)$ for every fixed $y \in Y$ and $A_x(y) = T(x, y)$ for every fixed $x \in X$. According to Dubei et al. in [17], T is said to be two-Lipschitz if A_x is Lipschitz for every fixed $x \in X$ and A_y is Lipschitz for every fixed $y \in Y$.

In the following proposition, we show that this definition with a requirement on the operator $x \longrightarrow A_x$ is equivalent to our definition (Definition 2.1).

Proposition 2.2. *For a mapping $T : X \times Y \longrightarrow E$, the following statements are equivalent.*

(i) $T \in BLip_0(X, Y; E)$.
(ii) $A_x \in Lip_0(Y, E)$ *for every fixed* $x \in X$, $A_y \in Lip_0(X, E)$ *for every fixed* $y \in Y$ *and* $G : x \longrightarrow A_x$ *belongs to* $Lip_0(X, Lip_0(Y, E))$.
(iii) $A_x \in Lip_0(Y, E)$ *for every fixed* $x \in X$, $A_y \in Lip_0(X, E)$ *for every fixed* $y \in Y$ *and* $H : y \longrightarrow A_y$ *belongs to* $Lip_0(Y, Lip_0(X, E))$.

Proof. $(i) \Longrightarrow (ii)$ For every fixed $x \in X$, starting from (2) take $x' = 0$ we obtain

$$\begin{aligned}\|A_x(y) - A_x(y')\| &= \|T(x, y) - T(x, y') - T(0, y) + T(0, y')\| \\ &\leq BLip(T) d(x, 0) d(y, y').\end{aligned}$$

By (3) we have $A_x(0) = 0$. It follows that, $A_x \in Lip_0(Y, E)$ and $Lip(A_x) \leq BLip(T) d(x, 0)$. Similarly, $A_y \in Lip_0(X, E)$ and $Lip(A_y) \leq BLip(T) d(y, 0)$ for every fixed $y \in Y$. Now for all $x, x' \in X$ we have

$$\begin{aligned}\|G(x) - G(x')\| &= Lip(A_x - A_{x'}) \\ &= \sup_{y \neq y'} \frac{\|(A_x - A_{x'})(y) - (A_x - A_{x'})(y')\|}{d(y, y')} \\ &= \sup_{y \neq y'} \frac{\|T(x, y) - T(x, y') - T(x', y) + T(x', y')\|}{d(y, y')} \\ &\leq BLip(T) d(x, x'),\end{aligned}$$

showing that $G \in Lip_0(X, Lip_0(Y, E))$.

$(ii) \Longrightarrow (i)$ The equalities (3) follow easily from that $A_x(0) = 0$ and $A_y(0) = 0$. The assumption $\|G(x) - G(x')\| \leq Lip(G) d(x, x')$ for all $x, x' \in X$, implies that T satisfies the inequality (2). The equivalence $(i) \Longleftrightarrow (iii)$ is proved in a similar way. □

By using simple calculation, we prove the following result.

Proposition 2.3. *Let X, Y, Z, W be pointed metric spaces and let E, F be Banach spaces. If $f \in Lip_0(Z, X)$, $g \in Lip_0(W, Y)$, $T \in BLip_0(X, Y; E)$ and $u \in \mathcal{L}(E, F)$ then $u \circ T \circ (f, g) \in BLip_0(Z, W; F)$, where $(f, g)(z, w) := (f(z), g(w))$, $z \in Z, w \in W$. Moreover,*

$$BLip(u \circ T \circ (f, g)) \leq \|u\| \, BLip(T) Lip(f) Lip(g). \tag{5}$$

Remark 2.4. If X and Y be Banach spaces. Then every bilinear operator $T : X \times Y \longrightarrow E$ is two-Lipschitz. Moreover, we have $BLip\,(T) = \|T\|$. In order to see this, for each $x, x' \in X$ and $y, y' \in Y$,

$$\begin{aligned} &\|T\,(x, y) - T\,(x, y') - T\,(x', y) + T\,(x', y')\| \\ &= \|T\,(x - x', y - y')\| \le \|T\|\,\|x - x'\|\,\|y - y'\|\,. \end{aligned}$$

Therefore, $BLip\,(T) \le \|T\|$. For the reverse inequality, we will write (4) for $x' = y' = 0$,

$$BLip\,(T) \ge \sup_{x \neq 0, y \neq 0} \frac{\|T\,(x, y)\|}{d_X\,(x, 0)\, d_Y\,(y, 0)} = \|T\|\,.$$

The next theorem and its proof are similar to the Lipschitz case (see [34, Proposition 1.6.2]).

Theorem 2.5. *$BLip_0\,(X, Y; E)$ is a Banach space under the norm $BLip\,(\cdot)$ defined by (4).*

In what follows, let (X, d_X) and (Y, d_Y) be metric spaces and consider the product metric space $X \times Y$ equipped with the metric

$$d\,((x, y)\,, (x', y')) = d_X\,(x, x') + d_Y\,(y, y')\,,$$

for all $x, x' \in X$ and $y, y' \in Y$. Also if E, F be Banach spaces, the product Banach space $E \times F$ is equipped with the norm

$$\|(x, y)\|_{E \times F} = \|x\|_E + \|y\|_F\,,$$

for all $x \in E$ and $y \in F$.

A simple calculation shows that the mapping $(\delta_X, \delta_Y) : X \times Y \longrightarrow$Æ$(X) \times$Æ$(Y)$ defined by

$$(\delta_X, \delta_Y)\,(x, y) := (\delta_X(x), \delta_Y(y)) = (m_{x0}, m_{y0})$$

isometrically embeds $X \times Y$ in Æ$(X) \times$Æ(Y).

For all two-Lipschitz operator $T : X \times Y \longrightarrow E$, we define a bilinear mapping $T_B : \mathcal{M}(X) \times \mathcal{M}(Y) \longrightarrow E$ by

$$T_B\,(m_{xx'}, m_{yy'}) = T\,(x, y) - T(x, y') - T\,(x', y) + T(x', y'), \tag{6}$$

for all $x, x' \in X$ and $y, y' \in Y$. Thus the two-Lipschitz operator T is associated with the bilinear mapping T_B.

Theorem 2.6. *For every two-Lipschitz operator $T \in BLip_0\,(X, Y; E)$ there exists a unique bilinear mapping T_B : Æ$(X) \times$Æ$(Y) \longrightarrow E$ satisfying (6) and*

$$T = T_B \circ (\delta_X, \delta_Y) : X \times Y \overset{(\delta_X, \delta_Y)}{\longrightarrow} \text{Æ}(X) \times \text{Æ}(Y) \overset{T_B}{\longrightarrow} E.$$

Furthermore $BLip\,(T) = \|T_B\|$. The bilinear mapping T_B is called bi-linearization of the two-Lipschitz operator T.

Proof. Let $(m_1, m_2) \in \mathcal{M}(X) \times \mathcal{M}(Y)$ and let $\varepsilon > 0$. Choose representations of m_1 and m_2 of the form

$$m_1 = \sum_{i=1}^{n} \alpha_i m_{x_i x_i'}, \quad m_2 = \sum_{j=1}^{r} \beta_j m_{y_j y_j'}$$

such that

$$\sum_{i=1}^{n} |\alpha_i| \, d(x_i, x_i') \leq \varepsilon + \|m_1\|_{\mathcal{M}(X)} \quad \text{and} \quad \sum_{j=1}^{r} |\beta_j| \, d(y_j, y_j') \leq \varepsilon + \|m_2\|_{\mathcal{M}(Y)} .$$

Then

$$\begin{aligned}\|T_B(m_1, m_2)\| &= \left\| \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{r} \beta_j \left(T(x_i, y_j) - T(x_i, y_j') - T(x_i', y_j) + T(x_i', y_j') \right) \right\| \\ &\leq BLip(T) \sum_{i=1}^{n} |\alpha_i| \, d(x_i, x_i') \sum_{j=1}^{r} |\beta_j| \, d\left(y_j, y_j'\right) \\ &\leq BLip(T) \left(\varepsilon + \|m_1\|_{\mathcal{M}(X)} \right) \left(\varepsilon + \|m_2\|_{\mathcal{M}(Y)} \right).\end{aligned}$$

Since this holds for every $\varepsilon > 0$ we obtain

$$\|T_B(m_1, m_2)\| \leq BLip(T) \, \|m_1\|_{\mathcal{M}(X)} \, \|m_2\|_{\mathcal{M}(Y)} .$$

Therefore, T_B is continuous and satisfies $\|T_B\| \leq BLip(T)$. On the other hand, by the bilinearity of T_B and taking into account that $T = T_B \circ (\delta_X, \delta_Y)$ we get

$$\begin{aligned}&\|T(x, y) - T(x, y') - T(x', y) + T(x', y')\| \\ &= \|T_B(\delta_X(x) - \delta_X(x'), \delta_Y(y) - \delta_Y(y'))\| \\ &\leq \|T_B\| \, \|\delta_X(x) - \delta_X(x')\| \, \|\delta_Y(y) - \delta_Y(y')\| \\ &= \|T_B\| \, d(x, x') \, d(y, y').\end{aligned}$$

It follows that $BLip(T) \leq \|T_B\|$. Therefore, $BLip(T) = \|T_B\|$. Now, the continuous bilinear mapping T_B has a unique extension to

$$\overline{\mathcal{M}(X)} \times \overline{\mathcal{M}(Y)} = Æ(X) \times Æ(Y),$$

denoted also by T_B, with $\|T_B\| = BLip(T)$.

For the uniqueness of the bi-linearization, suppose $S : \mathcal{M}(X) \times \mathcal{M}(Y) \longrightarrow E$ is a bilinear mapping such that $T = S \circ (\delta_X, \delta_Y)$. Thus, for any $m_1 = \sum_{i=1}^{n} \alpha_i m_{x_i x_i'} \in \mathcal{M}(X)$ and $m_2 = \sum_{j=1}^{r} \beta_j m_{y_j y_j'} \in \mathcal{M}(Y)$

$$\begin{aligned}S(m_1, m_2) &= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{r} \beta_j S\left(m_{x_i x_i'}, m_{y_j y_j'} \right) \\ &= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{r} \beta_j S\left(\delta_X(x_i) - \delta_X(x_i'), \delta_Y(y_j) - \delta_Y(y_j') \right) \\ &= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{r} \beta_j T_B\left(\delta_X(x_i) - \delta_X(x_i'), \delta_Y(y_j) - \delta_Y(y_j') \right) \\ &= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{r} \beta_j T_B\left(m_{x_i x_i'}, m_{y_j y_j'} \right) \\ &= T_B(m_1, m_2).\end{aligned}$$

This proves that $S = T_B$. $\square$

Remark 2.7. Note that the bilinear operator T_B admits a linearization

$$(T_B)_L : Æ(X) \widehat{\otimes}_\pi Æ(Y) \longrightarrow E$$

satisfies

$$T = T_B \circ (\delta_X, \delta_Y) = (T_B)_L \circ \sigma_2 \circ (\delta_X, \delta_Y),$$

where $\sigma_2 : Æ(X) \times Æ(Y) \longrightarrow Æ(X) \widehat{\otimes}_\pi Æ(Y)$ is the canonical bilinear operator defined by $\sigma_2(m_{x0}, m_{y0}) = m_{x0} \otimes m_{y0}$. In addition we have

$$BLip(T) = \|T_B\| = \|(T_B)_L\|.$$

The linear operator $(T_B)_L$ is referred to as the linearization of the two-Lipschitz operator T. For the simplification, write T_L instead of $(T_B)_L$.

Next we give a simple but crucial example of a two-Lipschitz operator. Let X, Y be pointed metric spaces and let E be Banach space. Consider non-zero Lipschitz functions $f \in X^\#$, $g \in Y^\#$ and $e \in E$. Define the mapping $f \cdot g \cdot e : X \times Y \longrightarrow E$ by

$$f \cdot g \cdot e(x, y) = f(x)g(y)e. \tag{7}$$

Then, an easy computation shows that this mapping is two-Lipschitz and

$$BLip(f \cdot g \cdot e) = Lip(f)Lip(g)\|e\|. \tag{8}$$

Definition 2.8. We denote by $BLip_{0\mathcal{F}}(X, Y; E)$, the linear subspace of all two-Lipschitz operators generated by the mappings of the special form (7). All elements T of this space are called of finite type. So, any $T \in BLip_{0,\mathcal{F}}(X, Y; E)$ admits a finite representation of the form

$$T = \sum_{i=1}^{n} f_i \cdot g_i \cdot e_i,$$

where $(f_i)_{i=1}^n \subset X^\#, (g_i)_{i=1}^n \subset Y^\#$ and $(e_i)_{i=1}^n \subset E$.

3. Two-Lipschitz operator ideals

We will follow the spirit of the definitions of multilinear operator ideals ([28] or [19]) and Lipschitz operator ideals [3], for defining the concept of two-Lipschitz operator ideals.

Definition 3.1. A two-Lipschitz operator ideal between pointed metric spaces and Banach spaces, $\mathcal{I}_{BLip}$, is a subclass of $BLip_0$ such that for every pointed metric spaces X, Y and every Banach space E the components

$$\mathcal{I}_{BLip}(X, Y; E) := BLip_0(X, Y; E) \cap \mathcal{I}_{BLip}$$

satisfy

(i) $\mathcal{I}_{BLip}(X, Y; E)$ is a linear subspace of $BLip_0(X, Y; E)$.
(ii) For any $f \in X^\#$, $g \in Y^\#$ and $e \in E$, the map $f \cdot g \cdot e$ belongs to $\mathcal{I}_{BLip}(X, Y; E)$.

(iii) The ideal property: if $f \in Lip_0(Z, X)$, $g \in Lip_0(W, Y)$, $T \in \mathcal{I}_{BLip}(X, Y; E)$ and $u \in \mathcal{L}(E, F)$, then the composition $u \circ T \circ (f, g)$ is in $\mathcal{I}_{BLip}(Z, W; F)$.

A two-Lipschitz operator ideal $\mathcal{I}_{BLip}$ is a normed (Banach) two-Lipschitz operator ideal if there is $\|\cdot\|_{\mathcal{I}_{BLip}} : \mathcal{I}_{BLip} \longrightarrow [0, +\infty[$ that satisfies

(i') For every pointed metric spaces X, Y and every Banach space E, the pair $(\mathcal{I}_{BLip}(X, Y; E), \|\cdot\|_{\mathcal{I}_{BLip}})$ is a normed (Banach) space and $BLip(T) \leq \|T\|_{\mathcal{I}_{BLip}}$ for all $T \in \mathcal{I}_{BLip}(X, Y; E)$.
(ii') $\|Id_{\mathbb{K}^2} : \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{K} : Id_{\mathbb{K}^2}(\alpha, \beta) = \alpha\beta\|_{\mathcal{I}_{BLip}} = 1$.
(iii') If $f \in Lip_0(Z, X)$, $g \in Lip_0(W, Y)$, $T \in \mathcal{I}_{BLip}(X, Y; E)$ and $u \in \mathcal{L}(E, F)$, the inequality $\|u \circ T \circ (f, g)\|_{\mathcal{I}_{BLip}} \leq \|u\| \, \|T\|_{\mathcal{I}_{BLip}} Lip(f) Lip(g)$ holds.

Of course the Banach spaces considered in this definition are all over the same fixed scalar field.

The two-Lipschitz operator ideal $\mathcal{I}_{BLip}$ is said to be closed if each $\mathcal{I}_{BLip}(X, Y; E)$ is a closed subspace of $BLip(X, Y; E)$ with the norm $BLip(\cdot)$.

Proposition 3.2. *Let $\mathcal{I}_{BLip}$ be a normed two-Lipschitz operator ideal, X, Y be pointed metric spaces and E be Banach space. Then*

$$\|f \cdot g \cdot e\|_{\mathcal{I}_{BLip}} = \|e\| \, Lip(f) Lip(g),$$

for any $f \in X^\#$, $g \in Y^\#$ and $e \in E$.

Proof. Let $f \in X^\#$, $g \in Y^\#$ and $e \in E$. We can write $f \cdot g \cdot e$ in the following way

$$f \cdot g \cdot e = id_{\mathbb{K}} e \circ Id_{\mathbb{K}^2} \circ (f, g).$$

By (ii'), (iii') and (8), we obtain directly

$$\begin{aligned} \|f \cdot g \cdot e\|_{\mathcal{I}_{BLip}} &\leq \|id_{\mathbb{K}} e\| \, \|Id_{\mathbb{K}^2}\|_{\mathcal{I}_{BLip}} Lip(f) Lip(g) \\ &= \|e\| \, Lip(f) Lip(g) = BLip(f \cdot g \cdot e) \\ &\leq \|f \cdot g \cdot e\|_{\mathcal{I}_{BLip}}, \end{aligned}$$

this gives, $\|f \cdot g \cdot e\|_{\mathcal{I}_{BLip}} = \|e\| \, Lip(f) Lip(g)$. □

Remark 3.3. By the above definition, the class $BLip_{0\mathcal{F}}$ is the smallest two-Lipschitz operator ideal and the class of all two-Lipschitz operators between arbitrary pointed metric spaces and Banach spaces, is the largest two-Lipschitz operator ideal.

We use techniques inspired by [11], we give a method (composition method) to build a two-Lipschitz operator ideal starting from a given operator ideal. The properties enjoyed by the linear operators in this ideal can be generalized to the two-Lipschitz case and the resulting classes of two-Lipschitz mappings happen to be a two-Lipschitz ideal called ideal of composition type.

Recall that, from [27], an operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components $\mathcal{I}(E, F) := \mathcal{L}(E, F) \cap \mathcal{I}$ satisfy

(i) $\mathcal{I}(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$ which contains the mappings of the form $x^* \otimes y : x \longmapsto \langle x, x^* \rangle \, y$ where $x \in E$, $x^* \in E^*$ and $y \in F$.

(ii) The ideal property: if $u \in \mathcal{L}(E,K)$, $v \in \mathcal{I}(K,G)$ and $w \in \mathcal{L}(G,F)$, then the composition $w \circ v \circ u$ is in $\mathcal{I}(E,F)$.

If $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \longrightarrow \mathbb{R}^+$ satisfies

(i') $(\mathcal{I}(E,F), \|\cdot\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces E and F,
(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$,
(iii') If $u \in \mathcal{L}(E,K)$, $v \in \mathcal{I}(K,G)$ and $w \in \mathcal{L}(G,F)$, $\|w \circ v \circ u\|_{\mathcal{I}} \leq \|w\| \, \|v\|_{\mathcal{I}} \, \|u\|$,

then $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal.

Definition 3.4. Let $\mathcal{I}$ be an operator ideal. A two-Lipschitz operator $T \in BLip_0(X,Y;E)$ belongs to the composition two-Lipschitz operator ideal $\mathcal{I} \circ BLip_0$, in this case we write $T \in \mathcal{I} \circ BLip_0(X,Y;E)$, if there is a Banach space F, a two-Lipschitz operator $S \in BLip_0(X,Y;F)$ and a linear operator $u \in \mathcal{I}(F,E)$ such that $T = u \circ S$. If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed operator ideal we write

$$\|T\|_{\mathcal{I} \circ BLip_0} = \inf \|u\|_{\mathcal{I}} \, BLip(S),$$

where the infimum is taken over all u, S as above.

Theorem 3.5. *Let $\mathcal{I}$ be an operator ideal. A two-Lipschitz operator $T \in BLip_0(X,Y;E)$ belongs to $\mathcal{I} \circ BLip_0(X,Y;E)$ if and only if its linearization T_L belongs to $\mathcal{I}(Æ(X) \widehat{\otimes}_\pi Æ(Y), E)$.*

Furthermore, if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed operator ideal, then

$$\|T\|_{\mathcal{I} \circ BLip_0} = \|T_L\|_{\mathcal{I}}, \tag{9}$$

and we have the isometric identification

$$(\mathcal{I} \circ BLip_0(X,Y;E), \|\cdot\|_{\mathcal{I} \circ BLip_0}) = \left(\mathcal{I}(Æ(X) \widehat{\otimes}_\pi Æ(Y), E), \|\cdot\|_{\mathcal{I}}\right). \tag{10}$$

Proof. For the "if" part, if $T_L \in \mathcal{I}(Æ(X) \widehat{\otimes}_\pi Æ(Y), E)$, consider the factorization of T given by $T = T_L \circ \sigma_2 \circ (\delta_X, \delta_Y)$. Since the canonical bilinear mapping σ_2 is also two-Lipschitz with $BLip(\sigma_2) = \|\sigma_2\| = 1$, then $T \in \mathcal{I} \circ BLip_0(X,Y;E)$. By (5), we get that $\|T\|_{\mathcal{I} \circ BLip_0} \leq \|T_L\| \leq \|T_L\|_{\mathcal{I}}$.

To prove the "only if" part, take $T \in \mathcal{I} \circ BLip_0(X,Y;E)$ and $\varepsilon > 0$. Choose a Banach space F, a two-Lipschitz operator $S \in BLip_0(X,Y;E)$ and a linear operator $u \in \mathcal{I}(F,E)$ such that $T = u \circ S$ with $\|u\|_{\mathcal{I}} \, BLip(S) \leq \varepsilon + \|T\|_{\mathcal{I} \circ BLip_0}$. The uniqueness of the linearization maps gives that $T_L = u \circ S_L$, so $T_L \in \mathcal{I}(Æ(X) \widehat{\otimes}_\pi Æ(Y), E)$ by the ideal property. Furthermore,

$$\|T_L\|_{\mathcal{I}} \leq \|u\|_{\mathcal{I}} \, \|S_L\| = \|u\|_{\mathcal{I}} \, BLip(S) \leq \varepsilon + \|T\|_{\mathcal{I} \circ BLip_0}.$$

To show the identification given in (10), just consider the correspondence $T \longleftrightarrow T_L$. □

Proposition 3.6. *If $\mathcal{I}$ is a (normed, closed, Banach) operator ideal then, $\mathcal{I} \circ BLip_0$ is a (respectively normed, closed, Banach) two-Lipschitz operator ideal.*

Proof. Let us check that $\mathcal{I} \circ BLip_0$ is a closed two-Lipschitz operator ideal whenever $\mathcal{I}$ is a closed operator ideal. Thanks to an argument detailed in [3, Corollary 3.3], we prove that $\mathcal{I} \circ BLip_0(X,Y;E)$ is a closed linear subspace of $BLip_0(X,Y;E)$ with the norm $BLip(\cdot)$ and that the ideal property holds. If $f \in X^\#$, $g \in Y^\#$ and $e \in E$, we can write

$$T = f \cdot g \cdot e = id_{\mathbb{K}} \otimes e \circ (f \cdot g) \in \mathcal{I} \circ BLip_0(X, Y; E).$$

An application of (9) reveals that $(\mathcal{I} \circ BLip_0(X, Y; E), \| \cdot \|_{\mathcal{I} \circ BLip_0})$ is a normed space. Also, for all $T \in \mathcal{I}_{BLip}(X, Y; E)$,

$$BLip(T) = \|T_L\| \leq \|T_L\|_{\mathcal{I}} = \|T\|_{\mathcal{I} \circ BLip_0}.$$

Since $Id_{\mathbb{K}^2} = id_{\mathbb{K}} \circ Id_{\mathbb{K}^2}$ and $id_{\mathbb{K}} \in \mathcal{I}(\mathbb{K}, \mathbb{K})$, it follows that $Id_{\mathbb{K}^2} \in \mathcal{I} \circ BLip_0(\mathbb{K}, \mathbb{K}; \mathbb{K})$ and

$$1 = BLip(Id_{\mathbb{K}^2}) \leq \|Id_{\mathbb{K}^2}\|_{\mathcal{I} \circ BLip_0} \leq \|id_{\mathbb{K}}\|_{\mathcal{I}} BLip(Id_{\mathbb{K}^2}) = 1.$$

Now, let $f \in Lip_0(Z, X)$, $g \in Lip_0(W, Y)$, $T \in \mathcal{I}_{BLip}(X, Y; E)$ and $u \in \mathcal{L}(E, F)$. Let $\widehat{f} \in \mathcal{L}(Æ(Z), Æ(X))$ and $\widehat{g} \in \mathcal{L}(Æ(W), Æ(Y))$ be the associated linear operators of f and g defined in (1). By [31, Proposition 2.3] we take $\widehat{f} \otimes \widehat{g}$ the unique linear operator defined from $Æ(Z) \widehat{\otimes}_\pi Æ(W)$ to $Æ(X) \widehat{\otimes}_\pi Æ(Y)$ by $\widehat{f} \otimes \widehat{g}(m \otimes m') = \widehat{f}(m) \otimes \widehat{g}(m')$, for all $m \in Æ(Z)$ and $m' \in Æ(W)$ with $\left\| \widehat{f} \otimes \widehat{g} \right\| = \left\| \widehat{f} \right\| \|\widehat{g}\|$ and consider the canonical bilinear mappings

$$\sigma_2 : Æ(X) \times Æ(Y) \longrightarrow Æ(X) \otimes Æ(Y),$$
$$\sigma_2' : Æ(Z) \times Æ(W) \longrightarrow Æ(Z) \otimes Æ(W).$$

We have

$$\sigma_2 \circ (\delta_X, \delta_Y) \circ (f, g) = \widehat{f} \otimes \widehat{g} \circ \sigma_2' \circ (\delta_Z, \delta_W).$$

Since $T = T_L \circ \sigma_2 \circ (\delta_X, \delta_Y)$,

$$\begin{aligned} u \circ T \circ (f, g) &= u \circ T_L \circ \sigma_2 \circ (\delta_X, \delta_Y) \circ (f, g) \\ &= u \circ T_L \circ \widehat{f} \otimes \widehat{g} \circ [\sigma_2' \circ (\delta_Z, \delta_W)]. \end{aligned}$$

The uniqueness of the linearization maps gives that

$$(u \circ T \circ (f, g))_L = u \circ T_L \circ \widehat{f} \otimes \widehat{g}.$$

By the ideal property concerning the operator ideal $\mathcal{I}$ and (9) we obtain

$$\begin{aligned} \|u \circ T \circ (f, g)\|_{\mathcal{I} \circ BLip_0} &= \left\| u \circ T_L \circ \widehat{f} \otimes \widehat{g} \right\|_{\mathcal{I}} \\ &\leq \|u\| \, \|T_L\|_{\mathcal{I}} \left\| \widehat{f} \otimes \widehat{g} \right\| \\ &= \|u\| \, \|T\|_{\mathcal{I} \circ BLip_0} Lip(f) Lip(g). \end{aligned}$$

Finally, it easily follows from the isometric identification (10) that if $(\mathcal{I}, \| \cdot \|_{\mathcal{I}})$ is a Banach operator ideal then, $(\mathcal{I} \circ BLip_0, \| \cdot \|_{\mathcal{I} \circ BLip_0})$ is a Banach two-Lipschitz operator ideal. □

Let $\mathcal{I}$ be an operator ideal and Y be a pointed metric space. In the next result we give a necessary and sufficient condition for assuring that every two-Lipschitz mappings $T : X_1 \times X_2 \longrightarrow Æ(Y)$ belong to the composition ideal $\mathcal{I} \circ BLip_0(X_1, X_2; Æ(Y))$, for all pointed metric spaces X_1 and X_2. In order to prove this result we need the following lemma.

Lemma 3.7. *Let $\mathcal{I}_1, \mathcal{I}_2$ be operator ideals, X_1, X_2 be pointed metric spaces and F be Banach space. If $\mathcal{I}_1 \circ BLip_0(X_1, X_2; F) \subset \mathcal{I}_2 \circ BLip_0(X_1, X_2; F)$, then $\mathcal{I}_1(Æ(X_i), F) \subset \mathcal{I}_2(Æ(X_i), F)$, for every $i = 1, 2$. In particular, if $\mathcal{I}_1 \circ BLip_0(X_1, X_2; F) = \mathcal{I}_2 \circ BLip_0(X_1, X_2; F)$, then $\mathcal{I}_1(Æ(X_i), F) = \mathcal{I}_2(Æ(X_i), F)$, for every $i = 1, 2$.*

Proof. Let $u \in \mathcal{I}_1(Æ(X_2), F)$. Fix $a^1 \in X_1$ and $\varphi_1 \in Æ(X_1)^*$ with $a^1 \neq 0$ and $\varphi_1(m_{a^1 0}) = 1$. For all $(x^1, x^2) \in X_1 \times X_2$ take $T(x^1, x^2) = \varphi_1(m_{x^1 0}) \, u(m_{x^2 0})$. It is clear that $T = u \circ R$ where $R(x^1, x^2) = \varphi_1(m_{x^1 0}) \, m_{x^2 0}$. An easy calculation shows that $R \in BLip_0(X_1, X_2; Æ(X_2))$ and hence $T \in \mathcal{I}_1 \circ BLip_0(X_1, X_2; F)$. Therefore, there is a Banach space G, a linear operator $v \in \mathcal{I}_2(G, F)$ and a two-Lipschitz operator $S \in BLip_0(X_1, X_2; G)$ such that $T = v \circ S$. Now if we consider S_B the bi-linearization of S, for all $x^2, x^{2\prime} \in X_2$ we obtain

$$\begin{aligned} u(m_{x^2 x^{2\prime}}) &= u(m_{x^2 0}) - u(m_{x^{2\prime} 0}) \\ &= T(a^1, x^2) - T(a^1, x^{2\prime}) \\ &= v \circ S(a^1, x^2) - v \circ S(a^1, x^{2\prime}) \\ &= v \circ S_B(m_{a^1 0}, m_{x^2 x^{2\prime}}). \end{aligned}$$

By linearity of u we conclude that $u(m) = v \circ w(m)$ for all $m \in \mathcal{M}(X_2)$, where $w : \mathcal{M}(X_2) \longrightarrow G$ is the bounded linear mapping defined by $w(m) = S_B(m_{a^1 0}, m)$ for all $m \in \mathcal{M}(X_2)$. Now, the operator w has a unique extension to a bounded linear mapping from $\overline{\mathcal{M}(X_2)} = Æ(X_2)$ to G, denoted also by w with $u = v \circ w$. Consequently, $u \in \mathcal{I}_2(Æ(X_2), F)$ by the ideal property. □

Proposition 3.8. *Let $\mathcal{I}$ be an operator ideal and Y be a pointed metric space. For all pointed metric spaces X_1, X_2, we have*

$$\mathcal{I} \circ BLip_0(X_1, X_2; Æ(Y)) = BLip_0(X_1, X_2; Æ(Y)),$$

if and only if the identity operator on $Æ(Y)$ belongs to $\mathcal{I}$.

Proof. For the sufficient condition, suppose that $id_{Æ(Y)} \in \mathcal{I}(Æ(Y), Æ(Y))$. If $T \in BLip_0(X_1, X_2; Æ(Y))$ then $T = id_{Æ(Y)} \circ T \in \mathcal{I} \circ BLip_0(X_1, X_2; Æ(Y))$. For the necessary condition, take $X_1 = X_2 = Y$ and applying the Lemma 3.7 for $\mathcal{I}_1 = \mathcal{I}$, $\mathcal{I}_2 = \mathcal{L}$ and $F = Æ(Y)$. □

4. Applications: some examples of two-Lipschitz operators ideals

4.1. Ideal of compact two-Lipschitz operators

We introduce the compactness concept for the two-Lipschitz operators. By showing that the new class of these operators is an ideal of the composition type, we see that the nature of this extension allows us to transfer some properties of the bilinear compact operators (and also the linear compact operators) to the two-Lipschitz case. Many papers were devoted to the concept of compactness for the bilinear mappings between Banach spaces (see [29], [30], [10], [22]).

Let E, F, G be Banach spaces and $T : E \times F \longrightarrow G$ be a bilinear operator. We call T compact, in symbols $T \in \mathcal{L}_{\mathcal{K}}(E, F; G)$ if $T(B_E \times B_F)$ is a relatively compact subset of G. This is equivalent to saying that T takes bounded sets into relatively compact sets.

Now we present the definition of compact two-Lipschitz operators. Let X, Y be pointed metric spaces, E be a Banach space and $T \in BLip_0(X, Y; E)$. As in the Lipschitz case ([20]), the two-Lipschitz image of T is the subset $Im_{BLip}(T) \subset E$ that consists of all elements of the form

$$\frac{T(x,y)-T(x',y)-T(x,y')+T(x',y')}{d(x,x')d(y,y')},$$

where $x, x' \in X, y, y' \in Y$ with $x \neq x'$ and $y \neq y'$.

It is easy to see that if $Im_{BLip}(T)$ is a bounded subset of E, then $T : X \times Y \longrightarrow E$ is a two-Lipschitz mapping, which motivates the following concept.

Definition 4.1. A two-Lipschitz operator $T \in BLip_0(X, Y; E)$ is said to be compact if $Im_{BLip}(T)$ is relatively compact in E. The vector space of these mappings is indicated by $BLip_{0\mathcal{K}}(X, Y; E)$.

Remark 4.2. Observe that the two-Lipschitz compact operators can be seen as an extension of the bilinear compact operators. Indeed, if X, Y, E are Banach spaces and $T : X \times Y \longrightarrow E$ is bilinear compact, it follows from $Im_{BLip}(T) = T(S_X \times S_Y)$ that T is two-Lipschitz compact.

The next result provides a shortcut for showing that the class of the two-Lipschitz compact operators is a closed two-Lipschitz operator ideal. Recall that the absolutely convex hull of the subset A of a Banach space E is defined to be

$$\Gamma(A) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N}, x_i \in A, \alpha_i \in \mathbb{R}, \sum_{i=1}^{n} |\alpha_i| \leq 1 \right\}.$$

Theorem 4.3. *Let X_1, X_2 be pointed metric spaces, E be a Banach space. For $T \in BLip_0(X_1, X_2; E)$, the following statements are equivalent.*

(i) *T is two-Lipschitz compact.*
(ii) *$T_B : Æ(X_1) \times Æ(X_2) \longrightarrow E$ is bilinear compact.*
(iii) *$T_L : Æ(X_1)\widehat{\otimes}_\pi Æ(X_2) \longrightarrow E$ is linear compact.*

Proof. Take $M_{X_i} = \left\{ \frac{m_{x_i 0} - m_{x'_i 0}}{d(x_i, x'_i)} : x_i, x'_i \in X_i, x_i \neq x'_i \right\}, i = 1, 2$. By [20, Lemma 1.1] we have $B_{Æ(X_i)} = \overline{\Gamma}(M_{X_i})$, $i = 1, 2$. The equivalence between (i) and (ii) follows from the inclusions

$$Im_{BLip}(T) \subset T_B\left(\overline{\Gamma}(M_{X_1}) \times \overline{\Gamma}(M_{X_2})\right) \subset \overline{\Gamma}\left(Im_{BLip}(T)\right),$$

and the fact that the closed absolutely convex hull of a relatively compact subset of a Banach space is compact. The equivalence $(ii) \Longleftrightarrow (iii)$ is proved in [22]. □

Since the class $\mathcal{K}$ of compact linear operators between Banach spaces is a closed Banach operator ideal (see [27]) and using the preceding theorem, Theorem 3.5 and Proposition 3.6 we obtain the following corollary.

Corollary 4.4. *The class $BLip_{0\mathcal{K}}$ is the closed Banach two-Lipschitz operator ideal generated by the composition method from $\mathcal{K}$, i.e.,*

$$BLip_{0\mathcal{K}}(X, Y; E) = \mathcal{K} \circ BLip_0(X, Y; E),$$

for all pointed metric spaces X, Y and Banach space E.

Remark 4.5. The same technique that we have shown above should provide also the corresponding result for the class of two-Lipschitz weakly compact operators. The mapping $T \in BLip_0(X, Y; E)$ is two-Lipschitz weakly compact if $Im_{BLip}(T)$ is relatively weakly compact in E.

4.2. Ideal of strongly two-Lipschitz operators

Cohen in [15] introduced, $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$, the operator ideal of strongly p-summing linear operators. For $1 < p \leq \infty$, recall that a linear operator $u : E \longrightarrow F$ belongs to $\mathcal{D}_p(E, F)$ if there is a positive constant C such that for all $n \in \mathbb{N},\ x_1, \dots, x_n \in E$ and $y_1^*, \dots, y_n^* \in F^*$ we have

$$\|(\langle u(x_i), y_i^* \rangle)_{i=1}^n\|_1 \leq C \, \|(x_i)_{i=1}^n\|_p \, \|(y_i^*)_{i=1}^n\|_{p^*,\omega} \,. \tag{11}$$

In this case, $\|u\|_{\mathcal{D}_p} := \inf \{C > 0 : \text{satisfying (11)}\}$.

The definition of Cohen strongly p-summing m-linear operators is due to Achour and Mezrag (see [2]) in order to generalize the concept of strongly p-summing linear operators.

Definition 4.6. For $1 < p \leq \infty$, a mapping $T \in \mathcal{L}(E, F; G)$ is Cohen strongly p-summing if there is a constant $C > 0$ such that for any $x_1, ..., x_n \in E$, $y_1, ..., y_n \in F$, and any $g_1^*, ..., g_n^* \in G^*$, we have

$$\|(\langle T(x_i, y_i), g_i^* \rangle)_{i=1}^n\|_1 \leq C \left(\sum_{i=1}^n \|x_i\|^p \|y_i\|^p \right)^{\frac{1}{p}} \|(g_i^*)_{i=1}^n\|_{p^*,\omega} \,.$$

The vector space of these mappings is indicated by $\mathcal{D}_p^2(E, F; G)$ and the smallest C satisfying the inequality above, by $\|T\|_{\mathcal{D}_p^2}$. This defines a norm on $\mathcal{D}_p^2(E, F; G)$ and $\left(\mathcal{D}_p^2, \|\cdot\|_{\mathcal{D}_p^2}\right)$ is a Banach bilinear ideal.

The next Lipschitz generalization of the concept of strongly p-summing linear operators was introduced by Yahi, Achour and Rueda in [36].

Definition 4.7. A map $T \in Lip_0(X, E)$ is *strongly Lipschitz p-summing* $(1 < p \leq \infty)$, if there are a Banach space F and an operator $u \in \mathcal{D}_p(F, E)$ such that

$$|\langle T(x) - T(x'), y^* \rangle| \leq d(x, x') \, \|u^*(y^*)\| \ \text{ for all } x, x' \in X, y^* \in E^*.$$

The infimum of all constants $\|u\|_{\mathcal{D}_p}$ is denoted $\|T\|_{\mathcal{D}_p^L}$. This class of mappings is denoted by $\mathcal{D}_p^L(X, E)$ and with the norm $\|T\|_{\mathcal{D}_p^L}$ it is a Banach space.

Now we are going to construct a new two-Lipschitz operator ideal by the composition method starting from the operator ideal $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$.

Definition 4.8. Let $1 < p \leq \infty$. A mapping $T \in BLip_0(X, Y; E)$ is strongly two-Lipschitz p-summing if there exist a Banach space G and a p^*-summing linear operator $S : E^* \longrightarrow G$ such that for all $x, x' \in X$, $y, y' \in Y$ and $e^* \in E^*$ we have

$$|\langle T(x, y) - T(x', y) - T(x, y') + T(x', y'), e^* \rangle| \leq d(x, x')d(y, y') \, \|S(e^*)\| \,. \tag{12}$$

We denoted by $\mathcal{D}_p^{BL}(X, Y; E)$ the set of all *strongly* two-Lipschitz p-summing mappings from $X \times Y$ to E. Moreover, if $T \in \mathcal{D}_p^{BL}(X, Y; E)$, then we set $\|T\|_{\mathcal{D}_p^{BL}} = \inf \{\pi_{p^*}(S)\}$. The infimum is taken over all Banach spaces G and operators S such that (12) holds.

Let us give an example of a strongly two-Lipschitz p-summing operator.

Example 4.9. Let $1 < p \leq \infty$ and $S : Y \longrightarrow E$ be a *strongly Lipschitz p-summing* operator and $f \in X^{\#}$. The mapping

$$T : X \times Y \longrightarrow E, \quad T(x, y) = f(x)S(y),$$

is a strongly two-Lipschitz p-summing operator with $\|T\|_{\mathcal{D}_p^{BL}} \leq Lip(f)\,\|S\|_{\mathcal{D}_p^L}$. Indeed, for $\varepsilon > 0$ choose a Banach space F and $u \in \mathcal{D}_p(F, E)$ such that $\|u\|_{\mathcal{D}_p} \leq \left(\varepsilon + \|S\|_{\mathcal{D}_p^L}\right)$ and for every $x, x' \in X$, $y, y' \in Y$, $e^* \in E^*$,

$$\begin{aligned} &|\langle T(x, y) - T(x', y) - T(x, y') + T(x', y'), e^* \rangle| \\ &= |f(x) - f(x')|\,|\langle S(y) - S(y'), e^* \rangle| \\ &\leq Lip(f)d(x, x')d(y, y')\,\|u^*(e^*)\|\,. \end{aligned}$$

Since $u^* : E^* \longrightarrow F^*$ is p^*-summing with $\|u\|_{\mathcal{D}_p} = \pi_{p^*}(u^*)$ (see [15, Theorem 2.2.2]), it follows that $T \in \mathcal{D}_p^{BL}(X, Y; E)$ and

$$\|T\|_{\mathcal{D}_p^{BL}} \leq Lip(f)\pi_{p^*}(u^*) \leq Lip(f)\left(\varepsilon + \|S\|_{\mathcal{D}_p^L}\right).$$

The following theorem justifies that the class under study is a true extension of the bilinear notion.

Theorem 4.10. *If $T \in \mathcal{L}(X, Y; E)$ is a bilinear operator between Banach spaces X, Y and E, then $T \in \mathcal{D}_p^{BL}(X, Y; E)$ if and only if $T \in \mathcal{D}_p^2(X, Y; E)$. Furthermore, $\|T\|_{\mathcal{D}_p^2} = \|T\|_{\mathcal{D}_p^{BL}}$.*

Proof. Suppose that $T \in \mathcal{D}_p^{BL}(X, Y; E)$. For each $\varepsilon > 0$, choose a Banach space G and a p^*-summing linear operator $S : E^* \longrightarrow G$ such that (12) holds and $\pi_{p^*}(S) \leq \|T\|_{\mathcal{D}_p^{BL}} + \varepsilon$. Let $(x_i)_{1\leq i\leq n} \subset X$, $(y_i)_{1\leq i\leq n} \subset Y$, and $(e_i^*)_{1\leq i\leq n} \subset E^*$. Then by (12) and Hölder's inequality we get

$$\begin{aligned} \left\|\left(\langle T(x_i, y_i), e_i^* \rangle\right)_{i=1}^n\right\|_1 &\leq \sum_{i=1}^n \|x_i\|\,\|y_i\|\,\|S(e_i^*)\| \\ &\leq \left(\sum_{i=1}^n \|x_i\|^p\,\|y_i\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|S(e_i^*)\|^{p^*}\right)^{\frac{1}{p^*}} \\ &\leq \pi_{p^*}(S)\left(\sum_{i=1}^n \|x_i\|^p\,\|y_i\|^p\right)^{\frac{1}{p}} \|(e_i^*)_{i=1}^n\|_{p^*,\omega}\,. \end{aligned}$$

It follows that T is Cohen strongly p-summing and

$$\|T\|_{\mathcal{D}_p^2} \leq \pi_{p^*}(S) \leq \|T\|_{\mathcal{D}_p^{BL}} + \varepsilon.$$

Conversely, suppose that $T \in \mathcal{D}_p^2(X, Y; E)$. By [2, Theorem 2.4] there is a regular Borel probability measure μ on $B_{E^{**}}$ such that for all $x, x' \in X$, $y, y' \in Y$ and $e^* \in E^*$, we have

$$\begin{aligned} &|\langle T(x, y) - T(x', y) - T(x, y') + T(x', y'), e^* \rangle| \\ &= |\langle T(x - x', y - y'), e^* \rangle| \\ &\leq \|T\|_{\mathcal{D}_p^2}\,\|x - x'\|\,\|y - y'\|\left(\int_{B_{E^{**}}} |\langle e^*, \phi \rangle|^{p^*}\,d\mu(\phi)\right)^{\frac{1}{p^*}}. \end{aligned}$$

Let A be the natural isometric embedding $E^* \longrightarrow C(B_{E^{**}})$ composed with the formal identity from $C(B_{E^{**}})$ into $L_\infty(\mu)$ given by $A(e^*)(\phi) = \langle e^*, \phi \rangle$, $e^* \in E^*, \phi \in B_{E^{**}}$. The canonical mapping $i_{p^*} : L_\infty(\mu) \longrightarrow$

$L_{p^*}(\mu)$ is p^*-summing with $\pi_{p^*}(i_{p^*}) = 1$ (see [16, Page 40]), it follows that $i_{p^*} \circ A$ is also p^*-summing with $\pi_{p^*}(i_{p^*} \circ A) \leq 1$. Therefore, $T \in \mathcal{D}_p^{BL}(X, Y; E)$ by taking $G = L_{p^*}(\mu)$ and $S = \|T\|_{\mathcal{D}_p^2}(i_{p^*} \circ A)$. In addition,

$$\|T\|_{\mathcal{D}_p^{BL}} \leq \pi_{p^*}(S) \leq \|T\|_{\mathcal{D}_p^2}. \quad \square$$

We show in what follows that $\mathcal{D}_p^{BL}$ is the two-Lipschitz operator ideal generated by the composition method from the linear operator ideal $\mathcal{D}_p$.

Proposition 4.11. *Let X, Y be pointed metric spaces and E be Banach space. For $1 < p \leq \infty$, we have $T \in \mathcal{D}_p^{BL}(X, Y; E)$ if and only if its bi-linearization $T_B \in \mathcal{D}_p^2(Æ(X), Æ(Y); E)$. In this case $\|T\|_{\mathcal{D}_p^{BL}} = \|T_B\|_{\mathcal{D}_p^2}$.*

Proof. Suppose that T is strongly two-Lipschitz p-summing operator. Let $m^1 \in \mathcal{M}(X)$, $m^2 \in \mathcal{M}(Y)$, with representations, $m^1 = \sum_{k=1}^n \alpha_k m_{x_k x'_k}$ and $m^2 = \sum_{j=1}^r \beta_j m_{y_j y'_j}$ and let $e^* \in E^*$. Then there exist a Banach space G and a p^*-summing linear operator $S : E^* \longrightarrow G$ such that,

$$\left|\left\langle T_B(m^1, m^2), e^* \right\rangle\right| \leq \sum_{k=1}^n |\alpha_k| \, d(x_k, x'_k) \sum_{j=1}^r |\beta_j| \, d(y_j, y'_j) \, \|S(e^*)\|.$$

Taking the infimum over all representations of m^1 and m^2 we get

$$\left|\left\langle T_B(m^1, m^2), e^* \right\rangle\right| \leq \left\|m^1\right\| \left\|m^2\right\| \|S(e^*)\|.$$

By using the last inequality and Hölder's inequality we obtain

$$\left\|\left(\left\langle T_B\left(m_i^1, m_i^2\right), e_i^*\right\rangle\right)_{i=1}^n\right\|_1 \leq \pi_{p^*}(S) \left(\sum_{i=1}^n \left\|m_i^1\right\|^p \left\|m_i^2\right\|^p\right)^{\frac{1}{p}} \|(e_i^*)_{i=1}^n\|_{p^*,\omega},$$

for any $m_1^1, ..., m_n^1 \in \mathcal{M}(X)$, $m_1^2, ..., m_n^2 \in \mathcal{M}(Y)$, and any $e_1^*, ..., e_n^* \in E^*$. Therefore, $T_B \in \mathcal{D}_p^2(Æ(X), Æ(Y); E)$ and $\|T_B\|_{\mathcal{D}_p^2} \leq \pi_{p^*}(S)$. Passing to the infimum over all G and S as above we arrive at $\|T_B\|_{\mathcal{D}_p^2} \leq \|T\|_{\mathcal{D}_p^{BL}}$.

Conversely, suppose that T_B is Cohen strongly p-summing. Let $x, x' \in X$, $y, y' \in Y$ and $e^* \in E^*$. By [2, Theorem 2.4] there is a regular Borel probability measure μ on $B_{E^{**}}$ such that for all $x, x' \in X$, $y, y' \in Y$ and $e^* \in E^*$, we have

$$\begin{aligned} & |\langle T(x,y) - T(x',y) - T(x,y') + T(x',y'), e^* \rangle| \\ &= |\langle T_B(m_{xx'}, m_{yy'}), e^* \rangle| \\ &\leq \|T_B\|_{\mathcal{D}_p^2} \|m_{xx'}\| \|m_{yy'}\| \left(\int_{B_{E^{**}}} |\langle e^*, \phi\rangle|^{p^*} \, d\mu(\phi)\right)^{\frac{1}{p^*}}. \end{aligned}$$

A similar analysis to that in the proof of the second implication of Theorem 4.10 shows that $T \in \mathcal{D}_p^{BL}(X, Y; E)$ and $\|T\|_{\mathcal{D}_p^{BL}} \leq \|T\|_{\mathcal{D}_p^2}$. $\square$

The proof of the following corollary is a consequence of [1, Theorem 3.6] and the previous proposition.

Corollary 4.12. *Let X, Y be pointed metric spaces and E be a Banach space. For $1 < p \leq \infty$, we have $T \in \mathcal{D}_p^{BL}(X, Y; E)$ if and only if its linearization $T_L \in \mathcal{D}_p(Æ(X) \widehat{\otimes}_\pi Æ(Y), E)$. In this case $\|T\|_{\mathcal{D}_p^{BL}} = \|T_L\|_{\mathcal{D}_p}$.*

As a consequence, we obtain the following corollary which is a straightforward consequence of the preceding corollary, Theorem 3.5 and Proposition 3.6.

Corollary 4.13. *The class $\mathcal{D}_p^{BL}$ is the Banach two-Lipschitz operator ideal generated by the composition method from the operator ideal $\mathcal{D}_p$, i.e.,*

$$\mathcal{D}_p^{BL}(X, Y; E) = \mathcal{D}_p \circ BLip_0(X, Y; E),$$

for all pointed metric spaces X, Y and Banach space E.

4.3. Ideal of two-Lipschitz $(p; p_1, p_2)$-summing operators

Farmer and Johnson introduced the Banach Lipschitz operator ideal of Lipschitz p-summing mappings [18], extending Π_p, the operator ideal of p-summing linear operators, to the Lipschitz case. A mapping $T \in Lip_0(X, E)$ is called Lipschitz p-summing, $1 \leq p < \infty$ if there exists a constant $C > 0$ such that regardless of the choice of points $x_1, \ldots, x_n, x_1', \ldots, x_n'$ in X,

$$\left\|(T(x_i) - T(x_i'))_{i=1}^n\right\|_p \leq C \sup_{f \in B_{X^\#}} \left\|(f(x_i) - f(x_i'))_{i=1}^n\right\|_p .$$

The definition of absolutely p-summing m-linear functionals is due to Pietsch [28]. In [23], Matos presented a definition for vector-valued mappings.

Definition 4.14. Let E, F, G be Banach spaces and let $1 \leq p, p_1, p_2 < \infty$, with $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$. A bilinear operator $T \in \mathcal{L}(E, F; G)$ is said to be absolutely $(p; p_1, p_2)$-summing if there is a constant $C > 0$ such that for any $x_1, ..., x_n \in E$ and $y_1, ..., y_n \in F$ we have

$$\|(T(x_i, y_i))_{i=1}^n\|_p \leq C \sup_{f \in B_{E^*}} \|(f(x_i))_{i=1}^n\|_{p_1} \sup_{g \in B_{F^*}} \|(g(y_i))_{i=1}^n\|_{p_2} .$$

The Banach space of these mappings is denoted by $\mathcal{L}_{as,(p;p_1,p_2)}(E, F; G)$ with the norm $\|T\|_{\mathcal{L}_{as,(p;p_1,p_2)}}$, which *is the smallest* C satisfying the above inequality.

We extend the definition of the class of absolutely $(p; p_1, p_2)$-summing bilinear operators to the case of two-Lipschitz operators, for which the resulting vector space of two-Lipschitz $(p; p_1, p_2)$-summing operators is a Banach two-Lipschitz operator ideal.

Definition 4.15. Let $1 \leq p, p_1, p_2 < \infty$ with $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$. A mapping $T \in BLip_0(X, Y; E)$ is called two-Lipschitz $(p; p_1, p_2)$-summing if there exists a constant $C > 0$ such that for any $x_1, \ldots, x_n, x_1', \ldots, x_n'$ in X and $y_1, \ldots, y_n, y_1', \ldots, y_n'$ in Y we have

$$\begin{aligned} &\left\|(T(x_i, y_i) - T(x_i', y_i) - T(x_i, y_i') + T(x_i', y_i'))_{i=1}^n\right\|_p \\ &\leq C \sup_{f \in B_{X^\#}} \left\|(f(x_i) - f(x_i'))_{i=1}^n\right\|_{p_1} \sup_{g \in B_{Y^\#}} \left\|(g(y_i) - g(y_i'))_{i=1}^n\right\|_{p_2} . \end{aligned} \tag{13}$$

We denote this class of two-Lipschitz operators by $BL_{as,(p;p_1,p_2)}(X, Y; E)$. In this case, we define

$$\|T\|_{BL_{as,(p;p_1,p_2)}} = \inf\{C : \text{satisfying } (13)\} .$$

We don't know if two-Lipschitz $(p;p_1,p_2)$-summability implies $(p;p_1,p_2)$-summability whenever the mapping T is bilinear. The converse is of course clearly true. If X, Y and E are Banach spaces and $T : X \times Y \longrightarrow E$ is bilinear $(p;p_1,p_2)$-summing, it follows from the inclusions $B_{X^*} \subset B_{X^\#}$ and $B_{Y^*} \subset B_{Y^\#}$ that T is two-Lipschitz $(p;p_1,p_2)$-summing and $\|T\|_{BL_{as,(p;p_1,p_2)}} \leq \|T\|_{\mathcal{L}_{as,(p;p_1,p_2)}}$.

Proposition 4.16. *The class* $\left(BL_{as,(p;p_1,p_2)}, \|\cdot\|_{BL_{as,(p;p_1,p_2)}}\right)$ *is a Banach two-Lipschitz operator ideal.*

Proof. The properties (ii), (ii') and (iii') of Definition 3.1 may be easily verified. So we only show that (i') holds. Let X and Y be pointed metric spaces and E be Banach space. It is easily seen that $\alpha T \in BL_{as,(p;p_1,p_2)}(X,Y;E) \subset Lip_0(X,Y;E)$, $\|\alpha T\|_{BL_{as,(p;p_1,p_2)}} = |\alpha| \|T\|_{BL_{as,(p;p_1,p_2)}}$ and $\|T\| \leq \|T\|_{BL_{as,(p;p_1,p_2)}}$ for every $T \in BL_{as,(p;p_1,p_2)}(X,Y;E)$ and $\alpha \in \mathbb{K}$.

Let $S,T \in BL_{as,(p;p_1,p_2)}(X,Y;E)$, and $(x_i)_{i=1}^n, (x_i')_{i=1}^n \subset X$ and $(y_i)_{i=1}^n, (y_i')_{i=1}^n \subset Y$. Then

$$\begin{aligned}
&\left\|((S+T)(x_i,y_i) - (S+T)(x_i',y_i) - (S+T)(x_i,y_i') + (S+T)(x_i',y_i'))_{i=1}^n\right\|_p \\
&\leq \left\|(S(x_i,y_i) - S(x_i',y_i) - S(x_i,y_i') + S(x_i',y_i'))_{i=1}^n\right\|_p \\
&\quad + \left\|(T(x_i,y_i) - T(x_i',y_i) - T(x_i,y_i') + T(x_i',y_i'))_{i=1}^n\right\|_p \\
&\leq \left(\|S\|_{BL_{as,(p;p_1,p_2)}} + \|T\|_{BL_{as,(p;p_1,p_2)}}\right) \sup_{f\in B_{X^\#}} \left\|(f(x_i) - f(x_i'))_{i=1}^n\right\|_{p_1} \\
&\quad \times \sup_{g\in B_{Y^\#}} \left\|(g(y_i) - g(y_i'))_{i=1}^n\right\|_{p_2},
\end{aligned}$$

which means that $S+T$ is in $BL_{as,(p;p_1,p_2)}(X,Y;E)$ and

$$\|S+T\|_{BL_{as,(p;p_1,p_2)}} \leq \|S\|_{BL_{as,(p;p_1,p_2)}} + \|T\|_{BL_{as,(p;p_1,p_2)}}.$$

Thus, we have shown that $(BL_{as,(p;p_1,p_2)}(X,Y;E), \|\cdot\|_{BL_{as,(p;p_1,p_2)}})$ is a normed linear subspace of $BLip_0(X,Y;E)$.

To prove the completeness of the space $BL_{as,(p;p_1,p_2)}(X,Y;E)$, take a Cauchy sequence $(T_n)_n \subset BL_{as,(p;p_1,p_2)}(X,Y;E)$. Hence for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$BLip(T_n - T_k) \leq \|T_n - T_k\|_{BL_{as,(p;p_1,p_2)}} < \varepsilon, \quad \text{for all } n,k \geq n_\varepsilon,$$

which means that $(T_n)_n$ is a Cauchy sequence in the Banach space $BLip_0(X,Y;E)$. Thus, it exists $T \in BLip_0(X,Y;E)$ such that $BLip(T_n - T) \longrightarrow 0$. Now, let $x_1,\dots,x_n,x_1',\dots,x_n'$ in X and $y_1,\dots,y_n,y_1',\dots,y_n'$ in Y. Since $T_n - T_k$ is two-Lipschitz $(p;p_1,p_2)$-summing, it follows that for every $n,k \geq n_\varepsilon$, we have

$$\begin{aligned}
&\left\|((T_n - T_k)(x_i,y_i) - (T_n - T_k)(x_i',y_i) - (T_n - T_k)(x_i,y_i') + (T_n - T_k)(x_i',y_i'))_{i=1}^n\right\|_p \\
&\leq \varepsilon \sup_{f\in B_{X^\#}} \left\|(f(x_i) - f(x_i'))_{i=1}^n\right\|_{p_1} \sup_{g\in B_{Y^\#}} \left\|(g(y_i) - g(y_i'))_{i=1}^n\right\|_{p_2}.
\end{aligned}$$

Since $T_n(x,y) \longrightarrow T(x,y)$ for all $x \in X, y \in Y$ and after passing to the limit for $k \to +\infty$, we obtain that for every $n \geq n_\varepsilon$,

$$\begin{aligned}
&\left\|((T_n - T)(x_i,y_i) - (T_n - T)(x_i',y_i) - (T_n - T)(x_i,y_i') + (T_n - T)(x_i',y_i'))_{i=1}^n\right\|_p \\
&< \varepsilon \sup_{f\in B_{X^\#}} \left\|(f(x_i) - f(x_i'))_{i=1}^n\right\|_{p_1} \sup_{g\in B_{Y^\#}} \left\|(g(y_i) - g(y_i'))_{i=1}^n\right\|_{p_2}
\end{aligned}$$

which means that $(T_n - T) \in BL_{as,(p;p_1,p_2)}(X,Y;E)$ and hence, $T \in BL_{as,(p;p_1,p_2)}(X,Y;E)$. In addition, $\|T_n - T\|_{BL_{as,(p;p_1,p_2)}} < \varepsilon$ for all $n \geq n_\varepsilon$, i.e., the sequence $(T_n)_n$ is convergent to $T \in BL_{as,(p;p_1,p_2)}(X,Y;E)$ with respect to the norm, $\|\cdot\|_{BL_{as,(p;p_1,p_2)}}$. $\square$

Let us show with an example that $BL_{as,(p;p_1,p_2)}$ is not of composition type, that is $BL_{as,(p;p_1,p_2)} \neq \Pi_p \circ BLip_0$

Example 4.17. Let $1 \leq p, p_1, p_2 < \infty$ with $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$. Consider the two-Lipschitz mapping

$$S : \mathbb{R} \times \mathbb{R} \longrightarrow Æ(\mathbb{R}) \widehat{\otimes}_\pi Æ(\mathbb{R}), \quad S(x,y) = m_{x0} \otimes m_{y0}.$$

Then S is two-Lipschitz $(p;p_1,p_2)$-summing. In order to see this, let $x_i, x_i', y_i, y_i' \in \mathbb{R}$ $(i = 1, ..., n)$. Then, using Hölder's inequality and taking into account that the mapping $\delta_\mathbb{R} : \mathbb{R} \longrightarrow Æ(\mathbb{R})$ is Lipschitz p-summing for all $p \geq 1$, we obtain

$$\begin{aligned}
&\left\|(S(x_i,y_i) - S(x_i',y_i) - S(x_i,y_i') + S(x_i',y_i'))_{i=1}^n\right\|_p \\
&= \left(\sum_{i=1}^n \pi\left(m_{x_ix_i'} \otimes m_{y_iy_i'}\right)^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^n \left\|m_{x_ix_i'}\right\|^p \left\|m_{y_iy_i'}\right\|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^n \left\|m_{x_ix_i'}\right\|^{p_1}\right)^{\frac{1}{p_1}} \left(\sum_{i=1}^n \left\|m_{y_iy_i'}\right\|^{p_2}\right)^{\frac{1}{p_2}} \\
&= \left(\sum_{i=1}^n \|\delta_\mathbb{R}(x_i) - \delta_\mathbb{R}(x_i')\|^{p_1}\right)^{\frac{1}{p_1}} \left(\sum_{i=1}^n \|\delta_\mathbb{R}(y_i) - \delta_\mathbb{R}(y_i')\|^{p_2}\right)^{\frac{1}{p_2}} \\
&\leq \sup_{f \in B_{X^\#}} \left\|(f(x_i) - f(x_i'))_{i=1}^n\right\|_{p_1} \sup_{g \in B_{Y^\#}} \left\|(g(y_i) - g(y_i'))_{i=1}^n\right\|_{p_2}.
\end{aligned}$$

On the other hand, a trivial verification shows that $S = \sigma_2 \circ (\delta_\mathbb{R}, \delta_\mathbb{R})$. The uniqueness of the linearization maps gives that S_L is the identity map on the infinite dimensional space $Æ(\mathbb{R}) \otimes Æ(\mathbb{R})$ and so, $S_L \notin \Pi_p$ (see [16, Page 50]). Finally, the Theorem 3.5 asserts that $S \notin \Pi_p \circ BLip_0$

4.4. Ideal of two-Lipschitz factorable p-dominated operators

The p-semi-integral multilinear mappings were introduced in [26] motivated by the work of Alencar and Matos [8]. Let E, F, G be Banach spaces and $1 \leq p < \infty$. A bilinear mapping $T \in \mathcal{L}(E,F;G)$ is p-semi-integral, in symbols $T \in \mathcal{L}_{si,p}(E,F;G)$, if there is a constant $C > 0$ such that

$$\left(\sum_{i=1}^n \|T(x_i,y_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\substack{\phi_1 \in B_{E^*} \\ \phi_2 \in B_{F^*}}} \left(\sum_{i=1}^n |\phi_1(x_i)\phi_2(y_i)|^p\right)^{\frac{1}{p}},$$

for any $(x_i)_{i=1}^n \subset E$ and $(y_i)_{i=1}^n \subset F$. In this case, take $\|T\|_{si,p}$ the infimum of all constants C working in the above inequality. We can found some details about this concept in [26], [14] and [13].

In the following definition we generalize the concept of factorable p-dominated bilinear operators (see [25, Definition 2.7]), to the two-Lipschitz case obtaining in this way a Banach ideal of two-Lipschitz operators.

Definition 4.18. Let $1 \leq p < \infty$. A mapping $T \in BLip_0(X,Y;E)$ is called two-Lipschitz factorable p-dominated if there exists a constant $C > 0$ such that for any $x_i^j, x_i'^j \in X$, $y_i^j, y_i'^j \in Y$, $\lambda_i^j \in \mathbb{K}, (1 \leq i \leq n, 1 \leq j \leq s)$ and all positive integers n, s we have

$$\left(\sum_{j=1}^{s}\left\|\sum_{i=1}^{n}\lambda_i^j\left(T(x_i^j,y_i^j)-T(x_i^j,y_i'^j)-T(x_i'^j,y_i^j)+T(x_i'^j,y_i'^j)\right)\right\|^p\right)^{\frac{1}{p}}$$
$$\leq C\sup_{\substack{f\in B_{X^\#}\\ g\in B_{Y^\#}}}\left(\sum_{j=1}^{s}\left|\sum_{i=1}^{n}\lambda_i^j\left(f(x_i^j)-f(x_i'^j)\right)\left(g(y_i^j)-g(y_i'^j)\right)\right|^p\right)^{\frac{1}{p}}.$$

The class of all two-Lipschitz factorable p-dominated operators is denoted by $BL_{f,p}(X,Y;E)$. In this case, we define $\|T\|_{BL_{f,p}}$ as the infimum of all constants C fulfilling the above inequality.

Remark 4.19. Note that if T is two-Lipschitz factorable p-dominated then taking $n=1$, by Hölder's inequality we have

$$\left(\sum_{j=1}^{s}\left\|\left(T(x^j,y^j)-T(x^j,y'^j)-T(x'^j,y^j)+T(x'^j,y'^j)\right)\right\|^p\right)^{\frac{1}{p}}$$
$$\leq\|T\|_{BL_{f,p}}\sup_{f\in B_{X^\#}}\left\|\left(f(x^j)-f(x'^j)\right)_{j=1}^{s}\right\|_{p_1}\sup_{g\in B_{Y^\#}}\left\|\left(g\left(y^j\right)-g(y'^j)\right)_{j=1}^{s}\right\|_{p_2},$$

for any $x^j,x'^j\in X$, $y^j,y'^j\in Y$ $(1\leq j\leq s)$, i.e., T is two-Lipschitz $(p;p_1,p_2)$-summing with $\frac{1}{p}\leq\frac{1}{p_1}+\frac{1}{p_2}$.

Now, we study the connection between a mapping belonging to $BL_{f,p}(X,Y;E)$ and its bi-linearization.

Theorem 4.20. *Let X,Y be pointed metric spaces and E be a Banach space. For $1\leq p<\infty$, we have $T\in BL_{f,p}(X,Y;E)$ if and only if its bi-linearization T_B is p-semi-integral from Æ$(X)\times$Æ(Y) to E. In this case*

$$\|T\|_{BL_{f,p}}=\|T_B\|_{si,p} \tag{14}$$

Proof. Suppose that $T\in BL_{f,p}(X,Y;E)$. Let $(m_j^1)_{j=1}^s\subset$ Æ(X) and $(m_j^2)_{j=1}^s\subset$Æ(Y), with $m_j^1=\sum_{i=1}^n\alpha_i^j m_{x_i^jx_i'^j}$ and $m_j^2=\sum_{k=1}^r\beta_k^j m_{y_k^jy_k'^j}$ then, we have

$$\begin{aligned}
&\left(\sum_{j=1}^{s}\left\|T_B(m_j^1,m_j^2)\right\|^p\right)^{\frac{1}{p}}\\
&=\left(\sum_{j=1}^{s}\left\|T_B\left(\sum_{i=1}^{n}\alpha_i^j m_{x_i^jx_i'^j},\sum_{k=1}^{r}\beta_k^j m_{y_k^jy_k'^j}\right)\right\|^p\right)^{\frac{1}{p}}\\
&=\left(\sum_{j=1}^{s}\left\|\sum_{i=1}^{n}\alpha_i^j\sum_{k=1}^{r}\beta_k^j T_B\left(m_{x_i^jx_i'^j},m_{y_k^jy_k'^j}\right)\right\|^p\right)^{\frac{1}{p}}\\
&=\left(\sum_{j=1}^{s}\left\|\sum_{i=1}^{n}\alpha_i^j\sum_{k=1}^{r}\beta_k^j\left(T(x_i^j,y_k^j)-T(x_i^j,y_k'^j)-T(x_i'^j,y_k^j)+T(x_i'^j,y_k'^j)\right)\right\|^p\right)^{\frac{1}{p}}\\
&\leq\|T\|_{BL_{f,p}}\sup_{\substack{f\in B_{X^\#}\\ g\in B_{Y^\#}}}\left(\sum_{j=1}^{s}\left|\sum_{i=1}^{n}\alpha_i^j\sum_{k=1}^{r}\beta_i^j\left(f(x_i^j)-f(x_i'^j)\right)\left(g(y_k^j)-g(y_k'^j)\right)\right|^p\right)^{\frac{1}{p}}=(*).
\end{aligned}$$

Since Æ$(Z)^*$ and $Z^\#$ $(Z=X$ or $Y)$ are isometrically isomorphic via the linearization, for all $h\in Z^\#$ there is $\varphi\in$ Æ$(Z)^*$ such that

$$\varphi(m_{zz'})=h_L(m_{zz'})=h(z)-h(z'),$$

for all $z, z' \in Z$, we obtain

$$
\begin{aligned}
(*) &= \|T\|_{BL_{f,p}} \sup_{\substack{\|\varphi_1\|\leq 1 \\ \|\varphi_2\|\leq 1}} \left(\sum_{j=1}^{s} \left| \sum_{i=1}^{n} \alpha_i^j \sum_{k=1}^{r} \beta_i^j \varphi_1(m_{x_i^j x_i'^j}) \varphi_2(m_{y_k^j y_k'^j}) \right|^p \right)^{\frac{1}{p}} \\
&= \|T\|_{BL_{f,p}} \sup_{\substack{\|\varphi_1\|\leq 1 \\ \|\varphi_2\|\leq 1}} \left(\sum_{j=1}^{s} \left| \varphi_1(m_j^1) \varphi_2(m_j^2) \right|^p \right)^{\frac{1}{p}}
\end{aligned}
$$

Therefore, $T_B \in \mathcal{L}_{si,p}(Æ(X), Æ(Y); E)$ and $\|T_B\|_{si,p} \leq \|T\|_{BL_{f,p}}$.

Conversely, suppose that $T_B \in \mathcal{L}_{si,p}(Æ(X),Æ(Y); E)$. Let $x_i^j, x_i'^j \in X$, $y_i^j, y_i'^j \in Y$, $\lambda_i^j \in \mathbb{K}$, $(1 \leq i \leq n, 1 \leq j \leq s)$, we have

$$
\begin{aligned}
&\left(\sum_{j=1}^{s} \left\| \sum_{i=1}^{n} \lambda_j^i \left(T(x_i^j, y_i^j) - T(x_i^j, y_i'^j) - T(x_i'^j, y_i^j) + T(x_i'^j, y_i'^j) \right) \right\|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{j=1}^{s} \left\| T_B \left(\sum_{i=1}^{n} \lambda_j^i m_{x_i^j x_i'^j}, m_{y_i^j y_i'^j} \right) \right\|^p \right)^{\frac{1}{p}} \\
&\leq \|T_B\|_{si,p} \sup_{\substack{\|\varphi_1\|\leq 1 \\ \|\varphi_2\|\leq 1}} \left(\sum_{j=1}^{s} \left| \sum_{i=1}^{n} \lambda_j^i \varphi_1(m_{x_i^j x_i'^j}) \varphi_1(m_{y_i^j y_i'^j}) \right|^p \right)^{\frac{1}{p}} \\
&= \|T_B\|_{si,p} \sup_{\substack{f \in B_{X^\#} \\ g \in B_{Y^\#}}} \left(\sum_{j=1}^{s} \left| \sum_{i=1}^{n} \lambda_i^j \left(f(x_i^j) - f(x_i'^j) \right) \left(g\left(y_i^i\right) - g(y_i'^j) \right) \right|^p \right)^{\frac{1}{p}}.
\end{aligned}
$$

Which means that $T \in BL_{f,p}(X, Y; E)$ and $\|T\|_{BL_{f,p}} \leq \|T_B\|_{si,p}$. □

Remark 4.21. If we consider $T \in Lip_0(X, E)$, we obtain a characterization of strictly Lipschitz p-summing operators that was introduced by Saadi in [32] i.e.,

$$
\begin{aligned}
&T \text{ is strictly } p\text{-summing} \\
&\Longleftrightarrow T_L \text{ is linear } p\text{-summing} \\
&\Longleftrightarrow T \text{ is factorable } p\text{-dominated,}
\end{aligned}
$$

where the first equivalence follows from [32, Theorem 3.5] and the second one from Theorem 4.20. So, our notion is really a generalization of strictly Lipschitz p-summing operators to the two-Lipschitz case.

Pellegrino in [26] proved that the class $(\mathcal{L}_{si,p}, \|\cdot\|_{si,p})$ is a Banach ideal of bilinear mappings (see also [14]). As a straightforward consequence of this result and the preceding proposition we have the following corollary.

Corollary 4.22. *The class $(BL_{f,p}, \|\cdot\|_{BL_{f,p}})$ is a Banach ideal of two-Lipschitz operators.*

Proof. The proof is based on the equality (14) and the following statements that we get directly from the uniqueness of the bi-linearization maps,
1) $(\alpha S + T)_B = \alpha S_B + T_B$, for all $S, T \in BL_{f,p}$ and $\alpha \in \mathbb{R}$.
2) $(f \cdot g \cdot e)_B = f_L \cdot g_L \cdot e$, for any $f \in X^\#$, $g \in Y^\#$ and $e \in E$, where $f_L \cdot g_L \cdot e \in \mathcal{L}(Æ(X),Æ(Y); E)$ is defined by $f_L \cdot g_L \cdot e(m, m') = f_L(m) g_L(m')e$.
3) $(u \circ T \circ (f, g))_B = u \circ T_B \circ (\widehat{f}, \widehat{g})$ for all $f \in Lip_0(Z, X)$, $g \in Lip_0(W, Y)$, $T \in BL_{f,p}(X, Y; E)$. □

Acknowledgments

We would like to thank the referee for his/her careful reading and useful suggestions. Also, we acknowledge with thanks the support of the General Direction of Scientific Research and Technological Development (DGRSDT), Algeria.

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