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Theme

Hyper nuclear multilinear operators

Presented by :

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In front of the jury composed of :

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Dahmane Achour	Prof,	University of Msila	**Supervisor.**
Douadi Drihem	Prof,	University of Msila	**Examiner.**

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Dedication

To my extraordinary family, on this significant day of my graduation, I am filled with heartfelt gratitude for your unwavering support, limitless love, and endless encouragement throughout my entire university journey. Today is not only a celebration of my personal achievements but also a tribute to the incredible strength and unity that defines our family. I dedicate this milestone to each and every one of you

Nekbil Noureddine

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Nekbil Noureddine

Notations	
$\mathcal{L}(X,Y)$	The space of all bounded linear operators from X to$Y$.
$\mathbb{R}_+$	The field of non negative real numbers
E^*	The topological dual of E
p^*	The conjugate of the number p $(1 \leq p \leq \infty)$, that is $\frac{1}{p}+\frac{1}{p^*}=1$
$\mathbb{K}$	The field of real or complex numbers.
$\mathcal{L}_f(X,Y)$	The space of fall finite-rank operators from X to Y.
ℓ_∞	The Banach space of bounded scalar sequences.
$\mathcal{L}(X;Y)$	The set of all continuous linear operators.
$L(X_1,\ldots,X_n;Y)$	The set of all multilinear operators.
$\mathcal{L}(X_1,\ldots,X_n;Y)$	The set of all continuous multilinear operators.
$\mathcal{L}_f(X_1,\ldots,X_n;Y)$	The set of all finite type multilinear operators.
$\mathcal{L}_{\mathcal{F}}(X_1,\ldots,X_n;Y)$	The class of finite rank multilinear operators
$\mathcal{L}_{\mathcal{N}}(X_1,\ldots,X_n;Y)$	The class of nuclear multilinear operators
$\mathcal{L}_{\mathcal{HN}_p^{right}}(X_1,\ldots,X_n;Y)$	The class of hyper right p-nuclear multilinear operators
$\mathcal{L}_{\mathcal{HN}}(X_1,\ldots,X_n;Y)$	The class of hyper-nuclear multilinear operators

Introduction

The theory of Banach (or s-Banach) ideals of bounded multilinear operators (see [15, 19]) has attracted the attention of many researchers, as evidenced by numerous articles published in this field. For example, nuclear multilinear operators ([16, 14]) an extension of p-nuclear linear operators introduced by Pietsch and Persson in [18] have been studied, along with other recent concepts such as the ideal of strongly p-summing multilinear operators [13], Cohen strongly p-summing multilinear operators [6], p-dominated multilinear operators [15], and others.

Researchers have also explored questions regarding the composition of multilinear operators and whether certain properties remain stable under composition, as discussed in Popa s work [22]. Popa provided examples demonstrating the validity of his questions. In an independent study, as part of Torres doctoral project [24] (published in collaboration with Botelho [8]), the researchers introduced a new definition of multilinear ideal operators, incorporating the composition property raised in Popa s question as one of the defining conditions. They called these operators "Hyper-Banach (or s-Banach) ideals of bounded multilinear operators," highlighting their key characteristics and supporting examples, while also clarifying the differences from classical Banach (or s-Banach) ideals of bounded multilinear operators. This work was followed by additional contributions from the same researchers [8, 9, 21], as well as by Achour [2, 3, 4, 5].

In this master s memory, we examine Botelho and Torres article entitled **"Hyper-ideal of multilinear operators"**, published in Journal **"Linear Algebra and its Applications"** to explore this concept and, as an example, study Hyper p-nuclear multilinear operators, with an attempt to provide a simple extension to this framework.

The master s memory is divided into three chapters: The first chapter covers fundamental concepts and results necessary for the thesis. The second chapter introduces the concept of Hyper-ideal multilinear operators, the motivation behind their definition, and some of their key properties. The third chapter focuses on the example of p-nuclear operators as a Hyper-ideal, examining their special properties such as satisfying the linearization theorem and proving that they admit a decomposition involving lp-spaces via a linear operator and a multilinear operator.

Chapter 1

Preliminaries

This chapter presents the fundamental concepts necessary for understanding and studying linear and multilinear operators in Banach spaces. We begin by reviewing sequence spaces of the form $\ell_p(X)$ and $\ell_{p,w}(X)$,, which are commonly used to characterize various types of operators and serve as foundational tools in this context.

The discussion then proceeds to the definition of operator ideals within the framework established by Pietsch, with a focus on two key properties: stability under composition and the inclusion of all finite-rank operators.

We then introduce the notion of multilinear operator ideals as a natural extension of linear operator ideals, outlining the structural conditions required to ensure their stability and coherence [1, 19].

The chapter concludes with the concept of nuclear multilinear operators introduced in [16]. Serving as a prelude to the study of *hyper-nuclear* operators within the broader framework of generalized operator ideals.

1.1 Absolutely summable sequence

For X a Banach space over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, B_X denotes the closed unit ball of X and X^* denotes the dual of X. The norm of a functional $x^* \in X^*$ is given by

$$\|x^*\| = \sup\{|\langle x, x^* \rangle| : x \in B_X\}.$$

For X, Y Banach spaces, we denote by $\mathcal{L}(X,Y)$ the Banach space of all continuous linear operators between X and Y with the norm

$$\|T\| = \sup_{x \in B_X} \|T(x)\|.$$

We write $\mathcal{L}(X)$ instead of $\mathcal{L}(X,X)$. If $T \in \mathcal{L}(X,Y)$, the continuous linear operator $T^* : Y^* \longrightarrow X^*$ defined as

$$T^*(y^*)(x) = y^*(T(x)),$$

for every $y^* \in Y^*$ and $x \in X$ is called the *adjoint operator of* T with $\|T\| = \|T^*\|$.

Let $1 \leq p \leq \infty$. The classical Banach sequence spaces ℓ_p, ℓ_∞ and c_0 are defined by

· $\ell_p = \left\{ (x_n)_n \subset \mathbb{K} : \| (x_n)_n \|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} < \infty \right\}$,

· $\ell_\infty = \{ (x_n)_n \subset \mathbb{K} : \| (x_n)_n \|_\infty = \sup_{n \in \mathbb{N}} |x_n| < \infty \}$, $p = \infty$.

· $c_0 = \{ (x_n)_n \subset \mathbb{K} : \lim_{n \to \infty} |x_n| = 0 \}$.

For $p > 1$, we note p^* the conjugate of p defined by the formula $\frac{1}{p} + \frac{1}{p^*} = 1$. We pose $p^* = \infty$ si $p = 1$.

The following fact is well known, which is discussed in [1, 12].

Definition 1.1.1 *1) A sequence $(x_n)_n$, of elements of X is said to be absolutely p-summable if*

$$\|(x_n)_n\|_p = \begin{cases} (\sum_n \|x_n\|^p)^{\frac{1}{p}} < \infty & , \text{ if } 1 \leq p < \infty \\ \sup_n \|x_n\| < \infty & , \text{ if } p = \infty \end{cases}.$$

When $p = 1$, it is said that $(x_n)_n$ is absolutely summable. We denote by $\ell_p(X)$ the vector space of all absolutely p-summable sequences of elements of X. $(\ell_p(X), \|\cdot\|_p)$ is a Banach space.

2) The spaces $c_0(X)$ of norm null sequences in X is Branch spaces with the norm given by

$$\|(x_n)_n\|_\infty = \sup_n \|x_n\|.$$

3) A sequence $(x_n)_n$ in X is said to be weakly p-summable if

$$\sum_{n=1}^{\infty} |x^*(x_n)|^p < \infty, \text{ for every } x^* \in X^*$$

We denoted by $\ell_{p,w}(X)$ the Banach spaces of weakly p-summable sequence in X when equipped with the norm given by

$$\|(x_n)_n\|_{p,w} = \sup\left\{\left(\sum_{n=1}^{\infty} |x^*(x_n)|^p\right)^{\frac{1}{p}} : x^* \in B_{X^*}\right\}.$$

Remark 1.1.1 *In the case $p = \infty$, then the spaces $\ell_{\infty,w}(X)$ of weakly bounded sequences coincide with the spaces $\ell_\infty(X)$,*

$$\|(x_n)_n\|_{\infty,w} = \|(x_n)_n\|_\infty .$$

1.2 Operator ideals

Recall that a linear operator $u \in \mathcal{L}(X, Y)$ is said to have finite rank if $u(X)$ is a finite dimensional subspace of Y. The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(X, Y)$. One can readily see that an operator $u \in \mathcal{L}(X, Y)$ has finite rank if, and only if, there exist $(x_i^*)_{i=1}^n \subset X^*$ and $(y_i)_{i=1}^n \subset Y$ such that

$$u(x) = \sum_{i=1}^{n} x_i^*(x) y_i,$$

for every $x \in X$.

Let us recall the definition of a Banach operator ideal, from [20] (see also [12]).

Definition 1.2.1 *An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces X and Y its components $\mathcal{I}(X, Y) := \mathcal{L}(X, Y) \cap \mathcal{I}$ satisfy*

(i) $\mathcal{I}(X, Y)$ is a linear subspace of $\mathcal{L}(X, Y)$ which contains the finite rank operators.

(ii) The ideal property: if $v \in \mathcal{L}(G, X)$, $u \in \mathcal{I}(X, Y)$ and $w \in \mathcal{L}(Y, H)$, then the composition $w \circ u \circ v$ is in $\mathcal{I}(G, H)$.

If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies

(i') $(\mathcal{I}(X, Y), \|.\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces X and Y .

(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1, id_{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$, $id_{\mathbb{K}}(\lambda) = \lambda$.

(iii') $\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \|v\|_{\mathcal{I}} \|u\|$.

Then $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal.

The operator ideal $\mathcal{I}$ is said to be *closed* if each $\mathcal{I}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$ for the sup norm.

Theorem 1.2.1 *(Series criterion) Let $\mathcal{I}$ be a subclass of $\mathcal{L}(X,Y)$ endowed with a non-negative function $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^{+}$. For Banach spaces X,Y, define*

$$\mathcal{I}(E,F) : = \mathcal{I} \cap \mathcal{L}(X,Y).$$

Then $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a Banach ideal of linear operators if and only if the following conditions hold :

(i) The linear operator $id_{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$, $id_{\mathbb{K}}(\lambda) = \lambda$, belongs to $\mathcal{I}$ and $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$.

(ii) If $S_1, S_2, ... \in \mathcal{I}(E,F)$ and $\sum_{k=1}^{\infty} \|S_k\|_{\mathcal{I}} < \infty$, then $S = \sum_{k=1}^{\infty} \|S_k\| \in \mathcal{I}(X,Y)$ and $\|S\|_{\mathcal{I}} \le \sum_{k=1}^{\infty} \|S_k\|$.

(iii) If $T \in \mathcal{L}(G,X)$, $S \in \mathcal{I}(X,Y)$ and $\in R \in \mathcal{L}(Y,H)$, then $R \circ S \circ u \in \mathcal{I}(G,H)$ and $\|R \circ S \circ T\|_{\mathcal{I}} \le \|R\| \|S\|_{\mathcal{I}} \|u\|$.

For every Banach ideal $\mathcal{I}$

$$\|x^*(\cdot)y\|_{\mathcal{I}} = \|x^*\| \|y\|.$$

where $x \in X^*, y \in Y$ and

$$x^*(\cdot)y : X \to Y, \quad x \mapsto x^*(x)y$$

Corollary 1.2.1 *If $S \in \mathcal{I}(X,Y)$, $\|S\| \le \|S\|_{\mathcal{I}}$.*

Example 1.2.1 *We now give a list of examples*

1) $\mathcal{L}$: Ideal of continuous operators;

2) $\mathcal{L}_f$: Ideal of finite rank operators;

3) $\overline{\mathcal{I}}$: The closure (with the usual operator norm) of an operator ideal $\mathcal{I}$;

4) ***Approximable operators.*** *An operator $u \in \mathcal{L}(X,Y)$ is called approximable operators if there are $u_n \in \mathcal{L}_f(X,Y)$, with*

$$\lim_n \|u - u_n\| = 0.$$

We denote by $\overline{\mathcal{L}_f(X,Y)}$ the ideal space of all approximable operators from X to Y.

1.3 Ideal of multilinear operators

Let $E_1, \ldots, E_n, F$ be normed spaces.

Definition 1.3.1 *Let $n \in \mathbb{N}$. A map $A : E_1 \times \cdots \times E_n \to F$ is called a multilinear map (or n-linear operator) if*

$$A(x_1, \ldots, \lambda x_j + \mu y_j, \ldots, x_n) = \lambda A(x_1, \ldots, x_j, \ldots, x_n) + \mu A(x_1, \ldots, y_j, \ldots, x_n)$$

for all $1 \leq j \leq n$, $x_j, y_j \in E_j$, and scalars $\lambda, \mu \in \mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

If $Y = \mathbb{K}$, then A is called a multilinear form.

We denote by $L(E_1, \ldots, E_n; F)$ the set of all multilinear operators from $X_1 \times \cdots \times X_n$ into Y.

Let us define the following linear operations:

· $(A_1 + A_2)(x_1, \ldots, x_n) = A_1(x_1, \ldots, x_n) + A_2(x_1, \ldots, x_n)$

· $(\lambda A)(x_1, \ldots, x_n) = \lambda A(x_1, \ldots, x_n)$

This gives the space $L(X_1, \ldots, X_n; Y)$ a vector space structure.

Proposition 1.3.1 *Let $A \in L(E_1, \ldots, E_n; F)$. The following statements are equivalent*

1. *The operator A is continuous*

2. *The operator A is continuous at the point $(0, \ldots, 0)$.*

3. *There exists a constant $C > 0$ such that*

$$\|A(x_1, \ldots, x_n)\| \leq C\|x_1\| \cdots \|x_n\| \quad \text{for all } (x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n.$$

In this case, we say that A is bounded, and we define:

$$\begin{aligned} \|A\| &= \sup_{\|x_j\| \leq 1,\ 1 \leq j \leq n} \|A(x_1, \ldots, x_n)\| \\ &= \inf\{C > 0 : \|A(x_1, \ldots, x_n)\| \leq C\|x_1\| \cdots \|x_n\|, \forall x_j \in E_j, 1 \leq j \leq n\} \end{aligned}$$

We denote by $\mathcal{L}(E_1, \ldots, E_n; F)$ the space of all continuous multilinear operators from $X_1 \times \cdots \times X_n$ to Y.

Corollary 1.3.1 *If $E_1, \ldots, E_n$ are finite-dimensional spaces, then every map $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ is continuous.*

Definition 1.3.2 *(Finite-type operator) We say that a multilinear mapping $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ is of finite type if there exist $k \in \mathbb{N}$, linear functionals $\varphi_j^{(i)} \in E_l^*$, and vectors $y_j \in F$ for $j = 1, \ldots, k$ and $i = 1, \ldots, n$, such that*

$$A(x_1, \ldots, x_n) = \sum_{j=1}^{k} \varphi_j^{(1)}(x_1) \cdots \varphi_j^{(n)}(x_n)\, y_j.$$

For every $(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n$, this n-linear mapping is also denoted by

$$\sum_{j=1}^{k} \varphi_j^{(1)} \otimes \cdots \otimes \varphi_j^{(n)} \otimes y_j.$$

The vector subspace of $\mathcal{L}(E_1, \ldots, E_n; F)$ consisting of all finite type mappings will be denoted by

$$\mathcal{L}_f(E_1, \ldots, E_n; F).$$

Definition 1.3.3 *[24] (Finite-rank operator) We say that a multilinear mapping $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ is of finite rank if the dimension of the vector subspace of F generated by the image of A is finite. That is, if there exist $k \in \mathbb{N}$, n-linear forms $T_j \in \mathcal{L}(E_1, \ldots, E_n)$, and vectors $y_j \in F$, for $j = 1, \ldots, k$, such that*

$$A(x_1, \ldots, x_n) = \sum_{j=1}^{k} T_j(x_1, \ldots, x_n)\, y_j.$$

For every $(x_1, \ldots, x_m) \in E_1 \times \cdots \times E_n$, this n-linear mapping A is also denoted by

$$\sum_{j=1}^{k} T_j \otimes y_j.$$

The vector subspace of $\mathcal{L}(E_1, \ldots, E_n; F)$ consisting of all finite rank mappings is denoted by $\mathcal{L}_{\mathcal{F}}(E_1, \ldots, E_n; F)$.

Remark 1.3.1 *Clearly, every multilinear mapping of finite type is also of finite rank. Therefore, we have the inclusion*

$$\mathcal{L}_f(E_1, \ldots, E_n; F) \subseteq \mathcal{L}_{\mathcal{F}}(E_1, \ldots, E_n; F).$$

Now we give an example of continuous multilnear operator that is not finite type

Example 1.3.1 *Consider the following bilinear form:*

$$A : \ell_2 \times \ell_2 \to \mathbb{K}, \quad A((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty}) = \sum_{i=1}^{\infty} x_i y_i.$$

This bilinear form does not belong to the class of finite type operators.

Proof. *Let us show that A is not of finite type. If it were, we could write*

$$A(x, y) = \sum_{j=1}^{k} \varphi_j^{(1)}(x) \cdot \varphi_j^{(2)}(y)$$

for all $x, y \in \ell_2$, where $\varphi_j^{(1)}, \varphi_j^{(2)} \in \ell_2^*$. By the duality $\ell_2^* = \ell_2$, for each $j = 1, \ldots, k$, there exist sequences $(a_i^{(j)})_{i=1}^{\infty}$, $(b_i^{(j)})_{i=1}^{\infty} \in \ell_2$ such that

$$\varphi_j^{(1)}(x) = \sum_{i=1}^{\infty} a_i^{(j)} x_i \quad \textit{and} \quad \varphi_j^{(2)}(y) = \sum_{i=1}^{\infty} b_i^{(j)} y_i$$

for all $x = (x_i)_{i=1}^{\infty}$, $y = (y_i)_{i=1}^{\infty} \in \ell_2$. Thus, we obtain

$$A(x, y) = \sum_{j=1}^{k} \left(\sum_{i=1}^{\infty} a_i^{(j)} x_i \right) \cdot \left(\sum_{i=1}^{\infty} b_i^{(j)} y_i \right)$$

for all $x, y \in \ell_2$. But by the definition of A, for each i, taking the i-th canonical unit vector e_i, we have

$$1 = A(e_i, e_i) = \sum_{j=1}^{k} a_i^{(j)} b_i^{(j)}$$

Note that the last term in the equality above tends to zero as $i \to \infty$, since it is a finite sum of products of general terms from absolutely convergent series. This contradiction shows that A is not of finite type. ■

Definition 1.3.4 *(Multilinear operators ideal). A multilinear ideal $\mathcal{M}$ is a class of continuous multilinear operators such that for every $n \in \mathbb{N}$, $E_1, \ldots, E_n$ and F are Banach spaces, we have*

(i) $\mathcal{M}(E_1, \ldots, E_n; F)$ is a subspace of $\mathcal{L}(E_1, \ldots, E_n; F)$ that contains $\mathcal{L}_f$.

(ii) Ideal Property: If $A \in \mathcal{M}(X_1, \ldots, X_n; Y)$*,* $u_j \in \mathcal{L}(E_j, X_j)$ *for* $j = 1, \ldots, n$*, and* $v \in \mathcal{L}(Y, F)$*, then* $v \circ A \circ (u_1, \ldots, u_n) \in \mathcal{M}(E_1, \ldots, E_n; F)$*.*

Moreover, if $\| \cdot \|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^+$ *satisfies*

(i') $(\mathcal{M}(E_1, \ldots, E_n; F), \| \cdot \|_{\mathcal{M}})$ *is a normed space (in fact, a Banach space).*

(ii') $\|I_{\mathbb{K}^n} : \mathbb{K}^n \to \mathbb{K}\|_{\mathcal{M}} = 1$*, where* $I_{\mathbb{K}^n}(\lambda_1, \ldots, \lambda_n) = \lambda_1 \cdots \lambda_n$*,*

(iii') If $A \in \mathcal{M}(X_1, \ldots, X_n; Y)$*,* $u_j \in \mathcal{L}(E_j, X_j)$ *for* $j = 1, \ldots, n$*, and* $v \in \mathcal{L}(Y, F)$*, then*

$$\|v \circ A \circ (u_1, \ldots, u_n)\|_{\mathcal{M}} \leq \|v\| \cdot \|A\|_{\mathcal{M}} \cdot \|u_1\| \cdots \|u_n\|.$$

Then $(\mathcal{M}, \| \cdot \|_{\mathcal{M}})$ *is called a normed (Banach) ideal of multilinear operators.*

Proposition 1.3.2 *Let* $(\mathcal{M}, \| \cdot \|_{\mathcal{M}})$ *be a normed ideal of multilinear operators. Then we have*

$$\|A\| \leq \|A\|_{\mathcal{M}} \quad \text{for all } A \in \mathcal{M}.$$

Example 1.3.2 $\mathcal{L}(E_1, \ldots, E_n; F)$ *and* $\mathcal{L}_f(E_1, \ldots, E_n; F)$ *are ideals of multilinear operators.*

Definition 1.3.5 *[16](Nuclear multilinear operators) An operator* $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ *is called multi-nuclear if there exist a sequences* $(\lambda_j)_{j=1}^{\infty} \in \ell_1, (\varphi_j^{(i)})_{j=1}^{\infty} \subset E_i^*$ *for each* $i = 1, \ldots, n$ *and* $(y_j)_{j=1}^{\infty} \subset F$ *such that for every* $(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n$*, the operator* A *can be represented as*

$$A(x_1, \ldots, x_n) = \sum_{j=1}^{\infty} \lambda_j \varphi_j^{(1)}(x_1) \cdots \varphi_j^{(n)}(x_n) y_j.$$

The expression above is called a nuclear representation for A *and the space of all nuclear* n*-linear operators from* $E_1 \times \cdots \times E_n$ *to* F *is denoted by* $\mathcal{L}_{\mathcal{N}}(E_1, \ldots, E_n; F)$*. The multi-nuclear norm of* A *is defined by*

$$\|A\|_{\mathcal{L}_{\mathcal{N}}} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \cdot \|\varphi_j^{(1)}\| \cdots \|\varphi_j^{(n)}\| \cdot \|y_j\| \right\}$$

where the infimum is taken over all such representations as above.

Proposition 1.3.3 $(\mathcal{L}_{\mathcal{N}}, \|.\|_{\mathcal{L}_{\mathcal{N}}})$ *is a Banach multilineare ideal*

Chapter 2

Hyper ideal of multilinear operators

In this chapter, we address the question posed by Bolteho and Torees: Can the ideal property be generalized to include composition with multilinear operators from the left, in addition to linear operators? they answered this question in their article entitled [**Hyper-ideal of multilinear operators**], published in the journal [**Linear Algebra and its Applications**]. In this chapter, we explore the concept of hyper-ideals, some of their related properties, and their relationship with *multi-ideals*.

2.1 Motivation

In the theory of linear operators, ideals are classes of operators closed under composition with other linear operators, facilitating the study of operator properties and their compositions. For multilinear operators, the concept of **multi-ideals** arises, where a class $\mathcal{M}$ of n-linear operators satisfies:

$$\text{If } A \in \mathcal{M} \quad \Rightarrow \quad v \circ A \circ (u_1, \ldots, u_n) \in \mathcal{M}$$

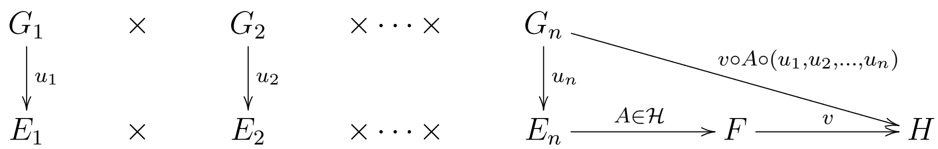
where

$$u_j \in \mathcal{L}(G_j, E_j), \quad \text{for each } j = 1, \ldots, n.$$

and

$$v \in \mathcal{L}(F, H).$$

depicted by the diagram:



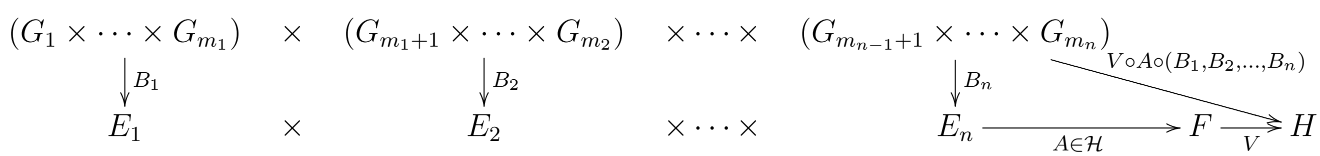
However, given the multilinear nature of these operators, a natural question arises: why not compose A also with multilinear operators on the left side? This leads to the concept of **hyper-ideals**, where a class $\mathcal{H}$ satisfies:

$$\text{If } A \in \mathcal{H} \quad \Rightarrow \quad v \circ A \circ (B_1, \ldots, B_n) \in \mathcal{H}$$

where

$$B_j \in \mathcal{L}(G_{m_{j-1}+1}, \ldots, G_{m_j}; E_j)$$

for all illustrated by the diagram:



This generalization enhances stability and allows for a broader study of classes that include more complex compositions among multilinear operators. It also reveals that some classical multi-ideals are not hyper-ideals, motivating the construction of new classes such as hyper-nuclear operators to fill these gaps.

2.2 Definitions and properties

Definition 2.2.1 *A* hyper-ideal *of multilinear operators, or simply a* hyper-ideal*, is a subclass $\mathcal{H}$ of the class of all continuous multilinear operators between Banach spaces such that for all $n \in \mathbb{N}$ and Banach spaces $E_1, \ldots, E_n$ and F, the components*

$$\mathcal{H}(E_1, \ldots, E_n; F) := \mathcal{L}(E_1, \ldots, E_n; F) \cap \mathcal{H}$$

satisfy:

(i) $\mathcal{H}(E_1, \ldots, E_n; F)$ *is a linear subspace of* $\mathcal{L}(E_1, \ldots, E_n; F)$ *.*

(ii) $\mathcal{L}_f(E_1, \ldots, E_n; F) \subset \mathcal{H}(E_1, \ldots, E_n; F)$

(iii) ***Hyper-ideal property:*** *Let n be a natural number and $1 \leq m_1 < \cdots < m_n$, and let $G_1, \ldots, G_{m_n}$, $E_1, \ldots, E_n$, F and H be Banach spaces. If*

$$B_1 \in \mathcal{L}(G_1, \ldots, G_{m_1}; E_1), \ldots, B_n \in \mathcal{L}(G_{m_{n-1}+1}, \ldots, G_{m_n}; E_n),$$

$v \in \mathcal{L}(F; H)$ and $A \in \mathcal{H}(E_1, \ldots, E_n; F)$, then

$$v \circ A \circ (B_1, \ldots, B_n) \in \mathcal{H}(G_1, \ldots, G_{m_n}; H).$$

If there exist $p \in (0, 1]$ and a map $\| \cdot \|_{\mathcal{H}} : \mathcal{H} \to [0, \infty)$ such that:

(i') The restriction of $\| \cdot \|_{\mathcal{H}}$ to each component $\mathcal{H}(E_1, \ldots, E_n; F)$ is a p-norm.

(ii') $\|I_{\mathbb{K}^n} : \mathbb{K}^n \to \mathbb{K},\ I_{\mathbb{K}^n}(\lambda_1, \ldots, \lambda_n) = \lambda_1 \cdots \lambda_n\|_{\mathcal{H}} = 1$ *for every n.*

(iii') ***Hyper-ideal inequality:*** *If $A \in \mathcal{H}(E_1, \ldots, E_n; F)$,*

$$B_1 \in \mathcal{L}(G_1, \ldots, G_{m_1}; E_1), \ldots, B_n \in \mathcal{L}(G_{m_{n-1}+1}, \ldots, G_{m_n}; E_n),$$

and $v \in \mathcal{L}(F; H)$, then

$$\|v \circ A \circ (B_1, \ldots, B_n)\|_{\mathcal{H}} \leq \|v\| \cdot \|A\|_{\mathcal{H}} \cdot \|B_1\| \cdots \|B_n\|. \tag{1}$$

Then $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$ is called a ***p-normed hyper-ideal****. If all components $\mathcal{H}(E_1, \ldots, E_n; F)$ are complete spaces with respect to the topology generated by $\| \cdot \|_{\mathcal{H}}$, then $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$ is called a* ***p-Banach hyper-ideal****.*

When $p = 1$, we say that $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$ is a ***normed hyper-ideal*** *or a* ***Banach hyper-ideal****.*

If $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$ is a p-normed (or p-Banach) hyper-ideal for some $p \in (0, 1]$, then we say that it is a ***quasi-normed hyper-ideal*** *(or* ***quasi-Banach hyper-ideal****).*

Proposition 2.2.1 *It is plain that every (normed, quasi-normed, Banach, quasi-Banach)hyper-ideal is a (normed, quasi-normed, Banach, quasi-Banach) multi-ideal. So, properties of multi-ideals are inherited by hyper ideals.*

Proposition 2.2.2 *Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ be a p-normed hyper-ideal. Then the following inequality holds*

$$\|\cdot\| \leq \|\cdot\|_{\mathcal{H}}. \tag{2}$$

Proof. Let $A \in \mathcal{H}(E_1, \ldots, E_n; F)$, and let $x_j \in E_j$ for each $j = 1, \ldots, n$. By the Hahn Banach Theorem, there exists a functional $\varphi \in F'$ such that $\|\varphi\| = 1$ and

$$\varphi\left(A(x_1, \ldots, x_n)\right) = \|A(x_1, \ldots, x_n)\|.$$

Let us consider the linear operators

$$1 \otimes x_i : \mathbb{K} \to E_i, \quad (1 \otimes x_i)(\lambda) = \lambda x_i, \quad \text{for } i = 1, \ldots, n,$$

and the functional $\varphi : F \to \mathbb{K}$.
Let

$$\begin{aligned}
\varphi \circ A \circ (1 \otimes x_1, \ldots, 1 \otimes x_n)(\lambda_1, \ldots, \lambda_n) &= \varphi \circ A(1 \otimes x_1(\lambda_1), \ldots, 1 \otimes x_n(\lambda_n)) \\
&= \varphi(A(\lambda_1 x_1, \ldots, \lambda_n x_n)) \\
&= \lambda_1 \cdots \lambda_n \, \varphi(A(x_1, \ldots, x_n)) \\
&= I_n(\lambda_1, \ldots, \lambda_n) \cdot \varphi(A(x_1, \ldots, x_n))
\end{aligned}$$

therefor

$$\varphi \circ A \circ (1 \otimes x_1, \ldots, 1 \otimes x_n) = \varphi(A(x_1, \ldots, x_n)) \cdot I_n$$

We conclude that

$$\begin{aligned}
\|A(x_1, \ldots, x_n)\| &= |\varphi(A(x_1, \ldots, x_n))| \\
&= |\varphi(A(x_1, \ldots, x_n))| \cdot \|I_n\|_{\mathcal{H}} \\
&= \|\varphi(A(x_1, \ldots, x_n)) \cdot I_n\|_{\mathcal{H}} \\
&= \|\varphi \circ A \circ (1 \otimes x_1, \ldots, 1 \otimes x_n)\|_{\mathcal{H}} \\
&\leq \|\varphi\| \cdot \|A\|_{\mathcal{H}} \cdot \|1 \otimes x_1\| \cdots \|1 \otimes x_n\| \\
&= \|A\|_{\mathcal{H}} \cdot \|x_1\| \cdots \|x_n\|.
\end{aligned}$$

Since the vectors $x_i \in E_i$, for $i = 1, \ldots, n$, are arbitrary, we obtain

$$\|A(x_1, \ldots, x_n)\| \leq \|A\|_{\mathcal{H}} \cdot \|x_1\| \cdots \|x_n\|.$$

Hence, we conclude that:

$$\|A\| \leq \|A\|_{\mathcal{H}}.$$

■

The closure of $\mathcal{H}$ is defined by

$$\overline{\mathcal{H}}(E_1, \ldots, E_n; F) := \overline{\mathcal{H}(E_1, \ldots, E_n; F)}^{\|\cdot\|},$$

for every $n \in \mathbb{N}$ and for arbitrary Banach spaces $E_1, \ldots, E_n$ and F. We denote this class by $\overline{\mathcal{H}}$. Moreover, $\mathcal{H}$ is said to be closed if $\overline{\mathcal{H}} = \mathcal{H}$; that is, if $\mathcal{H}(E_1, \ldots, E_n; F)$ is a closed subspace of $\mathcal{L}(E_1, \ldots, E_n; F)$ with respect to the uniform norm, for all Banach spaces $E_1, \ldots, E_n$ and F.

Proposition 2.2.3 *Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ be a quasi-normed hyper-ideal, n, $m_1 < \cdots < m_n$ be natural numbers, $T_1 \in \mathcal{L}(E_1, \ldots, E_{m_1})$, ..., $T_n \in \mathcal{L}(E_{m_{n-1}+1}, \ldots, E_{m_n})$ and $y \in F$. Consider the m_n-linear operator*

$$T_1 \otimes \cdots \otimes T_n \otimes y : E_1 \times \cdots \times E_{m_n} \to F$$

defined by

$$(T_1 \otimes \cdots \otimes T_n \otimes y)(x_1, \ldots, x_{m_n}) = T_1(x_1, \ldots, x_{m_1}) \cdots T_n(x_{m_{n-1}+1}, \ldots, x_{m_n}) \cdot y.$$

Then

$$T_1 \otimes \cdots \otimes T_n \otimes y \in \mathcal{H}(E_1, \ldots, E_{m_n}; F)$$

and

$$\|T_1 \otimes \cdots \otimes T_n \otimes y\|_{\mathcal{H}} = \|T_1 \otimes \cdots \otimes T_n \otimes y\| = \|T_1\| \cdots \|T_n\| \cdot \|y\|.$$

Proof. Considering the linear operator

$$1 \otimes y : \mathbb{K} \to F \text{ given by } (1 \otimes y)(\lambda) = \lambda \cdot y,$$

we have

$$(1 \otimes y) \circ \mathrm{I}_{m_n} \circ (T_1, \ldots, T_n) = T_1 \otimes \cdots \otimes T_n \otimes y.$$

As $I_{m_n} \in \mathcal{H}(\mathbb{K}^{m_n}; \mathbb{K})$, defined by $I_n(\lambda_1, \ldots, \lambda_n) = \lambda_1 \cdots \lambda_n$ for all $\lambda_i \in \mathbb{K}$

from the hyper-ideal property of $\mathcal{H}$ we conclude that

$$T_1 \otimes \cdots \otimes T_n \otimes y \in \mathcal{H}(E_1, \ldots, E_{m_n}; F).$$

Using first (2) and then (1), we get

$$\begin{aligned}
\|T_1 \otimes \cdots \otimes T_n \otimes y\| &\leq \|T_1 \otimes \cdots \otimes T_n \otimes y\|_{\mathcal{H}} \\
&= \|(1 \otimes y) \circ \mathrm{I}_{m_n} \circ (T_1, \ldots, T_n)\|_{\mathcal{H}} \\
&\leq \|1 \otimes y\| \cdot \|\mathrm{I}_{m_n}\|_{\mathcal{H}} \cdot \|T_1\| \cdots \|T_n\| \\
&= \|T_1\| \cdots \|T_n\| \cdot \|y\| \\
&= \|T_1 \otimes \cdots \otimes T_n \otimes y\|.
\end{aligned}$$

■

Proposition 2.2.4 *Let $\mathcal{H}$ be a hyper-ideal. Then $(\overline{\mathcal{H}}, \|\cdot\|)$ is a Banach hyper-ideal. Moreover, $\overline{\mathcal{H}}$ is the smallest closed hyper-ideal that contains $\mathcal{H}$.*

Proof.

(i) Let $\mathcal{H}(E_1, \ldots, E_n; F)$ be a subspace of the normed vector space $\mathcal{L}(E_1, \ldots, E_n; F)$. It is immediate that

$$\overline{\mathcal{H}}(E_1, \ldots, E_n; F)$$

is also a normed space. Moreover, since

$$\mathcal{H} \subseteq \overline{\mathcal{H}}$$

and $\mathcal{H}$ contains the finite type operators, $\overline{\mathcal{H}}$ also contains them.

(ii) Given $A \in \overline{\mathcal{H}}(E_1, \ldots, E_n; F)$, does there exist a sequence

$$(A_j)_{j=1}^{\infty} \subseteq \mathcal{H}(E_1, \ldots, E_n; F)$$

that converges to A in the uniform norm

Let $1 \leq m_1 < \cdots < m_n$,

$$B_1 \in \mathcal{L}(G_1, \ldots, G_{m_1}; E_1), \quad \ldots, \quad B_n \in \mathcal{L}(E_{m_{n-1}+1}, \ldots, E_{m_n}; E_n),$$

and

$$T \in \mathcal{L}(F; H).$$

Then

$$\begin{aligned}\|v \circ A \circ (B_1, \ldots, B_n) - v \circ A_j \circ (B_1, \ldots, B_n)\| &= \|v \circ (A - A_j) \circ (B_1, \ldots, B_n)\| \\ &\leq \|v\| \cdot \|A - A_j\| \cdot \|B_1\| \cdots \|B_n\|\end{aligned}$$

Therefore, the sequence

$$(v \circ A_j \circ (B_1, \ldots, B_n))_{j=1}^{\infty}$$

converges to

$$v \circ A \circ (B_1, \ldots, B_n).$$

Since $\mathcal{H}$ is a hyper-ideal, we have

$$v \circ A_j \circ (B_1, \ldots, B_n) \in \mathcal{H}(G_1, \ldots, G_{m_n}; H)$$

Hence,

$$v \circ A \circ (B_1, \ldots, B_n) \in \overline{\mathcal{H}}(G_1, \ldots, G_{m_n}; H)$$

Thus, $(\mathcal{H}; \|\cdot\|_{\mathcal{H}})$ is a normed hyper-ideal.

It is clear that $\overline{\mathcal{H}}(E_1, \ldots, E_n; F)$ is a closed subspace of the Banach space $\mathcal{L}(E_1, \ldots, E_n; F)$, hence $\overline{\mathcal{H}}(E_1, \ldots, E_n; F)$ is also a Banach space. Therefore, $(\mathcal{H}; \|\cdot\|_{\mathcal{H}})$ is a Banach hyper-ideal.

Let $\mathcal{G}$ be a smallest closed hyper-ideal containing $\mathcal{H}$. For any Banach spaces $E_1, \ldots, E_n$ and F, and for any $A \in \overline{\mathcal{H}}(E_1, \ldots, E_n; F)$, there exists a sequence $(A_j)_{j=1}^{\infty} \subseteq \mathcal{H}(E_1, \ldots, E_n; F) \subseteq \mathcal{G}(E_1, \ldots, E_n; F)$ such that

$$A_j \xrightarrow[j \to \infty]{} A \quad \text{in} \quad \mathcal{G}(E_1, \ldots, E_n; F).$$

Since $\mathcal{G}(E_1, \ldots, E_n; F)$ is closed, it follows that $A \in \mathcal{G}(E_1, \ldots, E_n; F)$. This proves that

$$\overline{\mathcal{H}}(E_1, \ldots, E_n; F) \subseteq \mathcal{G}(E_1, \ldots, E_n; F).$$

■

Theorem 2.2.1 *[24, Theorem 2.1.9]* ***(Series criterion).*** *Let $0 < p \leq 1$ and $\mathcal{H}$ be a subclass of the class of all continuous multilinear operators between Banach spaces, endowed with a map*

$$\|\cdot\|_{\mathcal{H}} : \mathcal{H} \to [0, +\infty).$$

Then $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a p-Banach hyper-ideal if and only if the following conditions are satisfied:

(i) For every $n \in \mathbb{N}$, the n-linear mapping

$$I_{\mathbb{K}^n} \in \mathcal{H}(\mathbb{K}^n; \mathbb{K}) \quad \textit{with} \quad \|I_{\mathbb{K}^n}\|_{\mathcal{H}} = 1,$$

where $I_{\mathbb{K}^n}(\lambda_1, \ldots, \lambda_n) = \lambda_1 \cdots \lambda_n$ for all $\lambda_i \in \mathbb{K}$.

(ii) If $(A_j)_{j=1}^{\infty} \subset \mathcal{H}(E_1, \ldots, E_n; F)$ is such that

$$\sum_{j=1}^{\infty} \|A_j\|_{\mathcal{H}}^p < \infty,$$

then the series $A := \sum_{j=1}^{\infty} A_j$ converges in $\mathcal{H}(E_1, \ldots, E_n; F)$, and

$$\|A\|_{\mathcal{H}}^p \leq \sum_{j=1}^{\infty} \|A_j\|_{\mathcal{H}}^p.$$

(iii) Let $n \in \mathbb{N}$, $1 \leq m_1 < \cdots < m_n$, and let $G_1, \ldots, G_{m_n}, E_1, \ldots, E_n, F, H$ be Banach spaces. Suppose

$$B_1 \in \mathcal{L}(G_1, \ldots, G_{m_1}; E_1), \ldots, B_n \in \mathcal{L}(G_{m_{n-1}+1}, \ldots, G_{m_n}; E_n),$$

$$A \in \mathcal{H}(E_1, \ldots, E_n; F), \quad \textit{and} \quad t \in \mathcal{L}(F; H).$$

Then the composition

$$v \circ A \circ (B_1, \ldots, B_n) \in \mathcal{H}(G_1, \ldots, G_{m_n}; H)$$

and

$$\|v \circ A \circ (B_1, \ldots, B_n)\|_{\mathcal{H}} \leq \|v\| \cdot \|A\|_{\mathcal{H}} \cdot \|B_1\| \cdots \|B_n\|.$$

Proposition 2.2.5 *[8, Proposition 2.6] Let $0 < p \leq 1$, $(\mathcal{G}, \|\cdot\|_{\mathcal{G}})$ and $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ be p-Banach hyper-ideals such that $\mathcal{G} \subseteq \mathcal{H}$. Then, for every $n \in \mathbb{N}$ there is a constant C_n, depending only on n, such that*

$$\|A\|_{\mathcal{H}} \leq C_n \|A\|_{\mathcal{G}},$$

for all Banach spaces $E_1, \ldots, E_n, F$ and $A \in \mathcal{G}(E_1, \ldots, E_n; F)$.

2.3 Examples

Example 2.3.1 *$\mathcal{L}_f$ is not a hyper ideal.*

Proof.

Consider the bilinear form

$$A : \ell_2 \times \ell_2 \longrightarrow \mathbb{K}, \quad A\left((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} x_i y_i.$$

It is well known in functional analysis that this bilinear form is not of finite type

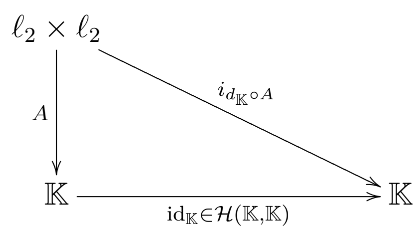
Now, assume for the sake of contradiction that $\mathcal{L}_f$ forms a hyper-ideal. Since the identity operator on the scalar field,

$$\mathrm{id}_{\mathbb{K}} \in \mathcal{L}_f(\mathbb{K}; \mathbb{K}),$$

the hyper-ideal property would imply that:

$$\mathrm{id}_{\mathbb{K}} \circ A \in \mathcal{L}_f(\ell_2 \times \ell_2; \mathbb{K}),$$

which means $A \in \mathcal{L}_f(\ell_2 \times \ell_2; \mathbb{K})$, a contradiction with the known fact that A is not of finite type.



we conclude that $\mathcal{L}_f$ is not a hyper-ideal.

■

Example 2.3.2 *Let us show that the class $\mathcal{L}_{\mathcal{N}}$ of nuclear multilinear mappings does not a hyper-ideal.*

First, note that the class of nuclear mappings is contained in the class of multilinear mappings approximable by finite type mappings. Indeed if $A \in \mathcal{L}_{\mathcal{N}}(E_1, \ldots, E_n; F)$, then we can consider sequences

$$(\varphi_j^{(l)})_{j=1}^{\infty} \subset E_l^* \quad \textit{for } l = 1, \ldots, n,$$

and

$$(\lambda_j)_{j=1}^{\infty} \in \ell_1, \quad (y_j)_{j=1}^{\infty} \subset F,$$

in such a way that we obtain a nuclear representation of A.

We can then define the sequence $(A_k)_{k=1}^{\infty}$ in $\mathcal{L}_f(E_1, \ldots, E_n; F)$ where each A_k is given by

$$A_k(x_1, \ldots, x_n) = \sum_{j=1}^{k} \lambda_j \varphi_j^{(1)}(x_1) \cdots \varphi_j^{(n)}(x_n) \cdot y_j,$$

for all $x_1 \in E_1, \ldots, x_n \in E_n$.

With this, the series

$$\sum_{j=1}^{\infty} |\lambda_j| \cdot \|\varphi_j^{(1)}\| \cdots \|\varphi_j^{(n)}\| \cdot \|y_j\|$$

is convergent, and therefore, given $\varepsilon > 0$, we can find $k_0 \in \mathbb{N}$ such that

$$\sum_{j=k_0}^{\infty} |\lambda_j| \cdot \|\varphi_j^{(1)}\| \cdots \|\varphi_j^{(n)}\| \cdot \|y_j\| < \varepsilon.$$

Thus,

$$\|A_k - A\| \leq \sum_{j=k_0}^{\infty} |\lambda_j| \cdot \|\varphi_j^{(1)}\| \cdots \|\varphi_j^{(n)}\| \cdot \|y_j\| < \varepsilon$$

for every $k \geq k_0$. Therefore, A is approximable by finite type mappings. The proof that $\mathcal{L}_{\mathcal{N}}$ is a multi-ideal can be found in [16, Proposition 2.2]. In particular, we have:

$$\|\cdot\| \leq \|\cdot\|_{\mathcal{L}_{\mathcal{N}}}.$$

We can now conclude that the class of nuclear mappings is not a hyper-ideal. To see this, it suffices to observe that the class of multilinear mappings approximable by finite type mappings does not contain the finite rank mappings. Therefore, the class of nuclear mappings also cannot contain them, that is, it does not satisfy condition $(1')$.

Example 2.3.3 *$\mathcal{L}_{\mathcal{F}}$ is the smallest hyper-ideal with a norm, meaning that every hyper-ideal containing the finite-rank operators has a norm greater than (or equal to) the natural norm.*

Proof.

Since condition (i) of Definition 2.2.1 is the same as in the case of multi-ideals, and we already know that $\mathcal{L}_{\mathcal{F}}$ is a multi-ideal, there is no need to check this condition.

Let us verify condition (ii). Let $1 \leq m_1 \leq \cdots \leq m_n$ be natural numbers, and let

$$A \in \mathcal{L}_F(E_1, \ldots, E_n; F), \quad B_1 \in \mathcal{L}(G_1, \ldots, G_{m_1}; E_1), \ldots, \quad B_n \in \mathcal{L}(G_{m_{n-1}+1}, \ldots, G_{m_n}; E_n), \quad v \in \mathcal{L}(F; H).$$

Since A is of finite rank, we can write it as

$$A = \sum_{j=1}^{k} T_j \otimes y_j,$$

where $T_j \in \mathcal{L}(E_1, \ldots, E_n)$ and $y_j \in F$. Then we have:

$$\begin{aligned} v \circ A \circ (B_1, \ldots, B_n)(x_1, \ldots, x_{m_n}) &= v\left(A(B_1(x_1, \ldots, x_{m_1}), \ldots, B_n(x_{m_{n-1}+1}, \ldots, x_{m_n}))\right) \\ &= v\left(\sum_{j=1}^{k} T_j(B_1(\cdots), \ldots, B_n(\cdots))y_j\right) \\ &= \sum_{j=1}^{k} T_j(B_1(\cdots), \ldots, B_n(\cdots))(v(y_j)) \\ &= \sum_{j=1}^{k} S_j \otimes z_j(x_1, \ldots, x_{m_n}), \end{aligned}$$

for all $x_i \in G_i$, where $S_j := T_j \circ (B_1, \ldots, B_n) \in \mathcal{L}(G_1, \ldots, G_{m_n})$, and $z_j := v(y_j) \in H$.

Thus,

$$v \circ A \circ (B_1, \ldots, B_n) \in \mathcal{L}_{\mathcal{F}}(G_1, \ldots, G_{m_n}; H),$$

proving that $\mathcal{L}_{\mathcal{F}}$ is a hyper-ideal.

Now we prove that $\mathcal{L}_{\mathcal{F}}$ is the smallest hyper-ideal. Let $\mathcal{H}$ be a hyper-ideal and let

$$A \in \mathcal{L}_{\mathcal{F}}(E_1, \ldots, E_n; F).$$

We need to show that

$$A \in \mathcal{H}(E_1, \ldots, E_n; F).$$

We may assume, without loss of generality, that

$$A = T \otimes y,$$

where $T \in \mathcal{L}(E_1, \ldots, E_n)$ and $y \in F$, since A can be written as

$$A = \sum_{j=1}^{k} T_j \otimes y_j,$$

with $T_j \in \mathcal{L}(E_1, \ldots, E_n)$ and $y_j \in F$, and $\mathcal{H}(E_1, \ldots, E_n; F)$ is a vector subspace $\mathcal{L}(E_1, \ldots, E_n; F)$ Let us consider the linear operator

$$1 \otimes y : \mathbb{K} \to F, \quad (1 \otimes y)(\lambda) = \lambda y.$$

We have

$$1 \otimes y \in \mathcal{L}_f(\mathbb{K}; F) \subseteq \mathcal{H}(\mathbb{K}; F).$$

From the hyper-ideal property, it follows that

$$A = T \otimes y = (1 \otimes y) \circ T \in \mathcal{H}(E_1, \ldots, E_n; F).$$

■

Proposition 2.3.1 *[24] Let $\mathcal{H}$ be a subclass of the class of multilinear mappings between Banach spaces that satisfies the hyper-ideal property. For every $n \in \mathbb{N}$ and for any Banach spaces $E_1, \ldots, E_n, F$, the following statements are equivalent:*

(i) *$\mathcal{H}(E_1, \ldots, E_n; F)$ is a vector subspace of $\mathcal{L}(E_1, \ldots, E_n; F)$ that contains all finite-type n-linear mappings.*

(i') *$\mathcal{H}(E_1, \ldots, E_n; F)$ is a vector subspace of $\mathcal{L}(E_1, \ldots, E_n; F)$ that contains all finite-rank n-linear mappings.*

Chapter 3

Hyper nuclear multilinear Operators

Before introducing the hyper nuclear multilinear operator ideal, we first recall the notion of a nuclear linear operator ideal. In the linear setting, an operator $S \in \mathcal{L}(E, F)$ is called p-nuclear [18, 20] if

$$S(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n,$$

where $(x_n^*)_n \in \ell_p(E^*)$ and $(y_n)_n \in \ell_{p^*,w}(F)$, and

$$\|S\|_{\mathcal{N}_p} = \inf \left\{ \|(x_n^*)_n\|_p \cdot \|(y_n)_n\|_{p^*,w} \right\},$$

where the infimum is taken over all so-called p-nuclear representations described above. The class of all p-nuclear operators with p-nuclear norm is denoted by $(\mathcal{N}_p, \|\cdot\|_{\mathcal{N}_p})$

An operator $R \in \mathcal{L}(E, F)$ is called right p-nuclear [17] if there are functionals $(x_j^*)_j \in \ell_{p^*,w}(E^*)$ and $(y_j)_j \in \ell_p(F)$ such that

$$R(x) = \sum_{j=1}^{\infty} x_j^*(x) y_j$$

and

$$\|R\|_{\mathcal{N}_p^{right}} = \inf \left\{ \|(x_j^*)_j\|_{p^*,w} \cdot \|(y_j)_j\|_p \right\},$$

where the infimum is taken over all so-called p-nuclear representations described above. The class of all p-nuclear operators with p-nuclear norm is denoted by $(\mathcal{N}_p^{right}, \|\cdot\|_{\mathcal{N}_p^{right}})$.

For $p = 1$, 1-nuclear operators are simply called nuclear operators. The class of all nuclear operators with nuclear norm is denoted by $(\mathcal{N}, \|\cdot\|_{\mathcal{N}})$

Botelho and Torres [8] extended the linear operator ideal of right p-nuclear operators, while Popa [21] independently developed an extension of the p-nuclear operator ideal. In this chapter, we study the hyper right p-nuclear multilinear operators, examining their special properties including their satisfaction of the linearization theorem and prove that they admit a factorization through ℓ^p-spaces via a composition of linear and multilinear operators.

3.1 The hyper right p-nuclear operators

We begin by proving the following property, which, though straightforward, we include for completeness.

Proposition 3.1.1 *Let $T : E \longrightarrow F$ be an operator. The following are equivalent:*

i) T is right p-nuclear;

ii) There are $(\lambda_n)_n \in \ell_p, (x_n^*)_n \in \ell_{p^*,\omega}(E^*)$ and $(z_n)_n \in \ell_\infty(F)$, such that

$$T = \sum_{n=1}^{\infty} \lambda_n x_n^*(x) z_n.$$

for $x \in E$. Moreover

$$\|T\|_{\mathcal{N}_p^{right}} = \inf \|(\lambda_n)_n\|_p . \|(x_n^*)_n\|_{p^*,\omega} . \|(z_n)_n\|_\infty$$

Proof. Let T is right p-nuclear, then there exist $(x_n^*)_n \in \ell_{p^*,\omega}(E^*)$ and $(y_n)_n \in \ell_p(F)$ such that

$$T = \sum_{n=1}^{\infty} x_n^*(x) y_n = \sum_{n=1}^{\infty} \lambda_n x_n^*(x) z_n$$

where $(z_n)_n = \left(\frac{y_n}{\|y_n\|}\right)_n \in \ell_\infty(F)$ and $(\lambda_n)_n = (\|y_n\|)_n \in \ell_p$.
So, $\inf \|(\lambda_n)_n\|_p . \|(x_n^*)_n\|_{p^*,\omega} . \|(z_n)_n\|_p \leq \inf \|(x_n^*)_n\|_{p^*,\omega} \|(y_n)_n\|_p = \|T\|_{\mathcal{N}_p^{right}}$.
Conversely, suppose that (ii) holds, then there exist $(\lambda_n)_n \in \ell_p$, $(x_n^*)_n \in \ell_{p^*,\omega}(E^*)$ and $(y_n)_n \in \ell_\infty(F)$, such that

$$T = \sum_{n=1}^{\infty} \lambda_n x_n^*(x) y_n = \sum_{n=1}^{\infty} x_n^*(x) z_n,$$

where $(z_n)_n = (\lambda_n y_n)_n \in \ell_p(F)$ so, T is right p-nuclear and

$$\|T\|_{\mathcal{N}_p^{right}} = \inf \|(x_n^*)_n\|_{p^*,\omega}\|(z_n)_n\|_p \leq \inf \|(\lambda_n)_n\|_p.\|(x_n^*)_n\|_{p^*,\omega}.\|(z_n)_n\|_p.$$

■

Let us see that, given sequences

$$(\lambda_j)_{j=1}^{\infty} \in \ell_p, \quad (T_j)_{j=1}^{\infty} \in \ell_{p^*,w}(\mathcal{L}(E_1,\dots,E_n)), \quad (y_j)_{j=1}^{\infty} \in \ell_{\infty}(F),$$

Then, the series

$$\sum_{j=1}^{\infty} \lambda_j T_j(x_1,\dots,x_n) y_j,$$

for all

$$(x_1,\dots,x_n) \in E_1 \times \cdots \times E_n,$$

is absolutely convergent, and therefore convergent. Indeed, consider the linear operator

$$\psi_{(x_1,\dots,x_n)} : \mathcal{L}(E_1,\dots,E_n) \to \mathbb{K}, \quad \psi_{(x_1,\dots,x_n)}(T) = T(x_1,\dots,x_n).$$

From

$$|\psi_{(x_1,\dots,x_n)}(T)| = |T(x_1,\dots,x_n)| \leq \|T\| \cdot \|x_1\| \cdots \|x_n\|,$$

we have $\psi_{(x_1,\dots,x_n)} \in \mathcal{L}(E_1,\dots,E_n)^*$ and $\|\psi_{(x_1,\dots,x_n)}\| \leq \|x_1\| \cdots \|x_n\|$. Thus,

$$\frac{\psi_{(x_1,\dots,x_n)}}{\|x_1\| \cdots \|x_n\|} \in B_{\mathcal{L}(E_1,\dots,E_n)^*}$$

whenever $x_i \neq 0$, for $i = 1,\dots,n$. In this case, by H lder s inequality, it follows that

$$\begin{aligned}\sum_{j=1}^{\infty} \|\lambda_j T_j(x_1,\dots,x_n) y_j\| &\leq \|(y_j)_{j=1}^{\infty}\|_{\infty} \cdot \left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p} \cdot \left(\sum_{j=1}^{\infty} \|T_j(x_1,\dots,x_n)\|^{p^*}\right)^{1/p^*} \\ &= \|(y_j)_{j=1}^{\infty}\|_{\infty} \cdot \|(\lambda_j)_{j=1}^{\infty}\|_p \cdot \left(\sum_{j=1}^{\infty} |\psi_{(x_1,\dots,x_n)}(T_j)|^{p^*}\right)^{1/p^*} \\ &\leq \|(y_j)_{j=1}^{\infty}\|_{\infty} \cdot \|(\lambda_j)_{j=1}^{\infty}\|_p \cdot \|(T_j)_{j=1}^{\infty}\|_{p^*,w} \cdot \|x_1\| \cdot \cdots \cdot \|x_n\| < \infty.\end{aligned}$$

Definition 3.1.1 *Let $p \in (0,\infty)$ and $p^* \in [1,\infty]$. An n-linear continuous operator $A \in \mathcal{L}(E_1,\dots,E_n;F)$ is called hyper right p-nuclear if there exist sequences $(\lambda_j)_{j=1}^{\infty} \in \ell_p$, $(T_j)_{j=1}^{\infty} \in \ell_{p^*,w}(\mathcal{L}(E_1,\dots,E_n))$, and $(y_j)_{j=1}^{\infty} \in \ell_{\infty}(F)$ such that*

$$A(x_1,\dots,x_n) = \sum_{j=1}^{\infty} \lambda_j T_j \otimes y_j(x_1,\dots,x_n) = \sum_{j=1}^{\infty} \lambda_j T_j(x_1,\dots,x_n) y_j, \tag{1}$$

for all $(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n$. *In this case, we write* $A \in \mathcal{L}_{\mathcal{HN}_p^{right}}(E_1, \ldots, E_n; F)$.

The hyper p*-nuclear norm*

$$\| \cdot \|_{\mathcal{L}_{\mathcal{HN}_p^{right}}} : \mathcal{L}_{\mathcal{HN}_p^{right}}(E_1, \ldots, E_n; F) \to [0, \infty)$$

is defined by

$$\|A\|_{\mathcal{L}_{\mathcal{HN}_p^{right}}} = \inf \left\{ \|(\lambda_j)\|_p \cdot \|(T_j)\|_{p^*,w} \cdot \|(y_j)\|_\infty \right\},$$

where the infimum is taken over all representations as in (1)

It is not difficult to prove that if $A \in \mathcal{L}(E_1, \cdots, E_n; F)$ then,
A is hyper right p-nuclear if and only if there are $(\psi_j)_j \in \ell_{p^*,w}(\mathcal{L}(E_1, \ldots, E_n))$ and $(y_j)_j \in \ell_p(F)$ such that

$$A(x_1, \cdots, x_n) = \sum_{j=1}^{\infty} \psi_j(x_1, \ldots, x_n) y_j$$

for all $(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n$. Moreover

$$\|A\|_{\mathcal{L}_{\mathcal{HN}_p^{right}}} = \inf \{ \|(\psi_j)_j\|_{p^*,w} \cdot \|(y_j)_j\|_p \},$$

Where the infimum is taken over all so called hiper right p-nuclear represents discribe above

Theorem 3.1.1 *Let* $p \in (0, \infty)$ *and* $p^* \in [1, \infty]$. *Then the class* $(\mathcal{HN}_p^{right}, \| \cdot \|_{\mathcal{HN}_p^{right}})$ *of hyper right* p*-nuclear multilinear operators is a* p*-Banach hyper-ideal*

Proof.

(i) It is plain that $\mathcal{I}_{\mathbb{K}^n} \in \mathcal{L}_{\mathcal{HN}_p^{right}}(\mathbb{K}^n; \mathbb{K})$. Regarding $\mathcal{I}_{\mathbb{K}^n}$ as a representation of itself it follows that $\|\mathcal{I}_{\mathbb{K}^n}\|_{\mathcal{L}_{\mathcal{HN}_p^{right}}} \leq \|\mathcal{I}_{\mathbb{K}^n}\|$. Assuming that $\|\mathcal{I}_{\mathbb{K}^n}\|_{\mathcal{L}_{\mathcal{HN}_p^{right}}} < 1$, there would exist a representation $\sum_{j=1}^{\infty} \lambda_j \otimes T_j$ of I_n with $\|(\lambda_j)_{j=1}^{\infty}\|_p \cdot \|(T_j)_{j=1}^{\infty}\|_{p^*,w} < 1$. As $\mathcal{L}_{\mathcal{HN}_p^{right}}(\mathbb{K}^n; \mathbb{K})$ is finit-dimensional, we have $\|(T_j)_{j=1}^{\infty}\|_{p^*,w} = \|(T_j)_{j=1}^{\infty}\|_{p^*}$. By holder's inequality

$$1 = |\mathcal{I}_{\mathbb{K}^n}(1, \ldots, 1)| \leq \sum_{j=1}^{\infty} |\lambda_j| \cdot \|T_j\| \cdot 1^n \leq \|(\lambda_j)_{j=1}^{\infty}\|_p \cdot \|(T_j)_{j=1}^{\infty}\|_{p^*} < 1,$$

a contradiction that gives $\|\mathcal{I}_{\mathbb{K}^n}\|_{\mathcal{L}_{\mathcal{HN}_p^{right}}} = 1$
(ii) Let $(A_j)_{j=1}^{\infty} \subset \mathcal{L}_{\mathcal{HN}_p^{right}}(E_1, \ldots, E_n; F)$. be such that $\sum_{j=1}^{\infty} \|A_j\|^p_{\mathcal{L}_{\mathcal{HN}_p^{right}}} < \infty$. Given

$\epsilon > 0$, for every $j \in \mathbb{N}$ ther are sequences $(\lambda_{jk})_{k=1}^{\infty} \in \ell_p$, $(T_{jk})_{k=1}^{\infty} \in \ell_{p^*,w}(\mathcal{L}(E_1,\dots,E_n))$ and $(y_{jk})_{k=1}^{\infty} \in \ell_{\infty}(F)$ such that $A_j = \sum_{j=1}^{\infty} \lambda_{jk} T_{jk} \otimes y_{jk}$ and

$$\|(\lambda_{jk})_{k=1}^{\infty}\|_p \cdot \|(T_{jk})_{k=1}^{\infty}\|_{p^*,w} \cdot \|(y_{jk})_{k=1}^{\infty}\|_{\infty} < (1+\varepsilon)\|A_j\|_{\mathcal{L}_{\mathcal{HN}_p^{right}}}.$$

We can assume, for each j, that $\|(y_{jk})_{k=1}^{\infty}\|_{\infty} = 1$ and

$$\|(\lambda_{jk})_{k=1}^{\infty}\|_p < ((1+\varepsilon)\|A_j\|_{\mathcal{L}_{\mathcal{HN}_p^{right}}})^{\frac{p}{s}}, \|(T_{jk})_{k=1}^{\infty}\|_{p^*,w} < ((1+\varepsilon)\|A_j\|_{\mathcal{L}_{\mathcal{HN}_p^{right}}})^{\frac{p}{r}}.$$

From

$$\sum_{j,k=1}^{\infty} |\lambda_{jk}|^p = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |\lambda_{jk}|^p = \sum_{j=1}^{\infty} \|(\lambda_{jk})_k\|_p^p < (1+\varepsilon)^p \cdot \sum_{j=1}^{\infty} \|A_j\|^p_{\mathcal{L}_{\mathcal{HN}_p^{right}}} < \infty \tag{2}$$

We conclude that $(\lambda_{jk})_{j,k}^{\infty} \in \ell_p$. For each linear functional $\varphi \in (\mathcal{L}(E_1,\dots,E_n))^*$ with $\|\varphi\| \le 1$, we have

$$\sum_{j,k=1}^{\infty} |\varphi(T_{jk})|^{p^*} \le \sum_{j=1}^{\infty} \|(T_{jk})_k\|^{p^*}_{p^*,w} < (1+\varepsilon)^p \sum_{j=1}^{\infty} \|A_j\|^p_{\mathcal{L}_{\mathcal{HN}_p^{right}}} < \infty \tag{3}$$

Therefor $(T_{jk})_{j,k=1}^{\infty} \in \ell_{p^*,w}(\mathcal{L}(E_1,\dots,E_n))$. We already know that $(y_{jk})_{j,k} \in \ell_{\infty}(F)$; so for all $x_1 \in E_1,\dots,x_n \in E_n$, the series

$$\sum_{j,k=1}^{\infty} \lambda_{jk} T_{jk} \otimes y_{jk}(x_1,\dots,x_n) \tag{4}$$

is absolutely converge in the banach space F. Then

$$\sum_{j,k=1}^{\infty} \lambda_{jk} T_{jk} \otimes y_{jk}(x_1,\dots,x_n) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \lambda_{jk} T_{jk} \otimes y_{jk}(x_1,\dots,x_n) = \sum_{j=1}^{\infty} A_j(x_1,\dots,x_n) =: A(x_1,\dots,x_n),$$

defines $A : E_1 \times \cdots \times E_n \to F$. and show that (4) it is representation of A as in (1) proving that A is hyper right p-nuclear. As $\|(y_{jk})_{j,k}\|_{\infty} = 1$, from (2) and (3) we get

$$\begin{aligned}
\|A\| &\le \|(\lambda_{jk})_{j,k=1}^{\infty}\|_p^p \cdot \|(T_{jk})_{j,k=1}^{\infty}\|^p_{p^*,w} \cdot \|(y_{jk})_{j,k=1}^{\infty}\|^p_{\infty} \\
&\le ((1+\epsilon)^p \cdot \sum_{j=1}^{\infty} \|A_j\|^p_{\mathcal{L}_{\mathcal{HN}_p^{right}}} \cdot)^{\frac{p}{p}} \cdot \left((1+\epsilon)^p \cdot \sum_{j=1}^{\infty} \|A_j\|^p_{\mathcal{L}_{\mathcal{HN}_p^{right}}} \cdot\right)^{\frac{p}{p^*}} \\
&= (1+\epsilon)^p \cdot \sum_{j=1}^{\infty} \|A_j\|^p_{\mathcal{L}_{\mathcal{HN}_p^{right}}}
\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain the desired inequality

(iii) Let $1 \leq m_1 < m_2 < \cdots < m_n$, and $A \in \mathcal{L}_{\mathcal{HN}_p^{right}}(E_1, \ldots, E_n; F)$, $B_1 \in \mathcal{L}(G_1, \ldots, G_{m_1}; E_1), \ldots, B_n \in \mathcal{L}(G_{m_{n-1}+1}, \ldots, G_{m_n}; E_n)$ and $v \in \mathcal{L}(F; H)$. We can write $A = \sum_{j=1}^{\infty} \lambda_j T_j \otimes y_j$, whith $(\lambda_j) \in \ell_p$ and $(T_j) \in \ell_{p^*,w}(\mathcal{L}(E_1, \ldots, E_n))$, $(y_j) \in \ell_\infty(F)$. Defining $S_j := T_j \circ (B_1, \ldots, B_n)$, and $z_j := v(y_j), \quad \forall j \in \mathbb{N}$

$$\Big(v \circ A \circ (B_1, \ldots, B_n)\Big)(x_1, \ldots, x_{m_n}) = \sum_{j=1}^{\infty} \lambda_j S_j \otimes z_j(x_1, \ldots, x_{m_n}),$$

for all $x_1, \in E_1, \ldots, x_n \in E_n$. It is clear that $(z_j)_j \in \ell_\infty(H)$ and $\|(z_j)\|_\infty \leq \|v\| \cdot \|(y_j)\|_\infty$. Giving a linear functional $\psi \in \mathcal{L}(G_1, \ldots, G_{m_n})^*$, considering the continuous linear operator

$$H_{(B_1,\ldots,B_n)} : \mathcal{L}(E_1, \ldots, E_n) \to \mathcal{L}(G_1, \ldots, G_{m_n})$$

defined by $\quad T \mapsto T \circ (B_1, \ldots, B_n)$.

as $\psi \circ H_{(B_1,\ldots,B_n)} \in \mathcal{L}(E_1, \ldots, E_n)^*$ and $(T_j) \in \ell_{p^*,w}(\mathcal{L}(E_1, \ldots, E_n))$, we have

$$\begin{aligned}
\sum_{j=1}^{\infty} |\psi(S_j)|^{p^*} &= \sum_{j=1}^{\infty} |\psi(T_j \circ (B_1, \ldots, B_n))|^{p^*} \\
&= \sum_{j=1}^{\infty} \left|\psi(H_{(B_1,\ldots,B_n)}(T_j)\right|^{p^*} \\
&= \sum_{j=1}^{\infty} \left|(\psi \circ H_{(B_1,\ldots,B_n)}(T_j)\right|^{p^*} \\
&\leq \|\psi \circ H_{(B_1,\ldots,B_n)}\| \cdot \left\|(T_j)_{j=1}^{\infty}\right\|_{p^*,w}^{p^*} < \infty.
\end{aligned}$$

This show that $(S_j) \in \ell_{p^*,w}(\mathcal{L}(G_1, \ldots, G_{m_n}))$. Hence $\sum_{j=1}^{\infty} \lambda_j S_j \otimes z_j$ is a representation of $v \circ A \circ (B_1, \ldots, B_n)$ as in (3). So $v \circ A \circ (B_1, \ldots, B_n)$ is hyper right p-nuclear and

$$\begin{aligned}
\|v \circ A \circ (B_1, \ldots, B_n)\|_{\mathcal{L}_{\mathcal{HN}_p^{right}}} &\leq \left\|(\lambda_j)_{j=1}^{\infty}\right\|_p \cdot \left\|(S_j)_{j=1}^{\infty}\right\|_{p^*,w} \cdot \left\|(z_j)_{j=1}^{\infty}\right\|_\infty \\
&\leq \left\|(\lambda_j)_{j=1}^{\infty}\right\|_p \cdot \left\|H_{(B_1,\ldots,B_n)}\right\| \cdot \left\|(T_j)_{j=1}^{\infty}\right\|_{p^*,w} \cdot \left\|(z_j)_{j=1}^{\infty}\right\|_\infty \\
&\leq \left\|(\lambda_j)_{j=1}^{\infty}\right\|_p \cdot \|B_1\| \cdots \|B_n\| \cdot \left\|(T_j)_{j=1}^{\infty}\right\|_{p^*,w} \cdot \left\|(z_j)_{j=1}^{\infty}\right\|_\infty \\
&\leq \|v\| \cdot \left(\left\|(\lambda_j)_{j=1}^{\infty}\right\|_p \cdot \left\|(T_j)_{j=1}^{\infty}\right\|_{p^*,w} \cdot \left\|(y_j)_{j=1}^{\infty}\right\|_\infty\right) \cdot \|B_1\| \cdots \|B_n\|.
\end{aligned}$$

Taking the infimum over all hyper right p-nuclear representations of A we have

$$\|v \circ A \circ (B_1, \ldots, B_n)\|_{\mathcal{L}_{\mathcal{HN}_p^{right}}} \leq \|v\| \cdot \|A\|_{\mathcal{L}_{\mathcal{HN}_p^{right}}} \cdot \|B_1\| \cdots \|B_n\|.$$

■

3.2 Hyper nuclear multilinear operators

By taking $p = 1$ in the general definition of right p-nuclear operators, we obtain the ideal of hyper-nuclear operators as follows.

Definition 3.2.1 *An n-linear continuous operator $A \in \mathcal{L}(E_1, \dots, E_n; F)$ is called hyper nuclear if there exist sequences $(\lambda_j)_{j=1}^{\infty} \in \ell_1$, $(T_j)_{j=1}^{\infty} \in \ell_\infty(\mathcal{L}(E_1, \dots, E_n))$, and $(y_j)_{j=1}^{\infty} \in \ell_\infty(F)$ such that*

$$A(x_1, \dots, x_n) = \sum_{j=1}^{\infty} \lambda_j T_j(x_1, \dots, x_n) y_j,$$

for all $(x_1, \dots, x_n) \in E_1 \times \dots \times E_n$. In this case, we write $A \in \mathcal{L}_{\mathcal{HN}}(E_1, \dots, E_n; F)$.

The hyper nuclear norm is defined by

$$\|A\|_{\mathcal{L}_{\mathcal{HN}}} = \inf \left\{ \|(\lambda_j)\|_1 \cdot \|(T_j)\|_\infty \cdot \|(y_j)\|_\infty \right\},$$

where the infimum is taken over all representations of A.

Corollary 3.2.1 *The class $(\mathcal{L}_{\mathcal{HN}}, \|\cdot\|_{\mathcal{HN}})$ of hyper-nuclear multilinear operators is a Banach hyper-ideal and $\mathcal{L}_{\mathcal{N}} \subset \mathcal{L}_{\mathcal{HN}}$.*

Proof. The inclusion is obvious and the classes are different because $\mathcal{L}_{\mathcal{N}}$ is not a hyper-ideal.
To show that $\mathcal{L}_{\mathcal{HN}}$ is the smallest Banach hyper-ideal we need the following characterization of the hyper-nuclear norm ■

Lemma 3.2.1 *For every operator $A \in \mathcal{L}_{\mathcal{HN}}(E_1, \dots, E_n; F)$,*

$$\|A\|_{\mathcal{L}_{\mathcal{HN}}} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \cdot \|T_j\| \cdot \|y_j\| \right\},$$

where the infimum is taken over all representations of A as in (1).

Proof. Let $A \in \mathcal{L}_{\mathcal{HN}}(E_1, \dots, E_n; F)$ be an operator admitting a representation of the form:

$$A = \sum_{j=1}^{\infty} \lambda_j T_j \otimes y_j,$$

where $\lambda_j \in \mathbb{K}$, $T_j \in \mathcal{L}(E_1, \ldots, E_n)$, and $y_j \in F$ for each $j \in \mathbb{N}$. Without loss of generality, assume that $T_j \neq 0$ and $y_j \neq 0$. We rewrite this representation in a normalized form by defining

$$\mu_j := \lambda_j \|T_j\| \cdot \|y_j\|, \quad S_j := \frac{T_j}{\|T_j\|}, \quad z_j := \frac{y_j}{\|y_j\|}.$$

These definitions are valid since $\|T_j\| \neq 0$ and $\|y_j\| \neq 0$ for all j. Then

$$\mu_j S_j \otimes z_j = \lambda_j \|T_j\| \cdot \|y_j\| \cdot \left(\frac{T_j}{\|T_j\|} \otimes \frac{y_j}{\|y_j\|} \right) = \lambda_j T_j \otimes y_j.$$

Therefore

$$\sum_{j=1}^{\infty} \mu_j S_j \otimes z_j = \sum_{j=1}^{\infty} \lambda_j T_j \otimes y_j = A,$$

which shows that the new representation produces the same operator A, and is thus a representation of the same type as in (1).

Note that S_j and z_j are normalized, that is

$$\|S_j\| = \left\| \frac{T_j}{\|T_j\|} \right\| = 1, \quad \|z_j\| = \left\| \frac{y_j}{\|y_j\|} \right\| = 1,$$

and therefore

$$\sup_j \|S_j\| = 1, \quad \sup_j \|z_j\| = 1.$$

Also,

$$\sum_{j=1}^{\infty} |\mu_j| = \sum_{j=1}^{\infty} |\lambda_j| \cdot \|T_j\| \cdot \|y_j\|.$$

Using the definition of the hyper-nuclear norm, we get

$$\|A\|_{\mathcal{L}_{\mathcal{HN}}} \leq \left(\sum_{j=1}^{\infty} |\mu_j| \right) \cdot \sup_j \|S_j\| \cdot \sup_j \|z_j\| = \sum_{j=1}^{\infty} |\lambda_j| \cdot \|T_j\| \cdot \|y_j\|.$$

Since this holds for every representation of the form (1), it follows that

$$\|A\|_{\mathcal{L}_{\mathcal{HN}}} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \cdot \|T_j\| \cdot \|y_j\| \right\},$$

where the infimum is taken over all possible representations of A as in (1). This completes the proof.

■

Theorem 3.2.1 *The class* $(\mathcal{L}_{\mathcal{HN}}, \|\cdot\|_{\mathcal{L}_{\mathcal{HN}}})$ *of hyper-nuclear multilinear operators is the smallest Banach hyper-ideal, in the sense that if* $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ *is a Banach hyper-ideal, then*

$$\mathcal{L}_{\mathcal{HN}} \subseteq \mathcal{H} \quad \text{and} \quad \|\cdot\|_{\mathcal{H}} \leq \|\cdot\|_{\mathcal{L}_{\mathcal{HN}}}.$$

Proof. Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ be a Banach hyper-ideal. Consider $A \in \mathcal{L}_{\mathcal{HN}}(E_1, \ldots, E_n; F)$ with a representation

$$A = \sum_{j=1}^{\infty} \lambda_j T_j \otimes y_j$$

as in (3). Given any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{j=k}^{\infty} |\lambda_j| \cdot \|T_j\| \cdot \|y_j\| < \varepsilon \tag{2}$$

for all $k \geq k_0$. Since $\mathcal{H}$ is a hyper-ideal, by Proposition 2.2.2 we have that

$$B_k := \sum_{j=1}^{k} \lambda_j T_j \otimes y_j \in \mathcal{L}_F(E_1, \ldots, E_n; F) \subseteq \mathcal{H}(E_1, \ldots, E_n; F)$$

for every $k \in \mathbb{N}$.

For every $k > i \geq k_0$, Proposition 2.2.3 yields

$$\begin{aligned} \|B_k - B_i\|_{\mathcal{H}} &= \left\| \sum_{j=i+1}^{k} \lambda_j T_j \otimes y_j \right\|_{\mathcal{H}} \\ &\leq \sum_{j=i+1}^{k} |\lambda_j| \cdot \|T_j \otimes y_j\|_{\mathcal{H}} \\ &= \sum_{j=i+1}^{k} |\lambda_j| \cdot \|T_j\| \cdot \|y_j\| \\ &< \varepsilon, \end{aligned}$$

Showing that $(B_k)_{k=1}^{\infty}$ is a Cauchy sequence in the Banach space $\mathcal{H}(E_1, \ldots, E_n; F)$. Then there is $B \in \mathcal{H}(E_1, \ldots, E_n; F)$ such that

$$B_k \xrightarrow{\|\cdot\|_{\mathcal{H}}} B.$$

From $\|\cdot\| \leq \|\cdot\|_{\mathcal{H}}$, so

$$B_k \to B \quad \text{in } \|\cdot\|.$$

By (2) it follows easily that

$$B_k \to A \quad \text{in } \|\cdot\|,$$

thus $A = B$. Hence

$$A \in \mathcal{H}(E_1, \ldots, E_n; F),$$

Proving that $\mathcal{L}_{\mathcal{HN}} \subseteq \mathcal{H}$. Using Proposition 2.2.3 once again.

$$\|A\|_{\mathcal{H}} \leq \sum_{j=1}^{\infty} \|\lambda_j T_j \otimes y_j\|_{\mathcal{H}} = \sum_{j=1}^{\infty} |\lambda_j| \cdot \|T_j \otimes y_j\|_{\mathcal{H}} = \sum_{j=1}^{\infty} |\lambda_j| \cdot \|T_j\| \cdot \|y_j\|.$$

Taking the infimum over all hyper-nuclear representations of A, from Lemma 3.2.1 it follows that

$$\|A\|_{\mathcal{H}} \leq \|A\|_{\mathcal{L}_{\mathcal{HN}}}.$$

■

3.3 Composition ideals and factorization theorem

Using the notion of composition, we show that many important multi-ideals are hyper-ideals, for example the huper right p-nuclear multilinear operators .

Definition 3.3.1 *(Composition ideal). Given an operator ideal $\mathcal{I}$, an n-linear operator*

$$A \in \mathcal{L}(E_1, \ldots, E_n; F)$$

belongs to the composition ideal $\mathcal{I} \circ \mathcal{L}$, in symbols, $A \in \mathcal{I} \circ \mathcal{L}(E_1, \ldots, E_n; F)$, if there exist a Banach space G, a linear operator $u \in \mathcal{I}(G; F)$, and an n-linear operator $B \in \mathcal{L}(E_1, \ldots, E_n; G)$ such that

$$A = u \circ B.$$

If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a p-normed operator ideal, $0 < p \leq 1$, we define

$$\|A\|_{\mathcal{I} \circ \mathcal{L}} = \inf\{\|u\|_{\mathcal{I}} \cdot \|B\|\},$$

where the infimum is taken over all factorizations $A = u \circ B$ with u belonging to $\mathcal{I}$.

It is well known [7] that $(\mathcal{I}\circ\mathcal{L}, \|\cdot\|_{\mathcal{I}\circ\mathcal{L}})$ is a p-normed (respectively, p-Banach) multi-ideal whenever $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a p-normed (respectively, p-Banach) operator ideal.

Theorem 3.3.1 *If $\mathcal{I}$ is an operator ideal, then $\mathcal{I}\circ\mathcal{L}$ is a hyper-ideal. If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a p-normed (respectively, p-Banach) operator ideal, then $(\mathcal{I}\circ\mathcal{L}, \|\cdot\|_{\mathcal{I}\circ\mathcal{L}})$ is a p-normed (respectively, p-Banach) hyper-ideal. In particular, if $\mathcal{I}$ is a closed operator ideal, then $\mathcal{I}\circ\mathcal{L}$ is a closed hyper-ideal.*

Proof. As we know that $\mathcal{I}\circ\mathcal{L}$ is a (p-normed, p-Banach) multi-ideal, all that is left to be checked is the hyper-ideal property. Let

$$B_1 \in \mathcal{L}(G_1,\ldots,G_{m_1};E_1), \quad \ldots, \quad B_n \in \mathcal{L}(G_{m_{n-1}+1},\ldots,G_{m_n};E_n),$$

where $m_1 < \cdots < m_n$,

$$A \in \mathcal{I}\circ\mathcal{L}(E_1,\ldots,E_n;F), \quad \text{and} \quad v \in \mathcal{L}(F;H).$$

We can write $A = v\circ C$,

where

$$C \in \mathcal{L}(E_1,\ldots,E_n;F_1) \quad \text{and} \quad w \in \mathcal{I}(F_1;F).$$

Defining

$$u := v\circ w \in \mathcal{I}(F_1;H) \quad \text{and} \quad D := C\circ(B_1,\ldots,B_n) \in \mathcal{L}(G_1,\ldots,G_{m_n};F_1),$$

it follows that

$$\begin{aligned} v\circ A\circ(B_1,\ldots,B_n) &= v\circ(w\circ C)\circ(B_1,\ldots,B_n) \\ &= (v\circ w)\circ\big(C\circ(B_1,\ldots,B_n)\big) \\ &= u\circ D, \end{aligned}$$

which proves that

$$v\circ A\circ(B_1,\ldots,B_n) \in \mathcal{I}\circ\mathcal{L}(G_1,\ldots,G_{m_n};H).$$

Furthermore,

$$\begin{aligned}\|v \circ A \circ (B_1, \ldots, B_n)\|_{\mathcal{I}\circ\mathcal{L}} &= \|(v \circ w) \circ (C \circ (B_1, \ldots, B_n))\|_{\mathcal{I}\circ\mathcal{L}} \\ &\leq \|v \circ w\|_{\mathcal{I}} \cdot \|C \circ (B_1, \ldots, B_n)\| \\ &\leq \|v\| \cdot \Big(\|w\|_{\mathcal{I}} \cdot \|C\|\Big) \cdot \|B_1\| \cdots \|B_n\|.\end{aligned}$$

Taking the infimum over all possible factorizations of $A = w \circ C$, we get

$$\|v \circ A \circ (B_1, \ldots, B_n)\|_{\mathcal{I}\circ\mathcal{L}} \leq \|v\| \cdot \|A\|_{\mathcal{I}\circ\mathcal{L}} \cdot \|B_1\| \cdots \|B_n\|.$$

The last assertion follows from the former ones and [7, Corollary 3.8].

■

We denote by $E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$ the completed projective tensor product of the Banach spaces $E_1, ..., E_n$. Given $S \in \mathcal{L}(E_1, ..., E_n; F)$, we consider its linearization $S_L \in \mathcal{L}(E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n; F)$, defined by

$$S_L(x_1 \otimes \cdots \otimes x_n) := S(x_1, ..., x_n), \quad \text{for all } (x_1, ..., x_n) \in E_1 \times \cdots \times E_n.$$

This linearization S_L is unique and satisfies $\|S\| = \|S_L\|$. In other words, the multilinear map S factors through the canonical continuous multilinear map

$$\sigma_n : E_1 \times \cdots \times E_n \to E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n, \quad \sigma_n(x_1, ..., x_n) := x_1 \otimes \cdots \otimes x_n y,$$

via $S = S_L \circ \sigma_n$. For more on the theory of topological tensor products, we refer the reader to (see [10, 23]).

Theorem 3.3.2

$$\mathcal{HN}_p^{right} = \mathcal{N}_p^{right} \circ \mathcal{L} \quad \textit{isometrically, and in particular,} \quad \mathcal{HN} = \mathcal{N} \circ \mathcal{L} \quad \textit{isometrically.}$$

Proof. Let $E_1, \ldots, E_n, F$ be Banach spaces and let $A \in \mathcal{HN}_p^{right} \circ \mathcal{L}(E_1, \ldots, E_n; F)$ and $\varepsilon > 0$. Then there exists a Banach space G and operators $B \in \mathcal{L}(E_1, \ldots, E_n; G)$ and $u \in \mathcal{N}_p^{right}(G; F)$ such that $A = u \circ B$. We can take sequences $(\lambda_j)_{j=1}^\infty \in \ell_p$, $(y_j)_{j=1}^\infty \in \ell_\infty(F)$, and $(w_j^*)_{j=1}^\infty \in \ell_{p^*,w}(G^*)$ such that

$$u = \sum_{j=1}^\infty \lambda_j w_j^* \otimes y_j.$$

and

$$\|(\lambda_j)_{j=1}^{\infty}\|_p \cdot \|(w_j^*)_{j=1}^{\infty}\|_{p^*,w} \cdot \|(y_j)_{j=1}^{\infty}\|_{\infty} \leq (1+\epsilon)\|u\|_{\mathcal{N}_p^{\text{right}}} \tag{3}$$

For any $(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n$, define

$$A(x_1, \ldots, x_n) = \sum_{j=1}^{\infty} \lambda_j \Big(w_j^* \circ B\Big) \otimes y_j(x_1, \ldots, x_n).$$

it is clear that

$$B^* : G^* \to \mathcal{L}(E_1, \ldots, E_n), \quad B^*(w^*)(x_1, \ldots, x_n) = w^*(B(x_1, \ldots, x_n)).$$

is a bounded linear operator and $\|B^*\| = \|B\|$. Therefore,

$$(w_j^* \circ B)_{j=1}^{\infty} = (B^*(w_j^*))_{j=1}^{\infty} \in \ell_{p^*,w}(\mathcal{L}(E_1, \ldots, E_n)),$$

and by [1, Exercice 6, page 27]

$$\|(w_j^* \circ B)_{j=1}^{\infty}\|_{p^*,w} \leq \|B\| \cdot \|(w_j^*)_{j=1}^{\infty}\|_{p^*,w} \quad .$$

It follows that

$$A = \sum_{j=1}^{\infty} \lambda_j (w_j^* \circ B) \otimes y_j$$

is a hyper right p-nuclear representation for A. Thus,

$$A \in \mathcal{HN}_p^{right}(E_1, \ldots, E_n; F),$$

and letting $\varepsilon \to 0$, we get

$$\|A\|_{\mathcal{HN}_p^{right}} \leq \|B\| \cdot \|u\|_{\mathcal{N}_p^{right}}.$$

Taking the infimum over all such factorizations $A = u \circ B$, we conclude that

$$\|A\|_{\mathcal{HN}_p^{right}} \leq \|A\|_{\mathcal{HN}_p^{right} \circ \mathcal{L}}$$

Conversely, given $A \in \mathcal{HN}_p^{right}(E_1, \ldots, E_n; F)$ and $\varepsilon > 0$, there exist sequences $(\lambda_j)_{j=1}^{\infty} \in \ell_p$, $(B_j)_{j=1}^{\infty} \in \ell_{p^*,w}(\mathcal{L}(E_1, \ldots, E_n))$, and $(y_j)_{j=1}^{\infty} \in \ell_{\infty}(F)$ such that

$$A = \sum_{j=1}^{\infty} \lambda_j B_j \otimes y_j,$$

and for all $(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n$,

$$A(x_1, \ldots, x_n) = \sum_{j=1}^{\infty} \lambda_j B_j(x_1, \ldots, x_n) \otimes y_j,$$

with

$$\|(\lambda_j)_{j=1}^{\infty}\|_p \cdot \|(B_j)_{j=1}^{\infty}\|_{p^*,w} \cdot \|(y_j)_{j=1}^{\infty}\|_{\infty} \leq (1+\varepsilon)\|A\|_{\mathcal{HN}_p^{right}}.$$

from

$$\begin{aligned}\left\| A - \sum_{j=1}^{m} \lambda_j B_j \otimes y_j \right\|_{\mathcal{HN}_p^{right}} &= \left\| \sum_{j=m+1}^{\infty} \lambda_j B_j \otimes y_j \right\|_{\mathcal{HN}_p^{right}} \\ &\leq \left\|(\lambda_j)_{j=m+1}^{\infty}\right\|_p \cdot \left\|(B_j)_{j=m+1}^{\infty}\right\|_{p^*,w} \cdot \left\|(y_j)_{j=m+1}^{\infty}\right\|_{\infty} \\ &\leq \left\|(B_j)_{j=1}^{\infty}\right\|_{p^*,w} \cdot \left\|(y_j)_{j=1}^{\infty}\right\|_{\infty} \cdot \left\|(\lambda_j)_{j=m+1}^{\infty}\right\|_p \\ &\xrightarrow[m\to\infty]{} 0\end{aligned}$$

We have that

$$A = \sum_{j=1}^{\infty} \lambda_j B_j \otimes y_j$$

in the norm $\|\cdot\|_{\mathcal{HN}_p^{right}}$. Since $\|\cdot\| \leq \|\cdot\|_{\mathcal{HN}_p^{right}}$ because $\mathcal{HN}_p^{right}$ is a Banach hyper-ideal, the convergence also occurs in the usual operator norm on $\mathcal{L}(E_1, \ldots, E_n)$. Since the correspondence

$$B \in \mathcal{L}(E_1, \ldots, E_n; F) \longmapsto B_L \in \mathcal{L}(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n; F)$$

is an isomorphism, hence linear and continuous, it follows that

$$A_L = \left(\sum_{j=1}^{\infty} \lambda_j B_j \otimes y_j \right)_L = \sum_{j=1}^{\infty} (\lambda_j B_j \otimes y_j)_L = \sum_{j=1}^{\infty} \lambda_j (B_j)_L \otimes y_j \tag{4}$$

In the space

$$\mathcal{L}(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n; F).$$

we now apply the isometric isomorphism

$$B \in \mathcal{L}(E_1, \ldots, E_n) \mapsto B_L \in (E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n)^*$$

and invoke once again the result from [1, Exercice 6, page 27] to obtain:

$$((B_j)_L)_{j=1}^{\infty} \in \ell_{p^*,w}((E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n)^*) \quad \text{and} \quad \|((B_j)_L)_{j=1}^{\infty}\|_{p^*,w} = \|(B_j)_j\|_{p^*,w}.$$

This proves that (4) is an right p-nuclear representation for A_L and

$$\begin{aligned}\|A_L\|_{\mathcal{N}_p^{right}} &\leq \|(\lambda_j)_{j=1}^{\infty}\|_p \cdot \|((B_j)_L)_{j=1}^{\infty}\|_{p^*,w} \cdot \|(y_j)_{j=1}^{\infty}\|_{\infty} \\ &= \|(\lambda_j)_{j=1}^{\infty}\|_p \cdot \|(B_j)_{j=1}^{\infty}\|_{p^*,w} \cdot \|(y_j)_{j=1}^{\infty}\|_{\infty} \\ &\leq (1+\varepsilon)\|A\|_{\mathcal{HN}_p^{right}}.\end{aligned}$$

Therefore,

$$A_L \in \mathcal{N}_p^{right}(E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n; F)$$

and, letting $\varepsilon \to 0$,

$$\|A_L\|_{\mathcal{N}_p^{right}} \leq \|A\|_{\mathcal{HN}_p^{right}}.$$

By [7, Propositions 3.2 and 3.7], we conclude that

$$A \in \mathcal{N}_p^{right} \circ \mathcal{L}(E_1, \ldots, E_n; F)$$

and

$$\|A\|_{\mathcal{N}_p^{right} \circ \mathcal{L}} \leq \|A\|_{\mathcal{HN}_p^{right}}.$$

is linear and continuous, This completes the desired dual inequality ■

Theorem 3.3.3 *[17, Page 215] Let E and F be a Banach spaces, $T \in \mathcal{N}_p^{right}(E, F)$ if and only if T has a factorization $T = RD_\lambda\tilde{S}$ such that the following diagram commutes:*

$$\begin{array}{ccc} E & \longrightarrow & F \\ \tilde{S}\downarrow & & \uparrow R \\ \ell_{p^*} & \underset{D_\lambda}{\longrightarrow} & \ell_1 \end{array} \qquad (3.3.1)$$

where $D_\lambda : \ell_{p^*} \longrightarrow \ell_1$ is the diagonal operator, $\tilde{S} \in \mathcal{L}(E, \ell_{p^*})$ and $R \in \mathcal{L}(\ell_1, F)$. Moreover,

$$\|T\|_{\mathcal{N}_p^{right}} := \inf \|R\|.\|\lambda\|_p.\|\tilde{S}\|,$$

where the infimum is taken over all the above factorizations.

From Theorem 3.3.3 and Theorem 3.3.2, we have the factorization theorem of hyper right p-nuclear multilinear operators.

Theorem 3.3.4 *Let $A \in \mathcal{L}(E_1, ..., E_n; F)$, $A \in \mathcal{HN}_p^{right}(E_1, ..., E_n; F)$ if and only if A has a factorization $A = RD_\lambda S$ such that the following diagram commutes:*

$$\begin{array}{ccc} E_1 \times \cdots \times E_n & \longrightarrow & F \\ S \downarrow & & \uparrow R \\ \ell_{p^*} & \underset{D_\lambda}{\longrightarrow} & \ell_1 \end{array} \tag{3.3.2}$$

where $D_\lambda : \ell_{p^*} \longrightarrow \ell_1$ is the diagonal operator, $R \in \mathcal{L}(E_1 \times \cdots \times E, \ell_{p^*})$ and $S \in \mathcal{L}(\ell_1, F)$. Moreover,

$$\|A\|_{\mathcal{HN}_p^{right}} := \inf \|R\|.\|\lambda\|_p.\|S\|,$$

where the infimum is taken over all the above factorizations.

Proof. Let $A \in \mathcal{HN}_p^{right}(E_1, \ldots, E_n; F)$. By Theorem 3.3.2, $A_L \in \mathcal{N}_p^{right}(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n; F)$. It follows from Theorem 3.3.3 that $A_L = R \circ D_\lambda \circ \widetilde{S}$, $\lambda = (\lambda_n)_n \in \ell_{p^*}$, $R \in \mathcal{L}(\ell_1, F)$ and $\widetilde{S} \in \mathcal{L}(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n, \ell_{p^*})$ such that for all $\epsilon > 0$, we get

$$\|R\|.\|\lambda\|_p.\|\widetilde{S}\| \leq (1 + \epsilon)\|A_L\|_{\mathcal{N}_p^{right}}.$$

Hence $A = A_L \circ \sigma_n = R \circ D_\lambda \circ S$ and $S = \widetilde{S} \circ \sigma_n \in \mathcal{L}(E_1, \ldots, E_n, \ell_{p^*})$ and

$$\|R\|.\|\lambda\|_p.\|S\| = \|R\|.\|\lambda\|_p.\|\widetilde{S}\| \leq (1 + \epsilon)\|A_L\|_{\mathcal{N}_p^{right}r} = (1 + \epsilon)\|A\|_{\mathcal{HN}_p^{right}}$$

Letting $\epsilon \longmapsto 0$, we get $\inf \|R\|.\|\lambda\|_p.\|S\| \leq \|A\|_{\mathcal{HN}_p^{right}}$.

Conversely, let A has a factorization $RD_\lambda S$, where $S \in \mathcal{L}(E_1, \ldots, E_n, \ell_{p^*})$ and $R \in \mathcal{L}(\ell_1, F)$. Then $S_L \in \mathcal{L}(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n, \ell_{p^*})$ and $\|S_L\| = \|S\|$. Therefore $A_L = RD_\lambda S_L$ is right p-nuclear and

$$\|A_L\|_{\mathcal{N}_p^{right}} \leq (1 + \epsilon)\|R\|.\|\lambda\|_p.\|S_L\| = (1 + \epsilon)\|R\|.\|\lambda\|_p.\|S\|, \text{for } \epsilon > 0$$

Letting $\epsilon \longmapsto 0$, we get $\|A\|_{\mathcal{HN}_p^{right}} \leq \inf \|R\|.\|\lambda\|_p.\|S\|$.

By Theorem 3.3.2, we conclude that

$$A \in \mathcal{HN}_p^{right}(E_1, \ldots, E_n; F)$$

and

$$\|A\|_{\mathcal{HN}_p^{right}} \leq \inf \|R\|.\|\lambda\|_p.\|S\|.$$

This completes the desired dual inequality ■

Conclusion

The study of Banach (or s-Banach) ideals of multilinear operators has evolved significantly, with notable contributions in nuclear, strongly summing, and dominated multilinear operators. Building on Popa s questions about operator composition, Botelho and Torres introduced Hyper-Banach ideals, a refined framework incorporating stability under composition. In this thesis, we examine Hyper p-nuclear multilinear operators as a key example, demonstrating their properties including linearization and ℓ^p-factorization while proposing a natural extension to the existing theory. Our three-chapter structure systematically reviews foundational concepts, develops the Hyper-ideal framework, and applies it to p-nuclear operators, bridging abstract theory with concrete analytical tools. This investigation not only clarifies distinctions from classical ideals but also opens avenues for further research in operator composition and factorization.

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ملخص

تدرس هذه المذكرة المُثُل العليا الفائقة للمُؤثرات متعددة الخطية، وهو تعريف وضعه بوتيلو وتوريس، والذي يوسّع المثُل الكلاسيكية المتعددة الخطية لبيتش من خلال دمج خصائص التركيب للمؤثرات المتعددة الخطية. ندرس نظريتهما العامة، ونُركز، كمثال رئيسي، على المُؤثرات الفائقة متعددة الخطية نيكلير. بالنسبة لهذه المؤثرات، نُبرهن نظرية الخطية ونُثبت تفكيكا متعدد الخطية عبر فضاءات المتتاليات الجمعية.

كلمات مفتاحية

المؤثرات متعددة الخطية،المُثُل العليا الفائقة ، مُثُل بيتش، المؤثرات الفائقة متعددة الخطية نيكلير.

Abstract

This memory examines hyper-ideals of multilinear operators, a framework introduced by Botelho and Torres that extends classical Pietsch multi ideals by incorporating composition properties. We study their general theory and, as a principal example, focus on hyper right p-nuclear multilinear operators. For these operators, we establish the linearization theorem and prove a decomposition characterized by mappings through ℓ^p-spaces via a linear operator followed by a multilinear operator.

Key words

Hyper-ideals, Multilinear operators, multi ideals-ideals, Hyper p-nuclear multilinear operators.

Résumé

Ce mémoire étudie les hyper-idéaux d'opérateurs multilinéaires, un cadre introduit par Botelho et Torres qui étend les idéaux multilinéaire classiques de Pietsch en incorporant des propriétés de composition. Nous examinons leur théorie générale et, comme exemple principal, nous concentrons sur les opérateurs multilinéaires hyper p-nucléaires. Pour ces opérateurs, nous établissons le théorème de linéarisation et démontrons une décomposition caractérisée par des applications à travers des espaces ℓ^p via un opérateur linéaire suivi d'un opérateur multilinéaire.

Mot-clés Hyper-idéal, Opérateurs multilinéaires, Idéaux multilinéaires , Hyper p-nucléaires.