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Theme

The $K_{p,q}$ -Compactness and $K_{p,q}$ -Null Sequences

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To my parents, whose unwavering support and encouragement have been my guiding light throughout this journey. Your sacrifices and love have made this achievement possible.

I dedicate this memory to my husband, who was a support to me. Your love and encouragement .
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To my friends and loved ones in this life.

To you, dear reader.

[Fairouz Bouziane]

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Notations

| | |
|-----------------------------|--|
| \mathbb{K} | The field of real or complex numbers. |
| \mathbb{R}_+ | The field of non negative real numbers. |
| p' | The conjugate of the number p ($1 \leq p \leq \infty$), that is $\frac{1}{p} + \frac{1}{p'} = 1$ |
| E^* | The topological dual of E . |
| B_E | The closed unit ball of E |
| $L(E; F)$ | The set of all linear operators. |
| $\mathcal{L}(E; F)$ | The sets of all continuous linear operators. |
| w | The weak topology. |
| w^* | The weak * topology. |
| \mathcal{I} | The ideal of all linear operator. |
| \mathcal{I}^{inj} | The injective of the linear operators ideal \mathcal{I} . |
| \mathcal{I}^{sur} | The surjective of the linear operators ideal \mathcal{I} . |
| $m_{\mathcal{I}}(K, X)$ | measure the size of \mathcal{I} -compactness |
| T^* | The adjoint operator of T . |
| \mathcal{K} | The set of all compact linear operators. |
| $\mathcal{K}_{\mathcal{I}}$ | The set of all Ideal compact linear operators. |
| \mathcal{W} | The set of all weakly compact linear operators. |
| \mathcal{L}_f | The set of all finite rank linear operators. |
| ℓ_p^u | unconditionally p -null sequences |
| $c_{0, \mathcal{I}}$ | \mathcal{I} -null sequences |

Introduction

In 1955, A. Grothendieck [9] described relatively compact subsets of a Banach space as sets contained in the closed absolute convex hull of a norm null sequence. Nowadays this result is called the Grothendieck compactness principle. A linear operator is said to be compact if it maps bounded sets to relatively compact ones.

Carl and Stephani(1984)[4] introduced a new type of compact sets called *relatively \mathcal{I} -compact set*. A subset B of a Banach space E is *relatively \mathcal{I} -compact* if there is a Banach space G and an operator $S \in \mathcal{I}(G, E)$ such that $B \subseteq S(M)$, where M is a compact subset of G . The new class of sets is formed by \mathcal{I} -compact sets. A linear operator T between Banach spaces E and F is called *\mathcal{I} -compact* if $T(B_E)$ is a relatively \mathcal{I} -compact subset of F .

The operator ideal formed by all linear \mathcal{I} -compact operators is denoted by $\mathcal{K}_{\mathcal{I}}$ The new class of sets is formed by \mathcal{I} -compact sets.

Lassalle and Turco [15] introduced a way to measure the size of \mathcal{I} -compact sets. For an \mathcal{I} -compact subset K of X ,

$$m_{\mathcal{I}}(K; X) := \inf \{ \|\mathcal{U}\|_{\mathcal{I}} : \mathcal{U} \in \mathcal{I}(Z, X), \text{relatively compact } C \subset B_Z, K \subset \mathcal{U}(C) \}.$$

where B_Z is the unit ball of a Banach space Z .

In 2002, a form of compactness was introduced by Sinha and Karn [19]. Let $1 \leq p \leq \infty$. A subset K of Banach space X is relatively p -compact if there exists $(x_n) \in \ell_p(X)$ such that

$$K \subset p\text{-conv}\{(x_n)_n\} = \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_p^*} \right\}$$

Galicer, Lassalle and Turco [8] introduced a notion to measure the size of p -compact sets. For a p -compact subset K of X :

$$m_p(K; X) := \inf \{ \|(x_n)_n\|_p : (x_n)_n \in \ell_p(X), K \subset p\text{-conv}\{(x_n)_n\} \}$$

and $m_p(K, X) = \infty$ if K is not p -compact. and it is shown that K is p -compact if and only if K is $K_{1,p}$ -compact. In this case, $m_p(K; X) = m_{K_{1,p}}(K; X)$.

Kim in [11] introduced the notion of unconditionally p -compact. A subset K of X is unconditionally p -compact when $\ell_p(X)$ is replaced by $\ell_p^u(X)$ in the definition of p -compact set. It was shown in [12](Proposition 5.2) that K is unconditionally p -compact if and only if K is $K_{p^*,p}$ -compact, in this case:

$$m_{K_{p^*,p}}(K; X) = \inf \left\{ \|(x_n)_n\|_p^w : (x_n)_n \in \ell_p^u(X), K \subset p\text{-conv}(x_n)_n \right\}$$

Let $0 < r \leq \infty, 1 \leq p, q \leq \infty$ and $1 + 1/r = 1/p + 1/q$. A linear operator $T \in \mathcal{L}(E, F)$ is called (p, q) -compact (see [17, page 253]) if

$$T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n$$

with $(\lambda_n) \in \ell_r, (x_n^*)_n \in \ell_{q^*}^w(E^*)$, and $(y_n)_n \in \ell_p^w(F)$. In the case $r = \infty$ let us suppose that $(\lambda_i) \in c_0$.

The class of all (p, q) -compact operators is denoted by $K_{p,q}$ and it is a Banach linear operator ideal. It is well known that $K_{p,1}$ is the ideal of p -nuclear operators and K_{p,p^*} is the ideal of p -compact operators.

Using the ideal of $K_{p,q}$ -compact operator Kim in [13] extended a new result to the $K_{p,q}$ -compactness. Let X be a Banach space, by $c_0(X)$ we denote the space of all null sequence that is

$$c_0(X) = \left\{ (x_n)_n, x_n \in X : \lim_{n \rightarrow \infty} \|x_n\| = 0 \right\}$$

Piñeiro and Delgado [18] introduced an extended concept of null sequences.

For $1 \leq p \leq \infty$, sequence $(x_n)_n$ in a Banach space, X is called p -null (in symbol $(x_n)_n \in c_{0,p}(X)$) if for every $\varepsilon > 0$, there exist a natural number n_ε and $(z_k)_k \in \varepsilon B_{\ell_p}(x)$ such that:

$$x_n \in p\text{-conv}(z_k)_k$$

for all $n \geq n_\varepsilon$.

For a Banach operator ideal \mathcal{I} and a Banach space X , a sequence $(x_n)_n$ in X is called \mathcal{I} -null if there exist a Banach space Z and $S \in \mathcal{I}(Z, X)$ such that for every $\varepsilon > 0$, there is a natural number

n_ε so that:

$$x_n \in \varepsilon S(B_Z)$$

for all $n \geq n_\varepsilon$. Let $c_{0,\mathcal{I}}(X)$ be the space of all \mathcal{I} -null sequences in X . Since \mathcal{I} -null sequences are \mathcal{I} -compact (see [4]), Lassalle and Turco [16] defined a norm on $c_{0,\mathcal{I}}(X)$ by:

$$\|(x_n)_n\|_{c_{0,\mathcal{I}}(X)} := m_{\mathcal{I}}(\{x_n\}_n; X).$$

Then $(c_{0,\mathcal{I}}(X), \|\cdot\|_{c_{0,\mathcal{I}}})$ is a Banach space (see[16]). Let $(c_0(X), \|\cdot\|_\infty)$ be the Banach space of all null sequences in X . Then, from ([15], Proposition 1.4), null sequences coincide with \mathcal{K} -null sequences and so $(c_0(X), \|\cdot\|_\infty) = (c_{0,\mathcal{K}}(X), \|\cdot\|_{c_{0,\mathcal{K}}})$

Ain and Oja [3] introduced a more general concept of the p -null and unconditionally p -null sequences. They showed that unconditionally p -null sequences and $K_{p^*,p}$ -null sequences coincide, and p -null sequences and $K_{1,p}$ -null sequences coincide in a more general setting. So the second goal of our work is to study the $K_{p,q}$ -null sequences extended by Kim [13]

This memory is organized into three chapters as follows

- The first chapter is a preliminaries which contains some basic concepts, as relatively compact sets, \mathcal{I} -compact sets, ideal of p -compact operators, and \mathcal{I} -compact operators.
- In the second chapter we present the ideal of (p, q) -compact operators and we present the main result obtained by [13] related to $K_{p,q}$ -compactness.
- Chapter three is devoted to present the $K_{p,q}$ -null sequences and we study some properties related to them.

Preliminaries

1.1 Sequence space

In this work the space of absolutely and weakly p -summable sequences are necessary. We introduce them in this section. Let X be a Banach space over \mathbb{K} , and $1 \leq p \leq \infty$. We start by The classical Banach sequence space ℓ_p , ℓ_∞ , and c_0 are defined by

$$\begin{aligned}\ell_p &= \ell_p(\mathbb{K}) = \left\{ (x_n)_n \subset \mathbb{K} : \|(x_n)_n\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\} \\ \ell_\infty &= \ell_\infty(\mathbb{K}) = \left\{ (x_n)_n \subset \mathbb{K} : \|(x_n)_n\|_\infty = \sup_n |x_n| < \infty \right\} \\ c_0 &= c_0(\mathbb{K}) = \left\{ (x_n)_n \subset \mathbb{K} : \lim_{n \rightarrow \infty} |x_n| = 0 \right\}\end{aligned}$$

Definition 1.1 Let $1 \leq p < \infty$. The space $\ell_p(X)$ of absolutely p -summable sequences in X becomes a Banach space when equipped with the norm given by

$$\|(x_n)_n\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}}.$$

Definition 1.2 The space $\ell_\infty(X)$ of bounded sequences in X , and the space $c_0(X)$ of norm null sequences in X , are Banach spaces with the norm given by

$$\|(x_n)_n\|_\infty = \sup_n \|x_n\|.$$

Definition 1.3 Let $1 \leq p < \infty$. A sequence $(x_n)_n$ in X is said to be weakly p -summable if

$$\sum_{n=1}^{\infty} |x^*(x_n)| < \infty,$$

for every $x^* \in B_{X^*}$.

We denote by $\ell_p^w(X)$ the Banach space of weakly p -summable sequences in X becomes a Banach space when equipped with the norm given by

$$\|(x_n)_n\|_p^w = \sup \left\{ \left(\sum_{n=1}^{\infty} |x^*(x_n)|^p \right)^{\frac{1}{p}} : x^* \in B_{X^*} \right\}.$$

Remark 1.1 In the case $p = \infty$. Then the space $\ell_\infty^w(X)$ of weakly bounded sequences coincide with the space $\ell_\infty(X)$ and

$$\|(x_n)_n\|_\infty^w = \|(x_n)_n\|_\infty.$$

Definition 1.4 A sequence $(x_n)_n$ in X is said to be weakly null if

$$\lim_{n \rightarrow \infty} |x^*(x_n)| = 0,$$

for every $x^* \in X^*$.

We denote by $c_0^w(X)$ the Banach space of weakly null sequences in X is a closed subspace of $\ell_\infty(X)$, therefore, it is a Banach space with the supremum norm of $\ell_\infty(X)$.

1.2 Linear operator ideals

Recall that a linear operator $T \in \mathcal{L}(E, F)$ is said to have finite rank if $T(E)$ is a finite dimensional subspace of F . The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(E, F)$. An operator has rank one if and only it has the form

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle y$$

i.e. if $u \in \mathcal{L}_f(E, F)$ we have

$$u = \sum_{i=1}^n x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset E^*$ and $(y_i)_{i=1}^n \subset F$ (see [17, Page 25]).

Definition 1.5 An operator ideal \mathcal{I} is a subclass of the class \mathcal{L} of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components

$\mathcal{I}(E, F) := \mathcal{L}(E, F) \cap \mathcal{I}$ satisfy:

- (i) $\mathcal{I}(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$ which contains the finite rank operators.
- (ii) The ideal property: if $v \in \mathcal{L}(G, E)$, $u \in \mathcal{I}(E, F)$ and $w \in \mathcal{L}(F, H)$, then the composition $w \circ v \circ u$ is in $\mathcal{I}(G, H)$.

If $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \rightarrow \mathbb{R}^+$ satisfies:

- (i') $(\mathcal{I}(E, F), \|\cdot\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces E and F ,
- (ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$,
- (iii') If $v \in \mathcal{L}(G, E)$, $u \in \mathcal{I}(E, F)$ and $w \in \mathcal{L}(F, H)$,

$$\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \|u\|_{\mathcal{I}} \|v\|,$$

then $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal.

The operator ideal \mathcal{I} is said to be closed if each $\mathcal{I}(E, F)$ is a closed subspace of $\mathcal{L}(E, F)$ for the sup norm.

Definition 1.6 (injective operator ideal)

The injective hull $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]^{inj}$ is defined as follows:

$$\mathcal{I}^{inj}(X, Y) := \{T \in \mathcal{L}(X, Y) : I_Y T \in \mathcal{I}(X, \ell_{\infty}(B_{Y^*}))\},$$

where $I_Y : Y \rightarrow \ell_{\infty}(B_{Y^*})$ is the canonical isometry, and $\|T\|_{\mathcal{I}^{inj}} := \|I_Y T\|_{\mathcal{I}}$ for $T \in \mathcal{I}^{inj}(X, Y)$

Definition 1.7 (surjective operator ideal)

The surjective hull $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]^{sur}$ is defined as follows:

$$\mathcal{I}^{sur}(X, Y) := \{T \in L(X, Y) : Tq_X \in \mathcal{I}(\ell_1(B_X), Y)\},$$

where $q_X : \ell_1(B_X) \rightarrow X$ is the canonical quotient map, and $\|T\|_{\mathcal{I}^{sur}} := \|Tq_X\|_{\mathcal{I}}$ for $T \in \mathcal{I}^{sur}(X, Y)$

Remark 1.2 The ideal \mathcal{L}_f of finite rank linear operators is the smallest operator ideal and \mathcal{L} the largest one [17, Theorem 1.2.2].

Definition 1.8 (The dual of an operators ideal) The dual ideal $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]^{dual}$ is defined as follows:

$$\mathcal{I}^{dual}(X, Y) := \{T \in \mathcal{L}(X, Y) : T^* \in \mathcal{I}(Y^*, X^*)\},$$

where T^* is the adjoint of T , and $\|T\|_{\mathcal{I}^{dual}} = \|T^*\|_{\mathcal{I}}$.

1.2.1 Ideal of compact operators.

Definition 1.9 We say that a bounded linear operator $T : E \rightarrow F$ is compact (weakly compact) if $T(B_E)$ is relatively compact (respectively, relatively weakly compact) in F . By $\mathcal{K}(E, F)$ and $\mathcal{W}(E, F)$ we denote the sets of compact linear operators and weakly compact linear operators from E to F , respectively.

Schauder theorem about the compactness of the adjoint of a compact linear operator between two Banach spaces is well known.

Theorem 1.1 Let $T \in \mathcal{L}(E, F)$. Then T is compact if and only if T^* is compact.

The weakly version is due to Gantmacher:

Proposition 1.1 ([17]) The classes \mathcal{K}, \mathcal{W} constitute closed injective Banach operator ideals, where the ideal norm is the operator norm.

1.3 (r, p, q) -nuclear operators

The following concepts (definitions) are based on the book of A. Pietsch [17]

Definition 1.10 Let $0 < r \leq \infty, 1 \leq p, q \leq \infty$ and $1 + 1/r \geq 1/p + 1/q$. An operator $T \in \mathcal{L}(E, F)$ is called (r, p, q) -nuclear if

$$T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n$$

with $(\lambda_n) \in \ell_r, (x_n^*)_n \in \ell_{q^*}^w(E^*)$, and $(y_n)_n \in \ell_p^w(F)$. In the case $r = \infty$ let us suppose that $(\lambda_i) \in c_0$

We put

$$\|T\|_{N_{(r,p,q)}} := \inf \left\{ \|(\lambda_n)_n\|_r \| (x_n^*)_n \|_{q^*}^w \| (y_n)_n \|_p^w : T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \right\},$$

where the infimum is taken over all so-called (r, p, q) -nuclear representations described above. The class of all (r, p, q) -nuclear operators is denoted by $N_{(r,p,q)}$.

1.4 (p, q) -compact operators

Definition 1.11 Let $1 \leq p, q \leq \infty$ and $1/p + 1/q \geq 1$. An operator $S \in \mathcal{L}(E, F)$ is called (p, q) -compact if it belongs to the normed ideal

$$[K_{(p,q)}, \|T\|_{K_{(p,q)}}] := [N_{(r,p,q)}, \|T\|_{N_{(r,p,q)}}],$$

where $1/r := 1/p + 1/q - 1$.

1.5 r -nuclear operators

Definition 1.12 let $1 \leq r \leq \infty$. An operator $T \in \mathcal{L}(E, F)$ is called r -nuclear if it belongs to the normed ideal

$$[N_r, \|T\|_{N_r}] := [N_{(r,r,1)}, \|T\|_{N_{(r,r,1)}}],$$

1.6 p -compact operators

Definition 1.13 let $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. An operator $T \in \mathcal{L}(E, F)$ is called p -compact if it belongs to the normed ideal

$$[K_p, \|T\|_{K_{(p,p^*)}}] := [K_{(p,p^*)}, \|T\|_{K_{(p,p^*)}}] := [N_{(\infty,p,p^*)}, \|T\|_{N_{(p,p^*)}}],$$

where $1 := 1/p + 1/q$.

Remark 1.3 It is well known that $K_{p,1}$ is the ideal of p -nuclear operators and K_{p,p^*} is the ideal of p -compact operators

$K_{p,q}$ -compactness

2.1 Compact sets

Definition 2.1 *The convex hull of a sequence $(x_n)_n \in c_0(X)$ is defined as*

$$\text{conv}\{(x_n)_n\} = \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_1} \right\}$$

Theorem 2.1 *(Grothendieck) A subset K of X is relatively compact if and only if there exists $(x_n) \in c_0(X)$ such that*

$$K \subset \overline{\left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_1} \right\}}$$

Definition 2.2 *Let $1 \leq p \leq \infty$. the p -convex hull of a sequence $(x_n)_n \in \ell_p(X)$ is defined as*

$$p\text{-conv}\{(x_n)_n\} = \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_{p^*}} \right\}$$

Moreover, the 1-convex hull of $(x_n)_n \in \ell_1(X)$ is given by

$$1\text{-conv}\{(x_n)_n\} = \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{c_0} \right\}$$

the ∞ -convex hull of $(x_n)_n \in \ell_{\infty}(X)$ is given by

$$\infty\text{-conv}\{(x_n)_n\} = \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_1} \right\}$$

2.2 p -compact set

In 2002, a form of compactness was introduced by Sinha and Karn [19]

Definition 2.3 (p -compact set) *Let $1 \leq p \leq \infty$. A subset K of Banach space X is relatively p -compact if there exists $(x_n) \in \ell_p(X)$ such that*

$$K \subset p\text{-conv}\{(x_n)_n\} = \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_p^*} \right\}$$

Remark 2.1 1. *Since $\infty\text{-conv}\{(x_i)_{i=1}^{\infty}\}$ represents the closed absolute convex hull of the sequence $\{(x_i)_{i=1}^{\infty}\}$. then compact sets are considered ∞ -compact sets*

2. *Since the sequence (x_n) in the definition of a relatively p -compact set K converges to zero, any p -compact set is compact*

2.3 Measure the size of p -compact set

Given a relatively p -compact set K in a Banach space E there exists a sequence $(x_n)_n \subset E$ so that $K \subset p\text{-conv}\{x_n\}$. such a sequence is not unique, Galicer, Lassalle and Turco [8] introduced a notion to measure the size of p -compact sets.

Definition 2.4 *For a p -compact subset K of X :*

$$m_p(K; X) := \inf \{ \|(x_n)_n\|_p : (x_n)_n \in \ell_p(X), K \subset p\text{-conv}\{(x_n)_n\} \}$$

and $m_p(K, X) = \infty$ if K is not p -compact.

We say that $m_p(K; E)$ measures the size of K as p -compact set of E .

Remark 2.2 1. *When K is relatively p -compact and $(x_n)_n \subset \ell_p(X)$ is a sequence such that $K \subset p\text{-conv}\{x_n\}$, any $x \in K$ has the form $x = \sum_n \alpha_n x_n$ with $(\alpha_n) \in B_{\ell_p^*}$. By Holder's inequality, we have that $\|x\| \leq \|(x_n)_n\|_{\ell_p(X)}$ and consequently,*

$$\|x\| \leq m_p(K) \text{ for all } x \in K.$$

2. *As compact sets can be considered p -compact sets for $p = \infty$ we have that any p -compact set is q -compact and $\sup_{x \in K} \|x\| = m_{\infty}(K; E) \leq m_q(K; E) \leq m_p(K; E)$, for $1 \leq p \leq q \leq \infty$.*

2.4 Relatively \mathcal{I} -compact sets

Definition 2.5 (*Carl and Stephani(1984)*)

- A subset B of a Banach space E is *relatively \mathcal{I} -compact* if there is a Banach space G and an operator $S \in \mathcal{I}(G, E)$ such that $B \subseteq S(M)$, where M is a compact subset of G .
- A linear operator T between Banach spaces E and F is called *\mathcal{I} -compact* if $T(B_E)$ is a relatively \mathcal{I} -compact subset of F .
- The operator ideal formed by all linear \mathcal{I} -compact operators is denoted by $\mathcal{K}_{\mathcal{I}}$.

Lassalle and Turco [15] introduced a way to measure the size of \mathcal{I} -compact sets. For an \mathcal{I} -compact subset K of X , let:

$$m_{\mathcal{I}}(K; X) := \inf \{ \|\mathcal{U}\|_{\mathcal{I}} : \mathcal{U} \in \mathcal{I}(Z, X), \text{relatively compact } C \subset B_Z, K \subset \mathcal{U}(C) \}.$$

where B_Z is the unit ball of a Banach space Z .

2.5 The linear ideal of $\mathcal{K}_{\mathcal{I}}$ operators

Definition 2.6 A linear map $T : Y \rightarrow X$ is said to be \mathcal{I} -compact if $T(B_Y)$ is an \mathcal{I} -compact subset of X . Let $\mathcal{K}_{\mathcal{I}}(Y, X)$ be the space of all \mathcal{I} -compact operators from Y to X . For $R \in \mathcal{K}_{\mathcal{I}}(Y, X)$, Let:

$$\|T\|_{\mathcal{K}_{\mathcal{I}}} := m_{\mathcal{I}}(T(B_Y); X).$$

Then $[\mathcal{K}_{\mathcal{I}}, \|\cdot\|_{\mathcal{K}_{\mathcal{I}}}]$ is a Banach operator ideal (see ([15], Section 2)).

Proposition 2.1 [15] Let $[\mathcal{K}, \|\cdot\|]$ be the ideal of classical compact operators. Then, the classical compactness coincides with the \mathcal{K} -compactness and for a relatively compact subset K of X , $m_{\mathcal{K}}(K; X) = \sup_{x \in K} \|x\|$. Consequently, $[\mathcal{K}_{\mathcal{K}}, \|\cdot\|_{\mathcal{K}_{\mathcal{K}}}] = [\mathcal{K}, \|\cdot\|]$.

Remark 2.3 According to ([15]/Remarks 1.3 and 1.7), K is p -compact if and only if K is $K_{1,p}$ -compact. In this case, $m_p(K; X) = m_{K_{1,p}}(K; X)$.

Definition 2.7 [11] We say that a subset K of X is unconditionally p -compact when $\ell_p(X)$ is replaced by $\ell_p^u(X)$ in the definition of p -compact set.

It was shown in [12](Proposition 5.2) that K is unconditionally p -compact if and only if K is $K_{p^*,p}$ -compact, in this case:

$$m_{K_{p^*,p}}(K; X) = \inf \left\{ \|(x_n)_n\|_p^w : (x_n)_n \in \ell_p^u(X), K \subset p\text{-conv}(x_n)_n \right\}$$

2.6 $K_{p,q}$ -compactness

Using the ideal of $K_{p,q}$ -compact operator Kim in [13] extended a new result to the $K_{p,q}$ -compactness. And that what we want to present it in this section

Lemma 2.1 ([12], Lemma 2.1). Let K be a collection of sequences of positive numbers.

If $\sup_{(K_j)_{j \in K}} \sum_{j=1}^{\infty} K_j < \infty$ and $\lim_{l \rightarrow \infty} \sup_{(K_j)_{j \in K}} \sum_{j \geq l} K_j = 0$, then for every $\varepsilon > 0$, there exists a sequence $(b_j)_j$ of real numbers with $b_j \nearrow \infty$ and $b_j > 1$ for all j such that:

$$\sup_{(K_j)_{j \in K}} \sum_{j=1}^{\infty} K_j b_j \leq (1 + \varepsilon) \sup_{(K_j)_{j \in K}} \sum_{j=1}^{\infty} K_j$$

and

$$\lim_{l \rightarrow \infty} \sup_{(K_j)_{j \in K}} \sum_{j \geq l} K_j b_j = 0$$

Theorem 2.2 Let K be a subset of a Banach space X and let $1 \leq p, q, r \leq \infty$ with $1/r = 1/p + 1/q - 1$. Then, the following statements are equivalent:

- (a) K is $K_{p,q}$ -compact.
- (b) There exist $(x_n)_n \in \ell_{p^*}^u(X)$ and $(\lambda_n)_n \in \ell_r$ such that:

$$K \subset q\text{-conv}(\lambda_n x_n)_n.$$

- (c) There exist a relatively compact $M \subset B_{\ell_{q^*}}(B_{c_0}$ when $q = 1$) and an $S \in K_{p,q}(\ell_{q^*}, X)$ such that $K \subset S(M)$.

In this case:

$$\begin{aligned} m_{K_{p,q}}(K, X) &= \inf \left\{ \|(x_n)_n\|_{p^*}^w \|(\lambda_n)_n\|_r : (x_n)_n \in \ell_{p^*}^u(X), (\lambda_n)_n \in \ell_r, K \subset q\text{-conv}(\lambda_n x_n)_n \right\} \\ &= \inf \left\{ \|S\|_{K_{p,q}} : \text{relatively compact } M \subset B_{\ell_{q^*}}, S \in K_{p,q}(\ell_{q^*}, X), K \subset S(M) \right\}. \end{aligned}$$

Proof 2.1 (a) \Rightarrow (b) : Let $\varepsilon > 0$ be given. By (a), there exist a Banach space $Z, T \in K_{p,q}(Z, X)$ and a relatively compact subset M of B_Z such that $K \subset T(M)$ and:

$$\|T\|_{K_{p,q}} \leq (1 + \varepsilon)m_{K_{p,q}}(K, X).$$

Since $T \in K_{p,q}(Z, X)$, there exist $(\lambda_n)_n \in \ell_r, (z_n^*)_n \in \ell_{q^*}^u(Z^*)$ and $(x_n)_n \in \ell_{p^*}^u(X)$ with $\|(\lambda_n)_n\|_r \| (z_n^*)_n \|_{q^*}^w \| (x_n)_n \|_{p^*}^w \leq (1 + \varepsilon) \|T\|_{K_{p,q}}$ such that:

$$T = \sum_{i=1}^{\infty} \lambda_n z_n^* \otimes x_n,$$

Then:

$$K \subset T(M) \subset T(B_Z) = \left\{ \sum_{n=1}^{\infty} \lambda_n z_n^*(z) x_n : z \in B_Z \right\} \subset q\text{-conv}(\lambda_n \| (z_k^*)_k \|_{q^*}^w x_n)_n$$

and the infimum:

$$\inf \| \cdot \|_{p^*}^w \| \cdot \|_r \leq \|(\lambda_n)_n\|_r \| (z_n^*)_n \|_{q^*}^w \| (x_n)_n \|_{p^*}^w \leq (1 + \varepsilon) \|T\|_{K_{p,q}} \leq (1 + \varepsilon)^2 m_{K_{p,q}}(K, X).$$

Since $\varepsilon > 0$ was arbitrary, $\inf \| \cdot \|_{p^*}^w \| \cdot \|_r \leq m_{K_{p,q}}(K, X)$.

(b) \Rightarrow (c): Let $\varepsilon > 0$ be given. By (b), there exist $(x_n)_n \in \ell_{p^*}^u(X)$ and $(\lambda_n)_n \in \ell_r$ such that $K \subset q\text{-conv}(\lambda_n x_n)_n$ and $\| (x_n)_n \|_{p^*}^w \| (\lambda_n)_n \|_r \leq (1 + \varepsilon) \inf \| \cdot \|_{p^*}^w \| \cdot \|_r$. By an application of Lemma 1, there exists a sequence $(r_n)_n$ of real numbers with $\lim_{n \rightarrow \infty} r_n = \infty$ and $r_n > 1$ such that $(r_n x_n)_n \in \ell_{p^*}^u(X)$ and $\| (r_n x_n)_n \|_{p^*}^w \leq (1 + \varepsilon) \| (x_n)_n \|_{p^*}^w$. We see that the following set:

$$M := \{ (\alpha_n r_n)_n : (\alpha_n)_n \in B_{\ell_{q^*}} \}$$

is a relatively compact subset of $B_{\ell_{q^*}}$ and:

$$S := \sum_{i=1}^{\infty} \lambda_n e_n \otimes r_n x_n \in K_{p,q}(\ell_{q^*}, X),$$

where each e_n is the standard unit vector in ℓ_q (c_0 when $q = \infty$).

Now:

$$K \subset q - \text{conv}(\lambda_n x_n)_n = \left\{ \sum_{i=1}^{\infty} \lambda_n \beta_n r_n x_n : (\beta_n)_n \in M \right\} = S(M)$$

and:

$$\begin{aligned} \inf \|\cdot\|_{K_{p,q}} &\leq \|S\|_{K_{p,q}} \leq \|(\lambda_n)_n\|_r \| (e_n)_n \|_{q^*}^w \| (r_n x_n)_n \|_{p^*}^w \\ &\leq (1 + \varepsilon) \|(\lambda_n)_n\|_r \| (x_n)_n \|_{p^*}^w \\ &\leq (1 + \varepsilon)^2 \inf \|\cdot\|_{p^*}^w \| \cdot \|_r. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\inf \|\cdot\|_{K_{p,q}} \leq \inf \|\cdot\|_{p^*}^w \| \cdot \|_r$.

(c) \Rightarrow (a) is trivial and $m_{K_{p,q}}(K, X) \leq \inf \|\cdot\|_{K_{p,q}}$.

Recall that $[K_{p,q'} \|\cdot\|_{K_{p,q}}]^{sur} = [K_{K_{p,q'}} \|\cdot\|_{K_{p,q}}]$. As an application of Theorem 1, we have:

Corollary 2.1 (cf. [14], Theorem 2.2). Let X and Y be Banach space and let $1 \leq p, q, r \leq \infty$ with $1/r = 1/p + 1/q - 1$. let $T : X \rightarrow Y$ be a linear map. Then the following statements are equivalent:

(a) $T \in K_{p,q}^{sur}(X, Y)$.

(b) There exist $(\lambda_n)_n \in \ell_r$ and $(y_n)_n \in \ell_{p^*}^u(Y)$ such that:

$$T(B_x) \subset q - \text{conv}(\lambda_n y_n).$$

(c) There exist a relatively compact $M \subset B_{\ell_{q^*}}(B_{c_0}$ when $q = 1$) and an $S \in K_{p,q}(\ell_{q^*}, Y)$, such that $T(B_x) \subset S(M)$.

In this case:

$$\begin{aligned} \|T\|_{K_{p,q}^{sur}} &= \inf \left\{ \| (y_n)_n \|_{p^*}^w \| (\lambda_n)_n \|_r : (y_n)_n \in \ell_{p^*}^u(Y), (\lambda_n)_n \in \ell_r, T(B_x) \subset q - \text{co}(\lambda_n y_n) \right\} \\ &= \inf \left\{ \|S\|_{K_{p,q}} : \text{relatively compact } M \subset B_{\ell_{q^*}}, S \in K_{p,q}(\ell_{q^*}, Y), T(B_x) \subset S(M) \right\}. \end{aligned}$$

Remark 2.4 One may refer to [1] for a different generalization of p -compactness.

$K_{p,q}$ -null sequences

3.1 p -null sequences

Definition 3.1 Let X be a Banach space, by $c_0(X)$ we denote the space of all null sequence that is

$$c_0(X) = \left\{ (x_n)_n, x_n \in X : \lim_{n \rightarrow \infty} \|x_n\| = 0 \right\}$$

Piñeiro and Delgado [18] introduced an extended concept of null sequences.

Definition 3.2 For $1 \leq p < \infty$, a sequence (x_n) in a Banach space X is said to be p -null if for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ and $(z_k) \in \ell_p(X)$ with $\|(z_k)\|_p \leq \epsilon$ such that $x_n \in p\text{-co}(\{z_k\})$ for all $n \geq N$. The collection of all p -null sequences in X is denoted by $c_{0,p}(X)$.

Let X be a Banach space. According to the Grothendieck compactness principle . Delgado and Piñeiro obtained the following Grothendieck -like result (see [[18],Theorem 2.5]).

Theorem 3.1 (Delgado-Piñeiro).let $1 \leq p < \infty$. A subset K of a Banach space X is relatively p -compact if and only if K is contained in the closed convex hull of a p -null sequence.

The p -null sequences can be characterized via p -compactness as follows.

Theorem 3.2 (Delgado-Pineiro-Oja). Let $1 \leq p < \infty$. A sequence (x_n) in a Banach space X is p -null if and only if (x_n) is null and relatively p -compact.

3.2 (p, r) -null sequences

The folowing concepts are based on The article of K. Ain and E. Oja. A [2]

Definition 3.3 Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. The (p, r) -convex hull of a sequence $(x_n)_n \in \ell_p(X)$ is defined as

$$(p, r) - \text{conv}\{(x_n)_n\} = \left\{ \sum_{n=1}^{+\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_r} \right\}.$$

Ain, K. and Oja, E. in[3] introduced the (p, r) -null sequences

Definition 3.4 Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. We call a sequence (x_n) in X (p, r) -null if for every $\varepsilon > 0$ there exist $n_\varepsilon \in \mathbb{N}$ and $(z_k) \in \ell_p(X)$ ($(z_k) \in c_0(X)$ when $p = \infty$) with $\|(z_k)\|_p \leq \varepsilon$ such that $x_n \in (p, r) - \text{conv}(z_k)$ for all $n \geq N$.

Remark 3.1 As $p - \text{conv}(z_k)$ coincides with the $(p, p^*) - \text{convex}$ hull of $(z_k) \in \ell_p(X)$, p -null sequences in X are precisely the (p, p^*) -null sequences.

Proposition 3.1 A sequence (x_n) in a Banach space X is $(\infty, 1)$ -null if and only if $x_n \rightarrow 0$.

Proof 3.1 Suppose that $(x_n) \subset X$ is $(\infty, 1)$ -null. Then for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $(z_k) \in c_0(X)$, $\|(z_k)\|_\infty \leq \varepsilon$, such that $x_n = \sum_{k=1}^{\infty} a_k^n z_k$, where $(a_k^n)_{k=1}^{\infty} \in B_{\ell_1}$, for all $n \geq N$.

$$\|x_n\| \leq \sum_{k=1}^{\infty} \|a_k^n z_k\| \leq \varepsilon \sum_{k=1}^{\infty} |a_k^n| \leq \varepsilon,$$

and there fore $x_n \rightarrow 0$.

let now $x_n \rightarrow 0$. This means that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_n\| \leq \varepsilon$ for all $n \geq N$. Now the sequence

$$z_k = \begin{cases} 0 & \text{if } k < N, \\ x_k & \text{if } k \geq N, \end{cases}$$

is in $c_0(X)$ with $\|(z_k)\|_\infty \leq \varepsilon$ and $x_n \in (\infty, 1) - \text{conv}(z_k)$ for all $n \geq N$.

Proposition 3.2 Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. If a sequence (x_n) in a Banach space X is (p, r) -null, then $x_n \rightarrow 0$

Proof 3.2 Assume $1 \leq p < \infty$ as for $p = \infty$ the assertion is clear from the above Proposition. Since (x_n) is (p, r) -null, for every $\varepsilon > 0$ there are $N \in \mathbb{N}$ and $(z_k) \in \ell_p(X)$, $\|(z_k)\|_p \leq \varepsilon$ such that $x_n = \sum_{k=1}^{\infty} a_k^n z_k$, where $(a_k^n)_{k=1}^{\infty} \in B_{\ell_r}$, for all $n \geq N$. Hence, for all $n \geq N$,

$$\|x_n\| \leq \sum_{k=1}^{\infty} \|a_k^n z_k\| \leq \|(a_k^n)_k\|_{p^*} \|(z_k)\|_p \|(a_k^n)_k\|_r \|(z_k)\|_p \leq \varepsilon,$$

and there fore $x_n \rightarrow 0$.

3.3 The unconditionally p -summable sequences

Between absolutely p -summable sequences $\ell_p(X)$ and weakly p -summable sequences $\ell_p^w(X)$, there is the Banach space $\ell_p^u(X)$ of unconditionally p -summable sequences. The space $\ell_p^u(X)$ was introduced and thoroughly studied by Fourie and Swart [6] in 1979. The space $\ell_{up}(X)$ is defined as the (closed) subspace of $\ell_p(X)^w$, formed by the $(x_n) \in \ell_p^w(X)$ satisfying

$$(x_n) = \lim_{N \rightarrow \infty} (x_1, \dots, x_N, 0, 0, \dots) \text{ in } \ell_p^w(X).$$

Recall that $\ell_\infty^u(X) = c_0(X)$ as Banach spaces (see [6, p. 351] or, e.g., [?, 8.2]) and recall that $\ell_\infty^w(X) = \ell_\infty(X)$. And it is well known that a sequence is in $\ell_1^u(Z)$ if and only if it is unconditionally summable.

Definition 3.5 *The closed subspace $\ell_p^u(Z)$ of $\ell_p^w(Z)$ consists of all sequences $(z_n)_n$ in Z satisfying that:*

$$\sup \sum_{n \geq m} |z^*(z_n)|^p \rightarrow 0$$

as $m \rightarrow \infty$ (cf. [?], Section 8.2 and [6, 7]).

The unconditionally p -summable sequences of X are characterized in [6, Lemma 1.2] as follows.

Lemma 3.1 (Fourie–Swart) *A sequence (x_n) in a Banach space X is an element of $\ell_p^u(X)$ if and only if there exist $(\epsilon_n) \in c_0$ and $(y_n) \in \ell_p^w(X)$ such that $x_n = \epsilon_n y_n$ for all $n \in \mathbb{N}$.*

Corollary 3.1 *If $(x_n) \in \ell_u^p(X)$, then there exists $1 \leq \lambda_n \rightarrow 1$ such that $\lambda_n x_n \in \ell_u^p(X)$.*

3.4 The unconditionally p -null sequences

In [11], the unconditionally p -null sequence was introduced by replacing $\ell_p(X)$ with $\ell_p^u(X)$.

Definition 3.6 *For $1 \leq p \leq \infty$, sequence $(x_n)_n$ in a Banach space, X is called unconditionally p -null (in symbol $(x_n)_n \in c_{0,up}(X)$) if for every $\epsilon > 0$, there exist a natural number n_ϵ and $(z_k)_k \in \epsilon B_{\ell_p^u(X)}$ such that:*

$$x_n \in p\text{-conv}(z_k)_k$$

for all $n \geq n_\epsilon$.

3.5 \mathcal{I} -null sequences

For a Banach operator ideal \mathcal{I} and a Banach space X , a sequence $(x_n)_n$ in X is called \mathcal{I} -null if there exist a Banach space Z and $S \in \mathcal{I}(Z, K)$ such that for every $\varepsilon > 0$, there is a natural number n_ε so that:

$$x_n \in \varepsilon S(B_Z)$$

for all $n \geq n_\varepsilon$. Let $c_{0,\mathcal{I}}(X)$ be the space of all \mathcal{I} -null sequences in X . Since \mathcal{I} -null sequences are \mathcal{I} -compact (see [4]), Lassalle and Turco [16] defined a norm on $c_{0,\mathcal{I}}(X)$ by:

$$\|(x_n)_n\|_{c_{0,\mathcal{I}}(X)} := m_{\mathcal{I}}(\{x_n\}_n; X).$$

Then $(c_{0,\mathcal{I}}(X), \|\cdot\|_{c_{0,\mathcal{I}}})$ is a Banach space (see[16]). Let $(c_0(X), \|\cdot\|_\infty)$ be the Banach space of all null sequences in X . Then, from ([15], Proposition 1.4), null sequences coincide with \mathcal{K} -null sequences and so $(c_0(X), \|\cdot\|_\infty) = (c_{0,\mathcal{K}}(X), \|\cdot\|_{c_{0,\mathcal{K}}})$

Remark 3.2 *It was shown in [15] (Proposition 1.5) that p -null sequences and $K_{1,p}$ -null sequences coincide.*

Ain and Oja [3] introduced a more general concept of the p -null and unconditionally p -null sequences. They showed that unconditionally p -null sequences and $K_{p^,p}$ -null sequences coincide, and p -null sequences and $K_{1,p}$ -null sequences coincide in a more general setting.*

3.6 $K_{p,q}$ -null sequences

Using the ideal of $K_{p,q}$ -compact operator Kim in [13] extended a new result to the $K_{p,q}$ -null sequence In the following theorem.

Theorem 3.3 *Let $(x_n)_n$ be a sequence in a Banach space X and let $1 < p, q, r < \infty$ with $1/p + 1/q - 1 = 1/r$. Then, the following statements are equivalent:*

- (a) $(x_n)_n$ is a $K_{p,q}$ -null sequence.
- (b) There exists $(z_k)_k \in \ell_{p^*}^u(X)$ and $(\lambda_k)_k \in \ell_r$, such that for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that:

$$x_n \in \varepsilon q - \text{conv}(\lambda_k z_k)_k$$

for every $n \geq N$.

(c) There exists an $S \in K_{p,q}(\ell_{q^*}, X)$ such that for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that:

$$x_n \in \varepsilon S(B_{\ell_{q^*}})$$

for every $n \geq N$.

Proof 3.3 (c) \Rightarrow (a) is trivial.

(a) \Rightarrow (b) : By (a), there exist a Banach space Z and $T \in K_{p,q}(Z, X) \subset K_{p,q}^{\text{sur}}(Z, X) = K_{K_{p,q}}(Z, X)$ such that for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $x_n \in \varepsilon T(B_Z)$ for every $n \geq N$. Since $T(B_Z)$ is $K_{p,q}$ -compact, by Theorem 2.2, there exist $(y_k)_k \in \ell_{p^*}^u(X)$ and $(\lambda_k)_k \in \ell_r$ such that:

$$T(B_Z) \subset q - \text{conv}(\lambda_k y_k)_k.$$

Hence, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that:

$$x_n \in \varepsilon q - \text{conv}(\lambda_k y_k)_k$$

for every $n \geq N$.

(b) \Rightarrow (c) : Let $(z_k)_k \in \ell_{p^*}^u(X)$ and $(\lambda_k)_k \in \ell_r$ be such that for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that:

$$x_n \in \varepsilon q - \text{conv}(\lambda_k z_k)_k$$

for every $n \geq N$. Consider the operator:

$$S := \sum_{i=1}^{\infty} \lambda_i e_i \otimes z_i \in K_{p,q}(\ell_{q^*}, X),$$

where each e_k is the standard unit vector in ℓ_q . Then, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that:

$$x_n \in \varepsilon q - \text{conv}(\lambda_k y_k)_k = \varepsilon S(B_{\ell_{q^*}})$$

for every $n \geq N$.

The prototype of the following corollary is described in [5] (Proposition 1.5).

Corollary 3.2 Let $(x_n)_n$ be a sequence in a Banach space X and let $1 \leq p, q, r \leq \infty$ with

$1/r = 1/p + 1/q - 1$. Then, the following statements are equivalent:

(a) $(x_n)_n$ is a $K_{p,q}$ -null sequence.

(b) For some $C > 0$, for every $\varepsilon > 0$, there exist a Banach space $Z, T \in K_{p,q}(Z, X)$ with $\|T\|_{k_{p,q}} \leq C$ and $N \in \mathbb{N}$ such that:

$$x_n \in \varepsilon T(B_Z)$$

for every $n \geq N$.

(c) For some $K > 0$, for every $\varepsilon > 0$, there exist $(z_k)_k \in \ell_{p^*}^u(X)$ and $(\lambda_k)_k \in \ell_r$ with $\|(z_k)_k\|_{p^*}^w \|(\lambda_k)_k\|_r \leq K$, and $N \in \mathbb{N}$ such that:

$$x_n \in \varepsilon q - \text{conv}(\lambda_k z_k)_k$$

for every $n \geq N$.

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Summary and Keywords

ملخص

في هذه الأطروحة، تناولنا واحدة من أهم مواضيع التحليل الدالي، وهي بعض اصناف المجموعات والمؤثرات المتراسة وخاصة التقريب . حاولنا قدر الإمكان، بجهد متواضع، أن نتعرف على التعريفات الأكثر أهمية، والنظريات والخصائص في جميع فصول هذا البحث. في الفصل الأول، نستعرض أهم المفاهيم المتعلقة بالمجموعات المتراسة ، المؤثرات المتراسة . الفصل الثاني يتناول بعض النتائج المتعلقة بالمجموعات $K-p, q$ متراسة نسبييا والمتتاليات $K-p, q$ المتقاربة نحو الصفر. الفصل الثالث يتمحور حول خاصية التقريب المتعلقة بالمؤثرات $K-p, q$ المتراسة الكلمات المفتاحية: المجموعات المتراسة ، المجموعات $K-p, q$ المتراسة ، والمتتاليات $K-p, q$ المتقاربة نحو الصفر ، المؤثرات $K-p, q$ المتراسة، خاصية التقريب

Abstract

In this memory, we deal with one of the most important topics in functional analysis, namely some classes of compact sets and operators, we endeavored, with modest effort, to identify the most important definitions, theorems, and properties throughout all chapters of this research.

In the first chapter, we review the key concepts related to compact sets and compact operators. The second chapter deals with some results related to relatively $K_{p,q}$ -compact sets and finally we present some results for $K_{p,q}$ -null sequence.