PEOPLE’s DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf university of M'sila
Faculty of Mathematics and computer sciences
Department of Mathematics

Graduation Thesis

Master's degree

Field : Mathematics and computer sciences

Branch : Mathematics

Option : Mathematics and numerical analysis

Theme

Legendre polynomial for linear ordinary differential equations

Presented by :

Ms Boumalia Yousra

Publicly defended on : 24/06/2022

By the jury :

Mr. Gagui Bachir	**MCA. Univ. M'sila**	**President**
Mr. LAKEHALI Belkacem	**MCA. Univ. M'sila**	**Supervisor**
Mr. Selt Omar	**Pr. Univ. M'sila**	**Examiner**

University year : 2021/2022

Acknowledgements

After giving thanks to God the almighty and merciful, my thanks go to my Supervisor

Mr. **LAKEHALI BELKACEM**

You give me the honour of presiding my work. For having ensured the good follow up of my work, by possing your experience, giving your advice and esprcially your availability be assured of my deep gratitude.

To all the members of the jury thank you for honored and agreed to evaluate this dissertation.

To all those who help to make this work successful.

Dedication

With the help of God who has traced the path of my life.

I humply dedicate this work with great pride :

To my dear father

For his advice and guidance you gave us everything. I hope one day I can make you proud.

To my mother

Your sacrifices, your support, your deep attachment allowed me to succeed in my studies.

You made me what I am.

To my brothers YACINE and MOHSSENE.

And sisters SANAA and MARAM.

You are part of my heart, my faithful companions in the most difficult moments.

To MOHAMED, HANAA, NOUR EL IMANE and DJANA.

To my friends, especially BESMA, NOUHA and FERIAL, for all these years of friendship and the beautiful memories.

For everyone I have ever loved.

Contents

Introduction

Differential equations arise in many areas of science and technology. A number of natural laws, including many from classical physics, chemistry, economics, and engineering, can be stated in the form of a differntial equation.

A differntial equation is a mathematical equation for an unknown function of one or several variables that relate the values of the function itself and its derivatives of various order.

Since most realistic differential equations do not have exact analytic solutions, therefore, approximate techniques are used. In this paper, we present an efficient numerical method to solve numerically the differential equation using Legendre polynomials, which was discovered by Adrien-Marie Legendre in 1784.

In this thesis we have

In chapter 1, Overview of Hilbert spaces and then in chapter 2, we shall present the notion of Legendre polynomial and its properties, and in chapter 3, we defined Ordinary Differentials Equations, and we focused on linear differential equations of the first and second order is the focus of the study. In chapter 4, we examined the results of applying Legendre-collocation method in the numerical solution of differential equations through examples in showing the difference between exact solutions and approximate solutions.

Chapter 1

Hilbert spaces

1.1 Vector spaces

A modern-seeming, axiomatic, definition of vector spaces goes back to the Italian mathematician Giuseppe Peano, in 1888. A vector space is an algebraic object, to introduce such analytic notions as convergence or continuity in a vector space we must provide our vector space with additional structure.

Definition 1 *A vector x of $\mathbb{R}^n$ is an ordered collection $x = (x_1, x_2, ..., x_n)'$ of n reals x_j, $j = 1, 2, ..., n$ called components of x. The number n is called the dimension of the n vector. The $\mathbb{R}^n$ space is the set of all collections of this type. It has two operations basic linear:*

1. **Addition:** The sum of the two vectors $x, y \in \mathbb{R}^n$ is a vector of $\mathbb{R}^n$, defined by:

$$x + y = (x_1 + y_1; x_2 + y_2; ...; x_n + y_n)'$$

2. **Multiplication by reals:** Let $x = (x_1, x_2, ..., x_n)'$ be a vector of $\mathbb{R}^n$. The multiplication of the vector x by the real λ is also a vector of $\mathbb{R}^n$, defined as follows:

$$\lambda x = (\lambda x_1, \lambda x_2, ..., \lambda x_n)'$$

The structure we get (the set of all $n-$dimensional vectors with the two operations that we have just defined) is called the real vector space $\mathbb{R}^n$ $n-$dimensional.

This brings us to the concept of a normed space, which is a vector space with a norm.

1.2 Normed spaces

It is an important class of metric spaces, of which Euclidean spaces are the model basic. In general, a vector normed space is a vector space in which there is a metric compatible with the vector space structure.

Definition 2 *Let X be a vector space over either the scalar field $\mathbb{R}$ of real numbers or the scalar field $\mathbb{C}$ of complex numbers. Suppose we have a function $\|\cdot\| : X \to [0, \infty)$ such that,*

1. $\|x\| = 0$ if and only if $x = 0$.
2. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.
3. $\|\alpha x\| = |\alpha| \, \|x\|$ for all scalars α and vectors x.

    (a) Property (2) is called the triangle inequality.

    (b) property (3) is referred to as homogeneity

Definition 3 *We call $(X, \|\cdot\|)$ a normed spaces.*

Example 4 *Let* $X = \mathbb{C}^n \equiv \{(z_1, z_2, ..., z_n) : z_j \in \mathbb{C}\}$ *with*

$$\|(z_1, z_2, ..., z_n)\| = (\sum_{j=1}^{n} |z_j|^2)^{\frac{1}{2}}$$

This is called the Euclidean norm. The Euclidean space $\mathbb{R}^n$ is similarly defined in this case we restrict to real scalars.

Proposition 5 *1. on $\mathbb{R}^n$ we can define several normed*

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$
$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$
$$\|x\|_\infty = \max_{1\leq i\leq n} \{|x_i|\}$$

are equivalent.

2. Vector space $\mathbb{C}([0,1]\,\mathbb{R})$ can be provided with standards:

$$\|x\|_1 = \int_0^1 |f(t)|\, dt$$
$$\|x\|_2 = \sqrt{\int_0^1 (f(t))^2 dt}$$
$$\|x\|_\infty = \max_{t\in[0,1]} |f(t)|$$

are not equivalent.

3. On the space of bounded numerical sequences (with value in $\mathbb{R}$ or $\mathbb{C}$) , we can define norm

$$\|u\| = \sup_{n\geq 0} |u_n|$$

4. **Product Norm:** if $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are two normed spaces, we can define a norm on the vector space $E \times F$ by

$$\forall (x,y) \in E \times F, \|(x,y)\| = \max \{\|x\|_E, \|y\|_F\}$$

Definition 6 *A metric space is a set X with a function $d(\cdot,\cdot) : X \times X \to [0,\infty)$ satisfying, for x, y, and z in X,*

1. $d(x,y) = 0$ if and only if $x = y$,

2. $d(x, y) = d(y, x)$, and

3. $d(x, y) + d(y, z) \geq d(x, z)$.

The third property is referred to as the triangle inequality.

Cauchy sequence

Definition 7 *Let X be a metric space. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if it has the following property: Given any $\varepsilon > 0$ there exists N such that if n, $m \geq N$, then $d(x_n, x_m) < \varepsilon$.*

Definition 8 *A metric space is said to be complete if every Cauchy sequence in X converges in X.*

1.3 Inner product spaces

Definition 9 *Let X be a vector space over $\mathbb{C}$. An inner product is a map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ satisfying, for x, y, and z in X and scalars $\alpha \in \mathbb{C}$,*

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all x, y in X,

2. $\langle x, x \rangle \geq 0$, with $\langle x, x \rangle = 0$ (if and) only if $x = 0$,

3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, and

4. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

Is called an inner product space .

Proposition 10 *:*

1. If $\langle \cdot, \cdot \rangle$ is an inner product on a vector space X, then for all x and y in X we have

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

   The cauchy-schwarz inequality.

2. If $\langle \cdot, \cdot \rangle$ is an inner product on a vector space X, then

$$\|x\| \equiv \langle x, x \rangle^{\frac{1}{2}}$$

is a norm on X.

1.4 Hilbert spaces

Definition 11 *A Hilbert space H is a vector space over $\mathbb{C}$ with an inner product such that H is complete in the metric*

$$d(x, y) = \|x - y\| = \langle x - y, x - y \rangle^{\frac{1}{2}}$$

Example 12 *On the Hilbert space*

- $\mathbb{C}^n$(or $\mathbb{R}$) for **Inner product**

$u = (u_1, ..., u_n) \in \mathbb{C}^n$(or $\mathbb{R}$)

$v = (v_1, ..., v_n) \in \mathbb{C}^n$(or $\mathbb{R}$)

$$\langle u, v \rangle = \sum_{1 \leq j \leq d} u_j \cdot \overline{v}_j$$

is a **Hilbert space**

- l^2 is a **Hilbert space**

1.5 Orthogonality

A Banach space is a complete normed linear space and a Hilbert space is a complete inner product space. The presence of an inner product permits the all-important geometric notion of orthogonality, which says in turn that Hilbert spaces behave in many ways as generalizations of finite-dimensional Euclidean space, where one can talk about angles and projections.

Definition 13 *Given vectors f, g in a Hilbert space H, we say that f is orthogonal to g, written $f \perp g$, if $\langle f, g\rangle = 0$. For sets A and B in H we write $A \perp B$ if $\langle f, g\rangle = 0$ for all $f \in A$ and $g \in B$. Finally, $A^{\perp}$ is the set of all vectors $f \in H$ such that $f \perp g$ for all g in A, for any set A this is always a subspace of H, moreover since $A^{\perp} = \cap_{\alpha \in A}\{a\}^{\perp}$, $A^{\perp}$ is a closed subspace by continuity of the inner product, see [2]*

Theorem 14 *(Weierstrass) Suppose f is a continuous real-valued function defined on the real interval $[a, b]$. For every $\varepsilon > 0$, there exists a polynomial $P_n(x)$ such that for all x in $[a, b]$, we have*

$$|f(x) - P_n(x)| < \varepsilon$$

Or equivalently, the supremum norm

$$\|f - P\| < \varepsilon$$

for all $x \in [a, b]$. In other words, any continuos function on a closed and bounded interval can be uniformly approximated on that interval by polynomials to any degree of accuracy.

Chapter 2

Legendre polynomials

2.1 Introduction

Legendre polynomials $P_n(x)$ arise from the orthogonalisation process for polynomials in the domain $(-1, 1)$ with a weight factor 1. They were the first set of orthogonal polynomials to be described.

They are a special case of the Jacobi polynomials with α and β both equal to zero. They are also equal to the Gegenbauer polynomials $C_n^{(1/2)}(x)$. That is

$$P_n(x) = C_n^{(1/2)}(x) = P_n^{(0/0)}(x) \tag{2.1}$$

Legendre polynomials can be obtained using the Gram-Schmidt orthogonalisation with a weight factor $w(x) = 1$ and then multiplying each of the resulting polynomials by a number such that its value at $x = 1$ is 1, see [1]

The first few Legendre polynomials are

$P_0(x) = 1.$

$P_1(x) = x.$

$P_2(x) = (3x^2 - 1)/2.$

$P_3(x) = (5x^3 - 3x)/2.$

$P_4(x) = (35x^4 - 30x^2 + 3)/8.$

$P_5(x) = (63x^5 - 70x^3 + 15x)/8.$

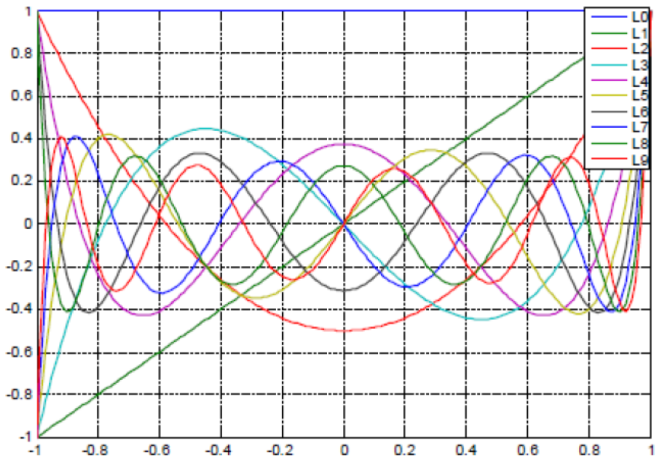
$P_6(x) = (231x^6 - 315x^4 + 105x^2 - 5)/16.$

$P_7(x) = (429x^7 - 693x^5 + 315x^3 - 35x)/16.$

$P_8(x) = (6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)/128.$

$P_9(x) = (12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x)/128.$

$P_{10}(x) = (46189x^{10} - 109395x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63)/256.$



Graphs of Legendre polynomials for n=1 to n=10

2.2 Differential Equations

The Legendre differential equations is homogrneous second order ordinary differential equation of the form

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \tag{2.2}$$

The point $x = 0$ is an ordinary point. This means that we can express the solution in the form of a power series, $y = \sum_{n=0}^{\infty} a_n x^n$. On substituting this power series into the differential equation, we find

$$(1 - x^2) \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=0}^{\infty} n a_n x^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n$$

If we equate the coefficients of x^n to zero we get

$$(n+2)(n+1)a_{n+2} + [\lambda - n(n+1)]\, a_n = 0$$

Leading to the recurrence relation

$$\frac{a_{n+2}}{a_n} = \frac{n(n+1) - \lambda}{(n+2)(n+1)}. \tag{2.3}$$

We see that there are two series solutions, one series contains only even powers of x, the other is an odd power series, see [1].

2.3 Orthogonality

The differential equation (2.2) for $R_m(x)$ can be written in Sturm Liouville form

$$\left((1 - x^2) R_m'(x)\right)' + \lambda_m R_m(x) = 0, \qquad \lambda_m = m(m+1) \tag{2.4}$$

Equivalent

$$(1 - x^2) R_m''(x) - 2x R_m'(x) + m(m+1) R_m(x) = 0$$

We then multiply this by $R_n(x)$, where $n \neq m$ and integrate from -1 to 1. The left-hand side becomes

$$\int_{-1}^{1} R_n(x) \frac{d}{dx} \left((1 - x^2) R'_m(x) \right) dx.$$

On integrating this by parts and noting that the integrated term vanishes at $x = \pm 1$ we find that

$$\int_{-1}^{1} (1 - x^2) R'_n(x) R'_m(x) dx = m(m+1) \int_{-1}^{1} R_n(x) R_m(x) dx$$

If we now follow the same procedure as above with m and n interchanged we arrive at an equation which is the same as that above on the left-hand side but with $n(n+1)$ on the right. If we take the difference of these equations, we get

$$[m(m+1) - n(n+1)] \int_{-1}^{1} R_n(x) R_m(x) dx = 0$$

Thus for $n \neq m$,

$$\int_{-1}^{1} R_n(x) R_m(x) dx = 0 \tag{2.5}$$

The polynomials $R_n(x)$ thus satisfy the same orthogonality relations as the Legendre polynomials and must therefore be multiples of them. If we divide $R_n(x)$ by $R_n(1)$ so that the resulting polynomial is equal to 1 at $x = 1$ we obtain the Legendre polynomial $P_n(x)$, see[1]

2.4 Rodrigues Formula

By treating $(1-x^2)^n$ as the product $(1-x)^n(1+x)^n$ and using the formula for differentiating the product uv m times, we see that if $n > m$ then the m th derivative of $(1-x^2)^n$ contains as a factor $(1-x^2)^{n-m}$ and so

$$\frac{d^m}{dx^m}(1-x^2)^n = \frac{d^m}{dx^m}\left[(1-x)^n(1+x)^n\right] = 0 \quad when \quad x = \pm 1$$

From this result, it follows that

$$\int_{-1}^{1} \frac{d^n}{dx^n}(1-x^2)^n dx = 0.$$

If we integrate by parts we see that

$$\int_{-1}^{1} x\frac{d^n}{dx^n}(1-x^2)^n dx = 0 \qquad provided\ that \quad n > 1$$

and integrating by parts m times

$$\int_{-1}^{1} x^m\frac{d^n}{dx^n}(1-x^2)^n dx = 0 \qquad provided\ that\ n > m \tag{2.6}$$

In other words, the n th order polynomial $Q_n(x) = d^n(1-x^2)^n/dx^n$ is orthogonal to x^m for all values of $m < n$. This means that $Q_n(x)$ is orthogonal to $Q_m(x)$ for all values of $m < n$. These polynomials must therefore be multiples of the Legendre polynomials we found earlier. We can find out what this multiple is by evaluating $Q_n(1)$ by writing

$$Q_n(1) = \lim_{x\to 1}\frac{d^n}{dx^n}\left\{(1-x)^n(1+x)^n\right\} = (-1)^n 2^n n!$$

therefore

$$P_n(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}(1-x^2)^n = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{2.7}$$

see[1]

2.5 Explicit expression

The Legendre polynomial has the expansion:

$$P_n(x) = \frac{1}{2^n} \sum_{m=0}^{[n/2]} \frac{(-1)^m (2n-2m)!}{m!(n-m)!(n-2m)!} x^{n-2m} \tag{2.8}$$

The coefficient of x^n in $P_n(x)$, and the leading coefficient k_n is

$$k_n = \frac{(2n)!}{2^n (n!)^2} \tag{2.9}$$

and of course $k'_n = 0$, see [9].

We can express x^n in terms of the Legendre polynomials

$$x^n = \sum_{m=0}^{n} a_m P_m(x)$$

2.6 Generating Function

We can show that

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{1-2xt\ +t^2}}\right]\right\} = -t\frac{\partial^2}{\partial t^2}\left[\frac{t}{\sqrt{1-2xt\ +t^2}}\right] \tag{2.10}$$

This is most easily done using a computer algebra package such as Maple. Let us define $\phi_n(x)$ by

$$\sum_{n=0}^{\infty} t^n \phi_n(x) = \frac{1}{\sqrt{1-2xt\ +t^2}} \qquad |t| < 1 \tag{2.11}$$

On expanding the right-hand side using the binomial theorem we see that the functions $\phi_n(x)$ are n th order polynomials in x. If we substitute the left hand side of (2.11) into

(2.10), we see that

$$\sum_{n=0}^{\infty} t^n \frac{d}{dx}\left\{(1-x^2)\frac{d\,\phi_n(x)}{dx}\right\} = -t\sum_{n=0}^{\infty}\phi_n(x)\frac{d^{\,2}}{dt^{\,2}}t^{n+1} = -\sum_{n=0}^{\infty} n(n+1)t^n\phi_n(x) \qquad (2.12)$$

If we now equate the coefficients of powers of t on both sides of the equation, we obtain

$$\frac{d}{dx}\left\{(1-x^2)\frac{d\,\phi_n(x)}{dx}\right\} + n(n+1)\phi_n(x) = 0 \qquad (2.13)$$

In other words $\phi_n(x)$ satisfies Legendre's equation and is therefore some multiple of the Legendre polynomial $P_n(x)$. If we put $x = 1$ in (2.11)

$$\sum_{n=0}^{\infty} t^n\phi_n(1) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

Thus we see that $\phi_n(1) = 1$ and that therefore $\phi_n(x) = P_n(x)$, the n th Legendre polynomial. Thus

$$\sum_{n=0}^{\infty} t^n P_n(x) = \frac{1}{\sqrt{1-2xt\ +t^2}} \qquad |t| < 1 \qquad (2.14)$$

We can use the generating function to derive the orthogonality relations. Consider

$$\begin{aligned}\sum_{n,m=0}^{\infty} s^m t^n \int_{-1}^{1} P_m(x)P_n(x) &= \int_{-1}^{1}\frac{1}{\sqrt{1-2sx\ +s^2}}\frac{1}{\sqrt{1-2tx\ +t^2}}dx = \frac{1}{\sqrt{st}}\ln\left\{\frac{1+\sqrt{st}}{1-\sqrt{st}}\right\}\\ &= 2+\frac{2}{3}st+\frac{2}{5}(st)^2+\ldots = 2\sum_{n=0}^{\infty} s^n t^n/(2n+1)\end{aligned}$$

From this we see that the coefficient of $s^m t^n$ on the left-hand side, that is $\int_{-1}^{1} P_m(x)P_n(x)dx$ is zero, confirming the orthogonality relations, and that for $n = m$, see [1]

$$h_m = \int_{-1}^{1} P_m^2(x)dx = \frac{2}{2m+1} \qquad (2.15)$$

2.7 Recurrence Relations

The recurrence relation for Legendre polynomials can be found from the generating function. If we differentiate (2.14) with respect to t, we get

$$\sum_{n=0}^{\infty} nt^{n-1}P_n(x) = \frac{t - x}{(1-2xt+t^2)^{3/2}}$$

and so

$$(1-2xt+t^2)\sum_{n=0}^{\infty} nt^{n-1}P_n(x) = \frac{t - x}{\sqrt{1-2xt+t^2}} = (t-x)\sum_{n=0}^{\infty} t^n P_n(x)$$

If we equate the coefficients of t^n on both sides of this equation we obtain the recurrence relation, see [1]

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{2.16}$$

2.8 Differential Relation

We can show that

$$(1-x^2)\frac{\partial}{\partial x}(\frac{1}{\sqrt{1-2xt+t^2}}) = t\frac{\partial}{\partial t}(\frac{t - x}{\sqrt{1-2xt+t^2}})$$

If we substitute the left-hand side of the generating function into the equation above and equate the coefficients of t^n on both sides of the equation we see that, see [1]

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x) \tag{2.17}$$

Chapter 3

Ordinary Differential Equations (ODE)

3.1 Introduction

An equation involving the derivatives of an unknown function y of a single variable x over an interval $x \in (I)$. More clearly and precisely speaking, a well defined ODEs must the following features:

- It can be written in the form:

$$F\left[x, y, y', y'', ..., y^n\right] = 0 \tag{3.1}$$

where the mathematical expression on the right hand side contains

1. variable x.
2. function y of x.
3. some derivatives of y with respect to x.

- The values of variables x, y must be specified in a certain number field, such as $\mathbb{N}$, $\mathbb{R}$, or $\mathbb{C}$.

- The variation region of variable x of Eq. must be specified, such as $x \in [a, b]$, see [7]

3.2 Linear Equations

If the function F is linear in the variables $u_0, u_1, ..., u_n$, which means every term in F is proportional to $u_0, u_1, ..., u_n$, the ODE is said to be linear. If, in addition, F is homogeneous then the ODE is said to be homogeneous. The first of the above examples above is linear are linear, the second is non-linear and the third is linear and homogeneous. The general n-th order linear ODE can be written, see [7]

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

3.3 Homogeneous Linear Equations

The linear DEs is homogeneous, if and only if $f(x) \equiv 0$. Linear homogeneous equations have the important property that linear combinations of solutions are also solutions. In other words, if $y_1, y_2, ..., y_m$ are solutions and $c_1, c_2, ..., c_m$ are constants then

$$y = c_1 y_1 + c_2 y_2 + ... + c_m y_m \tag{3.2}$$

is also a solution, see [7]

3.4 General Solution

It represents the set of all solutions, i.e., the set of all functions which satisfy the equation in the interval (I). For example, given the differential equation

$$y\prime = 3x^2$$

Its general solution is

$$y = x^3 + C.$$

where C is an arbitrary constant. To select a specific solution, one needs to determine the constant C with some additional conditions. For instance, the constant C can be determined by the value of y at $x = 0$. This condition is called the initial condition, which completely determines the solution. More generally, it will be shown in the future that given a, b there is a unique solution y of the differential equation with the initial condition $y(a) = b$. Geometrically, this means that the one-parameter family of curves $y = x^2 + C$ do not intersect one another and they fill up the plane $\mathbb{R}^2$, see [7]

3.5 The classification for Ordinary Differential Equations

3.5.1 First-Ordinary Differential Equations

Recall that the general form of an ODEs of order n is

$$f(x, y, y', y'', ..., y^{(n)}) = 0, \qquad x \in [a, b]\,, \tag{3.3}$$

If we choose $n = 1$, then the equation

$$f(x, y, y') = 0, \qquad x \in [a, b], \tag{3.4}$$

is a first-Order Differential Equations. Since (3.4) is of order one, we may guess that the solution (if it exists) contains a constant c. That is, the family of solutions will be of the form

$$F(x, y, c) = 0 \tag{3.5}$$

Let us write (3.4) as

$$M(x, y)dx + N(x, y)dy = 0 \tag{3.6}$$

see [4]

1. **Linear Equations**

A first-Order Linear Differential Equations has the form:

$$a_0(x)y\prime + a_1(x)y = f(x) \tag{3.7}$$

where $a_0(x)$, $a_1(x)$ and $f(x)$ are continuous functions of x on some interval (I).

To bring it to normal form $y\prime = f(x, y)$ we have to divide both sides of the equation by $a_0(x)$. This is possible only for those x where $a_0(x) \neq 0$. After possibly shrinking (I) we assume that $a_0(x) \neq 0$ on (I). So our equation has the form (standard form)

$$y\prime + p(x)y = q(x)$$

with

$$p(x) = a_1(x)/a_0(x), q(x) = b(x)/a_0(x),$$

both continuous on (I). Solving for $y\prime$ we get the normal form for a linear first order ODEs, namely

$$y\prime = q(x) - p(x)y.$$

see [7]

2. **Separable equations**

A separable first order ODEs has the form

$$M(x)dx + N(y)dy = 0 \tag{3.8}$$

Then we say the variables are separable, and the solution can be obtained by integrating both sides with respect to x. Thus,

$$\int M(x)dx + \int N(y(x))\frac{dy}{dx}dx = c,$$

or for simplicity one can write,

$$\int M(x)dx + \int N(y)dy = c,$$

see [4]

3. **Linear homogeneous equations**

Definition 15 *A first-Ordinary Differential Equations in standard form is homogeneous if* $q(x) = 0$

$$\frac{dy}{dx} + p(x)y = 0 \tag{3.9}$$

see [7]

3.5.2 Second-Order Differential Equations

Second-order ODEs explicitly contain a second derivative term, but no higher derivatives. These equations are of the form

$$f(x, y, y', y'') = 0 \tag{3.10}$$

The quantities x, y, y' may not appear explicitly in a second-order ODE such as in the equation, $y'' = 0$. a function satisfying the equation $f(x, y, y', y'') = 0$ is called a solution. It can be verified by substituting the solution into the ODE, see [5]

A second-order differential equation. For the first-order ODEs, we found that their solutions contain one arbitrary constant, but for the second-order ODEs, the solutions must contain two arbitrary constants because two integrations are required to obtain these solutions. In general, finding the family of solutions for (3.10) is difficult and in most cases impossible, in particular when the equation is nonlinear. For this reason and the fact that linear equations are very often used in engineering and in the applied sciences, in this chapter we will deal with linear ODEs, see [4]

1. **Linear Differential Equations**

A second-order ODEs is called linear if it is written

$$y'' + p(x)\, y' + q(x)\, y = f(x) \tag{3.11}$$

This equation is linear in y and its derivatives.

Here, p, q, and f are any given functions of x. If the coefficient of y'' is not 1, it is put into the standard form by dividing all terms by that coefficient, see [5]

2. **Linear Differential Equations with constant coefficients**

We have emphasized that are no general methods for solving second order linear differential equation. Howver, there are some special cases for which solution methods do exist. In this and the following sections we consider such a case, linear equations with constant coefficients.

Linear differential equation of 2^{nd} order with coefficients constants is a differential equation of the form

$$ay'' + by' + cy = f(x) \tag{3.12}$$

Where $a, b, c \in \mathbb{R}(a \neq 0)$ and $f \in \mathbb{C}^0(I)$.

3. **Homogeneous Equations**

(a) Consider a second-order homogeneous linear equation in the general form:

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{3.13}$$

where p and q are continuous functions on some interval I.

The trivial solution, $y = 0$, is a particular solution of the homogeneous linear equation, see [8]

Let $y_1(x), y_2(x)$ be a fundamental system of solutions (nontrivial linearly independent particular solutions) of equation (3.13).Then the general solution is given by:

$$y = C_1 y_1(x) + C_2 y_2(x) \tag{3.14}$$

where C_1 and C_2 are arbitrary constants.

b. Let $y_1 = y_1(x)$ be any nontrivial particular solution of equation (3.11). Then its general solution can be represented as:

$$y = y_1 \left(C_1 + C_2 \int \frac{e^{-F}}{y_1^2} dx \right), \text{ where } F = \int p(x)\, dx \tag{3.15}$$

4. **Nonhomogeneous Equations**

A second-order nonhomogeneous linear equation has the form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

where p, q, f are continuous functions on an interval I.

The objectives of this section are to determine the "structure" of the set of solutions of (3.11).

As we shall see, there is a close connection between equation (3.11) and

$$y'' + p(x)\, y' + q(x)\, y = 0$$

In this context, equation (3.13) is called the reduced equation of equation (3.11).

Remark 16 *This result illustrates why the emphasis is on linear homogeneous equations. To find the general solution of the nonhomogeneous equation* (3.11) *we need a fundamental set of solutions of the reduced equation* (3.13) *and one particular solution of* (3.11).

Chapter 4

Application

4.1 Legendre Collocation Method

The technique as adapted here involves constructing approximating trial solution of the form

$$u_N(x) = \sum_{n=0}^{N} c_n P_n(x) \tag{4.1}$$

N : is the degree of the trial solution.

c_n : are a coefficients (degree of freedom).

$P_n(x)$: is Legendre polynomial of order n.

The technique in this method, demands that (4.1) is substituted into the differential equations:

$$\begin{cases} \frac{d^2u}{dx^2} + \frac{du}{dx} + u = f(x), \qquad x \in [a,b] \\ \qquad u(a) = A, \; u(b) = B \end{cases} \tag{4.2}$$

to a points boundary value problems of the 2^{nd} order.

$$R_N(x) = \sum_{n=0}^{N} c_n P_n''(x) + q(x) \sum_{n=0}^{N} c_n P_n'(x) + r(x)P_n(x) - f(x) \neq 0 \tag{4.3}$$

we choose a point x_i called the collocation point in the domain $[a, b]$ for each undetermined parameter c_n. As is established in literatures, the point x_i can be located anywhere in the domain and on the boundary. There methode of selecting these points are considered in the following sub-section, see [6]

4.1.1 Collocation points at zeros of Legendre polynomials

It is important to first consider the following ideas behind location and interlacing of zeros of Legendre polynomial and their associated properties. Plotting $P_n(x)$ for the first few values of n we observed the following:

1. All zeros of $P_n(x)$lie in $-1 < x < 1$.
2. Between two consecutive zeros of $P_{n+1}(x)$ there is one of $P_n(x)$.
3. Between two consecutive zeros of $P_n(x)$ there is one of $P_{n+1}(x)$.
4. Between the smallest zero of $P_n(x)$ and -1 there is one zero of $P_{n+1}(x)$ and between the largest zero of $P_n(x)$ and +1, there is one zero of $P_{n+1}(x)$.

According to the idea of Lanczos (4.4), collocating at the zeros of orthogonal polynomials requires that at the zeros of relevant orthogonal polynomial, the residual equation (4.3) is satisfied, thus yielding a number of collocation equation of the form

$$\sum_{n=0}^{N} c_n P_n''(x_i) + q(x_i) \sum_{n=0}^{N} c_n P_n'(x_i) + r(x_i)P_n(x_i) - f(x_i) = 0 \tag{4.4}$$

It is worthy of note that the polynomial $P_{n-1}(x)$ is used in obtaining the Colloction points so as to yield $N-1$ zeros needed to collocate equation (4.3), see [6]

Example 17 *solve the boundary value problem*

$$\begin{cases} \frac{d^2u}{dx^2} - u = -4xe^x \\ u(0) = u(1) = 0 \end{cases}$$

The analytical solution is:

$$u(x) = x(1-x)e^x$$

x	Collocation point	$N = 4$
0	Points at zeros of P_{N-1}(x)	$3.2960e-17$
0.1	Points at zeros of P_{N-1}(x)	$7.3418e-4$
0.2	Points at zeros of P_{N-1}(x)	$1.7550e-3$
0.3	Points at zeros of P_{N-1}(x)	$1.9616e-3$
0.4	Points at zeros of P_{N-1}(x)	$1.0944e-3$

Table1: Table of Error

Example 18 *solve the second-order differential equation*

$$(1+x^2)\frac{d^2u}{dx^2}+4x\frac{du}{dx}+2u=0$$

Subject to boundary conditions

$$\begin{aligned} u(0) &= 1 \\ u(2) &= 0.2 \end{aligned}$$

The exact solution is:

$$u(x)=\frac{1}{1+x^2}$$

x	*Collocation point*	$N=4$
0	*Points at zeros of* $P_{N-1}(x)$	0
0.2	*Points at zeros of* $P_{N-1}(x)$	$5.2505e-3$
0.4	*Points at zeros of* $P_{N-1}(x)$	$1.1488e-2$
0.6	*Points at zeros of* $P_{N-1}(x)$	$1.9646e-2$
0.8	*Points at zeros of* $P_{N-1}(x)$	$3.2668e-2$

Table 2: Table of Error

Example 19 *Consider the differential equation*

$$x^2 \frac{d^2u}{dx^2} - 2u = -x$$

Subject to boundary conditions

$$u(2) = u(3) = 0$$

The analytical solution is

$$u(x) = \frac{1}{38}(19x - 5x^2 - \frac{36}{x})$$

x	*Collocation point*	$N = 4$
2.0	*Points at zeros of* $P_{N-1}(x)$	$2.2204e - 16$
2.1	*Points at zeros of* $P_{N-1}(x)$	$1.6734e - 5$
2.2	*Points at zeros of* $P_{N-1}(x)$	$4.0414e - 5$
2.3	*Points at zeros of* $P_{N-1}(x)$	$4.7938e - 5$
2.4	*Points at zeros of* $P_{N-1}(x)$	$3.6033e - 5$

Table 3: Table of Error

Conclution

In this thesis, we try to study the numerical solution of Ordinary Differential Equations, due to difficulty of analytical solutions. Methods of numerical accuracy of Ordinary Differential Equations play a very important role in various scientific fields. With the great advantage of this method is that the coefficients of the solution can be easily found by a computer.

We worked numerically on Differential Equations by the method of collocation using the Legendre polynomials. Where we clearly observ from the results obtaind that how close the solutions are. That's how we conclude that this technique is not only an effective numerical thecnique, but it is also an effective way to obtain an approximate solution that is close enough to the exact solution.

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ملخص:

في هذه المذكرة قمنا بتقديم طريقة عددية لحل المعادلات التفاضـــلية الخطية من الدرجة الاولى والثانية التي يصــعب حلها بالطرق العادية. وذلك باســتخدام كثيرات حدود لجندر بواسطة طريقة التجميع، مع ادراج امثلة توضـيحية لمقارنة الحل الدقيق والحل التقريبي للتحقق من دقة وفعالية الطريقة.

الكلمات المفتاحية: فضاء هلبرت، معادلات تفاضلية، كثيرات حدود لجندر، طريقة التجميع.

Résumé :

Ce mémoire, nous présentons une méthode numérique pour résoudre des équations différentielles linéaires du premier et du second ordre qui sant difficiles à résoudre explicitement (solution exacte). En utilisant des polynômes de legendre par la méthode collocation, avec des exemples illustratifs pour comparer la solution exacte avec la solution approchées pour vérifier l'exactitude et l'efficacité de la méthode.

Les mots clés : Espaces de Hilbert, équations différentielles, polynôme de Legendre, méthode de Collocation.

Abstract :

In this thesis, we presented an numerically method for approximate first-and second-order linear differential equations that are difficult to solve by explicite methods. Using legendre polynomials by the method of collocation, with illustrative examples to compare the exact solution and the approximate solution to verify the accuracy and effectiveness of the method.

Key words : Hilbert Spaces, differential equations, Legendre polynomials, collocation method.