PEOPLE's DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

**Mohamed Boudiaf university of Msila**
**Faculty of Mathematics and computer sciences**
**Department of Mathematics**

# *Master memory*

**Field** : Mathematics and computer sciences
**Speciality** : Mathematics and computer sciences
**Option** : Partial differential equation and applications

## Theme

# STUDY SOME IMPULSIVE SECOND-ORDER BOUNDARY PROBLEMS ON BOUNDED AND UNBOUNDED DOMAINS

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**Publicly supported on:** 11/06/2026.

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University year 2025/2026.

# Acknowledgement

♢ *We*
*thank God*
*Almighty for giv-*
*ing us health. and the*
*desire to start and finish*
*this dissertation. First of all, this*
*work would not be as rich and would*
*not have could have had the day without*
*the help and supervision of MR. Dahmane*
*BOUAFIA, we thank him for the qual-*
*ity of his supervision exceptional,*
*for his patience, his rigor and*
*his availability during*
*our preparation of*
*this disserta-*
*tion.*
♢

❖ *Nedjwa*

# Dedications

*In the name of Allah the merciful*

*I dedicate this work :*

*-To my father, whom God recommended to me in righteousness and kindness and gave me the years of my life .*

*-To my mother, the sea of love, tenderness, and pulsation that resides in my veins.*

*-To my respected teachers, who enlightened my path with knowledge and wisdom, and whose guidance greatly contributed to my academic formation.*

*-All my friends and family from the Mathematics department and the entire 2025/2026 class of de l'EDP.*

*-All my family "Lahouaou" who supported me, I couldn't name for fear to forget someone .*

*-To every heart that beat for me with love and fear .*

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# Notation

We introduce the necessary notations and definitions that are used later on.

$L^p(\Omega)$ $= \left\{u : \Omega \to \mathbb{R} \text{ measurable} : |u|^p \in L^1(\Omega)\right\}$ with $1 \leq p < \infty$.

$\|u\|_{L^p}$ $= \left(\int_\Omega |u(t)|^p dt\right)^{\frac{1}{p}}$.

$L^\infty(\Omega)$ $= \{u : \Omega \to \mathbb{R}$ measurable and $\exists C : |u(t)| \leq C$ *a.e on* $\Omega\}$.

$\|u\|_{L^\infty}$ $= \inf\{C : |u(t)| \leq C$ a.e on $\Omega\}$.

$C(\Omega)$ : space of continuous functions on $\Omega$.

$C_0^\infty(0, +\infty)$ : the space of infinitely differentiable functions with compact support in $(0, +\infty)$.

$H$ : Hilbert space.

$\langle ., . \rangle$ : duality bracket.

$(., .)$ : inner product.

$H^m(0, 1)$ $= \{u \in L^2(0, 1) : |u^{(i)}| \in L^2(0, 1) \quad \forall i \leq m\}$.

$H_0^m(0, 1)$ : the completion of $\mathcal{D}(0, 1)$ in $H^m(0, 1)$.

$C[0, T]$ $:= \left\{u : [0, T] \to \mathbb{R}\text{is continuous}\right\}$.

$\|u\|_\infty$ $= \sup_{t \in [0,T]} |u(t)|$.

$X^*$ : dual topology .

$D(A)$ : domain of definition of an unbounded operator $A$.

$R(A)$ : image of $A$ which is also noted by $ImA$

$\mathcal{L}(X, Y)$ : set of continuous linear applications.

$X \hookrightarrow Y$ : we write $X \hookrightarrow Y$ to mean that $X$ is included in $Y$ and the canonical injection of $X$ into $Y$ is continuous.

$X \hookrightarrow\hookrightarrow Y$ : to imply that $X$ is included in $Y$ and that the canonical injection of $X$ into $Y$ is compact.

$u_n \rightharpoonup u_0$ : $u_n$ converge weakly to $u_0$.

*i.e.* that is to say.

*a.e.* almost everywhere.

| | |
|---|---|
| $dF$ | derivative in the sense of Frechet which is also noted by $F'$. |
| $d_G F$ | derivative in the sense of Gateaux which sometimes is also noted by $F'$. |
| $\overline{B_\rho(0)}$ | the closed ball with center 0 and radius $\rho$. |
| $(P.S)$ | Palais-Smale condition. |
| $diamF$ | Diameter of a set $F$. |
| $l.s.c$ | lower Semi continuous . |
| $u.s.c$ | upper Semi continuous . |
| $w.l.s.c$ | Weakly lower semi continuous . |
| $w.l.s.c$ | Weakly upper Semi continuous . |

# Introduction

Variational methods are among the most important mathematical techniques used in the study of boundary value problems for impulsive differential equations, especially those involving parameters or mixed boundary conditions. These methods are based on the study of the energy functional associated with the problem and on transforming the existence of solutions into the search for critical points of such functionals by using tools from functional analysis and critical point theory.

Impulsive differential equations play an important role in modeling many real-world phenomena subjected to sudden changes at specific moments of time, such as mechanical systems, biology, population dynamics, control theory, and spacecraft motion. These systems are characterized by instantaneous perturbations that produce abrupt changes in the velocity or the state of the system, which makes their study very important in applied mathematics.

In recent years, many researchers have investigated second-order impulsive boundary value problems using variational methods. Among these works, *Jun Wang* and *Weigao Ge* studied the existence and multiplicity of solutions for second-order impulsive mixed boundary value problems on the bounded interval $[0, T]$. Their results were established by applying critical point theory[6], [15], and the Saddle Point Theorem [9].

In this memory, we consider the following impulsive Dirichlet boundary value problems

$$(P_1)\begin{cases} -u''(t) + g(t)u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = I_j(u(t_j)), \quad j = 1, 2, \ldots, p, \end{cases}$$

where

$$0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T,$$

with

$$g \in L^\infty([0, T]),$$

the function

$$f : [0, T] \times \mathbb{R} \to \mathbb{R}$$

is continuous, and

$$I_j : \mathbb{R} \to \mathbb{R}, \quad j = 1, 2, \ldots, p,$$

are continuous functions describing the impulsive effects and

$$(P_2) \begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), & \text{a.e. } t \in [0, +\infty),\ t \neq t_j, \\ \Delta u'(t_j) = g(t_j) I_j(u(t_j^-)), & j \in \mathbb{N}^*, \\ u(0) = u(+\infty) = 0, \end{cases}$$

where $\lambda$ is a positive parameter, the functions $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $I_j : \mathbb{R} \to \mathbb{R}$, $j \in \mathbb{N}^*$, and $g : [0, +\infty) \to [0, +\infty)$ are continuous with

$$\sum_{j=1}^{+\infty} g(t_j) < +\infty.$$

The aim of this work is to investigate the existence of weak solutions for this impulsive Dirichlet problems by using variational methods and critical point theory in suitable Sobolev spaces. Our analysis is based on some fundamental tools from functional analysis such as the Lax–Milgram theorem [15], the Mountain Pass Theorem [6], [15], and the Saddle Point Theorem[9].
This dissertation is organized into three chapters.

In the first chapter, we present some preliminary notions and basic results that will be used throughout this work, including Banach and Sobolev spaces, continuity and differentiability of operators, lower semicontinuous functionals, and some fundamental results from critical point theory.
The second chapter is devoted to the variational framework associated with the impulsive problem $(P_1)$. We establish the weak formulation of the problem and construct the corresponding energy functional.

Finally, in the third chapter, we prove the main existence results by applying variational techniques and critical point theory for the problem $(P_2)$. Some illustrative examples are also provided.

Chapter 1

# Some preliminaries

In this chapter we give a series of notions that will be useful for the rest of this memory. We recall some definitions and properites used in the remainder of the these thesis , this to allow a more rapid assimilation of our work . These consist essentially of functional analysis results . We thus find some properites for weak convergence , continuity and differentiability of operators in Banch space , critical points theory and Ekeland's variational principls .

## 1.1 Banach and Hilbert spaces

**Definition 1.1. (Normed spaces)**[1]
A vector space $X$ is said to be a normed space if to every $x \in X$ there is associated a nonnegative real number $||x||$, called the norm of $x$, such that

1. $||x|| \geq 0$ and $||x|| = 0$ if and only if $x = 0$,
2. $||\alpha x|| = |\alpha| \, ||x||$ for every scalar $\alpha$ and every $x \in X$,
3. $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$.

**Definition 1.2. (Bancach spaces)**[1] is a normed space which is complete in the metric defined by its norm; this means that every Cauchy sequence is required to converge.

**Definition 1.3. (Cauchy Sequence)**[1] Suppose $d$ is a metric on a set $X$. A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if for every $\varepsilon > 0$ there exists an integer $N$ such that

$$d(x_m, x_n) < \varepsilon \quad \text{whenever } m > N \text{ and } n > N.$$

If every Cauchy sequence in $X$ converges to a point of $X$, then the metric $d$ is said to be complete on $X$.

**Definition 1.4.** Hilbert space $H$ is an inner product space which is complete with respect to the metric induced by its inner product.

**Definition 1.5.** Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. A family of vectors $\mathcal{B} = (e_i)_{i \in I}$ in $H$ is a Hilbert basis (or complete orthonormal basis) if it satisfies the following two conditions:

1. **Orthonormal family:** For all $(i, j) \in I^2$,
$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
2. **Total family:** The vector subspace spanned by $(e_i)_{i \in I}$, denoted $\text{span}(\{e_i\}_{i \in I})$, is dense in $H$. That is to say:
$$\overline{\text{span}(\{e_i\}_{i \in I})} = H.$$
If $(e_i)_{i \in I}$ is a Hilbert basis, then every vector $x \in H$ can be uniquely written as a series sum:
$$x = \sum_{i \in I} \langle x, e_i \rangle e_i$$

**Proposition 1.1.** ***(Parseval identity)***
*If $(e_i)_{i \in I}$ is a Hilbert basis, then every vector $x \in H$ we have following equality holds:*
$$\|x\|^2 = \sum_{i \in I} |\langle x, e_i \rangle|^2,$$
*is called the Parseval equality holds.*

**Theorem 1.1.** ***(Compact spectral decomposition)*** *Let $H$ be a (separable) Hilbert space and $T : H \to H$ a compact self-adjoint operator. Then:*

1. *The eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ of $T$ are real and tend to $0$.*
2. *The eigenspaces $E_{\lambda_n} = \ker(T - \lambda_n I)$ are of finite dimension for $\lambda_n \neq 0$.*
3. *$H$ decomposes into an orthogonal direct sum:*
$$H = \ker(T) \oplus^{\perp} \left( \bigoplus_{\lambda_n \neq 0} E_{\lambda_n} \right)$$

*Consequently, there exists a Hilbert basis for $H$ consisting of eigenvectors of $T$.*

**Theorem 1.2.** ***(Hilbert-Schmidt)*** *Let $H$ be a separable Hilbert space and $T : H \to H$ be a compact self-adjoint operator. Let $\{\lambda_i\}_{i=1}^N$ $(N \leq \infty)$ be the non-zero eigenvalues of $T$ and $\{\phi_i\}_{i=1}^N$ be the corresponding orthonormal eigenfunctions. Then for any $x \in H$ :*
$$Tx = \sum_{i=1}^{N} \lambda_i \langle x, \phi_i \rangle \phi_i \tag{1.1}$$
*Furthermore, the expansion converges in the norm of $H$, and if $N = \infty$, then $\lim_{i \to \infty} \lambda_i = 0$.*

## 1.2 Operators on Banach spaces

### 1.2.1 Continuity, convexity and coercivity of operators

In the following, we state some definitions concerning continuity properties.

**Definition 1.6.** The operator $T : E \to F$ is called continuous at the point $x_0 \in E$ if for any sequence $(x_n) \subset E$ we have

$$x_n \to x_0 \text{ in } E \Rightarrow T(x_n) \to T(x_0) \text{ in } F$$

Or,

$$\|x_n - x_0\|_E \to 0 \Rightarrow \|T(x_n) - T(x_0)\|_F \to 0.$$

This definition is equivalent to

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in E : \|x - x_0\|_E \leq \delta \Rightarrow \|T(x) - T(x_0)\|_F \leq \varepsilon.$$

**Example 1.1.** Let $(E, \|\cdot\|)$ be a Banach space. We consider $N : E \to \mathbb{R}$ defined by $N(x) = \|x\|$. Then $N$ is a continuous functional. Indeed, for any $(x_n)_n \subset E$ such that $x_n \to x$ in $E$, since

$$|N(x_n) - N(x)| = |\,\|x_n\| - \|x\|\,| \leq \|x_n - x\|,$$

and $\|x_n - x\| \to 0$, we have

$$N(x_n) \to N(x) \text{ in } \mathbb{R}.$$

There are some other definitions that are related to the concept of weak convergence.

**Definition 1.7. (Weakly continuous)**[2] Let $T : E \to F$ be an operator. $T$ is called weakly continuous at $x_0 \in E$ if for every sequence $(x_n) \subset E$, we have

$$x_n \rightharpoonup x_0 \text{ in } E \ \Rightarrow\ T(x_n) \rightharpoonup T(x_0).$$

**Definition 1.8. (Compact operator)**[2] An operator $T : E \to F$ is called compact if it is continuous and $T(B)$ is a relatively compact set in $F$ (i.e., $T(B)$ is a compact set in $F$); here

$$B = \{x \in E : \|x\|_E < 1\}.$$

There is an equivalent definition using sequences.

**Definition 1.9. (Coercivity)**[2]

A functional $\varphi : E \to \mathbb{R}$ on a Banach space $E$ is called coercive if, for every sequence $(u_n)_{n\in\mathbb{N}} \subset E$,

$$\lim_{\|u_n\|\to+\infty} \varphi(u_n) = +\infty.$$

**Definition 1.10. (Dual space)**[2]

Let $A : E \to F$ be a linear operator. One says that $A$ is bounded (or continuous) if there exists a constant $c \geq 0$ such that

$$\|Au\|_F \leq c\,\|u\|_E, \qquad \forall\, u \in E.$$

**Definition 1.11. (Reflexive space)**[2]

The space $E$ is said to be reflexive if the canonical injection $i : E \to E''$ is surjective, i.e. $i(E) = E''$. When $E$ is reflexive, $E''$ is usually identified with $E$; $(E'' = E)$.

**Example 1.2.** 1. finite dimensional spaces are reflexive (since $\dim E = \dim E' = \dim E''$).

2. $L^p$ spaces are reflexive for $1 < p < \infty$. $L^1$ and $L^\infty$ are not reflexive.

3. Hilbert spaces are reflexive.

The following result shows an important compactness property in reflexive spaces (see [2]).

**Theorem 1.3.** *[2] Assume that $E$ is a reflexive Banach space and let $(x_n)$ be a bounded sequence in $E$. Then there exists a subsequence $(x_{n_k})$ that converges weakly to some $x$ in $E$.*

## 1.3 Lower and Upper Semi-continuity

The following concepts are often found in minimization theory of functionals.

**Definition 1.12.** A functional $\varphi : E \to \mathbb{R}$ is called lower semi-continuous at a point $x_0 \in E$ if for every sequence $(x_n) \subset E$, we have

$$x_n \to x_0 \quad \Rightarrow \quad \varphi(x_0) \leq \liminf_{n\to+\infty} \varphi(x_n).$$

**Example 1.3.** Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. The functional $N : H \to \mathbb{R}$ defined by

$$N(x) = \|x\|, \qquad \text{where } \|x\| = \sqrt{\langle x, x\rangle},$$

is weakly lower semicontinuous (w.l.s.c). Indeed, let $(x_n) \subset H$ and $x_0 \in H$ such that $x_n \rightharpoonup x_0$. Then

$$\langle x_n, y\rangle \to \langle x_0, y\rangle, \qquad \forall y \in H.$$

In particular, taking $y = x_0 \in H$, we obtain

$$\langle x_n, x_0\rangle \to \langle x_0, x_0\rangle = \|x_0\|^2. \tag{1.2}$$

By a direct calculation, we get

$$\|x_n - x_0\|^2 = \langle x_n - x_0, x_n - x_0\rangle = \|x_n\|^2 - 2\langle x_n, x_0\rangle + \|x_0\|^2.$$

Since $\|x_n - x_0\|^2 \geq 0$, we deduce that

$$\|x_n\|^2 \geq 2\langle x_n, x_0\rangle - \|x_0\|^2. \tag{1.3}$$

Passing to the limit in 1.2, and using1.3,we obtain

$$\liminf_{n\to+\infty} \|x_n\|^2 \geq 2\|x_0\|^2 - \|x_0\|^2 = \|x_0\|^2.$$

Hence, $N$ is weakly lower semicontinuous.

**Remark 1.1.** 1. It is easy to see that if $\varphi$ is continuous in $x_0$, then $\varphi$ is (u.s.c.) and (l.s.c.) and conversely.

2. It is easy to see that if $\varphi$ is weacly continuous in $x_0$, then $\varphi$ is (w.u.s.c.) and (w.l.s.c.) and conversely.

## 1.4 Differentiability in Banach spaces

### 1.4.1 Gâteaux Differentiability

**Definition 1.13.** [2] Let $E$ be a Banach space, $\Omega \subset E$ an open set, and let $\varphi : \Omega \to \mathbb{R}$ be a functional. We say that $\varphi$ is Gâteaux differentiable (G-differentiable) at $u \in \Omega$ if there exists $A \in E'$ (A linear and continuous functional), denoted by $\varphi'_G(u)$, such that for all $v \in E$,

$$\lim_{t\to 0} \frac{\varphi(u + tv) - \varphi(u)}{t} = \langle A, v\rangle. \tag{1.4}$$

If $\varphi$ is Gâteaux differentiable at $u$, there exists only one linear functional $A \in E'$ satisfying (1.4). It is called the Gâteaux differential of $\varphi$ at $u$ and is denoted by $\varphi'_G(u)$.

**Definition 1.14.** [2]

If the functional $\varphi$ is Gâteaux differentiable at every $u$ of an open set $U \subset E$, we say that $\varphi$ is Gâteaux differentiable on $U$. The map

$$\begin{aligned} \varphi'_G : E &\to E', \\ u &\mapsto \varphi'_G(u) \end{aligned} \tag{1.5}$$

is colled the Gâteaux derivative of $\varphi$.

**Example 1.4.** The functional defined by

$$\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$$

$$u \mapsto \varphi(u) = \int_\Omega |\nabla u|^p \, dx = \|u\|^p_{W_0^{1,p}}$$

is Gâteaux differentiable and we have

$$\langle \varphi'_G(u), v \rangle = p \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx.$$

In the particular case $p = 2$, the functional defined by

$$\varphi : H_0^1(\Omega) \to \mathbb{R}$$

$$u \mapsto \varphi(u) = \int_\Omega |\nabla u|^2 \, dx = \|u\|^2_{H_0^1}$$

is Gâteaux differentiable and we have

$$\langle \varphi'_G(u), v \rangle = 2 \int_\Omega \nabla u \cdot \nabla v \, dx.$$

### 1.4.2 Frechet differentiability

**Definition 1.15.** [2]

Let $E$ be a Banach space, $\Omega \subset E$ an open set and let $\varphi : \Omega \to \mathbb{R}$ be a functional. We say that $\varphi$ is Frechet differentiable at $u \in \Omega$ if there exists $A_u \in E'$ such that

$$\lim_{\|v\| \to 0} \frac{\varphi(u+v) - \varphi(u) - A_u v}{\|v\|} = 0. \tag{1.6}$$

Thus, for a Frechet differentiable functional $\varphi$, we have

$$\varphi(u+v) - \varphi(u) = A_u(v) + o(\|v\|),$$

where

$\lim_{\|v\| \to 0} \frac{o(\|v\|)}{\|v\|} = 0.$

If a functional $\varphi$ is differentiable at $u$, then $A_u$ is unique.

**Proposition 1.2.** *[2]*

*Suppose that $\Omega \subseteq E$ is on open set, such that $\varphi$ is G-differentiable in $\Omega$ and that*

$$\varphi'_G : \Omega \subset E \to E'$$

*is continuous, Then $\varphi$ is also Frechet differentiable at $u$, and we have*

$$\varphi'_G(u) = \varphi'(u).$$

*The proof of Proposition (1.2) is given in [2].*

**Proof.** Let $r \in (0, 1]$ be such that $B_r \subset U$. Since $d_G\varphi : U \to E^*$ is continuous at $u$, for every $\varepsilon > 0$ there exists $\delta \in (0, 1]$ such that

$$\|d_G\varphi(u + tv) - d_G\varphi(u)\| < \varepsilon \quad \text{for } \|v\| < \delta, |t| < 1. \tag{1.7}$$

On the other hand,for each $v \in B_r(0)$, the function

$$t \in (0, 1] \mapsto (d_G\varphi(u + tv), v),$$

is the derivative of the function

$$g(t) = \varphi(u + tv), \quad t \in [0, 1].$$

Consequently

$$\int_0^1 (d_G\varphi(u + tv), v)dt = g(1) - g(0) = \varphi(u + tv) - \varphi(u).$$

Hence

$$\varphi(u + v) - \varphi(u) - (d_G\varphi(u), v) = \int_0^1 (d_G\varphi(u + tv) - d_G\varphi(u), v)dt. \tag{1.8}$$

Let

$$R(u, v) = \int_0^1 (d_G\varphi(u + tv) - d_G\varphi(u), v)dt.$$

From (1.7) for all $v \in B_\delta(0)$ we have

$$\|R(u, v)\| \leq \int_0^1 \|d_G\varphi(u + tv) - d_G\varphi(u)\|\|v\|dt \leq \varepsilon\|v\|.$$

Hence $R(u, v) = o(\|v\|)$ when $\|v\| \to 0$. Then, (1.8) shows that $d_G\varphi(u)$ is the Frechet derivative of $\varphi$ at $u$. □

**Remark 1.2.** In applications, it is easier to verify differentiability in the sense of Gâteaux . If one wants to prove that $\varphi \in C^1$, it is enough to prove that it is Gâteaux differentiable ,then to verify that the differentiable $d_G\varphi$ is continuous.

**Definition 1.16.** Let $u_0 \in X$. We say that $u_0$ is a local minimum of $\varphi$ if there exists $\delta > 0$ such that

$$\varphi(u_0) \leq \varphi(v), \quad \forall v \in B(u_0, \delta),\ v \neq u_0.$$

**Proposition 1.3.** *Let $u_0 \in U \subset X$ be a local minimum (or maximum) of $\varphi$. Then if $\varphi$ is Gâteaux differentiable in $u_0$, then*

$$d_G\varphi(u_0) = 0.$$

**Proposition 1.4.** *[6]*

*Let X be a Banach space, and $\varphi$ be an application from X to $\mathbb{R}$,be $u \in X$.*

*Assume that*

$$\varphi(u) = \inf_{v \in X} \varphi(v), i.e., \varphi(u) \leq E(v), \quad \forall v \in X$$

*Then, if $\varphi$ is Gâteaux differentiable in u, We have*

$$d_G\varphi(u) = 0.$$

## 1.5 Functional spaces

### 1.5.1 $L^p$ Spaces

**Definition 1.17.** [2]

Let $p \in \mathbb{R}$ with $1 \leq p < \infty$, we set

$$L^p(\Omega) = \left\{ \varphi : \Omega \to \mathbb{R}^N;\ \varphi \text{ is measurable and } \int_\Omega |\varphi(x)|^p dx < \infty \right\}.$$

with

$$\|\varphi\|_{L^p} = \|\varphi\|_p = \left( \int_\Omega |\varphi(x)|^p dx \right)^{\frac{1}{p}}.$$

**Definition 1.18.** [2] We set

$$L^\infty(\Omega) = \left\{ \varphi : \Omega \to \mathbb{R}^N;\ \varphi \text{ is measurable and } |\varphi(x)| \leq c \text{ a.e. on } \Omega \right\},$$

with

$$\|\varphi\|_{L^\infty} = \|\varphi\|_\infty = \inf\{c :\ |\varphi(x)| \leq c \text{ a.e. on } \Omega\}.$$

with c is a constant.

If $\varphi \in L^\infty(\Omega)$ then we have

$$|\varphi(x)| \leq \|\varphi\|_\infty \quad \text{a.e. in } x \in \Omega.$$

**Theorem 1.4.** *[2] $(L^p(\Omega), \|\cdot\|_{L^p})$ is a Banach space for any $1 \leq p \leq \infty$, and reflexive for $1 < p < \infty$, and separable for $1 \leq p < \infty$.*

**Notation 1.1.** Let $1 \leq p \leq \infty$, we denote by $p'$ the conjugate exponent

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

**Theorem 1.5.** ***(Holder's inequality )****[2]*

*Assume that $f \in L^p$ and $g \in L^p$ with $1 \leq p \leq \infty$, then*

$$fg \in L^1$$

*and*

$$\int_\Omega |fg| dx \leq \|f\|_{L^p} \|g\|_{L^{p'}}.$$

### 1.5.2 Some convergence criteria

**Theorem 1.6.** ***(Lebesgue Dominated Convergence Theorem)****[2]*

*Let $(\varphi_n)$ be a sequence of functions of $L^1$. We suppose that*

1. *$\varphi_n(x) \to \varphi(x)$ a.e. $x \in \Omega$,*
2. *there exists a function $g \in L^1$ such that for every $n$,*

$$|\varphi_n(x)| \leq g(x) \quad \text{a.e. in } \Omega.$$

*We have $\varphi \in L^1(\Omega)$ and*

$$\|\varphi_n - \varphi\|_{L^1} \to 0.$$

**Lemma 1.1.** ***(Fatou Lemma)****[2]*

*Let $(\varphi_n)$ be a sequence of functions in $L^1$ that satisfy*

1. *for all $n \in \mathbb{N}$, $\varphi_n \geq 0$ a.e.,*
2. *$\sup_n \int \varphi_n < \infty$.*

*For almost $x \in \Omega$ we set*

$$\varphi(x) = \liminf_{n\to\infty} \varphi_n(x).$$

*Then $\varphi \in L^1$ and*

$$\int \varphi \leq \liminf_{n\to\infty} \int \varphi_n.$$

### 1.5.3 $W^{1,p}(\Omega)$ Spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

**Definition 1.19.** We denote by $D(\Omega)$ the set of functions of class $C^\infty(\Omega)$ with compact support compact include in $\Omega$. The Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega),\ \exists g_i \in L^p(\Omega) \text{ such that } : \int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega g_i \varphi dx, \quad \forall \varphi \in D(\Omega), \quad \forall i = 1, \ldots, N \right\}.$$

We set

$$H^1(\Omega) = W^{1,2}(\Omega).$$

For $u \in W^{1,p}(\Omega)$ we set

$$\frac{\partial u}{\partial x_i} = g_i$$

and we write

$$\nabla u = grad\, u = \left( \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N} \right).$$

The space $W^{1,p}$ is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|\nabla u\|_{L^p}.$$

**Proposition 1.5.** *$W^{1,p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$, reflexive for $1 < p < \infty$, and separable for $1 \leq p < \infty$.*
*In particular $H^1(\Omega)$ is reflexive, separable and Hilbert space.*

### 1.5.4 $W_0^{1,p}(\Omega)$ Space

**Definition 1.20.** Let $1 \leq p \leq \infty$, $W_0^{1,p}$ means the closing of $D(\Omega)$ in $W^{1,p}$, we notice

$$W_0^{1,p}(\Omega) = \overline{D(\Omega)}^{W^{1,p}} = \{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

and

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

The space $W_0^{1,p}(\Omega)$ provided with the norm induced by $W^{1,p}$. $H_0^1$ is a Hilbert space for the scalar product of $H^1$. We put $\|u\|_{H_0^1} = \|u\|$.

**Remark 1.3.** When $\Omega = \mathbb{R}^N$, we know that $D(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, and therefore

$$W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N).$$

**Proposition 1.6.** ***(Poincare inequality)****[2] Let $\Omega \subset \mathbb{R}^N$ be an open set. Then there exists a constant $c > 0$ such that*

$$\|u\|_{W^{1,p}(\Omega)} \leq c\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega).$$

*In other words, on $W_0^{1,p}(\Omega)$, the quantity $\|\nabla u\|_{L^p(\Omega)}$ is a norm equivalent to the $W^{1,p}$ norm.*

### 1.5.5 Some classical compactness theorem

**Theorem 1.7.** ***(Ascoli-Arzela)****[13]*

*Let $(\varphi_n)$ be a sequence of functions defined on a compact metric space $(X, d)$ and with real values. We assume that the sequence $(\varphi_n)$ has the following two properties:*

(1) *$(\varphi_n)$ is equicontinuous, i.e., $\forall \varepsilon > 0, \exists \eta > 0, \forall x, y \in X, \forall n \in \mathbb{N} : d(x, y) < \eta \Longrightarrow |\varphi_n(x) - \varphi_n(y)| < \varepsilon$.*

(2) *The sequence $(\varphi_n(x))_n$ is bounded for all $x \in X$.*

*Then there exists a subsequence of $(\varphi_n)$ that converges to a continuous function $\varphi$ in $C(X, \mathbb{R})$.*

**Lemma 1.2.** ***(C. Corduneanu)****[14]*

*Let $D \subset C_l([0, +\infty), \mathbb{R})$ be a bounded set. Then $D$ is relatively compact if the following conditions hold:*

*(a) $D$ is equicontinuous on any compact sub-interval of $\mathbb{R}^+$, i.e.*

$$\forall\, J \subset [0, +\infty) \text{ compact}, \forall\, \varepsilon > 0, \exists\, \delta > 0, \forall\, x_1, x_2 \in J : \\ |x_1 - x_2| < \delta \Longrightarrow |u(x_1) - u(x_2)| \leq \varepsilon, \forall\, u \in D,$$

*(b) $D$ is equiconvergent at $+\infty$ i.e.,*

$$\forall\, \varepsilon > 0, \exists\, T = T(\varepsilon) > 0 \text{ such that} \\ \forall\, x : x \geq T(\varepsilon) \Longrightarrow |u(x) - u(+\infty)| \leq \varepsilon, \forall\, u \in D.$$

## 1.6 Theory of Critical Points

**Definition 1.21. (Critical Points)**[6], [15]

Let $\varphi$ be a differentiable functional from $X$ to $\mathbb{R}$. A point $u \in X$ is said to be a critical point for $\varphi$ if and only if

$$d\varphi(u) = 0.$$

**Definition 1.22. (Critical Values)**[9] [15]

A number $c \in \mathbb{R}$ is called a critical value of the functional $\varphi$, of class $C^1$ defined on $X$, if there exists $u \in X$ such that

$$\varphi(u) = c, \quad d\varphi(u) = 0.$$

**Proposition 1.7.** *[6]*

*Let $X$ be a Banach space, and $\varphi$ a mapping from $X$ to $\mathbb{R}$, let $u \in X$. Suppose that*

$$\varphi(u) = \inf_{v \in X} \varphi(v),$$

*that is*

$$\varphi(u) \leq \varphi(v), \quad \forall v \in X.$$

*Then, if $\varphi$ is Gâteaux-differentiable at $u$, we have*

$$d_G\varphi(u) = 0.$$

**Proof.** Under the hypothesis of this proposition, suppose without loss of generality that $u_0$ is a minimum point (otherwise consider the function $-\varphi$ instead of $\varphi$). Since $u_0 \in U$ and $U$ is open, there exists a positive real number $r$ such that the open ball $B_r(u_0)$ is contained in $U$. Now, let $h \in X \setminus \{0\}$. Then for every $t$ such that

$$|t| \leq \frac{r}{||h||}$$

we have $\varphi(u_0 + th) \geq \varphi(u_0)$, and thus by the Gâteaux differentiability of $\varphi$ at $u_0$, we have

$$d_G\varphi(u_0)(h) = \lim_{t \to 0^+} \frac{1}{t} [\varphi(u_0 + th) - \varphi(u_0)] \geq 0.$$

It also follows that $d_G\varphi(u_0)(-h) \geq 0$, i.e.,

$$d_G\varphi(u_0)(h) \leq 0,$$

by the linearity of $d_G\varphi(u_0)$. Consequently, we obtain $d_G\varphi(u_0)(h) = 0$ for $h \in X$. Thus $d_G\varphi(u_0) = 0$. □

**Definition 1.23.** Let $u_0 \in X$. We say that $u_0$ is a local minimum for $\varphi$ if there exists $\delta > 0$ such that

$$\varphi(u_0) \leq \varphi(v), \quad \forall v \in B(u_0, \delta),\ v \neq u_0.$$

**Proposition 1.8.** *Let $u_0 \in X$ be a local minimum of $\varphi$. If $\varphi$ is Gâteaux differentiable at $u_0$, then*

$$d_G\varphi(u_0) = 0.$$

*For convex functionals (or convex spaces), a classical result is given by the following theorem.*

**Theorem 1.8.** *[6], [9]*

*Let $C$ be a closed convex subset of a reflexive Banach space $X$. Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$. Assume*

*(i) $\varphi$ is coercive and $\varphi \not\equiv +\infty$, that is*

$$\lim_{\|u\| \to +\infty} \varphi(u) = +\infty.$$

*(ii) $\varphi$ is lower semi-continuous for the weak convergence.*

*Then there exists $u \in C$ such that*

$$\varphi(u) = \inf_{v \in C} \varphi(v)\ (< +\infty).$$

*Moreover, if $\varphi$ is Gâteaux differentiable at $u$, then*

$$d_G\varphi(u) = 0.$$

## 1.7 Palais-Smale's sequence and condition

**Definition 1.24.** [15]

Let $X$ be a Banach space and $\varphi : X \to \mathbb{R}$ a $C^1$-functional. We say that $\varphi$ satisfies the Palais-Smale condition, denoted $(P.S)$, if any sequence $(u_n)$ in $X$ such that

$$(\varphi(u_n)) \text{ is bounded, and } \varphi'(u_n) \to 0, \text{ as } n \to +\infty \tag{1.9}$$

admits a convergent subsequence. Any sequence satisfying 1.9is called a Palais-Smale sequence.

**Definition 1.25.** Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$.We say that $\varphi$ is coercive if

$$\varphi(u) \to +\infty, \quad when \|u\| \to +\infty$$

i.e.,

$$\forall A > 0, \exists B > 0 \quad \text{such that } \|u\| \geq B \implies \varphi(u) \geq A.$$

**Theorem 1.9.** *[15], [6]*

*Let $C$ be a closed convex set in $X$, a reflexive Banach space.*

*Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$. We assume*

*i) $\varphi$ is coercive and $\varphi \not\equiv +\infty$, i.e. $\lim_{\|u\| \to +\infty} \varphi(u) = +\infty$,*

*ii) $\varphi$ satisfies the weak convergence condition (w.l.s.c.). Then there exists $u \in C$ such that*

$$\varphi(u) = \inf_{v \in C} \varphi(v) \quad (< +\infty).$$

*If, furthermore, $\varphi$ is Gâteaux-differentiable at $u$, then $d_G\varphi(u) = 0$.*

**Theorem 1.10.** *[6]*

*Let $X$ be a reflexive Banach space, and $\varphi$ a functional defined on $X$ such that*

1. *$\varphi$ is (w.l.s.c.),*
2. *The minimizing sequence of $\varphi$ is bounded on $X$,*

*Then $\varphi$ achieves its infimum on $X$.*

**Theorem 1.11.** ***(Lax Milgram)****[15] Let $H$ be a Hilbert space and $a : H \times H \to \mathbb{R}$ be a bounded bilinear form. If $a$ is coercive, i.e., there exists $\alpha > 0$ such that $a(u,u) \geq \alpha\|u\|^2$ for every $u \in H$, then for any $\sigma \in H'$ (the conjugate space of $H$),there exists a unique $u \in H$ such that*

$$a(u,v) = (\sigma, v), \quad \forall v \in H.$$

*Moreover, if $a$ is also symmetric, then the functional $\psi : H \to \mathbb{R}$ defined by*

$$\psi(v) = \frac{1}{2}a(v,v) - (\sigma, v)$$

*attains its minimum at $u$.*

## 1.8 Mountain Pass Theorem

**Theorem 1.12.** ***(Mountain Pass Theorem)****[6], [15]*

*Let $X$ be a real Banach space and $\varphi \in C^1(X, \mathbb{R})$ with $\varphi(0) = 0$. Suppose $\varphi(u)$ satisfies (P.S) condition and*

*(a) there are $\rho, \alpha > 0$ such that $\varphi(u) \geq \alpha$ when $\|u\|_X = \rho$,*

*(b) there is a* $\varphi \in X$, $\|e\|_X > \rho$ *such that* $\varphi(e) < 0$.

*Define*

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = 0, \gamma(1) = e\}.$$

*Then*

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} \varphi(\gamma(t)) \ge \alpha$$

*is a critical value of* $\varphi(u)$.

**Theorem 1.13.** ***(Saddle point theorem)****[9]*

*Let E be a Banach space. Let* $\varphi \in C^1(E, \mathbb{R})$ *be an even functional which satisfies the P.S. condition and* $\varphi(0) = 0$. *If* $E = V \oplus W$, *where V is finite dimensional, and* $\varphi$ *satisfies that*

*(I1) there are constants* $\rho, \alpha > 0$ *such that* $\varphi|_{\partial B_\rho \cap W} \ge \alpha$, $B_\rho = \{x \in E : \|x\| < \rho\}$,

*(I2) for each finite-dimensional subspace* $\tilde{E} \subset E$, *there is an* $R = R(\tilde{E})$ *such that* $\varphi \le 0$ *on* $\tilde{E} \setminus B_R(\tilde{E})$, *then* $\varphi$ *possesses an unbounded sequence of critical values.*

Chapter 2

# Study the existence of the solutions to an impulsive second-order boundary value problems on bounded interval

In this chapter, we discuss the existence of a solution for an impulsive second-order equation. To obtain our results, we use the variational method. In particular, we use a minimization theorem and the mountain pass lemma. We used as an example the article [59] which was published by authors Jun Wang and Weigao Ge in **(2009.)**

## 2.1 Introduction

We study the following impulsive problem :

$$\begin{cases} -u''(t) + g(t)u(t) = f(t, u(t)), & \text{a.e. } t \in [0, T]; \\ u(0) = u(T) = 0, & \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = I_j(u(t_j)), \quad j = 1, 2, \ldots, p, & \end{cases} \tag{2.1}$$

where $t_0 = 0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, $g \in L^\infty[0, T]$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous, and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \ldots, p$ are continuous.

### 2.1.1 The spaces provided for our problem

We now present the Hilbert space $H_0^1(0, T)$ which is suitable for the study of our problem. Let the space $H$ be such that:

$$H^1([0, T]) = \left\{ u \in L^2([0, T]) : u' \in L^2([0, T]) \right\},$$

$$C[0, T] = \{ u : [0, T] \to \mathbb{R}, \text{ telle que u continue } \},$$

muni par sa norme usuelle

$$\|u\|_\infty = \sup_{x \in [0,T]} |u(x)|.$$

**Remark 2.1.** we say that the space $H$ is space $H_0^1$

Define the space $H$ by:

$$H = \{u \in C([0,T]) : u'(.) \in L^2([0,T]), u(T) = 0\},$$

with the inner product

$$\langle u, v\rangle_{H_0^1(0,T)} = \int_0^T u'(t)v'(t)dt,$$

inducing the norm

$$\|u\| = \Big(\int_0^T |u'(t)|^2 dt\Big)^{\frac{1}{2}}.$$

### 2.1.2 The eigenvalues of the linear boundary value problem

Let $\lambda_1 = \dfrac{\pi^2}{T^2}$ be the first eigenvalue of the Dirichlet problem

$$\begin{cases} -u''(t) = \lambda u(t), & t \in [0,T]; \\ u(0) = u(T) = 0. \end{cases} \tag{2.2}$$

**Lemma 2.1.** *As is well known, (2.2)possesses a sequence of eigenvalues* $(\lambda_i)$*, where* $\lambda_i = \left(\frac{(2i-1)\pi}{2T}\right)^2$ *and* $\lambda_1 = \frac{\pi^2}{(2T)^2}$ *is the first nonzero eigenvalue of the problem (2.2) with*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots$$

**Proof.**

$$-r^2 = \lambda \Longrightarrow r = \pm i\sqrt{\lambda}$$

The solution to this problem is as follows:

$$u(t) = \alpha cos(\sqrt{\lambda}t) + \beta \sin(\sqrt{\lambda}t).$$

Conditions are applied to the limit One finds

$$u'(t) = -\alpha\sqrt{\lambda}\sin(\sqrt{\lambda}t) + \beta\sqrt{\lambda}\cos(\sqrt{\lambda}t)$$

$$u'(0) = 0 \Longrightarrow \beta = 0$$

So,

$$u(t) = \alpha\cos(\sqrt{\lambda}t)$$

we have,

$$u(T) = 0 \Longrightarrow \alpha \cos(\sqrt{\lambda}T) = 0.$$
$$\cos(\sqrt{\lambda}T) = 0 \Longrightarrow \lambda_i = (\frac{2i\pi - \pi}{2T})^2$$

□

**Remark 2.2.** The corresponding eigenfunctions are normalized so that :

$$\|\varphi_j\| = 1 = \lambda_j \int_0^T |\varphi_j(t)|^2 dt,$$

her

$$\varphi_j(t) = \sqrt{\frac{2}{T\lambda_j}} cos(\sqrt{\lambda_j}t), \quad j = 1, 2, .. \tag{2.3}$$

## 2.2 Preliminaries and statements

In this section, we recall some basic facts which will be used in the proofs of our main results. In order to apply the critical point theory, we make a variational structure. From this variational structure, we can reduce the problem of finding solutions of 2.1 to the one of seeking the critical points of a corresponding functional. In the Sobolev space $H_0^1(0,T)$, consider the inner product

$$\langle u, v\rangle_{H_0^1(0,T)} = \int_0^T u'(t)v'(t)\, dt,$$

inducing the norm

$$\|u\|_{H_0^1(0,T)} = \left(\int_0^T (u'(t))^2 dt\right)^{1/2}.$$

Throughout the paper, it will be assumed that $\operatorname{ess\,inf}_{t\in[0,T]} g(t) = m > -\lambda_1$. We also consider the inner product

$$\langle u, v\rangle = \int_0^T g(t)u(t)v(t)dt + \int_0^T u'(t)v'(t)dt, \quad \forall u, v \in H_0^1(0,T),$$

and the norm

$$\|u\| = \left(\int_0^T g(t)(u(t))^2 dt + \int_0^T (u'(t))^2 dt\right)^{1/2}, \quad \forall u \in H_0^1(0,T).$$

By Poincar's inequality:

$$\left(\int_0^T (u(t))^2 dt\right)^{1/2} \leq \frac{1}{\sqrt{\lambda_1}} \left(\int_0^T (u'(t))^2 dt\right)^{1/2},$$

**Proof**

The proof is given in (see [**?**]).

**Remark 2.3.** where $\lambda_1 = \dfrac{\pi^2}{T^2}$ is the first eigenvalue of the problem 2.2 We prove the following lemmas.

### 2.2.1 Auxiliary lemmas

**Lemma 2.2.** *If* $\operatorname{ess\,inf}_{t\in[0,T]} g(t) = m > -\lambda_1$, *then the norm* $\|\cdot\|$ *and the norm* $\|\cdot\|_{H_0^1(0,T)}$ *are equivalent.*

**Proof.** Since $\operatorname{ess\,inf}_{t\in[0,T]} g(t) = m > -\lambda_1$, there exists $C_1 \in (0,1)$ such that $-m \leq \lambda_1(1-C_1)$. Thus, by Poincar's inequality, one has

$$\begin{aligned}(1-C_1)\int_0^T (u'(t))^2 dt &\geq (1-C_1)\lambda_1 \int_0^T (u(t))^2 dt \\ &\geq -m\int_0^T (u(t))^2 dt\end{aligned}$$

for all $u \in H_0^1(0,T)$. Thereby, for every $u \in H_0^1(0,T)$, we have

$$\begin{aligned}\|u\|^2 &= \int_0^T g(t)(u(t))^2 dt + \int_0^T (u'(t))^2 dt \\ &\geq m\int_0^T (u(t))^2 dt + \int_0^T (u'(t))^2 dt \\ &\geq C_1 \int_0^T (u'(t))^2 d \\ &= C_1 \|u\|^2_{H_0^1(0,T)}.\end{aligned} \tag{2.4}$$

On the other hand, by Poincar's inequality, one has

$$\begin{aligned}\|u\|^2 &= \int_0^T g(t)(u(t))^2 dt + \int_0^T (u'(t))^2 dt \\ &\leq \|g\|_\infty \int_0^T (u(t))^2 dt + \int_0^T (u'(t))^2 dt \\ &\leq \left(\frac{\|g\|_\infty}{\lambda_1} + 1\right)\int_0^T (u'(t))^2 dt \\ &= \left(\frac{\|g\|_\infty}{\lambda_1} + 1\right)\|u\|^2_{H_0^1(0,T)}.\end{aligned} \tag{2.5}$$

Thus,for 2.4 and 2.5 we deduce that the norm $\|\cdot\|$ and the norm $\|\cdot\|_{H_0^1(0,T)}$ are equivalent. □

**Lemma 2.3.** *There exists* $C_2 > 0$ *such that, if* $u \in H_0^1(0,T)$, *then*

$$\|u\|_\infty \leq C_2\|u\|.$$

**Proof.** If $u \in H_0^1(0,T)$, it follows from the mean value theorem that

$$\frac{1}{T}\int_0^T u(s)ds = u(\tau)$$

for some $\tau \in (0,T)$. Hence, for $t \in [0,T]$, by Holder inequality and 2.5,

$$\begin{aligned}
|u(t)| &= \left|u(\tau) + \int_\tau^t u'(s)ds\right| \\
&\leq |u(\tau)| + \left|\int_0^T u'(s)ds\right| \\
&\leq \frac{\sqrt{T}}{T}\left(\int_0^T (u(s))^2 ds\right)^{1/2} + \sqrt{T}\left(\int_0^T (u'(s))^2 ds\right)^{1/2} \\
&\leq \left(\frac{\sqrt{T}}{T\sqrt{\lambda_1}} + \sqrt{T}\right)\left(\int_0^T (u'(s))^2 ds\right)^{1/2} \\
&\leq \left(\frac{\sqrt{T}}{T\sqrt{\lambda_1}} + \sqrt{T}\right)\frac{1}{\sqrt{C_1}}\|u\|.
\end{aligned}$$

Hence

$$\|u\|_\infty \leq \left(\frac{\sqrt{T}}{T\sqrt{\lambda_1}} + \sqrt{T}\right)\frac{1}{\sqrt{C_1}}\|u\| = C_2\|u\|. \quad \blacksquare$$

**Remark 2.4.** Based on theorem 1.1, if $u \in H^2(0,T)$, then $u$ and $u'$ are absolutely continuous and $u'' \in L^2(0,T)$. Hence,

$$\Delta u'(t) = 0 \quad \text{for all } t \in [0,T].$$

On the other hand, if $u \in H_0^1(0,T)$, then $u'$ may fail to be continuous, and thus

$$\Delta u'(t) \neq 0 \quad \text{for some } t \in (0,T),$$

which leads to impulsive effects.

Take $v \in H_0^1(0,T)$ and multiply the two sides of the equality

$$-u''(t) + g(t)u(t) = f(t,u(t)) \tag{2.6}$$

by $v$ and integrate from 0 to $T$, we have

$$-\int_0^T u''(t)v(t)dt + \int_0^T g(t)u(t)v(t)dt = \int_0^T f(t,u(t))v(t)dt.$$

Moreover,

$$
\begin{aligned}
-\int_0^T u''(t)v(t)\,dt &= -\sum_{j=0}^{p}\int_{t_j}^{t_{j+1}} u''(t)v(t)\,dt \\
&= -\sum_{j=0}^{p}\left(u'(t_{j+1}^-)v(t_{j+1}^-) - u'(t_j^+)v(t_j^+) - \int_{t_j}^{t_{j+1}} u'(t)v'(t)\,dt\right) \\
&= \sum_{j=1}^{p}\Delta u'(t_j)v(t_j) + u'(0)v(0) - u'(T)v(T) + \int_0^T u'(t)v'(t)\,dt \\
&= \sum_{j=1}^{p} I_j(u(t_j))v(t_j) + \int_0^T u'(t)v'(t)\,dt.
\end{aligned}
$$

Combining 2.6, we have

$$
\int_0^T u'(t)v'(t)dt + \int_0^T g(t)u(t)v(t)dt + \sum_{j=1}^{p} I_j(u(t_j))v(t_j) = \int_0^T f(t,u(t))v(t)dt.
$$

□

Considering the above, we introduce the following concept solution for problem 2.1.

### 2.2.2 The Euler–Lagrange functional

We define the functional $\varphi : H_0^1(0,T) \to \mathbb{R}$ by

$$
\varphi(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)\,ds - \int_0^T F(t,u(t))\,dt,
$$

where

$$
F(t,u) = \int_0^u f(t,s)\,ds,
$$

and

$$
\|u\|^2 = \int_0^T |u'(t)|^2 dt + \int_0^T g(t)|u(t)|^2 dt.
$$

**Proposition 2.1.** *The functional $\varphi$ is continuously differentiable on $H_0^1(0,T)$ and its Frchet derivative is given by*

$$
\langle \varphi'(u), v\rangle = \int_0^T u'(t)v'(t)\,dt + \int_0^T g(t)u(t)v(t)\,dt + \sum_{j=1}^{p} I_j(u(t_j))v(t_j) - \int_0^T f(t,u(t))v(t)\,dt,
$$

*for all $u, v \in H_0^1(0,T)$.*

**Proof. Step 1:** $\varphi$ is Gteaux differentiable.

Let $u, v \in H_0^1(0,T)$ and $\mu \in \mathbb{R}$. Then

$$\begin{aligned}\varphi(u+\mu v)-\varphi(u) &= \frac{1}{2}\|u+\mu v\|^2-\frac{1}{2}\|u\|^2 \\ &\quad+\sum_{j=1}^{p}\left(\int_0^{u(t_j)+\mu v(t_j)} I_j(s)\,ds-\int_0^{u(t_j)} I_j(s)\,ds\right) \\ &\quad-\int_0^T \Big(F(t,u+\mu v)-F(t,u)\Big)\,dt.\end{aligned}$$

Using the definition of the norm, we get

$$\begin{aligned}\frac{1}{2}\|u+\mu v\|^2-\frac{1}{2}\|u\|^2 &= \mu\int_0^T u'(t)v'(t)\,dt \\ &\quad+\mu\int_0^T g(t)u(t)v(t)\,dt \\ &\quad+\frac{\mu^2}{2}\|v\|^2.\end{aligned}$$

By the mean value theorem,

$$F(t,u+\mu v)-F(t,u)=\mu f(t,u+\theta\mu v)v,$$

for some $\theta \in (0,1)$.

Moreover,

$$\int_0^{u(t_j)+\mu v(t_j)} I_j(s)\,ds-\int_0^{u(t_j)} I_j(s)\,ds=\mu I_j(u(t_j)+\theta\mu v(t_j))v(t_j).$$

Hence,

$$\begin{aligned}\frac{\varphi(u+\mu v)-\varphi(u)}{\mu} &= \int_0^T u'(t)v'(t)\,dt+\int_0^T g(t)u(t)v(t)\,dt \\ &\quad+\frac{\mu}{2}\|v\|^2+\sum_{j=1}^{p} I_j(u(t_j)+\theta\mu v(t_j))v(t_j) \\ &\quad-\int_0^T f(t,u+\theta\mu v)v(t)\,dt.\end{aligned}$$

Letting $\mu \to 0$, we obtain

$$\langle\varphi'(u),v\rangle=\int_0^T u'(t)v'(t)\,dt+\int_0^T g(t)u(t)v(t)\,dt+\sum_{j=1}^{p} I_j(u(t_j))v(t_j)-\int_0^T f(t,u(t))v(t)\,dt.$$

Therefore, $\varphi$ is Gteaux differentiable.

**Step 2:** $\varphi'$ is continuous.

Let $(u_n) \subset H_0^1(0,T)$ such that $u_n \to u$ in $H_0^1(0,T)$. For every $v \in H_0^1(0,T)$, we have

$$\begin{aligned} & \langle \varphi'(u_n) - \varphi'(u), v \rangle \\ = & \int_0^T (u_n' - u')v' \, dt + \int_0^T g(t)(u_n - u)v \, dt \\ & + \sum_{j=1}^{p} \Big( I_j(u_n(t_j)) - I_j(u(t_j)) \Big) v(t_j) \\ & - \int_0^T \Big( f(t, u_n(t)) - f(t, u(t)) \Big) v(t) \, dt. \end{aligned}$$

Since $u_n \to u$ in $H_0^1(0,T)$ and $f$, $I_j$ are continuous, all terms converge to zero as $n \to +\infty$. Thus,

$$\varphi'(u_n) \to \varphi'(u),$$

which proves that $\varphi'$ is continuous. Therefore,

$$\varphi \in C^1(H_0^1(0,T), \mathbb{R}).$$

□

**Definition 2.1.** We say that a function $u \in H_0^1(0,T)$ is a mild solution of problem 2.1 if the identity

$$\int_0^T u'(t)v'(t)dt + \int_0^T g(t)u(t)v(t)dt + \sum_{j=1}^{p} I_j(u(t_j))v(t_j) = \int_0^T f(t, u(t))v(t)dt$$

holds for any $v \in H_0^1(0,T)$. Consider the functional $\varphi : H_0^1(0,T) \to \mathbb{R}$ defined by

$$\Big\{ \varphi(u) = \tfrac{1}{2}\|u\|^2 + \textstyle\sum_{j=1}^{p} \int_0^{u(t_j)} I_j(t)dt - \int_0^T F(t, u(t))dt, \tag{2.7}$$

where $F(t,s) = \int_0^s f(t,\tau)d\tau$. Using the continuity of $f$ and $I_j, j = 1, ..., 2, p$, one has that $\varphi \in C^1(H_0^1(0,T), \mathbb{R})$. For any $v \in H_0^1(0,T)$, we have

$$\varphi'(u)v = \int_0^T u'(t)v'(t)dt + \int_0^T g(t)u(t)v(t)dt + \sum_{j=1}^{p} I_j(u(t_j))v(t_j) - \int_0^T f(t, u(t))v(t)dt.$$

Thus, the solutions of problem 2.1 are the corresponding critical points of $\varphi$.

## 2.3 Existence of a unique solution

**Theorem 2.1.** *Let $dj$ $(j = 1, 2, \ldots, p)$ be fixed constants. If $f(t, u) = \sigma(t) \in L^2([0, T])$ and $I_j(t) = d_j$ $(j = 1, 2, \ldots, p)$, then problem 2.1 has a unique solution $u$ and $u$ minimizes the functional 2.7.*

**Proof.** We define the bilinear form

$$a : H \times H \longrightarrow \mathbb{R}, \quad a(u, v) = \int_0^T \Big(u'(x)v'(x) + g(x)u(x)v(x)\Big)\, dx,$$

and the linear operator

$$l : H \longrightarrow \mathbb{R}, \quad l(v) = \int_0^T \sigma(x)v(x)\, dx - \sum_{j=1}^{p} d_j v(x_j).$$

1. **Continuity of $a$:** For all $u, v \in H$, we have

$$\begin{aligned} |a(u, v)| &= \left| \int_0^T \Big(u'(x)v'(x) + g(x)u(x)v(x)\Big) dx \right| \\ &\leq \left| \int_0^T u'(x)v'(x) dx \right| + \left| \int_0^T g(x)u(x)v(x) dx \right| \end{aligned}$$

   Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} |a(u, v)| &\leq \|u'\|_2 \|v'\|_2 + \|g\|_\infty \|u\|_2 \|v\|_2 \\ &\leq \|u\|\|v\| + \|g\|_\infty \|u\|\|v\| \\ &\leq \|u\|\|v\| \Big(1 + \|g\|_\infty\Big) \\ &\leq C_1 \|u\|\|v\|, \end{aligned}$$

   where

$$C_1 = 1 + \|g\|_\infty.$$

   Thus, $a$ is continuous on $H \times H$.

2. **Symmetry of $a$:** For all $u, v \in H$, we have

$$\begin{aligned} a(u, v) &= \int_0^T \Big(u'(x)v'(x) + g(x)u(x)v(x)\Big) dx \\ &= \int_0^T \Big(v'(x)u'(x) + g(x)v(x)u(x)\Big) dx \\ &= a(v, u). \end{aligned}$$

   Hence, $a$ is symmetric.

3. **Coercivity of $a$:** For $u \in H$, we have

$$\begin{aligned} a(u,u) &= \int_0^T \Big(u'(x)^2 + g(x)u(x)^2\Big)dx \\ &= \|u\|^2 + \|g\|_\infty \|u\|^2 \\ &\geq \|u\|^2, \end{aligned}$$

Thus,

$$a(u,u) \geq C_2\|u\|^2, \quad \text{with } C_2 = 1.$$

Therefore, $a$ is coercive.

4. **Boundedness of $l$:** For $v \in H$, we have

$$|l(v)| = \left|\int_0^T \sigma(x)v(x)dx - \sum_{j=1}^p d_j v(x_j)\right| \leq \left|\int_0^T \sigma(x)v(x)dx\right| + \left|\sum_{j=1}^p d_j v(x_j)\right|,$$

Using the Cauchy–Schwarz inequality,

$$\begin{aligned} \left|\int_0^T \sigma(x)v(x)dx\right| &\leq \left(\int_0^T \sigma^2(x)dx\right)^{1/2}\left(\int_0^T v^2(x)dx\right)^{1/2} \\ &\leq \|\sigma\|_2\|v\|. \end{aligned}$$

Hence,

$$\begin{aligned} |l(v)| &\leq \|\sigma\|_2\|v\| + \left|\sum_{j=1}^p d_j v(x_j)\right| \\ &\leq \|\sigma\|_2\|v\| + \sum_{j=1}^p |d_j||v(x_j)| \\ &\leq \|\sigma\|_2\|v\| + \sup_{j\in[1,p]} |v(x_j)| \sum_{j=1}^p |d_j|. \end{aligned}$$

Let

$$C_3 = \|\sigma\|_2 + \sum_{j=1}^p |d_j|.$$

Then

$$|l(v)| \leq C_3\|v\|. \tag{2.8}$$

Thus, $l$ is bounded.

This implies that $l$ is bounded. Since $a(u,u) = \|u\|^2$, $a$ is coercive. By Theorem 1.11, problem 2.1 has a unique solution $u$ and $u$ minimizes the functional 2.7 . □

## 2.4 Existence of a nontrivial solution

**Theorem 2.2.** *Assume that the following conditions are satisfied:*

*(i) There exist* $a, b > 0$ *and* $\gamma \in [0, 1)$ *such that*

$$|f(t,u)| \le a + b|u|^{\gamma} \qquad \text{for every } (t,u) \in [0,T] \times \mathbb{R}.$$

*(ii) There exist* $a_j, b_j > 0$ *and* $\gamma_j \in [0,1)$ $(j = 1, 2, \dots, p.)$ *such that*

$$|I_j(u)| \le a_j + b_j|u|^{\gamma_j} \qquad \text{for every } u \in \mathbb{R}\ (j = 1, 2, \dots, p).$$

*Then problem 2.1 has at least one solution.*

**Proof.** Let $M_1 = \max_{(} a_1, a_2, \dots, a_p)$, $M_2 = \max_{(} b_1, b_2, \dots, b_p)$. In view of (i) and (ii) and Lemma 2.3 we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(t)\,dt - \int_0^T F(t,u(t))\,dt \\
&\ge \frac{1}{2}\|u\|^2 - \sum_{j=1}^{p} \int_0^{u(t_j)} (a_j + b_j|t|^{\gamma_j})\,ds - \int_0^T (a|u| + b|u|^{\gamma+1})\,dt \\
&\ge \frac{1}{2}\|u\|^2 - pM_1\|u\|_\infty - M_2 \sum_{j=1}^{p} \|u\|_\infty^{\gamma_j+1} - aT\|u\|_\infty - bT\|u\|_\infty^{\gamma+1} \\
&\ge \frac{1}{2}\|u\|^2 - pM_1C_2\|u\| - M_2 \sum_{j=1}^{p} C_2^{\gamma_j+1}\|u\|^{\gamma_j+1} - aTC_2\|u\| - bTC_2^{\gamma+1}\|u\|^{\gamma+1}.
\end{aligned}$$

for all $u \in H_0^1(0,T)$. This implies that $\lim_{\|u\|\to\infty} \varphi(u) = \infty$, and $\varphi$ is coercive. On the other hand, we Show that $\varphi$ is weakly lower semi-continuous. If $\{u_k\}_{k\in N} \subset H_0^1(0,T)$, $u_k \rightharpoonup u$, then we have that $\{u_k\}_{k\in N}$ converges uniformly to u on $[0,T]$, and $\liminf_{k\to\infty} \|u_k\| \ge \|u\|$. Thus

$$\begin{aligned}
\liminf_{k\to\infty} \varphi(u_k) &= \liminf_{k\to\infty} \Big(\frac{1}{2}\|u_k\|^2 + \sum_{j=1}^{p} \int_0^{u_k(t_j)} I_j(t)\,dt - \int_0^T F(t,u_k(t))\,dt\Big) \\
&\ge \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(t)\,dt - \int_0^T F(t,u(t))\,dt \\
&= \varphi(u).
\end{aligned}$$

By Theorem 1.8 of [51], $\varphi$ has a minimum point on $H_0^1(0,T)$, which is a critical point of $\varphi$. Hence, problem 2.1 has at least one solution. □

We readily have the following corollary.

**Corollary 2.1.** *Assume that $f$ is bounded and that the impulsive functions $I_j$ $(j = 1, 2, \ldots, p)$ are bounded. Then problem 2.1 has at least one solution.*

**Example 2.1.** Let $T = \frac{\pi}{2}$, $t_1 = \frac{\pi}{2}$. Consider the Dirichlet problem with impulse

$$\begin{cases} -u''(t) + (-1 - t^2)u(t) = t + \sqrt[3]{u(t)}, & \text{a.e. } t \in \left[0, \frac{\pi}{2}\right]; \\ u(0) = u\left(\frac{\pi}{2}\right) = 0, \\ \Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = 1 + \sqrt{u(t_1)}. \end{cases} \tag{2.9}$$

It is easy to see that conditions (i) and (ii) of Theorem2.2 hold. According to Theorem2.2problem2.9 has at least one solution.

## 2.5 Existence of a multiplicite solutions

**Theorem 2.3.** *Assume that the conditions (ii) of Theorem2.2 and the following conditions are satisfied:*
*(f1) $f(t, u)$ is odd in $u$.*
*(f2) There exist $r_1, r_2 > 0$ and $\mu \in (1, \infty)$ such that*

$$|f(t, u)| \leq r_1 + r_2|u|^{\mu} \qquad \text{for every } (t, u) \in [0, T] \times \mathbb{R}.$$

*(f3) There exist $R > 0$ and $\beta > 2$ such that for every $t \in [0, T]$ and every $u \in \mathbb{R}$ with $|u| \geq R$,*

$$0 < \beta F(t, u) \leq u f(t, u).$$

*Moreover, assume that $f(t, u) = o(u)$ as $u \to 0$ uniformly in $t$.*
*(f4) $I_j$ $(j = 1, 2, \ldots, p)$ are odd and nondecreasing. Then the problem2.1has an infinite number of nontrivial solutions.*

**Proof.** $\varphi \in C^1(H_0^1(0, T), \mathbb{R})$, by (f1) and (f4), $\varphi$ is an even functional and $\varphi(0) = 0$. We divide our proof into three parts in order to show Theorem 2.3 . Firstly, we show that $\varphi$ satisfies the P.S. condition. Let $\{u_n\}_{n \in N}$ be a sequence in $H_0^1(0, T)$ such that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$ as

$n \to \infty$. By (ii),(f3) and Lemma2.3 , we have

$$\begin{aligned}
\beta\varphi(u_n) - \langle \varphi'(u_n), u_n\rangle &= \left(\frac{\beta}{2} - 1\right)\|u_n\|^2 + \beta\sum_{j=1}^{p}\int_0^{u_n(t_j)} I_j(t)\,dt - \sum_{j=1}^{p} I_j(u_n(t_j))u_n(t_j) \\
&\quad + \int_0^T \Big(u_n f(t,u_n) - \beta F(t,u_n)\Big)\,dt \\
&\geq \left(\frac{\beta}{2} - 1\right)\|u_n\|^2 - (\beta+1)\left(pM_1C_2\|u_n\| + M_2\sum_{j=1}^{p} C_2^{\gamma_j+1}\|u_n\|^{\gamma_j+1}\right) \\
&\quad - \int_0^T \max_{t\in[0,T],|u|\leq R}\Big|u_n f(t,u_n) - \beta F(t,u_n)\Big|\,dt. \\
&\geq \left(\frac{\beta}{2} - 1\right)\|u_n\|^2 - (\beta+1)(pM_1C_2\|u_n\| + M_2\sum_{j=1}^{p} C_2^{\gamma_j+1}\|u_n\|^{\gamma_j+1}) + \ C_4.
\end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H_0^1(0,T)$. Hence there exists a subsequence of $\{u_n\}$ (for simplicity denoted again by $\{u_n\}$) such that

$$u_n \to u \quad \text{in } H_0^1(0,T), \tag{2.10}$$

$$u_n \to u \quad \text{uniformly in} C[0,T]. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}
\langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle &= \int_0^T |u_n'(t) - u'(t)|^2\,dt + \int_0^T g(t)|u_n(t) - u(t)|^2\,dt \\
&\quad + \sum_{j=1}^{p}\Big(I_j(u_n(t_j)) - I_j(u(t_j))\Big)\Big(u_n(t_j) - u(t_j)\Big). \\
&\quad + \int_0^T \big(f(t,u_n(t)) - f(t,u(t))\big)\big(u_n(t) - u(t)\big)\,dt
\end{aligned} \tag{2.12}$$

From (f2), 2.10,2.11 and 2.12, it follows that $u_n \to u$ in $H_0^1(0,T)$. Thus, $\varphi$ satisfies the P.S. condition. Secondly, we verify the condition $(I_2)$ of Theorem2.3.

$$\frac{\beta}{u} \leq \frac{f(x,u)}{F(x,u)}, \quad u \geq R, \tag{2.13}$$

$$\frac{\beta}{u} \geq \frac{f(x,u)}{F(x,u)}, \quad u \leq -R. \tag{2.14}$$

Integrating 2.13 and 2.14 for $u$ from $[R,u]$ and $[u,-R]$ respectively, we have

$$\beta \ln\frac{u}{R} \leq \ln\frac{F(x,u)}{F(x,R)}, \quad u \geq R,$$

$$\beta \ln \frac{R}{-u} \geq \ln \frac{F(x,u)}{F(x,-R)}, \quad u \leq -R.$$

That is

$$F(x,u) \geq F(x,R)\left(\frac{u}{R}\right)^{\beta}, \quad u \geq R, \tag{2.15}$$

$$F(x,u) \geq F(x,-R)\left(\frac{-u}{R}\right)^{\beta}, \quad u \leq -R. \tag{2.16}$$

Combining 2.15 and2.16, we have

$$F(x,u) \geq \alpha_1 |u|^{\beta}, \quad |u| \geq R, \tag{2.17}$$

where

$$\alpha_1 = R^{-\beta} \min\{\min_{x\in[0,T]} F(x,R), \min_{x\in[0,T]} F(x,-R)\} > 0.$$

On the other hand, by the continuity of $F(x,u)$, $F(x,u)$ is bounded on $[0,T]\times[-R,R]$, there exists $K > 0$ such that

$$F(x,u) \geq -K \geq \alpha_1 |u|^{\beta} - \alpha_1 R^{\beta} - K, \quad |u| \leq R. \tag{2.18}$$

Combining 2.17 and2.18, we have

$$F(t,u) \geq \alpha_1 |u|^{\beta} - \alpha_2, \quad \forall (t,u) \in [0,T]\times\mathbb{R}, \tag{2.19}$$

where $\alpha_2 = \alpha_1 + K$. For arbitrary finite-dimensional subspace $E_1 \subset H_0^1(0,T)$, and any $u \in E_1$, there exists $C_5 = C_5(E_1) > 0$ such that

$$\|u\|_{\beta} \geq C_5 \|u\|. \tag{2.20}$$

By (ii),2.19, 2.20 and Lemma2.3, we have

$$\begin{aligned}
\varphi(u) &\leq \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p} \int_0^{u(t_j)} (a_j + b_j |t|^{\gamma_j}) dt - \int_0^T (\alpha_1 |u|^{\beta} - \alpha_2) dt \\
&\leq \frac{1}{2}\|u\|^2 + pM_1 \|u\|_{\infty} + M_2 \sum_{j=1}^{p} \|u\|_{\infty}^{\gamma_j+1} - \alpha_1 \|u\|_{\beta}^{\beta} + \alpha_2 T \\
&\leq \frac{1}{2}\|u\|^2 + pM_1 C_2 \|u\| + M_2 \sum_{j=1}^{p} C_2^{\gamma_j+1} \|u\|^{\gamma_j+1} - \alpha_1 C_5^{\beta} \|u\|^{\beta} + \alpha_2 T
\end{aligned}$$

for every $u \in E_1$. This implies that $\varphi(u) \to -\infty$ as $u \in E_1$ and $\|u\| \to \infty$. So there exists $R(E_1) > 0$ such that $\varphi \leq 0$ on $E_1 \setminus B_{R(E_1)}$. Finally, we verify the condition $(I_1)$ of Theorem 2.1.

Let $V = X_1 \bigoplus X_2$, $W = \overline{\bigoplus_{i=k+1}^{\infty} X_i}$, then $H_0^1(0,T) = V + W$ and $V$ is finite dimensional. Using $(f_4)$, we have

$$\sum_{j=1}^{p} \int_0^{u(t_j)} I_j(t)dt \geq 0. \tag{2.21}$$

By $(f_3)$, we have,

$$\lim_{u \to 0} \frac{F(t,u)}{u^2} = 0.$$

Hence, for $\epsilon = \frac{1}{4C_3^2}$, there exists $\delta > 0$ such that for every $u$ with $|u| \leq \delta$,

$$|F(t,u)| \leq \frac{1}{4C_3^2} u^2. \tag{2.22}$$

Hence, for any $u \in W$ with $\|u\| \leq \frac{\delta}{C_2}$, $\|u\|_\infty \leq \delta$, by 2.8,2.21 and 2.22, we have

$$\begin{aligned}
\varphi(u) &\geq \frac{1}{2}\|u\|^2 - \int_0^T F(t,u(t))dt \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{4C_3^2}\|u\|_2^2 \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{4C_3^2} C_2^2 \|u\|^2 \\
&= \frac{1}{4}\|u\|^2.
\end{aligned}$$

Take $\alpha = \frac{1}{4}\frac{\delta^2}{C_2^2}$, $\rho = \frac{\delta}{C_2}$, then

$$\varphi(u) \geq \alpha, \quad \forall u \in W \cap \partial B_\rho.$$

By Theorem 2.1, $\varphi$ possesses infinite critical points, that is, problem 2.1 has infinite nontrivial solutions. □

**Example 2.2.** Let $T = 1, t_1 = \frac{1}{3}$. Consider the Dirichlet impulsive problem

$$\begin{cases} -u''(t) + (1+t^2)u(t) = (u(t))^3, \quad \text{a.e. } t \in [0,1]; \\ u(0) = u(1) = 0, \\ \Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = \sqrt[3]{u(t_1)}. \end{cases} \tag{2.23}$$

then, according to Theorem, problem 2.17 has infinite nontrivial solutions.

Chapter 3

# Existence of the solutions to an impulsive second-order boundary value problems on unbounded interval

In this chapter, we study impulsive boundary value problems on the half-line by using variational methods.

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, +\infty),\ t \neq t_j, \\ \Delta u'(t_j) = g(t_j) I_j(u(t_j^-)), \quad j \in \mathbb{N}^*, \\ u(0) = u(+\infty) = 0, \end{cases}$$

where $\lambda$ is a positive parameter, the functions $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $I_j : \mathbb{R} \to \mathbb{R}$, $j \in \mathbb{N}^*$, and $g : [0, +\infty) \to [0, +\infty)$ are continuous with

$$\sum_{j=1}^{+\infty} g(t_j) < +\infty.$$

The authors have extended some results of the work published in the article [7], to the half-line by using variational techniques.

## 3.1 Preliminaries

We cite some basic and celebrated theorems from critical point theory which are crucial tools in the proof of our main results. Let $H$ be a Hilbert space. Now, let us recall some necessary concepts that will be needed in our argument. Let us define the following reflexive Banach space

$$H_0^1(0, \infty) = \left\{ u : [0, \infty) \to \mathbb{R} \text{ is absolutely continuous, } u, u' \in L^2(0, \infty),\ u(0) = u(\infty) = 0 \right\},$$

equipped with the norm

$$\|u\| = \left( \int_0^{+\infty} |u(t)|^2 dt + \int_0^{+\infty} |u'(t)|^2 dt \right)^{1/2}.$$

Set the space

$$C_{l,p}[0,+\infty) = \left\{ u \in C([0,+\infty), \mathbb{R}) : \lim_{t\to+\infty} p(t)u(t) \text{ exists} \right\}$$

with the norm

$$\|u\|_{\infty,p} = \sup_{t\in[0,+\infty)} p(t)|u(t)|,$$

where the function $p : [0,+\infty) \to (0,+\infty)$ is continuously differentiable and bounded, satisfying

$$C = 2\max(\|p\|_{L^2}, \|p'\|_{L^2}) < +\infty.$$

Concerning the above spaces, we get the following vital embeddings.

### 3.1.1 Auxiliary Lemmas

**Lemma 3.1.** *[56]*

*The space $H_0^1(0,+\infty)$ embeds continuously in $C_{l,p}[0,+\infty)$, more precisely $\|u\|_{\infty,p} \le C\|u\|$ for every $u \in H_0^1(0,\infty)$.*

**Proof**

For $u \in H^1(0,+\infty)$, we have

$$\begin{aligned}
|p(t)u(t)| &= |p(+\infty)u(+\infty) - p(t)u(t)| = \left| \int_t^{+\infty} (pu)'(s)ds \right| \\
&\le \left| \int_t^{+\infty} p'(s)u(s)ds \right| + \left| \int_t^{+\infty} p(s)u'(s)ds \right| \\
&\le \left( \int_0^{+\infty} p'^2(s)ds \right)^{\frac{1}{2}} \left( \int_0^{+\infty} u^2(s)ds \right)^{\frac{1}{2}} + \left( \int_0^{+\infty} p^2(s)ds \right)^{\frac{1}{2}} \left( \int_0^{+\infty} u'^2(s)ds \right)^{\frac{1}{2}} \\
&\le \max(\|p'\|_{L^2}, \|p\|_{L^2})\|u\|.
\end{aligned}$$

Hence

$$\|u\|_{\infty,p} \le M\|u\|.$$

□

**Lemma 3.2.** *[56]*

*The embedding $H_0^1(0,+\infty) \hookrightarrow C_{l,p}[0,+\infty)$ is compact.*

**Proof**

Let $D \subset H^1(0,+\infty)$ be a bounded set. Then it is bounded in $C_{l,p}[0,+\infty)$ by Lemma 3.1. Let $R > 0$

be such that for all $u \in D$, $\|u\| \leq R$. We will apply Lemma 1.2.

(a) $D$ is equicontinuous on every compact interval of $[0, +\infty)$. Let $u \in D$ and $t_1, t_2 \in J \subset [0, +\infty)$ where $J$ is a compact sub-interval. Using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
|p(t_1)u(t_1) - p(t_2)u(t_2)| &= \left|\int_{t_2}^{t_1} (pu)'(s)ds\right| = \left|\int_{t_2}^{t_1} p'(s)u(s) + u'(s)p(s)ds\right| \\
&\leq \left(\int_{t_2}^{t_1} p'^2(s)ds\right)^{\frac{1}{2}} \left(\int_{t_2}^{t_1} u^2(s)ds\right)^{\frac{1}{2}} \\
&+ \left(\int_{t_2}^{t_1} p^2(s)ds\right)^{\frac{1}{2}} \left(\int_{t_2}^{t_1} u'^2(s)ds\right)^{\frac{1}{2}} \\
&\leq \max\left[\left(\int_{t_2}^{t_1} p'^2(s)ds\right)^{\frac{1}{2}}, \left(\int_{t_2}^{t_1} p^2(s)ds\right)^{\frac{1}{2}}\right] \|u\| \\
&\leq R \max\left[\left(\int_{t_2}^{t_1} p'^2(s)ds\right)^{\frac{1}{2}}, \left(\int_{t_2}^{t_1} p^2(s)ds\right)^{\frac{1}{2}}\right] \longrightarrow 0,
\end{aligned}$$

as $|t_1 - t_2| \to 0$.

(b) $D$ is equiconvergent at $+\infty$. For $t \in [0, +\infty)$ and $u \in D$, using the fact that $(pu)(+\infty) = 0$ (note $u(\infty) = 0$ and $p$ is bounded) and using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
|(pu)(t) - (pu)(+\infty)| &= \left|\int_{t}^{+\infty} (pu)'(s)ds\right| = \left|\int_{t}^{+\infty} p'(s)u(s) + u'(s)p(s)ds\right| \\
&\leq \max\left[\left(\int_{t}^{+\infty} p'^2(s)ds\right)^{\frac{1}{2}}, \left(\int_{t}^{+\infty} p^2(s)ds\right)^{\frac{1}{2}}\right] \|u\| \\
&\leq R \max\left[\left(\int_{t}^{+\infty} p'^2(s)ds\right)^{\frac{1}{2}}, \left(\int_{t}^{+\infty} p^2(s)ds\right)^{\frac{1}{2}}\right] \longrightarrow 0,
\end{aligned}$$

as $t \to +\infty$. □

Let $X$ be a real Banach space and $X^*$ its dual.

**Theorem 3.1.** *If $\varphi$ is weakly lower semi-continuous (w.l.s.c.) on a reflexive Banach space $X$ and has a bounded minimizing sequence, then $\varphi$ has a minimum on $X$.*

## 3.2 Impulsive linear problem

We study the following linear Dirichlet boundary value problem with impulses in the derivative at the prescribed instants $t_j$, $j \in \mathbb{N}^* = \{1, 2, 3, \dots\}$

$$\begin{cases} -u''(t) + \lambda u(t) = \sigma(t), & \text{a.e. } t \in [0, +\infty), t \neq t_j, \\ \Delta u'(t_j) = d(t_j), & j \in \mathbb{N}^*, \\ u(0) = u(+\infty) = 0, \end{cases} \tag{3.1}$$

where $\lambda \in \mathbb{R}$, $\sigma \in L^2(0, +\infty)$, $0 = t_0 < t_1 < t_2 < \cdots < t_j < \cdots < t_m \to +\infty$, as $m \to +\infty$, are the impulse points, $d : [0, \infty) \to \mathbb{R}$ satisfies $\sum_{j=1}^{+\infty} \frac{d(t_j)}{p(t_j)} < +\infty$ and $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$ for $u'(t_j^\pm) = \lim_{s \to t_j^\pm} u'(s)$.

**Remark 3.1.** multiply the equation in problem 3.1 by $v \in H_0^1(0, +\infty)$, and then integrate over $(0, +\infty)$, we obtain

$$-\int_0^{+\infty} u''v + \lambda \int_0^{+\infty} uv = \int_0^{+\infty} \sigma v.$$

We have

$$-\int_0^{+\infty} u''v = -\sum_{j=0}^{+\infty} \int_{t_j}^{t_{j+1}} u''v,$$

and

$$\int_{t_j}^{t_{j+1}} u''v = u'(t_{j+1}^-)v(t_{j+1}^-) - u'(t_j^+)v(t_j^+) - \int_{t_j}^{t_{j+1}} u'v'.$$

Consequently,

$$\begin{aligned} -\int_0^{+\infty} u''v &= \sum_{j=1}^{+\infty} \Delta u'(t_j)v(t_j) + u'(0)v(0) - u'(+\infty)v(+\infty) + \int_0^{+\infty} u'v' \\ &= \sum_{j=1}^{+\infty} d(t_j)v(t_j) + \int_0^{+\infty} u'v'. \end{aligned}$$

This leads to define the bilinear form $a : H_0^1(0, +\infty) \times H_0^1(0, +\infty) \to \mathbb{R}$, by

$$a(u, v) = \int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv, \tag{3.2}$$

and the linear operator $l : H_0^1(0, +\infty) \to \mathbb{R}$ by

$$l(v) = \int_0^{+\infty} \sigma v - \sum_{j=1}^{+\infty} d(t_j)v(t_j). \tag{3.3}$$

### 3.2.1 The Euler–Lagrange functional

We define the functional $\varphi : H_0^1(0,T) \to \mathbb{R}$ by

$$\varphi(u) = \frac{1}{2}\int_0^{+\infty} u'^2 + \frac{\lambda}{2}\int_0^{+\infty} u^2 - \int_0^{+\infty} \sigma u + \sum_{j=1}^{+\infty} d(t_j)u(t_j).$$

where

$$F(t,u) = \int_0^u f(t,s)\,ds,$$

**Definition 3.1.** *We say that a function $u$ is a weak solution of the impulsive problem (3.1) if $u \in H_0^1(0,+\infty)$ such that $a(u,v) = l(v)$ is valid for any $v \in H_0^1(0,+\infty)$.*

In what follows we refer to problem 3.1 as (*LP*). It is easily verified that $a$ and $l$ defined by 3.2,3.3 respectively are continuous, and $a$ is coercive if $\lambda > 0$. Consider the functional $\varphi : H_0^1(0,+\infty) \to \mathbb{R}$, defined by

$$\varphi(u) = \frac{1}{2}\int_0^{+\infty} u'^2 + \frac{\lambda}{2}\int_0^{+\infty} u^2 - \int_0^{+\infty} \sigma u + \sum_{j=1}^{+\infty} d(t_j)u(t_j). \tag{3.4}$$

It is clear that $\varphi$ is differentiable at any $u \in H_0^1(0,+\infty)$ and

$$\varphi'(u)v = \int_0^{+\infty} u'v' + \lambda\int_0^{+\infty} uv - \int_0^{+\infty} \sigma v + \sum_{j=1}^{+\infty} d(t_j)v(t_j) = a(u,v) - l(v).$$

Thus, a critical point of 3.4 gives us a weak solution of the problem (*LP*).

**Definition 3.2.** *We mean by a classical solution of the problem (LP) a function $u \in H^2(t_j, t_{j+1})$ for all $j \in \mathbb{N}^*$, where*

$$H^2(t_j,t_{j+1}) = \{u : [0,+\infty) \to \mathbb{R} \text{ is absolutely continuous, } u', u'' \in L^2(t_j,t_{j+1})\},$$

*and $u$ satisfies the first equation of (3.1) a.e. on $[0,+\infty)$ with $u(0) = u(+\infty) = 0$, the limits $u'(t_j^+)$, $u'(t_j^-)$, $j \in \mathbb{N}^*$ exist and the impulse conditions hold.*

### 3.2.2 Auxiliary lemmas

**Lemma 3.3.** *If $u \in H_0^1(0,+\infty)$ is a weak solution of (LP), then $u$ is a classical solution of (LP).*

**Proof.** Since $u \in H_0^1(0,+\infty)$, it is evident that $u(0) = u(+\infty) = 0$.

For $j \in \{1,2,\dots\}$, choose any $v \in H_0^1(0,+\infty)$ such that $v(t) = 0$ for $t \in [0,t_j] \cup [t_{j+1},+\infty)$. Then

$$\int_{t_j}^{t_{j+1}} u'v' + \lambda\int_{t_j}^{t_{j+1}} uv = \int_{t_j}^{t_{j+1}} \sigma v.$$

Hence, $-u'' + \lambda u = \sigma$ a.e. on $(t_j, t_{j+1})$. So, $u \in H^2(t_j, t_{j+1})$ and satisfies the previous equation a.e. on $[0, +\infty)$.

Multiplying $-u'' + \lambda u = \sigma$ by $v \in H_0^1(0, +\infty)$ and integrating over $[0, +\infty)$, we get

$$\sum_{j=1}^{+\infty} \Delta u'(t_j) v(t_j) = \sum_{j=1}^{+\infty} d(t_j) v(t_j).$$

Therefore, $\Delta u'(t_j) = d(t_j)$ for every $j \in \mathbb{N}^*$, and the impulsive conditions are satisfied. □

**Lemma 3.4.** *If $u \in H_0^1(0, +\infty)$ is a critical point of $\varphi$ defined by 3.4, then $u$ is a weak solution of the impulsive Dirichlet problem (LP).*

**Proof.** Let $u \in H_0^1(0, +\infty)$. The assumption that $u$ is a critical point of $\varphi$ means that $\varphi'(u)v = 0$, for all $v \in H_0^1(0, +\infty)$. Thus,

$$\int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv - \int_0^{+\infty} \sigma v + \sum_{j=1}^{+\infty} d(t_j) v(t_j) = 0, \quad \forall v \in H_0^1(0, +\infty).$$

Hence,

$$\int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv = \int_0^{+\infty} \sigma v - \sum_{j=1}^{+\infty} d(t_j) v(t_j), \quad \forall v \in H_0^1(0, +\infty).$$

This implies that $a(u, v) = l(v)$ is valid for any $v \in H_0^1(0, +\infty)$. As a result, $u$ is a weak solution of the $(LP)$. □

In view of Lax-Milgram theorem, we formulate the following main result.

**Theorem 3.2.** *If $\lambda > 0$, then the Dirichlet impulsive problem (LP) has a weak solution $u \in H_0^1(0, +\infty)$ for any $\sigma \in L^2(0, +\infty)$. Moreover, $u \in H^2(0, +\infty)$ and $u$ is a classical solution and minimizes the functional 3.4 and hence it is a critical point of 3.4.*

**Proof.** For $\lambda > 0$, it follows that the bilinear $a$ is coercive. The fact that $a$ is continuous, by applying Theorem 1.11, for any $\sigma \in L^2(0, +\infty)$, there exists a unique $u \in H_0^1(0, +\infty)$ such that $a(u, v) = l(v)$ for all $v \in H_0^1(0, +\infty)$. So, the problem $(LP)$ has a weak solution $u \in H_0^1(0, +\infty)$.

Owing to Lemma 3.3, a weak solution of $(LP)$ is a classical solution. In addition, $a$ is symmetric, then the functional $\varphi$ attains its minimum at $u$ which is exactly a critical point of $\varphi$ since it is differentiable. □

**Example 3.1.** *As an example, let* $\lambda = 1$ *and* $p(t) = \frac{1}{1+t^2}$.

*This impulsive boundary value problem*

$$\begin{cases} -u''(t) + u(t) = \frac{1}{1+t^2}, & \text{a.e. } t \in [0, +\infty), \\ \Delta u'(j) = e^{-j}, & j \in \mathbb{N}^*, \\ u(0) = u(+\infty) = 0, \end{cases} \tag{3.5}$$

*has a solution.*

## 3.3 Impulsive nonlinear problem

We study the following impulsive boundary value problem

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), & \text{a.e. } t \in [0, +\infty),\ t \neq t_j, \\ \Delta u'(t_j) = g(t_j) I_j(u(t_j^-)), & j \in \mathbb{N}^*, \\ u(0) = u(+\infty) = 0, \end{cases} \tag{3.6}$$

where $\lambda$ is a positive parameter, the functions $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $I_j : \mathbb{R} \to \mathbb{R}$, $j \in \mathbb{N}^*$, and $g : [0, +\infty) \to [0, +\infty)$ are continuous with

$$\sum_{j=1}^{+\infty} g(t_j) < +\infty.$$

We refer to problem 3.6 as $(NP)$.

### 3.3.1 The Euler–Lagrange functional

We define the functional $\varphi : H_0^1(0, T) \to \mathbb{R}$ by

$$\varphi(u) = \frac{1}{2} \int_0^{+\infty} u'^2(t) dt + \frac{\lambda}{2} \int_0^{+\infty} u^2(t) dt + \sum_{j=1}^{+\infty} g(t_j) \int_0^{u(t_j)} I_j(s) ds - \int_0^{+\infty} F(t, u(t)) dt.$$

where

$$F(t, u) = \int_0^u f(t, s)\, ds,$$

**Definition 3.3.** A weak solution of $(NP)$ is a function $u \in H_0^1(0, +\infty)$ such that

$$\int_0^{+\infty} u' v'\, dt + \lambda \int_0^{+\infty} uv\, dt + \sum_{j=1}^{+\infty} g(t_j) I_j(u(t_j)) v(t_j) - \int_0^{+\infty} f(t, u(t)) v(t)\, dt = 0,$$

for every $v \in H_0^1(0, +\infty)$. Setting $F(t, u) = \int_0^u f(t, s) ds$, we define the functional $\varphi : H_0^1(0, +\infty) \to \mathbb{R}$ by

$$\varphi(u) = \frac{1}{2} \int_0^{+\infty} u'^2(t) dt + \frac{\lambda}{2} \int_0^{+\infty} u^2(t) dt + \sum_{j=1}^{+\infty} g(t_j) \int_0^{u(t_j)} I_j(s) ds - \int_0^{+\infty} F(t, u(t)) dt. \tag{3.7}$$

Now we present our principal results for this part.

**Theorem 3.3.** *Suppose that the following conditions hold:*

$(H_1)$ *There exists a positive bounded function* $M \in L^1(0,+\infty)$ *with* $\frac{M}{p} \in L^1(0,+\infty)$ *such that*

$$|f(t,u)| \leq M(t) \quad \text{for} \quad (t,u) \in [0,+\infty) \times \mathbb{R}.$$

$(I_1)$ *There exist* $M_j > 0$, $j \in \mathbb{N}^*$, *satisfying* $\sum_{j=1}^{+\infty} M_j g(t_j) < +\infty$ *and* $\sum_{j=1}^{+\infty} \frac{M_j g(t_j)}{p(t_j)} < +\infty$, *such that the impulsive functions* $I_j$ *are bounded i.e.,*

$$|I_j(u)| \leq M_j \quad \text{for every} \quad u \in \mathbb{R}, j \in \{1,2,\dots\}.$$

*Then there is a critical point of* $\varphi$, *and* $(NP)$ *has at least one solution.*

**Proof.** *Claim 1.* $\varphi$ *is weakly lower semi-continuous (w.l.s.c).*
Let $(u_n) \subset H_0^1(0,+\infty)$ be a sequence such that $u_n \rightharpoonup u$ in $H_0^1(0,+\infty)$, when $n \to +\infty$. Then,

$$\|u\| \leq \liminf_{n \to +\infty} \|u_n\|,$$

and by Lemma 3.2 we have that $(u_n)$ converges to $u$ in $C_{l,p}[0,+\infty)$, hence $u_n(t)$ converges to $u(t)$ for all $t \in [0,+\infty)$.
From $(H_1)$ and $(I_1)$, using the continuity of $f$ and $I_j$, $j \in \mathbb{N}^*$, together with the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned}\liminf_{n \to +\infty} \varphi(u_n) &= \liminf_{n \to +\infty} \left[ \frac{1}{2} \int_0^{+\infty} u_n'^2 + \frac{\lambda}{2} \int_0^{+\infty} u_n^2 + \sum_{j=1}^{+\infty} g(t_j) \int_0^{u_n(t_j)} I_j(s)ds - \int_0^{+\infty} F(t,u_n(t))dt \right] \\ &\geq \frac{1}{2} \int_0^{+\infty} u'^2 + \frac{\lambda}{2} \int_0^{+\infty} u^2 + \sum_{j=1}^{+\infty} g(t_j) \int_0^{u(t_j)} I_j(s)ds - \int_0^{+\infty} F(t,u(t))dt = \varphi(u).\end{aligned}$$

Thus, $\varphi$ is (w.l.s.c). *Claim 2.* $\varphi$ *is coercive.*
For any $u \in H_0^1(0,+\infty)$, the fact that $\lambda > 0$, there exists $\alpha > 0$ such that

$$\varphi(u) \geq \alpha \|u\|^2 + \sum_{j=1}^{+\infty} g(t_j) \int_0^{u(t_j)} I_j(s)ds - \int_0^{+\infty} F(t,u(t))dt.$$

Using conditions $(H_1)$, $(I_1)$ and Lemma 3.1, we have

$$\begin{aligned}
\varphi(u) &\geq \alpha\|u\|^2 - \sum_{j=1}^{+\infty} \frac{M_j g(t_j)}{p(t_j)} p(t_j)|u(t_j)| - \int_0^{+\infty} \frac{M(t)}{p(t)} p(t)|u(t)|dt \\
&\geq \alpha\|u\|^2 - \|u\|_{+\infty,p} \sum_{j=1}^{\infty} \frac{M_j g(t_j)}{p(t_j)} - \|u\|_{+\infty,p} \int_0^{+\infty} \frac{M(t)}{p(t)} dt \\
&\geq \alpha\|u\|^2 - C\|u\| \sum_{j=1}^{+\infty} \frac{M_j g(t_j)}{p(t_j)} - C\|u\| \left\| \frac{M}{p} \right\|_{L^1} \\
&\geq \alpha\|u\|^2 - C\left( \sum_{j=1}^{+\infty} \frac{M_j g(t_j)}{p(t_j)} + \left\| \frac{M}{p} \right\|_{L^1} \right) \|u\|,
\end{aligned}$$

for some $C > 0$. Then, the above inequality implies that $\lim_{\|u\|\to+\infty} \varphi(u) = +\infty$. Hence, $\varphi$ is coercive. □

Applying Theorem 3.1, $\varphi$ possesses a minimum which is a critical point of $\varphi$. Finally, by $(H_1)$ and $(I_1)$, it is easy to check that $\varphi$ is continuous and differentiable for any $u \in H_0^1(0, +\infty)$ and that

$$\varphi'(u)v = \int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv + \sum_{j=1}^{+\infty} g(t_j) I_j(u(t_j)) v(t_j) dt - \int_0^{+\infty} f(t, u(t)) v(t) dt. \tag{3.8}$$

Therefore, a critical point of $\varphi$ is a weak solution of the problem $(NP)$. □

**Remark 3.2.** Assume $M \in L^2(0, +\infty)$ in (H1), then it is easy to see that a weak solution u is in $H^2(0, +\infty)$.

**Example 3.2.** Take $\lambda = 1, p(t) = e^{-t}, M(t) = e^{-2t}, g(t) = e^{-2t}, M_j = \frac{1}{j}$ and $I_j(s) = \frac{1}{j+s^2}$, $j \in \mathbb{N}^*$. The following IBVP:

$$\begin{cases} -u''(t) + u(t) = e^{-3t}, & \text{a.e. } t \in [0, +\infty), \\ \Delta u'(j) = \frac{e^{-2j}}{j+u^2(j)}, & j \in \mathbb{N}^*, \\ u(0) = u(+\infty) = 0, \end{cases}$$

has at least one solution.

**Theorem 3.4.** *Assume the following conditions are satisfied:*

$(H_2)$ *The function $f$ is sublinear i.e., there exist a constant $\gamma \in [0, 1)$ and positive functions $a, b \in L^1(0, +\infty)$ with $\frac{a}{p}, \frac{b}{p^{\gamma+1}} \in L^1(0, +\infty)$ such that*

$$|f(t, u)| \leq a(t) + b(t)|u|^\gamma \quad \text{for} \quad (t, u) \in [0, +\infty) \times \mathbb{R}.$$

($I_2$) *There exist constants* $\delta \in [0,1)$ *and* $a_j, b_j > 0, j \in \{1,2,\dots\}$ *with* $\sum_{j=1}^{+\infty} a_j g(t_j)$, $\sum_{j=1}^{+\infty} \frac{a_j g(t_j)}{p(t_j)}$, $\sum_{j=1}^{+\infty} \frac{b_j g(t_j)}{p^{\delta}(t_j)}$, $\sum_{j=1}^{\infty} \frac{b_j g(t_j)}{p^{\delta+1}(t_j)}$ *are convergent series, such that the impulsive functions* $I_j$ *have sublinear growths i.e.,*

$$|I_j(u)| \leq a_j + b_j|u|^{\delta} \quad \text{for every} \quad u \in \mathbb{R}, j \in \{1,2,\dots\}.$$

*Then there is a critical point of* $\varphi$*, and* $(NP)$ *has at least one solution.*

**Proof.** *Claim 1.* $\varphi$ *is weakly lower semi-continuous.*

Under $(H_2)$ and $(I_2)$, arguing analogously to the proof of Theorem 3.3, we find the weak lower semi-continuity of $\varphi$.

*Claim 2.* $\varphi$ *is coercive.*

In view of conditions $(H_2)$, $(I_2)$ and (3.7), for any $u \in H_0^1(0, + + \infty)$, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^{+\infty} u'^2 + \frac{\lambda}{2}\int_0^{+\infty} u^2 + \sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)} I_j(s)ds - \int_0^{+\infty} F(t,u(t))dt \\
&\geq \alpha\|u\|^2 - \sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)} (a_j + b_j|s|^{\delta})ds - \int_0^{+\infty}\left(a(t)|u(t)| + \frac{b(t)}{\gamma+1}|u(t)|^{\gamma+1}\right)dt \\
&\geq \alpha\|u\|^2 - \sum_{j=1}^{+\infty} g(t_j)\left(\frac{a_j}{p(t_j)}p(t_j)|u(t_j)| + \frac{b_j}{(\delta+1)p^{\delta+1}(t_j)}|p(t_j)u(t_j)|^{\delta+1}\right) \\
&\quad - \int_0^{+\infty}\frac{a(t)}{p(t)}p(t)|u(t)|dt - \frac{1}{(\gamma+1)}\int_0^{+\infty}\frac{b(t)}{p^{\gamma+1}(t)}|p(t)u(t)|^{\gamma+1}dt \\
&\geq \alpha\|u\|^2 - \|u\|_{++\infty,p}\sum_{j=1}^{+\infty}\frac{a_j g(t_j)}{p(t_j)} - \|u\|_{\infty,p}^{\delta+1}\sum_{j=1}^{+\infty}\frac{b_j g(t_j)}{(\delta+1)p^{\delta+1}(t_j)} - \|u\|_{\infty,p}\left\|\frac{a}{p}\right\|_{L^1} - \|u\|_{\infty,p}^{\gamma+1}\left\|\frac{b}{p^{\gamma+1}}\right\|_{L^1}
\end{aligned}$$

Hence, by Lemma 3.1, we get

$$\begin{aligned}
\varphi(u) &\geq \alpha\|u\|^2 - C\|u\|\sum_{j=1}^{+\infty}\frac{a_j g(t_j)}{p(t_j)} - C^{\delta+1}\|u\|^{\delta+1}\sum_{j=1}^{+\infty}\frac{b_j g(t_j)}{p^{\delta+1}(t_j)} - C\left\|\frac{a}{p}\right\|_{L^1}\|u\| - C^{\gamma+1}\left\|\frac{b}{p^{\gamma+1}}\right\|_{L^1}\|u\|^{\gamma+1} \\
&\geq \alpha\|u\|^2 - C\left(\left\|\frac{a}{p}\right\|_{L^1} + \sum_{j=1}^{+\infty}\frac{a_j g(t_j)}{p(t_j)}\right)\|u\| - C^{\delta+1}\left(\sum_{j=1}^{+\infty}\frac{b_j g(t_j)}{p^{\delta+1}(t_j)}\right)\|u\|^{\delta+1} - C^{\gamma+1}\left\|\frac{b}{p^{\gamma+1}}\right\|_{L^1}\|u\|^{\gamma+1}.
\end{aligned}$$

Since $\delta, \gamma \in [0,1)$, then $\lim_{\|u\|\to+\infty}\varphi(u) = +\infty$. This means, $\varphi$ is coercive.

Using Theorem 3.1, $\varphi$ has a minimum, which is a critical point of $\varphi$. Finally, from $(H_2)$ and $(I_2)$, we get the differentiability of $\varphi$ such that its differentiable is defined by 3.8. Consequently, $(NP)$ has at least one solution. □

**Example 3.3.** *Consider the following problem*

$$\begin{cases} -u''(t) + u(t) = e^{-2t}\sqrt{|u(t)|} + e^{-3t}, & \text{a.e. } t \in [0, +\infty), \\ \Delta u'(j) = e^{-2j}\left(\frac{1}{j^2} + \frac{|u|^{\frac{1}{4}}}{j}\right), & j \in \mathbb{N}^*, \\ u(0) = u(+\infty) = 0, \end{cases}$$

where $\lambda = 1, p(t) = e^{-t}, g(t) = e^{-2t}, a_j = \frac{1}{j^2}, b_j = \frac{1}{j}$ and $I_j(s) = \frac{1}{j^2} + \frac{|s|^{\frac{1}{4}}}{j}, j \in \mathbb{N}^*$. By simple calculations, all conditions in Theorem 3.4 are satisfied, then 3.6 has at least one solution.

# Conclusion

In this work, we studied some critical point theory and investigated a second-order impulsive boundary value problem using variational methods, minimization theorems, the Mountain Pass Theorem, and the Saddle Point Theorem.

In the first chapter, we presented the main tools required for this work, such as critical point theory, differentiability of operators in Banach spaces, the Mountain Pass Theorem, and several minimization results. We also recalled the compactness condition known as the Palais–Smale condition.

In the second chapter, we studied a second-order impulsive boundary value problem defined on a bounded domain $[0, T]$, given by:

$$\begin{cases} -u''(t) + g(t)u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = I_j(u(t_j)), \quad j = 1, 2, \ldots, p, \end{cases}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $I_j : \mathbb{R} \to \mathbb{R}$ are continuous functions.

In the third chapter, we extended the study to the same type of impulsive boundary value problem posed on unbounded domains (such as the half-line or the whole real line), where more advanced variational tools and compactness techniques are required.

Our approach is variational: solutions are obtained as critical points or minimizers of an associated energy functional.

Finally, this work opens the way for further extensions to more general impulsive problems on unbounded domains using more sophisticated analytical and variational methods.

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# Abstract

In this memory, we study two second-order impulsive boundary value problems : one on a bounded domain $[0,T]$ and the other on an unbounded domain $[0,+\infty[$, using variational methods and critical point theorems, such as the Mountain Pass Lemma, the Saddle Point Theorem, and the Lax-Milgram Theorem. This study is applied in Chapter 2 to establish the existence and multiplicity of solutions for the following second-order impulsive boundary value problem :

$$\begin{cases} -u''(t)+g(t)u(t)=f(t,u(t)), & a.e.t\in[0,T];\\ u(0)=u(T)=0,\\ \Delta u'(t_j)=u'(t_j^+)-u'(t_j^-)=I_j(u(t_j)), \quad j=1,2,\dots,p, \end{cases}$$

In the final chapter, we apply this study to obtain at least one solution for the following second-order impulsive boundary value problem :

$$\begin{cases} -u''(t)+\lambda u(t)=f(t,u(t)), \quad a.e.t\in[0,+\infty),\ t\neq t_j,\\ \Delta u'(t_j)=g(t_j)I_j(u(t_j^-)), \quad j\in N^*,\\ u(0)=u(+\infty)=0. \end{cases}$$

**Keywords :**

Critical point, Variational method, Existence of solutions, Second order, Impulsive differential equation, Mountain Pass Lemma, Saddle Point Theorem, Palais-Smale condition, Energy functional.

# ملخص

**في هذه المذكرة ، درسنا مسألتي قيم حدية من الدرجة الثانية إندفاعية على مجالين أحدهما محدود $[0,T]$ , و الآخر غير محدود $[0,+\infty[$ بإستخدام طرائق التغاير ونظريات النقاط الحرجة، كتوطئة ممر الجبل، ونظرية نقطة السرج. ونظرية لاكس-ميليغرام، وقد طبقنا هذه الدراسة في الفصل الثاني لإيجاد وتعدد الحلول لمسألة قيمة حدية من الدرجة الثانية إندفاعية الآتية:**

$$\begin{cases} -u''(t)+g(t)u(t)=f(t,u(t)), & a.e.t\in[0,T];\\ u(0)=u(T)=0,\\ \Delta u'(t_j)=u'(t_j^+)-u'(t_j^-)=I_j(u(t_j)), \quad j=1,2,\dots,p, \end{cases}$$

**أما في الفصل الأخير فقد طبقنا هذه الدراسة لإيجاد حل على الأقل لمسألة قيمة حدية من الدرجة الثانية إندفاعية الآتية:**

$$\begin{cases} -u''(t)+\lambda u(t)=f(t,u(t)), \quad a.e.t\in[0,+\infty),\ t\neq t_j,\\ \Delta u'(t_j)=g(t_j)I_j(u(t_j^-)), \quad j\in N^*,\\ u(0)=u(+\infty)=0, \end{cases}$$

**كلمات مفتاحية:** نقطة حرجة. طريقة التغاير. وجود الحلول. الدرجة الثانية. المعادلة التفاضلية المندفعة. توطئة ممر الجبل. نظرية نقطة السرج. شرط بالي-سمال. . دالة الطاقة.