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Numerical solution of free surface flow problem

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ملخص

تم تقديم طريقة تقريبية لحل مشكلة التدفق السطحي الحر الثابت لسائل مثالي على منحدر شبه لا نهائي في القاع. يتم استخدام تحويل شوارتز- كريستوفيل لرسم خريطة لمنطقة التدفق، في المستوى المحتمل المعقد، على النصف العلوي من المستوى. تم استخدام تحويل هيلبرت مع تقنية الاضطراب كأساس للحل التقريبي للمشكلة لعدد فرود كبير وزاوية ميل صغيرة. يتم الحصول على معادلات عامة، في شكل متكامل، لأي ترتيب تقريبي. تمت مناقشة وتوضيح النتائج حتى التقريب من الدرجة الأولى. الكلمات المفتاحية: مشاكل التدفق على السطح الحر، تحويل شوارتز- كريستوفيل، تحويل هيلبرت، تقنية الاضطراب، المعادلات التكاملية غير الخطية.

Résumé

Une méthode approximative est présentée pour résoudre le problème de l'écoulement régulier en surface libre d'un fluide idéal sur une rampe semi-infinie dans le fond. La transformation de Schwartz-Christoffel est utilisée pour cartographier la région d'écoulement, dans le plan de potentiel complexe, sur le demi-plan supérieur. La transformation de Hilbert ainsi que la technique de perturbation sont utilisées comme base pour la solution approximative du problème pour un grand nombre de Froude et un petit angle d'inclinaison de la rampe. Des équations générales, sous forme intégrale, pour tout ordre d'approximation sont obtenues. La solution jusqu'à l'approximation du premier ordre est discutée et illustrée.

Mots clés: Problèmes d'écoulement à surface libre, transformation de Schwartz-Christoffel, transformation de Hilbert, technique de perturbation, équations intégrales non linéaires.

Abstract

An approximate method is presented to solve the problem of steady free-surface flow of an ideal fluid over a semi-infinite ramp in the bottom. Schwartz-Christoffel transformation is used to map the region of flow, in the complex potential-plane, onto the upper half-plane. The Hilbert transformation as well as the perturbation technique are used as a basis for the approximate solution of the problem for large Froude number and small inclination angle of the ramp. General equations, in integral form, for any order of approximation are obtained. Solution up to first-order approximation is discussed and illustrated.

Keywords: Free-surface flow problems, Schwartz-Christoffel transformation, Hilbert transformation, perturbation technique, nonlinear integral equations.

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Notation

$\vec{v}$	velocity vector
φ	velocity potential
ψ	stream function
$f(z) = \varphi(x, y) + i\psi(x, y)$	complex potential function
$z = x + iy$	complex variable
p	fluid pressure
ρ	fluid density
g	gravity
S	the surface
$H(t)$	solution of the homogenous Hilbert problem in the upper half-plane
F	Froude number
rot	Rotationnal
$\overrightarrow{grad}$	gradient
$\frac{d}{dt}, \frac{D}{Dt}$	material derivative
$\frac{\partial}{\partial x}$	derivative relative to a x
Δ	the laplacien
ω	logarithmic hodograph variable
$W'(t)$	the mapping function
L'	length of the rampe
α	inclination angle
V	volume
$Q(t)$	solution of Hilbert problem in the upper half- plane
$R(t)$	real part of Q(t)
$S(t)$	imaginary part of Q(t)

Introduction

In this memory on considers the problem of steady free surface of a running two-dimensional, irrotational, inviscid and incompressible flow in an open chanel with a non uniform bottom in a shape of a ramp. This problem has many applications in the fields of coastal engineering and hydraulics. That's why this type of problems has received considerable attention throughout the history of fluid mechanics, during the past 30 years, and the literature on the topic is rich.

Taking into account the effect of gravity, the problem can only be solved by a numerical approach because of the non-linear term in the condition at the edge of the free surface. The solution is obtained using a Hilbert method and the perturbation technique introduced by S.N.Hanna and M.B.Abdel-Malek [11].

This work is composed of three chapters :

In the first chapter, we recall the fundamental notions of fluid mechanics, such as the types of flows and properties of fluids, and some equations of fluid mechanics (conservation of mass, Bernoulli, Euler), and we also define the complex potential and velocity potential.

In the second chapter, we will study a problem of two-dimensional, irrotational motion, i.e., possessing a velocity potential (formulation of the problem) of a flow on a shape of a ramp.

The third chapter, deals with the approximate solution of the problem, using perturbation techniques as a basis for the approximate solution, and concludes with a presentation of some numerical results and a discussion.

In this memory, the Hilbert method as well as the perturbation technique are used as a basis for the approximate solution of a two-dimensional problem. First, as customary, the physical and complex potential-planes are mapped onto an auxiliary upper half-plane. Then, nonlinear integral equations are found giving the solution of a mixed boundary-value problem. Lastly, several approximations are done based upon large Froude number and the perturbation technique is applied based upon small inclination angles of the ramp.

Chapter 1

Some preliminary notions and equations

In this chapter, we present the basic concepts of fluid mechanics : fluid properties, fundamental equations of fluid mechanics for a potential, two-dimensional and irrotational flow of an incompressible and non-viscous fluid.

1.1 Types of fluid flow

1.1.1 Irrotational flow

A flow is said to be irrotational if :

$$\vec{rot}(\vec{v}) = 0 \quad (1.1)$$

$$rot(\vec{v}) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \Lambda \begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \frac{\partial u}{\partial x} = 0 \\ \frac{\partial v}{\partial y} = 0 \\ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \end{cases} \quad (1.2)$$

(v the flow speed.)

1.1.2 Uniform flow

A type of flow in which velocity v , pressure p , density ρ , temperature T , etc. at any given time do not change with respect to space (i.e., length of direction of the flow) is called uniform flow.

Mathematically,

$$\frac{\partial v}{\partial s} = 0, \frac{\partial p}{\partial s} = 0 \quad (1.3)$$

1.1.3 Potential flow

The flow is said to be potential if its velocity vector is derived from a potential, i.e.,

$$\vec{v} = \nabla \varphi \quad (1.4)$$

The function $\varphi(x, y)$ is speed potential.

1.1.4 Steady flow

A flow is considered to be steady if fluid flow parameters such as velocity, pressure, density, temperature etc. At any point do not change with time. If any one of these parameters changes with time, the flow is said to be unsteady.

Mathematically,

$$\frac{\partial v}{\partial t} = \frac{\partial p}{\partial t} = \frac{\partial \rho}{\partial t} = 0 \quad (1.5)$$

1.1.5 Two dimensional flow

If the z component of the fluid velocity is zero, while the x and y components depend on x and y but not on z , we obtain a two-dimensional flow in the xy plane. The velocity vector lies in the xy plane at every point in this two-dimensional flow.

1.2 Properties of fluid

1.2.1 Incompressible fluid

A fluid is said to be incompressible if its volume does not vary according to the external pressure. liquids can be considered incompressible fluids (water, oil, etc).

1.2.2 Non viscous fluid

A fluid is said to be non viscous (or perfect) if tangential stresses can be neglected.

1.3 Stream function

The stream function is a mathematical function of the space coordinates and time which completely describes the pattern of stream lines. It is denoted by the Greek symbol ψ . In general, $\psi = f(x, y, z, t)$, but if the flow is steady the time variable t does not appear in the relation. A two dimensional stream function for steady flow may be defined as a continuous function of x and y such that its numerical value at any point is equal to the discharge per unit thickness perpendicular to the $X - Y$ plane across any line joining that point to the origin. Thus, referring to figure (1.1), the value of the stream function at A is the discharge per unit thickness across the line OA , which may be any straight or curved line.

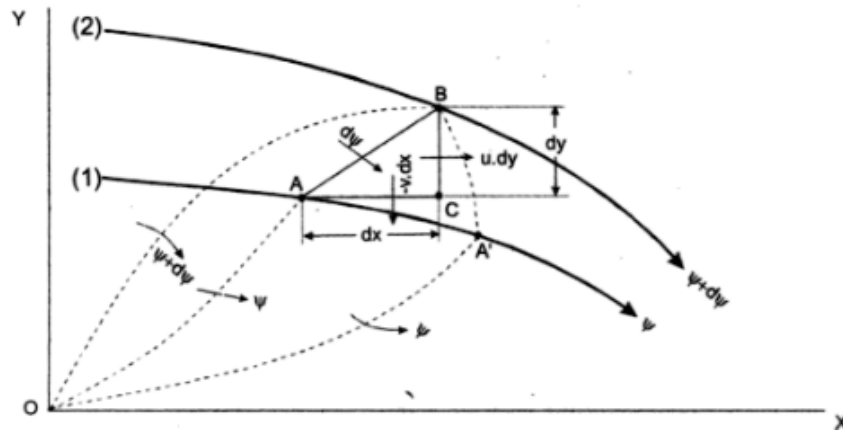


Figure 1.1: Stream function in two dimensional flow.

1.4 Velocity potential

The two-dimensional incompressible continuity equation.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1.6)$$

Guarantees the existence of a stream function ψ , from which the velocity components can be derived as

$$\begin{cases} u = \frac{\partial \psi}{\partial y} \\ v = -\frac{\partial \psi}{\partial x} \end{cases} \quad (1.7)$$

like wise, the condition of irrotationality

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad (1.8)$$

Guarantees the existence of another scalar function ϕ , called the velocity potential, which is related to the velocity components by

$$\begin{cases} u = \frac{\partial \phi}{\partial x} \\ v = \frac{\partial \phi}{\partial y} \end{cases} \quad (1.9)$$

Because a velocity potential must exist in all irrotational flows, such flows are frequently called potential flows. Equations (1.7) and (1.9) imply that the derivative of ψ gives the velocity component in a direction 90° clockwise from the direction of differentiation, where as the derivative of ϕ gives the velocity component in the direction of differentiation. Comparing equations (1.7) and (1.9) we obtain

$$\begin{aligned} \text{Cauchy-Riemann conditions} \quad & \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\ & \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{aligned} \quad (1.10)$$

From which one of the functions can be determined if the other is known. Equipotential lines (on which ϕ is constant) and streamlines are orthogonal, as equation (1.10) implies that

$$\nabla \phi \cdot \nabla \psi = \left(i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} \right) \cdot \left(i \frac{\partial \psi}{\partial x} + j \frac{\partial \psi}{\partial y} \right) = \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial y} = 0.$$

This demonstration fails at stagnation points where the velocity is zero.

The stream function and velocity potential satisfy the Laplace equations

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (1.11)$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (1.12)$$

1.5 Notions of complex potential and complex velocity

The complex potential is the function $W(z) = \varphi(x, y) + i\psi(x, y)$ where $z = x + iy$ is the complex variable associated with the complex potential function $W(z)$ (φ and ψ are the potential and stream function respectively).

The function W has the following properties:

- 1) $W(z)$ is a uniforme function, i.e. one value of z corresponds to a single value of W .
- 2) $W(z)$ is analytic, so its derivative must be defined every where, i.e.,

$$\begin{aligned} \frac{dW}{dz} &= \frac{\partial \varphi}{\partial x} + i \frac{\partial \psi}{\partial x} = -i \frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial y} \\ \Rightarrow &\begin{cases} \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} = v_x \\ \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} = v_y \end{cases} \end{aligned} \quad (1.13)$$

This system of equations constitutes the Cauchy-Riemann relations, which verify the relations found above.

For $W(z)$ to be analytic, φ and ψ must satisfy the Cauchy-Riemann relations.

The function $W(z)$ is called the complex velocity potential.

We have seen that for a flow to be described by a stream function ψ and a velocity potential φ , these two functions must satisfy the Laplace equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (1.14)$$

and

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad (1.15)$$

1.6 Free Surface

Physically, free surface is the bounding surface of a certain amount of fluid under consideration. From the mathematical point of view, we consider the free surface Σ to be a piecewise smooth. Orientable regular surface, the free surface is described by the relation $s(r, t)$, this free surface has to fulfill the so called free surface kinematic condition.

In the Lagrangian description this equation reads:

$$\frac{ds}{dt} = 0 \quad (1.16)$$

1.7 Dimensionless variables

Before the problem can be solved, it is necessary to use non dimensionlise the variables so as to reduce the number of free parameters. The dimensionless variables are defined by choosing U as the unit velocity and h as the unit length, the new dimensionless variables are:

$$z' = \frac{z}{h_1}, q' = \frac{q}{U_1}, W' = \frac{W}{\psi_1} \quad (1.17)$$

Where $\psi_1 = U_1 h_1$.

1.8 Fluid mechanics equations

1.8.1 Conservation of mass equation

The mass conservation laws says that the rate of change of mass in an arbitrary material volume $V(t)$ equals the rate of mass production in $V(t)$. This can be expressed as:

$$\frac{d}{dt} \int_{V(t)} \rho dV = \int_{V(t)} \sigma dV \quad (1.18)$$

Where $\rho(t, x)$ is the density of the material particle at time t and position x , and $\sigma(t, x)$ is the rate of mass production per volume. In practice $\sigma \neq 0$ only in multi phase flows, in which case (1.18) holds for each phase separately, we take $\sigma = 0$, and use the transport theorem to obtain

$$\int_{V(t)} \left(\frac{\partial \rho}{\partial t} + \text{div} \rho u \right) dV = 0 \quad (1.19)$$

Since this holds for every $V(t)$ the integrand must be zero:

$$\frac{\partial \rho}{\partial t} + \text{div} \rho u = 0 \quad (1.20)$$

This is the mass conservation law, also called the continuity equation.

1.8.2 Bernoulli Equation

Bernoulli's theory is an application of the conservation of energy to the case of moving fluids. In the case of an incompressible fluid, Bernoulli's equation is :

$$\frac{1}{2} q^2 + \frac{p}{\rho} + yg = \text{constant} \quad (1.21)$$

Where $q = \sqrt{u_x^2 + v_x^2}$ is the flow speed, p is the pressure of the fluid, ρ is the density of the fluid, g is gravity.

1.8.3 Euler Equation

The Euler equation is derived from the Navier-Stokes equation in the case of non viscous fluids, which is derived from the fundamental principle of dynamics (fpd) in the case of fluids:

$$\frac{dv}{dt} = -\frac{1}{\rho} \nabla p + X \quad (1.22)$$

v : is the velocity of the fluid;

t : is time;

ρ : is the mass density of the fluid;

X : the force of the body;

p : is the pressure in the fluid;

Chapter 2

Approximate solution of two dimensional flow problem

In this chapter, we are interested in the approximate solution of the two-dimensional flow problem considered, taking into account the force of gravity and we neglect the surface tension, using Hilbert's method and the perturbation technique.

2.1 Formulation of the problem

The study is concerned with two-dimensional, irrotational motion, i.e., possessing a velocity potential $\phi(x, y)$ satisfying the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (2.1)$$

Of a nonviscous and incompressible fluid, subject to the action of gravitational force and having a free upper-surface.

Consider a uniform flow with velocity U_1 in the positive x direction at depth h_1 flowing over a bottom which consists of a horizontal plane AB , inclined plane BC with an inclination angle α and length L , and a horizontal plane CD , where the bottom extends from $-\infty$ (point A) to $+\infty$ (point D), as shown in Figure (2.1).

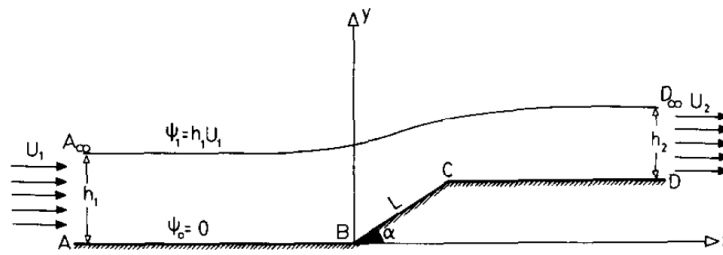


Figure 2.1: z -plane.

For convenience, we choose point B to be the origin in the z plane, the x axis from left to right and the y axis upwards. The complex potential

$$W(z) = \phi(x, y) + i\psi(x, y)$$

Is an analytic function of z within the region of flow, with complex conjugate velocity

$$\frac{dW(z)}{dz} = u(x, y) - v(x, y) = qe^{-i\theta} \quad (2.2)$$

For simplicity, we use the dimensionless analysis, then we choose U_1 and h_1 as the unit velocity and the unit length respectively, and the new non-dimensional variables are:

$$\begin{cases} z' &= \frac{z}{h_1} \\ q' &= \frac{q}{U_1} \\ W' &= \frac{W}{\psi_1} \end{cases} \quad (2.3)$$

Where $\psi_1 = U_1 h_1$.

The condition on the free-surface $A_\infty D_\infty$, where the pressure is uniform, the Bernoulli's equation take the form:

$$\begin{aligned}
& \frac{1}{2}q^2 + yg = \frac{1}{2}u^2 + hg \\
\Rightarrow & \frac{1}{2}q^2 + g(y - h) = \frac{1}{2}u^2 \\
\Rightarrow & \frac{1}{2}\left(\frac{q}{u}\right)^2 + \frac{g}{u^2}(y'h - h) = \frac{1}{2} \\
\Rightarrow & q'^2 + \frac{2gh}{u^2}(y' - 1) = 1
\end{aligned}$$

Hence

$$q'^2 + \frac{2}{F^2}(y' - 1) = 1 \quad (2.4)$$

Such that

$$\begin{cases} y' = \frac{y}{h_1} \\ q' = \frac{q}{U_1} \end{cases}$$

Where F is the Froude number defined by

$$F = \frac{U_1}{\sqrt{gh_1}} \quad (2.5)$$

Denote the dimensionless rampe height by ε , where

$$\varepsilon = \frac{L \sin \alpha}{h_1} = L' \sin \alpha \quad (2.6)$$

Now, using the dimensionless variables in (2.3), then

$$\xi = \frac{dW'}{dz'} = \frac{h_1}{\psi_1} \frac{dW}{dz} = \frac{q}{U_1} e^{-i\theta} = q' e^{-i\theta} \quad (2.7)$$

Let

$$\begin{aligned}
\omega &= \ln \xi = \ln q' - i\theta \\
\xi &= e^\omega
\end{aligned} \quad (2.8)$$

As

$$\xi = \frac{dW'}{dz'} = q' e^{-i\theta}$$

So,

$$\begin{aligned}
dz' &= \frac{1}{\xi} dW' \\
z' &= \int \frac{1}{e^\omega} dW'
\end{aligned}$$

Hence

$$z' = e^{-\omega} dW' \quad (2.9)$$

Where ω is called the logarithmic hodograph variable.

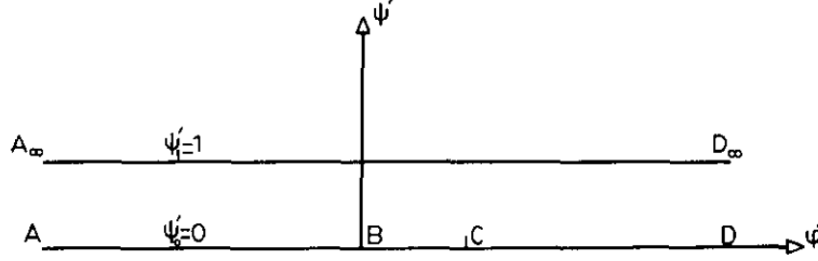


Figure 2.2: W' -plane

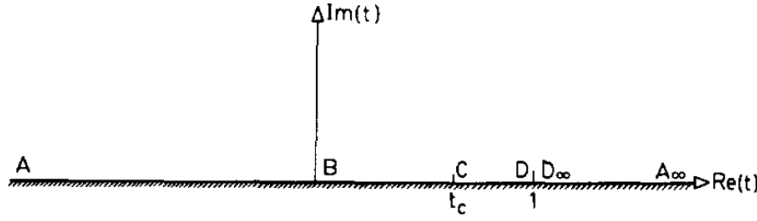


Figure 2.3: t -plane

Using the Schwartz-Christoffel transformation, we map the region of flow in the W' -plane onto the upper half of an auxiliary t -plane, so that the following points correspond (see figures (2.2) and (2.3))

Points	W'	t'
B	0	0
DD_∞	$+\infty$	1
AA_∞	$-\infty$	$+\infty$

The transformation of Schwartz-Christoffel defined by:

$$\frac{dW'}{dt} = M(t - t_1)^{\frac{\alpha_1}{\pi} - 1} (t - t_2)^{\frac{\alpha_2}{\pi} - 1} \dots (t - t_n)^{\frac{\alpha_n}{\pi} - 1} \quad (2.10)$$

Where α_i is the interior angles of polygon in the plane W' and M is complex constant. From (2.10) and figures (2.2), (2.3), we get the tranformation

$$W'(t) = \frac{-1}{\pi} \ln(1 - t), \quad 0 \leq \arg(1 - t) \leq \pi \quad (2.11)$$

For to express ω as a function of the single variable t , we introduce the Hilbert method for a mixed boundary-value problem in the upper half-plane.

Let $Q(t)$ be an analytic function defined in the upper half-plane, the t -plane, and suppose $Im(Q(t))$ satisfies the Hölder condition on the boundary, $Im(t) = 0$, of the t -plane.

If $Im(Q(t))$ is known on the boundary, then $Q(t)$ is given by the Poisson integral formula

$$Q(t) = \frac{1}{\pi} \oint_{-\infty}^{+\infty} \frac{Im(Q(v))}{v-t} dv + \sum_{j=0}^{\infty} A_j t^j \quad (2.12)$$

where A_j are real constant, and $\oint$ denotes a singular integral in the sense of Cauchy.

Now, we try to relate the function $\omega(t)$ to function $Q(t)$. From (2.12), we find that $Q(t)$ is expressed in terms of its imaginary parts along the real axis, the boundary of the t -plane.

Thus, we have to examine the value of $\omega(t)$ along the boundary of the t -plane we find that.

We have $\omega = \ln q' e^{-i\theta}$, hence

$$Im(\omega(t)) = -\theta(t) \quad (2.13)$$

And

$$\begin{aligned} q'^2 + \frac{2}{F^2}(y' - 1) &= 1 \\ \Rightarrow q'^2 &= 1 - \frac{2}{F^2}(y' - 1) \\ \Rightarrow q' &= \sqrt{1 - \frac{2}{F^2}(y' - 1)} \\ \Rightarrow \ln q' &= \frac{1}{2} \ln \left(1 - \frac{2}{F^2} \eta'(t) \right) \end{aligned}$$

So

$$\begin{aligned} Im(\omega(t)) &= -\theta(t) \\ Re(\omega(t)) &= \frac{1}{2} \ln \left(1 - \frac{2}{F^2} \eta'(t) \right) \end{aligned} \quad (2.14)$$

Where

$$\theta(t) = \begin{cases} 0, & t < 0 \\ \alpha, & 0 < t < t_c \\ 0, & t_c < t < 1 \\ \theta(t), & t > 1 \end{cases} \quad (2.15)$$

And

$$\eta'(t) = y'(t) - 1$$

This means that we know either the real or imaginary part of $\omega(t)$ along the boundary of the t -plane. Then we must construct an auxiliary function $H(t)$ which makes the imaginary part of the quotient $Q(t) = \omega(t)/H(t)$ known on the boundary of the t -plane. The general form of $H(t)$ is

$$H(t) = a\Pi_j(t - b_j)^{\mp\frac{1}{2}}$$

Where b_j are real constants, and $a = \mp\sqrt{\mp 1}$.

Song in [18] has shown that the final solution is independent of the particular choice of $H(t)$, we choose

$$H(t) = \begin{cases} -\sqrt{(1-t)}, & t < 1 \\ -i\sqrt{t-1}, & t > 1 \end{cases} \quad (2.16)$$

We have

$$Q(t) = \frac{\omega(t)}{H(t)} = \begin{cases} \frac{\ln q' - i\theta}{-\sqrt{(1-t)}}, & t < 1 \\ \frac{\ln q' - i\theta}{-i\sqrt{(t-1)}}, & t > 1 \end{cases} \quad (2.17)$$

Using (2.13) and (2.16), we obtain

$$\text{Im}(Q(t)) = \begin{cases} \frac{\theta(t)}{\sqrt{1-t}}, & t < 1 \\ \frac{\ln(1 - \frac{2}{F^2}\eta'(t))}{2\sqrt{(t-1)}}, & t > 1 \end{cases} \quad (2.18)$$

Next, we examine the upstream condition. As we approach point A_∞ ; i.e., as $t \rightarrow +\infty$, $H(t) \simeq -i\sqrt{t}$ and $\omega(t) \simeq \ln q'(t) = 0$. Therefore, $Q(t) = \omega(t)/(-i\sqrt{t}) = 0$, and from (2.12), $A_j = 0$ for $j = 0, 1, 2, \dots, n$. Thus, (2.12) takes the form

$$Q(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\text{Im}(Q(v))}{v-t} dv \quad (2.19)$$

Using (2.7), we obtain

$$\begin{aligned} Q(t) &= \frac{\omega(t)}{H(t)} = \begin{cases} \frac{-\ln q'(t)}{\sqrt{1-t}} + i\frac{\theta(t)}{\sqrt{1-t}}, & t < 1 \\ \frac{i\ln q'(t)}{\sqrt{t-1}} + \frac{\theta(t)}{\sqrt{t-1}}, & t > 1 \end{cases} \\ &= R(t) + iS(t) \end{aligned} \quad (2.20)$$

An equivalent form of (2.19) is

$$R(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{S(v)}{v-t} dv \quad (2.21)$$

$$S(t) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{R(v)}{v-t} dv \quad (2.22)$$

Writing (2.20) in details

t	$R(t)$	$S(t)$
$t < 0$	$\frac{-\ln q'_A(t)}{\sqrt{1-t}}$	0
$0 < t < t_c$	$\frac{-\ln q'_B(t)}{\sqrt{1-t}}$	$\frac{\alpha}{\sqrt{1-t}}$
$t_c < t < 1$	$\frac{-\ln q'_c(t)}{\sqrt{1-t}}$	0
$t > 1$	$\frac{\theta(t)}{\sqrt{t-1}}$	$\frac{\ln q'(t)}{\sqrt{t-1}}$

Using the Hilbert transformation relating the values on the real axis, of the real and imaginary parts of a function analytic in the upper half-plane, given by (2.21) and (2.22), we obtain a set of integral equation, and for our work we need only the following equations

$$\frac{\theta(t)}{\sqrt{t-1}} = \frac{1}{\pi} \int_1^{+\infty} \frac{\ln q'(v)}{(v-t)\sqrt{v-1}} dv + \frac{1}{\pi} \int_0^{t_c} \frac{\alpha}{(v-t)\sqrt{1-v}} dv, \quad t > 1 \quad (2.23)$$

$$\frac{\ln q'_B(t)}{\sqrt{1-t}} = \frac{-1}{\pi} \int_1^{+\infty} \frac{\ln q'(v)}{(v-t)\sqrt{v-1}} dv - \frac{1}{\pi} \int_0^{t_c} \frac{\alpha}{(v-t)\sqrt{1-v}} dv, \quad 0 < t < t_c \quad (2.24)$$

By integration, (2.23) and (2.24) takes the form:

$$\theta(t) = \frac{\sqrt{t-1}}{\pi} \int_1^{+\infty} \frac{\ln q'(v)}{(v-t)\sqrt{v-1}} dv + \frac{2\alpha}{\pi} \tan^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right), \quad t > 1 \quad (2.25)$$

And

$$\ln q'_B(t) = -\frac{\sqrt{1-t}}{\pi} \int_1^{+\infty} \frac{\ln q'(v)}{(v-t)\sqrt{v-1}} dv - \frac{2\alpha}{\pi} \tanh^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right), \quad 0 < t < t_c \quad (2.26)$$

Where

$$r = 1 - \sqrt{1-t_c}$$

And $q'_B(t)$ is the flow speed over the ramp.

The coordinates (x', y') of a point on the free-surface can be obtained using (2.8) and (2.9) as follows:

$$z' = \int e^{-\omega} dW'$$

Such that

$$\omega = \ln q' - i\theta$$

So

$$z' = \int e^{-\ln q' + i\theta} dW'$$

Using (2.11), we obtain

$$z'(t) = \frac{1}{\pi} \int \frac{e^{i\theta(v)}}{q'(v)} \times \frac{1}{1-v} dv + \text{const}$$

Hence

$$z'(t) = \frac{1}{\pi} \int_{\infty}^t \frac{e^{i\theta(v)}}{q'(v)(1-v)} dv + (x'_{\infty} + i)$$

Separating real and imaginary part, we get

$$x'(t) = x'_{\infty} + \frac{1}{\pi} \int_{\infty}^t \frac{\cos \theta(v)}{(1-v)q'(v)} dv, \quad t > 1 \quad (2.27)$$

$$y'(t) = 1 + \frac{1}{\pi} \int_{\infty}^t \frac{\sin \theta(v)}{(1-v)q'(v)} dv, \quad t > 1 \quad (2.28)$$

From (2.28) we may find the length of the ramp L' , we have:

$$y'(t) = 1 + \frac{1}{\pi} \int_{\infty}^t \frac{\sin \theta(v)}{(1-v)q'(v)} dv$$

This implies

$$1 - L' \sin \alpha = 1 - \frac{1}{\pi} \int_t^{\infty} \frac{\sin \theta(v)}{(1-v)q'(v)} dv$$

Along BC we have:

$$\theta = \alpha, \quad 0 < t < t_c, \quad q'(v) = q'_B(v).$$

$$\begin{aligned} 1 - L' \sin \alpha &= 1 - \frac{1}{\pi} \int_0^{t_c} \frac{\sin \alpha}{(1-v)q'_B(v)} dv \\ \implies L' &= \frac{1}{\pi} \int_0^{t_c} \frac{1}{(1-v)q'_B(v)} dv \end{aligned} \quad (2.29)$$

Where $q'_B(t)$ given by (2.26).

Hence, the system of equations (2.4), (2.25), (2.26), (2.27), (2.28) and (2.29) describes completely our problem.

2.2 The approximate equations

For large Froude number F and for small inclination angle α , in which case the change in $\theta(t)$ will be very small, we can approximate $\sin \theta(t)$ by $\theta(t)$ and $\cos \theta(t)$ by 1. Having done so, we proceed to drop the primes and the system of equations takes the form:

$$q(t) \simeq 1 - \frac{1}{F^2} \eta(t), \quad t > 1 \quad (2.30)$$

$$\eta(t) \simeq -\frac{1}{\pi} \int_t^\infty \frac{\theta(v)}{1-v} \left(1 + \frac{1}{F^2} \eta(v)\right) dv, \quad t > 1 \quad (2.31)$$

$$\theta(t) \simeq -\frac{\sqrt{t-1}}{\pi F^2} \int_1^\infty \frac{\eta(v)}{(v-t)\sqrt{v-1}} dv + \frac{2\alpha}{\pi} \tan^{-1} \left(\frac{r\sqrt{t-1}}{t-r} \right), \quad t > 1 \quad (2.32)$$

$$x(t) \simeq x_\infty - \frac{1}{\pi} \int_t^\infty \frac{1}{1-v} \left(1 + \frac{1}{F^2} \eta(v)\right) dv, \quad t > 1 \quad (2.33)$$

$$\ln q_B(t) \simeq \frac{\sqrt{1-t}}{\pi F^2} \int_1^\infty \frac{\eta(v)}{(v-t)\sqrt{v-1}} dv - \frac{2\alpha}{\pi} \tanh^{-1} \left(\frac{r\sqrt{t-1}}{t-r} \right), \quad 0 < t < t_c \quad (2.34)$$

$$L \simeq \frac{1}{\pi} \int_0^{t_c} \frac{1}{(1-v)q_B(v)} dv \quad (2.35)$$

2.3 Perturbation technique

Expand $q(t)$, $\eta(t)$, $\theta(t)$ and $x(t)$ in terms of the small parameter α ,

$$q(t) = q_0(t) + \alpha q_1(t) + \alpha^2 q_2(t) + \dots \quad (2.36)$$

$$\eta(t) = \eta_0(t) + \alpha \eta_1(t) + \alpha^2 \eta_2(t) + \dots \quad (2.37)$$

$$\theta(t) = \theta_0(t) + \alpha \theta_1(t) + \alpha^2 \theta_2(t) + \dots \quad (2.38)$$

$$x(t) = x_0(t) + \alpha x_1(t) + \alpha^2 x_2(t) + \dots \quad (2.39)$$

Upon employing the expansions (2.36 - 2.39) into the equations (2.30 - 2.33) and equating terms of similar power of α , we get the:

(i) Zero-order approximation

$$q_0(t) = 1 - \frac{1}{F^2} \eta_0(t) \quad (2.40)$$

$$\eta_0(t) = \frac{-1}{\pi} \int_t^\infty \frac{\theta_0(v)}{1-v} \left(1 + \frac{1}{F^2} \eta_0(v)\right) dv, \quad t > 1 \quad (2.41)$$

$$\theta_0(t) = -\frac{\sqrt{t-1}}{\pi F^2} \oint_1^\infty \frac{\eta_0(v)}{(v-t)\sqrt{v-1}} dv, \quad t > 1 \quad (2.42)$$

$$x_0(t) = x_\infty - \frac{1}{\pi} \int_t^\infty \frac{1}{1-v} \left(1 + \frac{1}{F^2} \eta_0(v)\right) dv, \quad t > 1 \quad (2.43)$$

This zero-order approximation corresponds to the case of flow over a flat bottom. The solution of the zero-order approximation in dimensionless variables $x' = \frac{x}{h_1}$, $y' = \frac{y}{h_1}$, $z' = \frac{z}{h_1}$, we have

$$q_0(t) = 1 - \frac{1}{F^2} \eta_0(t)$$

Such that

$$\begin{aligned} \eta_0(t) &= -\frac{1}{\pi} \int_t^\infty \frac{\theta_0(v)}{1-v} \left(1 + \frac{1}{F^2} \eta_0(v)\right) dv \\ &= y' - 1 \\ &= \frac{y}{h_1} - 1 \end{aligned}$$

Flow is over a flat bottom, so $y = h_1$. Hence

$$\eta_0(t) = 0 \implies q_0(t) = 1 \quad (2.44)$$

And so

$$\eta_0(t) = 0 \quad (2.45)$$

So

$$\theta_0(t) = 0 \quad (2.46)$$

And

$$x_0(t) = x_\infty - \frac{1}{\pi} \int_t^\infty \frac{1}{1-v} dv \quad (2.47)$$

On the other hand, at point B when $t = 0$, $x = 0$,

$$x_0(0) = 0 \implies x_\infty - \frac{1}{\pi} \int_0^\infty \frac{1}{1-v} dv = 0$$

Writing

$$x_\infty = \frac{1}{\pi} \int_0^\infty \frac{1}{1-v} dv \quad (2.48)$$

Hence, the equation (2.47) take the form:

$$\begin{aligned}
x_0(t) &= \frac{1}{\pi} \int_0^\infty \frac{1}{1-v} dv - \frac{1}{\pi} \int_t^\infty \frac{1}{1-v} dv \\
&= \frac{1}{\pi} \int_0^t \frac{1}{1-v} dv \\
&= -\frac{1}{\pi} \ln(t-1)
\end{aligned} \tag{2.49}$$

(ii) First-order approximation

Now, we find the first-order approximation of the non linear integral equations (2.30 - 2.33) by using development (2.36 - 2.39) and the zero-order approximation of the system.

Using the development (2.36 - 2.39), we can write

$$Q_1(t) = \frac{Q(t) - Q_0(t)}{\alpha} \tag{2.50}$$

Substituting (2.30) and (2.44) into (2.50) yields

$$\begin{aligned}
q_1(t) &= \frac{-\frac{1}{F^2}\eta(t)}{\alpha} \\
&= -\frac{1}{\alpha F^2}(\eta_0(t) + \alpha\eta_1(t))
\end{aligned} \tag{2.51}$$

But we know $\eta_0(t) = 0$. Thus

$$q_1(t) = -\frac{1}{F^2}\eta_1(t) \tag{2.52}$$

Then, from (2.31), (2.45) and (2.50), we get:

$$\begin{aligned}
\eta_1(t) &= -\frac{1}{\alpha\pi} \int_t^\infty \frac{\theta(v)}{1-v} (1 + \frac{1}{F^2}\eta(v)) dv \\
&= -\frac{1}{\pi} \int_t^\infty \frac{\theta_1(v)}{1-v} dv
\end{aligned} \tag{2.53}$$

And by using (2.32), (2.46) and (2.50) we get:

$$\theta_1(t) = -\frac{\sqrt{t-1}}{\alpha\pi F^2} \oint_1^\infty \frac{\eta(v)}{(v-t)\sqrt{v-1}} dv + \frac{2}{\pi} \tan^{-1} \left(\frac{r\sqrt{t-1}}{t-r} \right) \tag{2.54}$$

$$= -\frac{\sqrt{t-1}}{\alpha\pi F^2} \oint_1^\infty \frac{(\eta_0 + \alpha\eta_1)(v)}{(v-t)\sqrt{v-1}} dv + \frac{2}{\pi} \tan^{-1} \left(\frac{r\sqrt{t-1}}{t-r} \right)$$

$$= -\frac{\sqrt{t-1}}{\pi F^2} \oint_1^\infty \frac{\eta_1(v)}{(v-t)\sqrt{v-1}} dv + \frac{2}{\pi} \tan^{-1} \left(\frac{r\sqrt{t-1}}{t-r} \right) \tag{2.55}$$

On the other hand, from (2.33), (2.48), (2.49), and (2.50), we find

$$\begin{aligned}
x_1(t) &= -\frac{1}{\alpha\pi F^2} \int_t^\infty \frac{\eta(v)}{1-v} dv \\
&= -\frac{1}{\pi F^2} \int_t^\infty \frac{\eta_1(v)}{1-v} dv
\end{aligned} \tag{2.56}$$

(iii) In general, nth-order approximation

$$\begin{aligned}
q_n(t) &= -\frac{1}{F^2} \eta_n(t) \\
\eta_n(t) &= -\frac{1}{\pi} \int_t^\infty \frac{1}{1-v} (\theta_n(v) + \frac{1}{F^2} \sum_{j=1}^{n-1} \theta_j(v) \eta_{n-j}(v)) dv \\
\theta_n(t) &= -\frac{\sqrt{t-1}}{\pi F^2} \int_1^\infty \frac{\eta_n(v)}{(v-t)\sqrt{v-1}} dv \\
x_n(t) &= -\frac{1}{\pi F^2} \int_t^\infty \frac{\eta_n(v)}{1-v} dv
\end{aligned} \tag{2.57}$$

Where $n = 2, 3, 4, \dots$

2.4 Solution of the problem

The solution of the first-order approximation from (2.55), for very large Froude number F , we may neglect the first term with respect to the second one and one gets

$$\theta_1(t) \simeq \frac{2}{\pi} \tan^{-1} \left(\frac{r\sqrt{t-1}}{t-r} \right) \quad t > 1 \tag{2.58}$$

Upon substituting in (2.53) and carrying out integration, one finds

$$\eta_1(t) \simeq \frac{4r}{\pi^2 \sqrt{1-r}} \tan^{-1} \sqrt{\frac{1-r}{t-r}} \quad t > 1 \tag{2.59}$$

From (2.52) and (2.56), the equations $q_1(t)$ and $x_1(t)$ takes the form

$$q_1(t) \simeq -\frac{4r}{\pi^2 \sqrt{1-r} F^2} \tan^{-1} \sqrt{\frac{1-r}{t-r}} \quad t > 1 \tag{2.60}$$

And

$$x_1(t) \simeq \frac{8r}{\pi^3 F^2 \sqrt{t-1}} \quad t > 1 \tag{2.61}$$

For very large Froude number F , the first term in (2.34) may be neglected with respect to the second one and we get

$$\ln q_B(t) \simeq -\frac{2\alpha}{\pi} \tanh^{-1} \left(\frac{r\sqrt{t-1}}{t-r} \right) \quad 0 < t_c < 1 \quad (2.62)$$

From which we can find an approximate expression for the length of the ramp, using the equation (2.34)

$$L \simeq -\frac{1}{\pi} \ln(1 - t_c) \quad 0 < t_c < 1 \quad (2.63)$$

For small inclination angle α , we will consider only an approximate solution to an order of α , and we have

$$q(t) \simeq 1 - \alpha \frac{4r}{\pi^2 F^2 \sqrt{1-r}} \tan^{-1} \sqrt{\frac{1-r}{t-r}} + o(\alpha) \quad t > 1 \quad (2.64)$$

$$y(t) \simeq 1 + \alpha \frac{4r}{\pi^2 \sqrt{1-r}} \tan^{-1} \sqrt{\frac{1-r}{t-r}} + o(\alpha) \quad t > 1 \quad (2.65)$$

$$\theta(t) \simeq \frac{2\alpha}{\pi} \tan^{-1} \left(\frac{r\sqrt{t-1}}{t-r} \right) + o(\alpha) \quad t > 1 \quad (2.66)$$

$$x(t) \simeq -\frac{1}{\pi} \ln(t-1) + \alpha \frac{8r}{\pi^3 F^2 \sqrt{t-1}} + o(\alpha) \quad t > 1 \quad (2.67)$$

One quantity of physical interest is the drag force on the ramp caused by the fluid flow, which is horizontal component of the force acting on unit width of the ramp. In fact this drag force represents the change in momentum flux due to the change in the stream produced by the ramp. The drag force D made dimensionless by reference to the quantity $\rho g h_1^2$, where ρ is the fluid density, is given by

$$D = \frac{1}{2} F^2 \int_{y=0}^{\epsilon} (1 - q^2) dy \quad (2.68)$$

Where ϵ is given by (2.6). The equation (2.68) can be integrated most conveniently by using (2.64) and (2.65), we get

$$D = \frac{8\alpha \tan \alpha}{\pi^3} \left(e^{\pi \epsilon \cot \frac{1}{2} \alpha} - 1 \right) \left(1 - e^{-\pi \epsilon / 2 \sin \alpha} \right) \quad (2.69)$$

Chapter 3

Results and discussion

In this chapter, we give the results in curves showing angle α , height ε , and Froude number F , as well as the drag force D .

Approximate solutions for flows with large Froude number are found throughout a wide range of ramp heights. Results showing the free-surface elevation over a range of ramp heights ϵ , 0.002567 to 0.035513, for fixed $\alpha = \frac{1}{6}\pi$, $F^2 = 10.0$ and over a range of inclination angles α from $\frac{1}{12}\pi$ to $\frac{1}{6}\pi$, for fixed $\epsilon = 0.005012$, $F^2 = 10.0$ are given in the figures (3.1) and (3.2).

Effect of the Froude number F on the free-surface elevation for fixed $\alpha = \frac{1}{6}\pi$ and $\epsilon = 0.005012$ is given in the figures (3.3).

These results demonstrate the monotonic rise in free-surface elevation with increase of the ramp height, the inclination angle, and the Froude number.

From the figures (3.1) and (3.2) we are able to give an approximate formula for maximum elevation in terms of the ramp height and the inclination angle, namely

$$\frac{y_{\max}}{h_1} \simeq 1.0 + 1.045807\epsilon, \text{ for } F^2 = 10.0 \text{ and } \alpha = \frac{\pi}{6}.$$

And

$$\frac{y_{\max}}{h_1} \simeq 1.0 + 0.049636\alpha \text{ for } F^2 = 10.0 \text{ and } \epsilon = 0.005012$$

They are plotted in the figures (3.4) and (3.5), respectively.

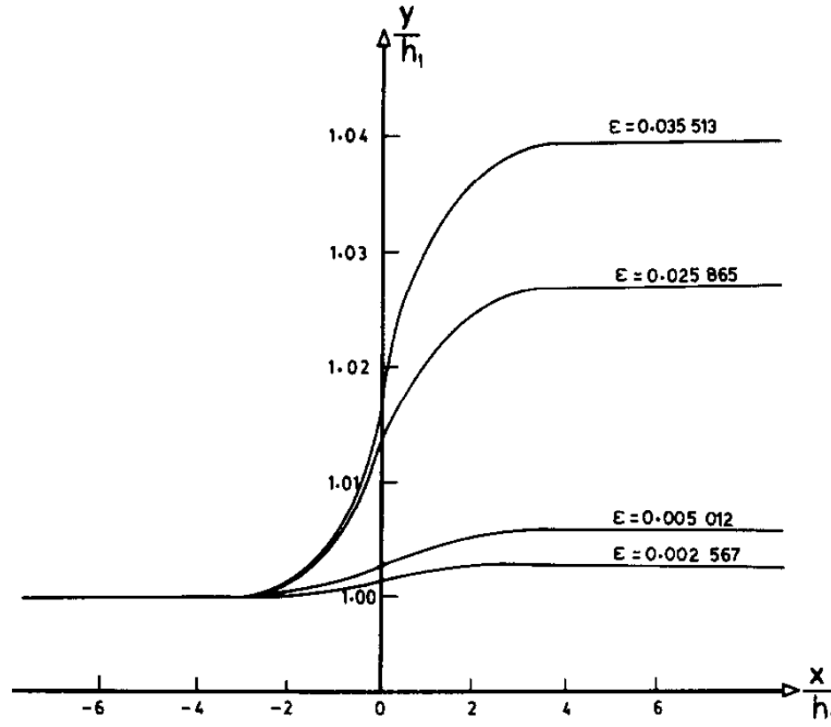


Figure 3.1: Effect of the ramp height ϵ on the free-surface profile for $\alpha = \frac{1}{6}\pi$ and $F^2 = 10.0$.

The drag force D , as shown in the figure (3.6) and given in table (3.1), increases monotonically with ramp height for different values of the inclination angle.

Forbes and Schwartz [8] considering the case of supercritical flows over a semi-circular obstruction with an otherwise flat bottom, concluded that the limiting form of the free-surface profile would have a sharp 120° -corner at its peak. A similar conclusion could be made with regard to the flow over a ramp. Behavior of curves, in the figure (3.1), for different ϵ , support this idea.

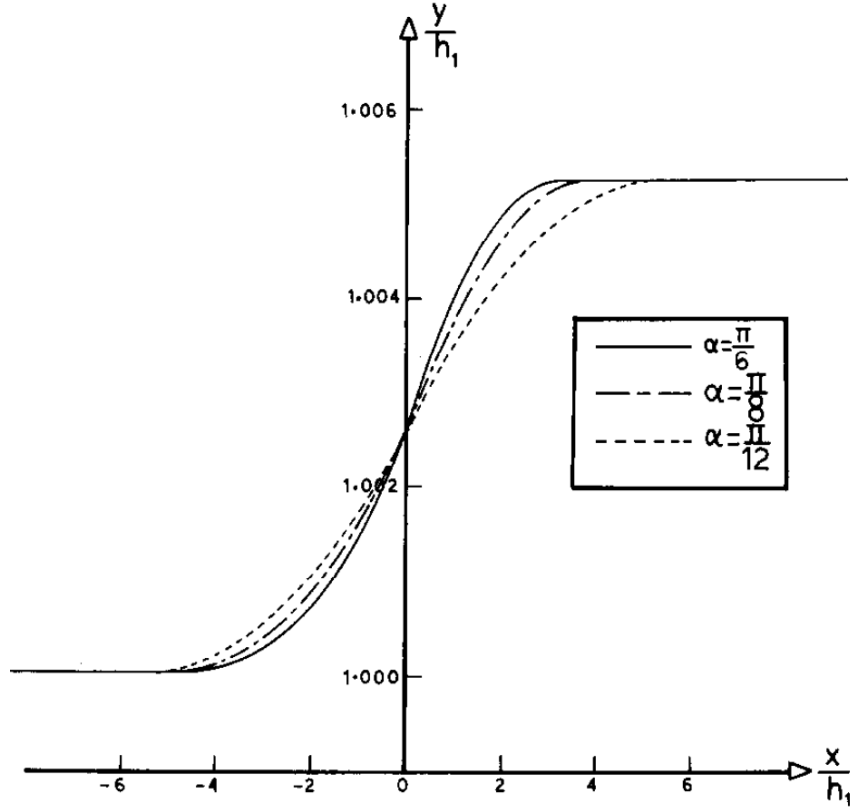


Figure 3.2: Effect of the inclination angle α the ramp on the free-surface profile for $\epsilon = 0.005012$ and $F^2 = 10.0$.

ϵ	$\alpha = \frac{1}{6}\pi$	$\alpha = \frac{1}{8}\pi$	$\alpha = \frac{1}{12}\pi$
0.005000	0.000017	0.000016	0.000016
0.025000	0.000415	0.000407	0.000405
0.050000	0.001652	0.001625	0.001619
0.060000	0.002375	0.002340	0.002333
0.070000	0.003229	0.003185	0.003177
0.080000	0.004213	0.004162	0.004155
0.090000	0.005328	0.005271	0.005266
0.100000	0.006572	0.006513	0.006512

Table 3.1: Effect of the ramp height on the drag force D for different inclination angles α .

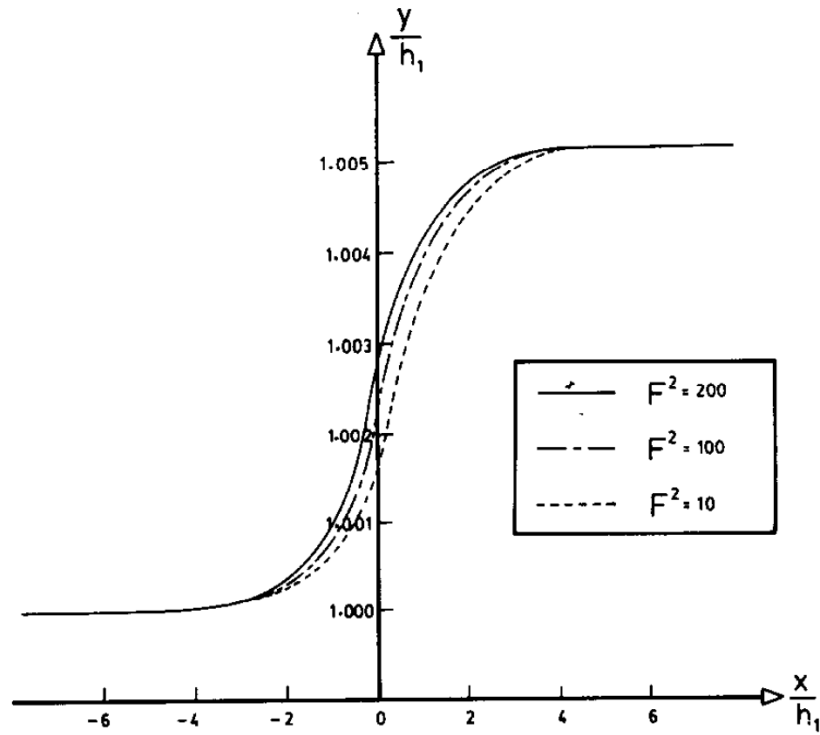


Figure 3.3: Effect of the Froude number on the free-surface profile for $\alpha = \frac{1}{6}\pi$ and $\epsilon = 0.005012$.

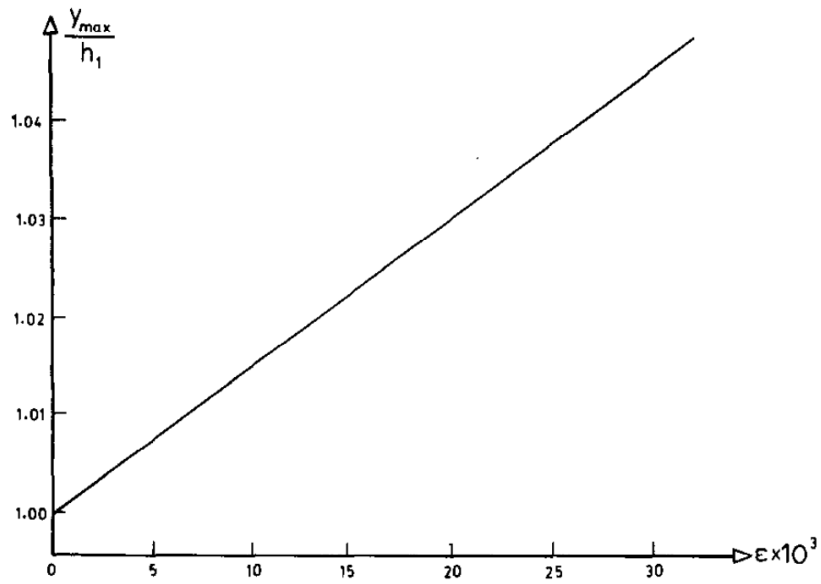


Figure 3.4: Normalized maximum free-surface height against the ramp height ϵ for $\alpha = \frac{1}{6}\pi$ and $F^2 = 10.0$.

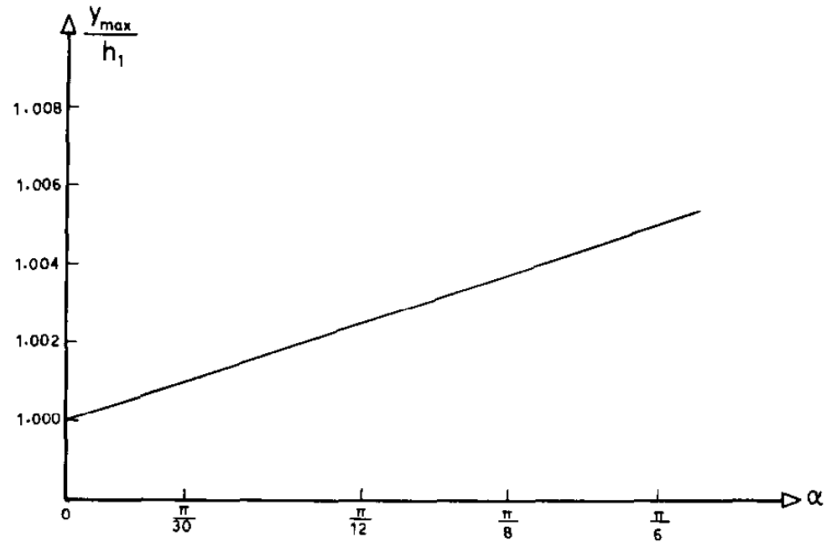


Figure 3.5: Normalized maximum free-surface height against the inclination angle α of the ramp for $\epsilon = 0.005012$ and $F^2 = 10.0$.

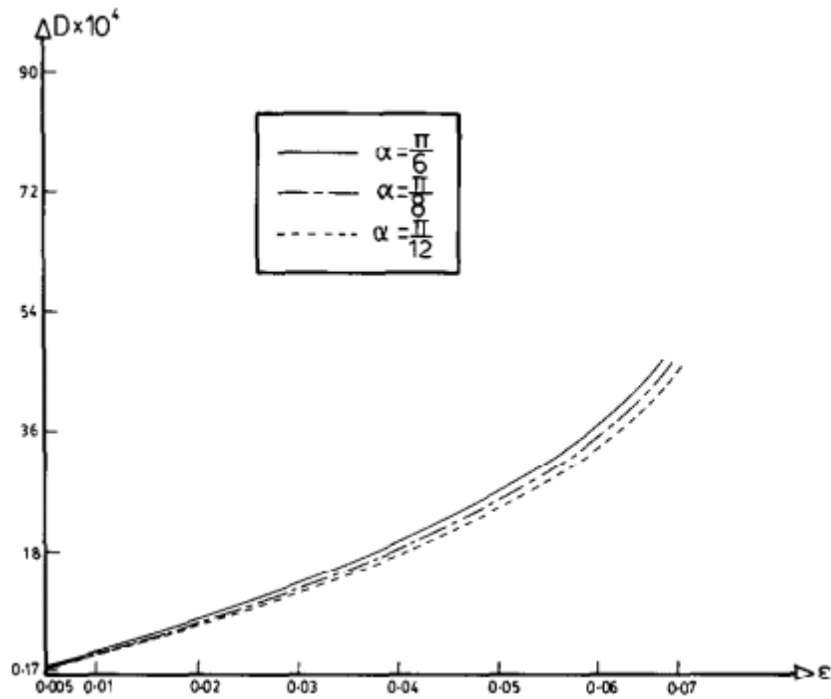


Figure 3.6: The drag force D on the ramp as a function of the ramp height ϵ for different values of the inclination angle α .

Conclusion

Free surface flow problems are very difficult and even impossible to solve explicitly, especially if the effects of gravity are included. This is due to the non-linearity of the edge condition of the Bernoulli equation on a surface of unknown shape. In this thesis, we study a two-dimensional incompressible and non-viscous fluid flow problem on a ramp using Hilbert's method and the perturbation technique.

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