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On intuitionistic Fuzzy Subgroups and their ideals

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شكر وإهداء :

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Introduction

Zadah[10] introduced the concept of fuzzy sets as an extension of the classical notion of set. Since its inception, the theory of fuzzy sets has developed in many directions and is finding applications in various fields.

In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition: an element either belongs to or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval $[0,1]$.

In abstract algebra, group theory studies the algebraic structures known as groups. The concept of a group is central to abstract algebra: other well-known algebraic structures, such as rings, fields, and vector spaces, can all be seen as groups endowed with additional operations and axioms. Groups recur throughout mathematics, and the methods of group theory have influenced many parts of algebra. Linear algebraic groups and Lie groups are two branches of group theory that have experienced advances and have become subject areas in their own right.

Atanassov [1] introduced the concept of intuitionistic fuzzy sets as an extension of fuzzy sets, where not only the degree of membership but also the degree of non-membership is considered. This development paved the way for the study of **intuitionistic fuzzy subgroups**, which are algebraic structures that combine classical group theory with intuitionistic fuzzy set theory. Furthermore, intuitionistic fuzzy ideals were introduced to extend classical ideal theory within the context of fuzzy sets and groups.

This thesis is divided into three chapters:

In the first chapter, we recall the basic concepts of crisp groups, fuzzy sets, and intuitionistic fuzzy sets, exploring their definitions and operations.

The second chapter is dedicated to the study of intuitionistic fuzzy sub-

groups, where we introduce their definitions, discuss their properties, and examine homomorphisms between these subgroups.

In the third chapter, we define the notion of **intuitionistic fuzzy ideals** and study some of their main properties, providing a deeper understanding of their role in group theory. We also explore the applications of fuzzy ideals in algebraic structures, extending the classical ideal theory to a fuzzy framework.

Chapter 1

Preliminary

In this chapter, we begin by recalling the basic concepts related to **crisp groups**, which form a fundamental part of algebraic structures. We then introduce the notion of **fuzzy sets**, as proposed by [10], and discuss their operations as a generalization of classical set theory. Finally, we present the concept of **intuitionistic fuzzy sets**, introduced by [1], which extend fuzzy sets by incorporating degrees of membership and non-membership.

1.1 Generalities on crisp groups

The definitions of group, abelian group, and subgroup are reviewed here, along with several instances.

1.1.1 Group structure

Definition 1.1 (Group [8]) *Let G be a non-empty set, and let $\cdot : G \times G \rightarrow G$ be a binary operation. The pair (G, \cdot) is called a group if it satisfies the following axioms:*

- (i) **Closure:** *For all $x, y \in G$, the element $x \cdot y \in G$.*
- (ii) **Associativity:** *For all $x, y, z \in G$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.*
- (iii) **Identity:** *There exists an element $e \in G$ such that $x \cdot e = e \cdot x = x$ for all $x \in G$.*
- (iv) **Inverses:** *For each $x \in G$, there exists $x^{-1} \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = e$.*

Definition 1.2 (Abelian group) Let (G, \cdot) be a group. Then

(G, \cdot) is called an abelian group, or commutative group, if $x \cdot y = y \cdot x$ for all $x, y \in G$.

Example 1.1 $(\mathbb{Q}, +), (\mathbb{C}^*, \times)$ are abelian groups.

Definition 1.3 (Subgroup [8]) Let (G, \cdot) be a group, and let $H \subseteq G$ be a non-empty subset.

Then H is called a subgroup of G if:

(i) $x, y \in H$ implies $x \cdot y \in H$,

(ii) $x \in H$ implies $x^{-1} \in H$.

Definition 1.4 (Ideal in a Crisp Group) Let (G, \cdot) be a group (i.e., a crisp group). A nonempty subset $I \subseteq G$ is called an ideal of G if it satisfies the following condition:

$$\forall g \in G, \forall i \in I, \quad g \cdot i \in I \quad \text{and} \quad i \cdot g \in I$$

Since G is a group (with inverses), this condition is equivalent to requiring that I is closed under conjugation, i.e.,

$$\forall g \in G, \quad gI g^{-1} \subseteq I$$

Thus, in the context of group theory, an ideal of a crisp group is simply a normal subgroup of the group.

Note: In classical group theory, the term “ideal” is not typically used for groups. However, in certain generalizations—such as fuzzy groups or algebraic structures inspired by ring theory—this term is used to emphasize the analogy with ring ideals. In this document, we adopt this terminology for conceptual consistency with later sections.

Example 1.2 1. Let $G = \mathbb{Z}$, the group of integers under addition, and let $I = \mathbb{Z}$ (i.e., the whole group).

Since addition is commutative, for any $g, i \in \mathbb{Z}$, we have:

$$g + i \in \mathbb{Z}, \quad i + g \in \mathbb{Z}$$

Thus, the first condition is satisfied.

This is a trivial example where $I = G$, yet it demonstrates the condition clearly.

2. Let $G = \mathbb{Z}$, and let $I = 3\mathbb{Z} = \{\dots, -6, -3, 0, 3, 6, \dots\}$, the set of all multiples of 3.

Since \mathbb{Z} is abelian, we have:

$$gig^{-1} = g + i - g = i \in I \quad \text{for all } g \in G, i \in I$$

So, $gIg^{-1} = I \subseteq I$, and the second condition holds.

Therefore, $I = 3\mathbb{Z}$ is a proper (nontrivial) ideal in \mathbb{Z} according to the conjugation condition.

In the following theorem, we present an equivalent definition of a subgroup [8].

Theorem 1.1 *A subset H of a group G is a subgroup of G if the following conditions are satisfied:*

- (i) H is a non-empty set.
- (ii) $x \in H$ and $y \in H$ imply $x \cdot y^{-1} \in H$.

Example 1.3 $(\mathbb{Q}, +)$ is a subgroup of $(\mathbb{R}, +)$.

Theorem 1.2 *Let $(G, *)$ be a group, and let H, H' be two subgroups of G . Then, $H \cap H'$ is also a subgroup of G .*

Proof. To prove that $H \cap H'$ is a subgroup of G , we check the subgroup conditions:

- **Identity:** Since H and H' are subgroups, they both contain the identity element e . Thus, $e \in H \cap H'$.
- **Closure:** Let $x, y \in H \cap H'$. Since $x, y \in H$ and H is closed under the operation $*$, we have $x * y \in H$. Similarly, $x * y \in H'$. Hence, $x * y \in H \cap H'$.
- **Inverses:** Let $x \in H \cap H'$. Then $x \in H$ and $x \in H'$. Since subgroups contain inverses, we get $x^{-1} \in H$ and $x^{-1} \in H'$. Thus, $x^{-1} \in H \cap H'$.

Since $H \cap H'$ satisfies the subgroup conditions, it is a subgroup of G . \square

Remark 1.1 *The union of two subgroups is not a subgroup. Indeed.*

Example 1.4 *Let $(\mathbb{Z}, +)$ be a group, and let $4\mathbb{Z}$ and $6\mathbb{Z}$ be two subgroups of \mathbb{Z} . The set $4\mathbb{Z} \cup 6\mathbb{Z}$ is not a subgroup of \mathbb{Z} , since $4, 6 \in 4\mathbb{Z} \cup 6\mathbb{Z}$ but*

$$6 - 4 = 2 \notin 4\mathbb{Z} \cup 6\mathbb{Z}.$$

1.1.2 Cosets and normal subgroup

Definition 1.5 (Cosets [2, 8]) *Let H be a subgroup of G and $g \in G$. The left coset of H with respect to g is defined by*

$$gH = \{g * h \mid h \in H\}.$$

The right coset is

$$Hg = \{h * g \mid h \in H\}.$$

and the conjugate (or middle coset) is

$$gHg^{-1} = \{g * h * g^{-1} \mid h \in H\}.$$

Definition 1.6 (Normal Subgroup [2, 8]) *A subgroup $N \leq G$ is said to be normal in G if*

$$gng^{-1} \in N \quad \text{for all } g \in G, n \in N,$$

or equivalently, if $gN = Ng$ for all $g \in G$. We write $N \triangleleft G$.

Remark 1.2 *Every subgroup of an abelian group is normal.*

Example 1.5 *Let's consider the permutation group S_4 of degree 4, and define the following permutations:*

$$\varepsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

Now, let us consider the subset

$$H = \{\varepsilon, \sigma_1\}.$$

We compute the left and right cosets of H in S_4 :

Right cosets

$$H = \{\varepsilon, \sigma_1\}$$

$$H\sigma_2 = \{\sigma_2, \sigma_1\sigma_2\} = \left\{ \sigma_2, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \right\}$$

$$H\sigma_3 = \{\sigma_3, \sigma_1\sigma_3\} = \left\{ \sigma_3, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \right\}$$

Left cosets

$$H = \{\varepsilon, \sigma_1\}$$

$$\sigma_2 H = \{\sigma_2, \sigma_2\sigma_1\} = \left\{ \sigma_2, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \right\}$$

$$\sigma_3 H = \{\sigma_3, \sigma_3\sigma_1\} = \left\{ \sigma_3, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \right\}$$

We observe that for each element $g \in S_4$, the left and right cosets satisfy:

$$gH = Hg,$$

which means the left and right cosets are equal. Therefore, the subgroup H is a **normal subgroup** of S_4 .

Definition 1.7 (Conjugate) Let $(G, *)$ be a group. Let A and B be two fuzzy subsets of G .

We say that A is conjugate to B if there exists an element $g \in G$ such that

$$B(x) = A(g^{-1}xg)$$

for all $x \in G$.

Example 1.6 Let $(G, *)$ be the group $\mathbb{Z}_3 = \{0, 1, 2\}$ under addition modulo 3.

Define the fuzzy subset A of G by

$$A(0) = 1, \quad A(1) = 0.5, \quad A(2) = 0.2.$$

Take $g = 1 \in G$. We define the fuzzy subset B by

$$B(x) = A(g^{-1} + x + g) \pmod{3}$$

for all $x \in G$.

Since in \mathbb{Z}_3 , $g^{-1} = 2$, we have:

$$B(0) = A(2 + 0 + 1) \pmod{3} = A(0) = 1,$$

$$B(1) = A(2 + 1 + 1) \pmod{3} = A(1) = 0.5,$$

$$B(2) = A(2 + 2 + 1) \pmod{3} = A(2) = 0.2.$$

Thus, A and B are conjugate via $g = 1$.

1.1.3 Morphism of groups

A **morphism** in group theory refers to a structure-preserving map between two groups. In the context of groups, this is called a **group homomorphism**. That is, morphism and homomorphism are used interchangeably in group theory. A map $f : G \rightarrow H$ is a homomorphism if for all $x, y \in G$, we have $f(xy) = f(x)f(y)$. A bijective group homomorphism is called a **group isomorphism**. If such a map exists between two groups G and H , then G and H are said to be **isomorphic**, written as $G \cong H$. The set of all group homomorphisms from G to H is denoted by $\text{Hom}(G, H)$; see [8].

Definition 1.8 Consider two groups, $(G, *)$ and (G', \circ) . A group morphism is a function $f : G \rightarrow G'$ if for every $x, y \in G$:

$$f(x * y) = f(x) \circ f(y).$$

An isomorphism of a group $f : G \rightarrow G'$ is a bijective group morphism; we write $G \cong G'$ and state that G and G' are isomorphism groups.

Example 1.7 Let $(\mathbb{C}, +)$ and (\mathbb{C}^*, \times) be two groups. Define a map $\psi : \mathbb{C} \rightarrow \mathbb{C}^*$ by $\psi(z) = e^z$.

We have:

$$\psi(0) = e^0 = 1.$$

$$\psi(z + w) = e^{z+w} = e^z \cdot e^w = \psi(z) \times \psi(w).$$

Thus, ψ is a group homomorphism.

Definition 1.9 (Monomorphism) A *monomorphism* is an injective group homomorphism, i.e., a function $f : G \rightarrow G'$ such that f is one-to-one and:

$$f(xy) = f(x)f(y), \quad \forall x, y \in G.$$

Example 1.8 The function $f : \mathbb{Z} \rightarrow \mathbb{Q}$ defined by:

$$f(n) = n$$

satisfies the following properties:

- **Preserves operation:** $f(a + b) = f(a) + f(b)$
- **Injective:** If $f(a) = f(b)$, then $a = b$

Thus, f is a monomorphism.

Definition 1.10 (Epimorphism) An *epimorphism* is a surjective group homomorphism, i.e., a function $f : G \longrightarrow G'$ such that every element in G' has a preimage in G .

Example 1.9 An example of an epimorphism is the map $f : \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ given by $f(n) = n \bmod 2$, which is surjective because every element in $\mathbb{Z}/2\mathbb{Z}$ has a preimage in \mathbb{Z} .

Definition 1.11 (Endomorphism) An *endomorphism* is a group homomorphism from a group to itself, i.e., $h : G \longrightarrow G$.

Example 1.10 The map $h : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $h(n) = 2n$ is an endomorphism because:

$$h(a + b) = 2(a + b) = 2a + 2b = h(a) + h(b).$$

Definition 1.12 (Automorphism) An *automorphism* is a bijective endomorphism $f : G \longrightarrow G$. The set of all automorphisms of a group G , with composition as the operation, forms a group called the *automorphism group* of G , denoted $\text{Aut}(G)$.

Example 1.11 The automorphism group of $(\mathbb{Z}, +)$ contains two elements: the identity map and the map $n \mapsto -n$. Thus, $\text{Aut}(\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z}, +)$.

Definition 1.13 (Kernel) If $f : G \longrightarrow G'$ is a group homomorphism, then the **kernel** of f , written $\ker(f)$, is the set of all elements in G whose image is the identity element $e_{G'}$ in G' . That is,

$$\ker(f) = \{x \in G \mid f(x) = e_{G'}\} = f^{-1}(e_{G'}).$$

Definition 1.14 (Image) The image of $f : G \longrightarrow G'$ is composed of the images of the elements under f , that is:

$$\text{Im}\{f\} = f(G) = \{f(x) \in G', x \in G\}.$$

Proposition 1.1 *Let (G_1, \cdot) and $(G_2, *)$ be two groups of neutral elements e_1, e_2 , and let f be a homomorphism from G to G' .*

(i) *If H is a subgroup of G_1 then $f(H)$ is a subgroup of G_2 .*

(ii) *If H is a subgroup of G_2 then $f^{-1}(H)$ is a subgroup of G_1 .*

Example 1.12 *Let g be an application given by:*

$$g : \mathbb{R} \longrightarrow \mathbb{R}^*, \quad x \longmapsto g(x) = e^x$$

The function g is a group homomorphism from $(\mathbb{R}, +)$ to (\mathbb{R}^, \times) , because*

$$\forall x, y \in \mathbb{R}, \quad g(x + y) = e^{x+y} = e^x \cdot e^y = g(x) \cdot g(y).$$

The kernel of g is given by:

$$\ker(g) = \{x \in \mathbb{R} \mid g(x) = 1\} = \{x \in \mathbb{R} \mid e^x = 1\} = \{0\}.$$

The image of g is:

$$\text{Im}(g) = \{g(x) \mid x \in \mathbb{R}\}.$$

Since for any $y > 0$, there exists $x = \ln y$ such that $g(x) = e^x = y$, we conclude that:

$$\text{Im}(g) = \mathbb{R}_+^*.$$

Proposition 1.2 [8] *Let f be a homomorphism from (G_1, \cdot) to $(G_2, *)$. Then:*

1. $\ker(f)$ is a subgroup of G_1 .
2. $\text{Im}(f)$ is a subgroup of G_2 .
3. f is injective if and only if $\ker(f) = \{e_1\}$.
4. f is surjective if and only if $\text{Im}(f) = G_2$.

Proposition 1.3 *The kernel of a group homomorphism is a normal subgroup of G and always contains the identity element of G .*

1.2 Generalities on Fuzzy sets

L. Zadeh first proposed the idea of fuzzy sets, an extension of classical sets, in [10]. Each element in a fuzzy set may have a degree of membership ranging from 0 to 1, where 1 denotes complete membership and 0 denotes no membership.

In this section, we will go over some of the basic ideas of fuzzy set theory. We'll start by defining fuzzy sets and going over some of the fundamental operations that go along with them, such complement, intersection, union, inclusion, etc. The features of fuzzy sets, such as support, kernel, α -cuts, strong α -cuts, etc.

1.2.1 Fuzzy sets

A crisp set is an unordered collection of distinct elements. A set can be represented in the following ways:

1. Enumerating its elements: The set is written as $A = \{a_1, a_2, \dots, a_n\}$, where the elements are listed explicitly.
2. Specifying conditions on its elements: The set is defined by a property or condition $P(x)$, and written as $A = \{x \mid P(x)\}$, where x satisfies the property $P(x)$.
3. Using a membership function: For a universal set X , a crisp set A is a subset of X . The membership function $\chi_A(x)$ is defined for any element $x \in X$ as follows:

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A; \\ 1 & \text{if } x \in A. \end{cases}$$

Definition 1.15 [10] *Let X be a non-empty set. A fuzzy subset $A = \{\langle x, \mu_A(x) \rangle \mid x \in X\}$ of X is characterized by a membership function $\mu_A : X \rightarrow [0, 1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for each $x \in X$.*

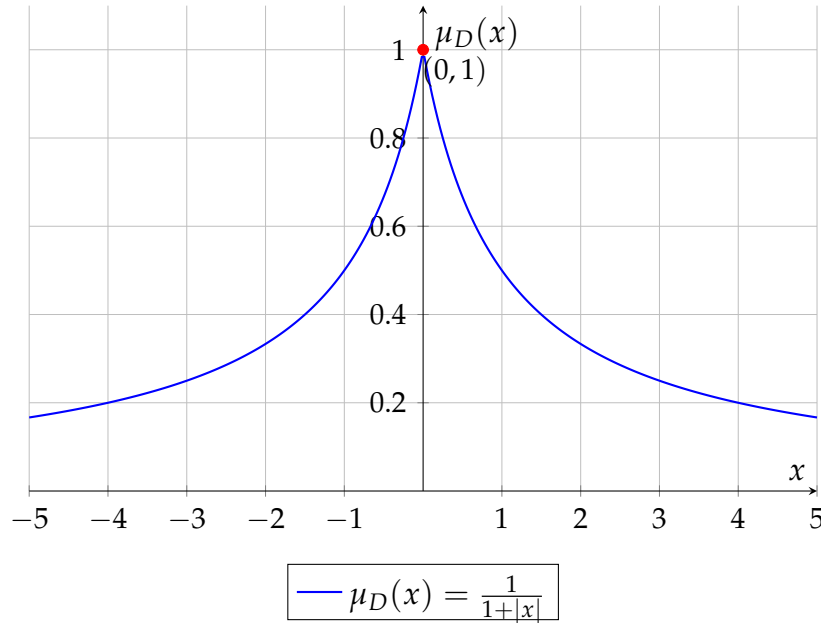
Example 1.13 (1) *Let $X = \{a, b, c\}$ be a universal set.*

$A = \{(a, 0.5), (b, 0.3), (c, 0.9)\}$ is a fuzzy subset of X .

(2) Let $X = \mathbb{R}$, and let D be a fuzzy subset of X defined by:

$$\mu_D(x) = \frac{1}{1 + |x|}.$$

The given graph represents the membership function of D .



1.2.2 Operations on fuzzy sets

The definitions of several operations on fuzzy sets, including equality, inclusion, intersection, union, and complement of a fuzzy set, will be provided in this section. We will conclude with an example.

Definition 1.16 (Complement) The complement of a fuzzy set A , denoted by $C(A)$, is defined as:

$$\mu_{C(A)}(x) = 1 - \mu_A(x), \text{ for all } x \in X.$$

Example 1.14 Let $X = \{a, b, c\}$, $A = \{(a, 0.3), (b, 0.7), (c, 0.9)\}$, $C(A) = \{(a, 0.7), (b, 0.3), (c, 0.1)\}$, and $C(C(A)) = \{(a, 0.3), (b, 0.7), (c, 0.9)\} = A$.

Definition 1.17 (Equality) Let A and B be two fuzzy subsets on X , and let x be a non-empty set. We say that $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$.

Example 1.15 Let $X = \{a, b, c\}$, and let

$$A = \{(a, 0.5), (b, 0.2), (c, 0.3)\}, B = \{(a, 0.5), (b, 0.2), (c, 0.3)\}.$$

Then $A = B$ for all $a, b, c \in X$.

Definition 1.18 (Inclusion) Let A and B be two fuzzy subsets on a non-empty set X . We say that $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$.

Example 1.16 Let $X = \{x, y, z\}$, $A = \{(x, 0.1), (y, 0.25), (z, 0.05)\}$, $B = \{(x, 0.2), (y, 0.3), (z, 0.1)\}$, then $A \subseteq B$ for all $x, y, z \in X$.

Definition 1.19 (Union) Let A and B be two fuzzy subsets on a non-empty set X . For any $x \in X$, the union of A and B is defined by:

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

Definition 1.20 (Intersection) Let A and B be two fuzzy subsets on X , and let x be a non-empty set. The intersection $A \cap B$ defined by

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \text{ for all } x \in X.$$

Example 1.17 Let A and B be two fuzzy subsets on $[0, 3]$, such that:

$$\mu_A(x) = \frac{1}{x+1}, \quad \mu_B(x) = \frac{x}{3}$$

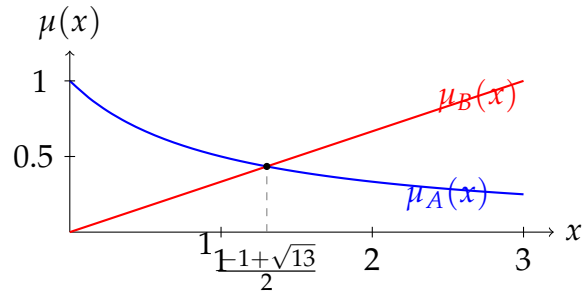
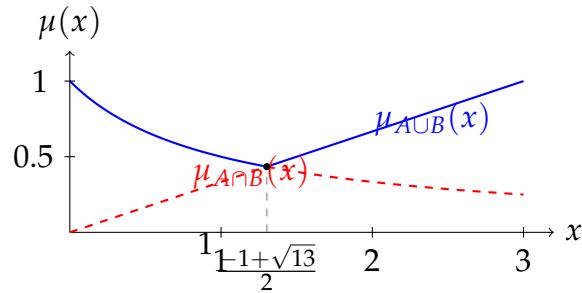
Then, since the intersection point of $\mu_A(x)$ and $\mu_B(x)$ is given by:

$$\frac{1}{x+1} = \frac{x}{3} \Rightarrow x^2 + x - 3 = 0 \Rightarrow x = \frac{-1 + \sqrt{13}}{2}$$

we get:

$$\mu_{A \cup B}(x) = \begin{cases} \frac{1}{x+1}, & \text{if } x \in \left[0, \frac{-1 + \sqrt{13}}{2}\right] \\ \frac{x}{3}, & \text{if } x \in \left(\frac{-1 + \sqrt{13}}{2}, 3\right] \end{cases}$$

$$\mu_{A \cap B}(x) = \begin{cases} \frac{x}{3}, & \text{if } x \in \left[0, \frac{-1 + \sqrt{13}}{2}\right] \\ \frac{1}{x+1}, & \text{if } x \in \left(\frac{-1 + \sqrt{13}}{2}, 3\right] \end{cases}$$

Figure 1.1: Graph of $\mu_A(x)$ and $\mu_B(x)$ Figure 1.2: Graph of $\mu_{A \cup B}(x)$ and $\mu_{A \cap B}(x)$

1.2.3 Characteristics Sets of fuzzy set

We shall define terms like support, kernel, height, and cardinality of a fuzzy set, as well as give an example, in the sections that follow

Definition 1.21 (α -cuts) Let A be a fuzzy subset on X , and let $\alpha \in]0, 1]$. The α -cut of A , denoted A_α , means all elements of X that belong to A to a degree of at least or equal to α . That is, A_α is a classical set defined by:

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

Example 1.18 Let $X = \{w, x, y, z\}$, and $A = \{(w, 0.5), (x, 0.2), (y, 0.8), (z, 0)\}$,

Thus, $A_{0.3} = \{w, y\}$.

Definition 1.22 (Strong α -cuts) Let A be a fuzzy subset on X , and let $\alpha \in [0, 1[$. The strong α -cut of A , denoted A_α^+ , means all elements of X that belong to A to a degree of strictly greater than α . That is, A_α^+ is a classical set defined by:

$$A_\alpha^+ = \{x \in X \mid \mu_A(x) > \alpha\}.$$

Example 1.19 Let $X = \{a, b, c, d\}$, and let $A = \{(a, 0.1), (b, 0.3), (c, 0.4), (d, 0)\}$; then $A_{0.2} = \{b, c\}$.

Definition 1.23 (line of degree α) Let A be a fuzzy subset on X , and let $\alpha \in [0, 1]$. The line of degree α of A , denoted $L_\alpha(A)$, means all elements of X that belong to A to a degree equal α . That is, $L_\alpha(A)$ is a classical subset defined by :

$$L_\alpha(A) = \{x \in X \mid \mu_A(x) = \alpha\}.$$

Example 1.20 Let $X = \{a, b, c, d, e\}$ and let A be a fuzzy subset of X defined as $A = \{(a, 0.2), (b, 0.5), (c, 0.7), (d, 0.5), (e, 0.9)\}$. For $\alpha = 0.5$, the line of degree α is $L_{0.5}(A) = \{b, d\}$.

Definition 1.24 (Support) Let A be a fuzzy subset on X . The support of A , denoted $Supp(A)$, means all elements of X that have a non-zero membership degree in fuzzy subset A . That is, $Supp(A)$ is a classical subset defined by :

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

Definition 1.25 (Kernel) Let A be a fuzzy set on X . The kernel of A , denoted $ker(A)$, means all elements of X that have a membership value equal to 1. That is, $ker(A)$ is a classical set defined by

$$ker(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

Example 1.21 Let $X = \{x, y, z, w\}$, and let $A = \{(x, 0), (y, 0.4), (z, 1), (w, 1)\}$.

$$supp(A) = \{y, z, w\} \quad ; \quad ker(A) = \{z, w\}$$

Definition 1.26 (Height) The height of a fuzzy subset A on X , denoted $H(A)$, is the largest membership degree of elements of A , i.e,

$$H(A) = \max_{x \in X} \{\mu_A(x)\}.$$

Example 1.22 Let $X = \{x, y, z, w\}$ and let the fuzzy subset:

$$A = \{(x, 0.85), (y, 0.62), (z, 0), (w, 0.27)\}.$$

Then, $H(A) = 0.85$.

Definition 1.27 (Cardinality) The cardinality of a finite fuzzy subset A on X , denoted by $|A|$, is defined as:

$$|A| = \sum_{x \in X} \mu_A(x).$$

For an infinite fuzzy set A , the cardinality is defined by:

$$|A| = \int_{x \in X} \mu_A(x) dx$$

Example 1.23 Finite case: Let $X = \{a, b, c, d, e\}$, and consider the fuzzy set:

$$A = \{(a, 0.2), (b, 0.5), (c, 0.1), (d, 0.3), (e, 0.4)\}.$$

Then, the cardinality of A is:

$$|A| = 0.2 + 0.5 + 0.1 + 0.3 + 0.4 = 1.5.$$

Example 1.24 Infinite case: Let $X = [1, 5]$, and define the fuzzy subset A by :

$$\mu_A(x) = \frac{x}{x+1}.$$

Then, the cardinality of A is computed as: $|A| = \int_1^5 \frac{x}{x+1} dx$.

Evaluating the integral:

$$|A| = \int_1^5 1 - \frac{1}{x+1} dx = [x - \ln(x+1)]_1^5 = (5 - \ln 6) - (1 - \ln 2) = 4 + \ln \frac{2}{6} = 4 - \ln 3.$$

Proposition 1.4 Let A, B be two fuzzy subsets on X , and let $\alpha, \beta \in]0, 1]$. Then

1. $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$,
2. $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$,

3. if $\alpha \leq \beta$, then $A_\beta \subseteq A_\alpha$,

4. $A_0 = X$,

5. $A_1 = \ker(A)$.

Proposition 1.5 *The kernel and the support of any fuzzy subset A satisfy the following properties:*

1. $\text{Supp}(C(A)) = X \setminus \ker(A)$,

2. $\ker(C(A)) = X \setminus \text{Supp}(A)$.

Example 1.25 *Let $X = \{x, y, z\}$ and $B = \{(x, 0.6), (y, 1), (z, 0.4)\}$ be a fuzzy subset on X .*

Then:

$$\begin{aligned} B_{0.4} &= \{x, y, z\}, & B_{0.4}^+ &= \{x, y\}, & L_{0.4}(B) &= \{z\}, \\ \text{Supp}(B) &= \{x, y, z\}, & \ker(B) &= \{y\}, & H(B) &= 1, & |B| &= 2. \end{aligned}$$

1.2.4 Cartesian product on fuzzy set

Definition 1.28 *Let $\mu_{A_1}, \mu_{A_2}, \dots, \mu_{A_n}$ be the membership functions of A_1, A_2, \dots, A_n . Then, the membership degree of $(x_1, x_2, \dots, x_n) \in X_1 \times X_2 \times \dots \times X_n$ on the fuzzy set $A_1 \times A_2 \times \dots \times A_n$ is:*

$$\mu_{A_1 \times A_2 \times \dots \times A_n}(x_1, x_2, \dots, x_n) = \min \{ \mu_{A_1}(x_1), \mu_{A_2}(x_2), \dots, \mu_{A_n}(x_n) \}.$$

Example 1.26 *Let $X_1 = \{p, q, r\}$, $X_2 = \{\delta, \epsilon\}$ and let A_1, A_2 be two fuzzy subsets respectively defined on X_1 and X_2 given by:*

$A_1 = \{(p, 0.3), (q, 0.5), (r, 0.9)\}; A_2 = \{(\delta, 0.4), (\epsilon, 0.7)\}$. *So, we get:*

$$A_1 \times A_2 = \{((p, \delta), 0.3), ((p, \epsilon), 0.3), ((q, \delta), 0.4), ((q, \epsilon), 0.5), ((r, \delta), 0.4), ((r, \epsilon), 0.7)\}.$$

1.3 Generalities on intuitionistic fuzzy sets

1.3.1 Intuitionistic Fuzzy Sets

Definition 1.29 [1] *Let X be a non-empty set. An intuitionistic fuzzy set (IFS, for short) A on X is an object of the form:*

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}.$$

where the functions:

$$\mu_A : X \rightarrow [0, 1] \quad \text{and} \quad \nu_A : X \rightarrow [0, 1]$$

define the degree of membership and the degree of non-membership of the element $x \in X$, respectively, and for every $x \in X$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Example 1.27 (1) *Let $X = \{a, b, c\}$ be a universal set. An intuitionistic fuzzy subset A of X is defined as:*

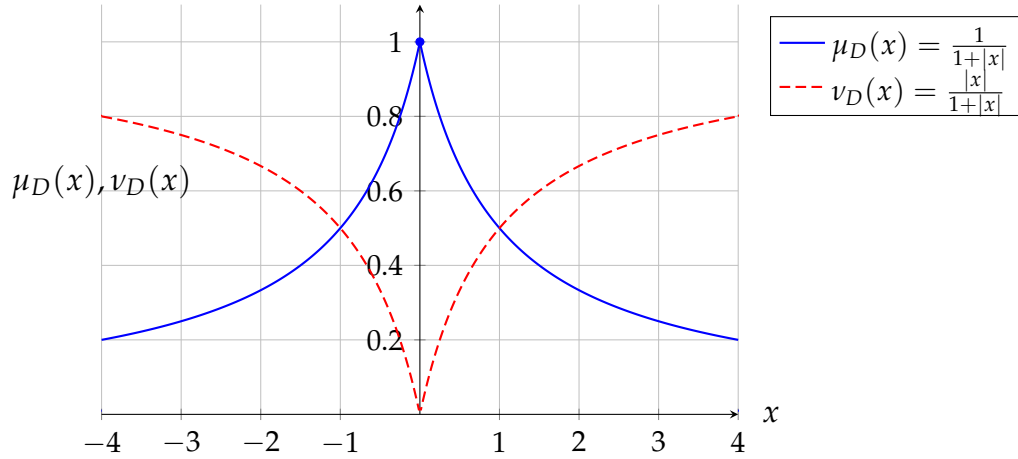
$$A = \{ (a, 0.5, 0.3), (b, 0.3, 0.4), (c, 0.9, 0.05) \}$$

where each element $(x, \mu_A(x), \nu_A(x))$ represents the **membership degree** $\mu_A(x)$ and **non-membership degree** $\nu_A(x)$.

(2) *Let $X = \mathbb{R}$, and let D be an intuitionistic fuzzy subset of X defined by:*

$$\mu_D(x) = \frac{1}{1 + |x|}, \quad \nu_D(x) = \frac{|x|}{1 + |x|}$$

The following graph represents both the membership function $\mu_D(x)$ and the non-membership function $\nu_D(x)$:



Definition 1.30 [4] Intuitionistic fuzzy sets can be seen as L^* -fuzzy sets by considering the lattice

$$L^* = \{(x, y) \in [0, 1]^2 \mid x + y \leq 1\}$$

where the inequality relation generating the lattice structure is defined by

$$(x, y) \leq_{L^*} (z, t) \iff x \leq z \text{ and } y \geq t.$$

Moreover, in this case, L^* is a complete distributive lattice.

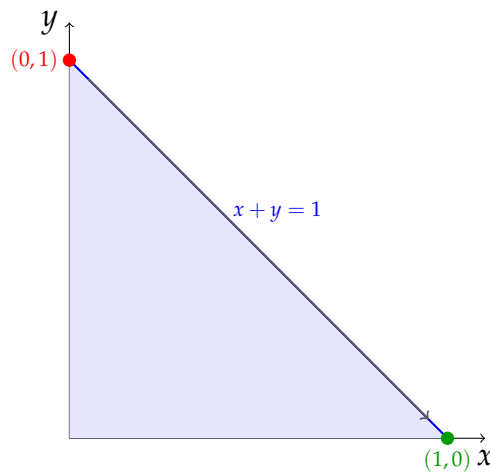


Figure 1.3: Graphical representation of the poset (L^*, \leq_{L^*}) .

Properties of the Lattice L^* :

1. The relation \leq_{L^*} is a *partial order* on L^* , because it satisfies:

- **Reflexivity:**

Let $(x, y) \in L^*$. Then clearly $x \leq x$ and $y \geq y$.

Therefore, by the definition of \leq_{L^*} , we have:

$$(x, y) \leq_{L^*} (x, y).$$

- **Antisymmetry:**

Assume $(x, y), (z, t) \in L^*$ such that:

$$(x, y) \leq_{L^*} (z, t) \quad \text{and} \quad (z, t) \leq_{L^*} (x, y).$$

Then by definition:

$$x \leq z \text{ and } y \geq t, \quad \text{and} \quad z \leq x \text{ and } t \geq y.$$

Combining both:

$$x = z \quad \text{and} \quad y = t \Rightarrow (x, y) = (z, t).$$

- **Transitivity:**

Assume $(x, y), (z, t), (u, v) \in L^*$ such that:

$$(x, y) \leq_{L^*} (z, t) \quad \text{and} \quad (z, t) \leq_{L^*} (u, v).$$

Then:

$$x \leq z \text{ and } y \geq t, \quad \text{and} \quad z \leq u \text{ and } t \geq v.$$

So:

$$x \leq u \quad \text{and} \quad y \geq v \Rightarrow (x, y) \leq_{L^*} (u, v).$$

2. The relation \leq_{L^*} is **not a total order**, since not every pair of elements in L^* is comparable. For example:

$$(0.2, 0.7), (0.7, 0.2) \in L^*, \quad \text{they are comparable.}$$

3. The **least element** of L^* is $\boxed{(0, 1)}$, since for any $(x, y) \in L^*$, we have $0 \leq x$ and $1 \geq y$.

4. The **greatest element** of L^* is $\boxed{(1, 0)}$, because for all $(x, y) \in L^*$, $x \leq 1$ and $y \geq 0$.

Let $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in L^*$. Then:

- $\alpha \leq_{L^*} \beta \iff \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \geq \beta_2.$
- If this condition holds, then α and β are **comparable**.
- Otherwise, α and β are **incomparable**. This occurs when neither condition of the partial order is satisfied:

$$\alpha_1 \leq \beta_1 \text{ and } \alpha_2 \geq \beta_2 \quad \text{nor} \quad \beta_1 \leq \alpha_1 \text{ and } \beta_2 \geq \alpha_2.$$

In other words, if:

$$\alpha_1 < \beta_1 \text{ and } \alpha_2 < \beta_2, \quad \text{or} \quad \alpha_1 > \beta_1 \text{ and } \alpha_2 > \beta_2,$$

then α and β are incomparable in L^* .

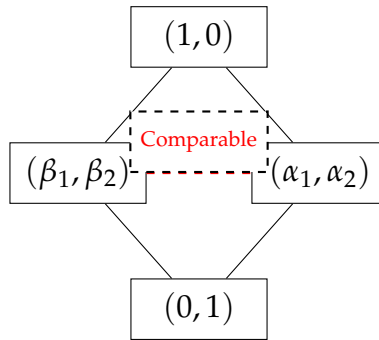


Figure 1.4: Hasse diagram of L^* showing elements $(0, 1)$, (α_1, α_2) , (β_1, β_2) , and $(1, 0)$. The dashed red line indicates that $\alpha \leq_{L^*} \beta$.

1.3.2 Operations On Intuitionistic Fuzzy Sets

For two intuitionistic fuzzy sets A and B on a set X , several operations are defined in the following way:[1]

Definition 1.31 (Inclusion)

$$A \subseteq B \text{ if } \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x), \text{ for any } x \in X.$$

Definition 1.32 (Union)

$$A \cup B = \{ \langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) \rangle \mid x \in X \}.$$

Definition 1.33 (Intersection)

$$A \cap B = \{ \langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) \rangle \mid x \in X \}.$$

Definition 1.34 (Equality)

$$A = B \text{ if } \mu_A(x) = \mu_B(x) \text{ and } \nu_A(x) = \nu_B(x), \text{ for any } x \in X.$$

Definition 1.35 (Complement)

$$A^c = \{ \langle x, \nu_A(x), \mu_A(x) \rangle \mid x \in X \}.$$

Definition 1.36 (Empty set)

$$A = \emptyset \text{ if } \mu_A(x) = 0 \text{ and } \nu_A(x) = 1.$$

Example 1.28 Consider a universe $X = \{x_1, x_2, x_3\}$ and two intuitionistic fuzzy sets A and B :

$$A = \{ \langle x_1, 0.5, 0.3 \rangle, \langle x_2, 0.7, 0.2 \rangle, \langle x_3, 0.4, 0.5 \rangle \}$$

$$B = \{ \langle x_1, 0.6, 0.2 \rangle, \langle x_2, 0.5, 0.4 \rangle, \langle x_3, 0.3, 0.6 \rangle \}$$

Then,

$$A \cap B = \{ \langle x_1, 0.5, 0.3 \rangle, \langle x_2, 0.5, 0.4 \rangle, \langle x_3, 0.3, 0.6 \rangle \}$$

$$A \cup B = \{ \langle x_1, 0.6, 0.2 \rangle, \langle x_2, 0.7, 0.2 \rangle, \langle x_3, 0.4, 0.5 \rangle \}$$

$$A^c = \{ \langle x_1, 0.3, 0.5 \rangle, \langle x_2, 0.2, 0.7 \rangle, \langle x_3, 0.5, 0.4 \rangle \}$$

$$B^c = \{ \langle x_1, 0.2, 0.6 \rangle, \langle x_2, 0.4, 0.5 \rangle, \langle x_3, 0.6, 0.3 \rangle \}$$

1.3.3 Necessity and possibility operators

Definition 1.37 (Necessity) [1] Let A be an intuitionistic fuzzy set on X , the necessity of A denoted by $\Box A$ is defined by:

$$\Box A(x) = \{ \langle x, \mu_A(x), \mu_A^c(x) \rangle \mid \mu_A^c(x) = 1 - \mu_A(x) \}$$

Example 1.29 Let $X = \{1, 2, 3\}$, and the intuitionistic fuzzy set A is given by:

$$A = \{(1, 0.65, 0.4), (2, 0.25, 0.50), (3, 0.35, 0.20)\}.$$

Then, the necessity of A , denoted by $\Box A(x)$, is:

$$\Box A(x) = \{(1, 0.65, 0.35), (2, 0.25, 0.75), (3, 0.35, 0.65)\}.$$

Definition 1.38 (Possibility) [1] Let A be an intuitionistic fuzzy set on X , the possibility of A denoted by $\Diamond A$ is defined by:

$$\Diamond A(x) = \{(x, v_A^c(x), v_A(x)) \mid v_A^c(x) = 1 - v_A(x)\}$$

Example 1.30 Let $X = \{a, b, c\}$, and let A be an intuitionistic fuzzy set given by:

$$A = \{(a, 0.6, 0.4), (b, 0.3, 0.85), (c, 0.9, 0.2)\}.$$

Then, the possibility of A , denoted by $\Diamond A(x)$, is:

$$\Diamond A(x) = \{(a, 0.6, 0.4), (b, 0.15, 0.85), (c, 0.8, 0.2)\}.$$

1.3.4 Characteristics Sets Of an Intuitionistic Fuzzy Set

Definition 1.39 (Support) [9] Let A be an intuitionistic fuzzy set on the universe X . The support of A is the crisp subset of X given by:

$$\text{Supp}(A) = \{x \in X \mid \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } v_A(x) < 1)\}.$$

Definition 1.40 (Kernel) [9] Let A be an intuitionistic fuzzy set on the universe X . The kernel of A is the crisp subset of X given by:

$$\text{Ker}(A) = \{x \in X \mid \mu_A(x) = 1 \text{ and } v_A(x) = 0\}.$$

Example 1.31 Let $X = \{x, y, z, w\}$, and let $A = \{(x, 0, 0.8), (y, 0.4, 0.4), (z, 1, 0), (w, 0, 1)\}$.

$$\text{supp}(A) = \{x, y, z\} \quad ; \quad \text{ker}(A) = \{z\}$$

Definition 1.41 ((α, β)-cuts) [1] Let A be an intuitionistic fuzzy set on a set X , and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. A set of (α, β)-level generated by an IFS A is defined as:

$$A_{\alpha, \beta} = \{x \in X \mid A(x) \geq_{L^*} (\alpha, \beta)\}$$

Example 1.32 Let $X = \{a, b, c\}$ and define an intuitionistic fuzzy set A on X by:

$$A(a) = (0.8, 0.1), \quad A(b) = (0.5, 0.4), \quad A(c) = (0.3, 0.6).$$

Let $\alpha = 0.6$ and $\beta = 0.3$. Then the (α, β)-cut of A is:

$$A_{\alpha, \beta} = \{x \in X \mid A(x) \geq_{L^*} (\alpha, \beta)\}.$$

We have:

- $A(a) = (0.8, 0.1)$ satisfies $\mu_A(a) \geq \alpha$ and $\nu_A(a) \leq \beta$, thus $A(a) \geq_{L^*} (\alpha, \beta)$, so $a \in A_{\alpha, \beta}$.
- $A(b) = (0.5, 0.4)$ does not satisfy $\mu_A(b) \geq \alpha$, thus $b \notin A_{\alpha, \beta}$.
- $A(c) = (0.3, 0.6)$ does not satisfy $\mu_A(c) \geq \alpha$, thus $c \notin A_{\alpha, \beta}$.

Therefore,

$$A_{\alpha, \beta} = \{a\}.$$

1.3.5 Cartesian Product on Intuitionistic Fuzzy Sets

Definition 1.42 The Cartesian product applied to n intuitionistic fuzzy sets can be defined as follows:

Let $\mu_{A_1}, \mu_{A_2}, \dots, \mu_{A_n}$ be membership functions of A_1, A_2, \dots, A_n . Then, the membership degree of $(x_1, x_2, \dots, x_n) \in X_1 \times X_2 \times \dots \times X_n$ on the intuitionistic fuzzy set $A_1 \times A_2 \times \dots \times A_n$ is:

$$\mu_{A_1 \times A_2 \times \dots \times A_n}(x_1, x_2, \dots, x_n) = \min \{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \dots, \mu_{A_n}(x_n)\}$$

and the non-membership degree is:

$$\nu_{A_1 \times A_2 \times \dots \times A_n}(x_1, x_2, \dots, x_n) = \max \{\nu_{A_1}(x_1), \nu_{A_2}(x_2), \dots, \nu_{A_n}(x_n)\}$$

Example 1.33 Let $X = \{x_1, x_2\}$, $Y = \{a, b\}$, and let A_1, A_2 be two intuitionistic fuzzy subsets respectively defined on X and Y given by:

$$A_1 = \{(x_1, 0.31, 0.2), (x_2, 0.54, 0.1)\}, \quad A_2 = \{(a, 0.02, 0.86), (b, 0.7, 0.53)\}.$$

So, we get:

$$A_1 \times A_2 = \{((x_1, a), 0.02, 0.86), ((x_1, b), 0.31, 0.53), ((x_2, a), 0.02, 0.86), ((x_2, b), 0.54, 0.53)\}.$$

Chapter 2

Intuitionistic Fuzzy Subgroups

As an extension of partial algebraic structures, this chapter explores **intuitionistic fuzzy subgroups** by introducing **definitions** and **examples** related to them, as well as examining their **characterizations** and **operations**. We will also study the concept of **homomorphism** between groups. Finally, we extend **Lagrange's theorem** in the context of **intuitionistic fuzzy logic**.

2.1 Definitions and properties

Definition 2.1 [3] Let $(G, *)$ be a crisp group, and let A be an intuitionistic fuzzy subset of G . Then A is said to be an intuitionistic fuzzy subgroup of G if:

- (i) $A(a * b) \geq_{L^*} A(a) \wedge A(b)$, for all $a, b \in G$.
- (ii) $A(a^{-1}) \geq_{L^*} A(a)$.

Example 2.1 Consider the crisp group $G = (\mathbb{Z}, +)$, and let A be an intuitionistic fuzzy subset defined as:

$$A(x) = \begin{cases} (0.35, 0.25) & \text{if } x \in 2\mathbb{Z}, \\ (0.2, 0.3) & \text{if } x \in 2\mathbb{Z} + 1. \end{cases}$$

x	y	$-x$	$A(x)$	$A(y)$	$A(-x)$	$x + y$	$A(x + y)$	$A(x) \wedge A(y)$	$A(x + y) \geq_{L^*} A(x) \wedge A(y)$	$A(-x) \geq_{L^*} A(x)$
$2\mathbb{Z}$	$2\mathbb{Z}$	$2\mathbb{Z}$	$(0.35, 0.25)$	$(0.35, 0.25)$	$(0.35, 0.25)$	$2\mathbb{Z}$	$(0.35, 0.25)$	$(0.35, 0.25)$	T	T
$2\mathbb{Z}$	$2\mathbb{Z} + 1$	$2\mathbb{Z}$	$(0.35, 0.25)$	$(0.2, 0.3)$	$(0.35, 0.25)$	$2\mathbb{Z} + 1$	$(0.2, 0.3)$	$(0.2, 0.3)$	T	T
$2\mathbb{Z} + 1$	$2\mathbb{Z}$	$2\mathbb{Z} + 1$	$(0.2, 0.3)$	$(0.35, 0.25)$	$(0.2, 0.3)$	$2\mathbb{Z} + 1$	$(0.2, 0.3)$	$(0.2, 0.3)$	T	T
$2\mathbb{Z} + 1$	$2\mathbb{Z} + 1$	$2\mathbb{Z} + 1$	$(0.2, 0.3)$	$(0.2, 0.3)$	$(0.2, 0.3)$	$2\mathbb{Z}$	$(0.35, 0.25)$	$(0.2, 0.3)$	T	T

Proposition 2.1 Let $(G, *)$ be a crisp group, and let A be an intuitionistic fuzzy subgroup of G . Then,

- (i) $A(e) \geq A(a)$, for all $a \in G$, where e is the identity of G ,
- (ii) $A(a^{-1}) = A(a)$, for all $a \in G$, where a^{-1} is the inverse of a in G .

Proof. Let $(G, *)$ be a group and let A be a intuitionistic fuzzy subgroup of G .

- (i) Let $a \in G$. We have $A(e) = A(a * a^{-1}) \geq A(a) \wedge A(a^{-1}) = A(a)$, since A is an intuitionistic fuzzy subgroup of G .
- (ii) Since A is intuitionistic fuzzy subgroup of G , therefore $A(a^{-1}) \geq A(a)$, for all $a \in G$. Replacing a by a^{-1} , it is obtained that $A(a) \geq A(a^{-1})$, for all $a \in G$. Thus, $A(a^{-1}) = A(a)$, for all $a \in G$.

□

Proposition 2.2 Let $(G, *)$ be a crisp (classical) group, and let A be an intuitionistic fuzzy subset in G . A is an intuitionistic fuzzy subgroup of G if and only if $A(a * b^{-1}) \geq_{L^*} A(a) \wedge A(b)$, for all $a, b \in G$.

Proof. Since A is an intuitionistic fuzzy subgroup of G , $A(a * b^{-1}) \geq_{L^*} A(a) \wedge A(b^{-1}) \geq_{L^*} A(a) \wedge A(b)$, for all $a, b \in G$.

Conversely, suppose that $A(a * b^{-1}) \geq_{L^*} A(a) \wedge A(b)$. Then,

$$A(e) = A(a * a^{-1}) \geq A(a) \wedge A(a) = A(a), \text{ for all } a \in G.$$

$$A(b^{-1}) = A(e * b^{-1}) \geq A(e) \wedge A(b) = A(b), \text{ for all } b \in G.$$

$$A(a * b) = A(a * (b^{-1})^{-1}) \geq A(a) \wedge A(b^{-1}) \geq A(a) \wedge A(b).$$

Consequently, A is an intuitionistic fuzzy subgroup of G . □

Proposition 2.3 Let $(G, *)$ be a crisp group,

If P and Q are two intuitionistic fuzzy subgroups in G , then $P \cap Q$ is an intuitionistic fuzzy subgroup in G .

Proof. Let $(G, *)$ be a group, and put $F = P \cap Q$,

$$F(a * b) = (P \cap Q)(a * b) = P(a * b) \wedge Q(a * b) \geq P(a) \wedge P(a) \wedge Q(b) \wedge Q(b) = P(a) \wedge Q(a) \wedge P(b) \wedge Q(b) = F(a) \wedge F(b).$$

$F(a^{-1}) = (P \cap Q)(a^{-1}) = P(a^{-1}) \wedge Q(a^{-1}) \geq P(a) \wedge Q(a) = F(a)$, for all $a \in G$. Since P and Q are Intuitionistic fuzzy subgroups of G . Consequently, $P \cap Q$ is a intuitionistic fuzzy subgroup of G . \square

Remark 2.1 *If P and Q are two intuitionistic fuzzy subgroups, $P \cup Q$ may not be an intuitionistic fuzzy subgroup. An example validates this observation.*

Example 2.2 *Let P and Q be two intuitionistic fuzzy subsets of the group $G = (\mathbb{Z}, +)$ defined as follows:*

$$P(a) = \begin{cases} (0.25, 0.20) & \text{if } a \in 7\mathbb{Z}, \\ (0.10, 0.21) & \text{otherwise,} \end{cases} \quad Q(a) = \begin{cases} (0.15, 0.35) & \text{if } a \in 5\mathbb{Z}, \\ (0.13, 0.17) & \text{otherwise.} \end{cases}$$

Then, the union $P \cup Q$ is defined pointwise by:

$$(P \cup Q)(a) = P(a) \vee Q(a) = \begin{cases} (0.25, 0.20) & \text{if } a \in 7\mathbb{Z} \cap 5\mathbb{Z}, \\ (0.25, 0.17) & \text{if } a \in 7\mathbb{Z} \setminus 5\mathbb{Z}, \\ (0.15, 0.21) & \text{if } a \in 5\mathbb{Z} \setminus 7\mathbb{Z}, \\ (0.13, 0.17) & \text{otherwise.} \end{cases}$$

Now consider $a = 2 = 7 + (-5)$. We compute:

$$(P \cup Q)(2) = P(2) \vee Q(2) = (0.13, 0.17),$$

but

$$(P \cup Q)(7) = P(7) \vee Q(7) = (0.25, 0.20), \quad (P \cup Q)(5) = P(5) \vee Q(5) = (0.15, 0.35),$$

so

$$(P \cup Q)(7) \wedge (P \cup Q)(5) = (0.25, 0.20) \wedge (0.15, 0.35) = (0.15, 0.20).$$

Since

$$(P \cup Q)(2) = (0.13, 0.17) \not\geq (0.15, 0.20),$$

this contradicts the required condition for being an intuitionistic fuzzy subgroup.

Therefore, $P \cup Q$ is not an intuitionistic fuzzy subgroup of G .

Proposition 2.4 *Let $(G, *)$ be a crisp group, if P and Q are two intuitionistic fuzzy subgroups of G , then $P \times Q$ is an intuitionistic fuzzy subgroup of $G \times G$.*

Proof. Let $(G, *)$ be a group.

We have $(P \times Q)(a, b) = P(a) \wedge Q(b)$, for all $(a, b) \in G \times G$. Then, $(P \times Q)((a, b) * (c, d)^{-1}) = (P \times Q)((a, b) * (c^{-1}, d^{-1})) = P(a * c^{-1}) \wedge Q(b * d^{-1}) \geq P(a) \wedge P(c) \wedge Q(b) \wedge Q(d) = (P(a) \wedge Q(b)) \wedge (P(c) \wedge Q(d)) = (P \times Q)((a, b)) \wedge (P \times Q)((c, d))$. Hence $(P \times Q)$ intuitionistic fuzzy subgroup of G . \square

Proposition 2.5 *Let $(G_1, *)$ and $(G_2, *)$ be two crisp groups, Let P and Q two intuitionistic fuzzy subgroups of G_1 and G_2 , respectively. Then $(P \times Q)((e_1, e_2)) \geq (P \times Q)(a_1, a_2)$, for all $(a_1, a_2) \in G_1 \times G_2$, where (e_1, e_2) is the identity in $G_1 \times G_2$.*

Proof. We have $(P \times Q)(e_1, e_2) = P(e_1) \wedge Q(e_2) \geq P(a_1) \wedge Q(a_2) = (P \times Q)(a_1, a_2)$, for all $a_1 \in G_1$ and for all $a_2 \in G_2$. Thus, $(P \times Q)(e_1, e_2) \geq (P \times Q)(a_1, a_2)$, for all $(a_1, a_2) \in G_1 \times G_2$. \square

Proposition 2.6 *Let $(G_1, *)$ and $(G_2, *)$ be two crisp groups and let P and Q be two intuitionistic fuzzy subgroups of G_1 and G_2 , respectively, such that $P \times Q$ is an intuitionistic fuzzy subgroup of $G_1 \times G_2$. Then, one of the following conditions must holds,*

- (i) $Q(e_2) < P(a)$, for all $a \in G_1$, where e_2 is the identity in G_2 ,
- (ii) $P(e_1) < Q(b)$, for all $b \in G_2$, where e_1 is the identity in G_1 .

Proof. Suppose that none of the conditions can be held. Then there exists $a \in G_1$ and $b \in G_2$. Then we have $(P \times Q)(a, b) \geq P(a) \wedge Q(b) \wedge Q(e_2) \wedge P(e_1) = (P \times Q)((e_1, e_2))$. Thus, $(P \times Q)((a, b)) > (P \times Q)((e_1, e_2))$, this is a contradiction since (e_1, e_2) is the identity in $G_1 \times G_2$, it is known that $(P \times Q)(e_1, e_2) \geq (P \times Q)((a_1, a_2))$, for all $(a_1, a_2) \in G_1 \times G_2$. Hence, one of the conditions must hold. \square

Proposition 2.7 *Let $(G, *)$ be a group.*

If A is an intuitionistic fuzzy subgroup of G , k a positive integer, then A^k is an intuitionistic fuzzy subgroup of G .

Proof. Suppose that A is a intuitionistic fuzzy subgroup, therefore

$$A^K(a * b^{-1}) = (A(a * b^{-1}))^k \geq (A(a) \wedge A(b))^k = (A(a))^k \wedge (A(b))^k = A^K(a) \wedge A^K(b),$$

for all $a, b \in G$.

Consequently, A^K is an intuitionistic fuzzy subgroup of G . \square

Proposition 2.8 *Let $(G, *)$ be a crisp group and let P, Q, R and S be four intuitionistic fuzzy subgroups of G , such that P is conjugate to R , and Q is conjugate to S . Then $P \times Q$ is conjugate to $R \times S$.*

Proof.

Since P is conjugate to R , $P(u_1) = R(a * u_1 * a^{-1})$, for all $a \in G$ and $u_1 \in G$. Since Q is conjugate to S , $Q(u_2) = S(b * u_2 * b^{-1})$, for all $b \in G$ and $u_2 \in G$.

Thus, $(P \times Q)(u_1, u_2) = P(u_1) \wedge Q(u_2) = R(a * u_1 * a^{-1}) \wedge S(b * u_2 * b^{-1}) = (R \times S)((a, b)(u_1, u_2)(a, b)^{-1})$, for some $(a, b) \in G \times G$ and for all $(u_1, u_2) \in G \times G$.

Therefore, $P \times Q$ is conjugate to $R \times S$. \square

2.2 Characterisation Of Intuitionistic Fuzzy Subgroups

Theorem 2.1 *Let $(G, *)$ be a group, and let A be an intuitionistic fuzzy subset of G .*

A is an intuitionistic fuzzy subgroup of G if and only if A_α are crisp groups for any $\alpha \in L_0^$.*

Proof. Suppose that A is an intuitionistic fuzzy subgroup of G .

1. If $x, y \in A_\alpha$, then $A(x) \geq \alpha$, and $A(y) \geq \alpha$. Thus, $A(x * y) \geq A(x) \wedge A(y) \geq \alpha$.

Hence, $x * y \in A_\alpha$.

2. We have $A(e) = A(x * x^{-1}) \geq A(x) \wedge A(x^{-1})$, then $A(e) \geq A(x) \geq \alpha$. Thus,

$e \in A_\alpha$

3. If $x \in A_\alpha$, i.e., $A(x) \geq \alpha$, then $A(x^{-1}) = A(x^{-1} * e) = A(x^{-1}) \wedge A(e) \geq A(x^{-1}) \wedge$

$A(x) \geq A(x) \geq \alpha$

Thus, A_α are crisp subgroup of G .

Conversely, A_α are a crisp groups for any $\alpha \in]0,1]$.

1. Put $A(x) \wedge A(y) = \alpha$, this implies $x \in A_\alpha$ and $y \in A_\alpha$. Then $x * y \in A_\alpha$, hence $A(x * y) \geq \alpha = A(x) \wedge A(y)$.
2. Put $A(x) = \alpha$, i.e., $x \in A_\alpha$, then $x^{-1} \in A_\alpha$, hence $A(x^{-1}) \geq \alpha = A(x)$.

Thus, A is an intuitionistic fuzzy subgroup of G . \square

Theorem 2.2 *Let $(G, *)$ be a group and let A be an intuitionistic fuzzy subgroup of G , Then $\text{Supp}(A)$ is a crisp subgroup of G .*

Proof. Recalling that $\text{Supp}(A) = \{x \in X, A(x) > 0_{L^*}\}$.

Suppose that A is an intuitionistic fuzzy subgroup of G ,

If $x, y \in \text{Supp}(A)$, that is, $A(x) > 0_{L^*}$ and $A(y) > 0_{L^*}$, then $A(x * y) \geq A(x) \wedge A(y) > 0_{L^*}$. Thus $x * y \in \text{Supp}(A)$.

If $x \in \text{Supp}(A)$, then $A(x^{-1}) \geq A(x) > 0_{L^*}$. Thus, $x^{-1} \in \text{Supp}(A)$.

Therefore, $\text{Supp}(A)$ is a crisp subgroup of G . \square

Remark 2.2 *The converse of the previous theorem is not valid, indeed.*

Example 2.3 *Let $G = (\mathbb{R}, +)$, and define an intuitionistic fuzzy subset A of \mathbb{R} by:*

$$A(x) = \begin{cases} (0.3, 0.5) & \text{if } x \in \mathbb{Z}, \\ (0.7, 0.2) & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}. \end{cases}$$

Then the support of A is:

$$\text{Supp}(A) = \{x \in \mathbb{R} \mid A(x) \succ 0_{L^*}\} = \mathbb{R},$$

which is a crisp subgroup of \mathbb{R} .

Let $x = 1.7, y = 0.7$, so that:

$$x - y = 1 \in \mathbb{Z}, \quad A(x - y) = A(1) = (0.3, 0.5),$$

$$A(x) = (0.7, 0.2), \quad A(y) = (0.7, 0.2).$$

Now compute the meet:

$$A(1.7) \wedge A(0.7) = (0.7, 0.2) \wedge (0.7, 0.2) = (0.7, 0.2).$$

Since:

$$A(1) = (0.3, 0.5) \not\geq_{L^*} (0.7, 0.2),$$

then $A(x - y) \not\geq_{L^*} A(x) \wedge A(y)$, so A is not an intuitionistic fuzzy subgroup of \mathbb{R} .

Theorem 2.3 Let $(G, *)$ be a group and let A be an intuitionistic fuzzy subset of G .

If A is an intuitionistic fuzzy subgroup of G , then $\text{Ker}(A)$ is a crisp subgroup of G .

Proof. Recalling that $\text{Ker}(A) = \{x \in G \mid A(x) = 1_{L^*}\}$.

Suppose that A is an intuitionistic fuzzy subgroup of G .

If $x, y \in \text{Ker}(A)$, then $A(x) = 1_{L^*}$ and $A(y) = 1_{L^*}$. Since A is a fuzzy subgroup, we have:

$$A(x * y) \geq A(x) \wedge A(y) = 1_{L^*}.$$

Thus, $x * y \in \text{Ker}(A)$.

If $x \in \text{Ker}(A)$, then $A(x^{-1}) \geq A(x) = 1_{L^*}$. Thus, $x^{-1} \in \text{Ker}(A)$.

Therefore, $\text{Ker}(A)$ is a crisp subgroup of G . \square

2.3 Homomorphism Of Intuitionistic Fuzzy Subgroups

Definition 2.2 Let $(G_1, *)$ and (G_2, \circ) be two crisp groups. Then a mapping $h : G_1 \rightarrow G_2$ is said to be a group homomorphism if $h(a * b) = h(a) \circ h(b)$ for all $a, b \in G_1$.

In the context of intuitionistic fuzzy subgroups, if A is an intuitionistic fuzzy subset of G_1 , then the image of A under h is the intuitionistic fuzzy subset $h(A)$ of G_2 defined by:

$$h(A)(b) = \sup\{A(a) \mid a \in h^{-1}(b)\}.$$

for all $b \in G_2$.

Definition 2.3 Let $(G_1, *)$ and (G_2, \circ) be two crisp groups, and let $h : G_1 \rightarrow G_2$ be a surjective group homomorphism. For an intuitionistic fuzzy subset A of G_1 , the image of A under h is the intuitionistic fuzzy subset $h(A)$ of G_2 defined by:

$$h(A)(b) = \sup\{A(a) \mid a \in h^{-1}(b)\}.$$

for all $b \in G_2$.

Proposition 2.9 *Let $(G_1, *)$ and (G_2, \circ) be two crisp groups and let $h : G_1 \longrightarrow G_2$ be a bijective group homomorphism. Then, for an intuitionistic fuzzy subgroup A in G_1 , the image $h(A)$ is an intuitionistic fuzzy subgroup of G_2 .*

Proof. Since h is bijective, for every $b \in G_2$, a unique $a \in G_1$ exists, such that $h(a) = b$. Therefore,

$$h(A)(b) = \sup_{x \in h^{-1}(b)} A(x) = A(a).$$

Let $b_1, b_2 \in G_2$, and let $a_1, a_2 \in G_1$ such that $h(a_1) = b_1$ and $h(a_2) = b_2$. Since h is a homomorphism,

$$h(a_1 * a_2^{-1}) = h(a_1) \circ h(a_2)^{-1} = b_1 \circ b_2^{-1}.$$

Then,

$$h(A)(b_1 \circ b_2^{-1}) = A(a_1 * a_2^{-1}) \geq A(a_1) \wedge A(a_2) = h(A)(b_1) \wedge h(A)(b_2).$$

Hence, $h(A)$ satisfies the conditions of an intuitionistic fuzzy subgroup of G_2 . \square

Definition 2.4 *Let $(G_1, *)$ and (G_2, \circ) be two crisp groups. A mapping $h : G_1 \longrightarrow G_2$ is said to be an anti-group homomorphism if*

$$h(a * b) = h(b) \circ h(a), \quad \text{for all } a, b \in G_1.$$

In the context of intuitionistic fuzzy subgroups, if A is an intuitionistic fuzzy subset of G_1 , then the image $h(A)$ is defined on G_2 by:

$$h(A)(b) = \sup\{A(a) \mid a \in h^{-1}(b)\}.$$

for all $b \in G_2$. If h is bijective, then $h(A)$ is also an intuitionistic fuzzy subgroup of G_2 under the anti-homomorphic structure.

Definition 2.5 *Let $(G_1, *)$ and (G_2, \circ) be two crisp groups, and let Q be an intuitionistic fuzzy subgroup of G_2 . For a mapping $h : G_1 \longrightarrow G_2$, the preimage $h^{-1}(Q)$ is defined as the intuitionistic fuzzy subset $h^{-1}(Q)$ of G_1 , where:*

$$h^{-1}(Q)(a) = Q(h(a)).$$

for all $a \in G_1$.

Proposition 2.10 *Let $(G_1, *)$ and (G_2, \circ) be two crisp groups, and let Q be an intuitionistic fuzzy subgroup of G_2 . Then for an anti-group homomorphism $h : G_1 \rightarrow G_2$, the preimage $h^{-1}(Q)$ is an intuitionistic fuzzy subgroup of G_1 .*

Proof. We define $h^{-1}(Q)$, where

$$h^{-1}(Q)(a) = Q(h(a)), \quad \text{for all } a \in G_1.$$

Let $a, b \in G_1$. Then,

$$h^{-1}(Q)(a * b^{-1}) = Q(h(a * b^{-1})) = Q(h(b^{-1}) \circ h(a)).$$

Since Q is an intuitionistic fuzzy subgroup of G_2 , we have:

$$Q(h(b)^{-1} \circ h(a)) \geq Q(h(a)) \wedge Q(h(b)) = h^{-1}(Q)(a) \wedge h^{-1}(Q)(b).$$

Therefore, $h^{-1}(Q)$ satisfies the conditions of an intuitionistic fuzzy subgroup of G_1 .

□

2.4 Intuitionistic Fuzzy Coset and Intuitionistic Fuzzy Normal Subgroup

Definition 2.6 *Let $(G, *)$ be a crisp group and let P be an intuitionistic fuzzy subgroup of G . Then, for any $a \in G$, the intuitionistic fuzzy left coset of P in G is the intuitionistic fuzzy subset aP defined by:*

$$aP(u) = P(a^{-1} * u), \text{ for all } u \in G.$$

Example 2.4 *Consider the group $G = \mathbb{Z}_6$ and the intuitionistic fuzzy subgroup $P = \{0, 3\}$. The left coset of P for $a = 1$ is $aP = \{1, 4\}$, as illustrated in the following diagram.*

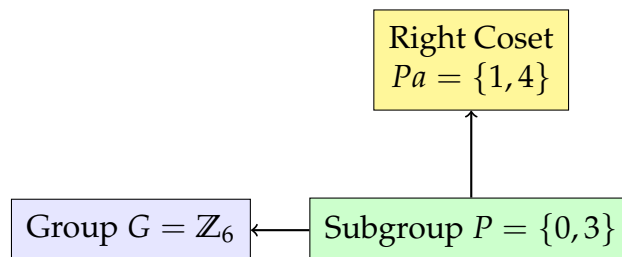


Figure 2.1: Illustration of Right Coset Pa for $a = 1$ in \mathbb{Z}_6

Definition 2.7 Let $(G, *)$ be a crisp group and let P be an intuitionistic fuzzy subgroup of G . Then for any $a \in G$, the intuitionistic fuzzy right coset of P in G is the intuitionistic fuzzy subset Pa defined by:

$$Pa(u) = P(u * a^{-1}).$$

for all $u \in G$.

Example 2.5 Consider the group $G = \mathbb{Z}_6$ and the intuitionistic fuzzy subgroup $P = \{0, 3\}$. The right coset of P for $a = 1$ is $Pa = \{1, 4\}$, as shown in the diagram below.

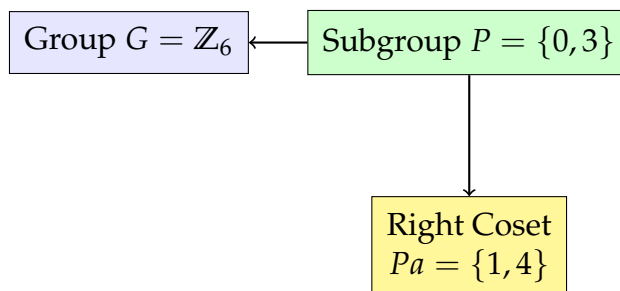


Figure 2.2: Illustration of Right Coset Pa for $a = 1$ in \mathbb{Z}_6

Definition 2.8 Let $(G, *)$ be a crisp group and let P be an intuitionistic fuzzy subgroup of G . Then, for any $a \in G$, the intuitionistic fuzzy middle coset of P in G is the intuitionistic fuzzy subset aPa^{-1} defined by:

$$aPa^{-1}(u) = P(a^{-1} * u * a),$$

for all $u \in G$.

Example 2.6 Consider the group $G = \mathbb{Z}_6$ and the intuitionistic fuzzy subgroup $P = \{0, 3\}$. As illustrated below the middle coset of P for $a = 1$ is $aPa^{-1} = \{1, 4\}$.

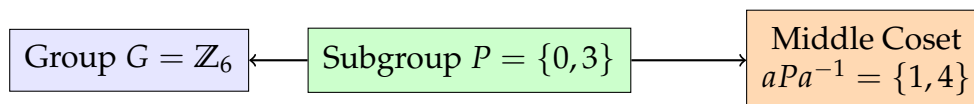


Figure 2.3: Illustration of Middle Coset aPa^{-1} for $a = 1$ in \mathbb{Z}_6

Definition 2.9 Let $(G, *)$ be a crisp group and let P be an intuitionistic fuzzy subgroup of G . Then P is called an **intuitionistic fuzzy normal subgroup** of G if:

$$Pa(u) = aP(u),$$

for all $a, u \in G$.

Example 2.7 Consider the group $G = \mathbb{Z}_6$ and the intuitionistic fuzzy subgroup $P = \{0, 3\}$. We verify whether P is an intuitionistic fuzzy normal subgroup by checking the condition:

$$\mu_{Pa}(u) = \mu_{aP}(u), \quad \nu_{Pa}(u) = \nu_{aP}(u).$$

The structure is illustrated in the diagram below.

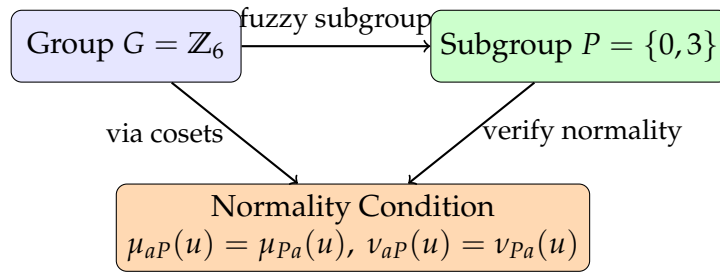


Figure 2.4: Verification of Intuitionistic Fuzzy Normal Subgroup in \mathbb{Z}_6

Table 2.1: Membership and Non-membership Degrees in aP for $a = 1$

$u \in G$	$\mu_{aP}(u)$	$\nu_{aP}(u)$
0	0.2	0.6
1	0.7	0.2
2	0.1	0.8
3	0.2	0.6
4	0.7	0.2
5	0.1	0.8

Proposition 2.11 Let $(G, *)$ be a crisp group and let P be an intuitionistic fuzzy subgroup of G . Then P is an intuitionistic fuzzy normal subgroup of G if and only if:

$$P(a * b) = P(b * a),$$

for all $a, b \in G$.

Proof. The proof follows similarly to the fuzzy case, replacing the membership functions with intuitionistic fuzzy membership and non-membership functions, which satisfy the conditions for normality of the intuitionistic fuzzy subgroup. \square

Proposition 2.12 *Let $(G, *)$ be a crisp group and let P be an intuitionistic fuzzy subgroup of G . Then P is an intuitionistic fuzzy normal subgroup of G if and only if:*

$$P(a * u * a^{-1}) = P(u),$$

for all $a, u \in G$.

Proof. The proof for intuitionistic fuzzy groups follows the same reasoning as the fuzzy case, ensuring the preservation of membership and non-membership under conjugation. \square

2.5 Lagrange's Theorem on Intuitionistic Fuzzy Subgroup

Definition 2.10 *Let $(G, *)$ be a crisp group and let P be an intuitionistic fuzzy subgroup of G . The order of the intuitionistic fuzzy subgroup P is denoted by $O(P)$ and is defined as the cardinality of the crisp set $H_P = \{u \in G : P(u) = P(e)\}$, where e is the identity element of G .*

Proposition 2.13 *Let $(G, *)$ be a crisp group and let P be an intuitionistic fuzzy normal subgroup of G . Then the order of P is equal to the order of the coset aPa^{-1} , i.e.,*

$$O(P) = O(aPa^{-1}),$$

for any $a \in G$.

Proof. The proof is similar to the fuzzy case, but now we work with both membership and non-membership functions. The sets H_P and $H_{aPa^{-1}}$ are equivalent under the transformation, thus showing that the orders are equal. \square

Theorem 2.4 (Lagrange's Theorem for Intuitionistic Fuzzy Subgroups) *Let $(G, *)$ be a crisp finite group and let P be an intuitionistic fuzzy subgroup of G . Then the order of P , $O(P)$, is a divisor of the order of G , $O(G)$.*

Proof. Since $O(P) = |H_P|$, where $H_P = \{u \in G : P(u) = P(e)\}$, and H_P is a crisp subgroup of G , by Lagrange's theorem for crisp groups, $|H_P|$ divides $O(G)$. Hence, $O(P)$ is a divisor of $O(G)$. \square

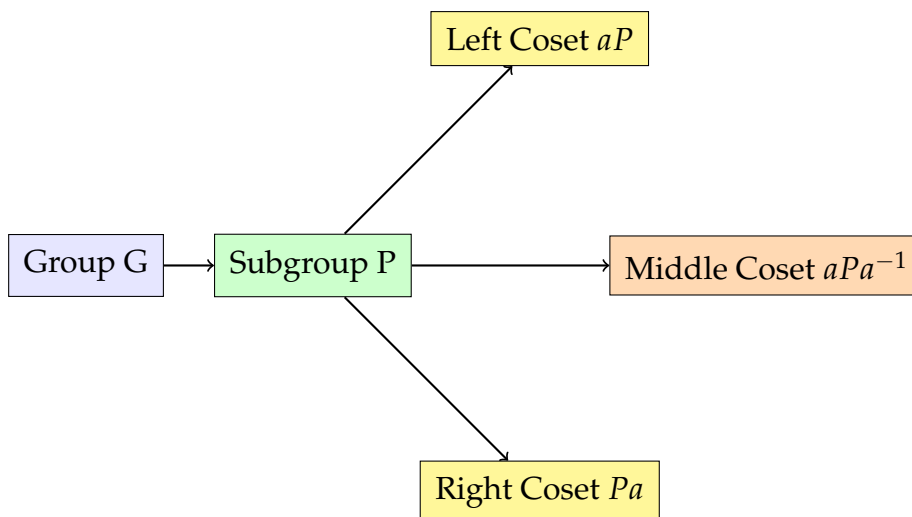


Figure 2.5: Illustration of Intuitionistic Fuzzy Cosets with Color-Coding for Clarity

Type	Membership Function	Non-Membership Function
Left Coset aP	$\mu_{aP}(u) = \mu_P(a^{-1}u)$	$\nu_{aP}(u) = \nu_P(a^{-1}u)$
Right Coset Pa	$\mu_{Pa}(u) = \mu_P(ua^{-1})$	$\nu_{Pa}(u) = \nu_P(ua^{-1})$
Middle Coset aPa^{-1}	$\mu_{aPa^{-1}}(u) = \mu_P(a^{-1}ua)$	$\nu_{aPa^{-1}}(u) = \nu_P(a^{-1}ua)$

Table 2.2: Membership and Non-Membership Functions for Intuitionistic Fuzzy Cosets

Chapter 3

Intuitionistic Fuzzy Ideals

In this final chapter, we will introduce the definition of **intuitionistic fuzzy ideals**, and study their **characterizations** and the **homomorphisms** between groups.

3.1 Definitions and properties

Definition 3.1 Let $(G, *)$ be an abelian group, and let A be an intuitionistic fuzzy subset of G . Then A is called an intuitionistic fuzzy ideal of G if:

$$A(x * y) \succeq A(x) \vee A(y).$$

for all $x, y \in G$.

Example 3.1 Consider the group (\mathbb{R}_+^*, \times) . Let A be an intuitionistic fuzzy subset of \mathbb{R}_+^* defined by:

$A(x) = \left(\frac{x}{x+1}, \frac{1}{x+1}\right)$, $A(y) = \left(\frac{y}{y+1}, \frac{1}{y+1}\right)$, then $A(xy) = \left(\frac{xy}{xy+1}, \frac{1}{xy+1}\right)$, we ask if $A(xy) \geq_{L^*} A(x) \vee A(y)$.

Firstly evaluate $A(x) - A(y) = \left(\frac{x(y-1)}{(x+1)(xy+1)}, \frac{x(y-1)}{(y+1)(xy+1)}\right) = \begin{cases} \succeq 0_{L^*} & \text{if } y > 1; \\ \preceq 0_{L^*} & \text{if } y \leq 1. \end{cases}$

Hence $A(x) \vee A(y) = \begin{cases} A(x) & \text{if } y > 1; \\ A(y) & \text{if } y \leq 1. \end{cases}$

It stil to show that $A(xy) \geq_{L^*} \begin{cases} A(x) & \text{if } y > 1; \\ A(y) & \text{if } y \leq 1. \end{cases}$

Subcase 1, $y > 1$

$\frac{xy}{xy+1} - \frac{x}{x+1} = \frac{x}{(xy+1)(x+1)}(y-1)$, this greater then 0, and

$\frac{1}{xy+1} - \frac{1}{x+1} = -\frac{x}{(xy+1)(x+1)}(y-1)$, this less then 0, so in this $A(xy) \geq_{L^*} A(x) \vee A(y)$.

Subcase 2, $y \leq 1$

$$\frac{xy}{xy+1} - \frac{y}{y+1} = \frac{y}{xy+1} \frac{x-1}{y+1} = \begin{cases} \leq 0 & \text{if } x \leq 1; \\ \geq 0 & \text{if } x \geq 1. \end{cases} \text{ Hence } A \text{ is not an ideal.}$$

Remark 3.1 From the previous example, an intuitionistic fuzzy ideal is not necessarily an intuitionistic fuzzy subgroup. Indeed, take $x = 2$, so that $x^{-1} = \frac{1}{2}$. Then

$$A(2) = \left(\frac{2}{3}, \frac{1}{3}\right), \quad A\left(\frac{1}{2}\right) = \left(\frac{1}{3}, \frac{2}{3}\right),$$

which implies

$$A\left(\frac{1}{2}\right) \neq A(2).$$

Hence, A is not an intuitionistic fuzzy subgroup.

Proposition 3.1 Let $(G, *)$ be a group, and let A and B be two intuitionistic fuzzy ideals of G . Then $A \cap B$ is an intuitionistic fuzzy ideal of G .

Proof. Let G be a group, and let A and B be two intuitionistic fuzzy ideals. Then

$$\begin{aligned} (A \cap B)(x * y) &= A(x * y) \wedge B(x * y) \\ &\geq [A(x) \vee A(y)] \wedge [B(x) \vee B(y)] \\ &= [A(x) \wedge B(x)] \vee [A(x) \wedge B(y)] \vee [A(y) \wedge B(x)] \vee [A(y) \wedge B(y)] \\ &\geq [A(x) \wedge B(x)] \vee [A(y) \wedge B(y)] \\ &= (A \cap B)(x) \vee (A \cap B)(y). \end{aligned}$$

Hence, $A \cap B$ is an intuitionistic fuzzy ideal of G . \square

Proposition 3.2 Let $(G, *)$ be a group, and let A, B be two intuitionistic fuzzy ideals in G .

Then $A \cup B$ is an intuitionistic fuzzy ideal in G .

Proof. Let G be a group, and let A and B be two intuitionistic fuzzy ideals. Then

$$\begin{aligned} (A \cup B)(x * y) &= A(x * y) \vee B(x * y) \\ &\geq [A(x) \vee A(y)] \vee [B(x) \vee B(y)] \\ &= [A(x) \vee B(x)] \vee [A(y) \vee B(y)] \\ &= (A \cup B)(x) \vee (A \cup B)(y). \end{aligned}$$

Hence, $A \cup B$ is an intuitionistic fuzzy ideal of G . \square

Proposition 3.3 Let $(G, *)$ be a group, and let A be an intuitionistic fuzzy ideal in G . Then A^k is an intuitionistic fuzzy ideal of G for a positive integer k .

Proof. Suppose that A is an intuitionistic fuzzy ideal. Then

$$\begin{aligned} A^K(x * y) &\geq (A(x) \vee A(y))^k \\ &= [A(x)^k \vee A(y)^k] \geq \max(A^k(x), A^k(y)), \end{aligned}$$

Hence, A^K is an intuitionistic fuzzy ideal of G . \square

Proposition 3.4 *Let $(G, *)$ be a group, and let A and B be two intuitionistic fuzzy ideals of G . Then $A \times B$ is an intuitionistic fuzzy ideal of G .*

Proof. Let G be a group, and suppose that A and B are two intuitionistic fuzzy ideals of G . Then

$$\begin{aligned} (A \times B)(x * y) &= A(x * y) \wedge B(x * y) \\ &\geq [A(x) \vee A(y)] \wedge [B(x) \vee B(y)] \\ &= [A(x) \wedge B(x)] \vee [A(x) \wedge B(y)] \vee [A(y) \wedge B(x)] \vee [A(y) \wedge B(y)] \\ &\geq [A(x) \wedge B(x)] \vee [A(y) \wedge B(y)] \\ &= (A \times B)(x) \vee (A \times B)(y), \end{aligned}$$

Hence, $A \times B$ is an intuitionistic fuzzy ideal of G . \square

3.2 Characterisation Of Intuitionistic Fuzzy Ideals

Theorem 3.1 *Let $(G, *)$ be a group, and let A be an intuitionistic fuzzy subset in G .*

If A is an intuitionistic fuzzy ideal in G , then A_α are crisp ideals for any $\alpha \in L^$.*

Proof. Suppose that A is an intuitionistic fuzzy ideal in G .

(i) If $x, y \in A_\alpha$, i.e., $A(x) \geq \alpha$ and $A(y) \geq \alpha$, then

$$A(x * y) \geq A(x) \vee A(y) \geq \alpha \vee \alpha = \alpha.$$

Thus, $x * y \in A_\alpha$.

(ii) If $x \in A_\alpha$, then $A(e) = A(x * x^{-1}) \geq A(x) \vee A(x^{-1})$, i.e.,

$$A(e) \geq A(x) \geq \alpha.$$

Hence, $e \in A_\alpha$.

(iii) If $x \in A_\alpha$, that is, $A(x) \geq \alpha$, then

$$A(x^{-1}) \geq A(x) \geq \alpha.$$

Thus, $x^{-1} \in A_\alpha$.

Hence, A_α is a crisp ideal of G . \square

Theorem 3.2 *Let $(G, *)$ be a group, and let A be an intuitionistic fuzzy subset in G .*

If A is an intuitionistic fuzzy ideal in G , then $Supp(A)$ is a crisp ideal in G .

Proof. Suppose that A is an intuitionistic fuzzy ideal in G .

(i) If $x, y \in Supp(A)$, i.e., $A(x) > 0_{L^*}$ and $A(y) > 0_{L^*}$, then

$$A(x * y) \geq A(x) \vee A(y) > 0_{L^*}.$$

Hence, $x * y \in Supp(A)$.

(ii) If $x \in Supp(A)$, then $A(e) = A(x * x^{-1}) \geq A(x) \vee A(x^{-1})$, i.e.,

$$A(e) \geq A(x) > 0_{L^*}.$$

Hence, $e \in Supp(A)$.

(iii) If $x \in Supp(A)$, that is, $A(x) > 0_{L^*}$, then

$$A(x^{-1}) \geq A(x) > 0_{L^*}.$$

Hence, $x^{-1} \in Supp(A)$.

Therefore, $Supp(A)$ is a crisp ideal of G . \square

Theorem 3.3 *Let $(G, *)$ be a group, and let A be an intuitionistic fuzzy subset in G .*

If A is an intuitionistic fuzzy ideal in G , then $ker(A) = \{x \in G \mid A(x) = 1_{L^}\}$ is a crisp ideal in G .*

Proof. Suppose that A is an intuitionistic fuzzy ideal in G .

(i) If $x, y \in ker(A)$, i.e, $A(x) = 1_{L^*}$ and $A(y) = 1_{L^*}$, then

$$A(x * y) \geq A(x) \vee A(y) = 1_{L^*}.$$

Hence, $x * y \in ker(A)$.

(ii) If $x \in \ker(A)$, then $A(e) \geq A(x) = 1_{L^*}$, i.e, $A(e) = 1_{L^*}$. Hence, $e \in \ker(A)$.

(iii) If $x \in \ker(A)$, that is, $A(x) = 1_{L^*}$, then

$$A(x^{-1}) \geq A(x) = 1_{L^*}, \text{ i.e, } A(x^{-1}) = 1_{L^*}.$$

Hence, $x^{-1} \in \ker(A)$.

Therefore, $\ker(A)$ is a crisp ideal of G . \square

Proposition 3.5 *Let $(G, *)$ and $(G', *)$ be two groups, and let f be a bijective homomorphism from G to G' .*

If A is an intuitionistic fuzzy ideal of G , then $f(A)$ is an intuitionistic fuzzy ideal of G' .

Proof. Suppose that A is an intuitionistic fuzzy ideal of G . Then, we have:

$$(f(A))(x * y) = \bigvee_{z \in f^{-1}(x * y)} A(z) = A(f^{-1}(x) * f^{-1}(y)) \quad (\text{since } f \text{ is a homomorphism})$$

Now, since A is a fuzzy ideal, we know:

$$A(f^{-1}(x) * f^{-1}(y)) \geq A(f^{-1}(x)) \vee A(f^{-1}(y)).$$

Thus, we get:

$$(f(A))(x * y) \geq (f(A))(x) \vee (f(A))(y).$$

Therefore, $f(A)$ is an intuitionistic fuzzy ideal of G' . \square

Proposition 3.6 *Let $(G, *)$ and $(G', *)$ be two groups, and let A be an intuitionistic fuzzy ideal of G' . Then $f^{-1}(A)$ is an intuitionistic fuzzy ideal of G .*

Proof. Suppose that A is an intuitionistic fuzzy ideal of G' . Then, we have:

$$(f^{-1}(A))(x * y) = A(f(x * y)) = A(f(x) * f(y)) \quad (\text{since } f \text{ is a homomorphism}).$$

Now, since A is a fuzzy ideal in G' , we know:

$$A(f(x) * f(y)) \geq A(f(x)) \vee A(f(y)).$$

Thus, we get:

$$(f^{-1}(A))(x * y) \geq (f^{-1}(A))(x) \vee (f^{-1}(A))(y).$$

Therefore, $f^{-1}(A)$ is an intuitionistic fuzzy ideal of G . \square

3.3 Homomorphism Of Intuitionistic Fuzzy Ideals

Proposition 3.7 *Let $(G_1, *)$ and (G_2, \circ) be two crisp groups, and let $h : G_1 \longrightarrow G_2$ be a bijective group homomorphism. Then, for an intuitionistic fuzzy ideal I in G_1 , the image $h(I)$ is an intuitionistic fuzzy ideal of G_2 .*

Proof. Since h is bijective, for every $b \in G_2$, there exists a unique $a \in G_1$ such that $h(a) = b$. Therefore,

$$h(I)(b) = \sup_{x \in h^{-1}(b)} I(x) = I(a).$$

Let $b_1, b_2 \in G_2$, and let $a_1, a_2 \in G_1$ such that $h(a_1) = b_1$ and $h(a_2) = b_2$. Since h is a homomorphism,

$$h(a_1 * a_2^{-1}) = h(a_1) \circ h(a_2)^{-1} = b_1 \circ b_2^{-1}.$$

Then,

$$h(I)(b_1 \circ b_2^{-1}) = I(a_1 * a_2^{-1}) \geq I(a_1) \vee I(a_2) = h(I)(b_1) \vee h(I)(b_2),$$

Hence, $h(I)$ satisfies the conditions of an intuitionistic fuzzy ideal of G_2 . \square

Proposition 3.8 *Let $(G_1, *)$ and (G_2, \circ) be two crisp groups, and let Q be an intuitionistic fuzzy ideal of G_2 . Then, for an anti group homomorphism $h : G_1 \rightarrow G_2$, the preimage $h^{-1}(Q)$ is an intuitionistic fuzzy ideal of G_1 .*

Proof. We define $h^{-1}(Q)$, where

$$h^{-1}(Q)(a) = Q(h(a)), \quad \text{for all } a \in G_1.$$

Let $a, b \in G_1$. Then,

$$h^{-1}(Q)(a * b^{-1}) = Q(h(a * b^{-1})) = Q(h(b^{-1}) \circ h(a)).$$

Since Q is an intuitionistic fuzzy ideal of G_2 , we have:

$$Q(h(b)^{-1} \circ h(a)) \geq Q(h(a)) \vee Q(h(b)) = h^{-1}(Q)(a) \vee h^{-1}(Q)(b).$$

Therefore, $h^{-1}(Q)$ satisfies the conditions of an intuitionistic fuzzy ideal of G_1 . \square

Conclusion

In this thesis, we have revisited some fundamental concepts and theorems related to group theory. We then introduced the definitions and basic concepts of fuzzy sets and intuitionistic fuzzy sets. We proceeded by studying intuitionistic fuzzy subgroups, exploring various types such as normal intuitionistic fuzzy subgroups, fuzzy cosets, homomorphisms, and Lagrange's theorem extended to intuitionistic fuzzy subgroups.

Finally, we examined fuzzy ideals in subgroups, providing a deeper understanding of their properties and applications within the context of intuitionistic fuzzy structures.

Bibliography

- [1] Atanassov K. Intuitionistic fuzzy sets, VII ITKR's Session, Sofia, June 1983 (Deposited in Central Sci. - Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulg.)
- [2] D. M. Burton, *Abstract Algebra*, Wm. C. Brown Publishers, 1988.
- [3] R. Biswas, Intuitionistic fuzzy subgroups, *Mathematical Forum*, 1 (1989), 37-46.
- [4] B. Bede, *Mathematics of Fuzzy Sets and Fuzzy Logic*, Springer-Verlag, Heidelberg/New York, (2013).
- [5] P. Burillo, H. Bustince, Intuitionistic fuzzy relations (Part I), *Mathware and Soft Computing*, 2 (1995), 5-38.
- [6] J. R. Durbin, *Modern algebra: An introduction*, John Wiley & Sons, 2008.
- [7] W. J. Liu, *Fuzzy invariant subgroups and fuzzy ideals*, *Fuzzy Sets and Systems*, 8(2) (1982), 133–139.
- [8] S. Lipschutz and M. Lipson, *Discrete Mathematics*, Schaum's Outlines, 1997.
- [9] Soheyb Milles (2017). On The Intuitionistic Fuzzy Ordered Sets, Doctoral thesis, Department of Mathematics, Med Boudiaf University of Msila, Algeria.
- [10] L. A. Zadeh, *Fuzzy sets*, *Information and Control*, 8(3) (1965), 338-353.

- [11] Zedam, L., A. Amroune, and B. Davvaz. "Szpilrajn theorem for intuitionistic fuzzy orderings." *Annals of Fuzzy Mathematics and Informatics* 9.5 (2015): 703–718.
- [12] A. Rosenfeld, Fuzzy groups, *Journal of mathematical analysis and applications*, **35(3)** (1971), 512-517.

ملخص

تستكشف هذه المذكرة المجموعات الفرعية الضبابية الحدسية ومثلها، بدءاً من استعراض نظرية الزمر الكلاسيكية، والمجموعات الضبابية، والمجموعات الضبابية الحدسية الضبابية، والعمليات ذات الصلة مثل الاتحاد والتقاطع والمكمل. كما نقدم أيضاً مفهوم مستويات ألفا القوية والضعيفة للمجموعات الضبابية الحدسية وذلك لدراسة الزمر الضبابية الحدسية الفرعية ومثلها. ويركز القسم الأخير على المثل الضبابية، ويناقش مجموعاتاً مميزة لها، والدعامات بالمثل، والنواة وخط ألفا الضبابي الحدسي. ويقدم العمل دراسة مهمة لخصائص وتطبيقات المجموعات الضبابية الحدسية الضبابية على الزمر الفرعية الضبابية الحدسية الضبابية ومثلها المرفقة.

كلمات مفتاحية

زمرة، مجموعة ضبابية، مجموعة ضبابية حدسية، زمرة فرعية ضبابية، زمرة فرعية ضبابية حدسية، مثالي زمرة فرعية ضبابية حدسية.

Abstract

This memory explores intuitionistic fuzzy subgroups and their ideals, starting with a review of classical group theory, fuzzy sets, intuitionistic fuzzy sets, and related operations such as union, intersection and completion. Also, we introduce the concept of strong and weak α -levels of intuitionistic fuzzy sets to study intuitionistic fuzzy subgroups and their ideals. The final section focuses on fuzzy ideals, discussing their characteristic sets, likewise supports, kernels, and intuitionistic fuzzy α -line. The work provides a comprehensive examination of the characteristics and applications of intuitionistic fuzzy sets to intuitionistic fuzzy subgroups and their corresponding ideals.

Key words

Group, fuzzy set, intuitionistic fuzzy set, fuzzy subgroup, intuitionistic fuzzy subgroup, intuitionistic fuzzy ideal.

Résumé

Ce mémoire étudié les sous-groupes flous intuitionnistes et leurs idéaux, commençant par une réview de la théorie des groupes classiques, des ensembles flous, des ensembles flous intuitionnistes et des opérations connexes telles que l'union, l'intersection et la complémentation. Nous introduisons également le concept de niveaux de flous α - forts et faibles des ensembles flous intuitionnistes pour étudier les sous-groupes flous intuitionnistes et leurs idéaux. La dernière section se concentre sur les idéaux flous, en discutant de leurs ensembles caractéristiques, tel que leurs supports, de leurs noyaux et de leur α -ligne de flous. Ce travail fournit un examen intéressant des caractéristiques et des applications des ensembles flous intuitionnistes aux sous-groupes flous intuitionnistes et à leurs idéaux correspondants.