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Dedications

Dedicated to:

my parents,

my brothers and sisters,

my wife,

my daughter, Ghussoun Al-Rihyan,

and my son, Omar Idris.

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Introduction

Fractional calculus generalizes classical calculus by allowing derivatives and integrals of non-integer order [36, 58]. Fractional operators, such as the fractional Laplacian and time-fractional derivatives, naturally model nonlocal interactions, memory effects, and anomalous diffusion that cannot be adequately described by integer-order differential equations. Consequently, fractional differential equations arise in many applications, including fluid mechanics, viscoelastic wave propagation, and stochastic processes with jumps [13, 21, 26, 29, 60].

A central class of models is provided by time-fractional diffusion equations, obtained by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0, 1]$. These equations describe subdiffusive transport in heterogeneous and fractal media [52]. More generally, fractional Cauchy problems offer a natural framework for the analysis of time-fractional evolution equations and anomalous diffusion processes [47, 51, 59, 69]. Fundamental analytical results-including existence, uniqueness, regularity, and asymptotic behavior of solutions-have been established using semigroup and spectral methods [6, 20, 37, 44–46, 50, 57].

The most commonly used fractional derivatives are the Riemann–Liouville and Caputo derivatives [36,58], together with variants such as the Hadamard and Caputo–Hadamard derivatives [35]. Recently, generalized fractional operators defined with respect to an arbitrary kernel function φ have been introduced [2,42]. Many analytical questions concerning time–space fractional equations involving such operators remain open, particularly for nonlinear problems and for fractional time derivatives of order greater than one. This thesis investigates well-posedness, regularity, qualitative properties, and finite-time blow-up for fractional Cauchy problems involving φ-Caputo derivatives and fractional Laplacian-type operators.

Main Contributions

The main contributions of this thesis can be summarized as follows:

- A unified analytical framework is developed for time–space fractional diffusion and superdiffusion equations involving generalized φ-Caputo fractional derivatives and the fractional Laplacian.
- Explicit formulas for the fundamental solutions are derived for a wide class of linear fractional evolution equations. These solutions are expressed in terms of Fox H-functions.
- Decay estimates in Lebesgue and Lorentz spaces are established for the fundamental solutions. These estimates describe the smoothing and dispersive effects associated with fractional diffusion.
- Local existence and uniqueness of mild solutions are proved for nonlinear fractional diffusion and superdiffusion equations using fixed-point arguments.
- Global well-posedness is obtained for selected nonlinear fractional diffusion models under suitable smallness assumptions on the initial data.
- For nonlinear superdiffusion equations with time-fractional order greater than one, both mild and weak solutions are constructed, and their equivalence is established.
- Conditions guaranteeing finite-time blow-up are derived for nonlinear fractional superdiffusion equations, highlighting the influence of the fractional orders in time and space.

This thesis is organized as follows:

Chapter 1, entitled *Functional spaces and special functions*, introduces the basic concepts used throughout this work. We first review weighted $L^p(\mathbb{R}^d)$ spaces, fractional Sobolev spaces $H^{2\delta}(\mathbb{R}^d)$, and Lorentz spaces $L^{r,s}(\mathbb{R}^d)$, together with their main properties. Next, we introduce several fundamental special functions, including the Gamma function, the Beta function, and the Fox H-function. We then present different forms of *generalized fractional calculus*, including φ-fractional integrals and fractional derivatives, along with their principal properties. Afterward, we briefly discuss the *generalized Laplace transform and convolution*. Finally, we introduce the *generalized fractional Laplacian* and the *Fourier transform of type* ϕ, and summarize their main properties.

In Chapter 2, we study the global existence of solutions to the following semilinear time–space fractional diffusion equation:

$$\begin{cases} {}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi} u + (-\Delta)_x^{\delta} u = a|u|^{p-1}u + b|\nabla u|^{q}, & (t,x) \in (0,+\infty) \times \mathbb{R}^d, \\ u(0) = u_0, \quad x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $d \geq 1$, $\alpha \in (0,1)$, $\delta \in (0,1)$, $p > 1$, and

$$q = \frac{2\delta p}{(p-1) + 2\delta}.$$

The constants $a, b \in \mathbb{R}$ are given, and the initial data satisfies $u_0 \in L^s(\mathbb{R}^d)$, with

$$s = \frac{d}{2\delta}(p-1) > 1.$$

Here, ${}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi}$ denotes the left φ-Caputo fractional derivative of order α, and $(-\Delta)_x^{\delta}$ the fractional Laplacian. We first derive the fundamental solution of the associated linear problem by means of the φ-Laplace transform and the Fox H-function. We then establish decay estimates in Lebesgue spaces. Finally, under a suitable smallness assumption on the initial datum, we prove global existence, uniqueness, and continuous dependence of mild solutions.

In Chapter 3, we establish the local existence and uniqueness of solutions to the Cauchy problem for the time-fractional diffusion equation

$$\begin{cases} {}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi} u + (-\Delta)_{\phi(x)}^{\delta} u = \mathrm{I}_{0,t}^{1-\gamma,\varphi}[e^u], & (t,x) \in (0,+\infty) \times \mathbb{R}^d, \\ u(0) = u_0, \quad x \in \mathbb{R}^d, \end{cases} \tag{2}$$

where the initial data satisfies

$$u_0 \in C_0(\mathbb{R}^d) = \left\{ v \in C(\mathbb{R}^d) \;\middle|\; \lim_{|x|\to\infty} v(x) = 0 \right\},$$

endowed with the norm

$$\|v\|_{C_0(\mathbb{R}^d)} = \|v\|_\infty = \sup_{x\in\mathbb{R}^d} |v(x)|.$$

In this setting, ${}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi}$ denotes the left φ-Caputo fractional derivative of order α, while $\mathrm{I}_{0,t}^{1-\gamma,\varphi}$ denotes the Riemann–Liouville fractional integral of order $1-\gamma$ with respect to φ. The parameters satisfy $\alpha, \delta, \gamma \in (0,1)$ and $d \in \mathbb{N}$, and $\varphi \in C^1((a,b),\mathbb{R})$ is a strictly increasing function. We first derive the fundamental solution of the associated linear problem by means of the φ-Laplace transform, the ϕ-Fourier transform, and Fox H-functions. We then prove the local existence and uniqueness of mild solutions.

The chapter further investigates the local existence of solutions to the Cauchy problem for the semilinear time-fractional superdiffusion equation

$$\begin{cases} {}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi} u + (-\Delta)_x^{\delta} u = (\varphi(t) - \varphi(0))^{\sigma}\, \theta(x) + |u|^p, \quad (t,x) \in (0,+\infty) \times \mathbb{R}^d, \\ u(0) = u_0, \quad x \in \mathbb{R}^d, \\ \partial_\varphi u(0) = u_1, \quad x \in \mathbb{R}^d, \end{cases} \tag{3}$$

where $\alpha \in (1,2)$, $\delta \in (0,1)$, $\sigma \in (-1,0)$, and $p > 1$. The operator ${}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi}$ denotes the φ-Caputo derivative, $(-\Delta)_x^{\delta}$ is the fractional Laplacian, and

$$\partial_\varphi u(t) := \frac{1}{\varphi'(t)} \partial_t u(t),$$

denotes the so-called φ-derivative. The functions u_0, u_1, and θ are prescribed initial data. The analytical framework and auxiliary results required for this problem are developed in detail in Chapter 4.

Finally, Chapter 5 addresses the Cauchy problem associated with the time–space fractional superdiffusion equation

$$\begin{cases} {}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi} u + (-\Delta)_x^{\delta} u = |u|^p, \quad (t,x) \in (0,+\infty) \times \mathbb{R}^d, \\ u(0) = u_0, \quad x \in \mathbb{R}^d, \\ \partial_\varphi u(0) = u_1, \quad x \in \mathbb{R}^d, \end{cases} \tag{4}$$

where $\alpha \in (1,2)$, $\delta \in (0,1)$, $p > 1$, and $d \in \mathbb{N}$. The initial data satisfy $u_0, u_1 \in C_0(\mathbb{R}^d)$. In this chapter, we introduce a weak solution formulated within a variational framework and show that every mild solution also satisfies the corresponding weak formulation. Moreover, we derive sufficient conditions guaranteeing that solutions blow up in finite time.

Most of the results presented in this thesis have already been either *published* or *accepted* for publication in peer-reviewed international journals. The results in Chapter 2 were *published* in [9], whereas the results in Chapters 3, 4, and 5 have been *accepted* for publication [4, 5, 10].

List of Notations

Chapter 1

Preliminaries

This chapter collects the basic material that will be used throughout this thesis. We begin by recalling several functional spaces, including weighted L^p spaces, fractional Sobolev spaces, and Lorentz spaces. We then introduce the special functions that appear repeatedly in the sequel, namely the Gamma, Beta, Mittag–Leffler, Fox H-, and Bessel functions. Next, we review the main notions of generalized fractional calculus with respect to a strictly increasing function φ, including the associated fractional integrals and fractional derivatives. Finally, we recall the generalized Laplace transform, the corresponding convolution structure, and a generalized fractional Laplacian together with the Fourier transform associated with a spatial change of variables ϕ.

1.1 Functional spaces and special functions

In this section, we recall the function spaces and special functions that will be used throughout the thesis.

Weighted L^p spaces

We first recall the classical Lebesgue spaces on $\mathbb{R}^d$. For $1 \leq p < \infty$, we define

$$L^p(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{ measurable} \;\middle|\; \int_{\mathbb{R}^d} |f(x)|^p \, \mathrm{d}x < \infty \right\}.$$

For $p = \infty$, we set

$$L^\infty(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{ measurable} \;\middle|\; \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f(x)| < \infty \right\}.$$

These spaces are Banach spaces endowed with their usual norms.

Let $g : \mathbb{R}^d \to [0, \infty)$ be a non-negative measurable weight, with $g(x) > 0$ a.e. in $\mathbb{R}^d$, and let $1 \leq p < \infty$. The weighted space $L^p_g(\mathbb{R}^d)$ is defined by

$$L^p_g(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{ measurable} \;\middle|\; \int_{\mathbb{R}^d} |f(x)|^p g(x) \, \mathrm{d}x < \infty \right\}.$$

Fractional Sobolev spaces

Let $0 < \delta < 1$. The fractional Sobolev space $H^{2\delta}(\mathbb{R}^d)$ is defined by

$$H^{2\delta}(\mathbb{R}^d) = \left\{ v \in \mathcal{S}'(\mathbb{R}^d) \ : \ (1 + |\xi|^2)^{\delta} \widehat{v}(\xi) \in L^2(\mathbb{R}^d) \right\}.$$

It is endowed with the norm

$$\|v\|_{H^{2\delta}(\mathbb{R}^d)} = \left(\|v\|^2_{L^2(\mathbb{R}^d)} + \|(-\Delta)^{\delta}_x v\|^2_{L^2(\mathbb{R}^d)} \right)^{1/2}.$$

Equivalently, $H^{2\delta}(\mathbb{R}^d)$ may be obtained as the completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to this norm.

Lorentz spaces and basic properties

For $1 < r < \infty$ and $1 \le s \le \infty$, the Lorentz space $L^{r,s}(\mathbb{R}^d)$ consists of all measurable functions ψ such that $\|\psi\|_{r,s} < \infty$, where the quasi-norm, equivalent to the standard Lorentz norm, is defined by

$$\|\psi\|_{r,s} := \begin{cases} \left(\displaystyle\int_0^\infty (\eta^{1/r} \psi^{**}(\eta))^s \frac{\mathrm{d}\eta}{\eta} \right)^{1/s}, & 1 \le s < \infty, \\ \sup\limits_{\eta > 0} \eta^{1/r} \psi^{**}(\eta), & s = \infty. \end{cases}$$

Here

$$\psi^{**}(\eta) = \frac{1}{\eta} \int_0^\eta \psi^*(\zeta) \, \mathrm{d}\zeta,$$

and ψ^* denotes the non-increasing rearrangement

$$\psi^*(\eta) = \inf\{\lambda > 0 : \mu(\{x \in \mathbb{R}^d : |\psi(x)| > \lambda\}) \le \eta\}, \quad \eta > 0,$$

where μ denotes the Lebesgue measure on $\mathbb{R}^d$.

It is well known (see, e.g., [11, 31]) that for $1 < r < \infty$,

$$L^{r,r}(\mathbb{R}^d) = L^r(\mathbb{R}^d)$$

with equivalent norms. Note that under the standard definition utilizing the maximal function ψ^{**}, if one sets $r = 1$, the space $L^{1,s}(\mathbb{R}^d)$ reduces to $\{0\}$ for all $1 \le s < \infty$, which implies that $L^{1,1}(\mathbb{R}^d) \ne L^1(\mathbb{R}^d)$. Therefore, to ensure a non-trivial framework and valid norms, we restrict our attention to the range $1 < r < \infty$. Within this setting, $L^{r,s}(\mathbb{R}^d)$ is a Banach space, and if $1 \le s_1 \le r \le s_2 \le \infty$, then we have the continuous embeddings

$$L^{r,1}(\mathbb{R}^d) \subset L^{r,s_1}(\mathbb{R}^d) \subset L^r(\mathbb{R}^d) \subset L^{r,s_2}(\mathbb{R}^d) \subset L^{r,\infty}(\mathbb{R}^d).$$

We also recall that, for $1 < p < \infty$ and $1 < r < \infty$,

$$\||\psi|^p\|_{r,s} = \|\psi\|^p_{rp,rs}.$$

If $1 \le q < s \le \infty$, then there exists a constant $C_{r,s,q} > 0$ such that

$$\|\psi\|_{r,s} \le C_{r,s,q}\|\psi\|_{r,q}.$$

We also recall the following interpolation property of Lorentz spaces. Let $1 < p_1 < p_2 \le \infty$, $0 < \vartheta < 1$, and let $1 \le r_1, r_2, r \le \infty$. If p is defined by

$$\frac{1}{p} = \frac{1-\vartheta}{p_1} + \frac{\vartheta}{p_2},$$

then

$$\Big(L^{p_1,r_1}(\mathbb{R}^d), L^{p_2,r_2}(\mathbb{R}^d)\Big)_{\vartheta,r} = L^{p,r}(\mathbb{R}^d).$$

Here $(\cdot,\cdot)_{\vartheta,r}$ denotes the real interpolation space obtained by the K-method. This result means that Lorentz spaces form an interpolation scale: an intermediate space between $L^{p_1,r_1}(\mathbb{R}^d)$ and $L^{p_2,r_2}(\mathbb{R}^d)$ is again a Lorentz space, whose integrability exponent p is determined by the convex relation above. The additional index r controls the fine scale of integrability inside the Lorentz class. In particular, this property will be useful when estimating nonlinear terms, since it allows one to pass from estimates in two different Lorentz spaces to an estimate in an intermediate Lorentz space. For further properties of Lorentz spaces and real interpolation, we refer to [11, 12, 28, 31].

Proposition 1.1 (Young's inequality in Lorentz spaces [31, 54])**.** *Let $1 < r_1, r_2, r < \infty$, $1 \le s_1, s_2, s \le \infty$ satisfy $1 + \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, and $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$. If $f \in L^{r_1,s_1}$ and $g \in L^{r_2,s_2}$, then*

$$\|f * g\|_{r,s} \le C\|f\|_{r_1,s_1}\|g\|_{r_2,s_2}.$$

Proposition 1.2 (Minkowski's inequality [48])**.** *For $1 \le r, s \le \infty$,*

$$\left\|\int_0^\infty f(\eta,\cdot)\,\mathrm{d}\eta\right\|_{r,s} \le \int_0^\infty \|f(\eta,\cdot)\|_{r,s}\,\mathrm{d}\eta.$$

Euler Gamma function

The Euler gamma function extends the factorial function from positive integers to complex arguments and plays a central role in fractional calculus, since it appears in the kernels of fractional integrals and derivatives.

Definition 1.1 ([36])**.** For $z \in \mathbb{C}$ with $\Re(z) > 0$, the Euler gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t.$$

Proposition 1.3. *The gamma function satisfies:*

1. $\Gamma(z+1) = z\,\Gamma(z)$.

2. *For* $n \in \mathbb{N}_0$, $\Gamma(n+1) = n!$.

3. *For all* $z > 0$, $\Gamma(z) > 0$.

4. Γ *admits an analytic continuation to* $\mathbb{C} \setminus \{0, -1, -2, \dots\}$.

Beta function

Definition 1.2 ([36]). For $z, w \in \mathbb{C}$ with $\Re(z) > 0$ and $\Re(w) > 0$, the beta function is defined by

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,\mathrm{d}t.$$

Proposition 1.4. *The beta function satisfies:*

1. $B(z,w) = \dfrac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$.

2. $B(z,w) = B(w,z)$.

Mittag–Leffler function

Solutions of many fractional differential equations are naturally expressed in terms of the Mittag–Leffler function.

Definition 1.3 ([36]). For $\Re(\alpha) > 0$ and $z \in \mathbb{C}$, the one-parameter Mittag–Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

For $\Re(\alpha) > 0$ and $\beta \in \mathbb{C}$, the two-parameter Mittag–Leffler function is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Proposition 1.5 (Basic properties of the Mittag–Leffler function)**.** *Let* $\Re(\alpha) > 0$. *Then:*

1. E_α *is an entire function.*

2. *Special cases include*

$$E_1(z) = e^z, \qquad E_{1,2}(z) = \frac{e^z - 1}{z}, \qquad E_{\alpha,1}(z) = E_\alpha(z).$$

3. E_α admits a Hankel-type contour representation:

$$E_\alpha(z) = \frac{1}{2\pi i}\int_\Omega \frac{t^{\alpha-1}e^t}{t^\alpha - z}\,\mathrm{d}t,$$

where Ω is a suitable Hankel contour.

Fox H-function

The Fox H-function is one of the most general classes of special functions and extends the Mellin–Barnes representation of the Meijer G-function introduced in the 1930s. Its connection with the φ-Caputo fractional derivative has made it an important tool for studying fractional diffusion models with memory and nonlocal effects.

Definition 1.4 ([49])**.** The Fox H-function is defined as follows:

$$H^{n,m}_{q,p}\left[z\Big|^{(a_1,\alpha_1),\ldots,(a_q,\alpha_q)}_{(b_1,\beta_1),\ldots,(b_p,\beta_p)}\right] = H^{n,m}_{q,p}\left[z\Big|^{(a_{\tilde{p}},\alpha_q)}_{(b_p,\beta_{\tilde{q}})}\right] = \frac{1}{2\pi i}\int_\Omega \pi(s) z^{-s}\mathrm{d}s,$$

where

$$\pi(s) = \frac{\prod_{j=1}^{n}\Gamma\left(b_j+\beta_j s\right)\prod_{k=1}^{m}\Gamma\left(1-a_k-\alpha_k s\right)}{\prod_{k=m+1}^{q}\Gamma\left(a_k+\alpha_k s\right)\prod_{j=n+1}^{p}\Gamma\left(1-b_j-\beta_j s\right)},$$

with $z \neq 0$ and $z^{-s} = \exp\left[-s\left\{\ln|z| + i\arg z\right\}\right]$. Here, $n,m,q,p \in \mathbb{N}$, with the constraints $0 \leq m \leq q$, $1 \leq n \leq p$, $\alpha_k, \beta_j \in \mathbb{R}^+$, and $a_k, b_j \in \mathbb{R}$ or $\mathbb{C}$, for $k = 1,\ldots,q$ and $j = 1,\ldots,p$.

The contour Ω starts at the point $c - i\infty$ and extends to $c + i\infty$, where $c \in \mathbb{R}$, and is defined by the distance between the poles of $\Gamma\left(b_j + \beta_j s\right)$, for $j = 1,\ldots,n$, and those of $\Gamma\left(1 - a_k - \alpha_k s\right)$, for $k = 1,\ldots,m$.

The conditions for the convergence of the integral are as follows: 1. $\alpha^* > 0$, $|\arg z| < \frac{1}{2}\pi\alpha^*$, and $z \neq 0$.

2. $\alpha^* = \rho = 0$, $\mathfrak{Re}(\varrho) < -1$, $\arg z = 0$, and $z \neq 0$, where

$$\alpha^* = \sum_{j=1}^{m}\alpha_j - \sum_{j=m+1}^{q}\alpha_j + \sum_{j=1}^{n}\beta_j - \sum_{j=n+1}^{p}\beta_j,$$

$$\rho = \sum_{j=1}^{p}\beta_j - \sum_{j=1}^{q}\alpha_j,$$

$$\varrho = \sum_{j=1}^{p} b_j - \sum_{j=1}^{q} a_j + \frac{q-p}{2}.$$

Proposition 1.6 ([49])**.** *The following inversion identity holds:*

$$H^{n,m}_{q,p}\left[z\Big|^{(a_q,\alpha_q)}_{(b_p,\beta_p)}\right] = H^{m,n}_{p,q}\left[\frac{1}{z}\Bigg|^{(1-b_p,\beta_p)}_{(1-a_q,\alpha_q)}\right]. \tag{1.1}$$

Bessel function

Definition 1.5 ([49]). The first type of Bessel function J_ι is defined as follows:

$$\mathrm{J}_\iota(w) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+\iota+k)k!} \left(\frac{w}{2}\right)^{\iota+2k},$$

where $w \in \mathbb{C} \setminus (-\infty, 0]$ and $\iota \in \mathbb{C}$.

Next, we assume that $\alpha^* > 0$ or $\alpha^* = \rho = 0$ and $\mathfrak{Re}(\varrho) < -1$. If $\epsilon \in \mathbb{C}$ and $\vartheta > 0$ satisfy the following conditions:

$$\mathfrak{Re}(\epsilon) + \mathfrak{Re}(\iota) + \vartheta \min_{1\leq j\leq n} \left[\frac{\mathfrak{Re}(b_j)}{\beta_j}\right] > -1,$$

and

$$\mathfrak{Re}(\epsilon) + \vartheta \max_{1\leq j\leq m} \left[\frac{1}{\alpha_j} - \frac{\mathfrak{Re}(a_j)}{\alpha_j}\right] < \frac{3}{2},$$

for $\mathfrak{Re}(\iota) > -\frac{1}{2}$, then for $a, b > 0$, the following formula holds:

$$\begin{aligned} &\int_0^\infty x^{\epsilon-1} \mathrm{J}_\iota(ax) H^{n,m}_{q,p}\left[bx^\vartheta \Big|^{(a_q,\alpha_q)}_{(b_p,\beta_p)}\right] \mathrm{d}x \\ &= \frac{2^{\epsilon-1}}{a^\epsilon} H^{n,m+1}_{q+2,p}\left[b\left(\frac{2}{a}\right)^\vartheta \Bigg|^{\left(1-\frac{(\epsilon+\iota)}{2},\frac{\vartheta}{2}\right),(a_q,A_q),\left(1-\frac{(\epsilon-\iota)}{2},\frac{\vartheta}{2}\right)}_{(b_p,B_p)}\right]. \end{aligned} \tag{1.2}$$

Proposition 1.7 ([68]). *For any $\alpha > 0$, $\beta \in \mathbb{C}$, $\vartheta > 0$, and $a \in \mathbb{R}$, the following identity holds:*

$$\int_{\mathbb{R}^d} e^{i<x,\zeta>} E^{(\mathrm{i})}_{\alpha,\beta}\left(-a|\zeta|^\vartheta\right) \mathrm{d}\zeta = (2\pi)^{\frac{d}{2}} |x|^{1-\frac{d}{2}} \int_0^\infty |\zeta|^{\frac{d}{2}} E^{(\mathrm{i})}_{\alpha,\beta}\left(-a|\zeta|^\vartheta\right) \mathrm{J}_{\frac{d}{2}-1}\left(|x||\zeta|\right) \mathrm{d}|\zeta|. \tag{1.3}$$

Here, $E^{(\mathrm{i})}_{\alpha,\beta}$ denotes the i-th derivative of the Mittag-Leffler function (ML function).
The H-function can be used to express the i-th derivative of the Mittag-Leffler function as:

$$E^{(\mathrm{i})}_{\alpha,\beta}(z) = H^{1,1}_{1,2}\left[-z\Big|^{(-\mathrm{i},1)}_{(0,1),(1-(\alpha\mathrm{i}+\beta),\alpha)}\right].$$

1.2 Generalized fractional calculus

In this section, we introduce the φ-fractional integrals and fractional derivatives and present some of their basic properties.

Let $(a,b)(-\infty \le a < b \le \infty)$ be a finite or infinite interval and $n-1 < \alpha < n \in \mathbb{N}$. Assume that $\varphi(t) \in C^n(a,b)$ is a strictly monotone increasing function with $\varphi \to +\infty$ as $t \to +\infty$, satisfying $\varphi'(t) > 0$. We define the function space $AC^n_{\partial_\varphi}[a,b]$ as

$$AC^n_{\partial_\varphi}[a,b] = \left\{ f : [a,b] \to \mathbb{R} \,\middle|\, \partial_\varphi^{n-1} f(t) \in AC[a,b] \right\},$$

where the operator ∂_φ is defined recursively by

$$\partial_\varphi f(t) = \left(\frac{1}{\varphi'(t)} \frac{\mathrm{d}}{\mathrm{d}t} \right) f(t), \quad \partial_\varphi^{n-1} f(t) = \partial_\varphi \left(\partial_\varphi^{n-2} f(t) \right), \quad \text{with} \quad \partial_\varphi^0 f(t) = f(t).$$

φ-fractional integrals

Definition 1.6 ([43, 55]). Let $f \in L^1(a,b)$. The left-sided φ-fractional integral of order $\alpha > 0$ is defined by

$$\mathrm{I}^{\alpha,\varphi}_{a,t} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \varphi'(\tau) \left(\varphi(t) - \varphi(\tau)\right)^{\alpha-1} f(\tau) \, \mathrm{d}\tau, \quad t > a.$$

Similarly, the right-sided φ-fractional integral is defined as

$$\mathrm{I}^{\alpha,\varphi}_{t,b} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \varphi'(\tau) \left(\varphi(\tau) - \varphi(t)\right)^{\alpha-1} f(\tau) \, \mathrm{d}\tau, \quad t < b.$$

Remark 1.1. Special cases include:

- $\varphi(t) = t$: yields the classical left- and right-sided Riemann–Liouville fractional integrals:

$$^{RL}\mathrm{I}^{\alpha}_{a,t} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) \, \mathrm{d}\tau,$$

  and

$$^{RL}\mathrm{I}^{\alpha}_{t,b} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau) \, \mathrm{d}\tau.$$

- $\varphi(t) = \log t$ with $a > 0$: gives the Hadamard fractional integrals:

$$^{H}\mathrm{I}^{\alpha}_{a,t} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} f(\tau) \frac{\mathrm{d}\tau}{\tau}, \quad t > a > 0.$$

and

$$^{H}\mathrm{I}^{\alpha}_{t,b}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\log\frac{\tau}{t}\right)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad b > t > a > 0.$$

- $\varphi(t) = t^{\sigma}$ with $a > 0$ and $\sigma \geq 0$: corresponds to the Erdélyi–Kober fractional integrals:

$$^{EK}\mathrm{I}^{\alpha}_{a,t}f(t) = \frac{\sigma}{\Gamma(\alpha)}\int_a^t (t^{\sigma} - \tau^{\sigma})^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\sigma}},$$

and

$$^{EK}\mathrm{I}^{\alpha}_{t,b}f(t) = \frac{\sigma}{\Gamma(\alpha)}\int_t^b (\tau^{\sigma} - t^{\sigma})^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\sigma}}.$$

Example 1.1. Let $\alpha > 0$ and $\beta > 0$. We have

1. $\mathrm{I}^{\alpha,\varphi}_{a,t}(\varphi(t) - \varphi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(\varphi(t) - \varphi(a))^{\alpha+\beta-1}$,

2. $\mathrm{I}^{\alpha,\varphi}_{t,b}(\varphi(b) - \varphi(t))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(\varphi(b) - \varphi(t))^{\alpha+\beta-1}$.

Proof. 1. Starting from the definition of the fractional integral,

$$\mathrm{I}^{\alpha,\varphi}_{a,t}(\varphi(t) - \varphi(a))^{\beta-1} = \frac{1}{\Gamma(\alpha)}\int_a^t \varphi'(\tau)\,(\varphi(t) - \varphi(\tau))^{\alpha-1}(\varphi(\tau) - \varphi(a))^{\beta-1}\,\mathrm{d}\tau,$$

we introduce the substitution

$$x = \frac{\varphi(\tau) - \varphi(a)}{\varphi(t) - \varphi(a)}.$$

This implies that $x = 0$ when $\tau = a$ and $x = 1$ when $\tau = t$, and

$$\varphi(\tau) - \varphi(a) = (\varphi(t) - \varphi(a))\,x, \qquad \varphi'(\tau)\,\mathrm{d}\tau = (\varphi(t) - \varphi(a))\,\mathrm{d}x,$$

together with

$$\varphi(t) - \varphi(\tau) = (\varphi(t) - \varphi(a))(1 - x).$$

Substituting these expressions into the integral yields

$$\mathrm{I}^{\alpha,\varphi}_{a,t}(\varphi(t) - \varphi(a))^{\beta-1} = \frac{1}{\Gamma(\alpha)}(\varphi(t) - \varphi(a))^{\alpha+\beta-1}\int_0^1 x^{\beta-1}(1 - x)^{\alpha-1}\,\mathrm{d}x.$$

The remaining integral is the Beta function,

$$\int_0^1 x^{\beta-1}(1 - x)^{\alpha-1}\,\mathrm{d}x = B(\beta, \alpha)$$

Hence,

$$\mathrm{I}^{\alpha,\varphi}_{a,t}(\varphi(t) - \varphi(a))^{\beta-1} = \frac{1}{\Gamma(\alpha)}(\varphi(t) - \varphi(a))^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Simplifying gives the asserted relation,

$$\mathrm{I}_{a,t}^{\alpha,\varphi}(\varphi(t)-\varphi(a))^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,(\varphi(t)-\varphi(a))^{\alpha+\beta-1}.$$

2. The same steps as in 1.

□

Remark 1.2. 1. When $\varphi(t)=t$, we obtain

$$\begin{aligned}{}^{RL}\mathrm{I}_{a,t}^{\alpha}(t-a)^{\beta-1}&=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,(t-a)^{\alpha+\beta-1},\\ {}^{RL}\mathrm{I}_{t,b}^{\alpha}(b-t)^{\beta-1}&=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,(b-t)^{\alpha+\beta-1}.\end{aligned}$$

2. When $\varphi(t)=\log t$, we obtain

$$\begin{aligned}{}^{H}\mathrm{I}_{a,t}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}&=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,\left(\log\frac{t}{a}\right)^{\alpha+\beta-1},\\ {}^{H}\mathrm{I}_{t,b}^{\alpha}\left(\log\frac{b}{t}\right)^{\beta-1}&=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,\left(\log\frac{b}{t}\right)^{\alpha+\beta-1}.\end{aligned}$$

3. When $\varphi(t)=t^{\sigma}$, we obtain

$$\begin{aligned}{}^{EK}\mathrm{I}_{a,t}^{\alpha}(t^{\sigma}-a^{\sigma})^{\beta-1}&=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,(t^{\sigma}-a^{\sigma})^{\alpha+\beta-1},\\ {}^{EK}\mathrm{I}_{t,b}^{\alpha}(b^{\sigma}-t^{\sigma})^{\beta-1}&=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,(b^{\sigma}-t^{\sigma})^{\alpha+\beta-1}.\end{aligned}$$

Proposition 1.8. *The following semigroup property is valid for fractional integrals: if $\alpha>0$ and $\beta>0$. then*

1. $\mathrm{I}_{a,t}^{\alpha,\varphi}\,\mathrm{I}_{a,t}^{\beta,\varphi}f(t)=\mathrm{I}_{a,t}^{\alpha+\beta,\varphi}f(t)$,
2. $\mathrm{I}_{t,b}^{\alpha,\varphi}\,\mathrm{I}_{t,b}^{\beta,\varphi}f(t)=\mathrm{I}_{t,b}^{\alpha+\beta,\varphi}f(t)$.

Proof. 1. We have

$$\mathrm{I}_{a,t}^{\alpha+\beta,\varphi}f(t)=\frac{1}{\Gamma(\alpha+\beta)}\int_a^t\varphi'(\tau)(\varphi(t)-\varphi(\tau))^{\alpha+\beta-1}\,f(\tau)\,\mathrm{d}\tau.$$

On the other hand,

$$\mathrm{I}_{a,t}^{\alpha,\varphi}\mathrm{I}_{a,t}^{\beta,\varphi}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\varphi'(\tau)(\varphi(t)-\varphi(\tau))^{\alpha-1}\,\mathrm{I}_{a,t}^{\beta,\varphi}f(\tau)\,\mathrm{d}\tau.$$

Substituting yields

$$\mathrm{I}_{a,t}^{\alpha,\varphi}\mathrm{I}_{a,t}^{\beta,\varphi}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(\varphi(t)-\varphi(\tau))^{\alpha-1}\varphi'(\tau)\frac{1}{\Gamma(\beta)}\int_a^\tau\varphi'(s)(\varphi(\tau)-\varphi(s))^{\beta-1}f(s)\,\mathrm{d}s\,\mathrm{d}\tau.$$

Thus,

$$\mathrm{I}_{a,t}^{\alpha,\varphi}\mathrm{I}_{a,t}^{\beta,\varphi}f(t)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^\tau\varphi'(\tau)\varphi'(s)(\varphi(t)-\varphi(\tau))^{\alpha-1}(\varphi(\tau)-\varphi(s))^{\beta-1}f(s)\,\mathrm{d}s\,\mathrm{d}\tau.$$

We now apply the Fubini-Tonelli theorem to interchange the order of integration:

$$\mathrm{I}_{a,t}^{\alpha,\varphi}\mathrm{I}_{a,t}^{\beta,\varphi}f(t)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_s^t\varphi'(\tau)\varphi'(s)(\varphi(t)-\varphi(\tau))^{\alpha-1}(\varphi(\tau)-\varphi(s))^{\beta-1}f(s)\,\mathrm{d}\tau\,\mathrm{d}s.$$

Equivalently,

$$\mathrm{I}_{a,t}^{\alpha,\varphi}\mathrm{I}_{a,t}^{\beta,\varphi}f(t)=\frac{1}{\Gamma(\beta)}\int_a^t\varphi'(s)\,f(s)\left(\frac{1}{\Gamma(\alpha)}\int_s^t(\varphi(t)-\varphi(\tau))^{\alpha-1}(\varphi(\tau)-\varphi(s))^{\beta-1}\varphi'(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Therefore,

$$\mathrm{I}_{a,t}^{\alpha,\varphi}\mathrm{I}_{a,t}^{\beta,\varphi}f(t)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\varphi'(s)\,f(s)\,\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(\varphi(t)-\varphi(s))^{\alpha+\beta-1}\,\mathrm{d}s,$$

which simplifies to

$$\mathrm{I}_{a,t}^{\alpha,\varphi}\mathrm{I}_{a,t}^{\beta,\varphi}f(t)=\frac{1}{\Gamma(\alpha+\beta)}\int_a^t(\varphi(t)-\varphi(s))^{\alpha+\beta-1}\varphi'(s)\,f(s)\,\mathrm{d}s.$$

This is precisely the definition of $I_{a,t}^{\alpha+\beta,\varphi}f(t)$, and thus

$$\mathrm{I}_{a,t}^{\alpha,\varphi}\mathrm{I}_{a,t}^{\beta,\varphi}f(t)=\mathrm{I}_{a,t}^{\alpha+\beta,\varphi}f(t),$$

as asserted. □

φ-fractional derivatives

Definition 1.7 ([43, 55])**.** Assume that a function $f \in AC^n_{\partial_\varphi}[a,b]$. The left-sided φ-Riemann–Liouville fractional derivative of order α is defined by

$$\mathrm{D}^{\alpha,\varphi}_{a,t} f = \frac{1}{\Gamma(n-\alpha)}\, \partial^n_\varphi \int_a^t \varphi'(\tau)(\varphi(t)-\varphi(\tau))^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau, \qquad t > a,$$

Similarly, the right-sided φ-Riemann–Liouville fractional derivative is defined as

$$\mathrm{D}^{\alpha,\varphi}_{t,b} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\, \partial^n_\varphi \int_t^b \varphi'(\tau)(\varphi(\tau)-\varphi(t))^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau, \qquad t < b.$$

Remark 1.3. Special cases include:

- $\varphi(t) = t$: yields the classical left- and right-sided Riemann–Liouville fractional derivatives:

$$^{RL}\mathrm{D}^{\alpha}_{a,t} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau,$$

  and

$$^{RL}\mathrm{D}^{\alpha}_{t,b} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau.$$

- $\varphi(t) = \log t$ with $a > 0$: gives the Hadamard fractional derivatives:

$$^{H}\mathrm{D}^{\alpha}_{a,t} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \left(\log\frac{t}{\tau}\right)^{n-\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad t > a > 0.$$

  and

$$^{H}\mathrm{D}^{\alpha}_{t,b} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b \left(\log\frac{\tau}{t}\right)^{n-\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad b > t > a > 0.$$

- $\varphi(t) = t^\sigma$ with $a > 0$ and $\sigma \geq 0$: corresponds to the Erdélyi–Kober fractional derivatives:

$$^{EK}\mathrm{D}^{\alpha}_{a,t} f(t) = \frac{\sigma}{\Gamma(n-\alpha)} \left(\sigma^{-1}t^{1-\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t^\sigma-\tau^\sigma)^{n-\alpha-1} f(\tau)\,\frac{\mathrm{d}\tau}{\tau^{1-\sigma}},$$

  and

$$^{EK}\mathrm{D}^{\alpha}_{t,b} f(t) = \frac{\sigma(-1)^n}{\Gamma(n-\alpha)} \left(\sigma^{-1}t^{1-\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b (\tau^\sigma-t^\sigma)^{n-\alpha-1} f(\tau)\,\frac{\mathrm{d}\tau}{\tau^{1-\sigma}}.$$

Example 1.2. Let $\alpha > 0$, $n = \lceil\alpha\rceil$, and $\beta > n$. Then

1. $\mathrm{D}^{\alpha,\varphi}_{a,t}\,(\varphi(t)-\varphi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\varphi(t)-\varphi(a))^{\beta-\alpha-1}$,

2. $\mathrm{D}^{\alpha,\varphi}_{t,b}\,(\varphi(b)-\varphi(t))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\varphi(b)-\varphi(t))^{\beta-\alpha-1}$.

Proof. proof of 1: We begin with

$$\mathrm{D}_{a,t}^{\alpha,\varphi}(\varphi(t)-\varphi(a))^{\beta-1}=\partial_{\varphi}^{n}\mathrm{I}_{a,t}^{n-\alpha,\varphi}(\varphi(t)-\varphi(a))^{\beta-1}.$$

We have

$$\mathrm{I}_{a,t}^{n-\alpha,\varphi}(\varphi(t)-\varphi(a))^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+n)}(\varphi(t)-\varphi(a))^{\beta-\alpha+n-1}.$$

Hence,

$$\mathrm{D}_{a,t}^{\alpha,\varphi}(\varphi(t)-\varphi(a))^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+n)}\partial_{\varphi}^{n}(\varphi(t)-\varphi(a))^{\beta-\alpha+n-1}.$$

Set

$$\partial_{\varphi}^{i}(\varphi(t)-\varphi(a))^{\beta-\alpha+n-1}\qquad \text{for } i=1,\dots,n.$$

Step $i=1$

$$\partial_{\varphi}(\varphi(t)-\varphi(a))^{\beta-\alpha+n-1}=\frac{1}{\varphi'(t)}(\beta-\alpha+n-1)(\varphi(t)-\varphi(a))^{\beta-\alpha+n-2}\varphi'(t),$$

so

$$\partial_{\varphi}(\varphi(t)-\varphi(a))^{\beta-\alpha+n-1}=(\beta-\alpha+n-1)(\varphi(t)-\varphi(a))^{\beta-\alpha+n-2}.$$

Step $i=2$

$$\partial_{\varphi}^{2}(\varphi(t)-\varphi(a))^{\beta-\alpha+n-1}=(\beta-\alpha+n-1)(\beta-\alpha+n-2)(\varphi(t)-\varphi(a))^{\beta-\alpha+n-3}.$$

Proceeding inductively, for $i=n$ we obtain

$$\partial_{\varphi}^{n}(\varphi(t)-\varphi(a))^{\beta-\alpha+n-1}=(\beta-\alpha+n-1)(\beta-\alpha+n-2)\cdots(\beta-\alpha)(\varphi(t)-\varphi(a))^{\beta-\alpha-1}.$$

Thus,

$$\partial_{\varphi}^{n}(\varphi(t)-\varphi(a))^{\beta-\alpha+n-1}=\frac{\Gamma(\beta-\alpha+n)}{\Gamma(\beta-\alpha)}(\varphi(t)-\varphi(a))^{\beta-\alpha-1}.$$

We may justify the preceding product formula by induction, using the identity

$$\Gamma(\beta-\alpha+n)=\Gamma\Big((\beta-\alpha+n-1)+1\Big)=(\beta-\alpha+n-1)\,\Gamma(\beta-\alpha+n-1).$$

Applying this repeatedly, we obtain

$$\Gamma(\beta-\alpha+n)=(\beta-\alpha+n-1)(\beta-\alpha+n-2)\cdots(\beta-\alpha)\,\Gamma(\beta-\alpha).$$

Hence,

$$(\beta-\alpha+n-1)\cdots(\beta-\alpha)=\frac{\Gamma(\beta-\alpha+n)}{\Gamma(\beta-\alpha)}.$$

Therefore,

$$\partial_{\varphi}^{n}(\varphi(t)-\varphi(a))^{\beta-\alpha+n-1}=\frac{\Gamma(\beta-\alpha+n)}{\Gamma(\beta-\alpha)}(\varphi(t)-\varphi(a))^{\beta-\alpha-1}.$$

Substituting this expression into the earlier formula yields

$$\mathrm{D}_{a,t}^{\alpha,\varphi}(\varphi(t)-\varphi(a))^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\varphi(t)-\varphi(a))^{\beta-\alpha-1},$$

which completes the proof. □

Remark 1.4. 1. When $\varphi(t)=t$, we obtain

$$\begin{aligned}{}^{RL}\mathrm{D}_{a,t}^{\alpha}\,(t-a)^{\beta-1}&=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\,(t-a)^{\beta-\alpha-1},\\ {}^{RL}\mathrm{D}_{t,b}^{\alpha}\,(b-t)^{\beta-1}&=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\,(b-t)^{\beta-\alpha-1}.\end{aligned}$$

2. When $\varphi(t)=\log t$, we obtain

$$\begin{aligned}{}^{H}\mathrm{D}_{a,t}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}&=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{t}{a}\right)^{\beta-\alpha-1},\\ {}^{H}\mathrm{D}_{t,b}^{\alpha}\left(\log\frac{b}{t}\right)^{\beta-1}&=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{b}{t}\right)^{\beta-\alpha-1}.\end{aligned}$$

3. When $\varphi(t)=t^{\sigma}$, we obtain

$$\begin{aligned}{}^{EK}\mathrm{D}_{a,t}^{\alpha}\,(t^{\sigma}-a^{\sigma})^{\beta-1}&=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\,(t^{\sigma}-a^{\sigma})^{\beta-\alpha-1},\\ {}^{EK}\mathrm{D}_{t,b}^{\alpha}\,(b^{\sigma}-t^{\sigma})^{\beta-1}&=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\,(b^{\sigma}-t^{\sigma})^{\beta-\alpha-1}.\end{aligned}$$

Definition 1.8. Suppose that $f\in AC^{n}_{\partial_{\varphi}}[a,b]$. The left-sided φ-Caputo fractional derivative of order α, with $n-1<\alpha<n$ and $n\in\mathbb{N}$, is defined by

$${}^{C}\mathrm{D}_{a,t}^{\alpha,\varphi}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}\varphi'(\tau)\,(\varphi(t)-\varphi(\tau))^{n-\alpha-1}\,\partial_{\varphi}^{n}f(\tau)\,\mathrm{d}\tau,\quad t>a,$$

Similarly, the right-sided φ-Caputo fractional derivative is defined as

$${}^{C}\mathrm{D}_{t,b}^{\alpha,\varphi}f(t)=\frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{t}^{b}\varphi'(\tau)\,(\varphi(\tau)-\varphi(t))^{n-\alpha-1}\,\partial_{\varphi}^{n}f(\tau)\,\mathrm{d}\tau,\quad t<b.$$

Remark 1.5. The left-sided φ-Caputo fractional derivative was originally introduced by Kilbas et al. [36] (see Equation (2.7.35), p. 115), although it was not explicitly named or analyzed under the condition $\varphi\to+\infty$ as $t\to+\infty$.

Remark 1.6. For specific choices of φ, the operators reduce to known fractional derivatives:

- $\varphi(t) = t$: yields the classical Caputo fractional derivatives [36, 58];

$$^{C}\mathrm{D}^{\alpha}_{a,t} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, \mathrm{d}\tau,$$

and

$$^{C}\mathrm{D}^{\alpha}_{t,b} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} f^{(n)}(\tau)\, \mathrm{d}\tau.$$

- $\varphi(t) = \log t$: gives the Caputo–Hadamard fractional derivatives [36, 58];

$$^{CH}\mathrm{D}^{\alpha}_{a,t} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log\frac{t}{\tau}\right)^{n-\alpha-1} \left(\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n f(\tau)\, \frac{\mathrm{d}\tau}{\tau},$$

and

$$^{CH}\mathrm{D}^{\alpha}_{t,b} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \left(\log\frac{\tau}{t}\right)^{n-\alpha-1} \left(\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n f(\tau)\, \frac{\mathrm{d}\tau}{\tau}.$$

- $\varphi(t) = e^t$: corresponds to the exponential Caputo derivatives as defined in [42].

$$^{Ce}\mathrm{D}^{\alpha}_{a,t} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (e^t - e^\tau)^{n-\alpha-1} \left(e^{-\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n f(\tau)\, \frac{\mathrm{d}\tau}{e^{-\tau}},$$

and

$$^{Ce}\mathrm{D}^{\alpha}_{t,b} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (e^\tau - e^t)^{n-\alpha-1} \left(e^{-\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n f(\tau)\, \frac{\mathrm{d}\tau}{e^{-\tau}}.$$

Example 1.3. Let $\alpha > 0$, $n = \lceil\alpha\rceil$, and $\beta > n$. Then

1. $^{C}\mathrm{D}^{\alpha,\varphi}_{a,t} (\varphi(t) - \varphi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\varphi(t) - \varphi(a))^{\beta-\alpha-1}$,

2. $^{C}\mathrm{D}^{\alpha,\varphi}_{t,b} (\varphi(b) - \varphi(t))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\varphi(b) - \varphi(t))^{\beta-\alpha-1}$.

Proposition 1.9. *Let $f \in C^n[a,b]$ and $\alpha > 0$. Then*

$$\mathrm{I}^{\alpha,\varphi}_{a,t}\, {}^{C}\mathrm{D}^{\alpha,\varphi}_{a,t} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{\partial^k_\varphi f(a)}{k!} (\varphi(t) - \varphi(a))^k,$$

and

$$\mathrm{I}^{\alpha,\varphi}_{t,b}\, {}^{C}\mathrm{D}^{\alpha,\varphi}_{t,b} f(t) = f(t) - \sum_{k=0}^{n-1} (-1)^k \frac{\partial^k_\varphi f(b)}{k!} (\varphi(b) - \varphi(t))^k.$$

Proof. Using the semigroup law and the integration by parts formula repeatedly, we get

$$\begin{aligned}
\mathrm{I}_{a,t}^{\alpha,\varphi}\, {}^{C}\mathrm{D}_{a,t}^{\alpha,\varphi} f(t) &= \mathrm{I}_{a,t}^{\alpha,\varphi}\mathrm{I}_{a,t}^{n-\alpha,\varphi}\partial_{\varphi}^{n} f(t) = \mathrm{I}_{a,t}^{n,\varphi}\partial_{\varphi}^{n} f(t) \\
&= \frac{1}{(n-1)!}\int_a^t \varphi'(\tau)\left(\varphi(t)-\varphi(\tau)\right)^{n-1}\partial_{\varphi}^{n} f(\tau)\,\mathrm{d}\tau \\
&= \frac{1}{(n-1)!}\int_a^t \left(\varphi(t)-\varphi(\tau)\right)^{n-1}\frac{\mathrm{d}}{\mathrm{d}\tau}\partial_{\varphi}^{n-1} f(\tau)\,\mathrm{d}\tau \\
&= \frac{1}{(n-2)!}\int_a^t \left(\varphi(t)-\varphi(\tau)\right)^{n-2}\frac{\mathrm{d}}{\mathrm{d}\tau}\partial_{\varphi}^{n-2} f(\tau)\,\mathrm{d}\tau \\
&\quad - \frac{\partial_{\varphi}^{n-1} f(a)}{(n-1)!}\left(\varphi(t)-\varphi(a)\right)^{n-1} \\
&= \frac{1}{(n-3)!}\int_a^t \left(\varphi(t)-\varphi(\tau)\right)^{n-3}\frac{\mathrm{d}}{\mathrm{d}\tau}\partial_{\varphi}^{n-3} f(\tau)\,\mathrm{d}\tau \\
&\quad - \sum_{k=n-2}^{n-1}\frac{\partial_{\varphi}^{k} f(a)}{k!}\left(\varphi(t)-\varphi(a)\right)^{k} \\
&= \ldots = \int_a^t \frac{\mathrm{d}}{\mathrm{d}\tau} f(\tau)\,\mathrm{d}\tau - \sum_{k=1}^{n-1}\frac{\partial_{\varphi}^{k} f(a)}{k!}\left(\varphi(t)-\varphi(a)\right)^{k} \\
&= f(t) - \sum_{k=0}^{n-1}\frac{\partial_{\varphi}^{k} f(a)}{k!}\left(\varphi(t)-\varphi(a)\right)^{k}.
\end{aligned}$$

□

1.3 Generalized Laplace transform

Definition 1.9 ([32])**.** Let $\varphi, f : [0,\infty) \to \mathbb{R}$ be real-valued functions such that φ is continuous and $\varphi' > 0$ for all $t \in [0,\infty)$. The generalized Laplace transform of f is defined as

$$\mathcal{L}_{\varphi}\{f(t)\}(\omega) = \int_0^{\infty} e^{-\omega(\varphi(t)-\varphi(0))} f(t)\varphi'(t)\,\mathrm{d}t$$

where the integral is valid for all values of ω. In particular, this definition coincides with the classical Laplace transform when $\varphi(t) = t$.

Corollary 1.1 (See [22])**.** *Let $f, \varphi : [0,\infty) \to \mathbb{R}$ be real-valued functions such that $\varphi(t)$ is continuous and $\varphi'(t) > 0$ for all $t \in [0,\infty)$. The inverse generalized Laplace transform is defined by*

$$\mathcal{L}_{\varphi}^{-1}\{F(\omega)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\omega(\varphi(t)-\varphi(0))} F(\omega)\,\mathrm{d}\omega,$$

where $F(\omega) = \mathcal{L}_{\varphi}\{f(t)\}(\omega)$.

Lemma 1.1 ([22]). *Let $\alpha > 0$, and $\left|\frac{\lambda}{w^\alpha}\right| < 1$. Then, the following relations hold:*

$$\mathcal{L}_\varphi \left\{E_{\alpha,1}\left(\lambda(\varphi(t)-\varphi(0))^\alpha\right)\right\} = \frac{w^{\alpha-1}}{w^\alpha - \lambda},$$

and

$$\mathcal{L}_\varphi \left\{(\varphi(t)-\varphi(0))^{\beta-1} E_{\alpha,\beta}\left(\lambda(\varphi(t)-\varphi(0))^\alpha\right)\right\} = \frac{w^{\alpha-\beta}}{w^\alpha - \lambda}.$$

Definition 1.10 ([32]). A function $f : [0,\infty) \to \mathbb{R}$ is said to be of φ-exponential order if there exist non-negative constants M, c, and T such that

$$|f(t)| \leq M e^{c\varphi(t)}$$

for all $t \geq T$.

Corollary 1.2 ([32]). *Let $\alpha > 0$, and $f \in AC_\varphi^n[a,b]$ with the derivatives ∂_φ^k , for $k = 0, 1, \ldots, n$, being of φ-exponential order. Then, the following holds:*

$$\mathcal{L}_\varphi \left\{\left({}^C\mathrm{D}_{0,t}^{\alpha,\varphi} f\right)(t)\right\}(w) = w^\alpha \left[\mathcal{L}_\varphi\{f(t)\}(w) - \sum_{k=0}^{n-1} w^{-k-1}\left(\partial_\varphi^k f\right)(0)\right].$$

Definition 1.11 ([32]). Let f and g be two functions that are piecewise continuous on each interval $[0,T]$ and of exponential order. The generalized convolution of f and g is defined by

$$(f *_\varphi g)(t) = \int_0^t f(\tau) g\left(\varphi^{-1}\left(\varphi(t) + \varphi(0) - \varphi(\tau)\right)\right)\varphi'(\tau)\,\mathrm{d}\tau.$$

Furthermore, the generalized convolution of two functions is commutative.

Theorem 1.1 ([32]). *Let f and g be two functions that are piecewise continuous on each interval $[0,T]$ and of exponential order. Then, the following relationship holds:*

$$\mathcal{L}_\varphi\{f *_\varphi g\} = \mathcal{L}_\varphi\{f\}\mathcal{L}_\varphi\{g\}.$$

Proposition 1.10 (The Young inequality for φ-convolutions, [23]). *If $s_1, s_2, s_3 \in [1,\infty]$ with $\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s_3} + 1$, $f \in L^{s_1}\left(\mathbb{R}^d\right)$ and $g \in L^{s_2}\left(\mathbb{R}^d\right)$, then $f *_\varphi g \in L^{s_3}\left(\mathbb{R}^d\right)$ with*

$$\|f *_\varphi g\|_{L^{s_3}\left(\mathbb{R}^d\right)} \leq \|f\|_{L^{s_1}\left(\mathbb{R}^d\right)} \|g\|_{L^{s_2}\left(\mathbb{R}^d\right)}.$$

Proposition 1.11 (Gagliardo-Nirenberg inequality, [53]). *Let $u \in W^{1,r}\left(\mathbb{R}^d\right)$. The inequality is valid*

$$\|u\|_{L^m\left(\mathbb{R}^d\right)} \leqslant M \|\nabla u\|_{L^r\left(\mathbb{R}^d\right)}^\theta \|u\|_{L^r\left(\mathbb{R}^d\right)}^{1-\theta}, \tag{1.4}$$

for $\frac{1}{m} = \frac{1}{r} - \frac{\theta}{d}$ and $\theta \in]0,1[$.

1.4 Generalized fractional Laplacian and Fourier transform

The d-dimensional fractional Laplacian with respect to a function ϕ is defined as [23]:

$$(-\Delta)^{\delta}_{\phi(x)} v(x) = C_{d,\delta}\,\mathrm{P.V.}\int_{\mathbb{R}^d} \frac{v(x)-v(y)}{|\phi(x)-\phi(y)|^{d+2\delta}}\,|J_\phi(y)|\,\mathrm{d}y,$$

where $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a bijection whose first-order partial derivatives exist almost everywhere, and J_ϕ denotes the Jacobian determinant. A commonly used constant is

$$C_{d,\delta} := \frac{\delta 2^{2\delta}\Gamma\left(\delta+\frac{d}{2}\right)}{\pi^{d/2}\Gamma(1-\delta)}.$$

Remark 1.7. when $\phi(x) = x$, this reduces to the standard fractional Laplacian.

The fractional Laplacian $(-\Delta)^{\delta}_{x}$ is defined as follows [19, 58]:

$$\begin{aligned}(-\Delta)^{\delta}_{x} v(x) &= C_{d,\delta}\,\mathrm{P.V.}\int_{\mathbb{R}^d} \frac{v(x)-v(y)}{|x-y|^{d+2\delta}}\,\mathrm{d}y\\ &= -\frac{1}{2}C_{d,\delta}\int_{\mathbb{R}^d} \frac{v(\mathrm{x}-\mathrm{y})-2v(x)+v(\mathrm{x}+\mathrm{y})}{|y|^{d+2\delta}}\,\mathrm{d}y, \quad x\in\mathbb{R}^d,\end{aligned}$$

where P.V. denotes the Cauchy principal value. One possible normalization is

$$C_{d,\delta} := \frac{\delta 2^{2\delta}\Gamma\left(\delta+\frac{d}{2}\right)}{\pi^{d/2}\Gamma(1-\delta)}.$$

Numerical illustrations of the fractional Laplacian are postponed to Appendix A.

Definition 1.12. (See [23].) Let $d \in \mathbb{N}$, and let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a bijection whose first-order partial derivatives exist almost everywhere. The d-dimensional Fourier transform with respect to ϕ of a function f is defined by

$$\widehat{f}(\zeta) = [\mathcal{F}_\phi f](\zeta) := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-i\langle\phi(x),\zeta\rangle} f(x)\,|J_\phi(x)|\,\mathrm{d}x,$$

provided that the integral exists. The inverse transform is

$$f(x) = \left[\mathcal{F}_\phi^{-1}\widehat{f}\right](x) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{i\langle\phi(x),\zeta\rangle}\widehat{f}(\zeta)\,\mathrm{d}\zeta, \quad x\in\mathbb{R}^d.$$

Remark 1.8. when $\phi(x) = x$, this reduces to the standard Fourier transform.

Definition 1.13. Let f and g be functions in $L^1\left(\mathbb{R}^d, \mathrm{d}\phi\right)$. The Fourier-type ϕ-convolution is defined by

$$(f *_\phi g)(x) = \int_{\mathbb{R}^d} f\left(\phi^{-1}\Big(\phi(x) - \phi(y)\Big)\right) g(y)\, |J_\phi(y)|\, \mathrm{d}y.$$

Theorem 1.2. *(See [23].) Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a bijection whose first-order partial derivatives exist almost everywhere. Then*

$$\mathcal{F}_\phi\left[(-\Delta)^\delta_{\phi(x)} f(x)\right] = |\zeta|^{2\delta}\, \widehat{f}(\zeta),$$

for any $f \in L^p\left(\mathbb{R}^d, \mathrm{d}\phi\right)$, where $0 < \delta < 1$ and $1 \leq p \leq 2$.

Chapter 2

Global existence for a generalized fractional diffusion equation with gradient nonlinearity

In this chapter, we study the global existence of solutions to the following semilinear time–space fractional diffusion equation:

$$\begin{cases} {}^C\mathrm{D}_{0,t}^{\alpha,\varphi} u + (-\Delta)_x^{\delta} u = a|u|^{p-1}u + b|\nabla u|^q, & (t,x) \in (0,+\infty) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{2.1}$$

where $d \geq 1$, $\alpha \in (0,1)$, $\delta \in (0,1)$, $p > 1$, and

$$q = \frac{2\delta p}{(p-1) + 2\delta}.$$

The constants $a, b \in \mathbb{R}$ are prescribed, and the initial data satisfies $u_0 \in L^s(\mathbb{R}^d)$, where

$$s = \frac{d}{2\delta}(p-1) > 1.$$

Problems of this type are closely related to the classical theory of semilinear heat equations. In the particular case $\alpha = 1$, $\varphi(t) = t$, $\delta = 1$, $b = 0$, and $a = 1$, problem (2.1) reduces to the well-known Fujita equation. Fujita [25] showed that the exponent $p = 1 + \frac{2}{d}$ plays a decisive role in the global existence theory. Later, Weissler [64] refined this approach by proving global existence for sufficiently small initial data in the critical Lebesgue space $L^{s_c}(\mathbb{R}^d)$, where $s_c = \frac{d}{2}(p-1) > 1$.

The presence of the gradient nonlinearity connects the present problem with the model studied by Snoussi and Tayachi [62], corresponding to the non-fractional case $\alpha = 1$, $\varphi(t) = t$, and $\delta = 1$. They established the existence of mild global solutions under suitable smallness assumptions on the initial data, with the critical gradient exponent $q_* = \frac{2p}{p+1}$. Thus, the equation considered here extends this framework by introducing both a generalized time-fractional derivative and a fractional Laplacian, while preserving the critical balance between the source term and the gradient term.

Several related models are also included as special cases of (2.1). For instance, when $a = 0$

and $b \neq 0$, the equation is related to the viscous Hamilton–Jacobi equation, which appears in stochastic control theory and various physical models [3, 8]. Further results on global solutions, asymptotic behavior, and critical gradient effects were obtained in [61, 63], while extensions to bounded domains and applications to population dynamics were discussed in [17, 18].

More recently, fractional diffusion equations with memory effects have attracted considerable attention. Viana and Gomes [27] studied nonlinear diffusion equations involving convolution-type memory terms and obtained sufficient conditions for global existence, positivity, symmetry, and long-time behavior of solutions. In a related direction, Li and Li [40] investigated a semilinear time–space fractional diffusion equation involving the Caputo–Hadamard derivative and the fractional Laplacian, proving global existence for small initial data in a critical Lebesgue space. These works highlight the importance of fractional operators, critical spaces, and small-data assumptions in the study of global solutions.

Motivated by these developments, we first derive the fundamental solution of the associated linear problem by means of the φ-Laplace transform and the Fox H-function. We then establish decay estimates in Lebesgue spaces. Finally, under a suitable smallness assumption on the initial datum, we prove global existence, uniqueness, and continuous dependence of mild solutions to (2.1).

2.1 Fundamental solution of the linear problem

In this section, we derive the fundamental solution of the linear nonhomogeneous problem associated with (2.1), namely

$$\begin{cases} {}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi} u + (-\Delta)_x^{\delta} u = g(t,x), & (t,x) \in (0,+\infty) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.2}$$

where u_0 is the prescribed initial datum and g is a given source term. Throughout this section, we assume that

$$u_0 \in \mathcal{S}(\mathbb{R}^d), \qquad g \in C\Big([0,\infty); \mathcal{S}(\mathbb{R}^d)\Big).$$

These assumptions ensure that the Fourier transform with respect to the spatial variable and the generalized φ-Laplace transform with respect to time are well defined. The resulting representation formula can then be extended to less regular data by density arguments.

Theorem 2.1. *Let $d \geq 1$, $\alpha \in (0,1)$, and $\delta \in (0,1)$.*

The solution of problem (2.2) is given by

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}^d} G_0^{\varphi}(t, x-y)\, u_0(y)\, \mathrm{d}y \\ &+ \int_0^t \int_{\mathbb{R}^d} G_1^{\varphi}\Big(\varphi^{-1}\Big(\varphi(t) + \varphi(0) - \varphi(\tau)\Big), x-y\Big)\, g(\tau,y)\, \mathrm{d}y\, \varphi'(\tau)\, \mathrm{d}\tau, \end{aligned}$$

where

$$G_0^{\varphi}(t,x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \zeta, x\rangle} E_{\alpha,1}\Big(-|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}\Big)\, \mathrm{d}\zeta, \tag{2.3}$$

and

$$G_1^{\varphi}(t,x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \zeta, x\rangle} (\varphi(t)-\varphi(0))^{\alpha-1} E_{\alpha,\alpha}\Big(-|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}\Big)\, \mathrm{d}\zeta. \tag{2.4}$$

Proof. We apply the Fourier transform with respect to the spatial variable x. Since

$$(-\Delta)_x^{\delta} u(x) = \mathcal{F}^{-1}\Big(|\zeta|^{2\delta}\widehat{u}(\zeta)\Big)(x), \qquad x \in \mathbb{R}^d, \quad 0<\delta<1,$$

problem (2.2) is transformed into

$$\begin{cases} {}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi}\widehat{u}(t,\zeta) + |\zeta|^{2\delta}\widehat{u}(t,\zeta) = \widehat{g}(t,\zeta), & (t,\zeta)\in(0,+\infty)\times\mathbb{R}^d, \\ \widehat{u}(0,\zeta) = \widehat{u}_0(\zeta). \end{cases} \tag{2.5}$$

Applying the generalized φ-Laplace transform to (2.5), and denoting

$$U(s,\zeta) = \mathcal{L}_{\varphi}\{\widehat{u}(t,\zeta)\}(s), \qquad G(s,\zeta) = \mathcal{L}_{\varphi}\{\widehat{g}(t,\zeta)\}(s),$$

we obtain

$$\mathcal{L}_{\varphi}\left\{{}^{C}D_{0,t}^{\alpha,\varphi}\widehat{u}(t,\zeta)\right\} + |\zeta|^{2\delta}\mathcal{L}_{\varphi}\{\widehat{u}(t,\zeta)\} = \mathcal{L}_{\varphi}\{\widehat{g}(t,\zeta)\}.$$

Using Corollary 1.2 with $n=1$, since $0<\alpha<1$, we have

$$\mathcal{L}_{\varphi}\left\{{}^{C}D_{0,t}^{\alpha,\varphi}\widehat{u}(t,\zeta)\right\} = s^{\alpha}U(s,\zeta) - s^{\alpha-1}\widehat{u}(0,\zeta).$$

Moreover, by the initial condition

$$\widehat{u}(0,\zeta) = \widehat{u}_0(\zeta),$$

and by the linearity of the generalized φ-Laplace transform,

$$\mathcal{L}_{\varphi}\left\{|\zeta|^{2\delta}\widehat{u}(t,\zeta)\right\} = |\zeta|^{2\delta}U(s,\zeta).$$

Therefore,

$$s^{\alpha}U(s,\zeta) - s^{\alpha-1}\widehat{u}_0(\zeta) + |\zeta|^{2\delta}U(s,\zeta) = G(s,\zeta).$$

Equivalently,

$$\Big(s^{\alpha} + |\zeta|^{2\delta}\Big)U(s,\zeta) = s^{\alpha-1}\widehat{u}_0(\zeta) + G(s,\zeta).$$

Solving for $U(s,\zeta)$, we get

$$U(s,\zeta) = \frac{s^{\alpha-1}}{s^{\alpha}+|\zeta|^{2\delta}}\,\widehat{u}_0(\zeta) + \frac{1}{s^{\alpha}+|\zeta|^{2\delta}}\,G(s,\zeta).$$

Hence,

$$\mathcal{L}_\varphi\{\widehat{u}(t,\zeta)\} = \frac{s^{\alpha-1}}{s^\alpha + |\zeta|^{2\delta}}\,\widehat{u}_0(\zeta) + \frac{1}{s^\alpha + |\zeta|^{2\delta}}\,\mathcal{L}_\varphi\{\widehat{g}(t,\zeta)\}.$$

Indeed, the inverse generalized φ-Laplace transform is applied term by term. Setting

$$\lambda = |\zeta|^{2\delta},$$

we use

$$\mathcal{L}_\varphi^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha+\lambda}\right\}(t) = E_{\alpha,1}\left(-\lambda(\varphi(t)-\varphi(0))^\alpha\right),$$

and

$$\mathcal{L}_\varphi^{-1}\left\{\frac{1}{s^\alpha+\lambda}\right\}(t) = (\varphi(t)-\varphi(0))^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda(\varphi(t)-\varphi(0))^\alpha\right).$$

Moreover, by the convolution theorem associated with the generalized φ-Laplace transform, we have

$$\mathcal{L}_\varphi^{-1}\left\{\frac{1}{s^\alpha+|\zeta|^{2\delta}}\mathcal{L}_\varphi\{\widehat{g}(t,\zeta)\}\right\} = \int_0^t (\varphi(t)-\varphi(\tau))^{\alpha-1}E_{\alpha,\alpha}\left(-|\zeta|^{2\delta}(\varphi(t)-\varphi(\tau))^\alpha\right)\widehat{g}(\tau,\zeta)\,\varphi'(\tau)\,d\tau.$$

Therefore,

$$\begin{aligned}\widehat{u}(t,\zeta) &= \widehat{u}_0(\zeta)E_{\alpha,1}\Big(-|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^\alpha\Big)\\ &\quad+\int_0^t(\varphi(t)-\varphi(\tau))^{\alpha-1}E_{\alpha,\alpha}\Big(-|\zeta|^{2\delta}(\varphi(t)-\varphi(\tau))^\alpha\Big)\,\widehat{g}(\tau,\zeta)\,\varphi'(\tau)\,d\tau. \end{aligned}\tag{2.6}$$

Taking the inverse Fourier transform to (2.6), we obtain

$$\begin{aligned}u(t,x) &= \int_{\mathbb{R}^d} G_0^\varphi(t,\mathrm{x}-\mathrm{y})\,u_0(\mathrm{y})\,d\mathrm{y}\\ &\quad+\int_0^t\int_{\mathbb{R}^d} G_1^\varphi\Big(\varphi^{-1}\Big(\varphi(t)+\varphi(0)-\varphi(\tau)\Big),\mathrm{x}-\mathrm{y}\Big)\,g(\tau,\mathrm{y})\,d\mathrm{y}\,\varphi'(\tau)\,d\tau.\end{aligned}$$

This completes the proof. □

The following lemma provides explicit expressions for the kernels G_0^φ and G_1^φ in terms of the Fox H-function.

Lemma 2.1. *The kernels G_0^φ and G_1^φ, defined by* (2.3)–(2.4)*, satisfy*

1.

$$G_0^\varphi(t,x) = \frac{1}{\pi^{d/2}|x|^d}H^{2,1}_{2,3}\left[\frac{|x|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^\alpha}\,\middle|\,\begin{matrix}(1,1),(1,\alpha)\\(1,1),\left(\frac{d}{2},\delta\right),(1,\delta)\end{matrix}\right],$$

2.
$$G_1^{\varphi}(t,x) = \frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\pi^{d/2}|x|^d} H^{2,1}_{2,3}\left[\frac{|x|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}} \Bigg|^{(1,1),(\alpha,\alpha)}_{(1,1),\left(\frac{d}{2},\delta\right),(1,\delta)} \right].$$

Proof. We prove only the first identity, since the second one is obtained in exactly the same way.

By (1.3), (1.2), and (1.1), we have

$$\begin{aligned} G_0^{\varphi}(t,x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \zeta, x\rangle} E_{\alpha,1}\Big(-|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}\Big)\, \mathrm{d}\zeta \\ &= \frac{1}{(2\pi)^{d/2}} |x|^{1-d/2} \int_0^{\infty} |\zeta|^{d/2} H^{1,1}_{1,2}\Big[|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha} \Big|^{(0,1)}_{(0,1),(0,\alpha)} \Big]\, \mathrm{J}_{\frac{d}{2}-1}(|x||\zeta|)\, \mathrm{d}|\zeta| \\ &= \frac{1}{\pi^{d/2}|x|^d} H^{2,1}_{2,3}\left[\frac{|x|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}} \Bigg|^{(1,1),(1,\alpha)}_{(1,1),\left(\frac{d}{2},\delta\right),(1,\delta)} \right]. \end{aligned}$$

This completes the proof. □

Example 2.1. Figure 2.1 illustrates the kernel G_0^{φ} corresponding to

$$\varphi(t) = \frac{t^{\rho}}{\rho}, \qquad \rho = 1,$$

for the two values $\alpha = \frac{1}{2}$ and $\alpha = 1$, in dimension $d = 1$, over the time interval $(0, 2]$.

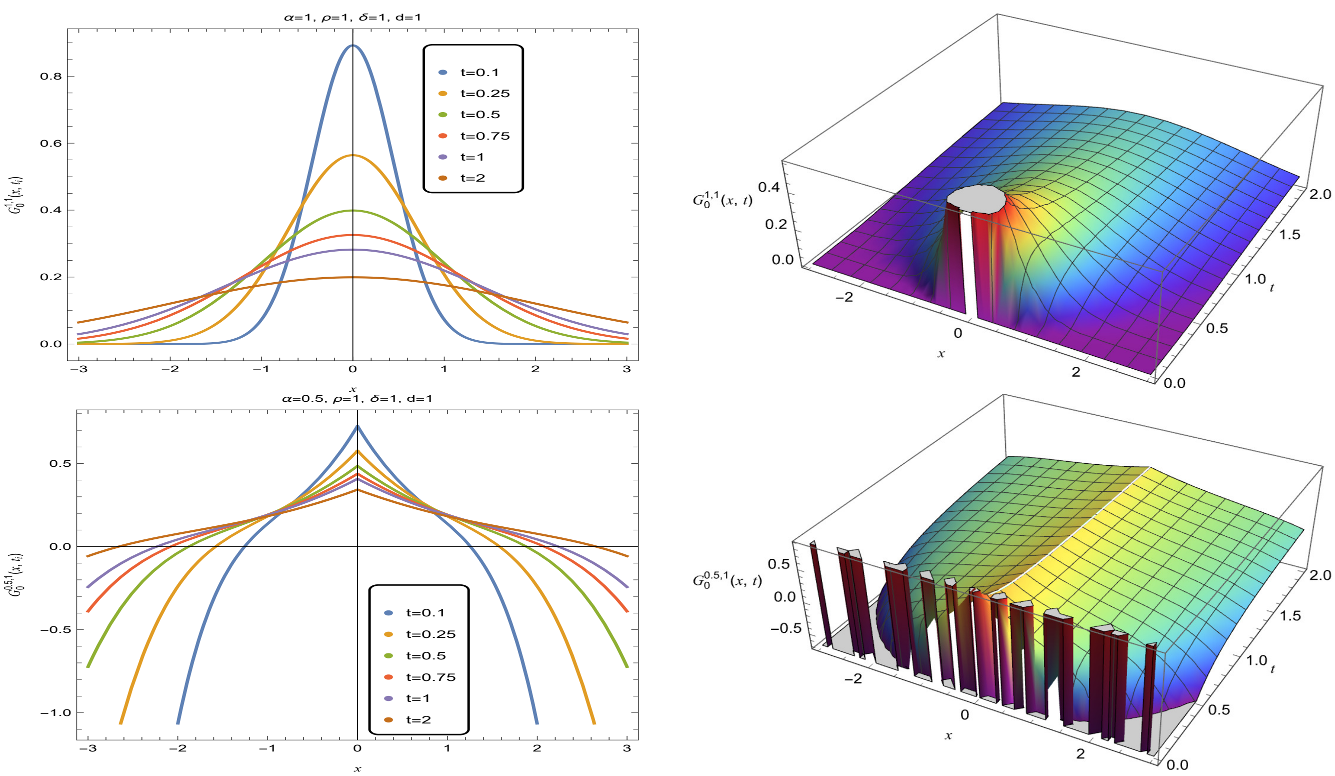

Figure 2.1: Profiles of the heat kernel G_0^{φ}. Top: $\alpha = 1$, $\rho = 1$, $\delta = 1$, $d = 1$. Bottom: $\alpha = 0.5$, $\rho = 1$, $\delta = 1$, $d = 1$.

2.2 Decay estimates

In this section, we establish decay estimates for solutions to (2.2) in Lebesgue spaces.

For convenience, we write

$$\|\cdot\|_{s'} := \|\cdot\|_{L^{s'}(\mathbb{R}^d)},$$

and define

$$\Lambda(d,\delta) = \begin{cases} \infty, & d \le 2\delta, \\ \dfrac{d}{d-2\delta}, & d > 2\delta, \end{cases} \qquad \Lambda^*(d,\delta) = \begin{cases} \infty, & d \le 4\delta, \\ \dfrac{d}{d-4\delta}, & d > 4\delta. \end{cases}$$

Arguing as in [38, Theorem 3.5] and [40, Lemma 2.2], we first obtain the following estimate.

Lemma 2.2. *Let $1 \le s \le r \le \infty$, and define*

$$S_0^{\varphi}(t)u_0 := \int_{\mathbb{R}^d} G_0^{\varphi}(t, \mathrm{x}-\mathrm{y})\, u_0(\mathrm{y})\, \mathrm{dy}.$$

Then

$$\|S_0^{\varphi}(t)u_0\|_r \le C_0(\varphi(t)-\varphi(0))^{-\frac{\alpha d}{2\delta}\left(\frac{1}{s}-\frac{1}{r}\right)}\|u_0\|_s \tag{2.7}$$

whenever

$$\frac{1}{s}-\frac{1}{r} < \min\left\{1, \frac{2\delta}{d}\right\}.$$

Remark 2.1. In particular, if $g \equiv 0$, then (2.7) yields

$$\|u(t,\cdot)\|_r \le C_0(\varphi(t)-\varphi(0))^{-\frac{\alpha d}{2\delta}\left(\frac{1}{s}-\frac{1}{r}\right)}\|u_0\|_s.$$

Theorem 2.2. *Let $d \in \mathbb{N}$, $0 < \alpha < 1$, and $0 < \delta < 1$. Assume that*

$$1 \le s' < \Lambda^*(d,\delta).$$

Then $G_1^{\varphi}(t,\cdot) \in L^{s'}(\mathbb{R}^d)$, and

$$\|G_1^{\varphi}(t,\cdot)\|_{s'} \le C(\varphi(t)-\varphi(0))^{\alpha-1-\frac{\alpha d}{2\delta}\left(1-\frac{1}{s'}\right)}, \qquad t > 0.$$

We next derive a decay estimate for the solution to (2.2) with vanishing initial data.

Theorem 2.3. *Let $1 \leq s' < \Lambda(d,\delta)$, $1 \leq s, r < \infty$, and assume that*

$$\frac{1}{s'} + \frac{1}{s} = 1 + \frac{1}{r}.$$

Suppose that $g(t,\cdot) \in L^s(\mathbb{R}^d)$ for every $t > 0$, and that

$$\|g(t,\cdot)\|_s \leq C\Big(1 + \varphi(t) - \varphi(0)\Big)^{-\vartheta}, \qquad t > 0, \tag{2.8}$$

for some $\vartheta > 0$. Then

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G_1^{\varphi}\Big(\varphi^{-1}(\varphi(t) + \varphi(0) - \varphi(\tau)), \mathrm{x} - \mathrm{y}\Big)\, g(\tau, \mathrm{y})\, \mathrm{dy}\, \varphi'(\tau)\, \mathrm{d}\tau$$

satisfies

$$\|u(t,\cdot)\|_r \leq C(\varphi(t) - \varphi(0))^{\alpha - \min\{1,\vartheta\} - \frac{\alpha d}{2\delta}\left(1 - \frac{1}{s'}\right)}, \qquad t > 0, \tag{2.9}$$

if $\vartheta \neq 1$, and

$$\|u(t,\cdot)\|_r \leq C(\varphi(t) - \varphi(0))^{\alpha - 1 - \frac{\alpha d}{2\delta}\left(1 - \frac{1}{s'}\right)} \log\Big(1 + \varphi(t) - \varphi(0)\Big), \qquad t > 0, \tag{2.10}$$

if $\vartheta = 1$.

Proof. We split the proof into two steps.

Step 1. Estimate of the first time interval. By Minkowski's inequality and young's inequality, splitting the time integral at

$$\varphi^{-1}\left(\frac{\varphi(t) + \varphi(0)}{2}\right),$$

we write

$$\|u(t,\cdot)\|_r \leq I + II.$$

Using Theorem 2.2 and (2.8), we obtain

$$\begin{aligned} I &\leq C(\varphi(t) - \varphi(0))^{\alpha - 1 - \frac{\alpha d}{2\delta}\left(1 - \frac{1}{s'}\right)} \int_0^t \Big(1 + \varphi(\tau) - \varphi(0)\Big)^{-\vartheta} \varphi'(\tau)\, \mathrm{d}\tau \\ &\leq C(\varphi(t) - \varphi(0))^{\alpha - \min\{1,\vartheta\} - \frac{\alpha d}{2\delta}\left(1 - \frac{1}{s'}\right)}. \end{aligned} \tag{2.11}$$

Step 2. Estimate of the second time interval. For the second term, the singularity of G_1^{φ} near $t = 0$ is integrable provided that

$$\alpha - 1 - \frac{\alpha d}{2\delta}\left(1 - \frac{1}{s'}\right) > -1. \tag{2.12}$$

Under this condition, we infer that

$$\begin{aligned} II &\le C(\varphi(t)-\varphi(0))^{-\vartheta}\int_0^{\varphi^{-1}\left(\frac{\varphi(t)+\varphi(0)}{2}\right)}(\varphi(\tau)-\varphi(0))^{\alpha-1-\frac{\alpha d}{2\delta}\left(1-\frac{1}{s'}\right)}\varphi'(\tau)\,\mathrm{d}\tau \\ &\le C(\varphi(t)-\varphi(0))^{\alpha-\vartheta-\frac{\alpha d}{2\delta}\left(1-\frac{1}{s'}\right)}. \end{aligned} \tag{2.13}$$

Conclusion. Combining (2.11) and (2.13), we obtain (2.9). The case $\vartheta = 1$ is handled in the same way and yields the logarithmic correction in (2.10). This completes the proof. □

2.3 Global existence and continuous dependence

In this section, we introduce the notion of mild solution to (2.1) and establish global existence and uniqueness by means of a contraction mapping argument.

For brevity, we shall sometimes write $u(t)$ instead of $u(t,x)$.

Definition 2.1 (Mild solution)**.** A continuous function

$$u : (0,\infty) \to W^{1,r}(\mathbb{R}^d)$$

is called a *mild solution* of problem (2.1) if it satisfies

$$u(t) = S_0^{\varphi}(t)u_0 + \int_0^t S_1^{\varphi}\Big(\varphi^{-1}\Big(\varphi(t)+\varphi(0)-\varphi(\tau)\Big)\Big)\Big[a|u(\tau)|^{p-1}u(\tau)+b|\nabla u(\tau)|^q\Big]\varphi'(\tau)\,\mathrm{d}\tau. \tag{2.14}$$

Here

$$S_1^{\varphi}(t)f := \int_{\mathbb{R}^d} G_1^{\varphi}(t,\mathrm{x}-\mathrm{y})f(\mathrm{y})\,\mathrm{dy}.$$

To prove global existence, we need suitable decay estimates for $S_1^{\varphi}(t)u_0$ and $\nabla S_1^{\varphi}(t)u_0$. Arguing as in [38, Theorem 3.9] and [40, Lemma 2.3], we obtain the following result.

Lemma 2.3. *Let* $1 \le s \le r \le \infty$*. Then*

$$\|S_1^{\varphi}(t)u_0\|_r \le C(\varphi(t)-\varphi(0))^{\alpha-1-\frac{\alpha d}{2\delta}\left(\frac{1}{s}-\frac{1}{r}\right)}\|u_0\|_s, \tag{2.15}$$

$$\|\nabla S_1^{\varphi}(t)u_0\|_r \le C'(\varphi(t)-\varphi(0))^{\alpha-1-\frac{\alpha}{2\delta}-\frac{\alpha d}{2\delta}\left(\frac{1}{s}-\frac{1}{r}\right)}\|u_0\|_s, \tag{2.16}$$

provided, respectively, that

$$\frac{1}{s}-\frac{1}{r} < \min\left\{1, \frac{4\delta}{d}\right\}, \qquad \frac{1}{s}-\frac{1}{r} < \min\left\{\frac{1}{2}, \frac{4\delta}{d}\right\}.$$

Lemma 2.4. *Let*

$$\nu = \frac{\alpha d}{2\delta}\left(\frac{1}{s} - \frac{1}{r}\right). \tag{2.17}$$

Assume that the parameters δ, d, p, q, s, r, m, with $m, r > 1$, satisfy

$$\max\left\{\frac{d(q-1)}{2\delta},\ q\right\} < r < qs, \tag{2.18}$$

$$\max\left\{r,\ p,\ \frac{rpd}{2\delta r + d}\right\} < m < ps. \tag{2.19}$$

Then the following assertions hold:

(i) $\dfrac{r}{q} > 1$ *and* $\dfrac{m}{p} > 1$,

(ii)

$$\frac{\alpha d}{2\delta}\left(\frac{p}{m} - \frac{1}{r}\right) - (\alpha - 1) < 1, \qquad \frac{\alpha d(q-1)}{2\delta r} - (\alpha - 1) < 1,$$

(iii)

$$p\nu + \frac{\alpha p d}{2\delta}\left(\frac{1}{r} - \frac{1}{m}\right) < 1, \qquad q\left(\frac{\alpha}{2\delta} + \nu\right) < 1,$$

(iv)

$$\nu + (\alpha - 1) - \frac{\alpha d}{2\delta}\left(\frac{p}{m} - \frac{1}{r}\right) - p\nu - \frac{p\alpha d}{2\delta}\left(\frac{1}{r} - \frac{1}{m}\right) + 1 = 0,$$

(v)

$$\nu + (\alpha - 1) - \frac{\alpha d(q-1)}{2\delta r} - q\left(\frac{\alpha}{2\delta} + \nu\right) + 1 = 0.$$

Remark 2.2. Corollary 2.4 provides the admissible exponents needed in the contraction argument below. More precisely, part (i) ensures that the nonlinear terms are well defined in suitable Lebesgue spaces, parts (ii) and (iii) guarantee the convergence of the relevant time integrals, while parts (iv) and (v) yield the cancellations required to derive uniform estimates on $(0, \infty)$.

Theorem 2.4. *Assume that* (2.18)–(2.19) *hold, and let* ν *be defined by* (2.17). *Suppose that* $L > 0$ *satisfies*

$$\kappa_1 L^{q-1} + \kappa_2 L^{p-1} < 1,$$

where κ_1 *and* κ_2 *are the positive constants defined in* (2.32) *and* (2.33). *Let* $R > 0$ *be such that*

$$R + \kappa_1 L^q + \kappa_2 L^p \leq L. \tag{2.20}$$

Let u_0 *be a tempered distribution satisfying*

$$\mathcal{N}(u_0) := \sup_{t>0} \Big[(\varphi(t) - \varphi(0))^{\nu} \|S_0^{\varphi}(t)u_0\|_r,\ (\varphi(t) - \varphi(0))^{\frac{\alpha}{2\delta}+\nu} \|\nabla S_0^{\varphi}(t)u_0\|_r \Big] \leq R. \tag{2.21}$$

Then problem (2.14) *admits a unique global mild solution* u *satisfying*

$$\sup_{t>0} \Big[(\varphi(t) - \varphi(0))^{\nu} \|u(t)\|_r,\ (\varphi(t) - \varphi(0))^{\frac{\alpha}{2\delta}+\nu} \|\nabla u(t)\|_r \Big] \leq L. \tag{2.22}$$

Moreover,

$$\begin{aligned}
&\text{(i)} \quad u(t) - S_0^{\varphi}(t)u_0 \in C\Big([0,\infty); L^{p'}(\mathbb{R}^d)\Big) \quad \textit{for } \max\left(\frac{m}{p}, \frac{r}{q}\right) \leq p' < \frac{d}{2\delta}(p-1),\\
&\text{(ii)} \quad u(t) - S_0^{\varphi}(t)u_0 \in L^{\infty}\Big([0,\infty); L^{p'}(\mathbb{R}^d)\Big) \quad \textit{if } p' = \frac{d}{2\delta}(p-1),\\
&\text{(iii)} \quad \lim_{t\to 0} u(t) = u_0 \quad \textit{in } \mathcal{S}'(\mathbb{R}^d).
\end{aligned}$$

In addition, if u_0 *and* v_0 *both satisfy* (2.21), *and if* u *and* v *are the corresponding solutions, then*

$$\begin{aligned}
&\sup_{t>0} \Big[(\varphi(t) - \varphi(0))^{\nu} \|u(t) - v(t)\|_r,\ (\varphi(t) - \varphi(0))^{\frac{\alpha}{2\delta}+\nu} \|\nabla u(t) - \nabla v(t)\|_r \Big] \\
&\qquad \leq \Big(1 - (\kappa_1 L^{q-1} + \kappa_2 L^{p-1})\Big)^{-1} \mathcal{N}(u_0 - v_0).
\end{aligned} \tag{2.23}$$

Proof. We divide the proof into several steps.

Step 1. The fixed-point operator. Let X be the set of all Bochner measurable functions

$$u : (0,\infty) \to W^{1,r}(\mathbb{R}^d)$$

such that

$$\|u\|_X = \sup_{t>0} \Big[(\varphi(t) - \varphi(0))^{\nu} \|u(t)\|_r,\ (\varphi(t) - \varphi(0))^{\nu+\frac{\alpha}{2\delta}} \|\nabla u(t)\|_r \Big] < \infty. \tag{2.24}$$

For $L > 0$, define

$$X_L := \{u \in X : \|u\|_X \leq L\}.$$

Equipped with the metric $d(u,v) = \|u-v\|_X$, the set X_L is a complete metric space.

Define $\Psi_{u_0} : X_L \to X$ by

$$\Psi_{u_0}(u)(t) = S_0^{\varphi}(t)u_0 + \int_0^t S_1^{\varphi}\Big(\varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\tau))\Big)\Big[a|u(\tau)|^{p-1}u(\tau) + b|\nabla u(\tau)|^q\Big]\varphi'(\tau)\,\mathrm{d}\tau.$$

We shall show that Ψ_{u_0} is a strict contraction on X_L.

Step 2. Estimate in the L^r-component.

Let u_0, v_0 satisfy (2.21), and let $u, v \in X_L$. Using (2.15), together with

$$\||\nabla u|^q - |\nabla v|^q\|_{r/q} \le q\Big(\|\nabla u\|_r^{q-1} + \|\nabla v\|_r^{q-1}\Big)\|\nabla u - \nabla v\|_r,$$

and

$$\||u|^{p-1}u - |v|^{p-1}v\|_{m/p} \le p\Big(\|u\|_m^{p-1} + \|v\|_m^{p-1}\Big)\|u-v\|_m,$$

as well as the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned}(\varphi(t)-\varphi(0))^{\nu}&\|\Psi_{u_0}(u)(t) - \Psi_{v_0}(v)(t)\|_r \\ &\le (\varphi(t)-\varphi(0))^{\nu}\|S_0^{\varphi}(t)(u_0-v_0)\|_r + \Big(C_1L^{q-1} + C_2L^{p-1}\Big)d(u,v),\end{aligned} \tag{2.25}$$

where

$$C_1 = 2C|b|q\int_0^1 (1-\tau)^{\alpha-1-\frac{\alpha d}{2\delta}(\frac{q}{r}-\frac{1}{r})}\tau^{-q(\frac{\alpha}{2\delta}+\nu)}\,\mathrm{d}\tau, \tag{2.26}$$

and

$$C_2 = 2C|a|pM^p\int_0^1 (1-\tau)^{\alpha-1-\frac{\alpha d}{2\delta}(\frac{p}{m}-\frac{1}{r})}\tau^{-p\nu-\frac{p\alpha d}{2\delta}(\frac{1}{r}-\frac{1}{m})}\,\mathrm{d}\tau. \tag{2.27}$$

Similarly, by (2.16), we obtain

$$\begin{aligned}(\varphi(t)-\varphi(0))^{\nu+\frac{\alpha}{2\delta}}&\|\nabla\Psi_{u_0}(u)(t) - \nabla\Psi_{v_0}(v)(t)\|_r \\ &\le (\varphi(t)-\varphi(0))^{\nu+\frac{\alpha}{2\delta}}\|\nabla S_0^{\varphi}(t)(u_0-v_0)\|_r + \Big(C_1'L^{q-1} + C_2'L^{p-1}\Big)d(u,v),\end{aligned} \tag{2.28}$$

where

$$C_1' = 2C'|b|q\int_0^1 (1-\tau)^{\alpha-1-\frac{\alpha}{2\delta}-\frac{\alpha d}{2\delta}(\frac{q}{r}-\frac{1}{r})}\tau^{-q(\frac{\alpha}{2\delta}+\nu)}\,\mathrm{d}\tau, \tag{2.29}$$

and

$$C_2' = 2C'|a|pM^p\int_0^1 (1-\tau)^{\alpha-1-\frac{\alpha}{2\delta}-\frac{\alpha d}{2\delta}(\frac{p}{m}-\frac{1}{r})}\tau^{-p\nu-\frac{p\alpha d}{2\delta}(\frac{1}{r}-\frac{1}{m})}\,\mathrm{d}\tau. \tag{2.30}$$

Step 3. Contraction estimate.

Combining (2.25) and (2.28), we obtain

$$d\Big(\Psi_{u_0}(u), \Psi_{v_0}(v)\Big) \le \mathcal{N}(u_0-v_0) + (\kappa_1 L^{q-1} + \kappa_2 L^{p-1})\,d(u,v), \tag{2.31}$$

where

$$\kappa_1 = \max\{C_1, C_1'\}, \tag{2.32}$$

$$\kappa_2 = \max\{C_2, C_2'\}. \tag{2.33}$$

By Corollary 2.4, all these constants are finite.

If $v_0 = v = 0$, then (2.31) yields

$$\|\Psi_{u_0}(u)\|_X \leq \mathcal{N}(u_0) + (\kappa_1 L^{q-1} + \kappa_2 L^{p-1})\|u\|_X.$$

Hence, by (2.20) and (2.21), the operator Ψ_{u_0} maps X_L into itself.

If $u_0 = v_0$, then (2.31) reduces to

$$d\Big(\Psi_{u_0}(u), \Psi_{u_0}(v)\Big) \leq (\kappa_1 L^{q-1} + \kappa_2 L^{p-1})\, d(u, v).$$

Since

$$\kappa_1 L^{q-1} + \kappa_2 L^{p-1} < 1,$$

the operator Ψ_{u_0} is a strict contraction on X_L. Therefore, by the Banach fixed-point theorem, it admits a unique fixed point in X_L, which is precisely the desired global mild solution.

Proof of (i) and (ii).

Let p' satisfy

$$\max\left(\frac{m}{p}, \frac{r}{q}\right) \leq p' < \frac{d}{2\delta}(p-1). \tag{2.34}$$

From the mild formulation, together with (2.15), the Gagliardo–Nirenberg inequality, and (2.22), we obtain

$$\begin{aligned} \|u(t) - S_0^{\varphi}(t)u_0\|_{p'} \leq{} & C_3(\varphi(t) - \varphi(0))^{(\alpha-1) - \frac{\alpha d}{2\delta}(\frac{p}{m} - \frac{1}{p'}) - p\left[\nu + \frac{\alpha d}{2\delta}\left(\frac{1}{r} - \frac{1}{m}\right)\right] + 1} \\ & + C_4(\varphi(t) - \varphi(0))^{(\alpha-1) - \frac{\alpha d}{2\delta}(\frac{q}{r} - \frac{1}{p'}) - q\left(\frac{\alpha}{2\delta} + \nu\right) + 1}. \end{aligned} \tag{2.35}$$

Using the definition of ν in (2.17), estimate (2.35) becomes

$$\|u(t) - S_0^{\varphi}(t)u_0\|_{p'} \leq (C_3 + C_4)(\varphi(t) - \varphi(0))^{\frac{\alpha d}{2\delta p'} - \frac{\alpha}{p-1}}. \tag{2.36}$$

Since the exponent on the right-hand side is positive for p' satisfying (2.34), it follows that

$$u(t) - S_0^{\varphi}(t)u_0 \to 0 \qquad \text{in } L^{p'}(\mathbb{R}^d) \qquad \text{as } t \to 0,$$

which proves (i). The endpoint case in (ii) follows by the same argument.

Proof of (iii) and continuous dependence.

Assertion (iii) follows from the convergence

$$S_0^{\varphi}(t)u_0 \to u_0 \qquad \text{in } \mathcal{S}'(\mathbb{R}^d) \qquad \text{as } t \to 0,$$

together with the fact that the nonlinear remainder tends to zero as $t \to 0$.

Finally, estimate (2.23) follows by setting $\Psi_{u_0}(u) = u$ and $\Psi_{v_0}(v) = v$ in (2.31). This completes the proof. □

Chapter 3

Local existence for a generalized fractional diffusion equation with fractional Laplacian and exponential nonlinearity

In this chapter, we establish the local existence and uniqueness of solutions to the Cauchy problem for the time-fractional diffusion equation

$$\begin{cases} {}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi} u + (-\Delta)^{\delta}_{\phi(x)} u = \mathrm{I}_{0,t}^{1-\gamma,\varphi}[e^{u}], \quad (t,x) \in (0,+\infty) \times \mathbb{R}^d, \\ u(0) = u_0, \quad x \in \mathbb{R}^d, \end{cases} \tag{3.1}$$

where $\alpha, \delta, \gamma \in (0,1)$ and $d \in \mathbb{N}$. The operator ${}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi}$ denotes the φ-Caputo fractional derivative, while $\mathrm{I}_{0,t}^{1-\gamma,\varphi}$ is the Riemann–Liouville fractional integral with respect to the function φ. The operator $(-\Delta)^{\delta}_{\phi(x)}$ represents the fractional Laplacian associated with the transformation ϕ, which allows the model to include spatial heterogeneity. The initial datum satisfies

$$u_0 \in C_0(\mathbb{R}^d) = \left\{ v \in C(\mathbb{R}^d) \;\middle|\; \lim_{|x|\to\infty} v(x) = 0 \right\}.$$

The study of diffusion equations with fractional operators has attracted considerable attention because such models provide a suitable framework for describing anomalous diffusion, memory effects, and nonlocal spatial interactions. In particular, equations involving time-fractional derivatives are useful for representing processes with hereditary properties, while fractional Laplacians are widely used to describe long-range diffusion phenomena. In the present chapter, the use of the φ-Caputo derivative and the ϕ-fractional Laplacian introduces a more general temporal and spatial structure than the standard fractional models.

The exponential nonlinearity appearing in (3.1) is also motivated by several works on reaction–diffusion equations with nonlinear memory. Cazenave et al. [16] investigated semilinear parabolic equations with nonlinear memory terms, while Ahmad et al. [1] studied reaction–diffusion models involving nonlocal time and space operators with exponential nonlinearities. These contributions show the relevance of nonlocal memory terms in the study of existence and qualitative behavior of solutions.

More recently, Kirane et al. [7] considered a time–space fractional evolution equation with a nonlocal exponential nonlinearity, where the time derivative and the spatial operator were taken in standard fractional forms. Borikhanov et al. [15] also studied an integro-differential diffusion equation involving a Riemann–Liouville type memory operator and an exponential source term. Compared with these works, problem (3.1) introduces two additional levels of generality through the φ-Caputo derivative and the ϕ-fractional Laplacian.

In a related direction, Benaissi et al. [9] studied nonlinear time–space fractional diffusion equations involving generalized fractional operators and established existence, uniqueness, well-posedness, and decay estimates for mild solutions in suitable functional spaces. These results provide important motivation for the present chapter, where the focus is placed on a generalized fractional diffusion equation with a nonlocal exponential nonlinearity.

Motivated by these developments, we first derive the fundamental solution of the associated linear problem by means of the φ-Laplace transform, the ϕ-Fourier transform, and Fox H-functions. We then prove the local existence and uniqueness of mild solutions to (3.1).

3.1 Fundamental solution of the linear problem

In this section, we derive the fundamental solution of the linear nonhomogeneous problem

$$\begin{cases} {}^{C}\mathrm{D}^{\alpha,\varphi}_{0,t}u + (-\Delta)^{\delta}_{\phi(x)}u = g(t,x), \quad (t,x) \in (0,+\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{3.2}$$

Theorem 3.1. *The solution to problem* (3.2) *is given by*

$$u(t,x) = G_0(t,\cdot) *_\phi u_0(x) + \int_0^t G_1\Big(\varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\tau)),\cdot\Big) *_\phi g(\tau,\cdot)(x)\,\varphi'(\tau)\,\mathrm{d}\tau,$$

where

$$G_0(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\langle\zeta,\phi(x)\rangle} E_{\alpha,1}\Big(-|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}\Big)\,\mathrm{d}\zeta, \tag{3.3}$$

and

$$G_1(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\langle\zeta,\phi(x)\rangle}(\varphi(t)-\varphi(0))^{\alpha-1} E_{\alpha,\alpha}\Big(-|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}\Big)\,\mathrm{d}\zeta. \tag{3.4}$$

Here $E_{\alpha,1}$ and $E_{\alpha,\alpha}$ denote the Mittag–Leffler functions.

Proof. Applying the ϕ-Fourier transform to (3.2) with respect to the spatial variable x, we obtain

$$\begin{cases} {}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi}\widehat{u}(t,\zeta) + |\zeta|^{2\delta}\widehat{u}(t,\zeta) = \widehat{g}(t,\zeta), \quad (t,\zeta) \in (0,+\infty)\times\mathbb{R}^d, \\ \widehat{u}(0,\zeta) = \widehat{u}_0(\zeta), \end{cases} \tag{3.5}$$

where

$$\widehat{u}(t,\zeta) = \mathcal{F}_{\phi}[u(t,\cdot)](\zeta), \qquad \widehat{g}(t,\zeta) = \mathcal{F}_{\phi}[g(t,\cdot)](\zeta).$$

Applying the generalized φ-Laplace transform to (3.5), and using Corollary 1.2 with $n = 1$ together with Lemma 1.1, we obtain

$$\mathcal{L}_{\varphi}\{\widehat{u}(t,\zeta)\} = \frac{w^{\alpha-1}}{w^{\alpha} + |\zeta|^{2\delta}}\widehat{u}_0(\zeta) + \frac{1}{w^{\alpha} + |\zeta|^{2\delta}}\mathcal{L}_{\varphi}\{\widehat{g}(t,\zeta)\}.$$

Hence,

$$\begin{aligned} \widehat{u}(t,\zeta) &= \widehat{u}_0(\zeta)E_{\alpha,1}\Big(-|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}\Big) \\ &\quad + \int_0^t (\varphi(t)-\varphi(\tau))^{\alpha-1}E_{\alpha,\alpha}\Big(-|\zeta|^{2\delta}(\varphi(t)-\varphi(\tau))^{\alpha}\Big)\,\widehat{g}(\tau,\zeta)\,\varphi'(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{3.6}$$

To justify the inversion procedure and the subsequent representation formula, we impose additional assumptions on the data. We assume that

$$u_0 \in C_0(\mathbb{R}^d) \cap L^1_{\phi}(\mathbb{R}^d) \quad \text{and} \quad g(\cdot,t) \in L^1_{\phi}(\mathbb{R}^d),\ \forall t \in [0,T].$$

Under these assumptions, the Fourier transform with respect to ϕ is well defined, and the kernels appearing in (3.3)–(3.3) are integrable in the spatial variable.

Moreover, from Lemma 3.2, the kernels G_0 and G_1 satisfy suitable L^1-bounds, which ensure that all convolution expressions below are finite.

Next, the application of the inverse generalized Laplace transform to (3.6) is justified by Lemma 1.1 together with the convolution theorem for the φ-Laplace transform. In particular, the second term in (3.6) corresponds to a φ-convolution in time.

Furthermore, the interchange between the inverse Fourier transform and the time integral is justified by Fubini's theorem. Indeed, the above assumptions and the kernel estimates imply that the integrand is absolutely integrable on $\mathbb{R}^d \times (0,T)$.

Applying the inverse ϕ-Fourier transform to (3.6), we deduce that

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} G_0\Big(t, \phi^{-1}(\phi(x)-\phi(y))\Big)\, u_0(y)\, |J_\phi(y)|\, \mathrm{d}y \\ &\quad + \int_0^t \int_{\mathbb{R}^d} G_1\Big(\varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\tau)), \phi^{-1}(\phi(x)-\phi(y))\Big)\, g(\tau,y)\, |J_\phi(y)|\, dy\, \varphi'(\tau)\, \mathrm{d}\tau \\ &= G_0(t,\cdot) *_\phi u_0(x) + \int_0^t G_1\Big(\varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\tau)), \cdot\Big) *_\phi g(\tau,\cdot)(x)\, \varphi'(\tau)\, \mathrm{d}\tau. \end{aligned}$$

This completes the proof. □

Lemma 3.1. *The kernels $G_0(t,x)$ and $G_1(t,x)$, defined by* (3.3) *and* (3.4)*, satisfy the following identities:*

1.

$$G_0(t,x) = \frac{1}{\pi^{d/2}|\phi(x)|^d} H^{2,1}_{2,3}\left[\frac{|\phi(x)|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^\alpha} \middle|^{(1,1),(1,\alpha)}_{(1,1),\left(\frac{d}{2},\delta\right),(1,\delta)} \right], \tag{3.7}$$

2.

$$G_1(t,x) = \frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\pi^{d/2}|\phi(x)|^d} H^{2,1}_{2,3}\left[\frac{|\phi(x)|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^\alpha} \middle|^{(1,1),(\alpha,\alpha)}_{(1,1),\left(\frac{d}{2},\delta\right),(1,\delta)} \right]. \tag{3.8}$$

Proof. We prove only the first identity, since the second is obtained in the same way.

Using (1.3), (1.2), and (1.1), we obtain

$$\begin{aligned} G_0(t,x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \zeta, \phi(x)\rangle} E_{\alpha,1}\Big(-|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^\alpha\Big)\, \mathrm{d}\zeta \\ &= \frac{1}{(2\pi)^{d/2}} |\phi(x)|^{1-d/2} \int_0^\infty |\zeta|^{d/2} H^{1,1}_{1,2}\left[|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^\alpha \middle|^{(0,1)}_{(0,1),(0,\alpha)} \right] \mathrm{J}_{\frac{d}{2}-1}(|\phi(x)||\zeta|)\, \mathrm{d}|\zeta| \\ &= \frac{1}{\pi^{d/2}|\phi(x)|^d} H^{2,1}_{2,3}\left[\frac{|\phi(x)|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^\alpha} \middle|^{(1,1),(1,\alpha)}_{(1,1),\left(\frac{d}{2},\delta\right),(1,\delta)} \right]. \end{aligned}$$

To ensure the validity of the above representation, we verify the convergence condition of the Fox H-function. A direct computation gives

$$\alpha^* = 2 - \alpha.$$

Since $\alpha \in (0,1)$, it follows that $\alpha^* > 0$. Therefore, the Fox H-function appearing above is well defined and the corresponding integral representation is convergent.

This completes the proof. □

Lemma 3.2. *For $G_0(t,x)$ and $G_1(t,x)$ given by* (3.7) *and* (3.8)*, the following properties hold:*

$$G_0(t,x) > 0, \quad \int_{\mathbb{R}^d} G_0(t,x)\, \mathrm{dx} = 1,$$

$$\int_{\mathbb{R}^d} G_1(t,x)\,\mathrm{d}x = \frac{1}{\Gamma(\alpha)}(\varphi(t)-\varphi(0))^{\alpha-1},$$

$$\mathrm{I}_{0,t}^{1-\alpha,\varphi} G_1(t,x) = G_0(t,x).$$

Proof. We first compute

$$\int_{\mathbb{R}^d} G_0(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^d} e^{i<\phi(x),0>} G_0(x,t)\,\mathrm{d}x = E_{\alpha,1}\Big(-|\omega|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}\Big)\Big|_{\omega=0} = 1,$$

and

$$\int_{\mathbb{R}^d} G_1(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^d} e^{i<\phi(x),0>} G_1(x,t)\,\mathrm{d}x$$

$$= (\varphi(t)-\varphi(0))^{\alpha-1} E_{\alpha,\alpha}\Big(-|\omega|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}\Big)\Big|_{\omega=0} = \frac{1}{\Gamma(\alpha)}(\varphi(t)-\varphi(0))^{\alpha-1},$$

where $E_{\alpha,1}(z)$ and $E_{\alpha,\alpha}(z)$ are the Mittag–Leffler functions.

Next, using Definition 1.6 and Definition 1.4, we obtain

$$\mathrm{I}_{0+}^{1-\alpha,\varphi} G_1(x,t)$$

$$= \mathrm{I}_{0+}^{1-\alpha,\varphi}\left[\frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\pi^{\frac{d}{2}}|\phi(x)|^d} H_{2,3}^{2,1}\left[\frac{|\phi(x)|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}}\middle|\begin{matrix}(1,1),(\alpha,\alpha)\\ (1,1),(\frac{d}{2},\delta),(1,\delta)\end{matrix}\right]\right].$$

Using the Mellin–Barnes representation of the H-function, we write

$$= \frac{1}{|\phi(x)|^d\pi^{d/2}}\frac{1}{2\pi i}\int_{\Omega}\frac{\Gamma(1+\tau)\Gamma(\frac{d}{2}+\delta\tau)\Gamma(-\tau)}{\Gamma(\alpha+\alpha\tau)\Gamma(-\delta\tau)}\left(\frac{|\phi(x)|^{2\delta}}{2^{2\delta}}\right)^{-\tau}\mathrm{I}_{0+}^{1-\alpha,\varphi}(\varphi(t)-\varphi(0))^{\alpha+\alpha\tau-1}\,\mathrm{d}\tau.$$

Applying the fractional integral gives

$$\mathrm{I}_{0+}^{1-\alpha,\varphi}(\varphi(t)-\varphi(0))^{\alpha+\alpha\tau-1} = \frac{\Gamma(\alpha+\alpha\tau)}{\Gamma(1+\alpha\tau)}(\varphi(t)-\varphi(0))^{\alpha\tau},$$

hence

$$= \frac{1}{|\phi(x)|^d\pi^{d/2}}\frac{1}{2\pi i}\int_{\Omega}\frac{\Gamma(1+\tau)\Gamma(\frac{d}{2}+\delta\tau)\Gamma(-\tau)\Gamma(\alpha+\alpha\tau)}{\Gamma(\alpha+\alpha\tau)\Gamma(-\delta\tau)\Gamma(1+\alpha\tau)}\left(\frac{|\phi(x)|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}}\right)^{-\tau}\mathrm{d}\tau.$$

Recognizing again the H-function, we obtain

$$= \frac{1}{\pi^{\frac{d}{2}}|\phi(x)|^d} H_{2,3}^{2,1}\left[\frac{|\phi(x)|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}}\middle|\begin{matrix}(1,1),(1,\alpha)\\ (1,1),(\frac{d}{2},\delta),(1,\delta)\end{matrix}\right] = G_0(x,t).$$

Moreover, since $0 < \alpha, \delta < 1$, and $\varphi(t) - \varphi(0) > 0$ for $t > 0$, the function

$$\zeta \mapsto E_{\alpha,1}\Big(-|\zeta|^{2\delta}(\varphi(t) - \varphi(0))^{\alpha}\Big)$$

is completely monotone with respect to $|\zeta|^{2\delta}$. Since the Mittag-Leffler function $E_{\alpha,1}(-r)$ is completely monotone for $r > 0$, it admits a subordination representation. Therefore, by the classical subordination principle for time-fractional diffusion equations (see [6]), its inverse Fourier transform is nonnegative. Hence, $G_0(x,t) \geq 0$ for all $x \in \mathbb{R}^d$, $t > 0$. Furthermore, since G_0 is the fundamental solution associated with the fractional diffusion operator and

$$\int_{\mathbb{R}^d} G_0(x,t)\,\mathrm{d}x = 1,$$

we have

$$G_0(x,t) > 0, \qquad x \in \mathbb{R}^d,\ t > 0.$$

This completes the proof. □

Lemma 3.3. *Let $d \in \mathbb{N}$, $\alpha \in (0,1)$, and $T > 0$. Assume that*

$$g \in L^q\Big((0,T), C_0(\mathbb{R}^d)\Big), \qquad q > 1.$$

Define

$$\Psi_g(t) = \int_0^t G_1\Big(\varphi^{-1}(\varphi(t) + \varphi(0) - \varphi(\tau)), \cdot\Big) *_{\phi} g(\tau,\cdot)\,\varphi'(\tau)\,\mathrm{d}\tau.$$

Then

$$\mathrm{I}_{0,t}^{1-\alpha,\varphi}\Psi_g(t) = \int_0^t G_0\Big(\varphi^{-1}(\varphi(t) + \varphi(0) - \varphi(\tau)), \cdot\Big) *_{\phi} g(\tau,\cdot)\,\varphi'(\tau)\,\mathrm{d}\tau.$$

Moreover, if $q\alpha > 1$, then

$$\Psi_g \in C\Big([0,T], C_0(\mathbb{R}^d)\Big).$$

Proof. We first prove the identity involving the fractional integral. By Definition 1.6 and Lemma 3.2, we have

$$\begin{aligned}
\mathrm{I}_{0,t}^{1-\alpha,\varphi}\Psi_g(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (\varphi(t) - \varphi(\tau))^{-\alpha}\varphi_g(\tau)\,\varphi'(\tau)\,\mathrm{d}\tau \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t\int_0^\tau (\varphi(t) - \varphi(\tau))^{-\alpha} G_1\Big(\varphi^{-1}(\varphi(\tau) + \varphi(0) - \varphi(s)), \cdot\Big) \\
&\qquad *_{\phi} g(s,\cdot)\,\varphi'(s)\,\mathrm{d}s\,\varphi'(\tau)\,\mathrm{d}\tau \\
&= \int_0^t G_0\Big(\varphi^{-1}(\varphi(t) + \varphi(0) - \varphi(s)), \cdot\Big) *_{\phi} g(s,\cdot)\,\varphi'(s)\,ds.
\end{aligned}$$

We next establish continuity. Set $Y = C_0(\mathbb{R}^d)$, let $h > 0$, and assume that $0 \leq t < t + h \leq T$.

We write

$$\Psi_g(t+h) - \Psi_g(t) = J_1 + J_2,$$

where

$$\begin{aligned} J_1 &= \int_t^{t+h} G_1\Big(\varphi^{-1}(\varphi(t+h) + \varphi(0) - \varphi(\tau)), \cdot\Big) *_\phi g(\tau, \cdot)\, \varphi'(\tau)\, \mathrm{d}\tau, \\ J_2 &= \int_0^t \Big[G_1\Big(\varphi^{-1}(\varphi(t+h) + \varphi(0) - \varphi(\tau)), \cdot\Big) - G_1\Big(\varphi^{-1}(\varphi(t) + \varphi(0) - \varphi(\tau)), \cdot\Big)\Big] \\ &\qquad *_\phi g(\tau, \cdot)\, \varphi'(\tau)\, \mathrm{d}\tau. \end{aligned}$$

By Lemma 3.2 and Holder's inequality, we estimate J_1 as follows

$$\begin{aligned} \|J_1\|_Y &\leq \frac{1}{\Gamma(\alpha)} \int_t^{t+h} (\psi(t+h) - \varphi(\tau))^{\alpha-1} \|g(\tau)\|_Y \varphi'(\tau) \mathrm{d}\tau \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_t^{t+h} \|g(\tau)\|_Y^q \varphi'(\tau) \mathrm{d}\tau \right)^{\frac{1}{q}} \left(\int_t^{t+h} (\varphi(t+h) - \varphi(\tau))^{\frac{q(\alpha-1)}{q-1}} \varphi'(\tau) \mathrm{d}\tau \right)^{\frac{q-1}{q}} \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\frac{q-1}{q\alpha - 1} \right)^{\frac{q-1}{q}} \|g\|_{L^q((0,T),Y)} (\varphi(T) - \varphi(0))^{\frac{1}{q}} (\varphi(t+h) - \varphi(t))^{\frac{q\alpha-1}{q}} \end{aligned}$$

To estimate J_2, we use the following standard splitting argument.

Set

$$h_\varphi = \varphi(t+h) - \varphi(t) > 0$$

and use the change of variables

$$r = \varphi(t) - \varphi(\tau).$$

Since φ is strictly increasing, we have

$$\varphi'(\tau)\, \mathrm{d}\tau = -\mathrm{d}r.$$

Therefore,

$$\begin{aligned} &\int_0^t \Big(\min\Big\{(\varphi(t) - \varphi(\tau))^{\alpha-1}, (\varphi(t) - \varphi(\tau))^{\alpha-2} h_\varphi\Big\}\Big)^{\frac{q}{q-1}} \varphi'(\tau)\, \mathrm{d}\tau \\ &= \int_0^{\varphi(t)-\varphi(0)} \Big(\min\Big\{r^{\alpha-1}, h_\varphi r^{\alpha-2}\Big\}\Big)^{\frac{q}{q-1}} \mathrm{d}r \\ &\leq \int_0^{\infty} \Big(\min\Big\{r^{\alpha-1}, h_\varphi r^{\alpha-2}\Big\}\Big)^{\frac{q}{q-1}} \mathrm{d}r. \end{aligned}$$

Splitting the last integral at $r = h_\varphi$, we obtain

$$\begin{aligned}
&\int_0^\infty \left(\min\left\{r^{\alpha-1}, h_\varphi r^{\alpha-2}\right\}\right)^{\frac{q}{q-1}} \mathrm{d}r \\
&= \int_0^{h_\varphi} r^{\frac{q(\alpha-1)}{q-1}} \mathrm{d}r + h_\varphi^{\frac{q}{q-1}} \int_{h_\varphi}^\infty r^{\frac{q(\alpha-2)}{q-1}} \mathrm{d}r \\
&= \frac{q-1}{q\alpha-1} h_\varphi^{\frac{q\alpha-1}{q-1}} + \frac{q-1}{q+1-q\alpha} h_\varphi^{\frac{q\alpha-1}{q-1}} \\
&= \frac{q(q-1)}{(q\alpha-1)(q+1-q\alpha)} h_\varphi^{\frac{q\alpha-1}{q-1}}.
\end{aligned}$$

Here the first integral is finite because $q\alpha > 1$, while the second integral is finite since

$$\frac{q(\alpha-2)}{q-1} < -1.$$

Consequently,

$$\|J_2\|_Y \leq C\|g\|_{L^q((0,T),Y)} \left(\varphi(t+h) - \varphi(t)\right)^{\frac{q\alpha-1}{q}}.$$

Since $q\alpha > 1$, both terms tend to zero as $h \to 0$. Therefore,

$$\Psi_g \in C\Big([0,T], C_0(\mathbb{R}^d)\Big).$$

This completes the proof. □

3.2 Local solution

In this section, we introduce the notion of mild solution to problem (3.1) and prove its local existence and uniqueness by means of a fixed-point argument.

Definition 3.1. Let $u_0 \in C_0(\mathbb{R}^d)$ and $T > 0$. A function

$$u \in C\Big([0,T], C_0(\mathbb{R}^d)\Big)$$

is called a mild solution of problem (3.1) if it satisfies

$$u(t) = G_0(t,\cdot) *_\phi u_0 + \int_0^t G_1\Big(\varphi^{-1}(\varphi(t) + \varphi(0) - \varphi(\tau)), \cdot\Big) *_\phi \Big(\mathrm{I}_{0,\tau}^{1-\gamma,\varphi}[e^u]\Big) \varphi'(\tau)\, \mathrm{d}\tau.$$

Theorem 3.2. *Let $d \in \mathbb{N}$, $\alpha \in (0,1)$, $\delta \in (0,1)$, and $u_0 \in C_0(\mathbb{R}^d)$. Then there exists a maximal time $T_{\max} > 0$ such that problem (3.1) admits a unique mild solution*

$$u \in C\Big([0,T_{\max}), C_0(\mathbb{R}^d)\Big).$$

Moreover, either $T_{\max} = \infty$*, or* $T_{\max} < \infty$ *and*

$$\|u\|_{L^\infty((0,T),L^\infty(\mathbb{R}^d))} \to \infty \qquad \text{as } t \uparrow T_{\max}.$$

In addition,

(i) if $u_0 \geq 0$ *and* $u_0 \not\equiv 0$*, then*

$$u(t,\cdot) > 0 \qquad \text{for all } 0 < t < T_{\max}.$$

(ii) if $u_0 \in L^r(\mathbb{R}^d)$ *for some* $1 \leq r < \infty$*, then*

$$u \in C\Big([0,T_{\max}), L^r(\mathbb{R}^d)\Big).$$

Proof. (i) We divide the proof into several steps.

Step 1. Definition of the fixed-point operator. For $T > 0$, define

$$E_{0,T} = \left\{ u \in C([0,T], C_0(\mathbb{R}^d)) \;\middle|\; \|u\|_{E_{0,T}} \leq 2\|u_0\|_{L^\infty(\mathbb{R}^d)} \right\},$$

where

$$\|u\|_{E_{0,T}} := \|u\|_{L^\infty((0,T),L^\infty(\mathbb{R}^d))}.$$

Since $C([0,T], C_0(\mathbb{R}^d))$ is a Banach space, the metric space $(E_{0,T}, d)$, endowed with

$$d(u,v) = \|u - v\|_{E_{0,T}},$$

is complete.

For $u \in E_{0,T}$, define

$$\Psi(u)(t) = G_0(t,\cdot) *_\phi u_0 + \int_0^t G_1\Big(\varphi^{-1}(\varphi(t) + \varphi(0) - \varphi(\tau)), \cdot\Big) *_\phi \left(\mathrm{I}_{0,\tau}^{1-\gamma,\varphi}[e^u]\right) \varphi'(\tau)\, d\tau.$$

We first show that the operator Ψ is well defined on $E_{0,T}$. Let $u \in E_{0,T}$. Since u is bounded, it follows that e^u is also bounded and belongs to $C([0,T], C_0(\mathbb{R}^d))$.

Moreover, the generalized fractional integral operator $\mathrm{I}_{0,\tau}^{1-\gamma,\varphi}$ preserves continuity in time. Combining this with the L^1-bounds of the kernels G_0 and G_1 and Young's inequality for the ϕ-convolution, we deduce that $\Psi(u)$ is continuous.

By Lemma 3.3, we have

$$\Psi(u) \in C([0,T], C_0(\mathbb{R}^d)).$$

We show that Ψ maps $E_{0,T}$ into itself for $T > 0$ sufficiently small. By Lemma 3.2,

$$\begin{aligned}
\|\Psi(u)(t)\|_{L^\infty(\mathbb{R}^d)} &\le \|G_0(t,\cdot) *_\phi u_0\|_{L^\infty(\mathbb{R}^d)} \\
&\quad + \int_0^t \left\| G_1\Big(\varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\tau)),\cdot\Big) *_\phi \left(\mathrm{I}_{0,\tau}^{1-\gamma,\varphi}[e^u]\right)\right\|_{L^\infty(\mathbb{R}^d)} \varphi'(\tau)\,\mathrm{d}\tau \\
&\le \|u_0\|_{L^\infty(\mathbb{R}^d)} \\
&\quad + \frac{e^{2\|u_0\|_{L^\infty(\mathbb{R}^d)}}}{\Gamma(\alpha)\Gamma(1-\gamma)} \int_0^t \int_0^\tau (\varphi(t)-\varphi(\tau))^{\alpha-1}(\varphi(\tau)-\varphi(s))^{-\gamma}\varphi'(s)\,ds\,\varphi'(\tau)\,\mathrm{d}\tau \\
&\le \|u_0\|_{L^\infty(\mathbb{R}^d)} + \frac{(\varphi(T)-\varphi(0))^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)} e^{2\|u_0\|_{L^\infty(\mathbb{R}^d)}}.
\end{aligned}$$

Hence, if $T > 0$ is chosen so that

$$\frac{(\varphi(T)-\varphi(0))^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)} e^{2\|u_0\|_{L^\infty(\mathbb{R}^d)}} \le \|u_0\|_{L^\infty(\mathbb{R}^d)},$$

then $\Psi(E_{0,T}) \subset E_{0,T}$.

Step 2. Contraction estimate.

Let $u, \widetilde{u} \in E_{0,T}$. By the mean value theorem applied to the exponential function, we have

$$|e^{u(\tau)} - e^{\widetilde{u}(\tau)}| \le e^{2\|u_0\|_{L^\infty(\mathbb{R}^d)}} |u(\tau) - \widetilde{u}(\tau)|.$$

Therefore,

$$\|\phi(u)(t) - \phi(\widetilde{u})(t)\|_{L^\infty(\mathbb{R}^d)} \le \frac{(\varphi(T)-\varphi(0))^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)} e^{2\|u_0\|_{L^\infty(\mathbb{R}^d)}} \|u-\widetilde{u}\|_{E_{0,T}}.$$

Choosing $T > 0$ smaller if necessary, we may assume that

$$\frac{(\varphi(T)-\varphi(0))^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)} e^{2\|u_0\|_{L^\infty(\mathbb{R}^d)}} \le \frac{1}{2}.$$

It follows that ϕ is a strict contraction on $E_{0,T}$. Hence, by Banach's fixed-point theorem, there exists a unique mild solution on $[0, T]$.

By standard continuation arguments, there exists a maximal existence time $T_{\max} \in (0, \infty]$ such that

$$u \in C([0, T_{\max}), C_0(\mathbb{R}^d)).$$

Step 3. Blow-up alternative.

Suppose that $T_{\max} < \infty$ and that there exists $M > 0$ such that

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq M \qquad \text{for all } 0 \leq t < T_{\max}.$$

Using the mild formulation and the continuity properties of the kernels, one verifies that $u(t)$ is a Cauchy sequence in $C_0(\mathbb{R}^d)$ as $t \uparrow T_{\max}$. Hence

$$u(t) \to u_{T_{\max}} \qquad \text{in } C_0(\mathbb{R}^d) \qquad \text{as } t \uparrow T_{\max}$$

for some $u_{T_{\max}} \in C_0(\mathbb{R}^d)$. Repeating the local fixed-point argument with initial value $u_{T_{\max}}$ at time $T_{\max}$, we extend the solution beyond $T_{\max}$, which contradicts the maximality of $T_{\max}$. Therefore, if $T_{\max} < \infty$, then necessarily

$$\|u\|_{L^\infty((0,T),L^\infty(\mathbb{R}^d))} \to \infty \qquad \text{as } t \uparrow T_{\max}.$$

Positivity.

If $u_0 \geq 0$ and $u_0 \not\equiv 0$, then the fixed-point argument may be carried out in the cone of nonnegative functions. Since $G_0(t,\cdot)$ is positive by Lemma 3.2, we obtain

$$u(t,\cdot) \geq G_0(t,\cdot) *_\phi u_0 > 0 \qquad \text{for } 0 < t < T_{\max}.$$

(ii) **L^r-regularity.**

Finally, assume that $u_0 \in C_0(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$, where $1 \leq r < \infty$. We define the space

$$E^r_{0,T} = \Big\{ u \in C([0,T], C_0(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)) : \|u\|_{L^\infty((0,T),L^\infty)} \leq 2\|u_0\|_{L^\infty}, \\ \|u\|_{L^\infty((0,T),L^r)} \leq 2\|u_0\|_{L^r} \Big\}.$$

Since the solution is already bounded in $L^\infty(\mathbb{R}^d)$, the exponential nonlinearity is locally Lipschitz, that is,

$$\|e^u - e^v\|_{L^r} \leq C\|u - v\|_{L^r}$$

for u, v in a bounded subset of $L^\infty(\mathbb{R}^d)$.

Moreover, using the fact that the kernels satisfy L^1-estimates and applying Young's inequality for the ϕ-convolution, we obtain that the operator Ψ maps $E^r_{0,T}$ into itself.

Repeating the fixed-point argument in this space, we deduce the existence of a unique solution

$$u \in C([0,T], L^r(\mathbb{R}^d)).$$

By continuation, this property extends to the maximal interval $[0, T_{\max})$.

This completes the proof. □

Example 3.1. We now illustrate the applicability of Theorem 3.2. Let

$$\psi(t) = t^2, \qquad \phi(x) = x,$$

and consider the nonlinear problem

$${}^{C}\mathrm{D}_{0+}^{\alpha,\psi} u(x,t) + (-\Delta)_x^{\delta} u(x,t) = \mathrm{I}_{0+}^{1-\gamma,\psi}[e^{u(x,t)}].$$

Assume that the initial data is given by

$$u_0(x) = \tfrac{1}{2}\, e^{-x^2}.$$

Clearly, $u_0 \in C_0(\mathbb{R})$ and $u_0 \geq 0$. Therefore, by Theorem 3.2, there exists a maximal time $T_{\max} > 0$ such that the problem admits a unique mild solution

$$u \in C\Big([0, T_{\max}), C_0(\mathbb{R}^d)\Big).$$

The corresponding mild solution is given by

$$u(x,t) = G_0(t) *_{\phi} u_0(x) + \int_0^t G_f\Big(\sqrt{t^2 - \tau^2}\Big) *_{\phi} \mathrm{I}_{0+}^{1-\gamma,\psi}[e^{u(x,\tau)}]\, 2\tau \, d\tau.$$

Moreover, the positivity property implies

$$u(x,t) > 0, \qquad x \in \mathbb{R},\ 0 < t < T_{\max}.$$

From the proof of Theorem 3.2, one may choose $T_0 > 0$ such that

$$\frac{(T_0^2)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)} e^{2\|u_0\|_\infty} \leq \frac{1}{2}.$$

Since $\|u_0\|_\infty = \frac{1}{2}$, this yields

$$T_{\max} \geq T_0 = \left(\frac{\Gamma(\alpha-\gamma+2)}{2e}\right)^{\frac{1}{2(\alpha-\gamma+1)}}.$$

This example shows that the abstract existence and uniqueness result applies to concrete initial data and yields an explicit lower bound for the lifespan of the solution. It also highlights the role of the nonlinear time scaling $\psi(t) = t^2$ in the evolution process.

Chapter 4

Well-posedness and Decay of a φ-Caputo Superdiffusion Equation in Lorentz Spaces

In this chapter, we investigate the local well-posedness and decay properties of solutions to the Cauchy problem for the semilinear time-fractional superdiffusion equation

$$\begin{cases} {}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi} u + (-\Delta)_x^{\delta} u = (\varphi(t)-\varphi(0))^{\sigma}\,\theta(x) + |u|^p, \quad (t,x) \in (0,+\infty)\times\mathbb{R}^d, \\ u(0) = u_0, \quad x \in \mathbb{R}^d, \\ \partial_{\varphi} u(0) = u_1, \quad x \in \mathbb{R}^d, \end{cases} \tag{4.1}$$

where $\alpha \in (1,2)$, $\delta \in (0,1)$, $\sigma \in (-1,0)$, $p > 1$, and $d \in \mathbb{N}$. The operator ${}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi}$ denotes the φ-Caputo derivative, $(-\Delta)_x^{\delta}$ is the fractional Laplacian, and

$$\partial_{\varphi} = \frac{1}{\varphi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}$$

denotes the so-called φ-derivative. The functions u_0 and u_1 represent the prescribed initial data, while θ is a given spatial source term. The analysis is carried out in Lorentz spaces $L^{r,s}(\mathbb{R}^d)$ and relies on decay estimates for the Green kernels associated with the corresponding linear problem.

The range $1 < \alpha < 2$ corresponds to the time-fractional superdiffusive regime. In contrast with the subdiffusive case $0 < \alpha < 1$, the present problem involves two initial conditions and combines diffusive and wave-like features. Moreover, the fractional Laplacian introduces spatial nonlocality, while the dependence on the function φ replaces the standard linear time scale by a more flexible one. Hence, problem (4.1) is nonlocal both in time and in space, and its analysis requires tools adapted to the generalized fractional framework.

The φ-fractional setting extends several classical operators. In particular, it includes the usual Caputo derivative when $\varphi(t) = t$, and the Caputo–Hadamard derivative when $\varphi(t) = \log t$ [36,58]. This generality is useful for describing evolution processes governed by non-uniform time scales. However, it also introduces additional analytical difficulties, especially in the derivation of the fundamental solution and the corresponding decay estimates.

The use of Lorentz spaces is another important feature of this chapter. These spaces refine the classical Lebesgue scale and are particularly suitable for convolution estimates involving singular kernels. In the present setting, the Green kernels generated by the linear problem exhibit both singular and decay behavior, for which Lorentz spaces provide a natural framework of analysis; see, for instance, [12, 14, 28, 54].

Several classical and fractional models are closely related to problem (4.1). The semilinear heat equation is governed by the Fujita critical exponent $p_c = 1 + \frac{2}{d}$ [56], while the semilinear wave equation is associated with a different critical threshold determined by the positive root of $(d-1)p^2 - (d+1)p - 2 = 0$ [33, 65]. For time-fractional diffusion equations with the classical Caputo derivative, Zhang and Sun [67] showed that the Fujita exponent remains $1 + \frac{2}{d}$ in the subdiffusive range $0 < \alpha < 1$. In the superdiffusive range $1 < \alpha < 2$, Zhang and Li [66] studied related semilinear problems with two initial conditions and obtained blow-up and global existence results reflecting the influence of the second datum.

Motivated by these works, we first derive explicit representations of the Green kernels for the corresponding linear problem by applying the Fourier transform in the spatial variable together with the φ-Laplace transform in time. These kernels are expressed in terms of Fox H-functions, and suitable decay estimates are established in Lorentz spaces. Using these estimates, we analyze the semilinear equation with source term $(\varphi(t) - \varphi(0))^{\sigma}\theta(x) + |u|^p$, where $\sigma \in (-1, 0)$ and $p > 1$. Through a fixed-point argument, we prove the local well-posedness of mild solutions. We also establish a blow-up alternative and discuss, at a formal level, the scaling condition suggested by the linear decay estimates. A complete global small-data theory is left as an open problem in this setting.

These results extend several classical studies on fractional diffusion equations to the broader framework of φ-fractional calculus in Lorentz spaces.

These estimates constitute the key analytical tool for the study of the nonlinear equation carried out in Section 4.2.

We first consider the linear Cauchy problem

$$\begin{cases} {}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi} u + (-\Delta)_x^{\delta} u = g(t,x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \\ \partial_{\varphi} u(0,x) = u_1(x), \end{cases} \tag{4.2}$$

where $1 < \alpha < 2$ and $0 < \delta < 1$. We assume first that

$$u_0, u_1 \in \mathcal{S}(\mathbb{R}^d), \qquad g \in C\Big([0,\infty); \mathcal{S}(\mathbb{R}^d)\Big).$$

The representation formula obtained under these regularity assumptions can later be extended to less regular data by density arguments.

The analysis proceeds in two steps. First, we derive a Green representation formula for the solution by combining the Fourier transform in the spatial variable with the φ-Laplace transform in time. Second, we study the structure of the resulting kernels and establish their decay properties in Lorentz spaces.

Theorem 4.1 (Green representation formula)**.** *et $d \geq 1$, $\alpha \in (1,2)$, and $\delta \in (0,1)$.*

The solution to problem (4.2) is given by

$$u(t,x) = \Big(G_0^{\alpha,\varphi}(t,\cdot) * u_0\Big)(x) + \Big(G_1^{\alpha,\varphi}(t,\cdot) * u_1\Big)(x) \\ + \int_0^t \Big(G_g^{\alpha,\varphi}\Big(\varphi^{-1}(\varphi(t) + \varphi(0) - \varphi(\eta)),\cdot\Big) * g(\eta,\cdot)\Big)(x)\, \varphi'(\eta)\, \mathrm{d}\eta,$$

where

$$G_0^{\alpha,\varphi}(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle x,\zeta\rangle} E_{\alpha,1}\Big(-|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}\Big)\, \mathrm{d}\zeta, \tag{4.3}$$

$$G_1^{\alpha,\varphi}(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle x,\zeta\rangle} (\varphi(t)-\varphi(0)) E_{\alpha,2}\Big(-|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}\Big)\, \mathrm{d}\zeta, \tag{4.4}$$

and

$$G_g^{\alpha,\varphi}(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle x,\zeta\rangle} (\varphi(t)-\varphi(0))^{\alpha-1} E_{\alpha,\alpha}\Big(-|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}\Big)\, \mathrm{d}\zeta. \tag{4.5}$$

The functions $G_0^{\alpha,\varphi}$, $G_1^{\alpha,\varphi}$, and $G_g^{\alpha,\varphi}$ are the corresponding Green kernels, and $E_{\alpha,1}$, $E_{\alpha,2}$, and $E_{\alpha,\alpha}$ denote the Mittag–Leffler functions.

Proof. The proof relies on the Fourier transform with respect to the spatial variable and the φ-Laplace transform in time.

Applying the Fourier transform in x to equation (4.2), we obtain

$${}^{C}D_{0,t}^{\alpha,\varphi}\widehat{u}(t,\zeta) + |\zeta|^{2\delta}\widehat{u}(t,\zeta) = \widehat{g}(t,\zeta), \tag{4.6}$$

with the initial conditions

$$\widehat{u}(0,\zeta) = \widehat{u_0}(\zeta), \qquad \partial_\varphi \widehat{u}(0,\zeta) = \widehat{u_1}(\zeta).$$

Thus, for each fixed $\zeta \in \mathbb{R}^d$, we obtain a fractional ordinary differential equation in time.

Applying the φ-Laplace transform to (3.5) and using Corollary 1.2, we obtain

$$\Big(w^{\alpha} + |\zeta|^{2\delta}\Big)\, \mathcal{L}_\varphi\{\widehat{u}(t,\zeta)\} = w^{\alpha-1}\widehat{u_0}(\zeta) + w^{\alpha-2}\widehat{u_1}(\zeta) + \mathcal{L}_\varphi\{\widehat{g}(t,\zeta)\}.$$

Therefore,

$$\mathcal{L}_\varphi\{\widehat{u}(t,\zeta)\} = \frac{w^{\alpha-1}}{w^{\alpha}+|\zeta|^{2\delta}}\widehat{u_0}(\zeta) + \frac{w^{\alpha-2}}{w^{\alpha}+|\zeta|^{2\delta}}\widehat{u_1}(\zeta) + \frac{1}{w^{\alpha}+|\zeta|^{2\delta}}\mathcal{L}_\varphi\{\widehat{g}(t,\zeta)\}.$$

Using the identities for the φ-Laplace transform of Mittag–Leffler functions (Lemma 1.1), we obtain

$$\widehat{u}(t,\zeta) = \widehat{u_0}(\zeta)E_{\alpha,1}\Big(-|\zeta|^{2\delta}\Phi(t)^{\alpha}\Big) + \widehat{u_1}(\zeta)\Phi(t)E_{\alpha,2}\Big(-|\zeta|^{2\delta}\Phi(t)^{\alpha}\Big)$$
$$+\int_0^t (\Phi(t)-\Phi(\eta))^{\alpha-1}E_{\alpha,\alpha}\Big(-|\zeta|^{2\delta}(\Phi(t)-\Phi(\eta))^{\alpha}\Big)\,\widehat{g}(\eta,\zeta)\varphi'(\eta)\,d\eta,$$

where $\Phi(t) = \varphi(t) - \varphi(0)$.

Applying the inverse Fourier transform with respect to ζ yields

$$\begin{aligned} u(t,x) &= \Big(G_0^{\alpha,\varphi}(t,\cdot) * u_0\Big)(x) + \Big(G_1^{\alpha,\varphi}(t,\cdot) * u_1\Big)(x) \\ &\quad + \int_0^t \Big(G_g^{\alpha,\varphi}\Big(\varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\eta)),\cdot\Big) * g(\eta,\cdot)\Big)(x)\,\varphi'(\eta)\,\mathrm{d}\eta, \end{aligned}$$

where the kernels $G_0^{\alpha,\varphi}$, $G_1^{\alpha,\varphi}$, and $G_g^{\alpha,\varphi}$ are defined by (4.3)–(4.5).

This completes the proof. □

In the following lemma, we express the Green kernels $G_0^{\alpha,\varphi}$, $G_1^{\alpha,\varphi}$, and $G_g^{\alpha,\varphi}$ in terms of Fox H-functions.

Lemma 4.1. *The Green kernels $G_0^{\alpha,\varphi}$, $G_1^{\alpha,\varphi}$, and $G_g^{\alpha,\varphi}$, defined in* (4.3)–(4.5)*, satisfy the following identities:*

1.

$$G_1^{\alpha,\varphi}(t,x) = \frac{\varphi(t)-\varphi(0)}{\pi^{\frac{d}{2}}|x|^d} H_{2,3}^{2,1}\left[\frac{|x|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}}\middle|\begin{matrix}(1,1),(2,\alpha)\\ \left(\frac{d}{2},\delta\right),(1,1),(1,\delta)\end{matrix}\right]. \tag{4.7}$$

2.

$$G_0^{\alpha,\varphi}(t,x) = \frac{1}{\pi^{\frac{d}{2}}|x|^d} H_{2,3}^{2,1}\left[\frac{|x|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}}\middle|\begin{matrix}(1,1),(1,\alpha)\\ \left(\frac{d}{2},\delta\right),(1,1),(1,\delta)\end{matrix}\right], \tag{4.8}$$

and

$$G_g^{\alpha,\varphi}(t,x) = \frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\pi^{\frac{d}{2}}|x|^d} H_{2,3}^{2,1}\left[\frac{|x|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}}\middle|\begin{matrix}(1,1),(\alpha,\alpha)\\ \left(\frac{d}{2},\delta\right),(1,1),(1,\delta)\end{matrix}\right]. \tag{4.9}$$

Proof. 1. We prove the first identity. By (1.3), (1.2), and (1.1), we have

$$\begin{aligned}
G_1^{\alpha,\varphi}(t,x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle x,\zeta\rangle}(\varphi(t)-\varphi(0))E_{\alpha,2}\left(-|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}\right)\mathrm{d}\zeta \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}}|x|^{1-\frac{d}{2}}\int_0^{\infty}|\zeta|^{\frac{d}{2}}(\varphi(t)-\varphi(0))H_{1,2}^{1,1}\left[|\zeta|^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}\Big|_{(0,1),(-1,\alpha)}^{(0,1)}\right] \\
&\quad\times \mathrm{J}_{\frac{d}{2}-1}(|x||\zeta|)\mathrm{d}|\zeta| \\
&= \frac{\varphi(t)-\varphi(0)}{\pi^{\frac{d}{2}}|x|^d}H_{3,2}^{1,2}\left[(\varphi(t)-\varphi(0))^{\alpha}\left(\frac{2}{|x|}\right)^{2\delta}\Bigg|_{(0,1),(-1,\alpha)}^{\left(1-\frac{d}{2},\delta\right),(0,1),(0,\delta)}\right] \\
&= \frac{\varphi(t)-\varphi(0)}{\pi^{\frac{d}{2}}|x|^d}H_{2,3}^{2,1}\left[\frac{|x|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}}\Bigg|_{\left(\frac{d}{2},\delta\right),(1,1),(1,\delta)}^{(1,1),(2,\alpha)}\right].
\end{aligned}$$

2. For parts (2) and (3), the proof follows exactly the same arguments as in part (1), replacing the Mittag–Leffler function $E_{\alpha,2}$ by $E_{\alpha,1}$ and $E_{\alpha,\alpha}$, respectively. Using the same integral representation and the standard properties of the Fox H-function, we obtain the corresponding formulas for $G_0^{\alpha,\varphi}$ and $G_g^{\alpha,\varphi}$. This completes the proof.

□

4.1 Asymptotic estimates for the Green kernels

Using the asymptotic properties of the corresponding Fox H-functions, we obtain the following pointwise estimates for the Green kernels $G_0^{\alpha,\varphi}$, $G_1^{\alpha,\varphi}$, and $G_g^{\alpha,\varphi}$. Throughout this section we fix $d \in \mathbb{N}$, $1 < \alpha < 2$, and $0 < \delta < 1$.

For simplicity, we set $\Phi(t) := \varphi(t) - \varphi(0), \qquad t > 0$. We denote

$$R = \Phi(t)^{-\alpha}|x|^{2\delta}.$$

The asymptotic estimates for the Green kernels follow from the known properties of the corresponding kernels in the classical Caputo case. Since the time variable appears through the quantity

$$\Phi(t) = \varphi(t) - \varphi(0),$$

the same arguments apply after a change of time scale. Consequently, the estimates established in [38, 39] remain valid in the present φ-fractional setting.

Theorem 4.2 ([38]). *1. If $R > 1$, then*

$$|G_0^{\alpha,\varphi}(t,x)| \le C\Phi(t)^{\alpha}|x|^{-d-2\delta}, \qquad \textit{for } d \ge 1.$$

2. If $R \leq 1$, then

$$|G_0^{\alpha,\varphi}(t,x)| \leq C\Phi(t)^{-\alpha}|x|^{-d+2\delta}, \qquad \text{for } d > 2\delta,$$

$$|G_0^{\alpha,\varphi}(t,x)| \leq C\Phi(t)^{-\alpha}\left(1 + \left|\log\left(\frac{1}{2}\Phi(t)^{-\alpha}|x|\right)\right|\right), \qquad \text{for } d = 2\delta,$$

$$|G_0^{\alpha,\varphi}(t,x)| \leq C\Phi(t)^{-\frac{\alpha}{2\delta}}, \qquad \text{for } d < 2\delta.$$

Theorem 4.3 ([39])**.** *1. If $R > 1$, then*

$$|G_1^{\alpha,\varphi}(t,x)| \leq C\Phi(t)^{\alpha+1}|x|^{-d-2\delta}, \qquad \text{for } d \geq 1.$$

2. If $R \leq 1$, then

$$|G_1^{\alpha,\varphi}(t,x)| \leq C\Phi(t)^{-\alpha+1}|x|^{-d+2\delta}, \qquad \text{for } d > 2\delta,$$

$$|G_1^{\alpha,\varphi}(t,x)| \leq C\Phi(t)^{-\alpha+1}\left(1 + \left|\log\left(\frac{1}{2}\Phi(t)^{-\alpha}|x|\right)\right|\right), \qquad \text{for } d = 2\delta,$$

$$|G_1^{\alpha,\varphi}(t,x)| \leq C\Phi(t)^{-\frac{\alpha}{2\delta}+1}, \qquad \text{for } d < 2\delta.$$

Theorem 4.4 ([38])**.** *1. If $R > 1$, then*

$$|G_g^{\alpha,\varphi}(t,x)| \leq C\Phi(t)^{2\alpha-1}|x|^{-d-2\delta}, \qquad \text{for } d \geq 1.$$

2. If $R \leq 1$, then

$$|G_g^{\alpha,\varphi}(t,x)| \leq C\Phi(t)^{-\alpha-1}|x|^{-d+4\delta}, \qquad \text{for } d > 4\delta,$$

$$|G_g^{\alpha,\varphi}(t,x)| \leq C\Phi(t)^{-\alpha-1}\left(1 + \left|\log\left(\left(\frac{|x|}{2}\right)^{2\delta}\Phi(t)^{-\alpha}\right)\right|\right), \qquad \text{for } d = 4\delta,$$

$$|G_g^{\alpha,\varphi}(t,x)| \leq C\Phi(t)^{\alpha-1-\frac{\alpha d}{2\delta}}, \qquad \text{for } d < 4\delta.$$

with $\alpha \in (1,2)$ and $\delta \in (0,1)$. Next, we investigate the $L^{r,s}(\mathbb{R}^d)$-estimates of the Green functions $G_0^{\alpha,\varphi}$, $G_1^{\alpha,\varphi}$, and $G_g^{\alpha,\varphi}$. For simplicity, we omit the variable x hereafter; that is, $G_0^{\alpha,\varphi}(t) = G_0^{\alpha,\varphi}(t,x)$, $G_1^{\alpha,\varphi}(t) = G_1^{\alpha,\varphi}(t,x)$, and $G_g^{\alpha,\varphi}(t) = G_g^{\alpha,\varphi}(t,x)$.

Theorem 4.5 (Lorentz estimates)**.** *Define*

$$\kappa(\delta,d) = \frac{d}{d-2\delta}.$$

Assume that $d > 2\delta$, $1 < r < \kappa(\delta, d)$ *and* $1 \leq s \leq \infty$. *Then:*

1.
$$G_0^{\alpha,\varphi}(t,\cdot) \in L^{r,s}(\mathbb{R}^d), \qquad \|G_0^{\alpha,\varphi}(t)\|_{r,s} \leq C_{r,s}\,(\Phi(t))^{\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)}, \quad t > 0.$$

2.
$$G_1^{\alpha,\varphi}(t,\cdot) \in L^{r,s}(\mathbb{R}^d), \qquad \|G_1^{\alpha,\varphi}(t)\|_{r,s} \leq C_{r,s}\,(\Phi(t))^{1+\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)}, \quad t > 0.$$

Proof. From the Fourier representations (4.3) and (4.4), we obtain the scaling forms

$$G_0^{\alpha,\varphi}(t,x) = \Phi(t)^{-\frac{\alpha d}{2\delta}} K_0\Big(x\Phi(t)^{-\frac{\alpha}{2\delta}}\Big),$$

and

$$G_1^{\alpha,\varphi}(t,x) = \Phi(t)^{1-\frac{\alpha d}{2\delta}} K_1\Big(x\Phi(t)^{-\frac{\alpha}{2\delta}}\Big),$$

where

$$K_0(x) := G_0^{\alpha,\varphi}(t,x)\Big|_{\Phi(t)=1}, \qquad K_1(x) := G_1^{\alpha,\varphi}(t,x)\Big|_{\Phi(t)=1}.$$

Step 1. Integrability of K_0 and K_1.

By Theorems 4.2 and 4.3, the kernels K_0 and K_1 satisfy

$$|K_j(x)| \leq C|x|^{-d-2\delta}, \qquad |x| > 1,$$

and

$$|K_j(x)| \leq C|x|^{-d+2\delta}, \qquad |x| \leq 1,$$

for $j = 0, 1$. Therefore,

$$K_0, K_1 \in L^p(\mathbb{R}^d) \qquad \text{for all } 1 \leq p < \kappa(\delta, d).$$

Step 2. Interpolation.

Fix $1 < r < \kappa(\delta, d)$ and $1 \leq s \leq \infty$. Choose r_0, r_1 such that

$$1 < r_0 < r < r_1 < \kappa(\delta, d).$$

Since

$$K_0, K_1 \in L^{r_0}(\mathbb{R}^d) \cap L^{r_1}(\mathbb{R}^d),$$

the real interpolation identity

$$(L^{r_0}, L^{r_1})_{\vartheta,s} = L^{r,s}$$

implies that

$$K_0, K_1 \in L^{r,s}(\mathbb{R}^d).$$

Step 3. Scaling.

Using the scaling property of Lorentz spaces,

$$\|f(\lambda\cdot)\|_{r,s} = \lambda^{-d/r}\|f\|_{r,s}, \qquad \lambda > 0,$$

we obtain

$$\|G_0^{\alpha,\varphi}(t)\|_{r,s} = \Phi(t)^{-\frac{\alpha d}{2\delta}} \left\|K_0\Big(\Phi(t)^{-\frac{\alpha}{2\delta}}\cdot\Big)\right\|_{r,s} \le C_{r,s}\Phi(t)^{\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)}.$$

Similarly,

$$\|G_1^{\alpha,\varphi}(t)\|_{r,s} \le C_{r,s}\Phi(t)^{1+\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)}.$$

This completes the proof. □

Lemma 4.2. *Define*

$$\varpi(\delta, d) = \frac{d}{d-4\delta}.$$

Assume that $d > 4\delta$, $1 < r < \varpi(\delta, d)$ *and* $1 \le s \le \infty$*. Then*

$$G_g^{\alpha,\varphi}(t,\cdot) \in L^{r,s}(\mathbb{R}^d),$$

and

$$\|G_g^{\alpha,\varphi}(t)\|_{r,s} \le C_{r,s}\,\Phi(t)^{\alpha-1+\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)}, \qquad t > 0.$$

Proof. From (4.3), we obtain

$$G_g^{\alpha,\varphi}(t,x) = \Phi(t)^{\alpha-1-\frac{\alpha d}{2\delta}} K_g\Big(x\Phi(t)^{-\frac{\alpha}{2\delta}}\Big),$$

where

$$K_g(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} E_{\alpha,\alpha}(-|\xi|^{2\delta})\,\mathrm{d}\xi.$$

By Theorem 4.4, the kernel K_g satisfies

$$|K_g(x)| \le C|x|^{-d-2\delta}, \qquad |x| > 1,$$

and

$$|K_g(x)| \le C|x|^{-d+4\delta}, \qquad |x| \le 1.$$

It follows that $K_g \in L^p(\mathbb{R}^d)$ for every

$$1 \le p < \varpi(\delta, d).$$

Indeed, the decay at infinity implies L^p-integrability for all $p \ge 1$, whereas near the origin the condition

$$|K_g(x)|^p \lesssim |x|^{-p(d-4\delta)}$$

is locally integrable if and only if

$$p(d-4\delta) < d,$$

that is,

$$p < \frac{d}{d-4\delta}$$

since $d > 4\delta$.

Now fix $1 < r < \varpi(\delta, d)$ and $1 \leq s \leq \infty$. Choose r_0, r_1 such that

$$1 < r_0 < r < r_1 < \varpi(\delta, d).$$

Since

$$K_g \in L^{r_0}(\mathbb{R}^d) \cap L^{r_1}(\mathbb{R}^d),$$

the real interpolation identity

$$(L^{r_0}, L^{r_1})_{\vartheta,s} = L^{r,s}$$

yields

$$K_g \in L^{r,s}(\mathbb{R}^d).$$

Using again the scaling property of Lorentz spaces, we obtain

$$\|G_g^{\alpha,\varphi}(t)\|_{r,s} = \Phi(t)^{\alpha-1-\frac{\alpha d}{2\delta}} \left\|K_g\Big(\Phi(t)^{-\frac{\alpha}{2\delta}}\cdot\Big)\right\|_{r,s},$$

and therefore

$$\|G_g^{\alpha,\varphi}(t)\|_{r,s} \leq C_{r,s}\,\Phi(t)^{\alpha-1+\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)}.$$

This completes the proof. □

Remark 4.1. When $s = r$, that is, in the Lebesgue space L^r, and $\varphi(t) = \log t$, the estimates of Theorem 4.5 and Lemma 4.2 reduce to the corresponding results obtained in [38, 40].

We next derive decay estimates for the Green operators in Lorentz spaces.

Lemma 4.3. *Let $1 < \rho < \tau < \infty$, $1 \leq q \leq s \leq \infty$, and assume that*

$$\frac{1}{\rho} - \frac{1}{\tau} < \min\left\{1, \frac{2\delta}{d}\right\}.$$

Then, for all $t > 0$,

$$\|G_0^{\alpha,\varphi}(t) * u_0\|_{\tau,s} \leq C\,\Phi(t)^{\frac{\alpha d}{2\delta}\left(\frac{1}{\tau}-\frac{1}{\rho}\right)}\|u_0\|_{\rho,q}, \tag{4.10}$$

and

$$\|G_1^{\alpha,\varphi}(t) * u_1\|_{\tau,s} \leq C\,\Phi(t)^{1+\frac{\alpha d}{2\delta}\left(\frac{1}{\tau}-\frac{1}{\rho}\right)}\|u_1\|_{\rho,q}. \tag{4.11}$$

Proof. Choose $r \in (1,\infty)$ such that

$$1+\frac{1}{\tau}=\frac{1}{\rho}+\frac{1}{r}.$$

The assumption

$$\frac{1}{\rho}-\frac{1}{\tau}<\min\left\{1,\frac{2\delta}{d}\right\}$$

ensures that $1<r<\kappa(\delta,d)$. Using Young's inequality in Lorentz spaces together with Theorem 4.5, we obtain

$$\|G_0^{\alpha,\varphi}(t) * u_0\|_{\tau,s} \leq C\|G_0^{\alpha,\varphi}(t)\|_{r,\infty}\|u_0\|_{\rho,q},$$

and similarly for $G_1^{\alpha,\varphi}$. The claimed estimates then follow from the bounds in Theorem 4.5. □

Lemma 4.4. *Let* $1<\rho<\tau<\infty$, $1\leq q\leq s\leq\infty$, *and assume that*

$$\frac{1}{\rho}-\frac{1}{\tau}<\min\left\{1,\frac{4\delta}{d}\right\}.$$

Then, for all $t>0$,

$$\|G_g^{\alpha,\varphi}(t) * f\|_{\tau,s} \leq C\,\Phi(t)^{\alpha-1+\frac{\alpha d}{2\delta}\left(\frac{1}{\tau}-\frac{1}{\rho}\right)}\|f\|_{\rho,q}. \tag{4.12}$$

Proof. Choose $r \in (1,\infty)$ such that

$$1+\frac{1}{\tau}=\frac{1}{\rho}+\frac{1}{r}.$$

The assumption

$$\frac{1}{\rho}-\frac{1}{\tau}<\min\left\{1,\frac{4\delta}{d}\right\}$$

ensures that $1<r<\varpi(\delta,d)$. Hence, by Young's inequality in Lorentz spaces and Lemma 4.2,

$$\|G_g^{\alpha,\varphi}(t) * f\|_{\tau,s} \leq C\|G_g^{\alpha,\varphi}(t)\|_{r,\infty}\|f\|_{\rho,q},$$

which yields (4.12). □

We now describe the decay behavior of solutions to (4.2) in the homogeneous case $g\equiv 0$.

Theorem 4.6. *Assume that* $g\equiv 0$, *and let*

$$u(t)=G_0^{\alpha,\varphi}(t) * u_0+G_1^{\alpha,\varphi}(t) * u_1.$$

Suppose that $1\leq q\leq s,s_1,s_2\leq\infty$, $1<\rho<\tau<\infty$, *and*

$$1+\frac{1}{\tau}=\frac{1}{\rho}+\frac{1}{r},\qquad \frac{1}{s}=\frac{1}{s_1}+\frac{1}{s_2},$$

for some $1 < r < \kappa(\delta, d)$*. If* $u_0, u_1 \in L^{\rho,q}(\mathbb{R}^d)$*, then, for all* $t > 0$*,*

$$\|u(t)\|_{\tau,s} \le C\Phi(t)^{\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)}\|u_0\|_{\rho,q} + C\Phi(t)^{1+\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)}\|u_1\|_{\rho,q}.$$

Proof. By Young's inequality in Lorentz spaces and Theorem 4.5,

$$\|u(t)\|_{\tau,s} \le \|G_0^{\alpha,\varphi}(t) * u_0\|_{\tau,s} + \|G_1^{\alpha,\varphi}(t) * u_1\|_{\tau,s}.$$

Using the convolution estimate,

$$\|G_0^{\alpha,\varphi}(t) * u_0\|_{\tau,s} \le C\|G_0^{\alpha,\varphi}(t)\|_{r,s_1}\|u_0\|_{\rho,s_2},$$

and similarly for the second term. Since $q \le s_2$, the monotonicity of Lorentz norms yields

$$\|u_j\|_{\rho,s_2} \le C\|u_j\|_{\rho,q}, \qquad j = 0, 1.$$

Combining these bounds with Theorem 4.5, we obtain the desired estimate. □

We next consider the inhomogeneous case with vanishing initial data.

Theorem 4.7. *Suppose that* $u_0 = u_1 \equiv 0$*, and let* $1 \le q \le s, s_1, s_2 \le \infty$*,* $1 < \rho < \tau < \infty$*, and*

$$1 + \frac{1}{\tau} = \frac{1}{\rho} + \frac{1}{r}, \qquad \frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2},$$

for some $1 < r < \varpi(\delta, d)$*. Assume that* $g(\cdot, t) \in L^{\rho,s_2}(\mathbb{R}^d)$ *for all* $t > 0$ *and that*

$$\|g(t)\|_{\rho,s_2} \le C\Big(1 + \Phi(t)\Big)^{-\gamma}, \tag{4.13}$$

for some $\gamma > 0$*. Then*

$$u(t) = \int_0^t G_g^{\alpha,\varphi}\Big(\varphi^{-1}(\varphi(t) + \varphi(0) - \varphi(\eta))\Big) * g(\eta)\, \varphi'(\eta)\, d\eta$$

satisfies the following estimates:

1. *If* $\gamma \ne 1$*, then*
$$\|u(t)\|_{\tau,s} \le C\Phi(t)^{\alpha-\min\{1,\gamma\}+\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)}. \tag{4.14}$$

2. *If* $\gamma = 1$*, then*
$$\|u(t)\|_{\tau,s} \le C\Phi(t)^{\alpha-1+\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)}\log\Big(1 + \Phi(t)\Big). \tag{4.15}$$

Proof. 1. By Minkowski's inequality and Young's inequality in Lorentz spaces,

$$\|u(t)\|_{\tau,s} \le C \int_0^t \left\| G_g^{\alpha,\varphi}\Big(\varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\eta))\Big)\right\|_{r,s_1} \|g(\eta)\|_{\rho,s_2}\, \varphi'(\eta)\, \mathrm{d}\eta.$$

We split the integral into two parts:

$$\|u(t)\|_{\tau,s} \le C(I + II),$$

where

$$I := \int_0^{\varphi^{-1}\left(\frac{\varphi(0)+\varphi(t)}{2}\right)} \left\| G_g^{\alpha,\varphi}\Big(\varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\eta))\Big)\right\|_{r,s_1} \|g(\eta)\|_{\rho,s_2}\, \varphi'(\eta)\, \mathrm{d}\eta,$$

and

$$II := \int_{\varphi^{-1}\left(\frac{\varphi(0)+\varphi(t)}{2}\right)}^t \left\| G_g^{\alpha,\varphi}\Big(\varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\eta))\Big)\right\|_{r,s_1} \|g(\eta)\|_{\rho,s_2}\, \varphi'(\eta)\, \mathrm{d}\eta.$$

For the first term, note that

$$\varphi^{-1}\left(\frac{\varphi(0)+\varphi(t)}{2}\right) \le \varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\eta)) \le t.$$

For the first term, let the time argument in the Green kernel be denoted by

$$z := \varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\eta)).$$

For $\eta \in \left[0, \varphi^{-1}\left(\frac{\varphi(0)+\varphi(t)}{2}\right)\right]$, we observe that $\Phi(\eta) \le \frac{\Phi(t)}{2}$. Consequently,

$$\Phi(z) = \varphi(t) - \varphi(\eta) \ge \frac{\varphi(t)-\varphi(0)}{2} = \frac{\Phi(t)}{2}.$$

Using Lemma 4.2, the norm of the Green kernel can be bounded as

$$\left\| G_g^{\alpha,\varphi}(z)\right\|_{r,s_1} \le C(\Phi(t))^{\alpha-1+\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)}.$$

Substituting this bound and the decay assumption (4.13) into the integral I, and applying the change of variables $w = \Phi(\eta)$ (so that $\mathrm{d}w = \varphi'(\eta)\,\mathrm{d}\eta$), we obtain

$$
\begin{aligned}
I &\le C(\Phi(t))^{\alpha-1+\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)} \int_0^{\frac{\Phi(t)}{2}} (1+w)^{-\gamma}\,\mathrm{d}w \\
&\le C(\Phi(t))^{\alpha-1+\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)} \times \begin{cases} C\left(1+\frac{\Phi(t)}{2}\right)^{1-\gamma}, & 0<\gamma<1, \\ C, & \gamma>1. \end{cases}
\end{aligned}
$$

By absorbing the constants and noting the asymptotic behavior for large t, we obtain, for $\gamma \neq 1$,

$$I \le C(\Phi(t))^{\alpha-\min\{1,\gamma\}+\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)}. \tag{4.16}$$

For the second term, we use the singular behavior of $G_g^{\alpha,\varphi}$ as the time variable approaches zero. By Lemma 4.2, local integrability near the upper limit requires

$$\alpha - 1 + \frac{\alpha d}{2\delta}\left(\frac{1}{r} - 1\right) > -1. \tag{4.17}$$

This condition is compatible with the range of exponents considered here; in particular, it is satisfied under the assumptions imposed on r in the statement of the theorem. Consequently,

$$II \le C(\Phi(t))^{\alpha-\gamma+\frac{\alpha d}{2\delta}\left(\frac{1}{r}-1\right)}. \tag{4.18}$$

Combining (4.16) and (4.18), we obtain (4.14).

2. If $\gamma = 1$, the borderline contribution produces a logarithmic correction, and the same splitting argument yields (4.15). This completes the proof.

□

The estimates established above constitute the main linear ingredients for the fixed-point analysis carried out in the next section.

4.2 Well-posedness and qualitative properties

Local well-posedness

Definition 4.1. Let $T > 0$. A function

$$u \in C([0,T], L^{\rho,s}(\mathbb{R}^d)) \cap C((0,T], L^{\tau,s}(\mathbb{R}^d))$$

is called a mild solution of problem (4.1) on $[0,T]$ if it satisfies the integral equation

$$\begin{aligned} u(t) &= G_0^{\alpha,\varphi}(t) * u_0 + G_1^{\alpha,\varphi}(t) * u_1 \\ &\quad + \int_0^t G_g^{\alpha,\varphi}\Big(\varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\eta))\Big) * \Big[(\varphi(\eta)-\varphi(0))^{\sigma}\theta + |u(\eta)|^p\Big]\, \varphi'(\eta) \mathrm{d}\eta. \end{aligned}$$

For $\tau \in [\rho, \infty]$, we set

$$\beta^{\tau} := \frac{\alpha d}{2\delta}\left(\frac{1}{\rho} - \frac{1}{\tau}\right).$$

Lemma 4.5. *Let $1 < \rho < \infty$. Assume that $\theta \in L^{\rho,s}(\mathbb{R}^d)$ and*

$$\sup_{t\in(0,T)} \Phi(t)^{\beta^{p\rho}} \|u(t)\|_{p\rho,ps} < \infty.$$

Define

$$\psi_u(t) = \int_0^t G_g^{\alpha,\varphi}\Big(\varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\eta))\Big) * \Big[(\varphi(\eta)-\varphi(0))^{\sigma}\theta + |u(\eta)|^p\Big]\varphi'(\eta)\, \mathrm{d}\eta.$$

If

$$0 \le \frac{1}{\rho} - \frac{1}{\tau} < \min\left\{1, \frac{2\delta}{d}\right\},$$

then

$$\psi_u \in C((0,T], L^{\tau,s}(\mathbb{R}^d)).$$

Moreover, if

$$\beta^{\tau} < \alpha + \sigma \qquad \text{and} \qquad \alpha - p\beta^{p\rho} - \beta^{\tau} > 0,$$

then

$$\psi_u \in C([0,T], L^{\tau,s}(\mathbb{R}^d)).$$

Proof. Let $t > 0$ and $0 < \varepsilon \leq T - t$. We write

$$\psi_u(t+\varepsilon) - \psi_u(t) = I_1 + I_2,$$

where

$$I_1 = \int_t^{t+\varepsilon} G_g^{\alpha,\varphi}\Big(\varphi^{-1}(\varphi(t+\varepsilon) + \varphi(0) - \varphi(\eta))\Big) * F(\eta)\, \varphi'(\eta)\, d\eta,$$

$$I_2 = \int_0^t \Big[K(t+\varepsilon,\eta) - K(t,\eta)\Big] * F(\eta)\, \varphi'(\eta)\, d\eta,$$

with

$$K(t,\eta) := G_g^{\alpha,\varphi}\Big(\varphi^{-1}(\varphi(t) + \varphi(0) - \varphi(\eta))\Big), \qquad F(\eta) = (\Phi(\eta))^{\sigma}\theta + |u(\eta)|^p.$$

Estimate of I_1.

Using Lemma 4.4 and the boundedness of $F(\eta)$ on $[t, t+\varepsilon]$, we obtain

$$\|I_1\|_{\tau,s} \leq C \int_t^{t+\varepsilon} (\Phi(t+\varepsilon) - \Phi(\eta))^{\alpha-\beta^{\tau}-1}\, \varphi'(\eta)\, d\eta.$$

A direct computation gives

$$\|I_1\|_{\tau,s} \leq C(\Phi(t+\varepsilon) - \Phi(t))^{\alpha-\beta^{\tau}},$$

which tends to zero as $\varepsilon \to 0^+$ since $\alpha - \beta^{\tau} > 0$.

Estimate of I_2.

For fixed $\eta < t$, we have

$$K(t+\varepsilon,\eta) \to K(t,\eta) \quad \text{as } \varepsilon \to 0^+.$$

Moreover, by Lemma 4.4,

$$\|K(t+\varepsilon,\eta)\|_{r,\infty} \leq C(\Phi(t) - \Phi(\eta))^{\alpha-\beta^{\tau}-1},$$

uniformly for ε small.

To be precise, applying Young's inequality in Lorentz spaces bounds the norm of the integrand of I_2 by a function that is integrable over the domain of integration and independent of ε. Since the integrand converges to zero pointwise almost everywhere as $\varepsilon \to 0^+$, the Lebesgue dominated convergence theorem justifies passing the limit inside the integral. Hence, we conclude that

$$\|I_2\|_{\tau,s} \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Continuity at $t = 0$.

Using Lemma 4.4, we obtain

$$\|\psi_u(t)\|_{\tau,s} \le C \int_0^t (\Phi(t) - \Phi(\eta))^{\alpha-\beta^\tau-1} \|F(\eta)\|_{\rho,s} \varphi'(\eta)\, \mathrm{d}\eta.$$

Estimating separately the contribution of the forcing term and that of the nonlinear term, we obtain

$$\|\psi_u(t)\|_{\tau,s} \le C\Phi(t)^{\alpha+\sigma-\beta^\tau} \|\theta\|_{\rho,s} + C\Phi(t)^{\alpha-p\beta^{p\rho}-\beta^\tau}.$$

Under the explicit assumptions of the lemma, both exponents $\alpha + \sigma - \beta^\tau$ and $\alpha - p\beta^{p\rho} - \beta^\tau$ are strictly positive, hence

$$\|\psi_u(t)\|_{\tau,s} \to 0 \quad \text{as } t \to 0^+.$$

This completes the proof. □

We are now in a position to state the main result concerning the local well-posedness of problem (4.1).

Theorem 4.8. *Let $1 < \alpha < 2$, $p > 1$, $\sigma \in (-1, 0)$, and $1 < \rho < \infty$. Assume that*

$$u_0 \in L^{\rho,s}(\mathbb{R}^d), \qquad u_1 \in L^{\rho,s}(\mathbb{R}^d), \qquad \theta \in L^{\rho,s}(\mathbb{R}^d).$$

Let $1 < \rho < \tau \le p\rho$ satisfy

$$0 < \frac{1}{\rho} - \frac{1}{\tau} < \min\left\{1, \frac{2\delta}{d}\right\},$$

and assume in addition that

$$\beta^\tau < \alpha + \sigma, \qquad \alpha - p\beta^{p\rho} - \beta^\tau > 0.$$

Then there exists $T > 0$ such that problem (4.1) admits a unique mild solution

$$u \in C([0, T], L^{\rho,s}(\mathbb{R}^d)) \cap C((0, T], L^{\tau,s}(\mathbb{R}^d)),$$

and

$$\sup_{0<t\le T} \Phi(t)^{\beta^\tau} \|u(t)\|_{\tau,s} < \infty.$$

Proof. The proof is based on a fixed-point argument in a suitable Lorentz space.

Step 1.

Let
$$X_T = \Big\{ u \in C([0,T], L^{\rho,s}) \cap C((0,T], L^{\tau,s}) : \, \|u\|_{X_T} < \infty \Big\},$$
where
$$\|u\|_{X_T} := \sup_{t\in[0,T]} \|u(t)\|_{\rho,s} + \sup_{0<t\leq T} \Phi(t)^{\beta^\tau} \|u(t)\|_{\tau,s}.$$

Equipped with this norm, X_T is a Banach space.

For $u \in X_T$ we define
$$(\mathcal{T}u)(t) = G_0^{\alpha,\varphi}(t) * u_0 + G_1^{\alpha,\varphi}(t) * u_1 + \psi_u(t),$$
where
$$\psi_u(t) = \int_0^t G_g^{\alpha,\varphi}\Big(\varphi^{-1}(\varphi(t)+\varphi(0)-\varphi(\eta))\Big) * \Big[(\varphi(\eta)-\varphi(0))^\sigma \theta + |u(\eta)|^p\Big] \varphi'(\eta)\, d\eta.$$

By Lemma 4.5, the map $u \mapsto \psi_u$ is well defined from X_T into $C([0,T], L^{\tau,s})$. Hence $\mathcal{T}$ maps X_T into itself.

Step 2. Estimates.

Using the Lorentz estimates of the kernels (Lemma 4.3), we obtain
$$\|G_0^{\alpha,\varphi}(t) * u_0\|_{\tau,s} \leq C\Phi(t)^{-\beta^\tau} \|u_0\|_{\rho,s},$$
and
$$\|G_1^{\alpha,\varphi}(t) * u_1\|_{\tau,s} \leq C\Phi(t)^{1-\beta^\tau} \|u_1\|_{\rho,s}.$$
Consequently
$$\sup_{t\in[0,T]} \Phi(t)^{\beta^\tau} \|G_0^{\alpha,\varphi}(t) * u_0\|_{\tau,s} \leq C\|u_0\|_{\rho,s},$$
and
$$\sup_{t\in[0,T]} \Phi(t)^{\beta^\tau} \|G_1^{\alpha,\varphi}(t) * u_1\|_{\tau,s} \leq C\Phi(T)\|u_1\|_{\rho,s}.$$

Using again the kernel estimates and the identity
$$\||u|^p\|_{\rho,s} = \|u\|^p_{p\rho,ps},$$
we obtain
$$\|\psi_u(t)\|_{\tau,s} \leq C\int_0^t (\Phi(t)-\Phi(\eta))^{\alpha-\beta^\tau-1}\Big(\Phi(\eta)^\sigma + \|u(\eta)\|^p_{p\rho,ps}\Big)\varphi'(\eta)\, d\eta.$$

Using the definition of the norm in X_T, this yields

$$\|\psi_u\|_{X_T} \le C\Big(\|\theta\|_{\rho,s} + \|u\|_{X_T}^p\Big)\Phi(T)^{\mu},$$

where $\mu = \min\Big\{\alpha + \sigma - \beta^{\tau},\ \alpha - p\beta^{p\rho} - \beta^{\tau}\Big\}$.

By the explicit assumptions of the theorem, both $\alpha + \sigma > \beta^{\tau}$ and $\alpha - p\beta^{p\rho} > \beta^{\tau}$, which rigorously guarantees that $\mu > 0$.

Let

$$B_R = \{u \in X_T : \|u\|_{X_T} \le R\}.$$

Combining the previous estimates gives

$$\|\mathcal{T}u\|_{X_T} \le C\Big(\|u_0\|_{\rho,s} + \|u_1\|_{\rho,s} + \|\theta\|_{\rho,s}\Big) + CR^p\Phi(T)^{\mu}.$$

Choosing R large enough and T sufficiently small, we ensure that

$$\mathcal{T}(B_R) \subset B_R.$$

Step 3. Contraction property.

For $u, v \in B_R$, we have

$$\||u|^p - |v|^p\| \le C(|u|^{p-1} + |v|^{p-1})|u - v|.$$

Using again the kernel estimates, we obtain

$$\|\mathcal{T}u - \mathcal{T}v\|_{X_T} \le CR^{p-1}\Phi(T)^{\mu}\|u - v\|_{X_T}.$$

Taking T sufficiently small, the constant on the right-hand side becomes strictly less than 1, and therefore $\mathcal{T}$ is a strict contraction on B_R.

By Banach's fixed-point theorem, $\mathcal{T}$ admits a unique fixed point in B_R. This fixed point is the unique mild solution of (4.1) in X_T.

The proof is complete. □

Corollary 4.1 (Continuous dependence on the data)**.** *Under the assumptions of Theorem 4.8, the mild solution depends continuously on the initial data* (u_0, u_1, θ) *in* $L^{\rho,s}(\mathbb{R}^d)$.

More precisely, let

$$(u_0^{(n)}, u_1^{(n)}, \theta^{(n)}) \to (u_0, u_1, \theta) \quad \text{in } L^{\rho,s}(\mathbb{R}^d) \times L^{\rho,s}(\mathbb{R}^d) \times L^{\rho,s}(\mathbb{R}^d),$$

and let $u^{(n)}$ *and* u *be the corresponding mild solutions on* $[0,T]$*. Then*

$$u^{(n)} \to u \quad \text{in } X_T,$$

where X_T *is the Banach space introduced in the proof of Theorem 4.8.*

In particular,

$$u^{(n)} \to u \quad \text{in } C([0,T], L^{\rho,s}(\mathbb{R}^d)).$$

Proof. The result follows from the contraction mapping argument used in the proof of Theorem 4.8. Indeed, the solution operator $\mathcal{T}$ depends continuously on the data (u_0, u_1, θ), and its unique fixed point in X_T therefore depends continuously on these data as well. □

Blow-up alternative

Theorem 4.9 (Blow-up alternative)**.** *Let the assumptions of Theorem 4.8 be satisfied, and let*

$$u \in C([0,T_{\max}), L^{\rho,s}(\mathbb{R}^d)) \cap C((0,T_{\max}), L^{p\rho,ps}(\mathbb{R}^d))$$

be the maximal mild solution of problem (4.1)*, defined on its maximal interval of existence* $[0,T_{\max})$*, where* $0 < T_{\max} \le \infty$*.*

If $T_{\max} < \infty$*, then*

$$\limsup_{t \uparrow T_{\max}} \left(\|u(t)\|_{\rho,s} + \Phi(t)^{\beta^{p\rho}} \|u(t)\|_{p\rho,ps} \right) = \infty,$$

where

$$\beta^{p\rho} = \frac{\alpha d}{2\delta} \left(\frac{1}{\rho} - \frac{1}{p\rho} \right).$$

Equivalently, if

$$\sup_{0<t<T_{\max}} \left(\|u(t)\|_{\rho,s} + \Phi(t)^{\beta^{p\rho}} \|u(t)\|_{p\rho,ps} \right) < \infty,$$

then the solution can be extended beyond $T_{\max}$*.*

Proof. Assume that $T_{\max} < \infty$ and

$$\sup_{0<t<T_{\max}} \left(\|u(t)\|_{\rho,s} + \Phi(t)^{\beta^{p\rho}} \|u(t)\|_{p\rho,ps} \right) \le M < \infty.$$

Let $T < T_{\max}$. By the mild formulation, the solution on $[0,T]$ satisfies the same integral equation as in the local existence result. Moreover, the uniform bound implies that the nonlinear term remains bounded in the corresponding weighted space.

Repeating the contraction argument used in Theorem 4.8, with constants depending only on M and $\Phi(T_{\max})$, we obtain that the solution can be extended beyond T by a fixed time step independent of T.

By iterating this extension procedure, we extend the solution beyond $T_{\max}$, which contradicts maximality.

Therefore,

$$\limsup_{t\uparrow T_{\max}} \left(\|u(t)\|_{\rho,s} + \Phi(t)^{\beta^{p\rho}} \|u(t)\|_{p\rho,ps}\right) = \infty.$$

□

Remarks on global behavior

The linear estimates derived in Section 4.1 suggest a natural framework for the study of global-in-time small-data solutions. In particular, the quantity

$$\beta^{p\rho} = \frac{\alpha d}{2\delta}\left(\frac{1}{\rho} - \frac{1}{p\rho}\right)$$

appears as the natural decay exponent associated with the nonlinear term in the weighted Lorentz-space setting.

However, in the present φ-fractional framework, the passage from the local fixed-point argument to a genuinely global small-data theory requires additional global-in-time estimates that are not completely captured by the local contraction scheme used above. This is especially delicate in the presence of the forcing term

$$\Phi(t)^{\sigma}\theta, \qquad \sigma \in (-1, 0),$$

whose long-time contribution is not automatically compatible with uniform decay.

For this reason, the rigorous results established in this thesis are the local well-posedness statement of Theorem 4.8, the continuous dependence result in Corollary 4.1, and the blow-up alternative in Theorem 4.9. The global behavior suggested by the linear decay estimates is discussed below at a formal level through the associated scaling condition.

Remark 4.2. A complete global small-data theory in the present setting would require an additional argument yielding uniform control of the nonlinear Duhamel term on $(0, \infty)$, possibly under stronger assumptions on the forcing profile θ, on the time scale φ, or in a different weighted solution space. We leave this question for future work.

Scaling insight and candidate critical exponent

We conclude with a formal discussion of the scaling suggested by the linear estimates.

For $\tau \in [\rho, p\rho]$, the linear analysis yields the decay profile

$$\|u(t)\|_{\tau,s} \lesssim \Phi(t)^{-\beta^\tau},$$

where

$$\beta^\tau = \frac{\alpha d}{2\delta}\left(\frac{1}{\rho} - \frac{1}{\tau}\right).$$

In particular, for $\tau = p\rho$,

$$\beta^{p\rho} = \frac{\alpha d}{2\delta}\left(\frac{1}{\rho} - \frac{1}{p\rho}\right) = \frac{\alpha d}{2\delta}\frac{p-1}{p\rho}.$$

Balancing the decay predicted by the linear part with the strength of the nonlinearity formally leads to the condition

$$p\beta^{p\rho} = 1.$$

Solving this relation for p gives

$$p_c(\rho) = 1 + \frac{2\delta\,\rho}{\alpha d}.$$

We emphasize that this exponent is obtained here as a *candidate threshold suggested by the weighted Lorentz-space scaling.* Establishing a sharp dichotomy between global existence and blow-up at this level of generality remains open in the present φ-fractional setting.

Chapter 5

Blow-up for the fractional superdiffusion equation

In this chapter, we investigate the finite-time blow-up phenomenon for solutions to the Cauchy problem associated with the time–space fractional superdiffusion equation

$$\begin{cases} {}^{C}\mathrm{D}^{\alpha,\varphi}_{0,t} u + (-\Delta)^{\delta}_{x} u = |u|^{p}, \quad (t,x) \in (0,+\infty) \times \mathbb{R}^{d}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^{d}, \\ \partial_{\varphi} u(0,x) = u_1(x), \quad x \in \mathbb{R}^{d}, \end{cases} \tag{5.1}$$

where $\alpha \in (1,2)$, $\delta \in (0,1)$, $p > 1$, and $d \in \mathbb{N}$. The initial data satisfy

$$u_0, u_1 \in C_0(\mathbb{R}^d).$$

The operator ${}^{C}\mathrm{D}^{\alpha,\varphi}_{0,t}$ denotes the φ-Caputo fractional derivative, $(-\Delta)^{\delta}_{x}$ is the fractional Laplacian, and

$$\partial_{\varphi} = \frac{1}{\varphi'(t)} \frac{\mathrm{d}}{\mathrm{d}t}$$

is the φ-derivative.

The equation combines a time-fractional dynamics of order $\alpha \in (1,2)$, a nonlocal spatial diffusion operator of order $\delta \in (0,1)$, and a power-type source term. Since $\alpha > 1$, the problem requires two initial conditions and exhibits a superdiffusive behavior that is qualitatively different from the subdiffusive case. The main purpose of this chapter is to analyze the blow-up mechanism for solutions and to derive sufficient conditions under which global-in-time solutions cannot exist.

The study of finite-time blow-up for nonlinear diffusion equations goes back to the classical work of Fujita [25], who investigated the semilinear heat equation with a power source term and identified a critical exponent separating global existence from blow-up phenomena. Since then, this topic has been extensively developed, especially for integer-order parabolic equations and related nonlinear evolution models; see, for instance, the survey work of Hu [30].

Further developments considered nonlocal and memory effects in diffusion equations. Cazenave et al. [16] studied a semilinear heat equation with nonlinear memory and obtained conditions for both finite-time blow-up and global existence. Later, Fino and Kirane [24] extended this type of analysis to models involving the fractional Laplacian, showing how spatial nonlocality influences the blow-up mechanism. These works form an important background for equations where the diffusion operator is no longer the classical Laplacian.

Time-fractional models have also attracted considerable attention in the study of blow-up. Zhang and Sun [67] investigated finite-time blow-up and global existence for a time-fractional semilinear heat equation involving the standard Caputo derivative. In the subdiffusive range $0 < \alpha < 1$, related time–space fractional equations with the φ-Caputo derivative and the fractional Laplacian were studied in [41], where blow-up of mild solutions was proved under suitable restrictions on the exponent p and nontrivial nonnegative initial data.

The superdiffusive case $1 < \alpha < 2$ is more delicate because of the presence of two initial data and the mixed diffusive–wave nature of the equation. In particular, when $\varphi(t) = t$ and $\delta = 1$, earlier results [66] established finite-time blow-up under different assumptions on the initial data u_0 and u_1, with critical ranges of the exponent p depending on the dimension and on the fractional order α. The present chapter extends this direction by considering the more general φ-Caputo framework together with the fractional Laplacian $(-\Delta)^\delta$.

The chapter is organized as follows. We first recall the representation formula for the associated linear problem together with the relevant kernel properties. We then introduce the notions of mild and weak solutions and show that every mild solution is also a weak solution. Finally, we establish several sufficient conditions for finite-time blow-up and conclude with a numerical example illustrating the qualitative behavior of the solutions.

5.1 Fundamental solution of the linear problem

We begin by recalling the representation formula obtained in Chapter 4 for the linear nonhomogeneous problem

$$\begin{cases} {}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi} u + (-\Delta)_x^{\delta} u = g(t,x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \\ \partial_{\varphi} u(0,x) = u_1(x). \end{cases} \tag{5.2}$$

Theorem 5.1. *The solution to problem* (5.2) *is given by*

$$\begin{aligned} u(t,x) &= G_0^{\alpha,\varphi}(t,\cdot) * u_0(x) + G_1^{\alpha,\varphi}(t,\cdot) * u_1(x) \\ &\quad + \int_0^t G_g^{\alpha,\varphi}\Big(\varphi^{-1}\Big(\varphi(t) + \varphi(0) - \varphi(\tau)\Big), \cdot\Big) * g(\tau,\cdot)(x)\, \varphi'(\tau)\, \mathrm{d}\tau, \end{aligned} \tag{5.3}$$

where

$$G_0^{\alpha,\varphi}(t,x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle x,\zeta\rangle} E_{\alpha,1}\Big(-|\zeta|^{2\delta}(\varphi(t) - \varphi(0))^{\alpha}\Big)\, \mathrm{d}\zeta, \tag{5.4}$$

$$G_1^{\alpha,\varphi}(t,x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle x,\zeta\rangle} (\varphi(t) - \varphi(0)) E_{\alpha,2}\Big(-|\zeta|^{2\delta}(\varphi(t) - \varphi(0))^{\alpha}\Big)\, \mathrm{d}\zeta, \tag{5.5}$$

and

$$G_g^{\alpha,\varphi}(t,x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle x,\zeta\rangle} (\varphi(t) - \varphi(0))^{\alpha-1} E_{\alpha,\alpha}\Big(-|\zeta|^{2\delta}(\varphi(t) - \varphi(0))^{\alpha}\Big)\, \mathrm{d}\zeta. \tag{5.6}$$

Here $E_{\alpha,1}$, $E_{\alpha,2}$, and $E_{\alpha,\alpha}$ denote the Mittag–Leffler functions.

Lemma 5.1. *The kernels $G_1^{\alpha,\varphi}$, $G_0^{\alpha,\varphi}$, and $G_g^{\alpha,\varphi}$, defined by* (5.4)–(5.6)*, satisfy*

1.
$$G_1^{\alpha,\varphi}(t,x) = \frac{\varphi(t) - \varphi(0)}{\pi^{d/2}|x|^d} H_{2,3}^{2,1}\left[\frac{|x|^{2\delta}}{2^{2\delta}(\varphi(t) - \varphi(0))^{\alpha}} \middle|_{\left(\frac{d}{2},\delta\right),(1,1),(1,\delta)}^{(1,1),(2,\alpha)} \right],$$

2.
$$G_0^{\alpha,\varphi}(t,x) = \frac{1}{\pi^{d/2}|x|^d} H_{2,3}^{2,1}\left[\frac{|x|^{2\delta}}{2^{2\delta}(\varphi(t) - \varphi(0))^{\alpha}} \middle|_{\left(\frac{d}{2},\delta\right),(1,1),(1,\delta)}^{(1,1),(1,\alpha)} \right],$$

3.
$$G_g^{\alpha,\varphi}(t,x) = \frac{(\varphi(t) - \varphi(0))^{\alpha-1}}{\pi^{d/2}|x|^d} H_{2,3}^{2,1}\left[\frac{|x|^{2\delta}}{2^{2\delta}(\varphi(t) - \varphi(0))^{\alpha}} \middle|_{\left(\frac{d}{2},\delta\right),(1,1),(1,\delta)}^{(1,1),(\alpha,\alpha)} \right].$$

5.2 Auxiliary lemmas

Lemma 5.2. *Let* $\alpha \in (1,2)$ *and* $\delta \in (0,1)$*. Then:*

1. $G_0^{\alpha,\varphi}(t,x) > 0$*,* $G_1^{\alpha,\varphi}(t,x) > 0$*, and* $G_g^{\alpha,\varphi}(t,x) > 0$*.*

2. $$\mathrm{I}_{0,t}^{2-\alpha,\varphi} G_g^{\alpha,\varphi}(t,x) = G_1^{\alpha,\varphi}(t,x).$$

Proof. We prove the two assertions separately.

1. The positivity of $G_0^{\alpha,\varphi}$, $G_1^{\alpha,\varphi}$, and $G_g^{\alpha,\varphi}$ follows from the same argument as in [34, Theorem 1].

2. Using the explicit Fox H-function representation of $G_g^{\alpha,\varphi}$ and the fractional integration formula for powers, we obtain

$$\begin{aligned}
\mathrm{I}_{0,t}^{2-\alpha,\varphi} G_g^{\alpha,\varphi}(t,x) &= \mathrm{I}_{0,t}^{2-\alpha,\varphi}\left[\frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\pi^{d/2}|x|^d} H_{2,3}^{2,1}\left[\frac{|x|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}}\middle|_{\left(\frac{d}{2},\delta\right),(1,1),(1,\delta)}^{(1,1),(\alpha,\alpha)}\right]\right] \\
&= \frac{\varphi(t)-\varphi(0)}{\pi^{d/2}|x|^d} H_{2,3}^{2,1}\left[\frac{|x|^{2\delta}}{2^{2\delta}(\varphi(t)-\varphi(0))^{\alpha}}\middle|_{\left(\frac{d}{2},\delta\right),(1,1),(1,\delta)}^{(1,1),(2,\alpha)}\right] \\
&= G_1^{\alpha,\varphi}(t,x).
\end{aligned}$$

This completes the proof. □

For notational convenience, we shall often omit the spatial variable and write

$$G_0^{\alpha,\varphi}(t) = G_0^{\alpha,\varphi}(t,x), \quad G_1^{\alpha,\varphi}(t) = G_1^{\alpha,\varphi}(t,x), \quad G_g^{\alpha,\varphi}(t) = G_g^{\alpha,\varphi}(t,x),$$

and similarly $u(t) = u(t,x)$, whenever no confusion is likely to arise.

Lemma 5.3. *Let* $\alpha \in (1,2)$*,* $\delta \in (0,1)$*, and* $u_0, u_1 \in C_0(\mathbb{R}^d)$*. Then, for every* $t > 0$*,*

$$G_0^{\alpha,\varphi}(t) * u_0 + G_1^{\alpha,\varphi}(t) * u_1 \in C_0(\mathbb{R}^d),$$

and

$${}^{C}\mathrm{D}_{0,t}^{\alpha,\varphi}\left(G_0^{\alpha,\varphi}(t) * u_0 + G_1^{\alpha,\varphi}(t) * u_1\right) = -(-\Delta)_x^{\delta}\left(G_0^{\alpha,\varphi}(t) * u_0 + G_1^{\alpha,\varphi}(t) * u_1\right), \qquad t > 0.$$

Proof. The continuity and vanishing at infinity follow from the continuity of the kernels together with the assumption $u_0, u_1 \in C_0(\mathbb{R}^d)$. Moreover, by the representation formula (5.3), the function

$$G_0^{\alpha,\varphi}(t) * u_0 + G_1^{\alpha,\varphi}(t) * u_1$$

is precisely the solution of (5.2) in the case $g \equiv 0$. Therefore, it satisfies the stated identity. □

Lemma 5.4. *Let* $\alpha \in (1,2)$, $\delta \in (0,1)$, *and* $k > 1$. *If*

$$f \in L^k\Big((0,T), C_0(\mathbb{R}^d)\Big), \qquad T > 0,$$

and

$$\hbar(t) = \int_0^t \varphi'(\tau)\, G_g^{\alpha,\varphi}\Big(\varphi^{-1}[\varphi(t)+\varphi(0)-\varphi(\tau)]\Big) * f(\tau)\,\mathrm{d}\tau,$$

then

$$\mathrm{I}_{0,t}^{2-\alpha,\varphi}\hbar(t) = \int_0^t \varphi'(\tau)\, G_1^{\alpha,\varphi}\Big(\varphi^{-1}[\varphi(t)+\varphi(0)-\varphi(\tau)]\Big) * f(\tau)\,\mathrm{d}\tau.$$

Proof. By Definition 1.6 and Lemma 5.2,

$$\begin{aligned}
\mathrm{I}_{0,t}^{2-\alpha,\varphi}\hbar(t) &= \frac{1}{\Gamma(2-\alpha)}\int_0^t (\varphi(t)-\varphi(\tau))^{1-\alpha}\hbar(\tau)\,\varphi'(\tau)\,d\tau \\
&= \frac{1}{\Gamma(2-\alpha)}\int_0^t\int_w^t (\varphi(t)-\varphi(\tau))^{1-\alpha}\varphi'(w)\varphi'(\tau)G_g^{\alpha,\varphi}\Big(\varphi^{-1}[\varphi(\tau)+\varphi(0)-\varphi(w)]\Big) \\
&\qquad * f(w)\,\mathrm{d}\tau\,\mathrm{d}w \\
&= \int_0^t \varphi'(w)G_1^{\alpha,\varphi}\Big(\varphi^{-1}[\varphi(t)+\varphi(0)-\varphi(w)]\Big) * f(w)\,\mathrm{d}w.
\end{aligned}$$

This proves the claim. □

The following fractional integration by parts formula will also be used:

$$\int_a^T \varphi'(t)\,g(t)\,{}^C\mathrm{D}_{a,t}^{\alpha,\varphi}f(t)\,\mathrm{d}t = \int_a^T \varphi'(t)\,(f(t)-(\varphi(t)-\varphi(a))\partial_\varphi f(a)-f(a))\,{}^C\mathrm{D}_{t,T}^{\alpha,\varphi}g(t)\,\mathrm{d}t, \tag{5.7}$$

provided that $1<\alpha<2$, the operators $\partial_\varphi f$ and $\partial_\varphi g$ are defined almost everywhere, and

$$g(T)=0, \qquad \partial_\varphi g(T)=0.$$

For a given $T>0$ and $k>0$, define

$$g(t) = \begin{cases} \left(1-\dfrac{\varphi(t)-\varphi(0)}{\varphi(T)-\varphi(0)}\right)^k, & t\le T, \\[2ex] 0, & t > T. \end{cases}$$

Then a direct computation yields, for $1<\alpha<2$,

$${}^C\mathrm{D}_{t,T}^{\alpha,\varphi}g(t) = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}(\varphi(T)-\varphi(0))^{-\alpha}\left(1-\frac{\varphi(t)-\varphi(0)}{\varphi(T)-\varphi(0)}\right)^{k-\alpha}, \qquad t\le T.$$

Since $(-\Delta)_x^{\delta}$ is self-adjoint; see [24], it satisfies

$$\int_{\mathbb{R}^d} \phi(x)\,(-\Delta)_x^{\delta}\psi(x)\,\mathrm{dx} = \int_{\mathbb{R}^d} \psi(x)\,(-\Delta)_x^{\delta}\phi(x)\,\mathrm{dx},$$

for any $\phi, \psi \in H^{2\delta}(\mathbb{R}^d)$. Moreover, for any $q \geq 1$ and any nonnegative $\chi \in \mathcal{S}(\mathbb{R}^d)$, one has

$$(-\Delta)_x^{\delta}\chi^q(x) \leq q\,\chi^{q-1}(x)\,(-\Delta)_x^{\delta}\chi(x),$$

(see [24]).

5.3 Blow-up

This section is devoted to the study of finite-time blow-up for solutions of Eq. (5.1). We first introduce the notions of mild and weak solutions and then derive several sufficient conditions for finite-time blow-up.

Definition 5.1. Let $\alpha \in (1,2)$, $\delta \in (0,1)$, $p > 1$, and suppose $u_0, u_1 \in C_0(\mathbb{R}^d)$. A function

$$u \in C\Big([0,T], C_0(\mathbb{R}^d)\Big), \qquad T > 0,$$

is called a *mild solution* of Eq. (5.1) if

$$\begin{aligned} u(t) = {} & G_0^{\alpha,\varphi}(t) * u_0 + G_1^{\alpha,\varphi}(t) * u_1 \\ & + \int_0^t \varphi'(\tau)\, G_g^{\alpha,\varphi}\Big(\varphi^{-1}[\varphi(t) + \varphi(0) - \varphi(\tau)]\Big) * |u(\tau)|^p\,\mathrm{d}\tau. \end{aligned}$$

Definition 5.2. Let $\alpha \in (1,2)$, $\delta \in (0,1)$, $p > 1$, and assume $u_0, u_1 \in L^2(\mathbb{R}^d)$. A function

$$u \in L^p\Big((0,T), L^{2p}(\mathbb{R}^d)\Big) \cap L^1\Big((0,T), L^2(\mathbb{R}^d)\Big)$$

is called a *weak solution* of Eq. (5.1) if

$$\begin{aligned} & \int_{\mathbb{R}^d}\int_0^T \varphi'(t)\Big[|u|^p\,\psi + \Big(u_0 + (\varphi(t) - \varphi(0))u_1\Big)\,{}^C\mathrm{D}_{t,T}^{\alpha,\varphi}\psi\Big]\,\mathrm{dt}\,\mathrm{dx} \\ & \quad = \int_{\mathbb{R}^d}\int_0^T \varphi'(t)\,u\,(-\Delta)_x^{\delta}\psi\,\mathrm{dt}\,\mathrm{dx} + \int_{\mathbb{R}^d}\int_0^T \varphi'(t)\,u\,{}^C\mathrm{D}_{t,T}^{\alpha,\varphi}\psi\,\mathrm{dt}\,\mathrm{dx} \end{aligned}$$

for every compactly supported function

$$\psi \in C^2\Big([0,T], H^{2\delta}(\mathbb{R}^d)\Big)$$

such that $\partial_\varphi \psi \in C^0([0,T], H^{2\delta}(\mathbb{R}^d))$ and

$$\psi(T,\cdot) \equiv 0, \qquad \partial_\varphi \psi(T,\cdot) \equiv 0.$$

Theorem 5.2. *Let $\alpha \in (1,2)$, $\delta \in (0,1)$, $p > 1$, and suppose $u_0, u_1 \in C_0(\mathbb{R}^d)$. If*

$$u \in C\Big([0,T], C_0(\mathbb{R}^d)\Big)$$

is a mild solution of Eq. (5.1)*, then u is also a weak solution of Eq.* (5.1).

Proof. Starting from Definition 5.1, we have

$$\begin{aligned} u - u_0 - (\varphi(t) - \varphi(0))u_1 &= G_0^{\alpha,\varphi}(t) * u_0 + G_1^{\alpha,\varphi}(t) * u_1 - u_0 - (\varphi(t) - \varphi(0))u_1 \\ &\quad + \int_0^t \varphi'(\tau)\, G_g^{\alpha,\varphi}\Big(\varphi^{-1}[\varphi(t) + \varphi(0) - \varphi(\tau)]\Big) * |u(\tau)|^p \,\mathrm{d}\tau. \end{aligned}$$

Applying $\mathrm{I}_{0,t}^{2-\alpha,\varphi}$ to both sides and using Lemma 5.4, we obtain

$$\begin{aligned} &\mathrm{I}_{0,t}^{2-\alpha,\varphi}\,(u - u_0 - (\varphi(t) - \varphi(0))u_1) \\ &\qquad = \mathrm{I}_{0,t}^{2-\alpha,\varphi}\,(G_0^{\alpha,\varphi}(t) * u_0 + G_1^{\alpha,\varphi}(t) * u_1 - u_0 - (\varphi(t) - \varphi(0))u_1) \\ &\qquad\quad + \int_0^t \varphi'(\tau)\, G_1^{\alpha,\varphi}\Big(\varphi^{-1}[\varphi(t) + \varphi(0) - \varphi(\tau)]\Big) * |u(\tau)|^p \,\mathrm{d}\tau. \end{aligned} \tag{5.8}$$

Let $\psi \in C^2([0,T], H^{2\delta}(\mathbb{R}^d))$ be compactly supported, with $\partial_\varphi \psi \in C^0([0,T], H^{2\delta}(\mathbb{R}^d))$, and satisfying

$$\psi(T,\cdot) \equiv 0, \qquad \partial_\varphi \psi(T,\cdot) \equiv 0.$$

Multiplying (5.8) by $\partial_\varphi \psi$, integrating over $\mathbb{R}^d$, and then using Lemmas 5.3 and 5.4, the fractional integration by parts formula, and the self-adjointness of $(-\Delta)_x^\delta$, we arrive at

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_0^T \varphi'(t) \Big[|u|^p\, \psi + \Big(u_0 + (\varphi(t) - \varphi(0))u_1\Big)\, {}^C\mathrm{D}_{t,T}^{\alpha,\varphi} \psi\Big] \,\mathrm{d}t\, dx \\ &\qquad = \int_{\mathbb{R}^d} \int_0^T \varphi'(t)\, u\, (-\Delta)_x^\delta \psi \,\mathrm{d}t\, dx + \int_{\mathbb{R}^d} \int_0^T \varphi'(t)\, u\, {}^C\mathrm{D}_{t,T}^{\alpha,\varphi} \psi \,\mathrm{d}t\, dx. \end{aligned}$$

Hence u is a weak solution. ∎

As usual, a solution u of Eq. (5.1) is said to blow up in finite time T if

$$\lim_{t \to T} \|u(t)\|_{L^\infty(\mathbb{R}^d)} = +\infty.$$

Theorem 5.3. *Let $d \in \mathbb{N}$, $\alpha \in (1,2)$, $\delta \in (0,1)$, and $p > 1$. Suppose that the initial data $u_0, u_1 \in C_0(\mathbb{R}^d)$ satisfy*

$$u_0, u_1 \geq 0,$$

and

$$\int_{\mathbb{R}^d} u_0(x)\eta(x)\,dx > 3^{\frac{1}{p-1}}, \qquad \int_{\mathbb{R}^d} u_1(x)\eta(x)\,dx \geq 0,$$

where

$$\eta(x) = \left(\int_{\mathbb{R}^d} e^{-\sqrt{|x|^2+d^2}}\,dx\right)^{-1} e^{-\sqrt{|x|^2+d^2}}.$$

Then the mild solution of Eq. (5.1) *blows up in finite time.*

Proof. Let ϕ satisfy

$$\phi \in C_0^\infty(\mathbb{R}^d), \qquad \phi(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| \geq 2, \end{cases} \qquad 0 \leq \phi \leq 1. \tag{5.9}$$

Define

$$\psi_n(x,t) = \eta(x)\phi_n(x)\tilde{\psi}(t), \qquad \phi_n(x) = \phi\left(\frac{|x|}{n}\right),$$

where

$$\tilde{\psi}(t) = \left(1 - \frac{\varphi(t)-\varphi(0)}{\varphi(T)-\varphi(0)}\right)^k,$$

with

$$k \geq \max\left\{2, \frac{p\alpha}{p-1}\right\}.$$

Since every mild solution is also a weak solution, Definition 5.2 yields

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_0^T \varphi'(t)\left[u^p\,\psi_n + \Big(u_0 + (\varphi(t)-\varphi(0))u_1\Big)\,{}^C\mathrm{D}_{t,T}^{\alpha,\varphi}\psi_n\right]\mathrm{dt}\,\mathrm{dx} \\ &\qquad = \int_{\mathbb{R}^d}\int_0^T \varphi'(t)\,u\,(-\Delta)_x^\delta\psi_n\,\mathrm{dt}\,dx + \int_{\mathbb{R}^d}\int_0^T \varphi'(t)\,u\,{}^C\mathrm{D}_{t,T}^{\alpha,\varphi}\psi_n\,\mathrm{dt}\,\mathrm{dx}. \end{aligned} \tag{5.10}$$

Using the inequality $(-\Delta)_x^\delta\eta \leq \eta$, see [40, Theorem 4.3], and passing to the limit as $n \to \infty$ by dominated convergence, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_0^T \varphi'(t)\,u^p\,\eta\,\tilde{\psi}\,\mathrm{dt}\,\mathrm{dx} + \int_{\mathbb{R}^d}\int_0^T \varphi'(t)\Big(u_0 + (\varphi(t)-\varphi(0))u_1\Big)\,\eta\,{}^C\mathrm{D}_{t,T}^{\alpha,\varphi}\tilde{\psi}\,\mathrm{dt}\,\mathrm{dx} \\ &\qquad \leq \int_{\mathbb{R}^d}\int_0^T \varphi'(t)\Big(3u\,\eta\,\tilde{\psi} + u\,\eta\,{}^C\mathrm{D}_{t,T}^{\alpha,\varphi}\tilde{\psi}\Big)\,\mathrm{dt}\,\mathrm{dx}. \end{aligned} \tag{5.11}$$

Applying Jensen's inequality and setting

$$g(t) = \int_{\mathbb{R}^d} u(t,x)\eta(x)\,\mathrm{dx},$$

we arrive at

$$\begin{aligned}
&\int_0^T \varphi'(t)\Big(g^p(t) - 3g(t)\Big)\widetilde{\psi}(t)\,\mathrm{dt} \\
&\qquad + \int_0^T \varphi'(t)\Big(g(0) + \tilde{g}(0)(\varphi(t) - \varphi(0))\Big)\,{}^C\mathrm{D}^{\alpha,\varphi}_{t,T}\widetilde{\psi}(t)\,\mathrm{dt} \\
&\leq \int_0^T \varphi'(t)\,g(t)\,{}^C\mathrm{D}^{\alpha,\varphi}_{t,T}\widetilde{\psi}(t)\,\mathrm{dt}.
\end{aligned} \tag{5.12}$$

A direct computation gives

$$\int_0^T \varphi'(t)\,g(0)\,{}^C\mathrm{D}^{\alpha,\varphi}_{t,T}\widetilde{\psi}(t)\,\mathrm{dt} = \frac{\Gamma(k+1)}{\Gamma(k+2-\alpha)} g(0)(\varphi(T) - \varphi(0))^{1-\alpha}. \tag{5.13}$$

Using Young's inequality, we also obtain

$$\int_0^T \varphi'(t)\,g(t)\,{}^C\mathrm{D}^{\alpha,\varphi}_{t,T}\widetilde{\psi}(t)\,\mathrm{dt} \leq \varepsilon \int_0^T \varphi'(t)\,g^p(t)\widetilde{\psi}(t)\,\mathrm{dt} + C(\varepsilon)(\varphi(T) - \varphi(0))^{1-\frac{\alpha p}{p-1}}. \tag{5.14}$$

Substituting (5.13) and (5.14) into (5.12), we obtain

$$\begin{aligned}
&\int_0^T \varphi'(t)\Big(g^p(t) - 3g(t)\Big)\widetilde{\psi}(t)\,\mathrm{dt} + \frac{\Gamma(k+1)}{\Gamma(k+2-\alpha)} g(0)(\varphi(T) - \varphi(0))^{1-\alpha} \\
&\qquad + \frac{\Gamma(k+1)}{\Gamma(k+2-\alpha)} g_1(0)(\varphi(T) - \varphi(0))^{2-\alpha} \\
&\leq \varepsilon \int_0^T \varphi'(t)\,g^p(t)\widetilde{\psi}(t)\,\mathrm{dt} + C(\varepsilon)(\varphi(T) - \varphi(0))^{1-\frac{\alpha p}{p-1}}.
\end{aligned} \tag{5.15}$$

Choosing $\varepsilon > 0$ sufficiently small and using the assumption

$$g(0) = \int_{\mathbb{R}^d} u_0(x)\eta(x)\,dx > 3^{\frac{1}{p-1}},$$

we infer that

$$g(0) \leq C(\varphi(T) - \varphi(0))^{\alpha-\frac{\alpha p}{p-1}}. \tag{5.16}$$

Assume now, by contradiction, that the solution is global. Letting $T \to \infty$ in (5.16) yields $g(0) = 0$, which contradicts the assumption on u_0. Hence the mild solution blows up in finite time. □

We next establish further sufficient conditions for finite-time blow-up. We first consider the case $u_1 \equiv 0$.

Theorem 5.4. *Let $d \in \mathbb{N}$, $\alpha \in (1,2)$, $\delta \in (0,1)$, and suppose that*

$$0 \le u_0 \in C_0(\mathbb{R}^d), \qquad u_0 \not\equiv 0.$$

Then the mild solution of Eq. (5.1) *blows up in finite time, provided that*

$$1 < p < 1 + \frac{2\delta}{d}.$$

Proof. Let ϕ be as in (5.9), and define

$$\chi(x) = \left(\phi\left((\varphi(T)-\varphi(0))^{-\frac{\alpha}{2\delta}}|x|\right)\right)^{\frac{2p}{p-1}}, \qquad \varpi(t) = \left(1 - \frac{\varphi(t)-\varphi(0)}{\varphi(T)-\varphi(0)}\right)^k,$$

where

$$k \ge \max\left\{2, \frac{p\alpha}{p-1}\right\}.$$

Using Theorem 5.2, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_0^T \varphi'(t)\left(u^p\chi\,\varpi + u_0\,\chi\,{}^C\mathrm{D}^{\alpha,\varphi}_{t,T}\varpi\right)\mathrm{dt}\,\mathrm{dx} \\ &\quad= \int_{\mathbb{R}^d}\int_0^T \varphi'(t)\left(u\,(-\Delta)^\delta\chi\,\varpi + u\,\chi\,{}^C\mathrm{D}^{\alpha,\varphi}_{t,T}\varpi\right)\mathrm{dt}\,\mathrm{dx}. \end{aligned} \tag{5.17}$$

By the inequality for $(-\Delta)^\delta_x\chi$ and the estimate for the fractional derivative of ϖ, there exist constants $C_1, C_2 > 0$, independent of T, such that

$$(-\Delta)^\delta_x\chi(x)\,\varpi(t) \le C_1(\varphi(T)-\varphi(0))^{-\alpha}\chi^{1/p}\varpi^{1/p}, \tag{5.18}$$

and

$$\chi\,{}^C\mathrm{D}^{\alpha,\varphi}_{t,T}\varpi(t) \le C_2(\varphi(T)-\varphi(0))^{-\alpha}\chi^{1/p}\varpi^{1/p}. \tag{5.19}$$

Substituting (5.18) and (5.19) into (5.17), and applying Hölder's and Young's inequalities, we obtain

$$\int_{\mathbb{R}^d} u_0(x)\chi(x)\,dx \le C(\alpha,p)(\varphi(T)-\varphi(0))^{\alpha+\frac{\alpha d}{2\delta}-\frac{\alpha p}{p-1}}.$$

In other words,

$$\int_{\mathbb{R}^d} u_0(x)\chi(x)\,dx \le C(\alpha,p)(\varphi(T)-\varphi(0))^{\alpha+\frac{\alpha d}{2\delta}-\frac{\alpha p}{p-1}}. \tag{5.20}$$

Since

$$1 < p < 1 + \frac{2\delta}{d} \quad\Longrightarrow\quad \alpha + \frac{\alpha d}{2\delta} - \frac{\alpha p}{p-1} < 0,$$

letting $T \to \infty$ in (5.20) yields $u_0 \equiv 0$, which contradicts the assumption. Hence the mild solution blows up in finite time. $\square$

We now turn to the case $u_1 \not\equiv 0$.

Theorem 5.5. *Let $d \geq 2$, and assume that*

$$0 \leq u_0, u_1 \in C_0(\mathbb{R}^d), \qquad u_1 \not\equiv 0.$$

If

$$1 < p < 1 + \frac{2\alpha\delta}{\alpha d - 2\delta},$$

then any mild solution of Eq. (5.1) *blows up in finite time.*

Proof. Proceeding exactly as in the proof of Theorem 5.4, but retaining the contribution of u_1, we obtain

$$\int_{\mathbb{R}^d} u_1(x)\chi(x)\,dx \leq C(\alpha,p)(\varphi(T)-\varphi(0))^{\alpha-1+\frac{\alpha d}{2\delta}-\frac{\alpha p}{p-1}}.$$

That is,

$$\int_{\mathbb{R}^d} u_1(x)\chi(x)\,dx \leq C(\alpha,p)(\varphi(T)-\varphi(0))^{\alpha-1+\frac{\alpha d}{2\delta}-\frac{\alpha p}{p-1}}. \tag{5.21}$$

Since

$$1 < p < 1 + \frac{2\alpha\delta}{\alpha d - 2\delta} \quad \Longrightarrow \quad \alpha - 1 + \frac{\alpha d}{2\delta} - \frac{\alpha p}{p-1} < 0,$$

letting $T \to \infty$ in (5.21) yields $u_1 \equiv 0$, which is a contradiction. Hence every mild solution blows up in finite time. $\square$

Example

We consider the nonlinear fractional superdiffusion problem

$$\begin{cases} {}^C D_{0,t}^{1.5,\varphi} u + (-\Delta)^{0.5} u = u^2, \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u(0,x) = u_0(x), \\ \partial_\varphi u(0,x) = u_1(x), \end{cases} \tag{5.22}$$

where

$$\varphi(t) = t^3.$$

Initial data. We choose Gaussian-type initial data

$$u_0(x) = \lambda_0 e^{-\beta x^2}, \tag{5.23}$$

where $\lambda_0 > 0$ and $\beta > 0$ are fixed parameters. This choice ensures that u_0 is smooth, nonnegative, rapidly decaying, and concentrated near the origin.

For the second initial condition, we consider two cases.

Case I: Zero initial velocity.

$$u_1(x) \equiv 0. \tag{5.24}$$

Case II: Nonzero nonnegative initial velocity.

$$u_1(x) = \kappa\, u_0(x), \tag{5.25}$$

where $\kappa > 0$ is a fixed parameter. This choice guarantees that $u_1 \geq 0$ and $u_1 \not\equiv 0$.

Blow-up behavior and comparison. Numerical simulations reveal a clear qualitative difference between the two cases. For the parameter values

$$\alpha = 1.5, \qquad \delta = 0.5, \qquad p = 2,$$

and for suitable choices of λ_0, β, and κ, the computed blow-up times satisfy

$$T^*_{u_1 \equiv 0} \approx 1.1118, \qquad T^*_{u_1 \not\equiv 0} \approx 1.4539.$$

Hence, in this numerical experiment, the case $u_1 \equiv 0$ leads to earlier blow-up, whereas the presence of a nonzero nonnegative second initial condition delays the onset of blow-up.

Figure 5.1 illustrates the qualitative difference between the two cases.

Influence of the second initial condition. For fractional evolution equations of order $\alpha \in (1, 2)$, the solution depends on both the initial state u_0 and the initial velocity $\partial_\varphi u(0) = u_1$. The numerical experiment above suggests that the second initial condition may significantly affect the early-stage dynamics. In the present example, a nonzero nonnegative initial velocity appears to delay the concentration mechanism responsible for blow-up.

Table 5.1: Numerical comparison of the blow-up time T^* for different second initial conditions in the φ-Caputo model.

α	δ	p	Second initial condition u_1	T^*	Behavior
1.5	0.5	2	$u_1 \equiv 0$	$T^* \approx 1.1118$	earlier blow-up
1.5	0.5	2	$u_1(x) = \kappa u_0(x)$	$T^* \approx 1.4539$	delayed blow-up

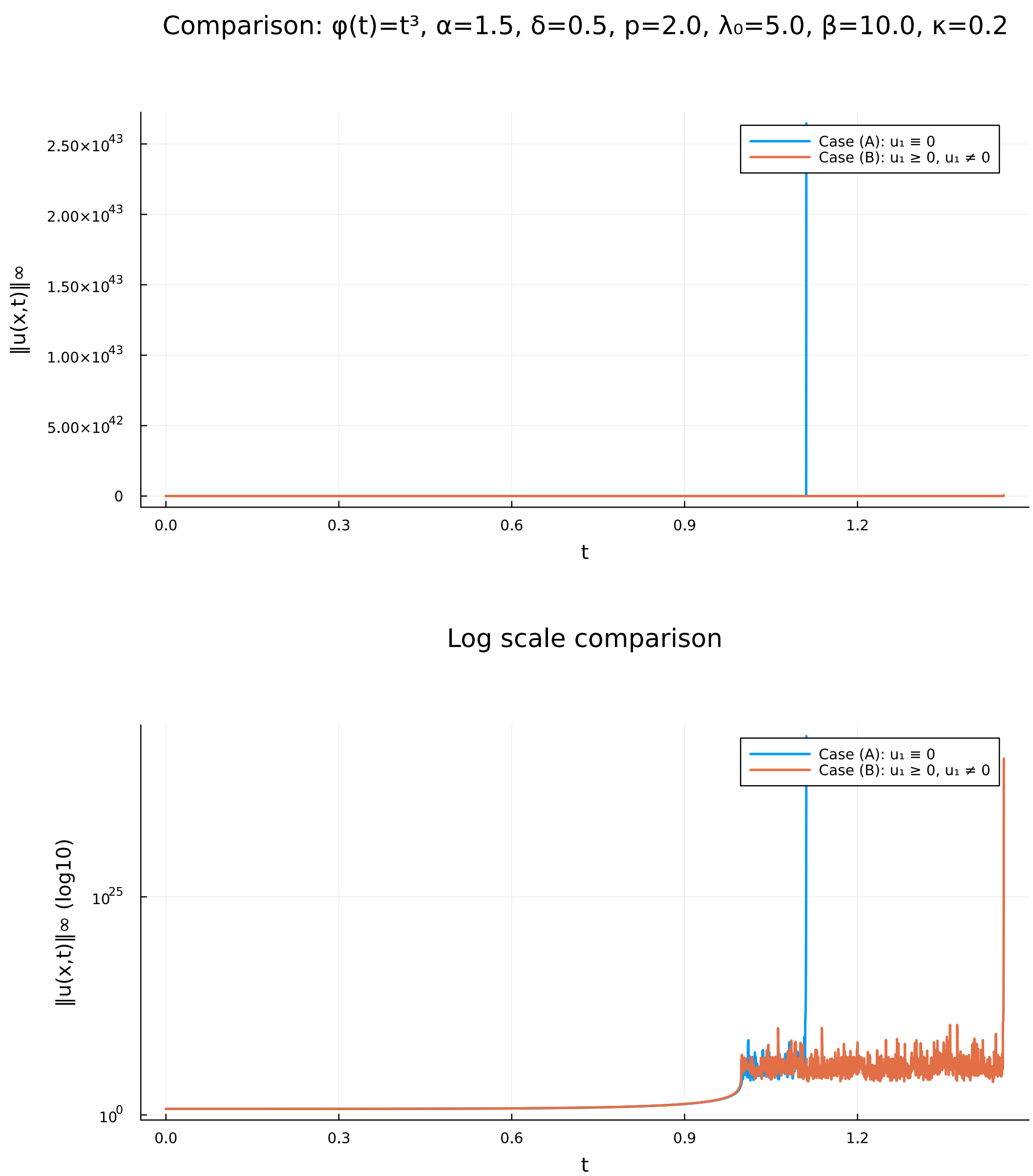

Figure 5.1: Comparison of the blow-up behavior for the φ-Caputo model with $\varphi(t) = t^3$. The figure shows the evolution of the L^∞-norm of the numerical solution for two different second initial conditions: Case (A) $u_1 \equiv 0$ and Case (B) $u_1(x) = \kappa u_0(x)$ with $\kappa > 0$. The parameters are $\alpha = 1.5$, $\delta = 0.5$, $p = 2$, and $u_0(x) = \lambda_0 e^{-\beta x^2}$.

Table 5.1 reports the numerically observed blow-up times for the two choices of u_1. In this simulation, the zero-velocity case blows up earlier, whereas the positive second initial condition delays blow-up. This illustrates the subtle interplay between memory effects and the second initial datum in fractional evolution equations of order $\alpha \in (1, 2)$.

Remark 5.1. The numerical value of the blow-up time T^* depends on the discretization parameters and on the prescribed blow-up threshold. Nevertheless, the qualitative distinction observed between the two cases remains stable under mesh refinement in our computations.

Numerical Illustrations of the Fractional Laplacian Operator

This appendix collects numerical illustrations related to the fractional Laplacian operator. These examples are provided to complement the theoretical developments presented in Chapter 1 and to offer additional insight into the behavior of the operator for specific test functions.

A.1 Fractional Laplacian in One Dimension

In the one-dimensional case ($d = 1$), explicit expressions for the fractional Laplacian can be obtained for certain polynomial-type functions. Table A.1 summarizes the values of $(-\Delta)_x^{\delta} u(x)$ for selected functions defined on the interval $(-1, 1)$.

$u(x)$	$(-\Delta)_x^{\delta} u(x)$
$(1-x^2)^{-1+\delta}$	0
$(1-x^2)^{\delta}$	$-\Gamma(2\delta+1)$
$(1-x^2)^{1+\delta}$	$-\Gamma(2\delta+1)\frac{2\delta+2}{2}\Big(1-(1+2\delta)x^2\Big)$
$(1-x^2)^{2+\delta}$	$-\Gamma(2\delta+1)\frac{2\delta+2}{2}\frac{2\delta+4}{4}\bigg(1-(4\delta+2)x^2+\Big(\frac{2\delta}{3}+1\Big)(2\delta+1)x^4\bigg)$

Table A.1: Fractional Laplacian of selected functions in one dimension.

A.2 Fractional Laplacian of a Gaussian Function

We consider the Gaussian function

$$u(x) = e^{-x^2}, \qquad x \in \mathbb{R},$$

and compute numerically the fractional Laplacian $(-\Delta)^{\delta} u(x)$ for several values of the fractional order $\delta \in (0, 1)$.

In the numerical experiments, the fractional orders

$$\delta \in \{0.1,\ 0.3,\ 0.5,\ 0.8,\ 0.9\}$$

are considered. To reduce boundary effects, the numerical results are displayed only on the interval $[-4, 4]$.

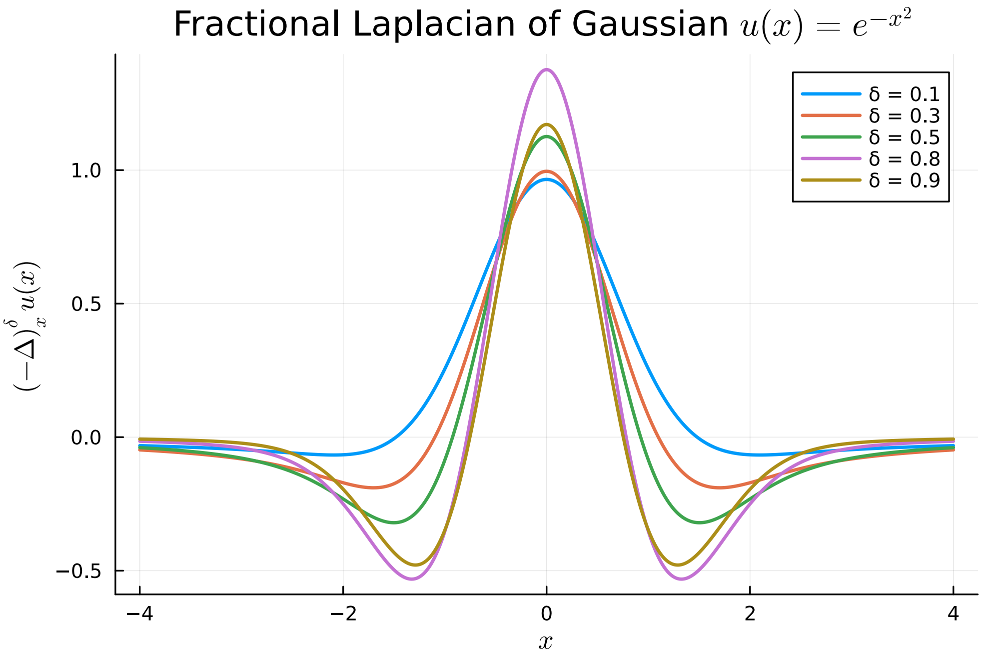

Figure A.1: Numerical approximation of $(-\Delta)^{\delta}_{x} u(x)$ for the Gaussian function $u(x) = e^{-x^2}$ with different values of δ.

A.3 Remarks

These numerical illustrations are not used directly in the analytical proofs developed in the thesis. They are included to provide intuition on the effect of the fractional order and to visualize the nonlocal behavior of the fractional Laplacian.

General conclusion

D In this thesis, we have undertaken a comprehensive study of nonlinear time–space fractional diffusion and superdiffusion equations involving generalized fractional derivatives in the sense of the φ-Caputo operator and spatial nonlocality generated by the fractional Laplacian. By employing advanced analytical tools such as generalized Laplace transforms, Fourier analysis with respect to appropriate kernel functions, and representations via Fox H-functions, we derived explicit fundamental solutions to several classes of associated linear problems. These representations allowed us to establish precise decay estimates in both Lebesgue and Lorentz spaces, thereby providing a deeper understanding of the dispersive and smoothing mechanisms inherent to fractional diffusion models.

Building upon the linear analysis, we developed a rigorous framework for studying nonlinear equations of superdiffusive type. Local existence and uniqueness of mild solutions were proved by combining the derived decay properties with fixed-point arguments in suitably chosen function spaces. Under additional smallness assumptions on the initial data and through the introduction of modified norms, we further demonstrated the global well-posedness of nonlinear fractional diffusion equations governed by φ-Caputo derivatives. These results highlight the delicate interplay between nonlinear effects, fractional time dynamics, and spatial nonlocality.

In the case of nonlinear superdiffusion equations with both fractional temporal and spatial operators, we constructed mild and weak solutions and established their equivalence. Moreover, we investigated conditions under which solutions exhibit finite-time blow-up, revealing that fractional orders in time and space significantly influence the long-term behavior of solutions.

Overall, the contributions of this thesis provide a unified analytical framework for fractional diffusion and superdiffusion equations with generalized fractional derivatives. The results enrich the existing theory by offering explicit fundamental solutions, refined decay estimates, and rigorous well-posedness and blow-up criteria. These findings may serve as a foundation for future research, including numerical analysis, stochastic interpretations, and applications in physics, biology, and anomalous transport phenomena.

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ملخص: يتناول هذا العمل دراسة تحليلية لمعادلات تفاضلية جزئية كسرية في الزمان والمكان، تتضمن مشتقات كسرية معمّمة ومؤثرات مكانية لا محلية. ويهدف البحث أساسًا إلى دراسة وجود ووحدانية الحلول والسلوك النوعي لها بالنسبة لمسائل كوشي الكسرية، حيث يُستعمل مشتق كسرّي زمني من نوع فاي-كابوتو مقترنًا باللابلاسي الكسرّي كمؤثر مكاني.تعتمد المنهجية المتبعة على بناء حلول أساسية صريحة باستعمال دوال أش فوكس، مما يسمح بتقديم تمثيل موحّد لنماذج الانتشار الكسري والانتشار الفائق. وقد مكّنت هذه التمثيلات من اشتقاق تقديرات دقيقة لتلاشي الحلول في فضاءات لوبيغ ولورنتز مبرزةً الخواص التشتتية والتنظيمية الملازمة للأنظمة الكسرية. وبالاعتماد على النظرية الخطية المتحصّل عليها، تم إثبات الوجود المحلي والعالمي للحلول المعتدلة لمعادلات الانتشار الكسري اللاخطية، وذلك باستعمال مبرهنات النقطة الثابتة في أطر دالية مناسبة. كما تم التوصل إلى شروط كافية لحدوث الانفجار في زمن منتهٍ بالنسبة لمعادلات الانتشار الفائق اللاخطية، حيث أظهرت النتائج أن التفاعل بين اللاخطية ورتب المشتقات الكسرية في الزمان والمكان يلعب دورًا حاسمًا في تحديد السلوك طويل الأمد للحلول.تُسهم النتائج المتحصل عليها في إثراء الإطار النظري لمعادلات التفاضل الجزئية الكسرية ذات المؤثرات المعممة، كما توفّر أساسًا تحليليًا صارمًا يوسّع عددًا من النماذج الكلاسيكية ويفتح آفاقًا بحثية مستقبلية في هذا المجال.

كلمات مفتاحية:المشتق الكسري من نوع فاي-كابوتو، اللابلاسي الكسري، دالة أش فوكس، الوجود العالمي، الانفجار في زمن منتهٍ.

This thesis is devoted to the analytical study of time–space Fractional Partial Differential Equations (FPDEs) involving generalized fractional derivatives and nonlocal spatial operators. The main objective is to investigate the existence, uniqueness, and qualitative behavior of solutions for a class of fractional Cauchy problems governed by the φ-Caputo fractional derivative in time and the fractional Laplacian in space.

The analysis is based on the construction of explicit fundamental solutions, expressed in terms of the Fox H-function, which provides a unified and powerful representation framework for fractional diffusion and superdiffusion models. These representations allow us to derive sharp decay estimates in Lebesgue and Lorentz spaces, highlighting the dispersive and smoothing effects induced by fractional dynamics.

Building on the linear theory, local and global existence results for nonlinear fractional diffusion equations are established using fixed-point arguments in suitable functional spaces. Moreover, sufficient conditions for finite-time blow-up are derived for nonlinear superdiffusion equations, showing that the interplay between nonlinear effects and fractional orders in time and space plays a decisive role in the long-time behavior of solutions. The results presented in this thesis contribute to the theoretical development of fractional partial differential equations with generalized fractional operators and provide a rigorous analytical framework that extends several classical models.

Keywords: φ-Caputo fractional derivative, Fractional Laplacian, Fox H-function, Global existence, Blow-up.

Cette thèse est consacrée à l'étude analytique des équations aux dérivées partielles fractionnaires (EDPF) en temps et en espace, faisant intervenir des dérivées fractionnaires généralisées et des opérateurs spatiaux non locaux. L'objectif principal est d'analyser l'existence, l'unicité et le comportement qualitatif des solutions pour une classe de problèmes de Cauchy fractionnaires gouvernés par la dérivée fractionnaire de type φ-Caputo en temps et par le Laplacien fractionnaire en espace.

L'approche adoptée repose sur la construction de solutions fondamentales explicites, exprimées à l'aide de la fonction de Fox H, offrant un cadre unifié pour l'étude des modèles de diffusion et de superdiffusion fractionnaires. Ces représentations permettent d'établir des estimations de décroissance fines dans les espaces de Lebesgue et de Lorentz, mettant en évidence les effets dispersifs et régularisants induits par la dynamique fractionnaire.

En s'appuyant sur cette analyse linéaire, des résultats d'existence locale et globale de solutions pour des équations de diffusion fractionnaire non linéaires sont obtenus à l'aide de méthodes de point fixe dans des espaces fonctionnels appropriés. De plus, des conditions suffisantes d'explosion en temps fini sont établies pour des équations de superdiffusion fractionnaire non linéaires, montrant que l'interaction entre les effets non linéaires et les ordres fractionnaires en temps et en espace joue un rôle déterminant dans le comportement à long terme des solutions.

Les résultats présentés dans cette thèse contribuent au développement théorique des équations aux dérivées partielles fractionnaires avec opérateurs fractionnaires généralisés et étendent plusieurs modèles classiques de la littérature.

Mots-Clés: dérivée fractionnaire de type φ-Caputo, Laplacien fractionnaire, fonction de Fox H, existence globale, explosion en temps fini.